Math 5600 5/29/14

Let f be $n+1$ times differentiable everywhere in $[a,b]$. Suppose that

$$a \le x_0 < x_1 < \dots < x_n \le b$$

and let P_n be the unique polynomial of degree n satisfying

$$P_n(x_i) = f(x_i) \quad i = 0, \dots, n$$

Then, for all x in (a,b) there exists ξ in $[\min(x, x_0), \max(x, x_n)]$ such that

$$f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi)$$

- certainly true if $x = x_i$
- Let $F(t) = f(t) - P_n(t) - \dfrac{(t-x_0)\dots(t-x_n)(f(x)-P_n(x))}{(x-x_0)\dots(x-x_n)}$

  suppose x does not equal any of the x_i

- Clearly, $F(x_i) = 0$

and also $F(x) = 0$

- F has $n+2$ real roots

- by Rolle's Thm $F^{(n+1)}$ has a root ξ in (a,b)

- $$F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{(n+1)!\,(f(x)-P_n(x))}{(x-x_0)\cdot\ldots\cdot(x-x_n)} = 0$$

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)\cdot\prod_{i=0}^{n}(x-x_i)}{(n+1)!}$$

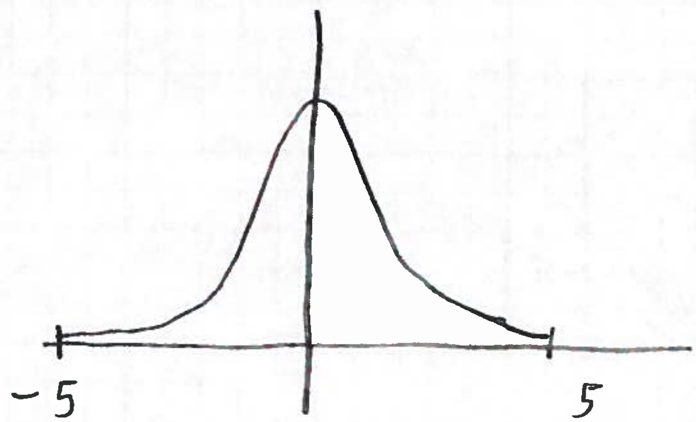
- This error goes to zero as $n \to \infty$ if the derivatives are bounded independently of n

- Example e^x, $\sin x$, $\cos x$

- On the other hand, the derivatives may grow too fast (or may not exist)

Runge Phenomenon

$$f(x) = \frac{1}{1+x^2}$$



- Interpolate at equally spaced points

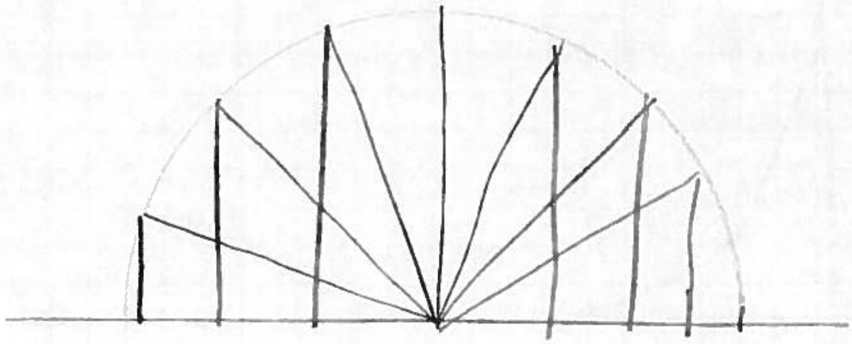
$$h = \frac{10}{n}$$

$$x_i = -5 + ih$$

- you get osillations towards the endpoints that grow as n increases

- poles at $\pm i$

- on the other hand, if the points are chosen suitably the error does go to zero



$$x_k = 5 \cos \frac{k\pi}{n}$$

- In general: Given any scheme of constructing data sites you can find an arbitrarily often differentiable function such that the max error goes to infinity, and for any (just) continuous function you can find a scheme to construct data sites so that the error goes to zero

- Bad idea to use high degree polynomials.