Math 5600 7/9/14

- Recall the QR algorithm

For $k = 1, 2, \ldots$

$A = QR$
$A = RQ$

equivalent to a simultaneous orthogonal iterations.

- If the eigenvalues are strictly ordered

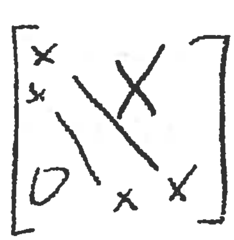
$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| \qquad (*)$$

this process converges to a triangular matrix that is similar to A and has the eigenvalues of A along the diagonal.

- However, we need to make the algorithm more efficient and we need to get it to work without the assumption $(*)$.

- We will start with transforming A to upper Hessenberg form.

A is u.H $\iff$ $i - j > 1 \Rightarrow a_{ij} = 0$



$$H_0 = U_0^T A U_0 \qquad U_0^T = U_0^{-1}$$

- How do we do this?

- 1^{st} column

$$H = \begin{bmatrix} 1 & 0 \\ 0 & \underbrace{I - 2vv^T}_{\text{Householder Reflector}} \end{bmatrix}$$

$$\begin{bmatrix} x & x & \cdots & x \\ a & & B & \end{bmatrix}$$

$$HA = \begin{bmatrix} x & \\ x & \\ 0 & X \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} x & x & \cdots & x \\ Ha & & HB & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \begin{array}{c} \begin{bmatrix} a_{11} & y^T \\ z & A_{22} \end{bmatrix} \end{array}\begin{array}{l} 1 \\ n-1 \end{array} = \begin{bmatrix} a_{11} & y^T \\ Hz & HA_{22} \end{bmatrix}$$

- pick H such that Hz is a multiple of e_1

- multiplying with H^T from the right (to get a similarity transform) does not undo this effect.

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 \\ 0 & H^T \end{bmatrix} \\ \begin{bmatrix} a_{11} & y^T \\ Hz & HA_{22} \end{bmatrix} & \begin{bmatrix} a_{11} & y^T H^T \\ Hz & HA_{22}H^T \end{bmatrix} \end{array}$$

- Then repeat the process on $HA_{22}H^T$ as usual.

- Note that creating zeros below the diagonal gets undone by multiplication from the right.

$$\begin{array}{cc} & \begin{bmatrix} x & x & x & x \\ x & & & \\ x & & & \\ x & & & \end{bmatrix} \\ \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} & \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \end{array}$$

- upper Hessenberg is the best we can do.

- of course if we could get triangular form we'd solve in a finite # of steps

- Does the QR iteration preserve the upper Hessenberg structure?

- yes

$H = QR$ $\qquad$ Q is upper Hessenberg

$$\begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix} \quad R$$

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$Q \qquad\qquad H$$

use induction on the columns

RQ:

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

- so now we have the algorithm

$$H_0 = U_0^T A U_0 \quad \text{upper Hessenberg}$$

For $k = 1, 2, \dots$

$$H_{k-1} = U_k R_k \qquad QR, \ U_k \ \text{u.H.}$$

$$H_k = R_k U_k$$

- We would like this to converge to upper quasi-triangular form

- quasi-triangular = block triangular with 1×1 and 2×2 triangular blocks

$$Q^T A Q = \begin{bmatrix} R_{11} & & & R_{1m} \\ & R_{22} & & \vdots \\ & & \ddots & \\ 0 & & & R_{mm} \end{bmatrix}$$

R_{ii} 1×1 or 2×2 blocks

- For this to be useful, the QR factorization of an u.H. matrix should take only $O(n^2)$ operations.

- Use Givens Rotations.

- explain for 2x2

$$G = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \qquad \begin{aligned} c &= \cos\alpha \\ s &= \sin\alpha \end{aligned}$$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

- rotates by α
- $G^{-1} = G^T$ since $\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} c^2+s^2 & 0 \\ 0 & c^2+s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- pick α such that $\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$

$$-sx + cy = 0$$

$$\frac{s}{c} = \frac{y}{x}$$

$$\tan\alpha = \frac{y}{x} \qquad \alpha = \tan^{-1}\frac{y}{x}$$

So we embedd Givens Rotations into the diagonal as usual

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$\begin{bmatrix} x & x & & \\ x & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & x & x & \\ & x & x & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & x & x \\ & & x & x \end{bmatrix} \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}$$

QR factorization

Q: product of Givens Rotations.

general G.R.

$$\begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & s & \\ & & & & \ddots & & \\ & & & -s & & c & \\ & & & & & & \ddots & \\ 0 & & & & & & & 1 \end{bmatrix}$$

- clearly, one QR Factorization requires only $O(n^2)$ operations.

- critical aspect of QR algorithm.

Decoupling.

$$H = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}$$

H_{11}, H_{22} upper Hessenberg

- An uH matrix is <u>unreduced</u> if all its subdiagonal entries are non-zero

- Decoupling can be accomplished by shifts of origin

$$H - \mu I = UR \qquad QR \text{ fact.}$$

$$H = RU + \mu I$$

- This generates a sequence of similar matrices since

$$RU + \mu I = U^T(UR + \mu I)U = U^T H U$$

- How do we pick μ

- decoupling occurs when μ is an eigenvalue. Suppose H is unreduced.

$$H - \mu I = UR \qquad \text{is singular}$$

$$U \text{ is orthogonal} \Rightarrow R \text{ is singular}$$

- In fact, $r_{nn} = 0$

- why?

- For an <u>unreduced</u> upper Hessenberg matrix H, the first $n-1$ columns of $H-\mu I$ are linearly independent

- So the n-th column of $H-\mu I$ must be a linear combination of the first $n-1$, since $H-\mu I$ is singular

$$H-\mu I = U R$$

- the first $n-1$ columns of $H-\mu I$ are linear combinations of the first $n-1$ columns of U, so is the n-th column

- r_{nn} is zero