Math 5600

7/7/14

- Start with eigenvalue problems.

$$Ax = \lambda x \qquad x \neq 0 \qquad A \; n \times n$$

λ eigenvalue $\quad x$ corresponding eigenvector

- eigenvectors are determined only up to a constant factor

- main difference ~~between~~ $Ax = b$ and $Ax = \lambda x$?

- $Ax = \lambda x \iff Ax - \lambda x = 0$

$$\iff (Ax - \lambda I)x = 0$$

$$\iff Ax - \lambda I \text{ is singular}$$

$$\iff \det(A - \lambda I) = 0$$

$$\det A = \sum_{\sigma} \operatorname{sign} \sigma \prod_{i=1}^{n} a_{i\sigma_i}$$

$$\Rightarrow \det(A - \lambda I) = (-\lambda)^n + LOT$$

characteristic polynomial of A

- However, it is not a good idea to find eigenvalues by finding roots of the characteristic polynomial.

- The reason is that the roots of a polynomial are very sensitive with respect to small perturbations of the coefficients

- It's better to go the other way: to find the roots of a polynomial set up and solve an appropriate eigenvalue problem

- $p(x) = (-1)^n \left[\lambda^n - \sum_{i=0}^{n-1} \alpha_i \lambda^i \right]$

$p(\lambda) = \det(C - \lambda I)$

C is the companion matrix of p

$$C = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & 0 \end{bmatrix}$$

Proof: Good exercise.

- A matrix may or may not have n linearly independent eigenvectors.

- If it does, it's <u>diagonalizable</u>

- If it does not it's non-diagonalizable or <u>defective</u>

- Suppose $Ax_i = \lambda_i x_i \qquad i=1,\dots,n$

$$X = [x_1, x_2, \dots, x_n] \quad \text{invertible}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$AX = X\Lambda$$

$$\Lambda = X^{-1}AX$$

- This is an example of a similarity transform.

- A and B are similar if

$$B = T^{-1} A T$$

for some matrix T.

- similar matrices have the same eigenvalues, Suppose $Ax = \lambda x$

$$B(T^{-1}x) = T^{-1} A T T^{-1} x$$
$$= T^{-1} A x$$
$$= \lambda (T^{-1} x)$$

- Note that eigenvalues may be complex.

- If they are they occur in conjugate complex pairs.

- But matrices may be defective.

Ex.: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

eigenvalues are $0, 0$

$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad y = 0$

- only eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- This is not an artifact of A being singular!

- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ evalues: $1, 1$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow y = 0$

In fact, singularity and defectiveness are unrelated

	singular	non-singular
defective	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
diagonalizable	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- ill conditioning with respect to $Ax = b$ is governed by $\|A\| \|A^{-1}\|$

- ill-conditioning with respect to $Ax = \lambda x$ is governed by $\|X\| \|X^{-1}\|$ where X is the matrix of eigenvectors
- Left eigenvector $y^T A = \lambda y^T$
- Hugely useful fact: Left and right eigenvectors corresponding to different eigenvalues are orthogonal.

$Ax = \lambda x \qquad y^T A = \lambda y^T$

$A\hat{x} = \mu \hat{x} \qquad \hat{y}^T A = \mu \hat{y}^T \qquad \lambda \neq \mu$

$$\hat{y}^T A x = \lambda \hat{y}^T x$$

$$\hat{y}^T A y = \mu \hat{y}^T x$$

$$0 = (\lambda - \mu)\hat{y}^T x \Rightarrow \hat{y}^T x = 0$$

- Application

- suppose $\quad z = \sum_{i=1}^{n} \alpha_i x_i \qquad Ax_i = \lambda_i x_i$

$$y_1^T z = \alpha_1 y_1^T x_1$$

- so the coefficient of x_1 can be computed without knowing the other eigenvectors.

- Eigenvalues of matrix functions

A	λ
A^{-1}	$1/\lambda$
A^n	λ^n

- Basic idea of eigenvalue computations: Apply similarity transforms to reduce the problem to one involving a simpler matrix

- Like a diagonal or triangular one.

- would be nice in particular if we could use orthogonal transform

$$Q^T Q = I \qquad B = Q^T A Q$$

- Basic idea of power method