Math 5600 6/5/14

①

- Recall $f(x) = \sum_{i=0}^{n} \alpha_i b_i(x) = p(x)$

- The coefficients $\alpha_0, \dots, \alpha_n$ are chosen such that

$$\|f(x) - p(x)\|^2 = (f-p, f-p) = \min$$

when $(\ ,\ )$ is an inner product with the properties:

$$(f,g) = (g,f)$$
$$(f+g, h) = (f,h) + (g,h)$$
$$(f,f) \geq 0$$
$$(f,f) = 0 \Rightarrow f = 0$$
$$(cf, g) = c(f,g)$$

$$\|f\| = \sqrt{(f,f)}$$

- Example $(f,g) = \int_a^b w(x) f(x) g(x)\,dx$

- We get the linear system

$$\Big[(b_i, b_j)\Big]_{i,j=0,\dots,n} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix} = \Big[(f, b_i)\Big]_{i=0,\dots,n}$$

- If $i \neq j \Rightarrow (b_i, b_j) = 0$ the b_i are orthogonal (with respect to the given inner product)

- we can use the Gram-Schmidt Process to construct orthogonal basis vectors.

- you may have seen it in terms of ordinary vectors with $(v, w) = v^T w = v \cdot w$

- we are given basis functions

$$b_0, b_1, b_2, \ldots$$

we want to construct a new sequence

$$q_0, q_1, q_2, \ldots$$

- such that

$$(q_i, q_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\text{span}\{b_0, \ldots, b_k\} = \text{span}\{p_0, \ldots, p_k\}$$

for all $k = 0, 1, 2, \ldots$

We start with

$$q_0 = \frac{b_0}{\|b_0\|}$$

- Then, for $k = 1, 2, \ldots$

define

$$z_k = b_k - \sum_{i=0}^{k-1} (b_k, q_i)\, q_i$$

$$q_k = \frac{z_k}{\|z_k\|}$$

This works since

$$(z_k, q_j) = (b_k, q_j) - \sum_{i=0}^{k-1} (b_k, q_i)(q_i, q_j)$$

$$= (b_k, q_j) - (b_k, q_j) = 0$$

- Let's do an example

$$b_k(x) = x^k \qquad (f,g) = \int_0^1 f(t)g(t)\,dx$$

$$q_0(x) = \frac{1}{\sqrt{\int_0^1 1 \cdot 1\,dt}} = 1$$

$$z_1(x) = x - \int_0^1 t \cdot 1\,dt = x - \frac{1}{2}$$

$$q_1(x) = \frac{x - 1/2}{\left(\int_0^1 (t - 1/2)^2\,dt\right)^{1/2}} = 2\sqrt{3}\,(x - 1/2)$$

$$z_2 = x^2 - \int_0^1 t^2 \cdot 1\,dt \cdot 1 - \int_0^1 2\sqrt{3}\,(t - 1/2)\,t^2\,dt \cdot 2\sqrt{3}\,(x - 1/2)$$

$$= x^2 - x + 1/6$$

$$q_2 = \frac{x^2 - x + 1/6}{\left(\int_0^1 (t^2 - 1 + 1/6)^2\,dt\right)^{1/2}} = 6 \cdot \sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$$

- The Gram-Schmidt process works for any inner product and any set of basis functions.

- However, for the special case that $b_i = x^i$ and

$$(f,g) = \int_a^b w(x)f(x)g(x)dx \qquad w(x) > 0$$

the resulting polynomials are usually normalised so that their leading coefficient is 1. The Gram-Schmidt process simplifies to the "three-term recurrence relation":

$$Q_n = Q_n(x) = x^n + L.O.T$$

$$Q_0 = 1$$

$$Q_1 = x - a_1$$

$$Q_n = (x - a_n) Q_{n-1} - b_n Q_{n-2}$$

$$a_n = \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})} \qquad b_n = \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}$$

- Note that for arbitrary a_n and b_n this creates a sequence of polynomials with leading coefficient 1.

- Also note that the denominators are non-zero

$$(f,g) = \int_0^1 f(x)g(x)\,dx$$

Example
Shifted Legendre

$$Q_n = (x - a_n)\,Q_{n-1} - b_n\,Q_{n-2}$$

$$a_n = \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})} \qquad b_n = \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}$$

$$Q_0 = 1 \qquad a_1 = \frac{\int_0^1 t\,dt}{\int_0^1 1\,dt} = 1/2$$

$$Q_1 = x - a_1 = x - 1/2$$

$$a_2 = \frac{\int_0^1 t(t-1/2)^2\,dt}{\int_0^1 (t-1/2)^2\,dt} = \frac{1}{2} \quad (\text{not } 0)$$

$$b_2 = \frac{\int_0^1 t(t-1/2)\,dt}{\int_0^1 1^2\,dt} = \frac{1}{12}$$

$$Q_2 = (x - \tfrac{1}{2})(x - \tfrac{1}{2}) - \frac{1}{12}$$

$$= x^2 - x + \frac{1}{4} - \frac{1}{12}$$

$$= x^2 - x + \frac{1}{6}$$ as with Gram-Schmidt

Proof by induction

$$(Q_1, Q_0) = \left(x - \frac{(x,1)}{(1,1)}, 1 \right) = (x,1) - \frac{(x,1)(1,1)}{(1,1)} = 0$$

suppose $Q_0, Q_1, \cdots, Q_{n-1}$ are orthogonal, and $k<n$

$$(Q_n, Q_k) = \left(\left(x - \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})}\right) Q_{n-1} - \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})} Q_{n-2}, Q_k \right)$$

$$= (xQ_{n-1}, Q_k) - \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})}(Q_{n-1}, Q_k) - \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}(Q_{n-2}, Q_k)$$

Three cases:

$k=n-1$: $= (xQ_{n-1}, Q_{n-1}) - (xQ_{n-1}, Q_{n-1}) = 0$

$k=n-2$: $= (xQ_{n-1}, Q_{n-2}) - (xQ_{n-1}, Q_{n-2}) = 0$

$k<n-2$ $= (xQ_{n-1}, Q_k)$

$= (Q_{n-1}, xQ_k)$ The key step!

$= \left(Q_{n-1}, \sum_{i=0}^{n-2} \alpha_i Q_i\right) = \sum_{i=0}^{n-2} \alpha_i (Q_{n-1}, Q_i) = 0$

22.2. Orthogonality Relations

	$f_n(x)$	Name of Polynomial	a	b	$w(x)$	Standardization	h_n	Remarks
22.2.1	$P_n^{(\alpha,\beta)}(x)$	Jacobi	-1	1	$(1-x)^\alpha(1+x)^\beta$	$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$	$\alpha>-1,\ \beta>-1$
22.2.2	$G_n(p,q,x)$	Jacobi	0	1	$(1-x)^{p-q}x^{q-1}$	$k_n=1$	$\frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}$	$p-q>-1,\ q>0$
22.2.3	$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer)	-1	1	$(1-x^2)^{\alpha-\frac{1}{2}}$	$C_n^{(\alpha)}(1)=\binom{n+2\alpha-1}{n}$ $(\alpha\neq0)$	$\frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}$ $\alpha\neq0$	$\alpha>-\frac{1}{2}$
						$C_n^{(0)}(1)=\frac{2}{n}$, $C_0^{(0)}(1)=1$	$\frac{2\pi}{n^2}$ $\alpha=0$	
22.2.4	$T_n(x)$	Chebyshev of the first kind	-1	1	$(1-x^2)^{-\frac{1}{2}}$	$T_n(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq0\\ \pi & n=0\end{cases}$	
22.2.5	$U_n(x)$	Chebyshev of the second kind	-1	1	$(1-x^2)^{\frac{1}{2}}$	$U_n(1)=n+1$	$\frac{\pi}{2}$	
22.2.6	$S_n(x)$	Chebyshev of the first kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$	$S_n(2)=n+1$	$\begin{cases}4\pi & n\neq0\\ 8\pi & n=0\end{cases}$	
22.2.7	$C_n(x)$	Chebyshev of the second kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}$	$C_n(2)=2$	4π	
22.2.8	$T_n^*(x)$	Shifted Chebyshev of the first kind	0	1	$(x-x^2)^{-\frac{1}{2}}$	$T_n^*(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq0\\ \pi & n=0\end{cases}$	
22.2.9	$U_n^*(x)$	Shifted Chebyshev of the second kind	0	1	$(x-x^2)^{\frac{1}{2}}$	$U_n^*(1)=n+1$	$\frac{\pi}{8}$ *	
22.2.10	$P_n(x)$	Legendre (Spherical)	-1	1	1	$P_n(1)=1$	$\frac{2}{2n+1}$	
22.2.11	$P_n^*(x)$	Shifted Legendre	0	1	1		$\frac{1}{2n+1}$	

*See page II.

22.2. Orthogonality Relations—Continued

22.2.12	$L_n^{(\alpha)}(x)$	Generalized Laguerre	0	∞	$e^{-x}x^{\alpha}$	$k_n=\frac{(-1)^n}{n!}$	$\frac{\Gamma(\alpha+n+1)}{n!}$	$\alpha>-1$
22.2.13	$L_n(x)$	Laguerre	0	∞	e^{-x}	$k_n=\frac{(-1)^n}{n!}$	1	
22.2.14	$H_n(x)$	Hermite	$-\infty$	∞	e^{-x^2}	$a_n=(-1)^n$	$\sqrt{\pi}2^n n!$	
22.2.15	$He_n(x)$	Hermite	$-\infty$	∞	$e^{-\frac{x^2}{2}}$	$a_n=(-1)^n$	$\sqrt{2\pi}n!$	

22.3. Explicit Expressions

$$f_n(x)=d_n\sum_{m=0}^{N}c_m g_m(x)$$

	$f_n(x)$	N	d_n	c_m	$g_m(x)$	k_n	Remarks
22.3.1	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{1}{2^n}$	$\binom{n+\alpha}{m}\binom{n+\beta}{n-m}$	$(x-1)^{n-m}(x+1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.2	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}$	$\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)}$	$(x-1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.3	$G_n(p,q,x)$	n	$\frac{\Gamma(q+n)}{\Gamma(p+2n)}$	$(-1)^m\binom{n}{m}\frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)}$	x^{n-m}	1	$p-q>-1,\ q>0$
22.3.4	$C_n^{(\alpha)}(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{\Gamma(\alpha)}$	$(-1)^m\frac{\Gamma(\alpha+n-m)}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$	$\alpha>-\frac{1}{2},\ \alpha\neq0$
22.3.5	$C_n^{(0)}(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n}\quad n\neq0$	$n\neq0,\ C_0^{(0)}(1)=1$
22.3.6	$T_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{n}{2}$	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^{n-1}	
22.3.7	$U_n(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	
22.3.8	$P_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{2^n}$	$(-1)^m\binom{n}{m}\binom{2n-2m}{n}$	x^{n-2m}	$\frac{(2n)!}{2^n(n!)^2}$	
22.3.9	$L_n^{(\alpha)}(x)$	n	1	$(-1)^m\binom{n+\alpha}{n-m}\frac{1}{m!}$	x^m	$\frac{(-1)^n}{n!}$	$\alpha>-1$
22.3.10	$H_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	see 22.11
22.3.11	$He_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!2^m(n-2m)!}$	x^{n-2m}	1	