Math 5600 5/13/14

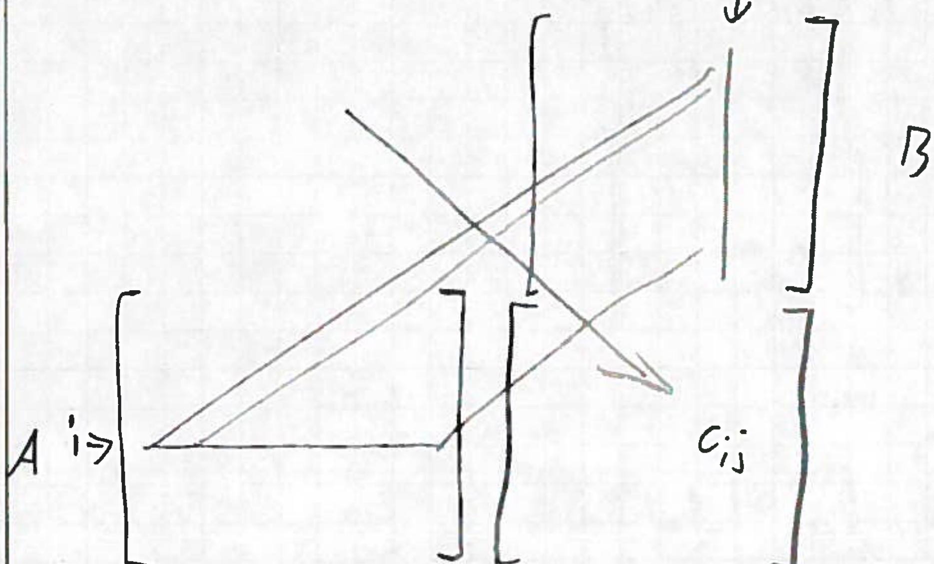
Recall matrix multiplication:

$A \quad m \times p$

$B \quad p \times n$

$C = AB \quad m \times n$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$




$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 10 & 12 \\ 3 & 7 & 8 \\ 7 & 17 & 20 \end{bmatrix}$$

- Block Matrices
- Matrices whose entries are matrices
- Block Matrices are multiplied just like ordinary matrices assuming the dimensions match.

$$\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \end{array} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \quad \begin{array}{cc} \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \\ & \begin{array}{cc} n_1 & n_2 \end{array} \end{array}$$

$$C_{11} = A_{11} B_{11} + A_{12} B_{21}$$

$$C_{12} = A_{11} B_{12} + A_{12} B_{22}$$

$$C_{21} = A_{21} B_{11} + A_{22} B_{21}$$

$$C_{22} = A_{21} B_{12} + A_{22} B_{22}$$

- rank of a matrix = number of linearly independent rows
  = number of linearly independent columns
- square linear system $Ax = b$ $m = n$
- This system has a unique solution

$\iff$ rank $A = n$

$\iff$ $A$ is non-singular

$\iff$ $\det A \neq 0$

$\iff$ $A$ has an inverse $A^{-1}$ $\quad A A^{-1} = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\iff$ $Ax = c$ has a solution for every $c$

$\iff$ $Ax = 0$ has only the solution $x = 0$

$\iff$ All eigenvalues of $A$ are non-zero

$$\det A = \sum_{\sigma} \operatorname{sign} \sigma \prod_{i=1}^{n} a_{i\sigma_i} \qquad (*)$$

$\sigma$: a permutation of $\{1, 2, \dots, n\} \to \{\sigma_1, \sigma_2, \dots, \sigma_n\}$

$$\operatorname{sign} \sigma = \pm 1 = (-1)^m = \begin{cases} 1 & \text{if } m \text{ is even} \\ -1 & \text{if } m \text{ is odd} \end{cases}$$

$m$ is the number of transpositions (switches of neighboring elements) to get $\sigma$

- This formula does not provide a good way to compute det A.
- suppose we can compute and sum $10^9$ products per second.
- T: time required to compute det A using (*):

| n | T |
|---|---|
| 1 | $10^{-9}$ sec |
| 10 | 0.003 sec |
| 20 | 77 years |
| 25 | 0.5 billion years |
| 26 | 12.7 billion years (age of universe) |

Nonetheless, (*) is a conceptually useful formula.

- It implies, for example, that the determinant of a triangular matrix is the product of the diagonal entries.

- Use simple row operations to reduce to triangular form

| operation | Determinant |
|---|---|
| multiply a single row with a constant $k$ | multiply $\det A$ with $k$ |
| switch two rows | multiply $\det A$ with $-1$ |
| Add multiple of some row to another row | no change |

$$\det A^T = \det A$$
$$\det AB = \det A \det B$$
$$\det A = \text{product of eigenvalues.}$$

eigenvalues and vectors

eigen = "own"

$Ax = \lambda x \qquad x \neq 0$

$x$ eigenvector, $\lambda$ corresponding eigenvalue

$Ax = \lambda x$

$Ax - \lambda x = (A - \lambda I)x = 0$

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \underbrace{(-\lambda)^n + \sum_{j=0}^{n-1} \alpha_j \lambda^j}_{\text{characteristic polynomial of } A}$$

- The eigenvalues are roots of the characteristic polynomial
- The eigenvalues of a real matrix may be complex.
- However, the eigenvalues of a symmetric real matrix are real.

- Finding the eigenvalues of a matrix by computing the roots of the characteristic polynomial is not a good way

- In fact it's better to go the other way.

companion matrix of

- $$\det\left(\begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ 0 & & & 1 & \end{bmatrix} - \lambda I\right)$$

$$= (-1)^n \left(\lambda^n - \sum_{j=0}^{n-1} \alpha_j \lambda^j\right)$$

- One way (a good one!) to find the roots of a polynomial is to compute the eigenvalues of its companion matrix.

- That's exactly what the matlab "roots" command does.

- The following result maximizes the ratio $\frac{\text{utility}}{\text{notoriety}}$

Gershgorin Theorem.

Suppose $Ax = \lambda x$ for some $x \neq 0$

Then, for some $i$

$$|a_{ii} - \lambda| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

- Proof: Suppose $\max_j |x_j| = 1 = x_i$

This is the definition of $i$. We can assume this since eigenvectors are determined only up to a constant factor.

- Then look at the $i$-th equation of (*)

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i = x_i$$

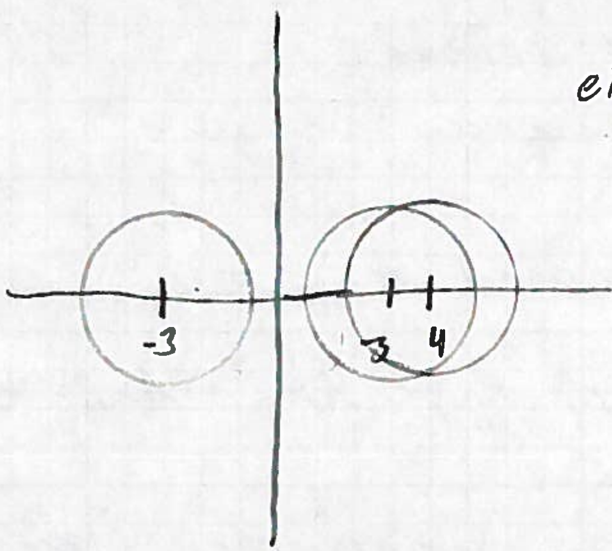
$$a_{ii} x_i - \lambda x_i = a_{ii} - \lambda = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j$$

$$|a_{ii} - \lambda| \leq \sum_{j \neq i} |a_{ij}| |x_j| \leq \sum_{j \neq i} |a_{ij}|$$

- Example.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & -3 \end{bmatrix}$$

- Gershgorin circles:




eigenvalues are

$-3.259$

$2.349$

$4.860$

- simple fact, but an entire book:

R.S. Varga, Gershgorin and His Circles
Springer, 2010, ISBN 9783642059285

- Major Principle: The general solution of a linear problem equals any particular solution plus the general solution of the homogeneous version of that problem.

- Illustrate with linear system

$Ax = b$ original problem

$Ax = 0$ homogeneous version

suppose $Ax_p = b$ and $Ax_h = 0$

then $A(x_p + x_h) = Ax_p + Ax_h = b + 0 = b$

on the other hand, if we have two solutions, $x_p$ and $\hat{x}_p$ of the original problem

$$Ax_p = A\hat{x}_p = b$$

then they differ by a solution of the homogeneous version

$$A(x_p - \hat{x}_p) = Ax_p - A\hat{x}_p = b - b = 0$$

Many types of matrices. Here are just a few. Assume $A$ is $m \times n$ real

square $\quad m = n$

rectangular $\quad m \neq n$

$\left.\begin{array}{l}\text{upper}\\ \text{lower}\end{array}\right\}$ triangular $\left\{\begin{array}{l} i > j \Rightarrow a_{ij} = 0 \\ i < j \Rightarrow a_{ij} = 0 \end{array}\right.$

symmetric $\quad A = A^T$

diagonal $\quad i \neq j \Rightarrow a_{ij} = 0$

tridiagonal $\quad |i-j| > 1 \Rightarrow a_{ij} = 0$

$\left.\begin{array}{l}\text{upper}\\ \text{lower}\end{array}\right\}$ Hessenberg $\left\{\begin{array}{l} i > j+1 \Rightarrow a_{ij} = 0 \\ j > i+1 \Rightarrow a_{ij} = 0 \end{array}\right.$

orthogonal $\quad A^{-1} = A^T \qquad m = n$

$\left.\begin{array}{ll}\text{positive definite} & x \neq 0 \Rightarrow x^T A x > 0 \\ \text{negative definite} & x \neq 0 \Rightarrow x^T A x < 0\end{array}\right\} m = n$

full rank $\quad \text{rank } A = \min\{m, n\}$

rank-deficient $\quad \text{rank } A < \min\{m, n\}$

defective: $m = n \quad \dim \text{span}\{\text{eigenvectors}\} < n$

$\left.\begin{array}{l}\text{singular (or non-invertible)}\\ \text{non-singular (or invertible)}\end{array}\right\} m = n$

see pg 3