Math 5600 6/9/14

- periodic functions occur in many applications, e.g., signal processing

- f is periodic, of period, or periodicity, p if
$$f(t) = f(t+p) \quad \text{for all } t \in \mathbb{R}$$

- If f is periodic of period p it is also periodic of period kp for all integers k.

- For example, $\sin t$ and $\cos t$ are periodic of period 2π. $\tan t$ is periodic of period π, hence also of period 2π

- More to the point, $\sin kt$ and $\cos kt$ are periodic of period $\frac{2\pi}{k}$, hence they are also periodic of period 2π

- If we have a problem of periodicity p we can convert it to one of periodicity 2π by a linear change of variables
$$s = \frac{2\pi t}{p}$$
$$t = p \iff s = 2\pi$$

- we already did this in the term project!

- So suppose we wish to approximate a 2π-periodic function f by a linear combination of sin and cos functions.

- The result is a (truncated) Fourier Series

- Baron de Jean Baptiste Joseph Fourier 1768 – 1830

Our approximation will be of the form

$$F_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \quad (*)$$

- Reason for dividing a_0 by 2 will become apparent later.

- If n is replaced by ∞ we have the full "Fourier Series" or "Fourier Expansion" of f.

- F_n is the n-th partial sum of the Fourier Series.

- How do we pick the coefficients a_k and b_k

- How about by the requirement:

$$\int_{-\pi}^{\pi} \left(f(x) - F_n(x)\right)^2 dx = \min$$

- we know where to go from here. Differentiate with respect to the a_k and b_k, set to zero, and solve the linear system.

- Have to compute integrals like

$$\int_{-\pi}^{\pi} \sin mt \cos nt \, dt \quad \text{etc.}$$

- Remarkably, the given basis functions are already orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \, dt$$

- Contrast this with the polynomial case where the ordinary basis functions $1, x, x^2, \ldots$ are anything but orthogonal.

- Exercise: verify orthogonality. We'll look at just a couple of cases here

- Recall integration by parts

$$\int u'v = uv - \int uv'$$

$$I = \int_{-\pi}^{\pi} \underset{u'}{\sin mt}\, \underset{v}{\sin nt}\, dt$$

$$= \underbrace{-\frac{1}{m}\cos mt \sin nt \Big|_{-\pi}^{\pi}}_{=0} + \frac{n}{m}\int_{-\pi}^{\pi} \underset{u'}{\cos mt}\, \underset{v}{\cos nt}\, dt$$

$$= \frac{n}{m}\left[\frac{1}{m}\sin mt \cos nt \Big|_{-\pi}^{\pi} + \frac{n}{m}\int_{-\pi}^{\pi} \sin mt \cos nt\, dt\right]$$

$$= \frac{n^2}{m^2} I \implies I = 0 \text{ if } n \neq m$$

- what if $n = m$? We get

$$\int_{-\pi}^{\pi} \sin^2 nt\, dt = \int_{-\pi}^{\pi} \cos^2 nt\, dt = \pi$$

since $\sin^2 nt + \cos^2 nt = 1$ and we are integrating over a period.

- Similarly

$$\int_{-\pi}^{\pi} 1 \, dt = 2\pi$$

$$\int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0$$

$$\int_{-\pi}^{\pi} \cos nt \cos mt = 0 \quad \text{if } n \neq m$$

- so we get the linear system

$$\begin{bmatrix} 2\pi & & & \\ & \pi & & \\ & & \ddots & \\ & & & \pi \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{bmatrix} = \begin{bmatrix} \int_{-\pi}^{\pi} 1 \cdot f(t) \, dt \\ \vdots \\ \int_{-\pi}^{\pi} \sin nt \, f(t) \, dt \end{bmatrix}$$

- Hence

$$a_k = \int_{-\pi}^{\pi} f(t) \cos kt \, dt \quad (\text{including } k = 0)$$

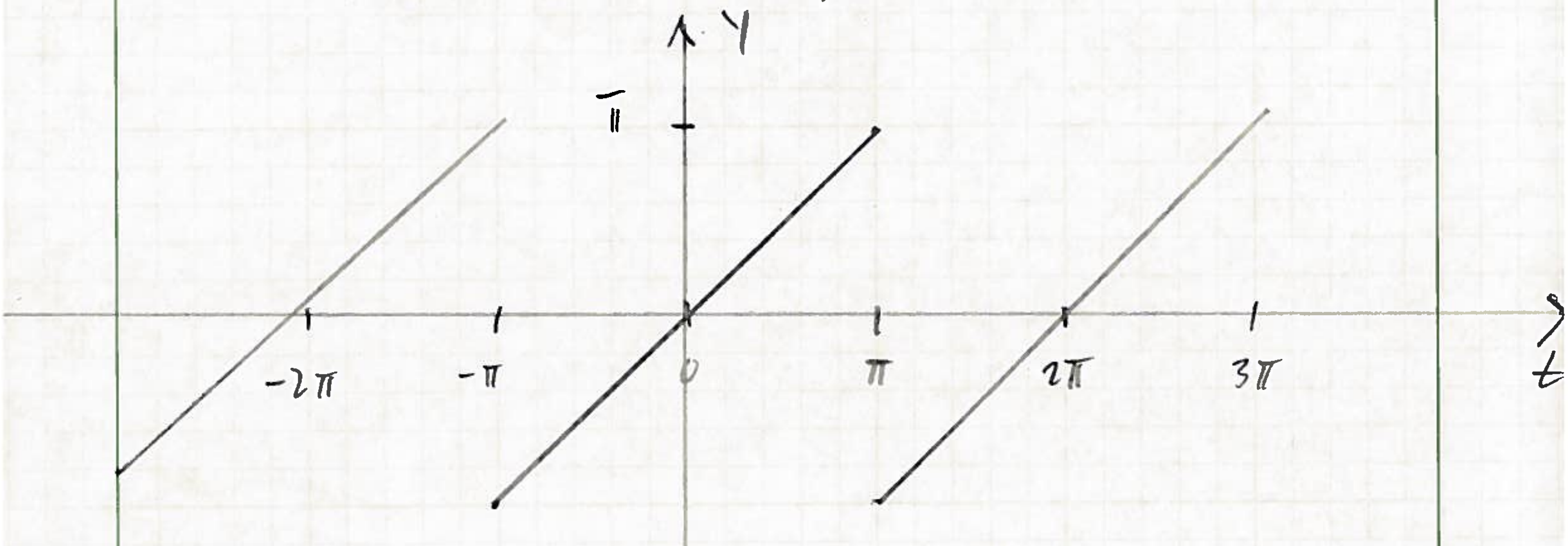
and

$$b_k = \int_{-\pi}^{\pi} f(t) \sin kt \, dt$$

what could be simpler!

- Let's do an example!

- Suppose $f(t) = t \quad -\pi \le t \le \pi$ + 2π-periodic
- This is a "sawtooth" function.



- Note that his function is discontinuous
- That sort of thing occurs in many applications
- we get:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos kt \, dt = 0$$ since the integrand is odd

- The b_k are more complicated.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{u}{t}\, \underset{v'}{\sin kt}\, dt = \frac{1}{\pi}\left[-\frac{t}{k}\cos kt\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{k}\cos kt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{k}\cos k\pi - \frac{\pi}{k}\cos(-k\pi) + \underbrace{\frac{1}{k^2}\sin kt\Big|_{-\pi}^{\pi}}_{=0}\right]$$

$$= \frac{-2}{k}\cos k\pi = \begin{cases} -2/k & \text{if } k \text{ is even} \\ +2/k & \text{if } k \text{ is odd}\end{cases}$$

$$f_n(t) = 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots \frac{1}{n}\sin nt\right)$$

- The next few pages show the graphs of f_n for $n = 5, 10, 20, 40, 80, 160, 320$.
- Notice the oscillations at the discontinuities
- As n goes to infinity they diminish in width but not in amplitude
- That's the Gibbs phenomenon

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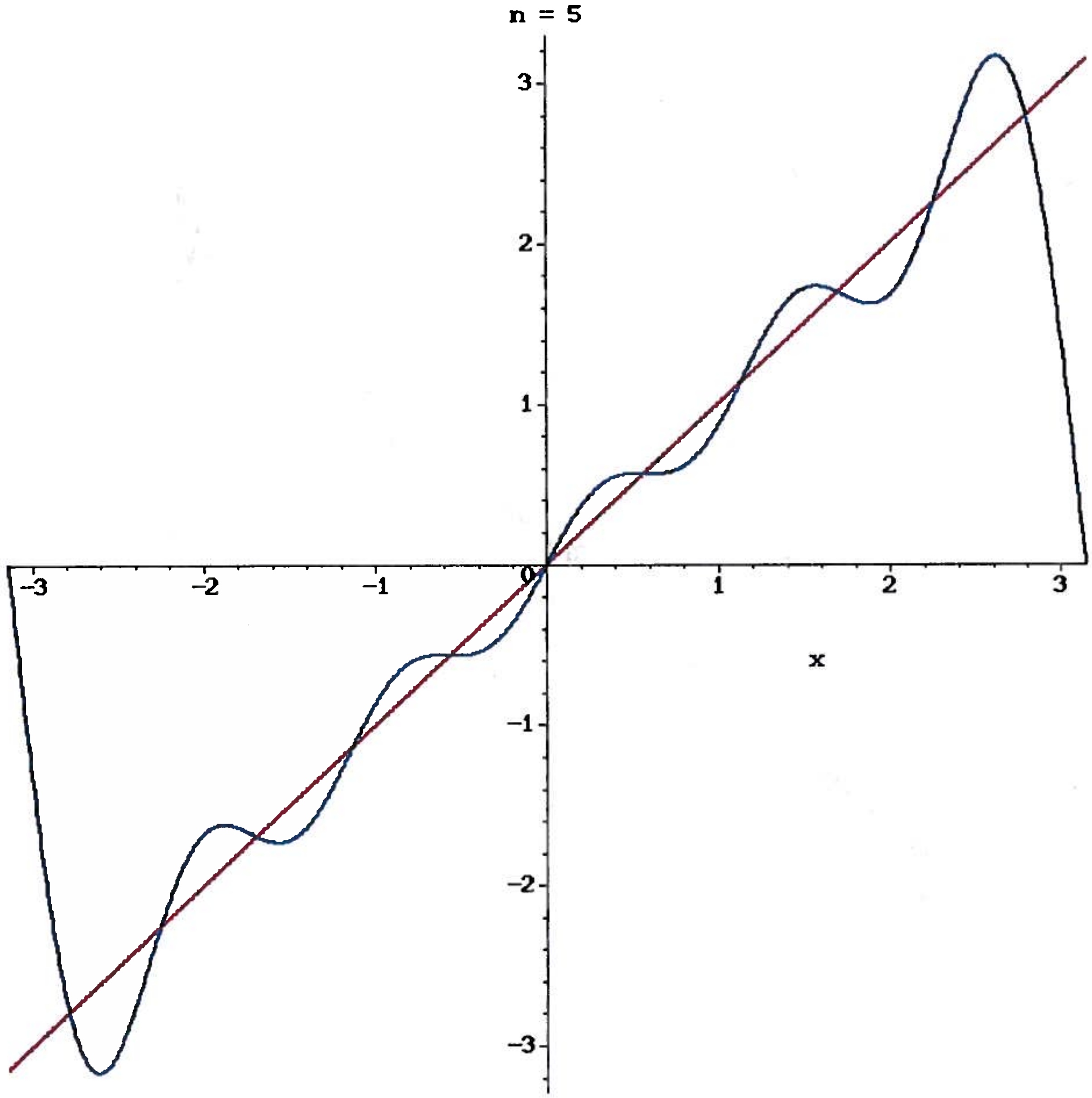

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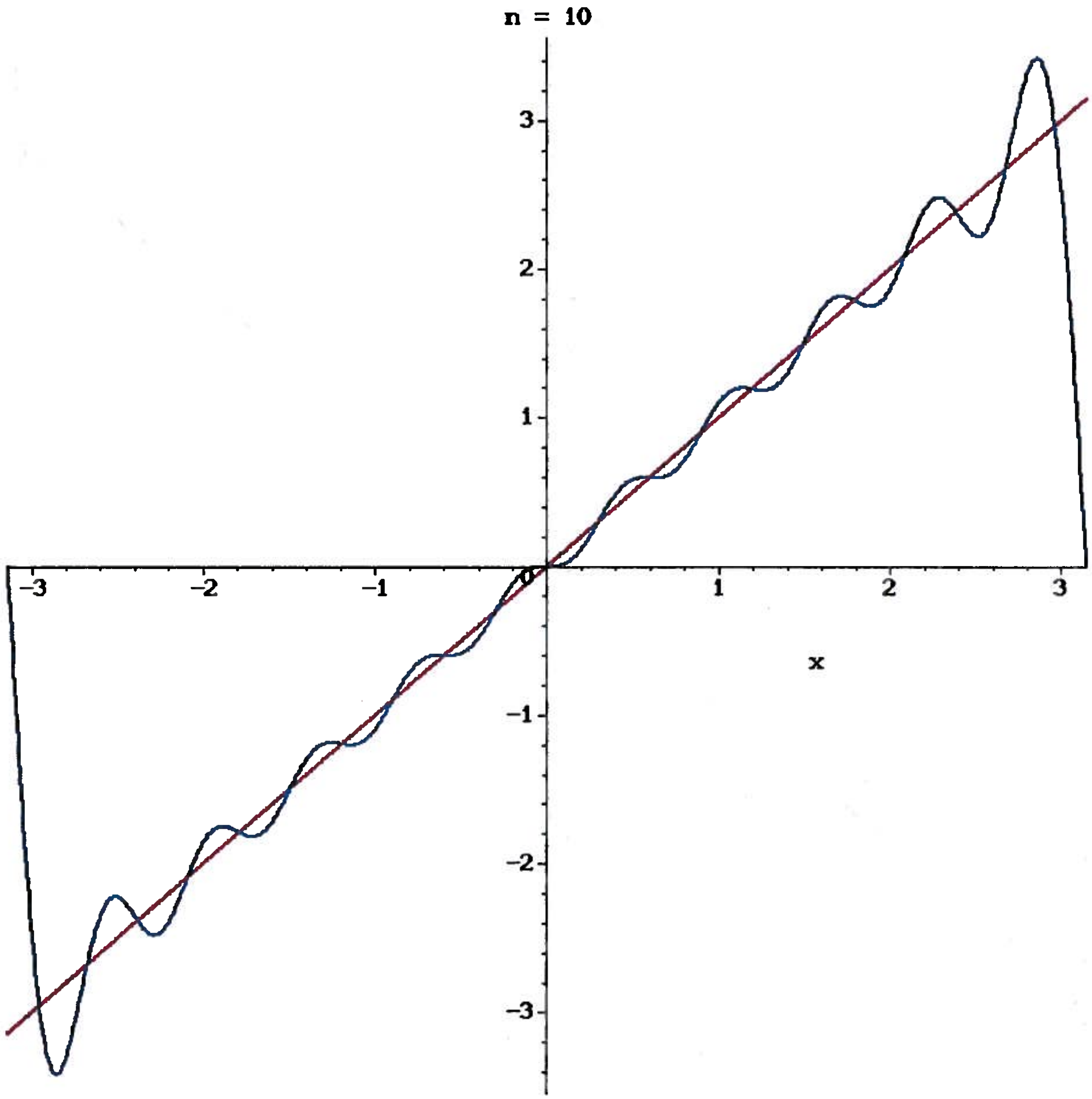
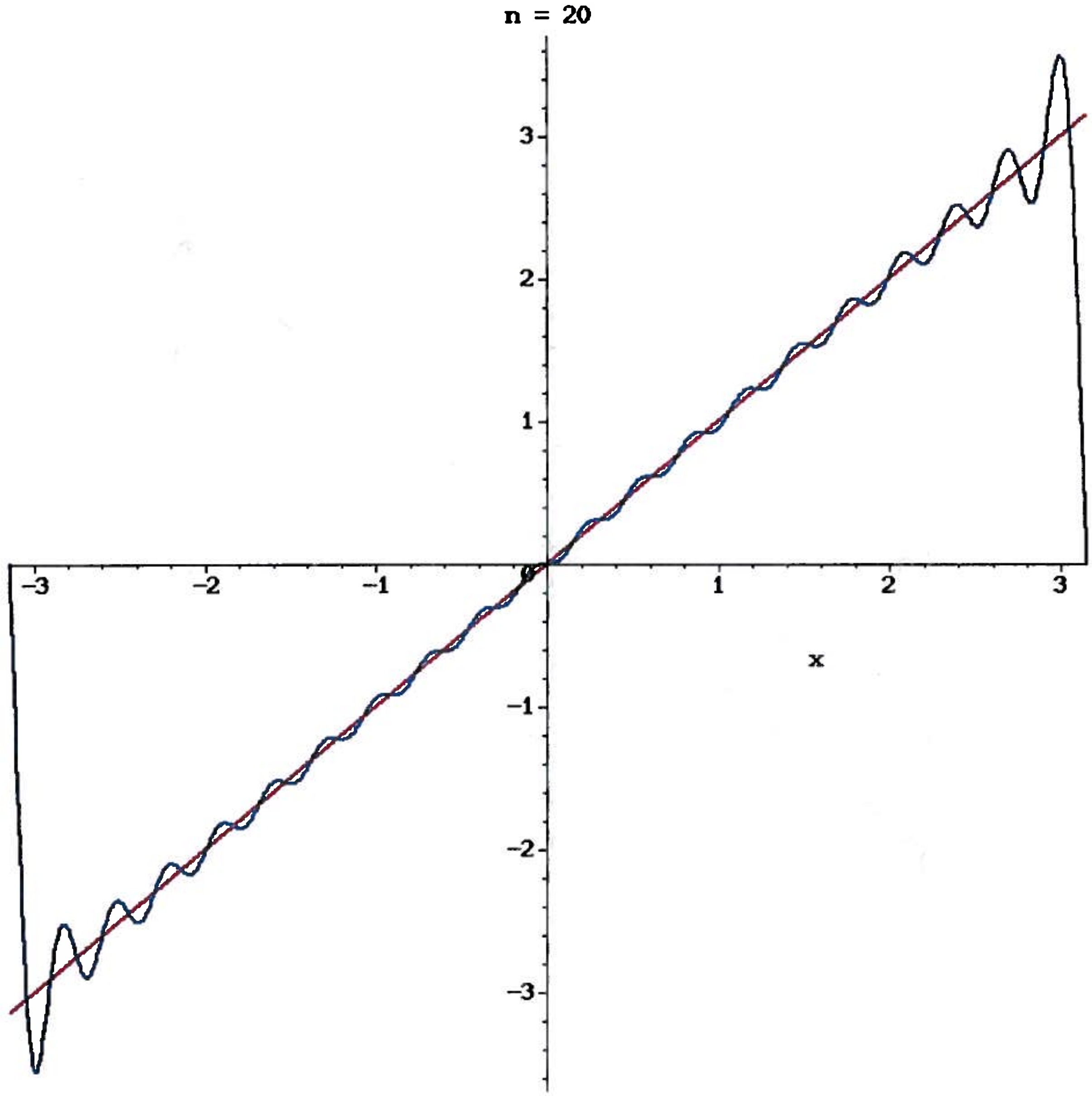
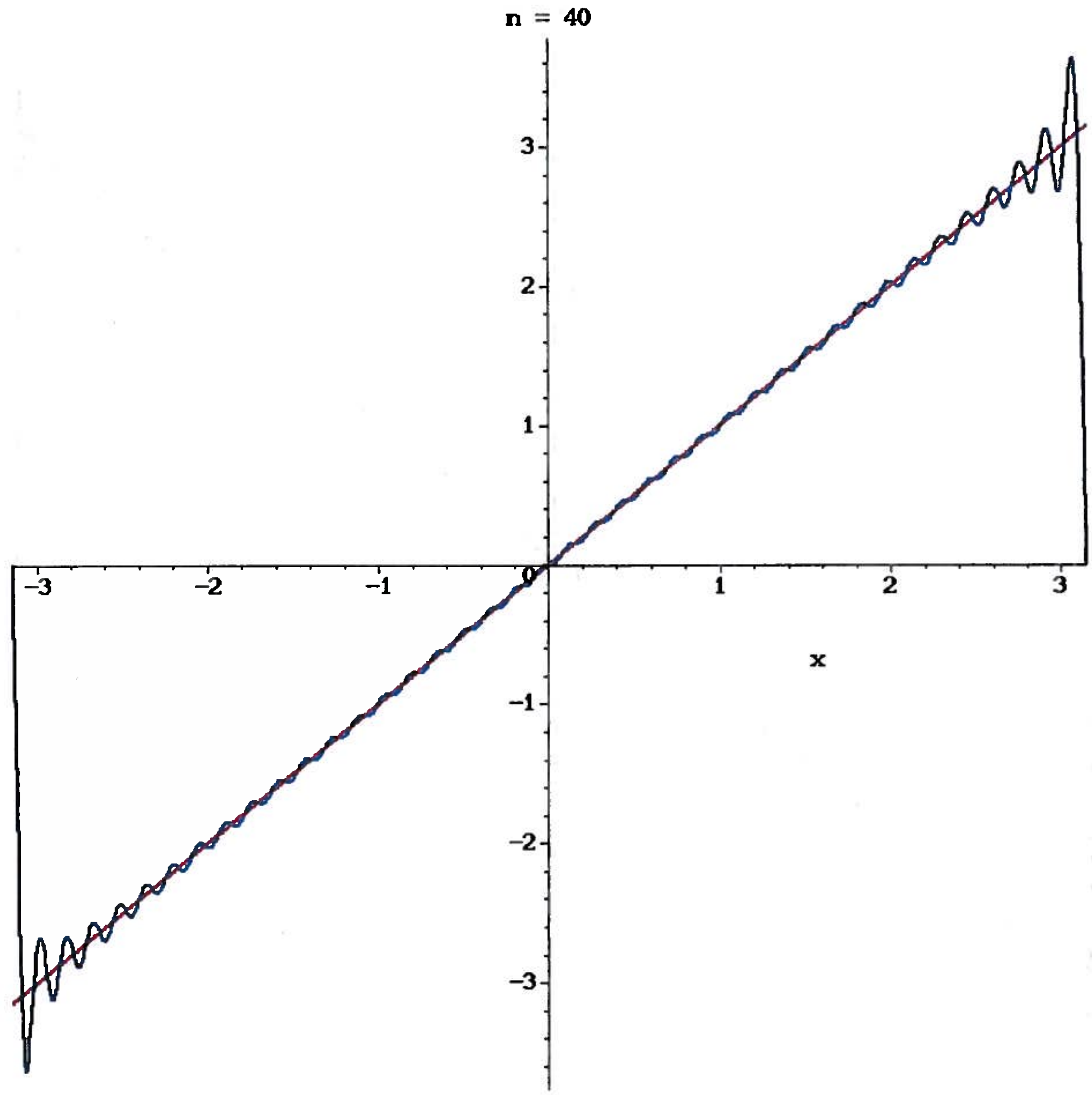



n = 20
3
2
1
−1
−2
−3
−3
−2
−1
0
1
2
3
x

(11)



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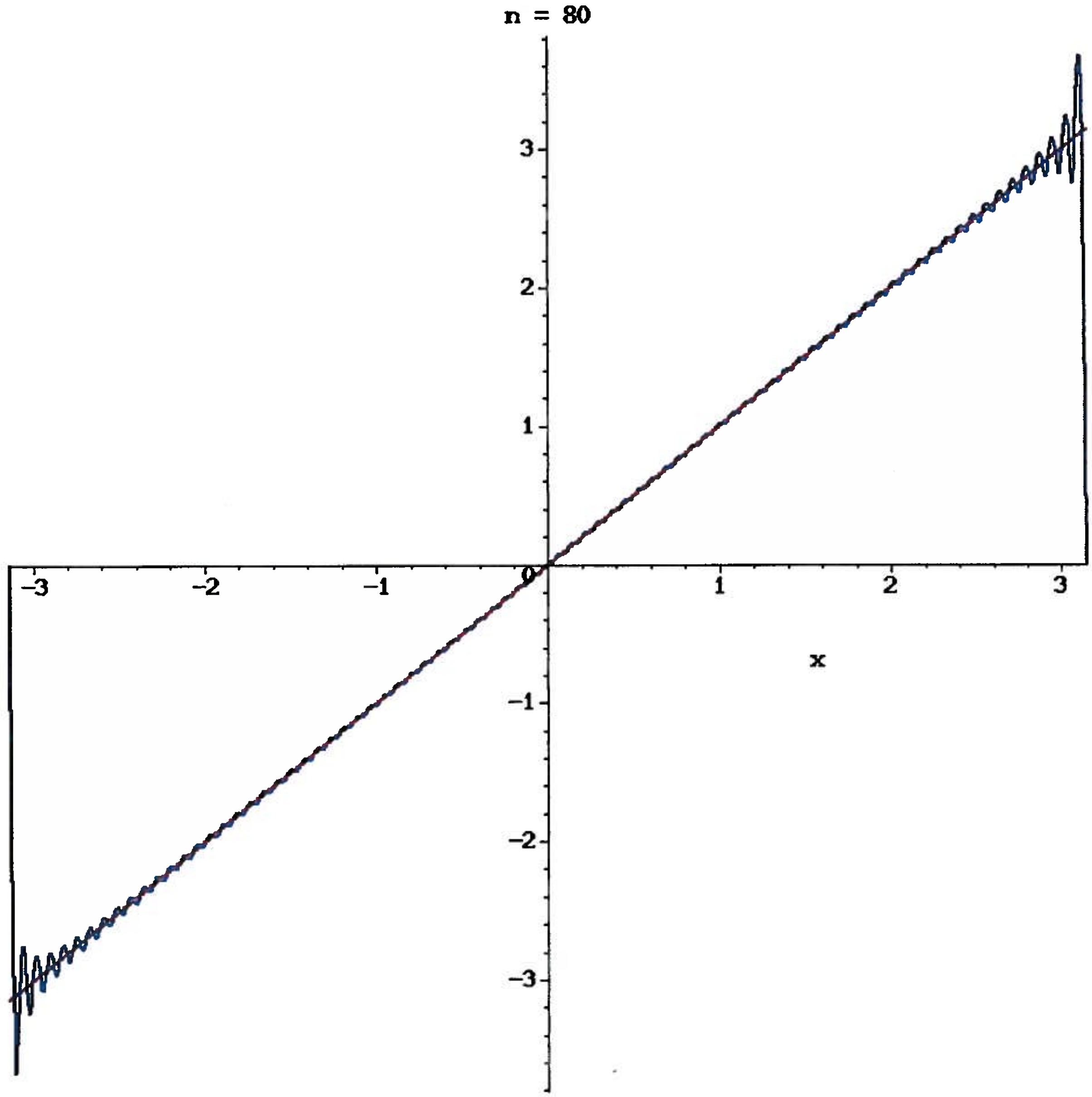
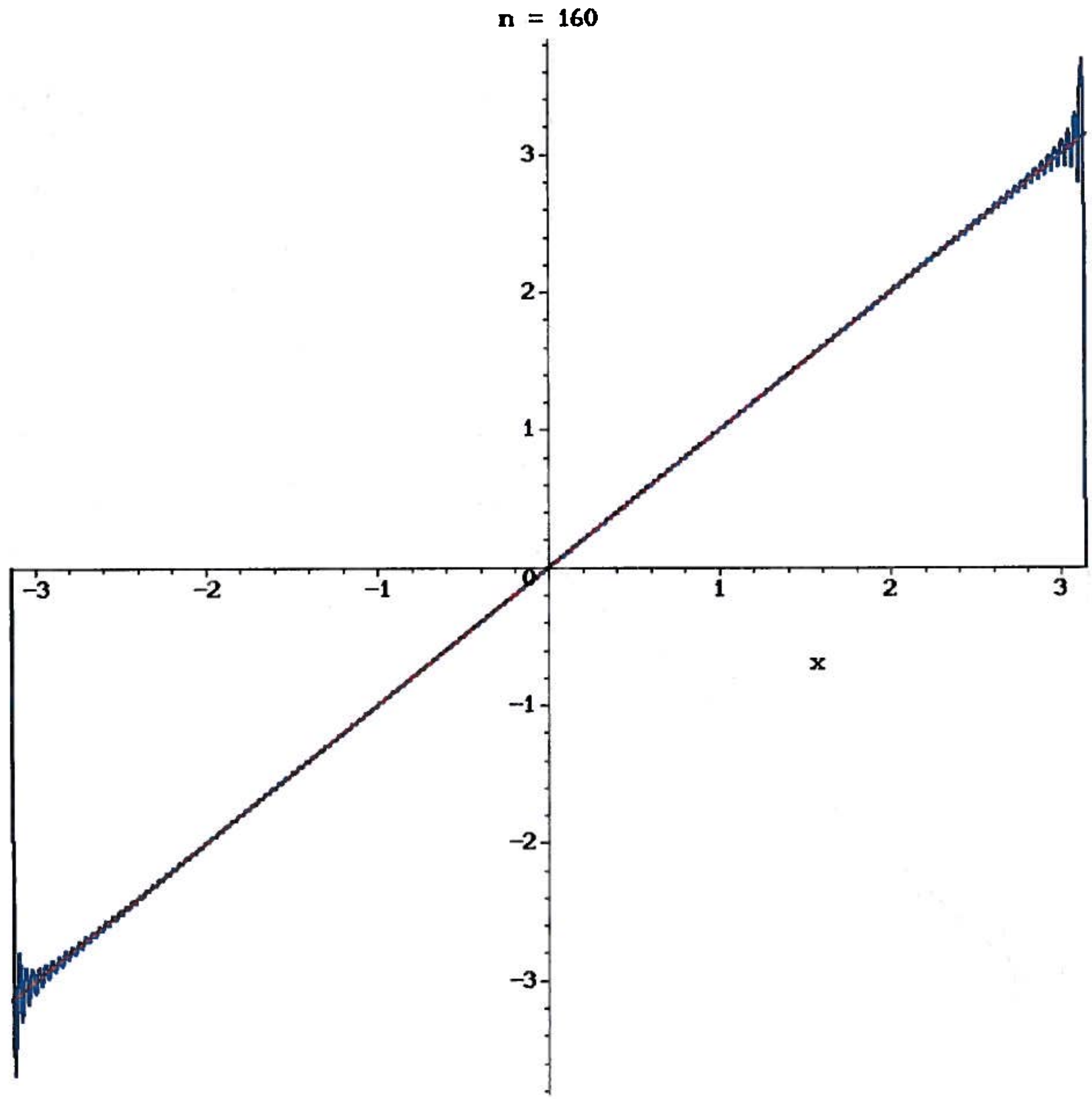
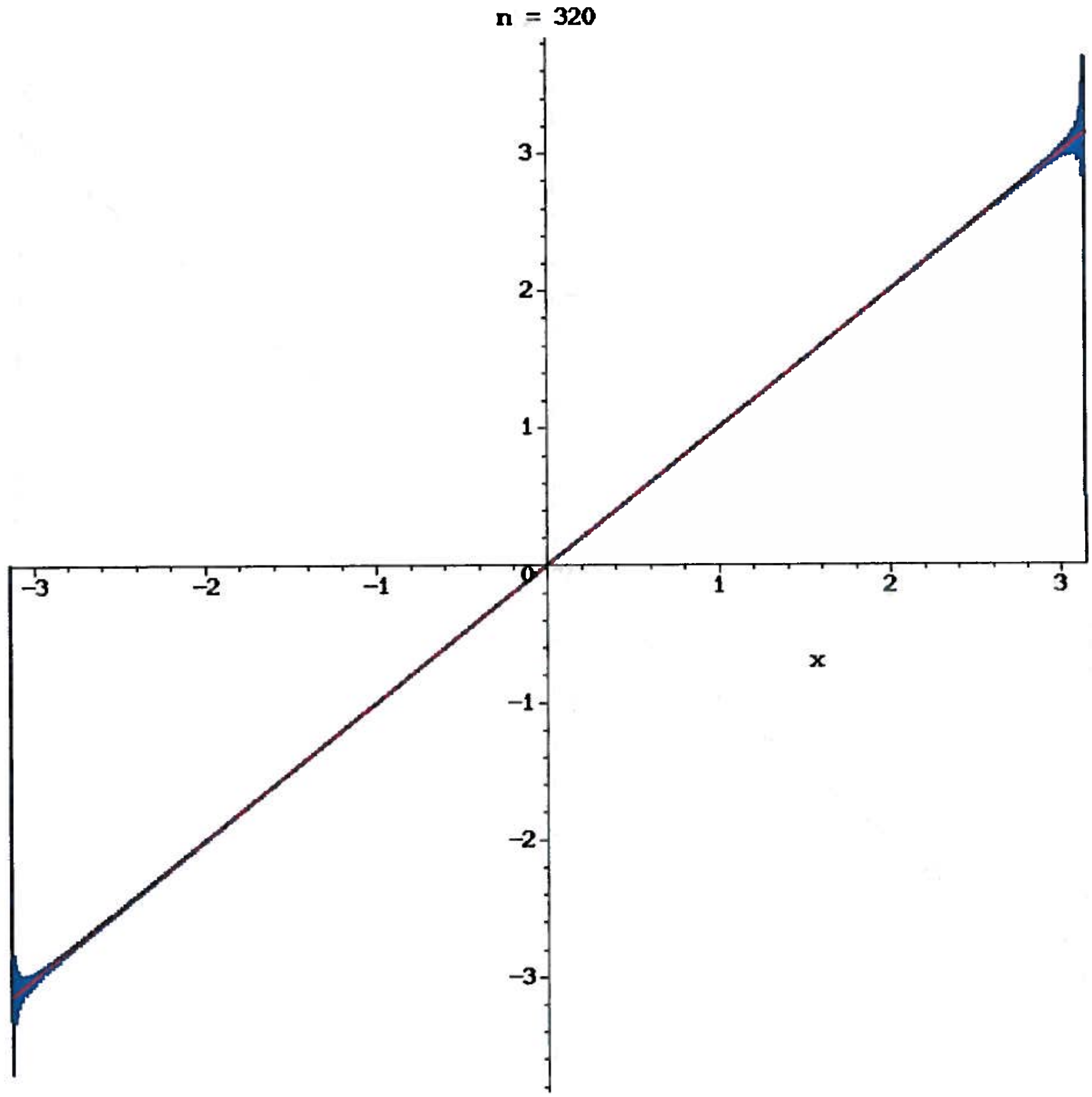



n = 160
3
2
1
0
-1
-2
-3
-3
-2
-1
1
2
3
x


n = 320
3
2
1
0
-1
-2
-3
-3
-2
-1
1
2
3
x

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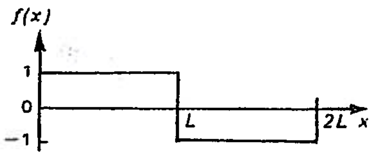
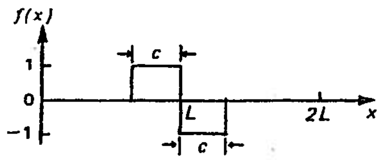
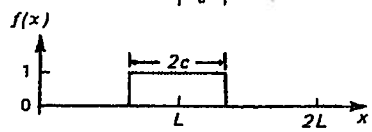
FOURIER EXPANSIONS FOR BASIC PERIODIC FUNCTIONS



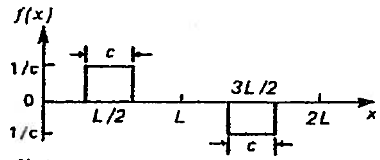
$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \sin \frac{n\pi x}{L}$$



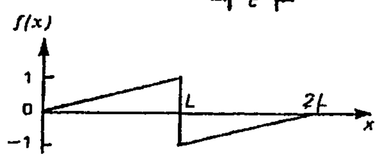
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\cos \frac{n\pi c}{L} - 1\right) \sin \frac{n\pi x}{L}$$



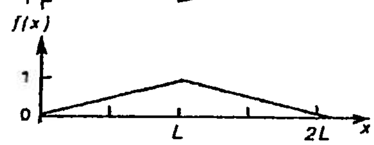
$$f(x) = \frac{c}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi c}{L} \cos \frac{n\pi x}{L}$$



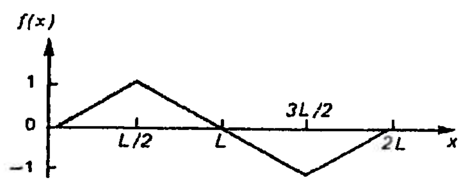
$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \frac{\sin\left(\frac{1}{2}n\pi c/L\right)}{\frac{1}{2}n\pi c/L} \sin \frac{n\pi x}{L}$$



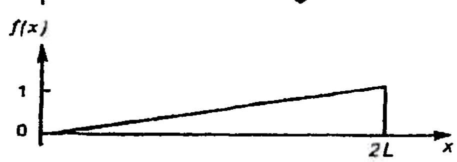
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}$$



$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5\ldots} \frac{1}{n^2} \cos \frac{n\pi x}{L}$$

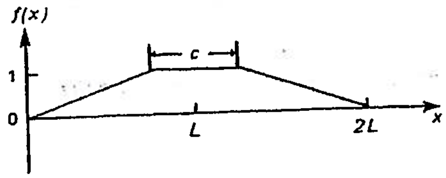


$$f(x) = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin \frac{n\pi x}{L}$$

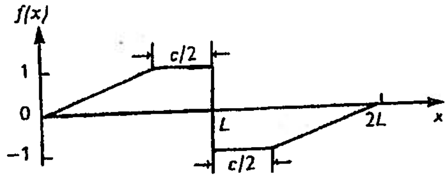


$$f(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L}$$

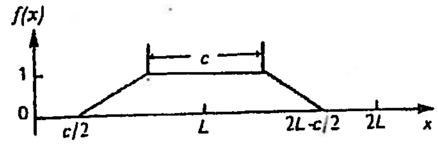
FOURIER EXPANSIONS FOR BASIC PERIODIC FUNCTIONS (Continued)



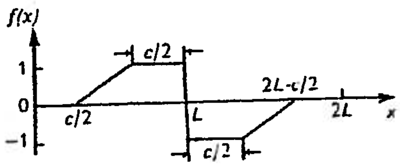
$$f(x) = \frac{1}{2}(1+a) + \frac{2}{\pi^2(1-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n \cos n\pi a - 1] \cos\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



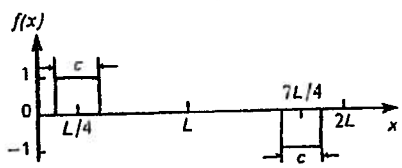
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[1 + \frac{\sin n\pi a}{n\pi(1-a)}\right] \sin\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



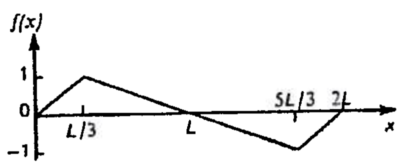
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2(1-2a)} \sum_{n=1,3,5\ldots} \frac{1}{n^2} \cos n\pi a \cos\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



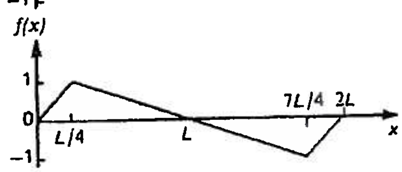
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[1 + \frac{1+(-1)^n}{n\pi(1-2a)} \sin n\pi a\right] \sin\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



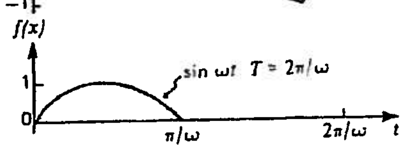
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi}{4} \sin n\pi a \sin\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



$$f(x) = \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



$$f(x) = \frac{32}{3\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{4} \sin\frac{n\pi x}{L} ; \left(a = \frac{c}{2L}\right)$$



$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{\pi} \sum_{n=2,4,6\ldots} \frac{1}{n^2-1} \cos n\omega t$$

Extracted from graphs and formulas, pages 372, 373, Differential Equations in Engineering Problems, Salvadori and Schwarz, published by Prentice-Hall, Inc., 1954.

- A major variation on this theme is Fourier computations for discrete data
- Suppose we know $f(t)$ for N uniformly spaced points in $[-\pi, \pi]$
- This is what digital equipment does...
- The integrals are replaced with summations.
- To compute N coefficients we'd need to carry out N^2 multiplications
- The Fast Fourier Transform accomplishes the same task with $O(n \log n)$ mults
- It's based on Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- very cunning, but beyond our scope