Homework #3

Paul English

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1. **Inner Products.** Let (f, g) denote an inner product on a suitable function space S, and let f be a given function in S. Suppose we want to approximate f by a function

$$s = \sum_{i=1}^{n} \alpha_i \phi_i$$

also in S. Recall that we have to solve a linear system with a coefficient matrix A whose i, j entry is

$$a_{i,j} = (\phi_i, \phi_j).$$

Show that A is positive definite.

2. **Example for Gram Schmidt Process.** Use the Gram-Schmidt Process to find a basis of

$$\text{span}\{1, x, e^x\}$$

that is orthonormal with respect to the inner product

$$(f, g) = \int_0^1 f(x)g(x)\mathrm{d}x.$$

$$b = \text{span}\{1, x, e^x\}$$

$$z_k = b_k - \sum_{i=0}^{k-1} (b_k, q_i) q_i$$

$$q_k = \frac{z_k}{||z_k||}$$

$$z_0 = 1$$

$$q_0 = \frac{b_0}{||b_0||} = \frac{1}{||1||} = 1$$

$$
\begin{aligned}
z_1 &= x - \int_0^1 t\mathrm{d}t \\
&= x - \frac{1}{2} \\
q_1 &= \frac{x - \frac{1}{2}}{(\int_0^1 (t - \frac{1}{2})^2 \mathrm{d}t)^{\frac{1}{2}}} \\
&= \frac{x - \frac{1}{2}}{(\int_0^1 t^2 - t + \frac{1}{4} \mathrm{d}t)^{\frac{1}{2}}} \\
&= \frac{x - \frac{1}{2}}{(\frac{1}{12})^{\frac{1}{2}}} \\
&= \frac{x - \frac{1}{2}}{\frac{1}{2\sqrt{3}}} \\
&= \sqrt{3}(2x - 1)
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= e^x - \int_0^1 e^t \mathrm{d}t - \int_0^1 e^t \sqrt{3}(2t-1)\mathrm{d}t(\sqrt{e}(2x-1)) \\
&= e^x - (e-1) - \sqrt{3}\int_0^1 e^t 2t - e^t \mathrm{d}t(\sqrt{3}(2x-1)) \\
&= e^x - e + 1 - (6x-3)(2\int_0^1 e^t t\mathrm{d}t - (e-1)) \\
&= e^x - e + 1 - (6x-3)(2(e^t t\Big|_0^1 - \int_0^1 e^t \mathrm{d}t) - (e-1)) \\
&= e^x - e + 1 - (6x-3)(2e - 2(e-1) - (e-1)) \\
&= e^x - e + 1 - (6x-3)(2e - 2e + 2 - e + 1) \\
&= e^x - e + 1 - (6x-3)(3-e) \\
&= e^x - e + 1 - (18x - 6ex - 9 + 3e) \\
&= e^x - e + 1 - 18x + 6ex + 9 - 3e \\
&= e^x - 4e - 18x + 6ex + 10
\end{aligned}
$$

$$q_2 = \frac{e^x - 4e - 18x + 6ex + 10}{||e^x - 4e - 18x + 6ex + 10||}$$

$$q_2 = \frac{e^x - 4e - 18x + 6ex + 10}{(\int_0^1 e^x - 4e - 18x + 6ex + 10\mathrm{d}t)^{\frac{1}{2}}}$$

$$q_2 = \frac{e^x - 4e - 1}{(\int_0^1 e^{2x} - 4e^xe - 18e^xx + 6e^xex + 10e^x - 4e^xe + 16e^2 + 72ex - 24e^2x - 40e - 18e^xx + 72ex + 324x}$$

$$q_2 = \frac{e^x - 4e - 18x + 6ex + 10}{(\int_0^1 e^{2x} - 360x - 80e + 16e^2 + 20e^x + 264xe - 48xe^2 - 216x^2e + 36x^2e^2 - 8ee^x - 36xe^x + 324x^2 +}$$

$$q_2 = \frac{e^x - 4e - 18x + 6ex + 10}{\sqrt{20e - \frac{7e^2}{2} - \frac{57}{2}}}$$

3. **The Three Term Recurrence Relation** Let the inner product (f, g) be defined by

$$(f, g) = \int_a^b w(x)f(x)g(x)\mathrm{d}x$$

(where w is a positive weight function). Prove that the sequence of polynomials defined by

$$
\begin{aligned}
Q_n &= (x - a_n)Q_{n-1} - b_n Q_{n-2} \\
Q_0 &= 1 \\
Q_1 &= x - a_n \\
a_n &= \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})} \\
b_n &= \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}
\end{aligned}
$$

is orthogonal with respect to (3). Note that the proof of this fact uses the property

$$(xf, g) = (f, xg)$$

of (3).

4. **Recurrence Relation.** Consider the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

Use the recurrence relation (4) to compute Q_i for $i = 0, 1, 2, 3, 4, 5$.

5. **More on the Recurrence Relation.** Remember that a key property of the inner products for which we established the three term relation was that $(xf, g) = (f, xg)$. Find an inner product that violates that rule, and for which the recurrence relation does indeed fail to yield orthogonal polynomials. (Thus use the recurrence relation to construct the first few polynomials, until you find two that are not orthogonal.)

6. **Fourier Series.** Compute the Fourier series of the function

$$f(x) = \begin{cases} 1 & \text{if } t \in (-\pi, 0) \\ -1 & \text{if } t \in [0, \pi] \end{cases}$$

where you assume that f is 2π periodic, i.e. $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. Draw the truncated Fourier series for some values of n and comment on your plots.

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \cos(nt)\mathrm{d}t - \int_{0}^{\pi} \cos(nt)\mathrm{d}t\right) \\
&= \frac{1}{\pi}\left(\frac{\sin(nt)}{n}\Big|_{-\pi}^{0} - \frac{\sin(nt)}{n}\Big|_{0}^{\pi}\right) \\
&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \sin(nt)\mathrm{d}t - \int_{0}^{\pi} \sin(nt)\mathrm{d}t\right) \\
&= \frac{1}{\pi}\left(\frac{-\cos(nt)}{n}\Big|_{-\pi}^{0} + \frac{\cos(nt)}{n}\Big|_{0}^{\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{-1+\cos(\pi n)}{n} + \frac{\cos(\pi n)-1}{n}\right) \\
&= \frac{2}{\pi n}\left(\cos(\pi n)-1\right) \\
&= \frac{2}{\pi n}\left((-1)^n-1\right) \\
\Rightarrow F(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt) \\
&= \sum_{n-\mathrm{odd}} \frac{-4}{\pi n}\sin(nt)
\end{aligned}
$$

7. **More on Fourier Series.** Calculate the Fourier series of

$$f(x) = \cos(x+1).$$

Hint: Before you embark on the computation of a bunch of integrals think about what you would expect the Fourier series to be. Perhaps you can find it without doing any integrals!

8. **Spline versus Cubic Hermite Interpolation.** Let the function $s(x)$ be defined by

$$s(x) = \begin{cases} (\gamma-1)(x^3-x^2)+x+1 & \text{if } x\in[0,1] \\ \gamma x^3 - 5\gamma x^2 + 8\gamma x - 4\gamma + 2 & \text{if } x \in [1,2] \end{cases}$$

(a) Show that s is piecewise cubic Hermite interpolant to the data:

$$s(0)=1,\quad s(1)=s(2)=2,\quad s'(0)=1,\quad s'(1)=\gamma,\quad s'(2)=0$$

(b) For what value of γ does s become a cubic spline?

9. **The Benstein Bézier Form.** With the notation given in our handout, show that every univariate polynomial of degree d can be written uniquely in Bernstein-Bézier form.

10. **The interpolant to symmetric data is symmetric.** Suppose you are given symmetric data

$$(x_i, y_i), \quad i = -n, -n+1, \ldots, n-1, n$$

such that

$$x_{-i} = -x_i, \quad \text{and} \quad y_{-i} = -y_i \quad i = 0, 1, \ldots, n.$$

What is the required degree of the interpolating polynomial p? Show that the interpolating polynomial is odd, i.e.

$$p(x) = -p(-x)$$

for all real numbers x.