

# Homework #4

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1. **(Iterative Solution of Linear Systems.)** Consider the  $n \times n$  linear system  $Ax = b$  and recall the three iterative techniques we discussed in class:

**Gauss-Jacobi:**

$$x_i^{[k+1]} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[k]} - \sum_{j=i+1}^n a_{ij}x_j^{[k]} \right)$$

**Gauss-Seidel:**

$$x_i^{[k+1]} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[k]} - \sum_{j=i+1}^n a_{ij}x_j^{[k+1]} \right)$$

**Successive Overrelaxation:**

$$x_i^{[k+1]} = (1 - w)x_i^{[k]} + \frac{w}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[k]} - \sum_{j=i+1}^n a_{ij}x_j^{[k+1]} \right)$$

As we discussed in class, think of  $A$  as being split as

$$A = L + D + U$$

where  $L$  contains the entries of  $A$  below the diagonal,  $U$  contains the entries above the diagonal, and  $D$  contains the entries on the diagonal. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = L + D + U = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Write the above three iterations as

$$x^{[k+1]} = T_x^{[k]} + c$$

and for each of the three methods give  $T$  in terms of  $L$ ,  $D$ , and  $U$ , and  $c$  in terms of  $L$ ,  $D$ ,  $U$  and  $b$ .

Note: in class we derived the correct expressions for the Gauss-Jacobi method. There was a mistake in the in-class discussion of the Gauss-Seidel method, and I left the SOR method as an exercise.

**Gauss-Jacobi: Gauss-Seidel: Successive Overrelaxation:**

2. **(Positive Definite Matrices.)** A *principal submatrix*  $A_I$  of an  $n \times n$  matrix  $A$  is obtained by picking a set  $I \subset \{1, 2, \dots, n\}$  and crossing out all rows and columns whose indices are not in  $I$ . For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

then the principal submatrix corresponding to the set  $\{2, 4\}$  is

$$A_{\{2,4\}} = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}$$

Show that every principal submatrix of a positive definite matrix is positive definite.

3. **(The  $UL$  factorization.)** Show how to compute the factorization  $A = UL$  where  $U$  is upper triangular with 1s along the diagonal and  $L$  is lower triangular. Show how this relates to a way of solving  $Ax = b$  by transforming the system into an equivalent system with a lower triangular matrix. (In other words, show that what we did for the  $LU$  factorization also works for a  $UL$  factorization.) Note: For the purposes of this exercise you may assume that no pivoting is required. This is of course unrealistic but pivoting would only distract from the point of this exercise (which is that conceptually there is no difference between an  $LU$  and a  $UL$  factorization).
4. **(Spectral Radius.)** We saw that the spectral radius of a (square) matrix  $A$  never exceeds an induced matrix norm of  $\|A\|$ . It can be shown that for any particular matrix  $A$  one can find a vector norm such that the induced matrix norm of  $A$  is arbitrarily close to the spectral radius of  $A$ . Does the spectral radius itself define a norm? Why, or why not?
5. **(Inequalities are sharp.)** Explain the meaning of

$$\frac{1}{\|A\|\|A^{-1}\|} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|}$$

and show that how we derived these inequalities in class.

- For a general matrix  $A$ , and  $\|\cdot\| = \|\cdot\|_2$ , show that there are non-trivial examples (i.e.  $x \neq 0 \neq e$ ) where the right hand inequality is satisfied with equality in (3). Do the same for the left hand inequality in (3).
- Repeat part a. for  $\|\cdot\| = \|\cdot\|_\infty$ .