Math 5600 7/01/14

- Let's start with a large linear system -

- Laplace Equation

$$u_{xx} + u_{yy} = 0 \qquad 0 \le x, y \le 1$$

$$u = u(x,y)$$

$$u(x,y) = g(x,y)$$ on the boundary of the unit square

- heat distribution in a plate
- Discretization

$$x_n = y_n = nh \qquad h = \frac{1}{N+1}$$

$$U_{mn} \approx u(x_m, y_n)$$

$$u_{xx}(x_m, y_n) \approx \frac{u(x_{m+1}, y_n) - 2u(x_m, y_n) + u(x_{m-1}, y_n)}{h^2}$$

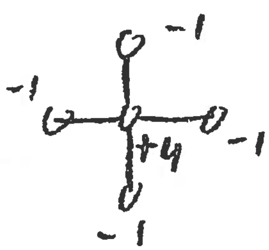
$$u_{yy}(x_m, y_n) \approx \frac{u(x_m, y_{n+1}) - 2u(x_m, y_n) + u(x_m, y_{n-1})}{h^2}$$

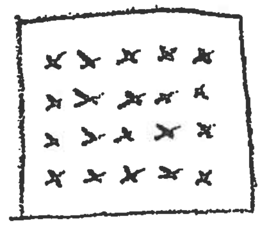
- Discretization

$$\frac{u_{m+1,n}-2u_{mn}+u_{m-1,n}}{h^2}+\frac{u_{m,n+1}-2u_{mn}+u_{m,n-1}}{h^2}=0$$

- Rewrite as

$$4u_{mn}-u_{m+1,n}-u_{m-1,n}-u_{m,n+1}-u_{m,n-1}=0 \qquad (*)$$

- 



$N=4$

- This is a linear system of N^2 equations in N^2 unknowns.

- It's not homogeneous despite its appearance, because of the boundary condition.

- Huge literature on how to solve systems like this.

- We'll just look at some basic ideas.

- Describe them in terms of a general linear system, but keep going back to (*) for illustration

$$Ax = b \qquad A \ n\times n \qquad x, b \in \mathbb{R}^n$$

$$\sum a_{ij} x_j = b_i$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j \right)$$

- Start with an initial solution $x^{[0]}$

$$x_i^{[k+1]} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[k]} - \sum_{j=i+1}^{n} a_{ij} x_j^{[k]} \right)$$

Jacobi Method (or iteration)

$$x_i^{[k+1]} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[k+1]} - \sum_{j=i+1}^{n} a_{ij} x_j^{[k]} \right)$$

Gauss-Seidel Method

$$x_i^{[k+1]} = (1-\omega) x_i^{[k]} + \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[k+1]} - \sum_{j=i+1}^{n} a_{ij} x_j^{[k]} \right)$$

Successive Overrelaxation SOR

ω relaxation parameter

Next step: think of this iteration in terms of matrices.

$$A = L + D + U \qquad \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{U}$$

- The Jacobi Method becomes

$$x^{[k+1]} = D^{-1}(b - (L+U)\, x^{[k]})$$
$$= -D^{-1}(L+U)\, x^{[k]} + D^{-1} b$$

- The GS method becomes:

$$x^{[k+1]} = D^{-1} b - L x^{[k+1]} - U x^{[k]}$$
$$(I+L) x^{[k+1]} = D^{-1} b - U x^{[k]}$$
$$x^{[k+1]} = (I+L)^{-1}(D^{-1} b - U x^{[k]})$$
$$= -(I+L)^{-1} U x^{[k]} + (D(I+L))^{-1} b$$
$$= B x^{[k]} + Z$$

- SOR is

$$x^{[k+1]} = (1-w)\, x^{[k]} + w\, (B x^{[k]} + c)$$
$$= ((1-w) I + w B)\, x^{[k]} + w Z$$

- In all cases:

$$x^{[k+1]} = T x^{[k]} + c \qquad (1)$$

for some T and c.

- So when does this iteration converge?

- Notice that the way the iteration is set up, the true solution x satisfies

$$x = Tx + c \qquad (2)$$

- Subtracting (2) - (1) gives:

$$e^{[k+1]} = T x^{[k]}$$

where $e^{[k]} = x - x^{[k]}$

- So when does $e^{[k+1]} = T e^{[k]}$ converge to zero.

- Let's review some basic linear algebra

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

- induced matrix norm.

- For example

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ $\|A\|_1 = \max_j \sum_i |a_{ij}|$

$\|x\|_\infty = \max_i |x_i|$ $\|A\|_\infty = \max_i \sum_j |a_{ij}|$

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ $\|A\|_2 = \sqrt{\rho(A^TA)}$

- major property of induced matrix norm.

$$\|Az\| = \|A \frac{z}{\|z\|}\| \, \|z\|$$

$$\leq \|A\| \, \|z\|$$

for all vectors z

- So if we can find a matrix norm $\| \ \|$ such that in the induced matrix norm

$$\|T\| < 1$$

then we have convergence.

- Example: Jacobi Method.

$$T_J = -D^{-1}(L+U)$$

- Then we will have $\|T_J\| < 1$
  if A is diagonally dominant

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

- our linear system $\begin{bmatrix} -4 & & \\ 1 & \ddots & 1 \\ & & -4 \end{bmatrix}$

  Only gives $\|T_J\| \leq 1$ but actually
  it is indeed true then $\|T_J\| < 1$

- Any induced matrix norm is no less
  than the larges eigenvalue.

$$Ax = \lambda x$$
$$\|Ax\| = |\lambda| \|x\| \leq \|A\| \|x\|$$

- The spectral radius itself is not
  a norm.

- why not?

- But we can find a norm that
  is arbitrarily close.

- So we want to minimize the
  spectral radius.

- For our Poisson Equation

$$\rho(T_J) = \cos h$$

$$\rho(T_{GS}) = \cos^2 h$$

$$\rho(T_\omega) < 1 \quad \text{for} \quad 0 < \omega < 2$$

$$\omega_{best} = \frac{2}{1 + \sin h}$$

$$\rho(T_{\omega_{best}}) = \frac{\cos^2 h}{(1 + \sin h)^2}$$

- Books by Varga and Young.

```fortran
      implicit double precision(a-h,o-z)
      n = 10
      write (20,10000)
      pi = dacos(0.0d0)*2.0d0
           do 100 i = 1,18
           n = idint(dfloat(n)*1.2d+00)
           h = pi/dfloat(n+1)
           omb = 2/(1.d+00+dsin(h))
           sj = dcos(h)
           sgs = sj**2
           sor = sgs/(1.d+00+dsin(h))**2
           nj = dlog(.1d+00)/dlog(sj)
           ngs = dlog(.1d+00)/dlog(sgs)
           nsor = dlog(.1d+00)/dlog(sor)
           annsor = 4.*float(nsor)*float(n)**2
           ange = float(n)**6/3
           ratio = ange/annsor
           write (20,20000) n,omb,sj,nj,sgs,ngs,sor,nsor,
     +        annsor,ange,ratio
  100      continue
      stop
10000 FORMAT(/'  n',' best omega',6X,' Jacobi    ',7X,
     X' Gauss-Seidel ',11X ' SOR ',
     X'    flops(sor) flops(LU)   ratio '//)
20000 FORMAT(I4,X,F8.5,3(3X,F8.5,I8),X,3(1PD10.2))
      end
```

n	best omega	Jacobi		Gauss-Seidel		SOR		flops(sor)	flops(LU)	ratio
12	1.61379	0.97094	78	0.94273	39	0.61379	4	2.30D+03	9.95D+05	4.32D+02
14	1.65575	0.97815	104	0.95677	52	0.65575	5	3.92D+03	2.51D+06	6.40D+02
16	1.68955	0.98297	134	0.96624	67	0.68955	6	6.14D+03	5.59D+06	9.10D+02
19	1.72945	0.98769	185	0.97553	92	0.72945	7	1.01D+04	1.57D+07	1.55D+03
22	1.76031	0.99069	246	0.98146	123	0.76031	8	1.55D+04	3.78D+07	2.44D+03
26	1.79197	0.99324	339	0.98652	169	0.79197	9	2.43D+04	1.03D+08	4.23D+03
31	1.82147	0.99518	477	0.99039	238	0.82147	11	4.23D+04	2.96D+08	7.00D+03
37	1.84744	0.99658	673	0.99318	336	0.84744	13	7.12D+04	8.55D+08	1.20D+04
44	1.86958	0.99756	944	0.99513	472	0.86958	16	1.24D+05	2.42D+09	1.95D+04
52	1.88815	0.99824	1309	0.99649	654	0.88815	19	2.06D+05	6.59D+09	3.21D+04
62	1.90504	0.99876	1851	0.99752	925	0.90504	23	3.54D+05	1.89D+10	5.35D+04
74	1.91961	0.99912	2623	0.99825	1311	0.91961	27	5.91D+05	5.47D+10	9.26D+04
88	1.93182	0.99938	3695	0.99875	1847	0.93182	32	9.91D+05	1.55D+11	1.56D+05
105	1.94244	0.99956	5241	0.99912	2620	0.94244	38	1.68D+06	4.47D+11	2.67D+05
126	1.95173	0.99969	7525	0.99939	3762	0.95173	46	2.92D+06	1.33D+12	4.57D+05
151	1.95950	0.99979	10779	0.99957	5389	0.95950	55	5.02D+06	3.95D+12	7.88D+05
181	1.96606	0.99985	15454	0.99970	7727	0.96606	66	8.65D+06	1.17D+13	1.36D+06
217	1.97159	0.99990	22173	0.99979	11086	0.97159	79	1.49D+07	3.48D+13	2.34D+06