Homework #2

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1. **Taylor Series:** Let $f(x) = e^x$ and $g(x) = \ln(x+1)$, and let p_n and q_n be the Taylor polynomials of degree n for f and g, repectively, about $x_0 = 0$.

   Plot the graphs of f, g, p_n and q_n, for some small values of n, and comment on your results. Discuss in particular how well f and g are approximated by their Taylor polynomials. Explain your observations in terms of a suitable expression for the error in the approximation.

$$
\begin{aligned}
f(x) = e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + R_n(x) \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!}
\end{aligned}
$$

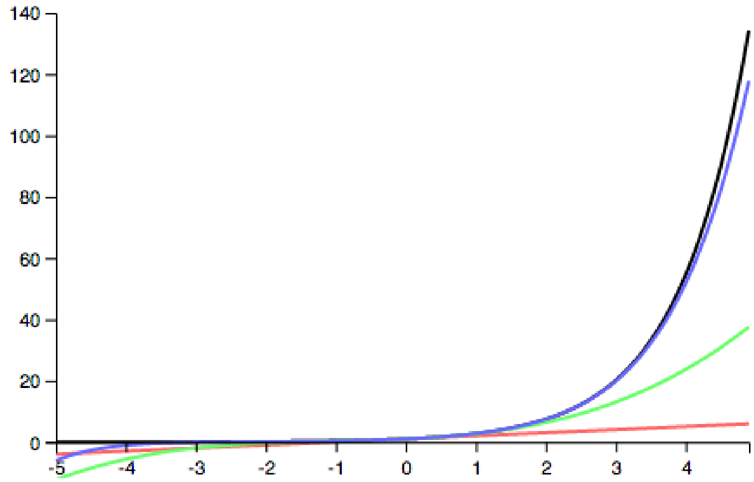

Figure 1: The Taylor polynomial of f plotted for values of $n = 2$ (red), $n = 4$ (green), and $n = 8$ (blue). $f(x)$ is plotted in black. The 2-degree Taylor polynomial is equivalent to the tangent line of f at the point x_0. As the degree n increases we have better and better approximation of the original function.

$$
\begin{aligned}
g(x) = \ln(x+1) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots + R_n(x) \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}
\end{aligned}
$$

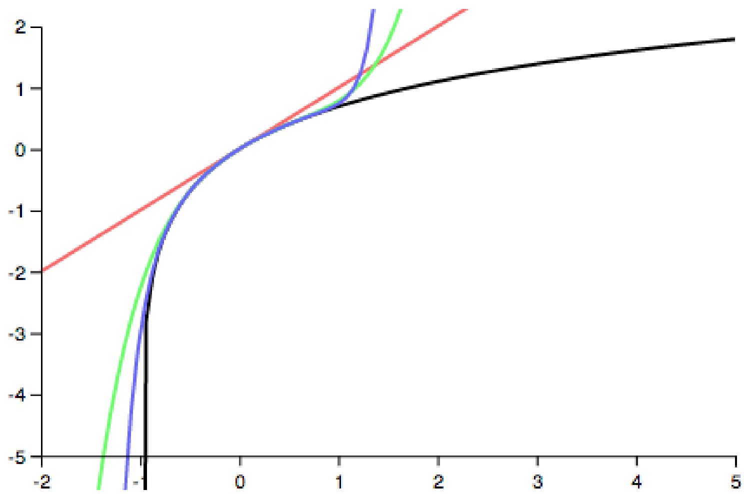

Figure 2: The Taylor polynomial of g plotted for values of $n = 2$ (red), $n = 6$ (green), and $n = 12$ (blue). $g(x)$ is plotted in black. Once again when $n = 2$ we have the tangent line of g at x_0. The Taylor polynomials for g appear to be very accurate close to the center, but diverge quickly outside of this neighborhood.

The error for a Taylor polynomial can be bounded by the remainder function.

$$|f(x) - T_n(x)| \leq R_n(x) = e^x \frac{x^n}{n!}$$

$$|g(x) - T_n(x)| \leq R_n(x) = \ln(x+1)(-1)^{n+1} \frac{x^n}{n}$$

Where $T_n(x)$ is the Taylor polynomial for f or g respectively. Here as $n \to \infty$ the $|f(x) - T_n(x)| \to 0$ and $|g(x) - T_n(x)| \to 0$. For small values of n the error is small around to the center point $x_0 = 0$ and grows as we move away from the center.

2. **A "simple" program:** Write a program that reads n and the entries $x_1, x_2, \cdots, x_n$ of a vector $x \in \mathbb{R}^n$ from standard input and prints

$$||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

to standard output.

```
(defn norm [x]
  {:pre [(coll? x)]}
```

```
3   (->> x
4        (map #(Math/pow % 2))
5        (apply +)
6        Math/sqrt))
7
8 ;; Examples
9 (norm [1 2 3]) => 3.7416573867739413
10 (norm (range 20)) => 49.69909455915671
11 (norm [1 1 1]) => 1.7320508075688772
12 (norm [1 0 0]) => 1.0
13 (norm [3 4]) => 5.0
```

Explanation: We define a function that takes one argument a vector x, and asserts that it is a collection (in this environment any "collection" can be treated like a vector) on line 2. We use the thread last operator *->>* to nest the evaluation of each statement through the end of the next function, similar to a composition. We start with x and apply the x_i^2 operation to each value in the vector using a *map* operation, we sum all the elements up, and lastly apply the *sqrt* operation. The result of this is implicitly set as the return value of the function.

3. **Some Iteration:** Consider the iteration $x_{n+1} = F(x_n) = \sin x_n$, $x_0 = 1$ (where of course the angle is measured in radians). What does our theory tell us about convergence? Show that the iteration does converge! What is the limit? How fast does the iteration converge? **Carefully explain the effects of rounding errors.**

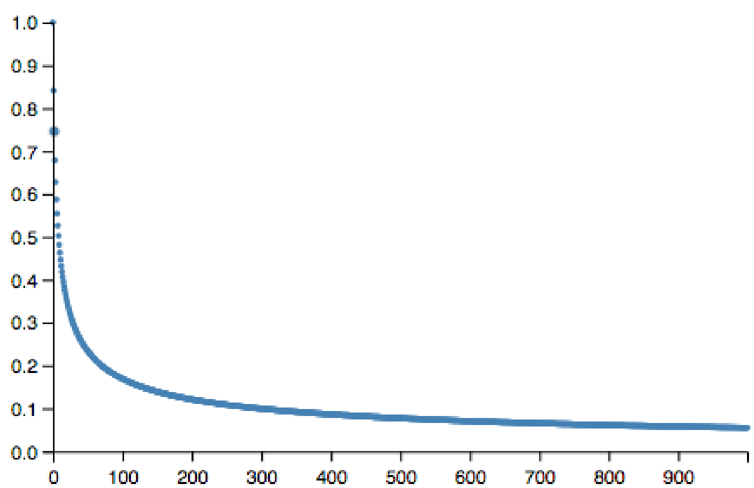

Figure 3: The iteration of $F(x_n)$ visually appears to converge.

We have the function $F(x_n) = \sin x_n = x_{n+1}$ and we want to know if it converges such that $\sin\alpha = \alpha$ where α is a fixed point of the iteration $F(x_n)$.

We know that a function has a fixed point if it intersects with the identity function $x = y$ in a specified interval. Consider $g(x) = \sin(x) - x = 0$ on the interval $[-1, 1]$. At the left bounds $x = -1$ we have $g(-1) > 0$, and at the right we have $g(1) < 0$, thus by the intermediate value theorem we know that it must have a root in the interval $[-1, 1]$. Therefore there exists a value of x such that $\sin x = x$ within this interval. It's also obvious that $x = 0$ is the fixed point such that $\sin x = x$.

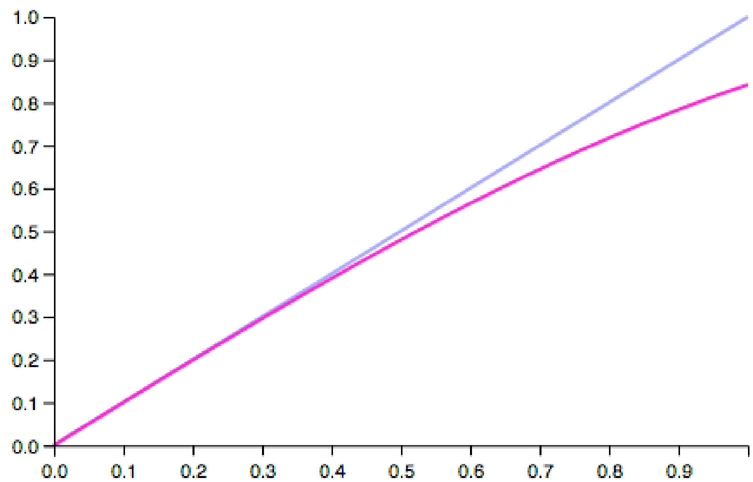

Figure 4: $\sin x = y$ crosses the function $x = y$ at $x = 0$, which is a fixed point of $\sin(x)$.

The derivative of $\sin' x = \cos x$ is between $[0, 1]$ on the interval $x \in [0, 1]$. So $\sin x$ is a monotone decreasing function on this interval and we know that it has no other fixed points and must converge to 0. Therefore iterating from $x_0 = 1$ will eventually converge to 0.

With Newton's method the convergence of $F(x_n)$ is very slow. The value for the Newton's step $\frac{F(x)}{F'(x)}$ becomes very small as we move closer to 0 from above, becuase the rate of change approaches 1. This can be seen visually as the curve for $F(x)$ gets very close to $x = y$ but only touches it at $x = 0$. For example, when coded with double precision, after 100000 iterations we arrived at $x = 0.00547696985405864$. Successive iterations would continue to decrease, but would do so slower and slower.

```
(first (drop 100000 (iterate #(Math/sin %) 1)))
=> 0.00547696985405864
```

As the iteration converges we will have to deal with very small and precise numbers. The precision of these numbers will quickly grow beyond the bounds of traditionaly floating point or double arithmetic operations. It's possible that trying to converge to 0 may never go below a precise threshold, or may skip around 0 with once the iteration of $F(x_n)$ becomes arbitrarily small.

4. **Newton's Method:** Suppose f has a root of multiplicity $p > 1$ at $x = \alpha$, i.e.

$$f(\alpha) = f'(\alpha) = \cdots = f^{(p-1)}(\alpha) = 0,\ f^{(p)}(\alpha) \neq 0$$

. (a) Show that Newton's method applied to $f(x) = 0$ converges linearly to α. (b) Show that this modification of Newton's method:

$$x_{k+1} = x_k - p\frac{f(x_k)}{f'(x_k)}$$

converges quadratically to α. **Hint:** You probably are thinking of using L'Hôpital, but the problem is much easier if you think of f as being defined by $f(x) = (x-\alpha)^p F(x)$ where $F(\alpha) \neq 0$.

(a) We have

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Then

$$g'(x) = \frac{f(x)\,f''(x)}{f'(x)^2}$$

If $f(x) = (x-\alpha)^p F(x)$ then we have that

$$\begin{aligned}
f(x) &= (x-\alpha)g_0(x) \\
f'(x) &= p(x-\alpha)^{p-1}F(x) + (x-\alpha)^p F'(x) \\
&= (x-\alpha)^{p-1}g_1(x) \\
f''(x) &= p(p-1)(x-\alpha)^{p-2}F(x) + 2p(x-\alpha)^{p-1}F'(x) + (x-\alpha)^p F''(x) \\
&= (x-\alpha)^{p-2}g_2(x)
\end{aligned}$$

where $g_0(x)$, $g_1(x)$, and $g_2(x)$ are the successive iterations of $g(x)$.
We can now plug these into the function for g' as it approaches α

$$\begin{aligned}
\lim_{x\to\alpha} g'(x) &= \frac{(x-\alpha)g_0(x)(x-\alpha)^{p-2}g_2(x)}{\left[(x-\alpha)^{p-1}g_1(x)\right]^2} \\
&= \frac{g_0(x)g_2(x)}{g_1(x)^2} \\
&= \frac{p(p-1)F(\alpha)^2}{p^2F(\alpha)^2} \\
&= \frac{p-1}{p} < 1
\end{aligned}$$

Meaning that our function will linearly converge to α since $\frac{p-1}{p} < 1$.

(b) We define an error term $e_n = x_n - \alpha$, and we have that

We can then expand the Taylor series of $f(x_n)$ and eliminate many of the initial terms because $f(\alpha) = f'(\alpha) = \cdots = f^{(p-1)}(\alpha) = 0$

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) \\ &= f(\alpha) + e_n f'(\alpha) + \cdots + \frac{e_n^{(p-1)} f^{(p-1)}(\alpha)}{(p-1)!} + \frac{e_n^p f^{(p)}(\alpha)}{p!} + \frac{e_n^{p+1} f^{(p+1)}(\xi_n)}{(p+1)!} \\ \Rightarrow f(x_n) &= \frac{e_n^p f^{(p)}(\alpha)}{p!} + \frac{e_n^{p+1} f^{(p+1)}(\xi_n)}{(p+1)!} \end{aligned}$$

We then differntiate $f(x_n)$

$$f'(x_n) = \frac{e_n^{p-1} f^p(\alpha)}{(p-1)!} + \frac{e_n^p f^{(p+1)}(\eta_n)}{p!}$$

Now if we calculate the next error term in our sequence we can see that there is quadratic convergence.

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha \\ &= x_n - \alpha - \frac{p f(x_n)}{f'(x_n)} \\ &= e_n - \frac{\left(\frac{e_n^p f^{(p)}(\alpha)}{(p-1)!} + \frac{e_n^{p+1} f^{(p+1)}(\xi_n)}{(p+1)(p-1)!}\right)}{\left(\frac{e_n^{p-1} f^{(p)}(\alpha)}{(p-1)!} + \frac{e_n^p f^{(p+1)}(\eta_n)}{p!}\right)} \\ &= e_n^2 \frac{\left(\frac{f^{(p+1)}(\eta_n)}{p!} - \frac{f^{(p+1)}(\eta_n)}{(p+1)(p-1)!}\right)}{\left(\frac{f^{(p)}(\alpha)}{(p-1)!} + \frac{e_n f^{(p+1)}(\eta_n)}{p!}\right)} \end{aligned}$$

Since the term e_{n+1} has a leading coefficient e_n^2 we know that it converges quadratically.

5. **Division without division:** Suppose you have a computer or calculator that has no built-in division. Come up with a fixed point iteration that converges to $1/r$ for any given non-zero number r, and that only uses addition, subtraction, and multiplication. **Hint:** Write down an equation satisfied by $1/r$, apply Newton's method to that equation, and then modify Newton's method so that it doesn't use division. Your resulting method should converge of order 2.

We can

$$
\begin{aligned}
f(x) = \frac{1}{r} &= x \\
\Rightarrow 1 &= xr \\
\Rightarrow \frac{1}{x} &= r \\
\Rightarrow \frac{1}{x} - r &= 0
\end{aligned}
$$

$$f'(x) = \frac{-1}{x^2}$$

$$
\begin{aligned}
F(x) &= x - \frac{f(x)}{f'(x)} \\
&= x - \frac{\frac{1}{x} - r}{\frac{-1}{x^2}} \\
&= x + \left(\frac{1}{x} - r\right) x^2 \\
&= x(2 - xr)
\end{aligned}
$$

```
(defn reciprocal [r]
  (let [convergent? (fn [{:keys [steps error] :or {steps 0
                                                   error 1}}]
                      (or (< (Math/abs (double error)) 0.00001)
                          (> steps 20)))
        next-guess (fn [{:keys [x last steps] :or {last 0
                                                   steps 0}}]
                     (let [guess (* x (- 2 (* x r)))]
                       {:steps (inc steps)
                        :error (- last guess)
                        :last x
                        :x guess}))
        answer (first (drop-while #(not (convergent? %))
                                  (iterate #(next-guess %)
                                           {:x (if (= 20 r) 0.01 0.1)})))]
    (println answer)
    (:x answer)))

(reciprocal 3) => 0.33333333333333337
(reciprocal 4) => 0.24999999999999997
```

Explanation: We define a function that uses a few local functions to perform Newton's method. On line 2, we locally define the `convergent?` function which checks if our answer is below a threshold or the iteration has exceeded a maximum number of steps. On line 6, we define a function called `next-guess` that performs the actual Newton's step operation $x(2 - xr)$ and returns it's value with additional metadata to help with iteration. On line 13 we perform the calculation, starting at a suitably small value and treating our function `next-guess` as a sequence of iterations we take only the first convergent value we receive.

6. **A cubically convergent method:** Consider the iteration

$$x_{k+1} = g(x_k) \text{ where } g(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2}\frac{f^2(x)f''(x)}{(f'(x))^3}$$

. (We assume f is sufficiently differentiable, and $f'(x) \neq 0$.) Suppose that $g(\alpha) = \alpha$. Show that

$$g'(\alpha) = g''(\alpha) = 0$$

. (Thus the fixed point method will converge of order at least 3 if we start sufficiently close to α.)

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f(x)^2\, f''(x)}{2\, f'(x)^3}$$

$$g'(x) = \frac{3\, f(x)^2\, f''(x)^2}{2\, f'(x)^4} - \frac{f(x)^2\, f^{(3)}(x)}{2\, f'(x)^3}$$

$$\begin{aligned} g'(\alpha) &= \frac{3\, f(\alpha)^2\, f''(\alpha)^2}{2\, f'(\alpha)^4} - \frac{f(\alpha)^2\, f^{(3)}(\alpha)}{2\, f'(\alpha)^3} \\ &= \frac{3\,(0)^2\, f''(\alpha)^2}{2\, f'(\alpha)^4} - \frac{(0)^2\, f^{(3)}(\alpha)}{2\, f'(\alpha)^3} \\ &= \frac{0}{2\, f'(\alpha)^4} - \frac{0}{2\, f'(\alpha)^3} \\ &= 0 \end{aligned}$$

Likewise when differentiating $g'(x)$ we will have a function that is 0 when $x = \alpha$ since $f(\alpha) = 0$,

$$g''(x) = \frac{3\, f(x)\, f''(x)^2}{f'(x)^3} - \frac{6\, f(x)^2\, f''(x)^3}{f'(x)^5} - \frac{f(x)^2\, f^{(4)}(x)}{2\, f'(x)^3} - \frac{f(x)\, f^{(3)}(x)}{f'(x)^2} + \frac{9\, f(x)^2\, f^{(3)}(x)\, f''(x)}{2\, f'(x)^4}$$

$$g''(\alpha) = 0$$

7. **Polynomial Interpolation:** Suppose you want to interpolate to the data (x_i, y_i), $i = 0, \ldots, n$ by a polynomial of degree n. Recall that the interpolating polynomial p can be written in its Lagrange form as

$$p(x) = \sum_{i=0}^{n} y_i L_i(x) \text{ where } L_i(x) = \frac{\prod_{i \neq j}(x - x_j)}{\prod_{i \neq j}(x_i - x_j)}.$$

Show that

$$\sum_{i=0}^{n} x_i^j L_i(x) = x^j \text{ for } j = 0, \ldots, n.$$

If $y_i = x_i^j$ then we have,

$$\begin{aligned}
p(x) &= \sum_{i=0}^{n} x_i^j L_i(x) \\
&= \sum_{i=0}^{n} x_i^j \frac{\prod_{i \neq j}(x - x_j)}{\prod_{i \neq j}(x_i - x_j)} \\
&= \sum_{i=0}^{n} x_i^j \frac{(x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x_i - x_n)} \\
&= \sum_{i=0}^{n} x_i^j \frac{(x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x_i - x_n)} \frac{(x_i - x_j)}{(x_i - x_j)} \\
&= \sum_{i=0}^{n} \frac{(x_i^{j+1} - x_i^j x_j)(x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{n(x_i - x_j)(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x_i - x_n)} \\
&= \sum_{i=0}^{n} \frac{x^j(x_i - x_j)(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x - x_n)}{n(x_i - x_j)(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x_i - x_n)} \\
&= x^j \sum_{i=0}^{n} \frac{(x_i - x_j)(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x - x_n)}{n(x_i - x_j)(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})(x_i - x_{j+1})\cdots(x_i - x_n)} \\
&= x^j \sum_{i=0}^{n} \frac{1}{n} \\
&= x^j
\end{aligned}$$

8. **Uniqueness of the interpolating polynomial:** Assume you are given the data

$$\begin{aligned}
x_i &: \; 1 \; 2 \; 4 \; 8 \\
y_i &: \; 1 \; 2 \; 3 \; 4
\end{aligned}$$

Construct the interpolating polynomial using

(a) the power form

(b) the Lagrange form

(c) the Newton form,

and show that they all yield the same polynomial.

(a) the power form

$$
\begin{aligned}
p(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
\Rightarrow p(x) &= V_n \vec{a} = \vec{y} \\
p(x) &= \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
p(x) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \\
p(x) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \end{pmatrix} \begin{pmatrix} -\frac{10}{21} \\ \frac{7}{4} \\ -\frac{7}{24} \\ \frac{1}{56} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \\
p(x) &= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}
\end{aligned}
$$

(b) the Lagrange form

$$
L(x) = \frac{\prod_{i \neq j}(x - x_j)}{\prod_{i \neq j}(x_i - x_j)}
$$

$$
\begin{aligned}
p(x) &= \sum_{i=0}^{n} y_i L(x) \\
p(x) &= 1\frac{(x-2)(x-4)(x-8)}{(1-2)(1-4)(1-8)} + 2\frac{(x-1)(x-4)(x-8)}{(2-1)(2-4)(2-8)} \\
&\quad + 3\frac{(x-1)(x-2)(x-8)}{(4-1)(4-2)(4-8)} + 4\frac{(x-1)(x-2)(x-4)}{(8-1)(8-2)(8-4)} \\
p(x) &= \frac{x^3 - 14\,x^2 + 56\,x - 64}{-21} + \frac{x^3 - 13\,x^2 + 44\,x - 32}{12} \\
&\quad + \frac{x^3 - 11\,x^2 + 26\,x - 16}{-24} + \frac{x^3 - 7\,x^2 + 14\,x - 8}{168} \\
p(x) &= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}
\end{aligned}
$$

(c) the Newton form

$$
\begin{aligned}
&p_k(x) = \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1}(x - x_j) \\
\Rightarrow\ &p_0(x) = c_0 \\
&p_1(x) = p_0(x) + c_1(x - x_0) \\
&p_2(x) = p_1(x) + c_2(x - x_0)(x - x_1) \\
&p_3(x) = p_2(x) + c_3(x - x_0)(x - x_1)(x - x_2) \\
\Rightarrow\ &p_0(x) = y_0 \\
&p_1(x) = p_0(x) + \frac{y_1 - P_0(x_1)}{(x_1 - x_0)}(x - x_0) \\
&p_2(x) = p_1(x) + \frac{y_2 - P_1(x_2)}{(x_2 - x_0)(x_2 - x_1)}(x - x_0)(x - x_1) \\
&p_3(x) = p_2(x) + \frac{y_3 - P_2(x_3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}(x - x_0)(x - x_1)(x - x_2) \\
\Rightarrow\ &p_0(x) = 1 \\
&p_1(x) = p_0(x) + \frac{2 - P_0(2)}{(2 - 1)}(x - 1) \\
&p_2(x) = p_1(x) + \frac{3 - P_1(4)}{(4 - 1)(4 - 2)}(x - 1)(x - 2) \\
&p_3(x) = p_2(x) + \frac{4 - P_2(8)}{(8 - 1)(8 - 2)(8 - 4)}(x - 1)(x - 2)(x - 4)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow p_0(x) &= 1 \\
p_1(x) &= 1 + \frac{2-1}{(2-1)}(x-1) \\
&= 1 + \frac{1}{1}(x-1) \\
&= x \\
p_2(x) &= x + \frac{3-4}{(4-1)(4-2)}(x-1)(x-2) \\
&= x - \frac{1}{6}(x-1)(x-2) \\
&= x - \frac{x^2-3x+2}{6} \\
&= -\frac{x^2}{6} + \frac{3\,x}{2} - \frac{1}{3} \\
p_3(x) &= -\frac{x^2}{6} + \frac{3\,x}{2} - \frac{1}{3} + \frac{4-1}{(8-1)(8-2)(8-4)}(x-1)(x-2)(x-4) \\
&= -\frac{x^2}{6} + \frac{3\,x}{2} - \frac{1}{3} + \frac{3}{168}(x-1)(x-2)(x-4) \\
&= -\frac{x^2}{6} + \frac{3\,x}{2} - \frac{1}{3} + \frac{3}{168}\left(x^3 - 7\,x^2 + 14\,x - 8\right) \\
&= -\frac{x^2}{6} + \frac{3\,x}{2} - \frac{1}{3} + \frac{3\,x^3 - 21\,x^2 + 42\,x - 24}{168} \\
&= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21} \\
\Rightarrow P(x) &= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{10}{21}
\end{aligned}
$$

9. **The infamous Runge-Phenomenon:** It is not generally true that higher degree interpolation polynomials yield more accurate approximations. This is illustrated in this problem. Let

$$f(x) = \frac{1}{1+x^2} \text{ and } x_j = -5 + jh,\ j = 0, 1, \ldots, n,\ h = \frac{10}{n}.$$

For

$$n = 1, 2, 3, \ldots, 20$$

plot the graph (in the interval $[-5, 5]$) of the interpolant

$$p(x) = \sum_{i=0}^{n} \alpha_i x^i$$

defined by

$$p(x_i) = f(x_i),\ i = 0, 1, \ldots, n.$$

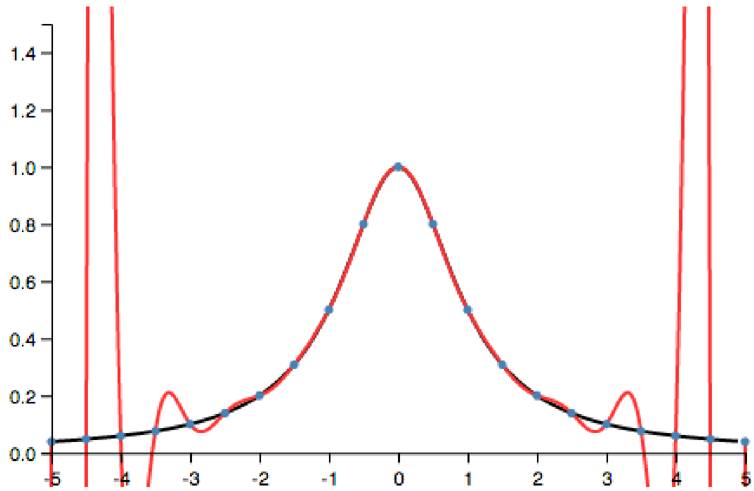

Figure 5: We interpolate points obtained from the Runge function across an equidistant interval with $n = 20$ in red. The original Runge function is plotted in black, along with the points we've interpolated from. The interpolated polynomial fluctuates wildly at either end of our domain.

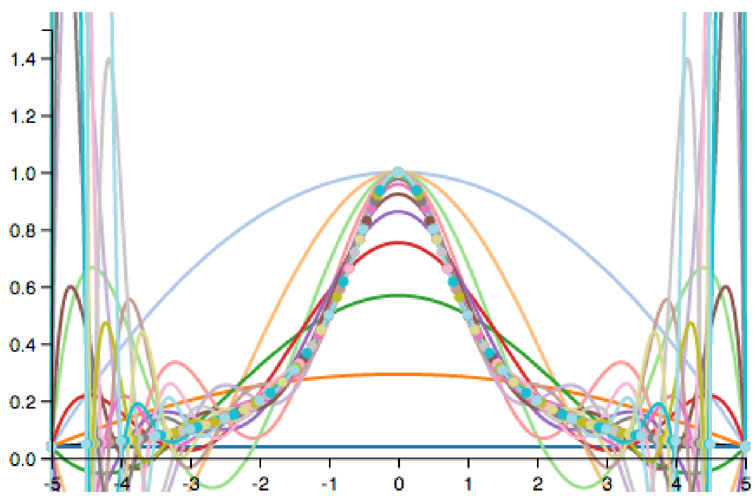

Figure 6: Interpolating several functions for $n = 1, \ldots, 20$

We can see that as we interpolate more points the center of our function gets approximated better, but the end points become more unstable. They fluctuate greatly trying to interpolate all points on the interval.

10. **Judicious interpolation:** Repeat the above except that you interpolate at the roots of the Chebycheff polynomials, i.e.

$$x_i = 5 \cos \frac{i\pi}{n}, \quad i = 0, 1, \ldots, n.$$

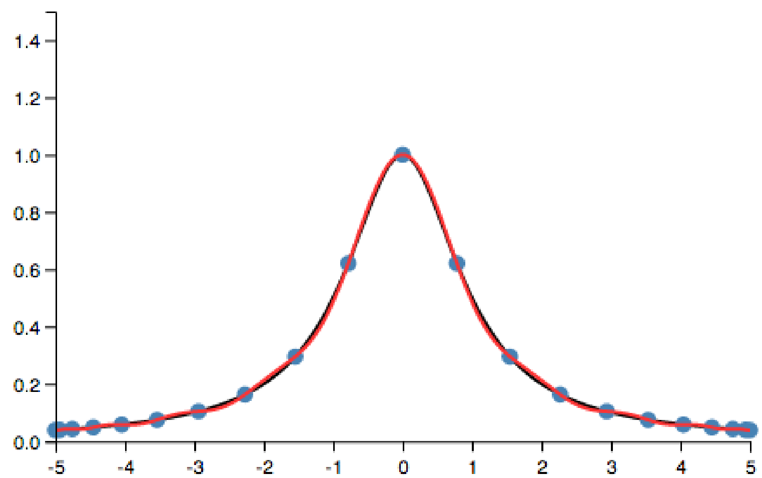

Figure 7: We interpolate points obtained from the Runge function across an interval of points with $n = 20$ made up from the Chebycheff roots in red. The original Runge function is plotted in black, along with the points we've interpolated from. Some minor fluctuation is visible, however it's overall much more accurate in our domain.

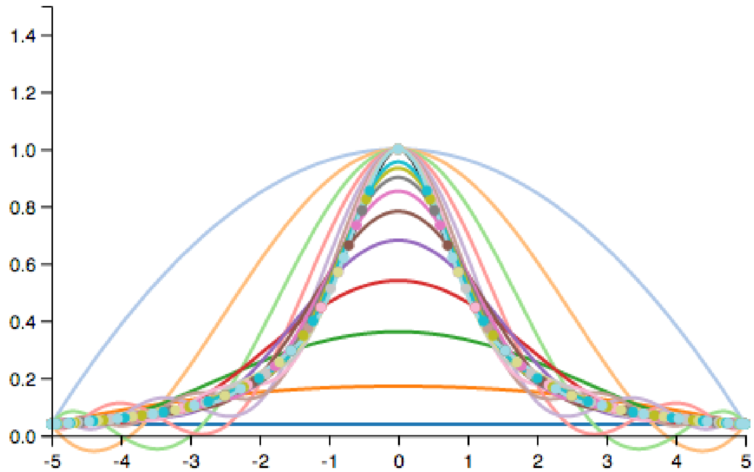

Figure 8: Once again plotting several interpolations with $n = 1, \ldots, 20$ we can see that all the functions interpolated with the Chebycheff roots behave much better than the previous problem with equidistant points.

Here we have that our interpolated polynomial still fluctuates near the boundaries of our interpolation, but it behaves much better as the points are distributed more at either end, and thus interpolated better.

11. **Least Squares approximation of functions:** Find a linear function $l(x)$ such that

$$\int_0^1 (\mathrm{e}^x - l(x))^2 \mathrm{d}x = \min .$$

$$f = (\mathrm{e}^x - a\,x - b)^2$$

$$\nabla f = \begin{pmatrix} 2\,x\,(\mathrm{e}^x - a\,x - b) \\ 2\,b - 2\,\mathrm{e}^x + 2\,a\,x \end{pmatrix}$$

$$\int_0^1 \nabla f \mathrm{d}x = \begin{pmatrix} \frac{2\,a}{3} + b - 2 \\ a + 2\,b - 2\,\mathrm{e} + 2 \end{pmatrix}$$

$$\Rightarrow a = 18 - 6\,\mathrm{e},\ b = 4\,\mathrm{e} - 10$$

$$\Rightarrow l(x) = (18 - 6\,\mathrm{e})x + (4\,\mathrm{e} - 10)$$

12. **An alternative approximation problem:** Find a linear function $l(x)$ such that

$$\int_0^1 |\mathrm{e}^x - l(x)| \mathrm{d}x = \min .$$

$$\begin{aligned} \min &= \int_0^1 |\mathrm{e}^x - a\,x - b|\,\mathrm{d}x \\ &= \int_0^c \mathrm{e}^x - a\,x - b\,\mathrm{d}x + \int_c^d -\mathrm{e}^x + a\,x + b\,\mathrm{d}x + \int_d^1 \mathrm{e}^x - a\,x - b\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} f &= \mathrm{e}^x - a\,x - b \\ g &= -\mathrm{e}^x + a\,x + b \end{aligned}$$

$$\nabla f = \begin{pmatrix} -x \\ -1 \end{pmatrix}, \ \nabla g = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$\begin{aligned} \int_0^c \nabla f \mathrm{d}x + \int_c^d \nabla g \mathrm{d}x + \int_d^1 \nabla f \mathrm{d}x &= \begin{pmatrix} -\frac{c^2}{2} \\ -c \end{pmatrix} + \begin{pmatrix} \frac{d^2}{2} - \frac{c^2}{2} \\ d - c \end{pmatrix} + \begin{pmatrix} \frac{d^2}{2} - \frac{1}{2} \\ d - 1 \end{pmatrix} \\ &= \begin{pmatrix} -c^2 + d^2 - \frac{1}{2} \\ 2\,d - 2\,c - 1 \end{pmatrix} \end{aligned}$$

$$\Rightarrow c = \frac{1}{4}, \ d = \frac{3}{4}$$

$$\begin{aligned} &\Rightarrow \begin{pmatrix} a\,c + b = e^c \\ a\,d + b = e^d \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{4} + b = \mathrm{e}^{\frac{1}{4}} \\ \frac{3\,a}{4} + b = \mathrm{e}^{\frac{3}{4}} \end{pmatrix} \end{aligned}$$

$$a = 2\,\mathrm{e}^{\frac{3}{4}} - 2\,\mathrm{e}^{\frac{1}{4}}, b = \frac{3\,\mathrm{e}^{\frac{1}{4}}}{2} - \frac{\mathrm{e}^{\frac{3}{4}}}{2}$$

$$l(x) = \left(2\,\mathrm{e}^{\frac{3}{4}} - 2\,\mathrm{e}^{\frac{1}{4}}\right) x + \frac{3\,\mathrm{e}^{\frac{1}{4}}}{2} - \frac{\mathrm{e}^{\frac{3}{4}}}{2}$$

13. **Another alternative approximation problem:** Find a linear function $l(x)$ such that

$$\max_{0\le x\le 1} |\mathrm{e}^x - l(x)| = \min .$$

We find the values for $f(x) = \mathrm{e}^x - a\,x - b$ where the difference is large.

$$\begin{aligned} f(x) &= |\mathrm{e}^x - a\,x - b| \\ f(0) &= |1 - b| \\ f(1) &= |\mathrm{e} - a - b| \\ f(\log a) &= |\mathrm{e}^{\log a} - a\,\log a - b| \\ &= |a - a\,\log a - b| \end{aligned}$$

By setting these equations equal to each other, we can eliminate variables, and find the values for a and b that minimize $f(x)$.

$$\begin{aligned} 1 - b &= \mathrm{e} - a - b \\ \Rightarrow a &= \mathrm{e} - 1 \end{aligned}$$

$$\begin{aligned} b - 1 &= a - a\log a - b \\ 2b &= 1 + a - a\log a \\ &= 1 + (\mathrm{e}-1) - (\mathrm{e}-1)\log(\mathrm{e}-1) \\ &= \mathrm{e} - (\mathrm{e}-1)\log(\mathrm{e}-1) \\ \Rightarrow b &= \frac{\mathrm{e} - (\mathrm{e}-1)\log(\mathrm{e}-1)}{2} \end{aligned}$$

$$l(x) = (\mathrm{e}-1)\;x + \frac{\mathrm{e} - (\mathrm{e}-1)\log(\mathrm{e}-1)}{2}$$

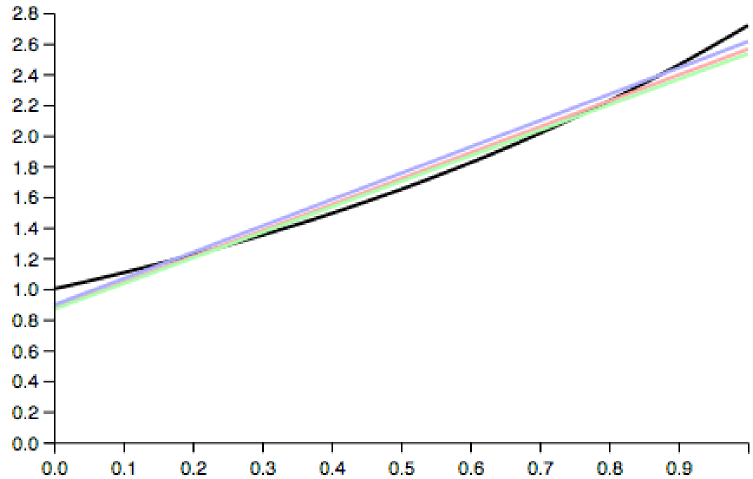

Figure 9: We plot e^x in black, with the solutions to exercises 11 (red), 12 (green), and 13 (blue). We can visually see that our solutions all linearly approximate e^x on $[0, 1]$.