Math 5600 6/26/2014

- Major Theme: Use of special structure.

- If your problem has special structure it's probably worthwhile, or indeed necessary, to utilize it.

- Examples $Ax = b$

A is
- Tridiagonal
- symmetric
- positive definite
- banded
- triangular
- Hessenberg
- sparse

(mostly zero, a huge area!)

- Mike Hohn recently wrote a MS thesis on systems of the form

$$A = \begin{bmatrix} A & G^T \\ G & 0 \end{bmatrix} \qquad (*)$$

where A is symmetric

- A is a "block matrix"

- I once spent a lot of time with a bunch of problems of the form (*) where A was positive semi-definite and rank deficient, in addition to being sparse.

- Recall LU factorization requires $\frac{n^3}{3}$ Flops

- Let's now focus on positive definite systems

$$A = A^T \qquad x^T A x > 0 \quad \text{if } x \neq 0$$

- what does this mean for 1×1 matrices?
- A is non-singular (why?)

- The Cholesky Decomposition

$$A = LL^T \; (= GG^T)$$

L is lower triangular
(but not necessarily or usually unit)

- if $A = LL^T$, and L is non-singular, then A is pos. def.

- It's symmetric and

$$x^T A x = x^T L^T L x = (Lx)^T Lx > 0 \text{ if } x \neq 0$$

- The Choleski Decomposition exists!
- proof by induction (Wilkinson)
- suppose we have shown this for $(n-1)\times(n-1)$ matrices
- It's trivial for 1×1 matrices.
- A $n\times n$ positive definite
- write

$$A = \begin{bmatrix} A_{n-1} & b \\ b & a_{nn} \end{bmatrix}$$

where A_{n-1} is $(n-1)\times(n-1)$, $b\in\mathbb{R}^{n-1}$, $a_{nn}\in\mathbb{R}$

- A_{n-1} is pos. definite. why?
- a_{nn} is positive. why?
- $A_{n-1} = L_{n-1} L_{n-1}^T$
- Let

$$L = \begin{bmatrix} L_{n-1} & 0 \\ c^T & x \end{bmatrix}$$

where we need to find c and x such that $LL^T = A$.

$$\begin{bmatrix} L_{n-1} & 0 \\ c^T & x \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} L_{n-1} & 0 \\ c^T & x \end{bmatrix}}_{L} \begin{bmatrix} L_{n-1} L_{n-1}^T & L_{n-1} c \\ c^T L_{n-1}^T & c^T c + x^2 \end{bmatrix} \quad (*)$$

- we need to have

$$L_{n-1} c = b$$

- such a c exists since L_{n-1} is non-singular
- it's non-singular since A_{n-1} is pos. def.

- Now consider $x^2 = a_{nn} - c^T c$
- we have to show that x is real, i.e., $a_{nn} - c^T c > 0$
- Take determinants is $(*)$

$$|L| = |L_{n-1}|^2 x^2 = \det A > 0$$

since A pos def.

- so $x^2 = a_{nn} - c^T c > 0 \Rightarrow x$ positive and real

- we now know that the Choleski decomposition exists.
- How do we compute it?
- we actually could build an algorithm based on our proof.
- But there is a better way

$$A = L L^T$$

$$a_{ij} = \sum_{k=1}^{n} l_{ik} l_{jk} = \sum_{k=1}^{\min\{i,j\}} l_{ik} l_{jk}$$

- $a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}}$
- how do we know that $a_{11} > 0$?

$$a_{i1} = l_{11} l_{i1} \qquad l_{i1} = \frac{a_{i1}}{l_{11}} \qquad i = 2, \dots, n$$

- we can continue in this fashion and compute L column by column

Here is the algorithm:

For $k = 1, \ldots, n$

$$l_{kk} = \sqrt{a_{kk} - \sum_{i=1}^{k-1} l_{ki}^2} \quad \left(\text{since } a_{kk} = \sum_{i=1}^{k} l_{ki}^2\right)$$

For $i = k+1, \ldots, n$

$$l_{ik} = \frac{a_{ik} - \sum_{j=1}^{k-1} l_{ij} l_{kj}}{l_{kk}} \quad i = k+1, \ldots, n$$

- The effort in this procedure is $\frac{n^3}{6} + O(n^2)$

- Notice that

$$a_{kk} = \sum_{i=1}^{k} l_{ki}^2$$

- This means the entries of L are bounded by $\sqrt{a_{kk}}$

- As a consequence we don't need to pivot!

- Of course we would use – symmetric pivoting, matching row and column interchanges, to preserve symmetry!
- major idea: if we don't have to pivot for stability, can we pivot for some other purpose?
- yes! reduce, or minimize, sparsity!

$$\begin{matrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & & \times & & \\ \times & & & \times & \\ \times & & & & \times \end{matrix}$$

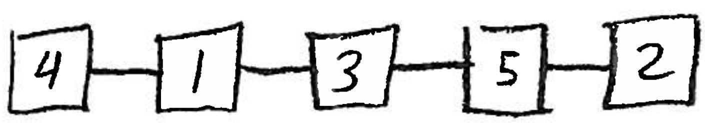
complete fill-in

but exchange 1^{st} and last rows and columns

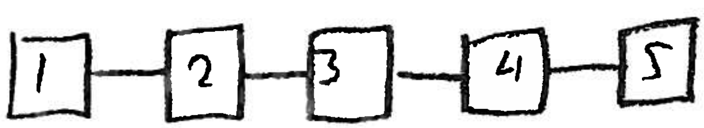
$$\begin{matrix} \times & & & & \times \\ & \times & & & \times \\ & & \times & & \times \\ & & & \times & \times \\ \times & \times & \times & \times & \times \end{matrix}$$

zero fill-in

- Graph of a matrix



$$\begin{matrix} \times & \cdot & \times & \times & \\ & \times & & & \times \\ \times & & \times & & \times \\ \times & & & \times & \\ & \times & \times & & \times \end{matrix}$$



$$\begin{matrix} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \times & \times & \times \\ & & & \times & \times \end{matrix}$$