Math 5600

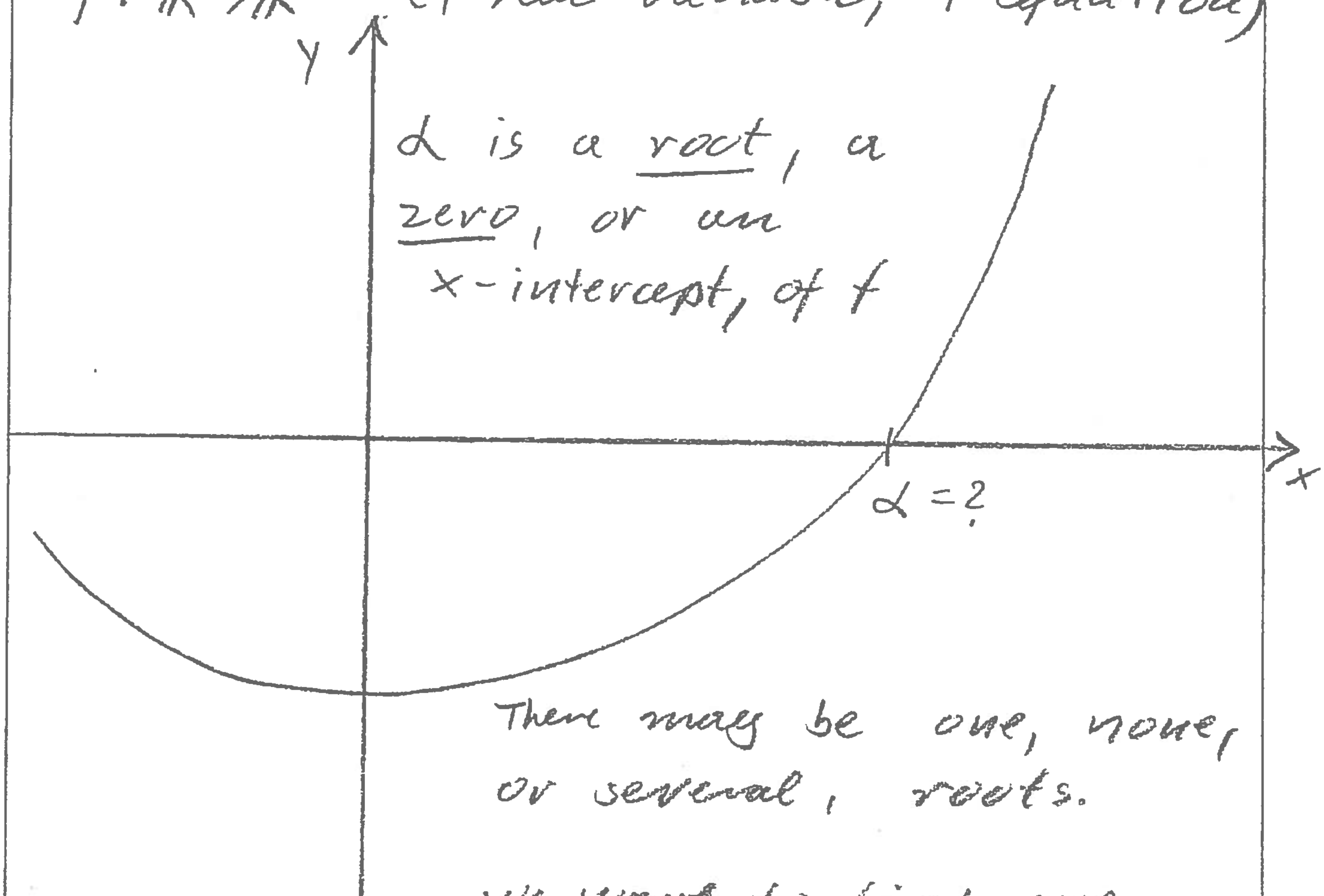
5/20/14

Nonlinear Equations (in 1 variable)

- Suppose we want to solve

$$y = f(x) = 0$$

$f: \mathbb{R} \to \mathbb{R}$ (1 real variable, 1 equation)



There may be one, none, or several, roots.

We want to find one

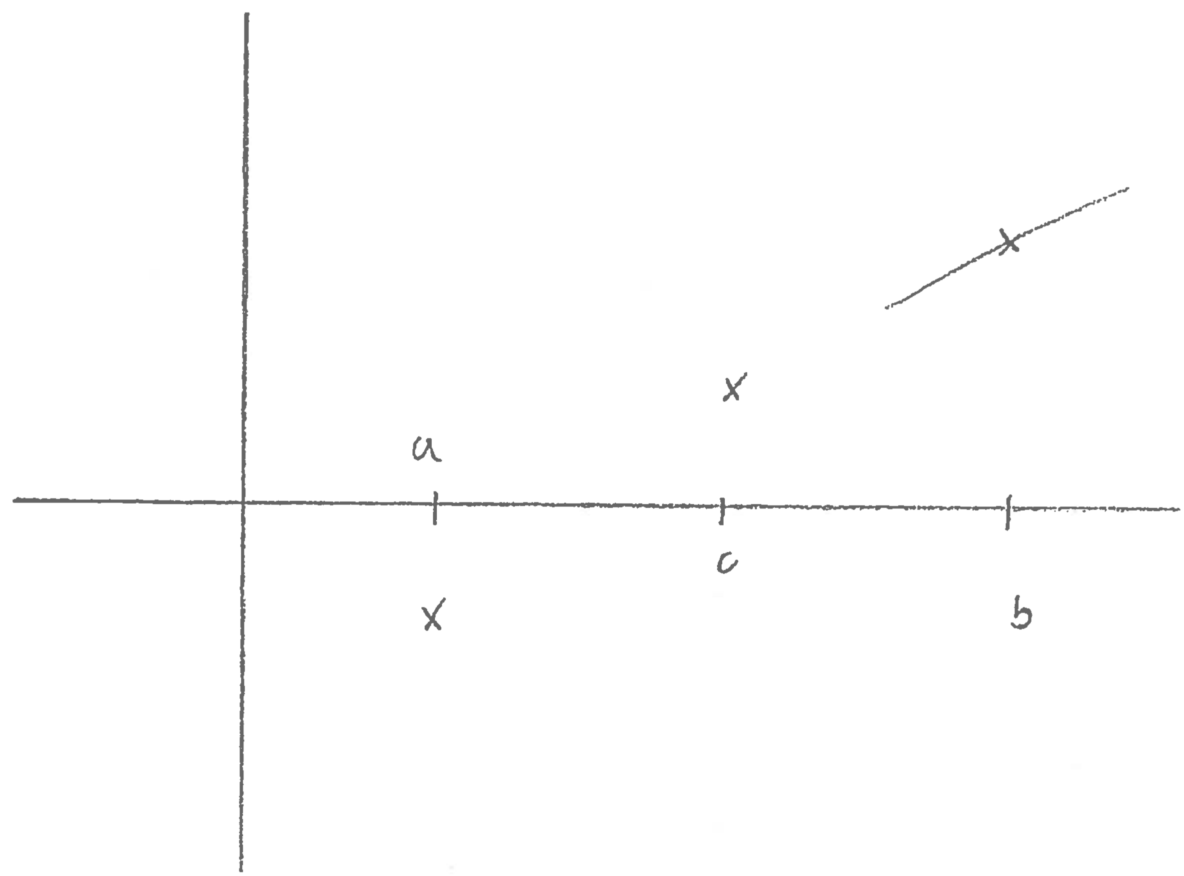
- Most frequently used: Newton's Method.

- But there are alternatives.

- Bisection: suppose $f(a)$ and $f(b)$ have opposite signs

$$f(a)f(b) < 0$$

- Also suppose f is continuous
- WE ALWAYS ASSUME SUFFICIENT SMOOTHNESS
- Then let $c = \dfrac{a+b}{2}$
- Evaluate $f(c)$ and throw away a or b. (unless $f(c) = 0$, then stop)



- At every step the interval gets halved.
- Proceed until the interval is sufficiently small
- It's not really an iteration since we know beforehand how few to go.
- Bisection is also a frequently used debugging method.
- Continuity is of course essential
- consider $f(x) = \frac{1}{x}$ $a = -1$ $b = 1$
- no root in $[-1, 1]$, and $f\left(\frac{-1+1}{2}\right)$ is undefined.

- Let's review Newton's Method.

x_0 given. $x_0, x_1, x_2, \longrightarrow \alpha$

converges hopefully

- Step from x_k to x_{k+1}

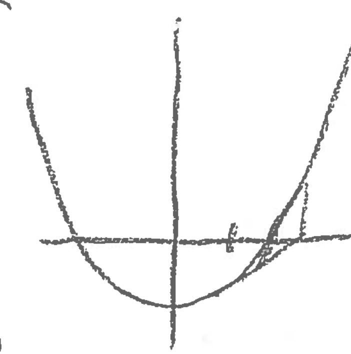
$$0 \approx f(x_{k+1}) = f(x_k) + f'(x_k)(x_{k+1} - x_k) + HOT$$

$$f(x_k) + f'(x_k)(x_{k+1} - x_k) = 0$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots}$$

- Example $f(x) = x^2 - 2$ $\quad x_0 = 1$

k	x_k	$\sqrt{2} - x_k$
0	1	0.414
1	1.5	-0.09
2	1.417	-0.002
3	1.414216	-0.000002
4	1.41421568627	-1.6 E-12



- Observations?

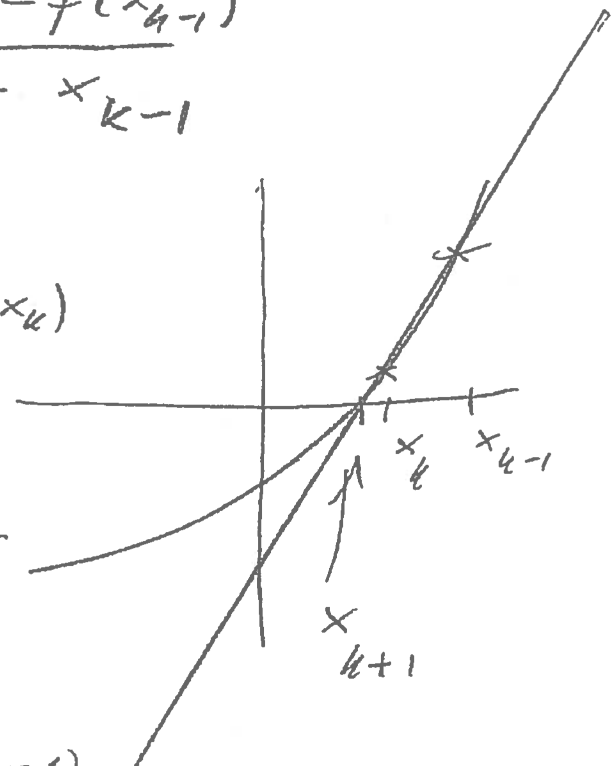
- Biggest Drawback of NM: It requires Derivatives.

- Idea: use the last two points and approximate the slope of the tangent with the slope of the secant.

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

$$= \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$



- The Secant Method
- Exercise: run SM on $x^2 - 2 = 0$

- what can go wrong?

- So far we have done the first part of numerical analysis: design an algorithm
- 2nd part analyze its properties
- we'll do those two parts many times
-

	starting	speed	convergence	Complex	Systems	[illegible]
Bisection	1	slow	yes	complicated	complic.	[illegible]
NM		faster	usually	works	easy	[illegible]
SM		fast	usually	works	very subtle	[illegible]

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- All 3 methods have a starting problem
  - Bisection: need opposite signs
  - NM, SM: may not converge

    will converge if "sufficiently close"

- Let's look closer at NM. It's an example of a fixed point iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = g(x_k)$$

$f(x) = 0$ $\Longleftrightarrow$ $x = g(x)$

root finding problem — fixed point problem

- Fixed point iteration: x_0 given $x_{k+1} = g(x_k)$
- There are many ways to convert a root finding problem to a fixed point problem.

- Exercise 9, hw 1 (tp) (satellite program)

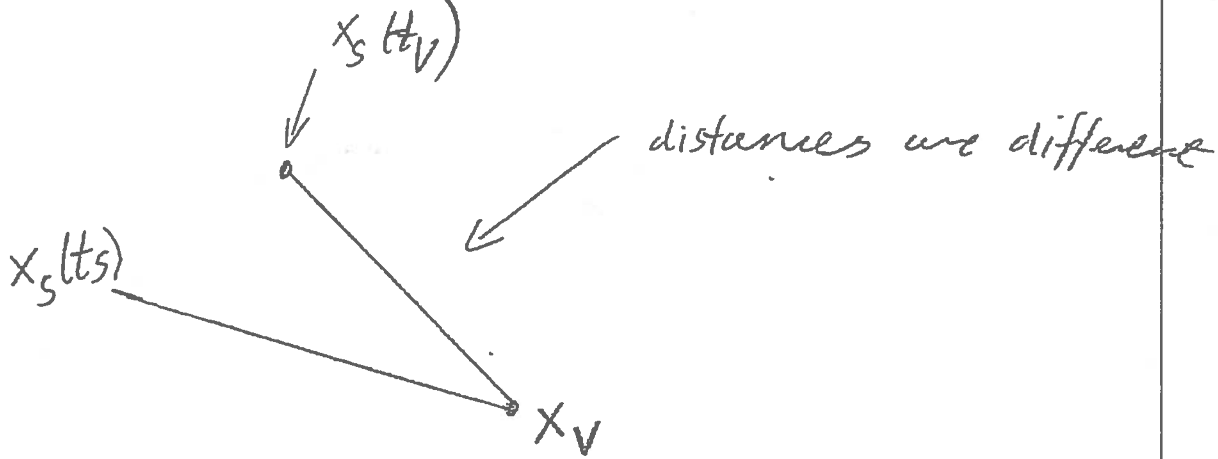
t_V: time vehicle receives satellite signal

x_V: location of vehicle at time t_V

t_S: time satellite sends signal

$x_S(t_S)$: location of satellite

t_S is unknown



$$x_s(t) = (R+h)\left[u\cos\left(\frac{2\pi t}{P}+\theta\right) + v\sin\left(\frac{2\pi t}{P}+\theta\right)\right]$$

formula (20) in tp.

- we need to solve the equation

$$F(t_s) = t_V - \frac{\|x_s(t_s) - x_V\|}{c} - t_s = 0 \qquad (*)$$

- we know everything but t_s
- could use Newton's Method. (exercise)

- Here is a much easier approach.
- write $t = t_s$ and rewrite (*) as

$$t = t_V - \frac{\|x_s(t) - x_V\|}{c}$$

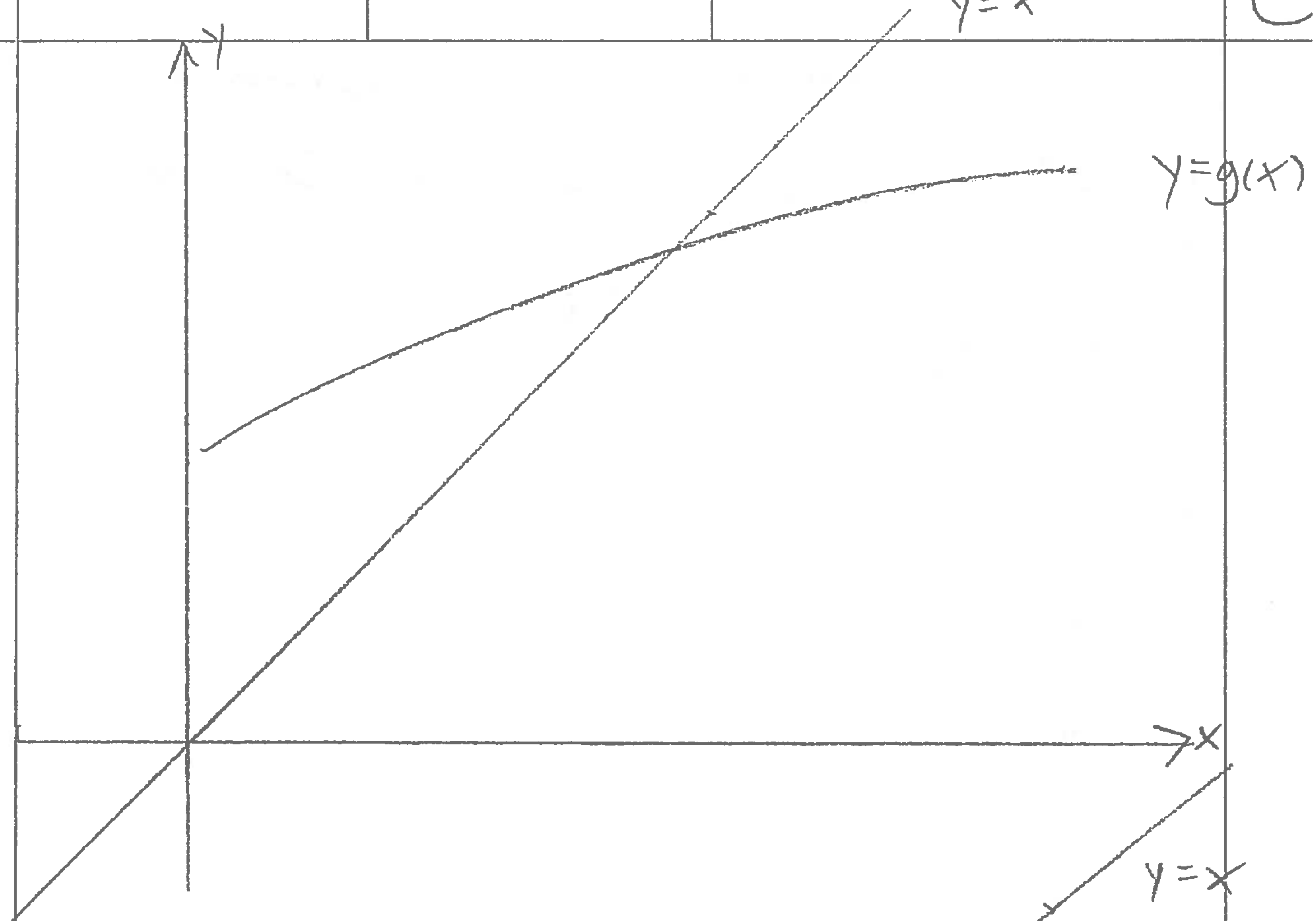
- Iterate:

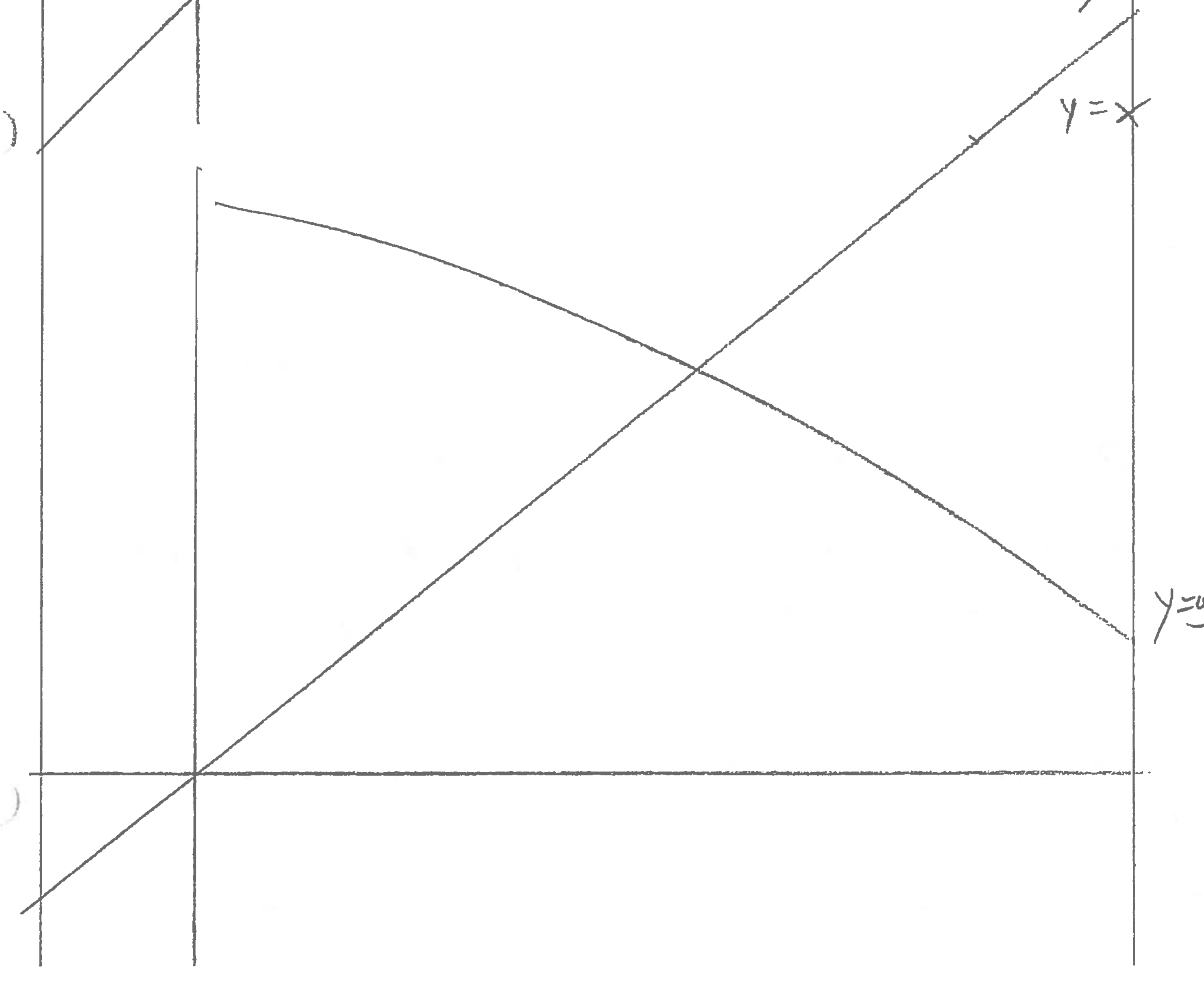
$$t_0 = t_V$$

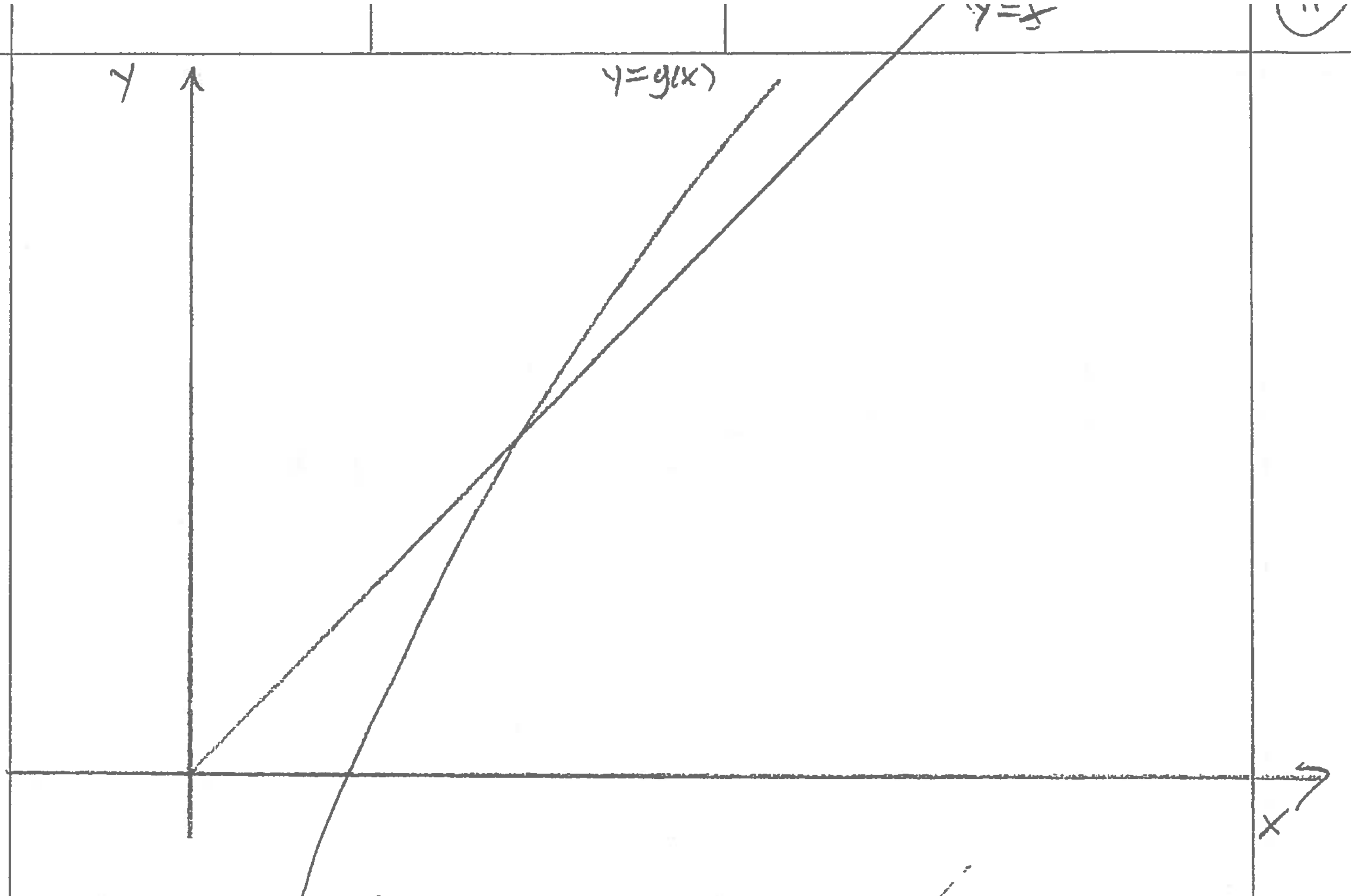
For $k = 0, 1, 2, \ldots$

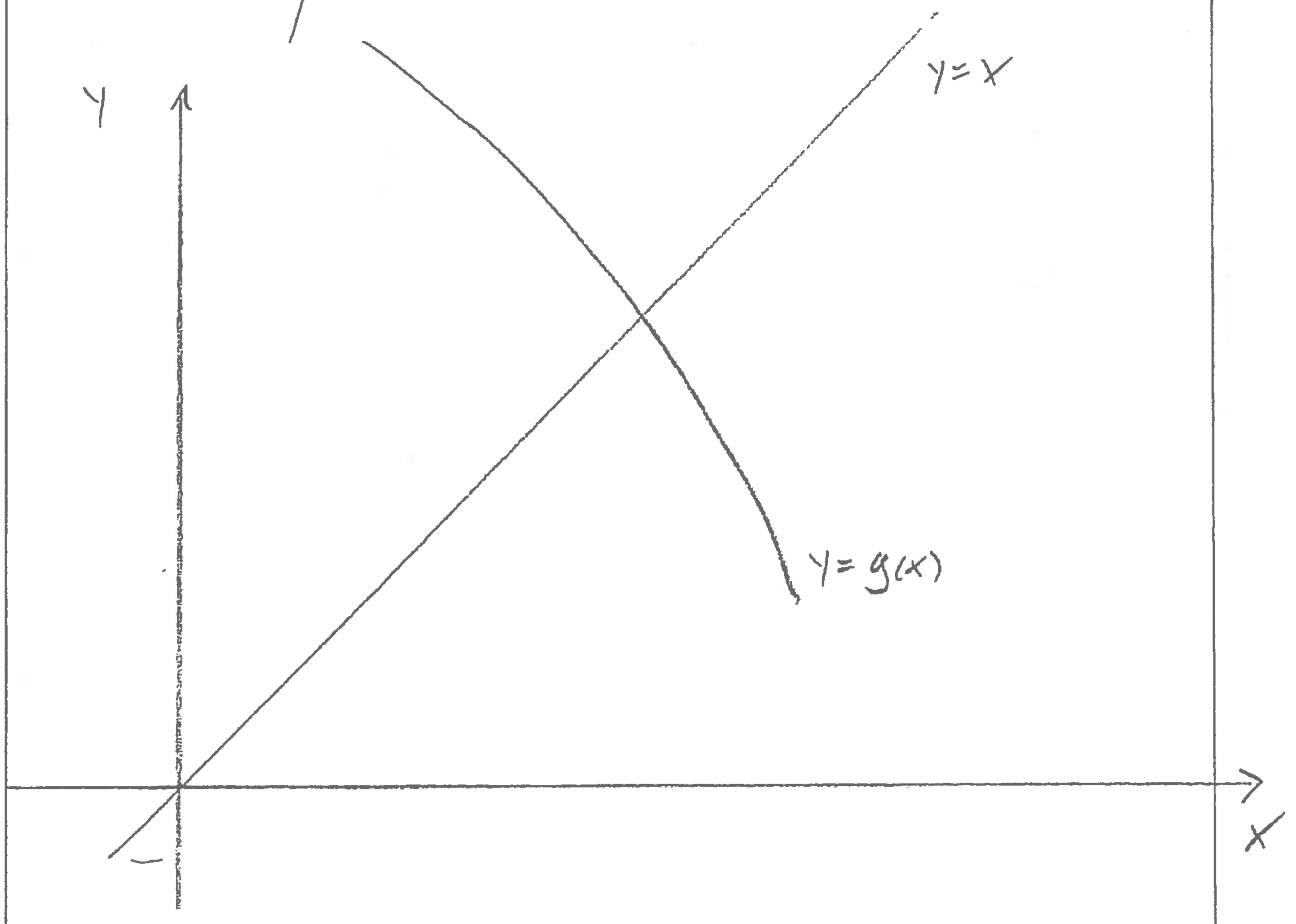
$$t_{k+1} = t_V - \frac{\|x_s(t_k) - x_V\|}{c}$$

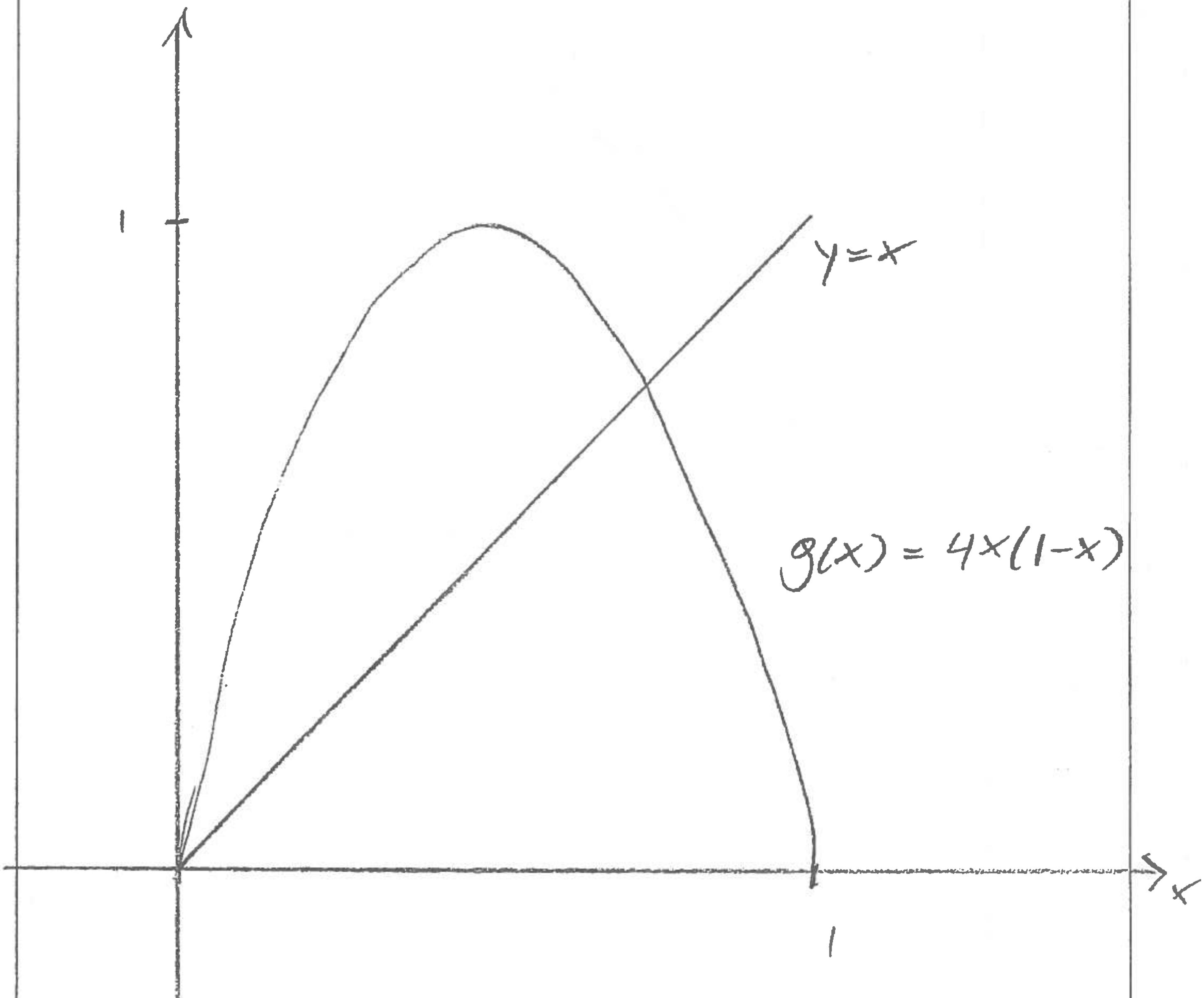
- stop when $c\,|t_{k+1} - t_k| < 10^{-2}$ (1 cm)
- Does not require derivatives.
- Typical: starting point, termination criterion, actual iteration, depend on the problem.
- We don't just apply a formula.
- When does $x_{k+1} = g(x_k)$ converge


y
y = x
y = g(x)
x


y = x
y = g


y = x
y = g(x)
y
x


y = x
y
y = g(x)
x


1
y=x
g(x) = 4x(1-x)
1
x