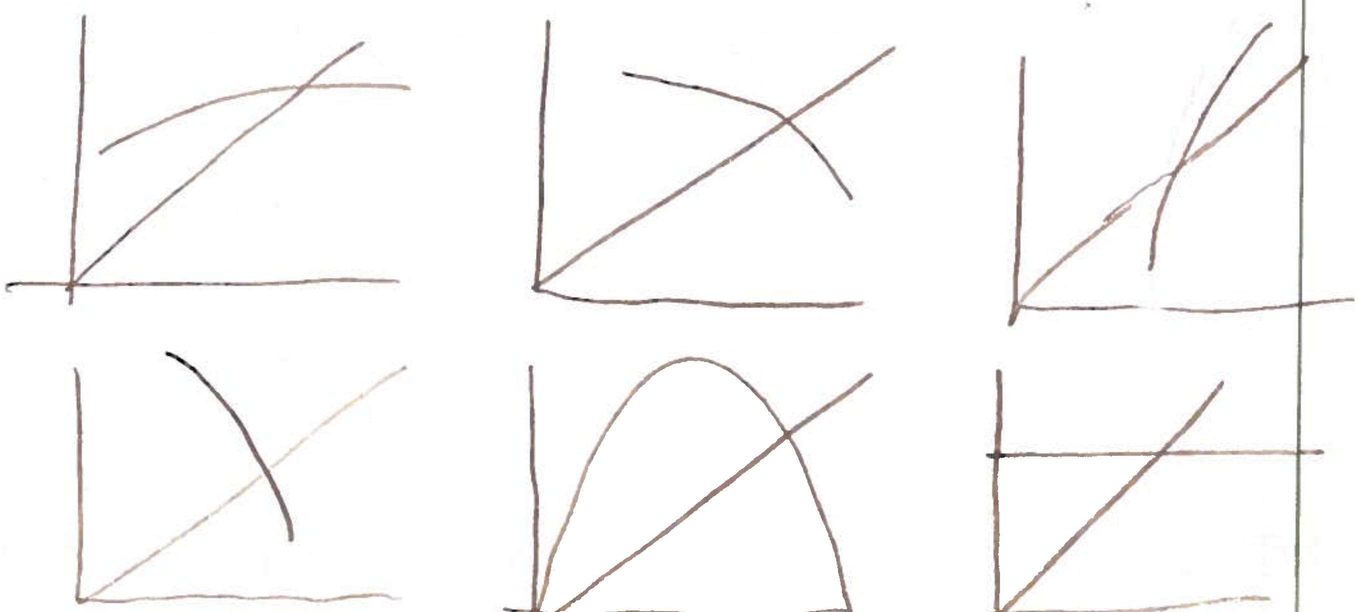
Math 5600

~~5/20/13~~
5/21/14

- Recall Fixed Point iteration

$$x_{k+1} = g(x_k)$$

- α is the fixed point $\alpha = g(\alpha)$
- we had 5 pictures.



- The crucial ingredient seems to be $g'(\alpha)$

- $$\alpha = g(\alpha)$$
$$x_{k+1} = g(x_k)$$

$$e_{k+1} = \alpha - x_{k+1} = g(\alpha) - g(x_k) = g'(c)(\alpha - x_k)$$
new error
$$= g'(c)\, e_k$$

- If $|g'(\alpha)|$ is small close to α we multiply the error by a small factor.

- The best possible value for $g'(\alpha)$ is of course 0

- Return to Newton's Method

$$g(x) = x - \frac{f(x)}{f'(x)} \qquad f(\alpha) = 0$$

- I am going to evaluate at α, and omit "(α)"

$$g' = 1 - \frac{f'^2 - ff''}{f'^2} = 1 - 1 + \frac{ff''}{f'^2} = 0$$

- That's why Newton's Method works so well!

- Suppose $g'(x) < L < 1$ for x in some interval containing α in its interior.

- Then we will get convergence if we start in that interval

- If g' is continuous and $g'(\alpha) < 1$ then such an interval will exist.

- Thus if $|g'(\alpha)| < 1$ and we start sufficiently close to α we will get convergence.

- Return to Newton's Method.

$$g''(x) = \frac{f(x)f''(x)}{f'^2(x)}$$

- what if $f'(\alpha) = 0$

- Apply the Rule of L'Hopital

$$\frac{ff''}{f'^2} \to \frac{f'f'' + ff'''}{2f'f''}$$

$$\xrightarrow[\text{again}]{} \frac{f''^2 + f'f''' + f'f''' + ff^{IV}}{2(f''^2 + f'f'''}$$

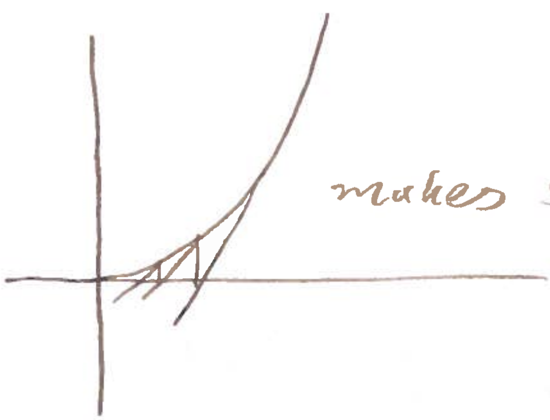
$$\xrightarrow[f=f'=0]{} \frac{f''^2}{2f''^2} = \frac{1}{2}$$

- So we multiply the error with about $\frac{1}{2}$ at each step.
- we still get convergence!
- Let's try it out

$$f(x) = x^2 = 0 \qquad f'(x) = 2x \quad f'(0) = f(0) = 0$$

$$e_{k+1} = x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2}{2x_k} = \frac{x_k}{2} = \frac{e_k}{2}$$

$$\alpha = 0$$



makes sense.

- Newton's Method may be in trouble if we have multiple roots (or several close single roots)

- Fixed point iteration may actually converge faster than Newton's

- More insight can be gained by expanding into a Taylor series.

- Again, evaluation at α

$$
\begin{aligned}
e_{k+1} &= \alpha - x_{k+1} \\
&= \alpha - g(x_k) \\
&= \alpha - \left(\alpha - \sum_{j=0}^{\infty} g^{(j)}(\alpha) \frac{(x_k - \alpha)^j}{j!}\right) \\
&= \sum_{j=1}^{\infty} g^{(j)}(\alpha) \frac{(-e_k)^j}{j!}
\end{aligned}
$$

- The Fixed point iteration

$$x_{k+1} = g(x_k)$$

is said to be "convergent of order p" if

$$g(\alpha) = \alpha \quad g'(\alpha) = \dots = g^{(p-1)}(\alpha) = 0$$

$$\text{and } g^{(p)}(\alpha) \neq 0$$

- In that case $e_{k+1} = c\, e_k^p + H.O.T. = O(e_k^p)$

convergence is said to be

linear $p = 1$ $|g'(\alpha)| < 1$

quadratic $p = 2$

cubic $p = 3$

- Newton's method converges quadratically (unless $f'(\alpha) = 0$)

- $e_{k+1} \approx c e_k^2$

explains why the number of correct digits roughly doubles at each step.

- could Newton's Method be of order greater than 2

$$g'(x) = \frac{f(x) f''(x)}{(f'(x))^2} \qquad g'(\alpha) = 0$$

$$g''(x) = \frac{(f'(x) f''(x) + f(x) f'''(x)) f'(x)^2 - 2 f(x) f''(x) f'(x) f''(x)}{(f'(x))^4}$$

$$g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} = 0 \text{ if } f''(\alpha) = 0$$

- $f''(\alpha) = 0$ means that the linear Taylor approximation of f is the same as the quadratic.

- How can we construct fixed point iterations of high order?

- one way is inverse interpolation

$$f(x) = 0 \qquad F(f(x)) = x$$

F inverse function

- Then $\alpha = F(0)$

suppose $y_k = f(x_k)$

$$\alpha = F(y_k) - y_k F'(y_k) + \frac{1}{2} y_k^2 F''(y_k) + \cdots \qquad (*)$$

Taylor Expansion.

- But what are those derivatives?

- Get them by implicit differentiation

$$F(y_k) = y_k = f(x_k)$$

$$F(f(x)) = x$$

$$F'(f(x))f'(x) = 1 \qquad (**)$$

$$F'(y_k) = \frac{1}{f'(x_k)}$$

- so using just the first two terms in (*) gives

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad \text{Newton's Method}$$

- But we can go on

- Differentiating in (**) gives

$$F''(f(x))f'(x)^2 + F'(f(x))f''(x) = 0$$

$$F''(f(x)) = \frac{-F'(f(x))f''(x)}{(f'(x))^2} = \frac{-f''(x)}{(f'(x))^3}$$

- Using the first 3 terms in (*) gives the third order method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2} f^2(x_k) \frac{f''(x_k)}{f'(x_k)^3}$$

- Example

$$f(x) = x^2 - 2$$

$$f'(x) = 2x$$

$$f''(x) = 2$$

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} - \frac{1}{2} \frac{(x_k^2-2)^2 \cdot 2}{8x_k^3}$$

Carry out with many digits

errors do get cubed at every step.