Homework #3

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June 25, 2014

1. **Inner Products.** Let (f, g) denote an inner product on a suitable function space S, and let f be a given function in S. Suppose we want to approximate f by a function

$$s = \sum_{i=1}^{n} \alpha_i \phi_i$$

also in S. Recall that we have to solve a linear system with a coefficient matrix A whose i, j entry is

$$a_{i,j} = (\phi_i, \phi_j).$$

Show that A is positive definite.

To show that A is positive definite, we must prove that for some $x \neq 0$ we have $x^\intercal Ax > 0$. If $x_i = \alpha_i$, then

$$\begin{aligned}
x^\intercal Ax &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i (\phi_i, \phi_j) \alpha_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j (\phi_i, \phi_j) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \alpha_j (\phi_j, \phi_i) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} (\alpha_j \phi_j, \phi_i) \\
&= \sum_{i=1}^{n} \alpha_i (\sum_{j=1}^{n} \alpha_j \phi_j, \phi_i) \\
&= \sum_{i=1}^{n} \alpha_i (s, \phi_i)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \alpha_i (\phi_i, s) \\
&= (\sum_{i=1}^{n} \alpha_i \phi_i, s) \\
&= (s, s) > 0
\end{aligned}
$$

Therefore we have that $x^{\intercal} A x > 0$ and our matrix A is positive definite.

2. **Example for Gram Schmidt Process.** Use the Gram-Schmidt Process to find a basis of

$$\text{span}\{1, x, e^x\}$$

that is orthonormal with respect to the inner product

$$(f, g) = \int_0^1 f(x) g(x) \mathrm{d}x.$$

$$
\begin{aligned}
z_k &= b_k - \sum_{i=0}^{k-1} (b_k, q_i) q_i \\
q_k &= \frac{z_k}{||z_k||}
\end{aligned}
$$

$$
\begin{aligned}
z_0 &= 1 \\
q_0 &= \frac{b_0}{||b_0||} = \frac{1}{||1||} = \boxed{1} \\
z_1 &= x - \int_0^1 t \mathrm{d}t = x - \frac{1}{2} \\
q_1 &= \frac{x - \frac{1}{2}}{(\int_0^1 (t - \frac{1}{2})^2 \mathrm{d}t)^{\frac{1}{2}}} \\
&= \frac{x - \frac{1}{2}}{(\int_0^1 t^2 - t + \frac{1}{4} \mathrm{d}t)^{\frac{1}{2}}} \\
&= \frac{x - \frac{1}{2}}{(\frac{1}{12})^{\frac{1}{2}}} \\
&= \frac{x - \frac{1}{2}}{\frac{1}{2\sqrt{3}}}
\end{aligned}
$$

$$
\begin{aligned}
&\boxed{= \sqrt{3}(2x-1)}\\
z_2 &= e^x - \int_0^1 e^t \mathrm{d}t - \int_0^1 e^t \sqrt{3}(2t-1)\mathrm{d}t(\sqrt{3}(2x-1))\\
&= e^x - (e-1) - \sqrt{3}\int_0^1 e^t 2t - e^t \mathrm{d}t(\sqrt{3}(2x-1))\\
&= e^x - e + 1 - (6x-3)(2\int_0^1 e^t t\mathrm{d}t - (e-1))\\
&= e^x - e + 1 - (6x-3)(2(e^t t\Big|_0^1 - \int_0^1 e^t \mathrm{d}t) - (e-1))\\
&= e^x - e + 1 - (6x-3)(2e - 2(e-1) - (e-1))\\
&= e^x - e + 1 - (6x-3)(2e - 2e + 2 - e + 1)\\
&= e^x - e + 1 - (6x-3)(3-e)\\
&= e^x - e + 1 - (18x - 6ex - 9 + 3e)\\
&= e^x - e + 1 - 18x + 6ex + 9 - 3e\\
&= e^x - 4e - 18x + 6ex + 10\\
q_2 &= \frac{e^x - 4e - 18x + 6ex + 10}{||e^x - 4e - 18x + 6ex + 10||}\\
&= \frac{e^x - 4e - 18x + 6ex + 10}{(\int_0^1 (e^x - 4e - 18x + 6ex + 10)^2 \mathrm{d}t)^{\frac{1}{2}}}\\
&= \frac{e^x - 4e - 1}{(\int_0^1 e^{2x} - 4e^x e - 18e^x x + 6e^x ex + 10e^x - 4e^x e + 16e^2 + 72ex - 24e^2x - 40e - 18e^x x + 72ex + 324x}\\
&= \frac{e^x - 4e - 18x + 6ex + 10}{(\int_0^1 e^{2x} - 360x - 80e + 16e^2 + 20e^x + 264xe - 48xe^2 - 216x^2e + 36x^2e^2 - 8ee^x - 36xe^x + 324x^2 +}\\
&\boxed{= \frac{e^x - 4e - 18x + 6ex + 10}{\sqrt{20e - \frac{7e^2}{2} - \frac{57}{2}}}}
\end{aligned}
$$

3. **The Three Term Recurrence Relation** Let the inner product (f, g) be defined by

$$(f, g) = \int_a^b w(x)f(x)g(x)\mathrm{d}x$$

(where w is a positive weight function). Prove that the sequence of polynomials defined by

$$
\begin{aligned}
Q_n &= (x - a_n)Q_{n-1} - b_n Q_{n-2} \\
Q_0 &= 1 \\
Q_1 &= x - a_n \\
a_n &= \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})} \\
b_n &= \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}
\end{aligned}
$$

is orthogonal with respect to (3). Note that the proof of this fact uses the property

$$(xf, g) = (f, xg)$$

of (3).

Proof by induction

$$
\begin{aligned}
(Q_0, Q_1) &= (1, x - \frac{(x,1)}{(1,1)}) \\
&= (1, x) - (1, \frac{(x,1)}{(1,1)}) \\
&= (1, x) - (1,1)\frac{(x,1)}{(1,1)} \\
&= (1, x) - (1, x) = 0.
\end{aligned}
$$

Assume that $Q_0, Q_1, \ldots, Q_{n-1}$ are all orthogonal to each other, and that $k < n$. Then,

$$
\begin{aligned}
(Q_n, Q_k) &= \left(\left(x - \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})}\right) Q_{n-1} - \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})} Q_{n-2}, Q_k\right) \\
&= (xQ_{n-1}, Q_k) - \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})}(Q_{n-1}, Q_k) - \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}(Q_{n-2}, Q_k)
\end{aligned}
$$

Here we have three cases: $k = n-1$, $k = n-2$, $k < n-2$. Checking each of these,

$$
\begin{aligned}
k = n-1 :&= (xQ_{n-1}, Q_{n-1}) - (xQ_{n-1}, Q_{n-1}) = 0 \\
k = n-2 :&= (xQ_{n-1}, Q_{n-2}) - (xQ_{n-1}, Q_{n-2}) = 0 \\
k < n-2 :&= (xQ_{n-1}, Q_k) \\
&= (Q_{n-1}, xQ_k) \\
&= \left(Q_{n-1}, \sum_{i=0}^{n-2} \alpha_i Q_i\right) = \sum_{i=0}^{n-2} \alpha_i (Q_{n-1}, Q_i) = 0
\end{aligned}
$$

4. **Recurrence Relation.** Consider the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

Use the recurrence relation (4) to compute Q_i for $i = 0, 1, 2, 3, 4, 5$.

For the interval $[-1, 1]$ the dot production of any odd and even polynomial will result in the integral of an odd function which will be zero, thus $a_i = 0$, for $i = 0, 1, 2, 3, 5$. We'll still include these for verification.

$$
\begin{aligned}
Q_0 &\boxed{= 1} \\
a_1 &= \frac{(xQ_0, Q_0)}{(Q_0, Q_0)} = \frac{(x, 1)}{(1, 1)} \\
Q_1 &= x - \frac{(x, 1)}{(1, 1)} \\
&\boxed{= x} \\
a_2 &= \frac{(xQ_1, Q_1)}{(Q_1, Q_1)} = \frac{(x, 1)}{(1, 1)} = \frac{(x^2, x)}{(x, x)} = 0 \\
b_2 &= \frac{(xQ_1, Q_0)}{(Q_0, Q_0)} = \frac{(x^2, 1)}{(1, 1)} = \frac{1}{3} \\
Q_2 &= (x - a_2)Q_1 - b_2 Q_0 \\
&= (x - a_2)x - b_2(1) \\
&\boxed{= x^2 - \frac{1}{3}} \\
a_3 &= \frac{(xQ_2, Q_2)}{(Q_2, Q_2)} = \frac{(x^3 - \frac{1}{3}x, x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} = \frac{\int_{-1}^{1}(x^3 - \frac{1}{3}x)(x^2 - \frac{1}{3})\mathrm{d}x}{\int_{-1}^{1}(x^2 - \frac{1}{3})^2\mathrm{d}x} = \frac{0}{8/45} = 0 \\
b_3 &= \frac{(xQ_2, Q_1)}{(Q_1, Q_1)} = \frac{(x^3 - \frac{1}{3}x, x)}{(x, x)} = \frac{8/105}{2/3} = \frac{4}{15} \\
Q_3 &= x(x^2 - \frac{1}{3}) - \frac{4}{15}x \\
&= x^3 - \frac{1}{3}x - \frac{4}{15}x \\
&\boxed{= x^3 - \frac{3}{5}x} \\
a_4 &= \frac{(xQ_3, Q_3)}{(Q_3, Q_3)} = \frac{\int_{-1}^{1} x(x^3 - \frac{3}{5}x)^2\mathrm{d}x}{\int_{-1}^{1}(x^3 - \frac{3}{5}x)^2\mathrm{d}x} = \frac{0}{8/175} = 0
\end{aligned}
$$

$$\begin{aligned}
b_4 &= \frac{(xQ_3, Q_2)}{(Q_2, Q_2)} = \frac{\int_{-1}^{1} x(x^3 - \frac{3}{5}x)(x^2 - \frac{1}{3})\mathrm{d}x}{\int_{-1}^{1}(x^2 - \frac{1}{3})^2\mathrm{d}x} = \frac{8/175}{8/45} = \frac{9}{35} \\
Q_4 &= (x - a_4)Q_3 - b_4 Q_2 \\
&= x(x^3 - \frac{3}{5}x) - \frac{9}{35}(x^2 - \frac{1}{3}) \\
&= x^4 - \frac{3}{5}x^2 - \frac{9}{35}x^2 - \frac{1}{3}(\frac{9}{35}) \\
&\boxed{= x^4 - \frac{6}{7}x^2 - \frac{3}{35}} \\
a_5 &= \frac{(xQ_4, Q_4)}{(Q_4, Q_4)} = \frac{\int_{-1}^{1} x(x^4 - \frac{6}{7}x^2 - \frac{3}{35})^2\mathrm{d}x}{\int_{-1}^{1}(x^4 - \frac{6}{7}x^2 - \frac{3}{35})^2\mathrm{d}x} = \frac{0}{128/11025} \\
b_5 &= \frac{(xQ_4, Q_3)}{(Q_3, Q_3)} = \frac{\int_{-1}^{1} x(x^4 - \frac{6}{7}x^2 - \frac{3}{35})(x^3 - \frac{3}{5}x)\mathrm{d}x}{\int_{-1}^{1}(x^3 - \frac{3}{5}x)^2\mathrm{d}x} = \frac{128/11025}{8/175} = \frac{16}{63} \\
Q_5 &= (x - a_5)Q_4 - b_5 Q_3 \\
&= x(x^4 - \frac{6}{7}x^2 - \frac{3}{35}) - \frac{16}{63}(x^3 - \frac{3}{5}x) \\
&= x^5 - \frac{5}{7}x^3 - \frac{8}{105}x - \frac{16}{63}x^3 - \frac{3}{5}(\frac{16}{63})x \\
&\boxed{= x^5 - \frac{10}{9}x^3 - \frac{5}{21}x}
\end{aligned}$$

5. **More on the Recurrence Relation.** Remember that a key property of the inner products for which we established the three term relation was that $(xf, g) = (f, xg)$. Find an inner product that violates that rule, and for which the recurrence relation does indeed fail to yield orthogonal polynomials. (Thus use the recurrence relation to construct the first few polynomials, until you find two that are not orthogonal.)

   Let the inner product (f, g) be defined as

$$(f, g) = \int_{-1}^{1} f(x)g(x) + f'(x)g'(x)\mathrm{d}x$$

   Then the following properties hold

$$\begin{aligned}
(f, g) &= \int_{-1}^{1} f(x)g(x) + f'(x)g'(x)\mathrm{d}x \\
&= \int_{-1}^{1} g(x)f(x) + g'(x)f'(x)\mathrm{d}x \\
&= (g, f)
\end{aligned}$$

$$
\begin{aligned}
(f+g,h) &= \int_{-1}^{1}(f(x)+g(x))h(x)+(f(x)+g(x))'h'(x)\mathrm{d}x \\
&= \int_{-1}^{1} f(x)h(x)+g(x)h(x)+(f'(x)+g'(x))h'(x)\mathrm{d}x \\
&= \int_{-1}^{1} f(x)h(x)+g(x)h(x)+f'(x)h'(x)+g'(x)h'(x)\mathrm{d}x \\
&= \int_{-1}^{1} f(x)h(x)+f'(x)h'(x)+g(x)h(x)+g'(x)h'(x)\mathrm{d}x \\
&= \int_{-1}^{1} f(x)h(x)+f'(x)h'(x)\mathrm{d}x+\int_{-1}^{1} g(x)h(x)+g'(x)h'(x)\mathrm{d}x \\
&= (f,h)+(g,h) \\
(f,f) &= \int_{-1}^{1} f(x)f(x)+f'(x)f'(x)\mathrm{d}x \\
&\geq 0
\end{aligned}
$$

However the property which our recurrence relation relies on $(xf,g)=(f,xg)$ does not,

$$
\begin{aligned}
(xf,g) &= \int_{-1}^{1} xf(x)g(x)+(xf(x))'g'(x)\mathrm{d}x \\
(f,xg) &= \int_{-1}^{1} xf(x)g(x)+f'(x)(xg(x))'\mathrm{d}x \\
(xf(x))' &\neq (xg(x))' \\
\Rightarrow (xf,g) &\neq (f,xg)
\end{aligned}
$$

If we test out our recurrence relation we find the following polynomials,

$$
\begin{aligned}
Q_0 &= 1 \\
Q_1 &= x \\
a_2 &= \frac{(xQ_1,Q_1)}{(Q_1,Q_1)} = 0 \\
b_2 &= \frac{(xQ_1,Q_0)}{(Q_0,Q_0)} = \frac{1}{3} \\
Q_2 &= x^2-\frac{1}{3} \\
a_3 &= \frac{(xQ_2,Q_2)}{(Q_2,Q_2)} = 0
\end{aligned}
$$

$$\begin{aligned}
b_3 &= \frac{(xQ_2, Q_1)}{(Q_1, Q_1)} = \frac{17}{30} \\
Q_3 &= x^3 - \frac{9}{10} \\
a_4 &= \frac{(xQ_3, Q_3)}{(Q_3, Q_3)} = 0 \\
b_4 &= \frac{(xQ_3, Q_2)}{(Q_2, Q_2)} = \frac{39}{140} \\
Q_4 &= x^4 - \frac{33}{28}x^2 + \frac{13}{140}
\end{aligned}$$

Checking the orthogonality of these polynomials we find that our fourth polynomial is no longer orthogonal to our first, and we can see that our recurrence relation has failed to find orthogonal polynomials.

$$\begin{aligned}
(Q_0, Q_1) &= 0 \\
(Q_2, Q_0) &= 0 \\
(Q_2, Q_1) &= 0 \\
(Q_3, Q_0) &= 0 \\
(Q_3, Q_1) &= 0 \\
(Q_3, Q_2) &= 0 \\
(Q_4, Q_0) &\boxed{= -\frac{1}{5} \neq 0}
\end{aligned}$$

6. **Fourier Series.** Compute the Fourier series of the function

$$f(t) = \begin{cases} 1 & \text{if } t \in (-\pi, 0) \\ -1 & \text{if } t \in [0, \pi] \end{cases}$$

where you assume that f is 2π periodic, i.e. $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. Draw the truncated Fourier series for some values of n and comment on your plots.

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}\cos(nt)\mathrm{d}t - \int_{0}^{\pi}\cos(nt)\mathrm{d}t\right) \\
&= \frac{1}{\pi}\left(\frac{\sin(nt)}{n}\bigg|_{-\pi}^{0} - \frac{\sin(nt)}{n}\bigg|_{0}^{\pi}\right) \\
&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}\sin(nt)\mathrm{d}t - \int_{0}^{\pi}\sin(nt)\mathrm{d}t\right) \\
&= \frac{1}{\pi}\left(\frac{-\cos(nt)}{n}\bigg|_{-\pi}^{0} + \frac{\cos(nt)}{n}\bigg|_{0}^{\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{-1+\cos(\pi n)}{n} + \frac{\cos(\pi n)-1}{n}\right) \\
&= \frac{2}{\pi n}\left(\cos(\pi n)-1\right) \\
&= \frac{2}{\pi n}\left((-1)^n-1\right) \\
\Rightarrow F(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt) \\
&= \sum_{n-\mathrm{odd}} \frac{-4}{\pi n}\sin(nt)
\end{aligned}
$$

7. **More on Fourier Series.** Calculate the Fourier series of

$$f(x) = \cos(x+1).$$

Hint: Before you embark on the computation of a bunch of integrals think about what you would expect the Fourier series to be. Perhaps you can find it without doing any integrals!

$$
\begin{aligned}
f(t) &= \cos(t+1) \\
&= \cos(t)\cos(1) - \sin(t)\sin(1)
\end{aligned}
$$

Here the trigonometric identity of $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ gives us a form equivalent to the Fourier series.

Solving for a_n and b_n we would find that

$$a_n = \begin{cases} \cos(1) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

and

$$b_n = \begin{cases} -\sin(1) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

Thus the Fourier series for $f(t)$ is

$$F(t) = \cos(1)\cos(t) - \sin(1)\sin(t)$$

8. **Spline versus Cubic Hermite Interpolation.** Let the function $s(x)$ be defined by

$$s(x) = \begin{cases} (\gamma - 1)(x^3 - x^2) + x + 1 & \text{if } x \in [0,1] \\ \gamma x^3 - 5\gamma x^2 + 8\gamma x - 4\gamma + 2 & \text{if } x \in [1,2] \end{cases}$$

(a) Show that s is piecewise cubic Hermite interpolant to the data:

$$s(0) = 1, \quad s(1) = s(2) = 2, \quad s'(0) = 1, \quad s'(1) = \gamma, \quad s'(2) = 0$$

(b) For what value of γ does s become a cubic spline?

$$s'(x) = \begin{cases} (\gamma - 1)(3x^2 - 2x) + 1 & \text{if } x \in [0,1] \\ 3\gamma x^2 - 10\gamma x + 8\gamma & \text{if } x \in [1,2] \end{cases}$$

(a)

$$\begin{aligned} s(0) &= 1 \\ s(1) &= s(2) = 2 \\ s'(0) &= 1 \\ s'(1) &= \gamma \\ s'(2) &= 0 \end{aligned}$$

(b)

9. **The Benstein Bézier Form.** With the notation given in our handout, show that every univariate polynomial of degree d can be written uniquely in Bernstein-Bézier form.

   For any univariate polynomial the interpolated polynomial for the Bernstein Bézier form will be the unique polynomial of degree d, and it will interpolate to the univariate polynomial.

10. **The interpolant to symmetric data is symmetric.** Suppose you are given symmetric data

$$(x_i, y_i), \quad i = -n, -n+1, \ldots, n-1, n$$

such that

$$x_{-i} = -x_i, \quad \text{and} \ \ y_{-i} = -y_i \quad i = 0, 1, \ldots, n.$$

What is the required degree of the interpolating polynomial p? Show that the interpolating polynomial is odd, i.e.

$$p(x) = -p(-x)$$

for all real numbers x.

Since we have a symmetric dataset the number of points is even and equals $2n$. Thus the interpolated polynomial will be of degree $2n - 1$ which is odd.

Interpolating symmetric data will give us an odd polynomial $p_n(x_i) = y_i$ which passes through all points in our data set x_i, and y_i.

This means that we have

$$\begin{aligned}
p_n(x_i) &= y_i \\
&= -(-y_i) \\
&= -y_{-i} \\
&= -p_n(x_{-i}) \\
&= -p_n(-x_i) \\
\Rightarrow p(x) &= -p(x) \ \text{ for all } i = 0, 1, 2, \ldots, n
\end{aligned}$$