Math 5600

~~5/24/13~~
5/22/14

Recall fixed point iteration

$$x_{k+1} = g(x_k) \qquad x_0 \text{ given}$$

where $f(x) = 0 \iff x = g(x)$

root   fixed point

suppose the root (or fixed point) is α

- we saw that, with suitable smoothness assumptions, and $e_k = x_k - \alpha$

$$e_{k+1} = x_{k+1} - \alpha = \frac{g^{(p)}(\alpha)}{p!} e_k^p + HOT$$

where

$$g(\alpha) = \alpha, \quad g'(\alpha) = \dots = g^{(p-1)}(\alpha) = 0 \qquad g^{(p)}(\alpha) \neq 0$$

- p is the order of the method
- The iteration will converge to α if we start sufficiently close to α and $p > 1$ or $p = 1$ and $|g'(\alpha)| < 1$.
- we also saw a way of getting methods with arbitrarily large p by inverse interpolation

- That approach requires that we know derivatives of f.

- They may not be available, and it may be unreasonable to have to provide them.

- There are other ways to construct higher order methods.

- Aitken Acceleration (or Extrapolation)

- we know that

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to 0} \frac{g(\alpha) - g(x_n)}{\alpha - x_n} = g'(\alpha)$$

for a convergent iteration.

- suppose $g'(\alpha) \neq 0$ $\quad |g'(\alpha)| < 1$

- can we compute, or approximate, $g'(\alpha)$ and use our approximation to get a better sequence?

Let:

$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} = \frac{g(x_{n-1}) - g(x_{n-2})}{x_{n-1} - x_{n-2}} \longrightarrow g'(\alpha)$$

since $x_n \to \alpha$

- we can use λ_n to improve convergence

$$\alpha - x_n \approx \lambda_n (\alpha - x_{n-1}) \qquad (*)$$

since

$$\lambda_n \approx g'(\alpha) \approx \frac{g(\alpha) - g(x_{n-1})}{\alpha - x_{n-1}} = \frac{\alpha - x_n}{\alpha - x_{n-1}}$$

solving $(*)$ for α gives

$$(1-\lambda_n)\alpha \approx x_n - \lambda_n x_{n-1}$$

$$\alpha \approx \frac{x_n - \lambda_n x_{n-1}}{1 - \lambda_n} = x_n + \frac{\lambda_n}{1-\lambda_n}(x_n - x_{n-1})$$

- This suggests to define

$$\hat{x}_n = x_n + \frac{\lambda_n}{1-\lambda_n}(x_n - x_{n-1}) \qquad (*)$$

- We can use this formula to convert a linearly convergent sequence

$$x_0, x_1, x_2, \ldots$$

into a more rapidly converging sequence

$$\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots$$

with the same limit

- (*) can be rewritten using $\lambda_n = \dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$

as

$$\hat{x}_n = x_n + \frac{\dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}}{1 - \dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}} (x_n - x_{n-1})$$

$$= x_n - \frac{(x_n - x_{n-1})^2}{(x_n - x_{n-1}) - (x_{n-1} - x_{n-2})}$$

$$= x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n}$$

$\Delta x_n = x_n - x_{n-1}$

$\Delta^{k+1} x_n = \Delta^k x_n - \Delta^k x_{n-1}$

- Because of this last form (*) is also called Aitken's Δ^2 process

- of course, to use the method as described we have to have that linearly convergent sequence $x_0, x_1, x_2, \ldots$

- we can also do the conversion on the fly, however.

- we compute $\hat{x}_n$ and then start fresh from there.

- writing
$$z_n = x_{n-2}$$
$$g(z_n) = x_{n-1}$$
$$g(g(z_n)) = x_n$$
$$z_{n+1} = \hat{x}_n$$

we get
$$z_{n+1} = g(g(z_n)) - \frac{\left(g(g(z_n)) - g(z_n)\right)^2}{g(g(z_n)) - 2g(z_n) + z_n}$$

- This method converges of order 2 (exercise)

- how do we know when to stop?
- we want to stop when $|x_n - \alpha| < \varepsilon$ for some user specified ε (like 10^{-2} m or 10^{-11} sec) in term project.

$$x_n - \alpha \approx \lambda_n (x_{n-1} - \alpha)$$

$$= \lambda_n (x_{n-1} - x_n + x_n - \alpha)$$

$$(1-\lambda_n)(x_n - \alpha) \approx \lambda_n (x_{n-1} - x_n)$$

- so stop when

$$|x_n - \alpha| \approx \left| \frac{\lambda_n}{1-\lambda_n}(x_{n-1} - x_n) \right| < \varepsilon$$

- simplify $\quad \lambda_n = \dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$

$$|x_n - \alpha| \approx \left| \frac{\frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}}{1 - \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}} (x_{n-1} - x_n) \right|$$

$$= \left| \frac{(x_n - x_{n-1})^2}{x_n - 2x_{n-1} + x_{n-2}} \right| < \varepsilon$$

- suppose the iteration converges of order $p > 1$
- Then stop when $|x_n - x_{n-1}| < \varepsilon$

- we can motivate this as follows:

$$e_{n-1} = x_{n-1} - \alpha$$

$$= x_{n-1} - x_n + x_n - \alpha$$

$$= x_{n-1} - x_n + c\, e_{n-1}^p$$

very small relative to e_{n-1}, ignore.

- so $|e_{n-1}| \approx |x_{n-1} - x_n|$

- our actual error is e_n which is smaller

- Query

$$e_n = x_n - \alpha$$
$$= x_n - x_{n-1} + x_{n-1} - \alpha$$
$$= x_n - x_{n-1} + \sqrt{e_n}$$

$$e_{n-1} = x_{n-1} - \alpha$$
$$= x_{n-1} - x_n + x_n - \alpha$$
$$= x_{n-1} - x_n + e_{n-1}^2$$

↑ ignore relative to e_n

$$e_{n-1} \approx x_{n-1} - x_n$$

Query: consider the iterations

$$x_{n+1} = \sin x_n \qquad x_0 = 1$$

$$y_{n+1} = \tan y_n \qquad y_0 = 1$$

what does our theory tell us?

what will actually happen

- Newton's Method applied to polynomials
- we need to evaluate $p(x_n)$ and $p'(x_n)$
- Let me illustrate the ideas first with an example.

Suppose $p(x) = 2x^3 - 3x^2 + x - 4$

$p(2) = 2 \cdot 8 - 3 \cdot 4 + 2 - 4 = 2$

easier $p(x) = ((2x-3)x+1)x-4$

4 1 2 3 6 2

- can be written like this:

$x=2$	2	-3	1	-4
		4	2	6
	2	1	3	2
		4	10	
	2	5	13	

- $p'(x) = 6x^2 - 6x + 1$ $p'(2) = 24 - 12 + 1 = 13$

coincidence!
not at all

Nested Multiplication
Synthetic Division
Horner's Scheme

- In general

$$p(x) = \sum_{k=0}^{n} \alpha_k x^k$$

- Evaluate at x_0

- Then we can do this by this recursion.

$$\beta_n = \alpha_n$$

For $k = n-1, n-2, \ldots, 0$

$$\beta_k = \beta_{k+1} x_0 + \alpha_k$$

$$p(x_0) = \beta_0$$

- Now consider synthetic division

$$p(x) = (x - x_0) q(x) + p(x_0)$$

the same

- Then $q(x) = \sum_{k=1}^{n} \beta_k x^{k-1}$

- To see this note that

$$\beta_n = \alpha_n$$

and $\alpha_k = \beta_k - x_0 \beta_{k-1}$

- Now note that $p'(x) = q(x) + (x - x_0) q'(x)$

and hence $p'(x_0) = q(x_0)$