Math 5600 5/27/14

- Summary from last week
- Termination criteria
- $g(x) = x \iff f(x) = 0$

  solution is α

  Fixed Point iteration: $x_{n+1} = g(x_n)$

  we want $|\alpha - x_n| < \varepsilon$

- when do we stop.
- Case 1. order of convergence > 1 (Newton)

  stop when $|x_n - x_{n-1}| < \varepsilon$

- Case 2. linear convergence

  estimate $g'(\alpha) \approx \lambda_n = \dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$

  stop when $\left| \dfrac{\lambda_n}{1-\lambda_n} (x_{n-1} - x_n) \right| < \varepsilon$

- consider accelerating the convergence

$$z_0 = x_0 \quad z_{n+1} = g(g(z_n)) - \frac{\left(g(g(z_n)) - g(z_n)\right)^2}{g(g(z_n)) - 2g(z_n) + z_n}$$

go on to polynomials ...

evaluate polynomials by

- Nested Multiplication
- Synthetic Division
- Horner's Scheme

Ex.: $p(x) = 2x^3 - 3x^2 + x - 4$

$p(2) = 2 \cdot 8 - 3 \cdot 4 + 2 - 4 = 2$

$p(x) = \big((2x-3)x+1\big)x - 4$

we can write this as

	2	-3	1	-4	
		4	2	6	
$x=2$	2	1	3	2	$= p(2)$
		4	10		
$x=2$	2	5	13	$= p'(2)$	

This always works!

In general

$$p(x) = \sum_{k=0}^{n} \alpha_k x^k$$

- Then we can evaluate $p(x)$ for a specific value of x by the recursion

$$\beta_n = \alpha_n$$

For $k = n-1, n-2, \ldots, 0$

(*) $$\beta_k = x_0 \beta_{k+1} + \alpha_k \qquad x = x_0, \text{say}$$

$$p(x_0) = \beta_0$$

- Now consider (long) division by $(x - x_0)$

(**) $$p(x) = (x - x_0) q(x) + p(x_0)$$

Then $$q(x) = \sum_{k=1}^{n} \beta_k x^{k-1}$$

To see this note that

$$\alpha_k = \beta_k - x_0 \beta_{k-1}$$

which is the same as (*)

- To see that $p'(x_0) = q(x_0)$ differentiate in (**)

$$p'(x) = q(x) + (x - x_0) q'(x) \Rightarrow p'(x_0) = q(x_0)$$

- Note that in general $q(x)$ does not equal $p'(x)$
- using nested multiplication twice is particularly convenient when we run Newton's Method to find roots of a polynomial.
- what if we continue nested multiplication? Do we get the second derivative
- For above example

$$p'(x) = 6x^2 - 6x + 4$$

$$\text{and} \quad p''(x) = 12x - 6 \qquad p''(2) = 18$$

- but using nested multiplication on the last row gives

$$\begin{array}{ll} 2 & 5 \\ & 4 \\ \hline 2 & 9 \end{array} = \frac{p''(2)}{2}$$

- in general we get $\dfrac{p^{(k)}(x_0)}{k!}$ exercise!

- Next subject: interpolation.
- interpolation = exact representation of data (as opposed to "approximation", approximate representation of data)
- we've seen one example: Taylor polynomial, exact representation of $f(x_0), f'(x_0), \ldots, f^{(n)}(x_0)$
- Another example: **Lagrange Interpolation**

  Given (x_i, y_i) $\quad i = 0, \ldots, n$

  Find a polynomial p of degree n such that

$$p(x_i) = y_i \qquad i = 0, \ldots, n$$

- One way to approach this problem is to solve the linear system

$$p(x_i) = \sum_{j=0}^{n} \alpha_j x_i^j = y_i \qquad i = 0, \ldots, n$$

- This can be written in matrix form:

The x_i are called knots, nodes, abscissas, x-values, or data sites

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n \end{bmatrix}}_{V_n} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}}_{y}$$

- let's use a specific example for all techniques of constructing p (we'll see several

i	x_i	y_i
0	1	5
1	2	10
2	4	32

$p(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$

$p(1) = \alpha_2 + \alpha_1 + \alpha_0 = 5$

$p(2) = 4\alpha_2 + 2\alpha_1 + \alpha_0 = 10$

$p(4) = 16\alpha_2 + 4\alpha_1 + \alpha_0 = 32$

- Linear system

$(x_0, y_0) = (1, 5)$

$(x_1, y_1) = (2, 10)$

$(x_2, y_2) = (4, 32)$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 32 \end{bmatrix}$$

- system is easy to solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 32 \end{bmatrix}$$

$p(x) = 4 - x + 2x^2$
$= 2x^2 - x + 4$

- The coefficient matrix is call the
  Vandermonde Matrix

in this context.

- Questions:
  - when does the system have a solution
  - is the solution unique?

- From Linear Algebra: the interpolation problem has a unique solution if and only if A is non-singular which is true if and only if the determinant of A is non-zero.

- we will see momentarily that

$$|V_n| = \prod_{j>i} (x_j - x_i) \qquad (*)$$

$\prod$ = product

- in our example

$$|V_n| = (4-1)(4-2)(2-1) = 6$$

- clearly the product is non-zero if and only if the knots are distinct, i.e.

$$i \neq j \Rightarrow x_i \neq x_j$$

- proof of (*) by induction (very pretty)

$n=0$ $V_0 = [1]$ det $V_0 = 1$ (which is the empty product)

$n=1$ $V_1 = \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}$ $|V_1| = x_1 - x_0$

- For the induction step expand the determinant about the last row

- You get a polynomial of degree n in x_n

$$\det |V_n| = \sum_{k=0}^{n} A_k \, x_n^k$$

- The factor of $x_n^n = |V_{n-1}|$

- moreover, $|V_n| = 0$ if x_n equals one of $x_0, \ldots, x_{n-1}$ since in that case V_n has two identical rows.

- Thus ~~V_n~~

$$\det V_n = \det V_{n-1} (x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})$$

$$= \prod_{n > j > i} (x_j - x_i)(x_n - x_0) \cdots (x_n - x_{n-1})$$

$$= \prod_{n \geq j > i} (x_j - x_i)$$

- As mentioned before, this shows that the interpolation problem has a unique solution iff the knots are distinct.

- You might think this is trivial since we have as many equations as parameters
- Here are a couple of interpolation problems where we have as many equations as parameters, but no unique solution.

1. Find a quadratic q such that

$$q(-1) = A \qquad q'(0) = B \qquad q(1) = C$$

Example $A = C = 0 \Rightarrow q'(0) = 0$

if $B = 0$ there are infinitely many solutions, if $B \neq 0$ there are none.

if we write q in standard form,

$$q(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$$
$$q'(x) = 2\alpha_2 x + \alpha_1$$

we get the Vandermonde

$$V = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

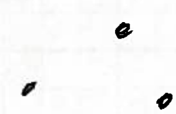
where $\det V = 0$

- Another example. 2 variables
- Given (x_i, y_i, z_i) $i = 1, 2, 3$, find

$$L(x, y) = ax + by + c$$

such that $L(x_i, y_i) = z_i$ $i = 1, 2, 3$

- If the data sites (x_i, y_i) form a (non-degenerate) triangle there is a unique solution



if they lie on a line

there is no solution, or infinitely many.

- Back to the Lagrange problem.

We have constructed the interpolant (the interpolating polynomial) in standard, or power, form.

- There are other forms of the same polynomial.

- The Lagrange, or cardinal, form

- idea: use part of the data as coefficients

$$p(x) = \sum_{i=0}^{n} y_i L_i(x)$$

- where $L_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

Kronecker Delta

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

$$= \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}$$

- For our running example, $(1,5), (2,10), (4,32)$

$$p(x) = 5\frac{(x-2)(x-4)}{(1-2)(1-4)} + 10\frac{(x-1)(x-4)}{(2-1)(2-4)} + 32\frac{(x-1)(x-2)}{(4-1)(4-2)}$$

$$= 2x^2 - x + 4 \quad \text{(exercise)}$$

- The Newton Form:

Basic idea: add one data point at a time

$P_k(x)$ interpolates at $x_0, \ldots, x_k$

$$P_0(x) = y_0$$

$$P_1(x) = P_0 + \gamma_1 (x - x_0) \qquad \gamma_1 = ?$$

$$P_1(x_1) = P_0 + \gamma_1 (x_1 - x_0) = y_1$$

$$\gamma_1 = \frac{y_1 - P_0}{x_1 - x_0} \quad \text{divided difference}$$

In our example $\quad (x_0, y_0) = (1,5)$
$(x_1, y_1) = (2,10)$
$(x_2, y_2) = (4,32)$

$$P_0(x) = 5$$

$$P_1(x) = 5 + \frac{10-5}{2-1}(x-1)$$

$$= 5 + 5(x-1) = 5x$$

In general,

$$P_{k+1}(x) = P_k(x) + \gamma_{k+1} \prod_{i=0}^{k} (x - x_i) \qquad \gamma_{k+1} = ?$$

$$P_{k+1}(x_{k+1}) = y_{k+1} = P_k(x_{k+1}) + \gamma_{k+1} \prod_{i=0}^{k} (x_{k+1} - x_i)$$

$$\gamma_{k+1} = \frac{y_{k+1} - P_k(x_{k+1})}{\prod_{i=0}^{k} (x_{k+1} - x_i)}$$

- In our example $\qquad P_1(4) = 20$

$$\gamma_2 = \frac{32 - 20}{(4-1)(4-2)} = \frac{12}{6} = 2$$

$$\begin{aligned} P_2(x) &= 5x + 2(x-1)(x-2) \\ &= 5x + 2(x^2 - 3x + 2) \\ &= 2x^2 - x + 4 \quad \checkmark \end{aligned}$$

- The γ_k are "divided differences", a big subject in classical Numerical Analysis.

Iterated Interpolation, or Blending

combine, blend, 2 interpolants to get a new interpolant whose interpolation properties are the union of the properties of the two ingredients.

p_0 interpolates to $(x_0, y_0) \cdots (x_k, y_k)$

p_1 interpolates to $(x_1, y_1) \cdots (x_{k+1}, y_{k+1})$

p interpolates to $(x_0, y_0) \cdots (x_{k+1}, y_{k+1})$

we want to express p in terms of p_0 and y_0

$$p(x) = \frac{x - x_{k+1}}{x_0 - x_{k+1}} \, p_0(x) + \frac{x - x_0}{x_{k+1} - x_0} \, p_1(x)$$

- clearly $p(x_0) = p_0(x_0) = y_0$

  $p(x_{k+1}) = p_1(x_{k+1}) = y_{k+1}$

- what about the other points?

$$P_0(x_i) = P_1(x_i) = y_i$$

$$P(x_i) = \left[\frac{x_i - x_{k+1}}{x_0 - x_{k+1}} + \frac{x_i - x_0}{x_{k+1} - x_0}\right] y_i$$

$$= \left[\frac{x_{k+1} - x_i + x_i - x_0}{x_{k+1} - x_0}\right] y_i = y_i$$

- Then we can obtain an interpolant as follow:

$$\begin{array}{llll} (x_0, y_0) & y_0 & & \\ & & y_{01} & \\ (x_1, y_1) & y_1 & & y_{012} \quad \text{etc.} \\ & & y_{12} & \\ (x_2, y_2) & y_2 & & \end{array}$$

In our example

x_i	y_i		
1	5		
		$5x$	
2	10		$2x^2 - x + 4$ (exercise)
		$11x - 12$	
4	32		

$$y_{01} = \frac{x-2}{1-2} 5 + \frac{x-1}{2-1} 10 = 5x$$

$$y_{12} = \frac{x-4}{2-4} 10 + \frac{x-2}{4-2} 32 = 11x - 12$$

$-5x + 16x$

$20 - 32$

Important Point: All approaches give the same interpolant. They give different *forms* of the interpolant

- we can also show existence and uniqueness in a simpler nonconstructive way.

- consider the linear system $V_n a = y$

- V_n is non-singular if $V_n a = 0$ has only the trivial solution

- Any solution of that system $V_n a = 0$ would give a polynomial of degree n that has $n+1$ roots $x_0, \ldots, x_n$

- The only such polynomial is the zero polynomial which has $a = 0$.