Math 5600 6/4/14

- Continuous Least Squares

- so far we have approximated functions by interpolation

- There are alternatives

$$f(x) \approx \sum_{i=0}^{n} \alpha_i b_i(x) = p(x)$$

- we approximate f by a linear combination of basis functions.

- The b_i may be polynomial, $b_i(x) = x^i$ but need not be

- How do we quantify $f(x) \approx p(x)$

- There are various ways of measuring the error

Examples $\int_a^b |f(x) - p(x)|\,dx = \min$

$$\max_{a \le x \le b} |f(x) - p(x)| = \min$$

$$\int_a^b (f(x) - p(x))^2\,dx = \min$$

- The last is called (continuous, linear) Least squares.

- "continuous" means we have infinitely many points, $a \le x \le b$ (The alternative is "discrete" least squares)

- "Linear" means we use a linear combination of basis functions and we will get a linear system to solve.

- Examples for nonlinear LS..

$$f(x) \approx \alpha \sin(\beta x + \gamma) \qquad f(x) \approx \frac{\alpha_0 + \alpha_1 x}{\beta_0 + x}$$

$$f(x) \approx \alpha e^{\beta x}$$

- Let's see how to solve the linear problem.

- Example- $b_i(x) = x^i \qquad 0 \le x \le 1$

$$\int_0^1 \left(f(x) - \sum_{j=0}^n \alpha_j x^j\right)^2 dx = \min$$

- $$F(\alpha_0, \dots, \alpha_n) = \int_0^1 \left(f(x) - \sum_{j=0}^n \alpha_j x^j\right)^2 dx = \min$$

- set the gradient to zero and solve

$$\frac{\partial F}{\partial \alpha_i} = -2\int_0^1 \left(f(x) - \sum_{j=0}^n \alpha_j x^j\right) x^i \, dx = 0$$

- This becomes

$$\sum_{j=0}^n \alpha_j \int_0^1 x^j x^i \, dx = \int_0^1 f(x) x^i \, dx$$

$$\int_0^1 x^j x^i \, dx = \int_0^1 x^{i+j} \, dx = \frac{1}{i+j+1} \qquad i,j = 0, \dots, n$$

- so we get the system

$$\sum_{j=0}^n \frac{1}{i+j+1} \alpha_j = \int_0^1 f(x) x^i \, dx \qquad i = 0, \dots, n$$

$$\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{bmatrix}}_{(n\times 1)\times(n+1)\ \text{Hilbert Matrix}} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x)\,dx \\ \vdots \\ \int_0^1 x^n f(x)\,dx \end{bmatrix}$$

– Example $n=1$ $f(x)=e^x$

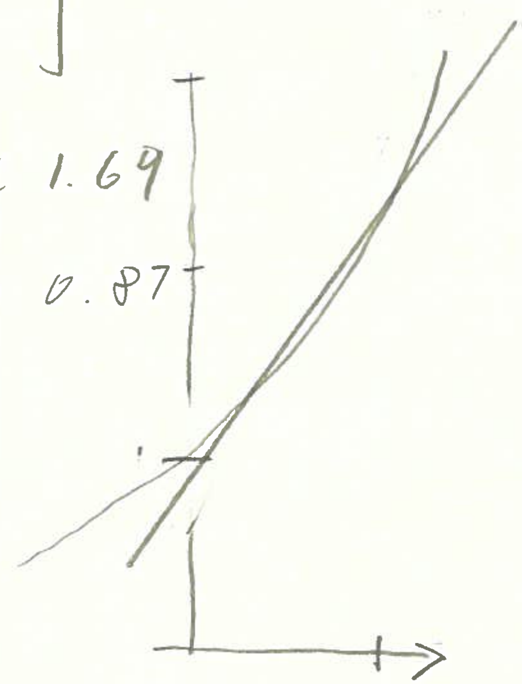
$$\int_0^1 e^x = e-1 \qquad \int_0^1 x e^x = 1$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \end{bmatrix}$$

$$\alpha_0 = 18-6e \approx 1.69$$

$$\alpha_1 = 4e-10 \approx 0.87$$

makes sense.



- Properties of the Hilbert Matrix:

- Let's redo the problem in general.

$$F(\alpha_0, \ldots, \alpha_n) = \int_a^b \left(f(x) - \sum_{j=0}^n \alpha_j b_j(x)\right)^2 dx = \min$$

$$\frac{\partial F}{\partial \alpha_i} = -2 \int_a^b \left(f(x) - \sum_{j=0}^n \alpha_j b_j(x)\right) b_i(x)\, dx = 0$$

- we get the linear system

$$\sum_{j=0}^n \alpha_j \int_a^b b_i(x) b_j(x)\, dx = \int_a^b f(x) b_i(x)\, dx$$

- we don't have to write a polynomial in standard form. How about

$$b_0(x) = 1$$
$$b_1(x) = 2x - 1$$
$$b_2(x) = 6x^2 - 6x + 1$$

$$\left[\int_0^1 b_i(x) b_j(x)\, dx\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

- The matrix is diagonal!
- much nicer!

- Does $\int_a^b b_i(x)\, b_j(x)\, dx$

remind you of anything?

$$\int_a^b u(x)\, v(x)\, dx \quad \text{versus} \quad u^T v$$

u, v vectors

- We have an inner product (or dot product)!

(f, g) is an inner product if

$$(f,g) = (g,f)$$

$$(f+g, h) = (f,h) + (g,h)$$

$$(f,f) \geq 0$$

$$(f,f) = 0 \Rightarrow f = 0$$

(say we restrict ourselves to continuous functions f and g)

- Examples:

$$(f,g) = \int_a^b w(x) f(x) g(x)\, dx \qquad w(x) > 0$$

$$(f,g) = \int_a^b f(x)g(x) + f'(x)g'(x)\, dx$$

$$(f,g) = \int_a^b f(x)g(x)\,dx + f(c)g(c)$$

- Two functions f and g are orthogonal with respect to a given inner product if

$$(f,g) = 0$$

- Let's do our linear system again in terms of inner products.

$$F(\alpha_0, \cdots, \alpha_n) = \left(f - \sum_{j=0}^{n} \alpha_j b_j, f - \sum_{j=0}^{n} \alpha_j b_j\right)$$

$$= (f, f) - 2 \sum_{j=0}^{n} \alpha_j (b_j, f) + \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j (b_i, b_j)$$

$$\frac{\partial F}{\partial \alpha_i} = -2(b_i, f) + 2 \sum_{j=0}^{n} \alpha_j (b_i, b_j) = 0$$

- same system as before.
- It would be nice if the linear system was diagonal, i.e., the basis functions are orthogonal.