

Homework #4

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1. **(Iterative Solution of Linear Systems.)** Consider the $n \times n$ linear system $Ax = b$ and recall the three iterative techniques we discussed in class:

Gauss-Jacobi:

$$x_i^{[k+1]} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[k]} - \sum_{j=i+1}^n a_{ij} x_j^{[k]} \right)$$

Gauss-Seidel:

$$x_i^{[k+1]} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[k]} - \sum_{j=i+1}^n a_{ij} x_j^{[k+1]} \right)$$

Successive Overrelaxation:

$$x_i^{[k+1]} = (1 - w) x_i^{[k]} + \frac{w}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[k]} - \sum_{j=i+1}^n a_{ij} x_j^{[k+1]} \right)$$

As we discussed in class, think of A as being split as

$$A = L + D + U$$

where L contains the entries of A below the diagonal, U contains the entries above the diagonal, and D contains the entries on the diagonal. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = L + D + U = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Write the above three iterations as

$$x^{[k+1]} = T_x^{[k]} + c$$

and for each of the three methods give T in terms of L , D , and U , and c in terms of L , D , U and b .

Note: in class we derived the correct expressions for the Gauss-Jacobi method. There was a mistake in the in-class discussion of the Gauss-Seidel method, and I left the SOR method as an exercise.

Gauss-Jacobi:

$$\begin{aligned}x_i^{[k+1]} &= \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[k]} - \sum_{j=i+1}^n a_{ij}x_j^{[k]} \right) \\x^{[k+1]} &= D^{-1} \left(b - Lx^{[k]} - Ux^{[k]} \right) \\&= D^{-1} \left(b - (L + U)x^{[k]} \right) \\&= -D^{-1} (L + U)x^{[k]} + D^{-1}b\end{aligned}$$

$$\begin{aligned}\Rightarrow T &= -D^{-1} (L + U) \\c &= D^{-1}b\end{aligned}$$

Gauss-Seidel:

$$\begin{aligned}x_i^{[k+1]} &= \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[k]} - \sum_{j=i+1}^n a_{ij}x_j^{[k+1]} \right) \\x^{[k+1]} &= D^{-1} \left(b - Lx^{[k]} - Ux^{[k+1]} \right) \\x^{[k+1]} &= D^{-1}b - D^{-1}Lx^{[k]} - D^{-1}Ux^{[k+1]} \\(I + D^{-1}U)x^{[k+1]} &= D^{-1}b - D^{-1}Lx^{[k]} \\x^{[k+1]} &= -(I + D^{-1}U)^{-1}D^{-1}Lx^{[k]} + (I + D^{-1}U)^{-1}D^{-1}b\end{aligned}$$

$$\begin{aligned}\Rightarrow T &= -(I + D^{-1}U)^{-1}D^{-1}L \\c &= (I + D^{-1}U)^{-1}D^{-1}b\end{aligned}$$

Successive Overrelaxation:

$$\begin{aligned}
x_i^{[k+1]} &= (1 - w)x_i^{[k]} + \frac{w}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[k]} - \sum_{j=i+1}^n a_{ij}x_j^{[k+1]} \right) \\
x^{[k+1]} &= (1 - \omega)x^{[k]} + \omega D^{-1} (b - Lx^{[k]} - Ux^{[k+1]}) \\
&= (1 - \omega)x^{[k]} + \omega D^{-1}b - \omega D^{-1}Lx^{[k]} - \omega D^{-1}Ux^{[k+1]} \\
(I + \omega D^{-1}U)x^{[k+1]} &= ((1 - \omega) - \omega D^{-1}L)x^{[k]} + \omega D^{-1}b \\
x^{[k+1]} &= (I + \omega D^{-1}U)^{-1} ((1 - \omega) - \omega D^{-1}L)x^{[k]} + (I + \omega D^{-1}U)^{-1} \omega D^{-1}b \\
\Rightarrow T &= (I + \omega D^{-1}U)^{-1} ((1 - \omega) - \omega D^{-1}L) \\
c &= (I + \omega D^{-1}U)^{-1} \omega D^{-1}b
\end{aligned}$$

2. **(Positive Definite Matrices.)** A *principal submatrix* A_I of an $n \times n$ matrix A is obtained by picking a set $I \subset \{1, 2, \dots, n\}$ and crossing out all rows and columns whose indices are not in I . For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

then the principal submatrix corresponding to the set $\{2, 4\}$ is

$$A_{\{2,4\}} = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}$$

Show that every principal submatrix of a positive definite matrix is positive definite.

3. **(The UL factorization.)** Show how to compute the factorization $A = UL$ where U is upper triangular with 1s along the diagonal and L is lower triangular. Show how this relates to a way of solving $Ax = b$ by transforming the system into an equivalent system with a lower triangular matrix. (In other words, show that what we did for the LU factorization also works for a UL factorization.) Note: For the purposes of this exercise you may assume that no pivoting is required. This is of course unrealistic but pivoting would only distract from the point of this exercise (which is that conceptually there is no difference between an LU and a UL factorization).

4. **(Spectral Radius.)** We saw that the spectral radius of a (square) matrix A never exceeds an induced matrix norm of $\|A\|$. It can be shown that for any particular matrix A one can find a vector norm such that the induced matrix norm of A is arbitrarily close to the spectral radius of A . Does the spectral radius itself define a norm? Why, or why not?

$$\text{Consider } 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{and } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Clearly, $A \neq 0$, the eigenvalues of the zero matrix 0 and A are 0 and 0 respectively.

$$\Rightarrow \rho(A) = \rho(0)$$

$$= \max\{0, 0\}$$

By definition, a norm requires the property $\|A\| = 0 \Rightarrow A = 0$. Here, we have $\rho(A) = 0$, but $A \neq 0$. Therefore, ρ is *not* a matrix norm.

5. **(Inequalities are sharp.)** Explain the meaning of

$$\frac{1}{\|A\|\|A^{-1}\|} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|}$$

and show that how we derived these inequalities in class.

- For a general matrix A , and $\|\cdot\| = \|\cdot\|_2$, show that there are non-trivial examples (i.e. $x \neq 0 \neq e$) where the right hand inequality is satisfied with equality in (3). Do the same for the left hand inequality in (3).
 - Repeat part a. for $\|\cdot\| = \|\cdot\|_\infty$.
6. **(Backward Error Analysis.)** This problem explores the effects of a perturbation in the coefficient matrix (rather than the right hand side) of the linear system

$$Ax = b$$

Suppose we solve instead of (4) the system

$$(A - E)(x - e) = b$$

where E is a perturbation of A that causes an error e in the solution x . Show that

$$\frac{\|e\|}{\|x - e\|} \leq \|A\| \|A^{-1}\| \frac{\|E\|}{\|A\|}$$

$$Ax = b \tag{1}$$

$$Ae = r, e = A^{-1}r \tag{2}$$

$$(A - E)(x - e) = b \tag{3}$$

$$\Rightarrow A(x - e) - E(x - e) = b \tag{4}$$

$$\Rightarrow Ax - Ae - b = E(x - e) \tag{5}$$

$$\Rightarrow -r = E(x - e) \tag{6}$$

$$\|e\| \leq \|A^{-1}\| \|r\| \tag{7}$$

$$\|r\| \leq \|E\| \|x - e\| \tag{8}$$

$$\Rightarrow \frac{\|e\|}{\|E\| \|x - e\|} \leq \frac{\|A^{-1}\| \|r\|}{\|r\|} \tag{9}$$

$$\Rightarrow \frac{\|e\|}{\|x - e\|} \leq \|A^{-1}\| \|E\| = \|A\| \|A^{-1}\| \frac{\|E\|}{\|A\|} \tag{10}$$

$$\Rightarrow \frac{\|e\|}{\|x - e\|} \leq \|A\| \|A^{-1}\| \frac{\|E\|}{\|A\|} \tag{11}$$

7. **(First Order Systems.)** Write the second order initial value problem

$$y'' = xy^2 \tag{12}$$

$$y(0) = 1 \tag{13}$$

$$y'(0) = 2 \tag{14}$$

as an autonomous first order system

$$y' = f(y) \tag{15}$$

$$y(a) = y_0. \tag{16}$$

(In other words, specify y , f , a , and y_0 such that the two problems are equivalent. Of course, y will have different meanings for the two problems.)

$$y' = xy^2$$

$$y(0) = 1$$

$$y'(0) = 2$$

Let $z = y'$ and $z(0) = 2$

$$\Rightarrow z' = xy^2, y(0) = 1, z(0) = 2$$

$$\begin{aligned}\frac{\partial y}{\partial x} &= z, y(0) = 1 \\ \frac{\partial z}{\partial x} &= xy^2, z(0) = 2\end{aligned}$$

Define $x = x(t)$ s.t. $\frac{\partial x}{\partial t} = 1$ and $x(0) = 0$

$$\begin{aligned}\Rightarrow \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial x} \frac{\partial x}{\partial t} = z \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} = x(t)y^2(t) \\ \Rightarrow \frac{\partial x}{\partial t} &= 1, x(0) = 0 \\ \frac{\partial y}{\partial t} &= z(t), y(x(0)) = 1 \\ \frac{\partial z}{\partial t} &= x(t)y^2(t), z(x(0)) = 2 \\ v &= \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, f(v) = \begin{bmatrix} v \\ ve_3 \\ (ve_1)(ve_2)^2 \end{bmatrix}\end{aligned}$$

Where e_1, e_2, e_3 are the basis vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

respectively

$$\frac{\partial v}{\partial t} = f(v), v(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

8. **(Improper Integrals.)** Let

$$I = \int_{-1}^1 \frac{\ln(x^2 + 1)}{\sqrt{1 - x^2}} dx = 2\pi \ln \left[\frac{1 + \sqrt{2}}{2} \right].$$

This integral is improper because the integrand approaches infinity as x approaches ± 1 . Attempt to approximate this integral by using Simpson's Rule (or any other Newton-Cotes formula) on the interval $[-1 + \epsilon, 1 - \epsilon]$ for small $\epsilon > 0$. Describe the results of your efforts. Then, for $n = 4, 5, 6, 7, 8$, use the Gaussian Quadrature formula

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1 - x^2}} dx = \frac{\pi}{n} \sum_{i=1}^n f \left(\cos \left(\frac{(2i - 1)\pi}{2n} \right) \right)$$

which we discussed in class. Compute the error and discuss your results.

9. **(Numerical Comparison.)** For Euler's Method, the Trapezoidal Rule, Simpson's Rule, and the standard 4-th order Runge-Kutta Method solve the initial value problem

$$y' = y \tag{17}$$

$$y(0) = 1 \tag{18}$$

$$x \in [0, 1] \tag{19}$$

with step-sizes $h = 2^{-s}$ where $s = 3, 4, \dots, 15$. Plot the error at the right end-point, i.e. $y(1) - y_n$ (where $n = 1/h$), against h . You may wish to superimpose the plots. If you like use appropriate logarithmic graph paper. Comment on your results. (For Simpson's rule use the exact starting value $y_1 = y(h)$.) Send me your program by e-mail.

The "standard" Runge-Kutta method is:

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = f(x_n, y_n) \tag{20}$$

$$K_2 = f \left(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1 \right) \quad K_3 = f \left(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2 \right) \quad K_4 = f(x_n + h, y_n + hK_3) \tag{21}$$

(a) **Euler's Method**

$$y_{n+1} - y_n = hf_n \quad (22)$$

(b) **Trapezoidal Rule**

$$y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n) \quad (23)$$

(c) **Simpons's Rule**

$$y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n) \quad (24)$$

(d) **Successive Overrelaxation:**