The B-Form

Math 5600

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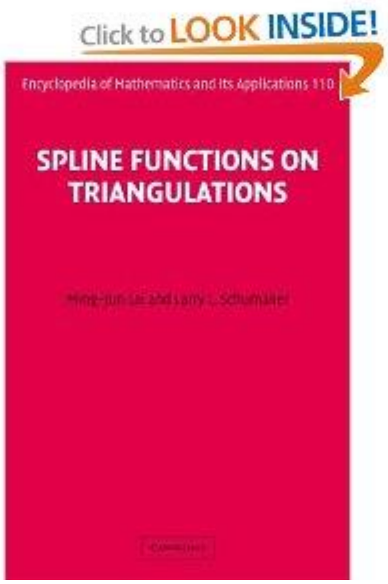
More information and software on www.math.utah.edu/~pa/
Pdf version of slides on www.math.utah.edu/~pa/mvs.pdf

State of the Art

Ming-Jun Lai and Larry L. Schumaker: *Spline Functions on Triangulations.* Cambridge University Press, 2007. ISBN 0521875927.



What's in a name?

- "Spline" = smooth piecewise polynomial function.
- "smooth" means so many times differentiable.
- "Univariate" = one independent variable.
- Classic subject, used ubiquitously in numerical analysis for approximating data and functions.
- Example: cubic splines, pass an elastic wire through a bunch of points.

Univariate Splines

partition: $[a, b]$ into N subintervals

$a = x_0 < x_1 < \ldots < x_N = b$

$S_d^r = \{s \in C^r[a, b] : s|_{I_i} \in P_d, \quad i = 1, \ldots, N\}$

where P_d is the space of univariate polynomials of degree d.

$\dim S_d^r = (d+1) + (N-1)\big((d+1) - (r+1)\big)$

What could be **simpler**?

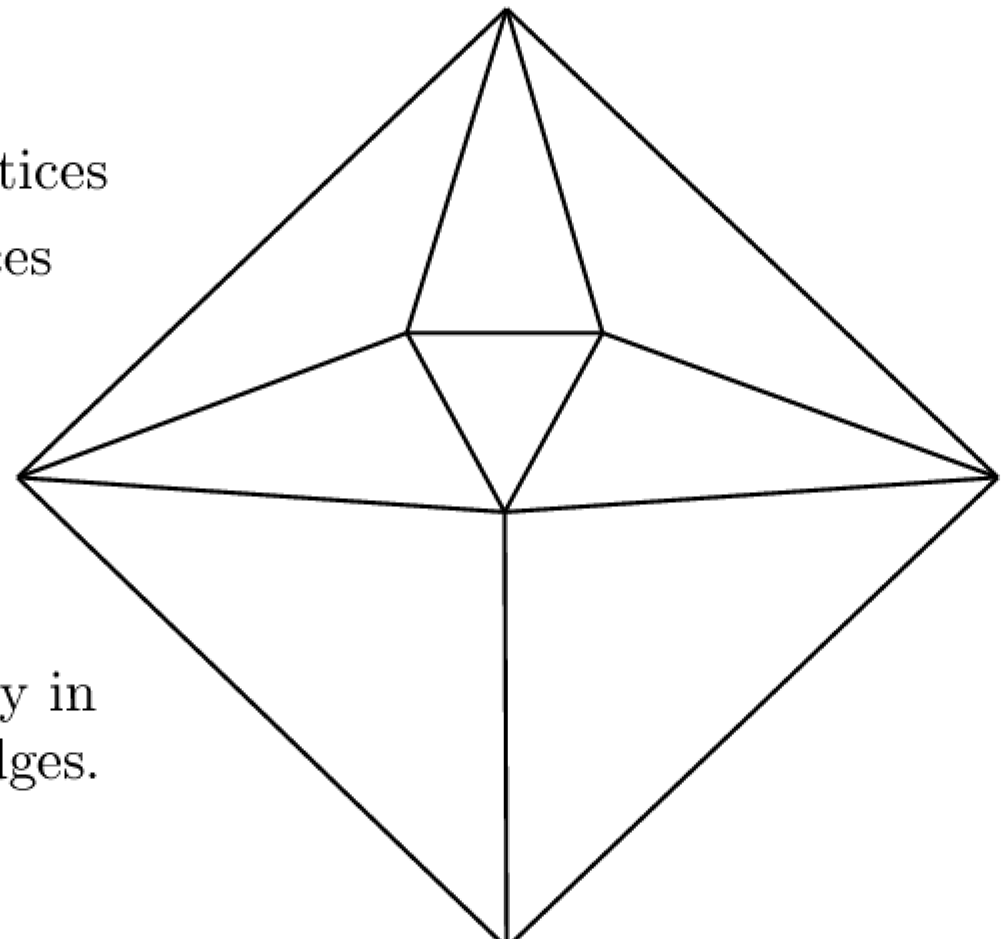
Triangulations

Tessellate by triangles connecting scattered points.

$N = 8$ triangles

$V_B = 4$ boundary vertices

$V_I = 3$ interior vertices



Triangles intersect only in common vertices or edges.

Bivariate Splines

- Triangulations are the natural generalizations of interval partitions because they can handle *scattered* (i.e., arbitrarily distributed) points.
- Key object: The set of all functions that are r times differentiable, and that on each triangle can be written as a bivariate polynomial of degree d.
- It's a linear space, and as such it has a dimension.
- Formally:

Natural bivariate analog of univariate S_d^r:

V Vertices: $v_1, v_2, \ldots, v_V$

N Triangles: $T_1, T_2, \ldots, T_N$

polynomial of degree d: $p(x, y) = \sum_{i+j \leq d} x^i y^j$

domain: $\Omega = \bigcup_{i=1,\ldots N} T_i$

$S_d^r = \{s \in C^r(\Omega) : s_{T_i} \text{ is polynomial of degree } d.\}$

Barycentric Coordinates

Let Δ be a triangle with vertices v_1, v_2, and v_3. For $x \in \mathbb{R}^2$, define its barycentric coordinates b_1, b_2, b_3 by:

$$x = \sum_{i=1}^{3} b_i v_i \quad \text{where} \quad \sum_{i=1}^{3} b_i = 1.$$

Note that barycentric coordinates are linear functions of x.

Bernstein-Bezier Form

Any polynomial $p \in P_d$ can be written uniquely in its *Bernstein-Bézier form* as:

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k.$$

where

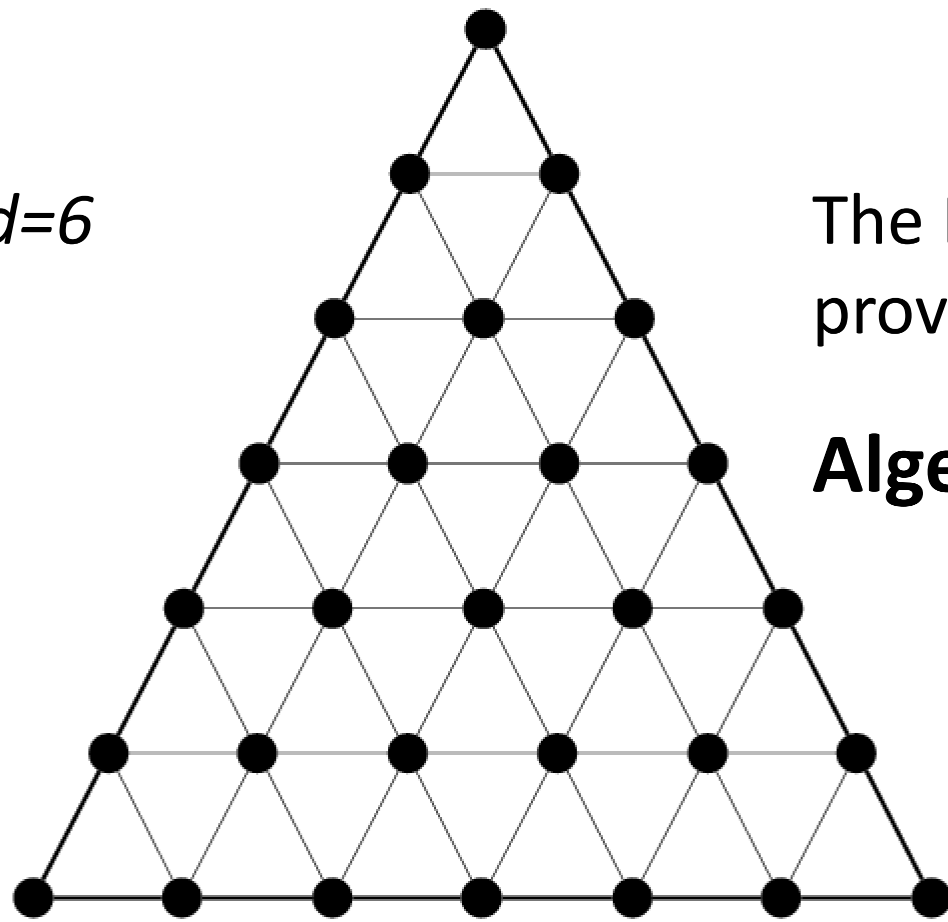
c_{ijk} : *Bézier ordinates*

$P_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d} \in \mathbb{R}^2$: *Domain Points*

$(P_{ijk}, c_{ijk}) \in \mathbb{R}^3$: *Bézier control points*

Bézier Control Net

d=6



The Bernstein-Bézier form provides a link between

Algebra ↔ Geometry

- The control points at the vertices lie on the graph of the polynomial. This is because

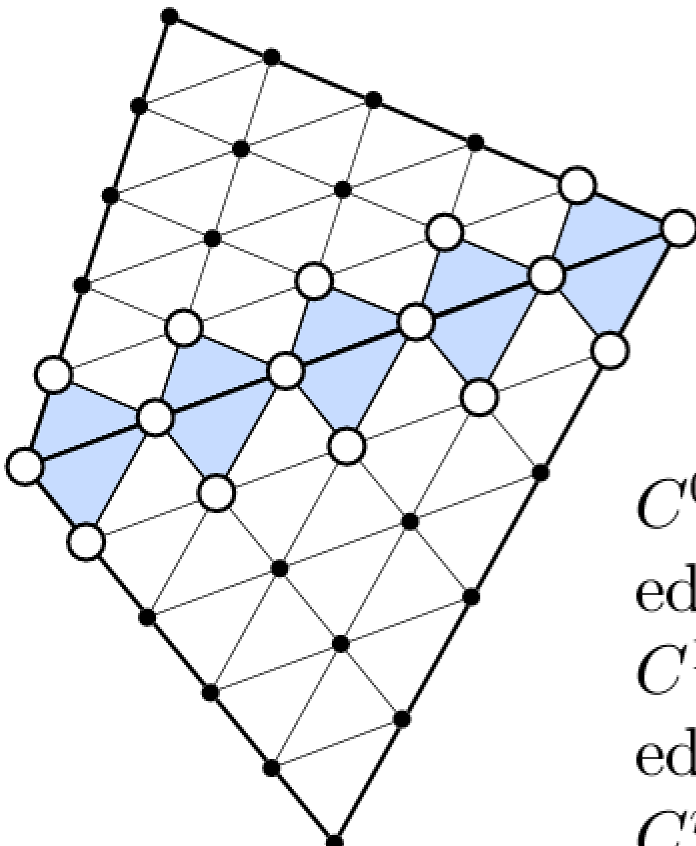
$$p(v_1) = c_{d00}, \quad p(v_2) = c_{0d0}, \quad p(v_3) = c_{0d0}$$

- The control points in the 1-disk, i.e.,

$$(P_{d00}, c_{d00}), \quad (P_{d-1,1,0}, c_{d-1,1,0}), \quad (P_{d-1,0,1}, c_{d-1,0,1})$$

  lie in the tangent plane of p at v_1. Similarly for v_2 and v_3.
- The control points along an edge determine the values of the polynomial along that edge.
- The control points along an edge, and in the first row parallel to the edge, determine the values of first derivatives of the polynomial along that edge.

Smoothness Conditions



d = 5, r = 1

C^0: Control points along edges coincide
C^1: Quadrilaterals across edges are planar in $\mathbb{R}^3$.
C^r: Evaluate subpolynomials.

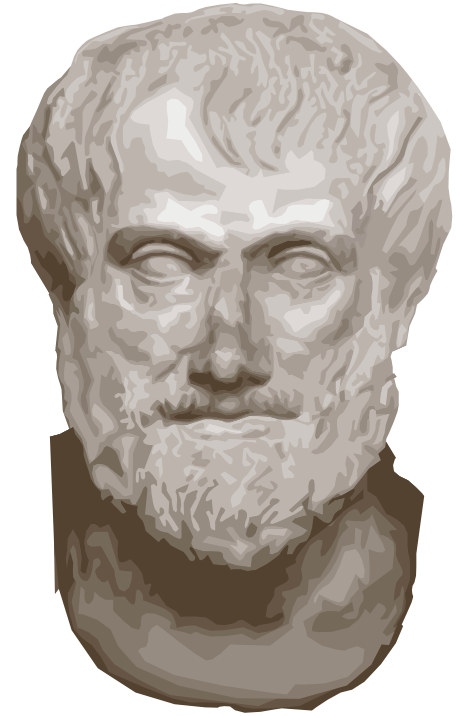
Idea of a particular proof: Let D be a first order directional derivative operator.

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k, \qquad \text{as before}$$

On a neighboring triangle we have

$$\tilde{p}(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} \tilde{c}_{ijk} b_1^i b_2^j b_4^k.$$

For continuity, we require $c_{ij0} = \tilde{c}_{ij0}$, $i+j=d$. We get:



There must be a proof in any math talk.

Aristotle, 346BC

$$Dp(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} \left(i b_1^{i-1} D b_1 b_2^j b_3^k + b_1^i j b_2^{j-1} D b_2 b_3^k + b_1^i b_2^j k b_3^{k-1} D b_k \right)$$

$$= \sum_{i+j+k=d-1} \frac{(d-1)!}{i!j!k!} \hat{c}_{ijk} b_1^i b_2^j b_3^k$$

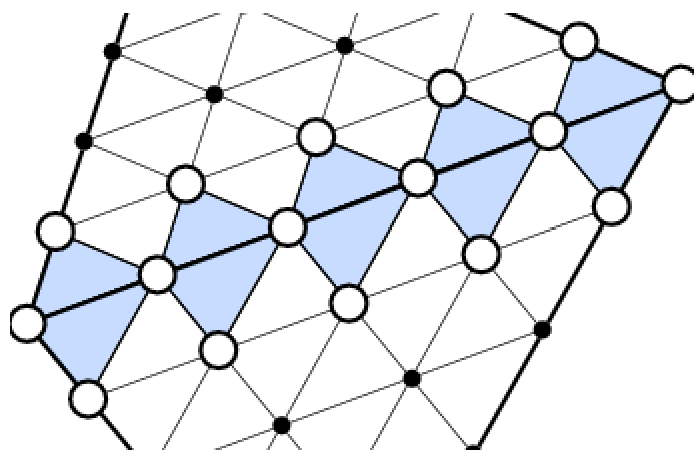
where

$$\hat{c}_{ijk} = d(c_{i+1,j,k}Db_1 + c_{i,j+1,k}Db_2 + c_{i,j,k+1}Db_3)$$

Note that Db_1, Db_2, and Db_3 are *constant.* Differentiating on both triangles, restricting the derivative to the common edge, equating coefficients, and dividing by d gives the condition

$$a_1 c_{i+1,j0} + a_2 c_{ij+10} + a_3 c_{ij1} + a_4 \tilde{c}_{ij1} = 0 \qquad (*)$$

for $i + j = d - 1$ where a_1, a_2, a_3, and a_4 are *independent* of i, j, and d.

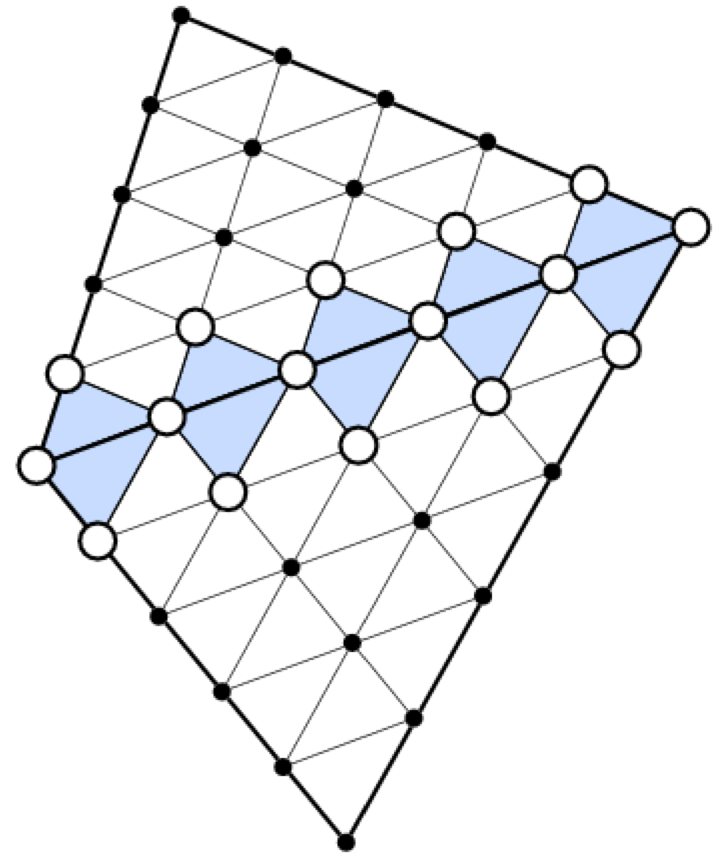


- Thus the equations $(*)$ are *independent* of the degree d and the particular quadrilateral along that edge.

- In particular, we obtain the same condition for the case $d = 1$. In that case, the C^1 condition means that the piecewise linear function be in fact linear.

- This means that in the large quadrilateral, formed by the two triangles, for $d = 1$, the four control points lie *in the same plane.*

- Since the small quadrilaterals are *similar* to the large quadrilateral the algebraic relation (*) has the same meaning: the quadrilateral in 3-space must be planar.



Algebra ↔ Geometry

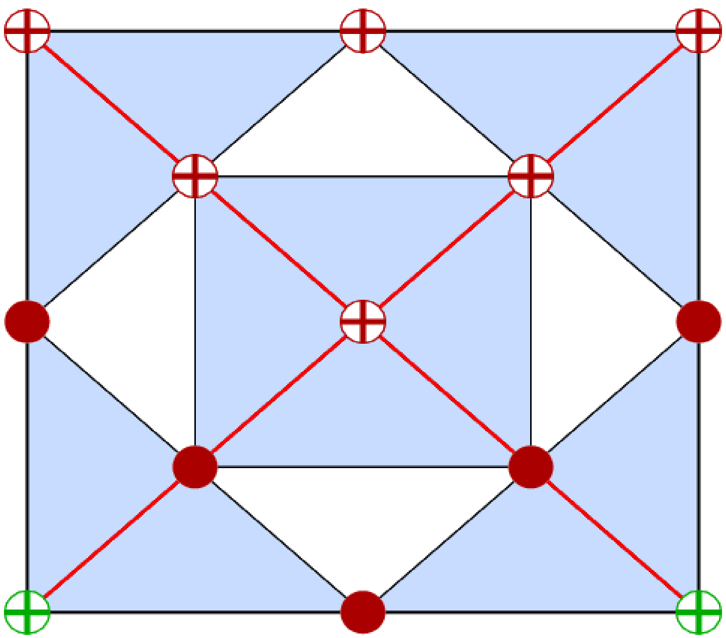
Major Difficulty (major!)

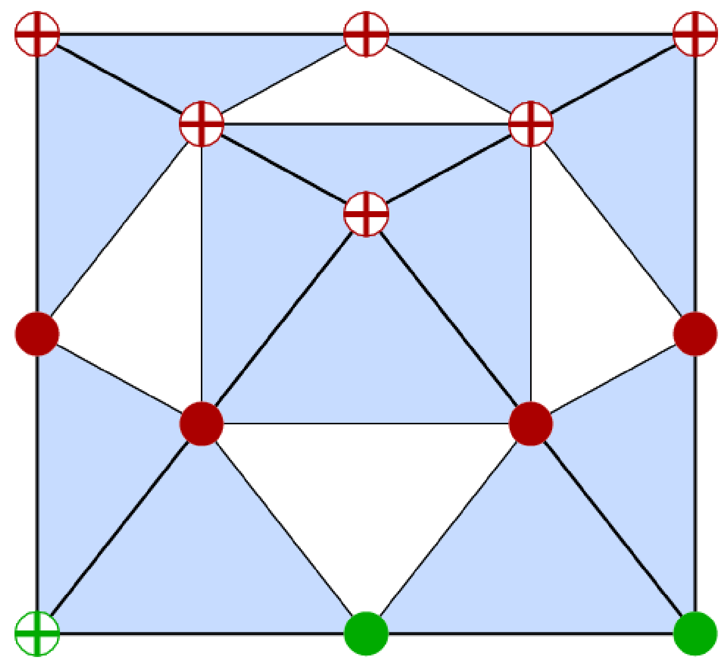
The dimension of S_d^r depends not just on the **combinatorics** of a triangulation, but also on its **geometry**.

Simplest Example: **Singular Vertex**

Four triangles meeting at an interior vertex

$\dim S_2^1 = 8$ $\dim S_2^1 = 7$

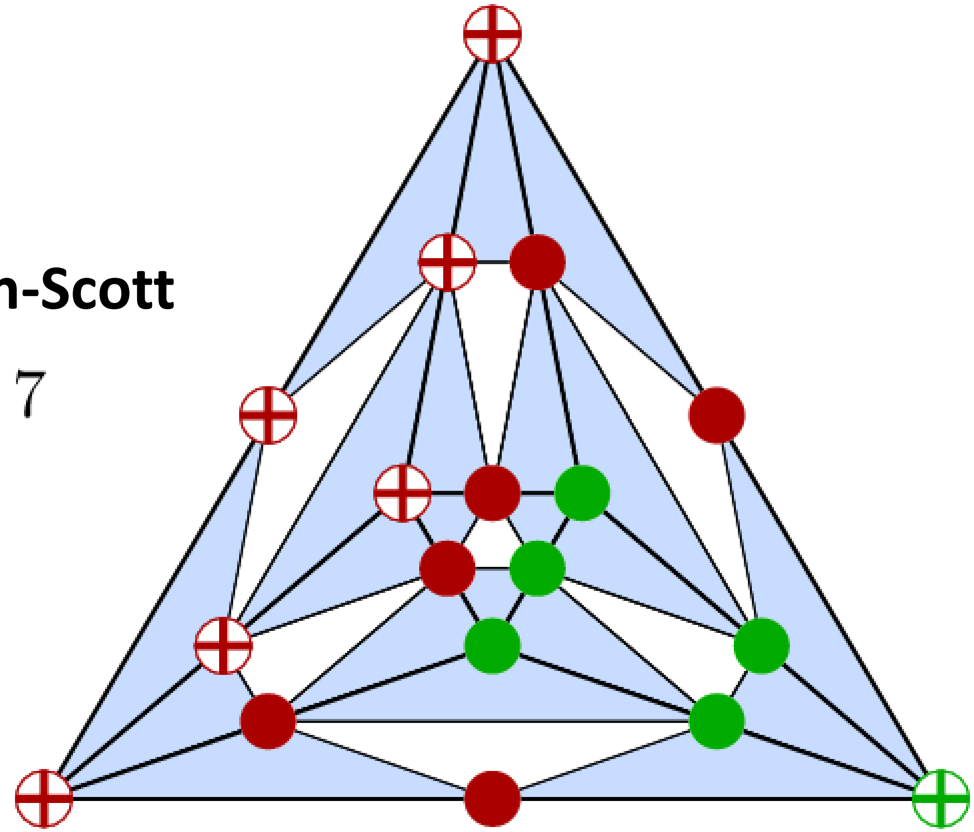


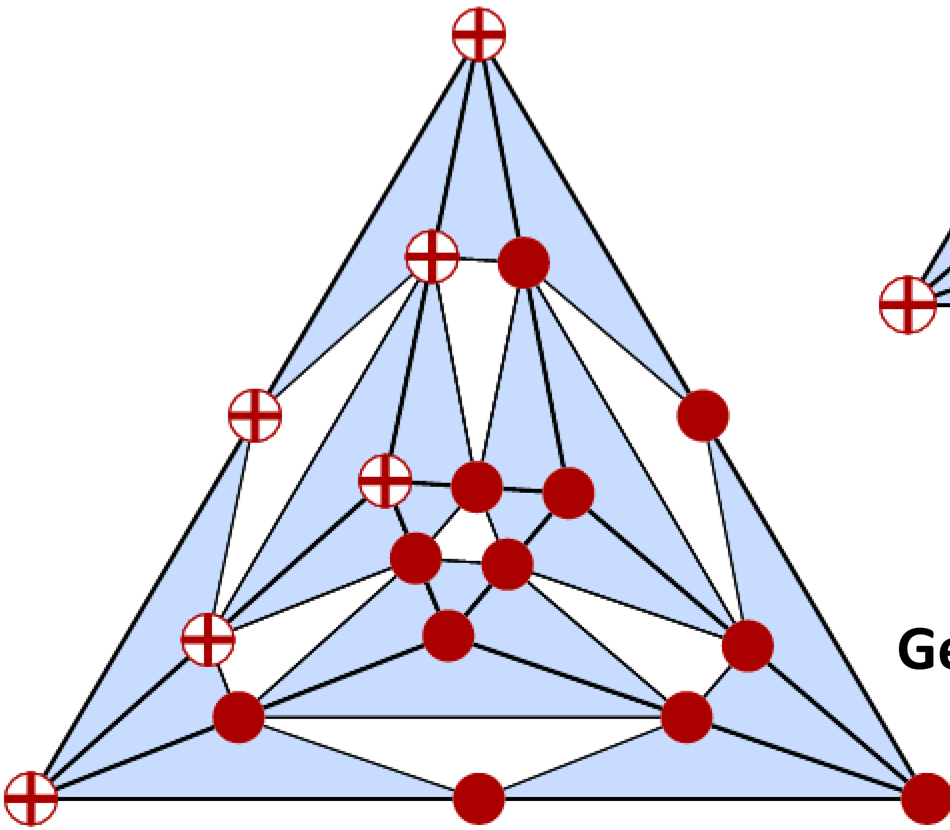


It's not just triangulations with one interior vertex!

Symmetric Morgan-Scott

$$\dim S_2^1 = 7$$

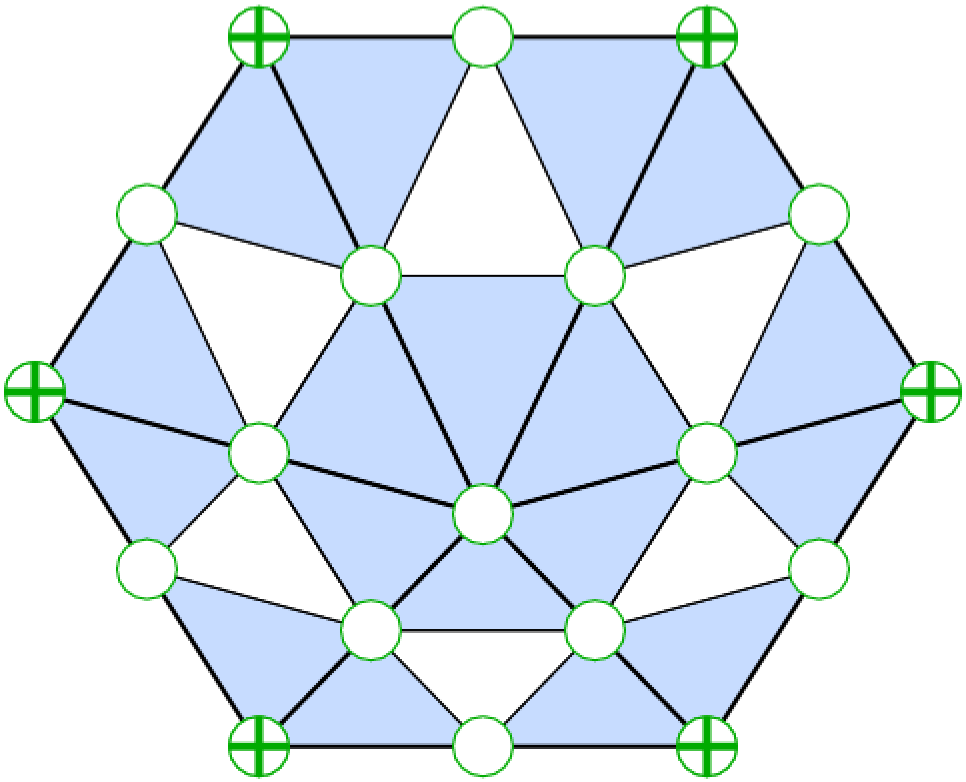




Generic Morgan-Scott

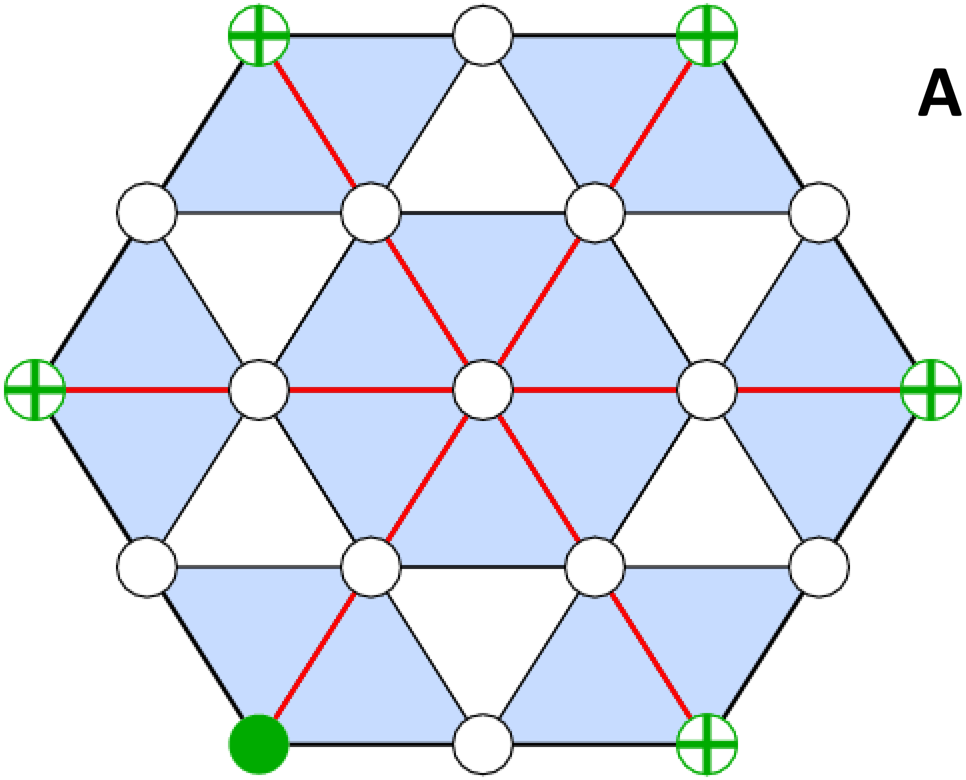
$$\dim S_2^1 = 6$$

It’s not just the dimension!



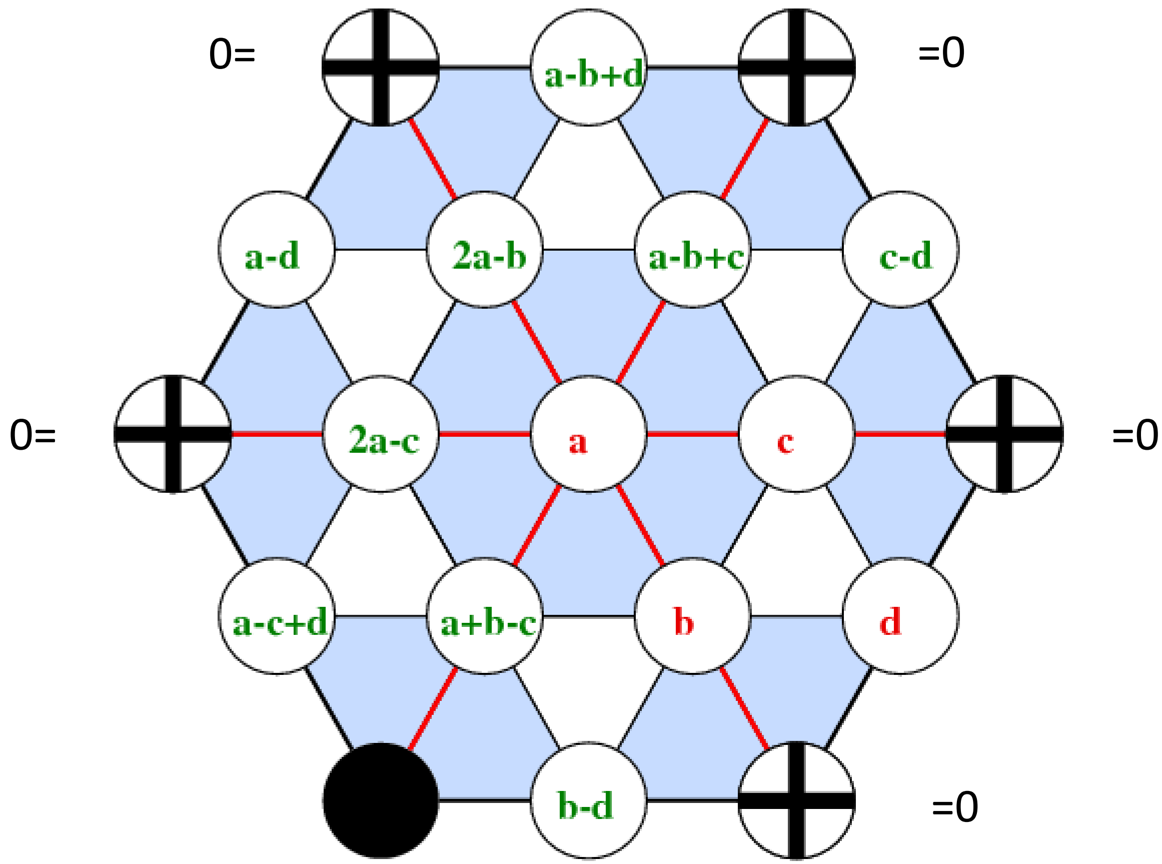
Here you can

Geometry affects Interpolation!



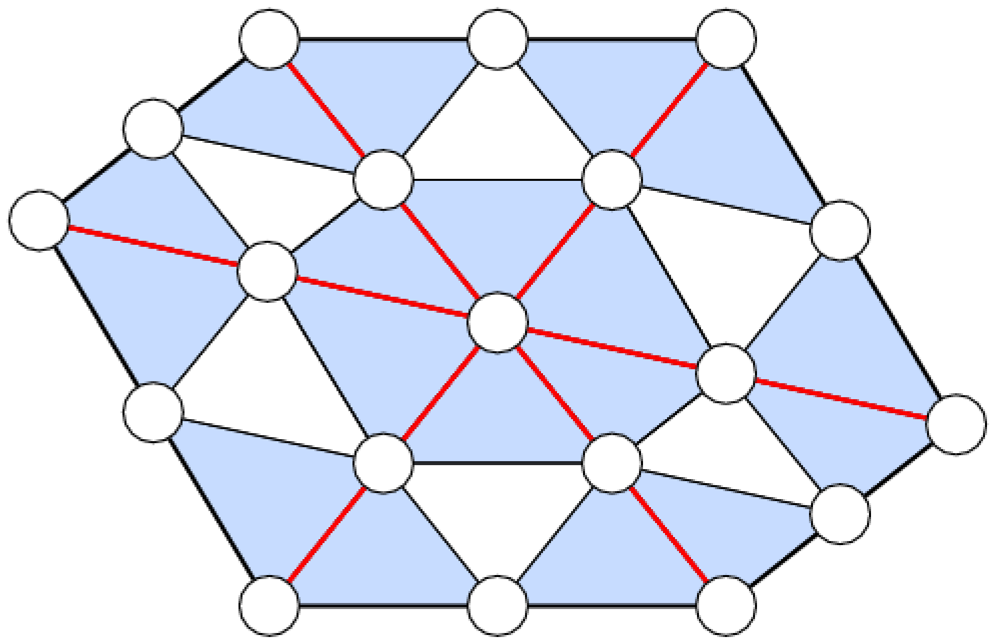
And here you can't!

What's happening?



z=(b-d)-(a+b-c)+(a-c+d) = 0
independent of a,b,c,d

What do you expect to happen on a defective 6-star?



Can, or can't you interpolate?

Generic Dimension

Every spline space S has a generic dimension. If the dimension of S does not equal its generic value then there is an arbitrarily small perturbation of the location of the vertices such that the dimension of S does equal the generic value. Any other dimensions can only be larger than the generic dimension.

Proof: Let $S = S_d^r$ be the subspace of S_d^0 with a coefficient vector c that satisfies the smoothness conditions

$$Ac = 0.$$

The entries of A are rational functions of the location of the vertices of the underlying triangulation.

Let D be the minimum dimension of S. Then

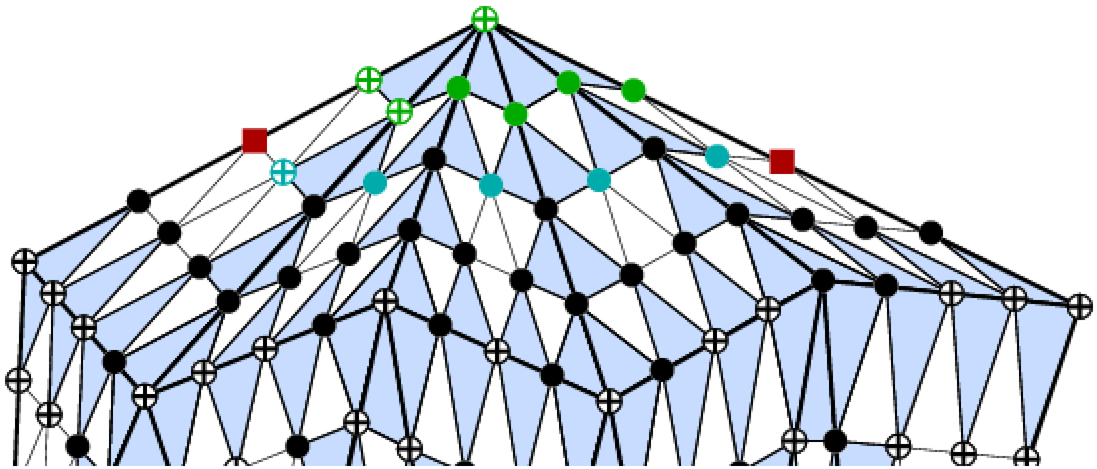
$$D = \dim S_d^0 - R,$$

with $R = \operatorname{rank} A$, and where (without loss of generality)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with A_{11} being a non-singular $R \times R$ matrix.

The expression $\det A_{11}$ is a non-zero rational function of the locations of the vertices, and can vanish only on a set of measure zero.



The generic dimension of S_4^1 is $6V-3$. Proof by Induction

Add the star of a boundary vertex to the growing triangulation T.

Black points are determined by the spline on T.

The green, red, and cyan points are newly imposed.

3 green, 2 red, and 1 cyan point are newly assigned.

This argument also shows that one can interpolate generically to function and gradient at vertices.

- Things get easier as the polynomial degree increases.
- Exact dimension known if $r = 1$ and $d \geq 4$, or $r > 1$ and $d \geq 3r + 2$.
- Generic dimension known for S_2^1 and S_3^1, and for $d = 3r+1$ when $r > 1$.
- Dimensions and many other facts known on many types of special triangulations.
- Most famous outstanding problem:

  V_B: number of boundary vertices

  V_I: number of interior vertices

  σ: number of singular vertices

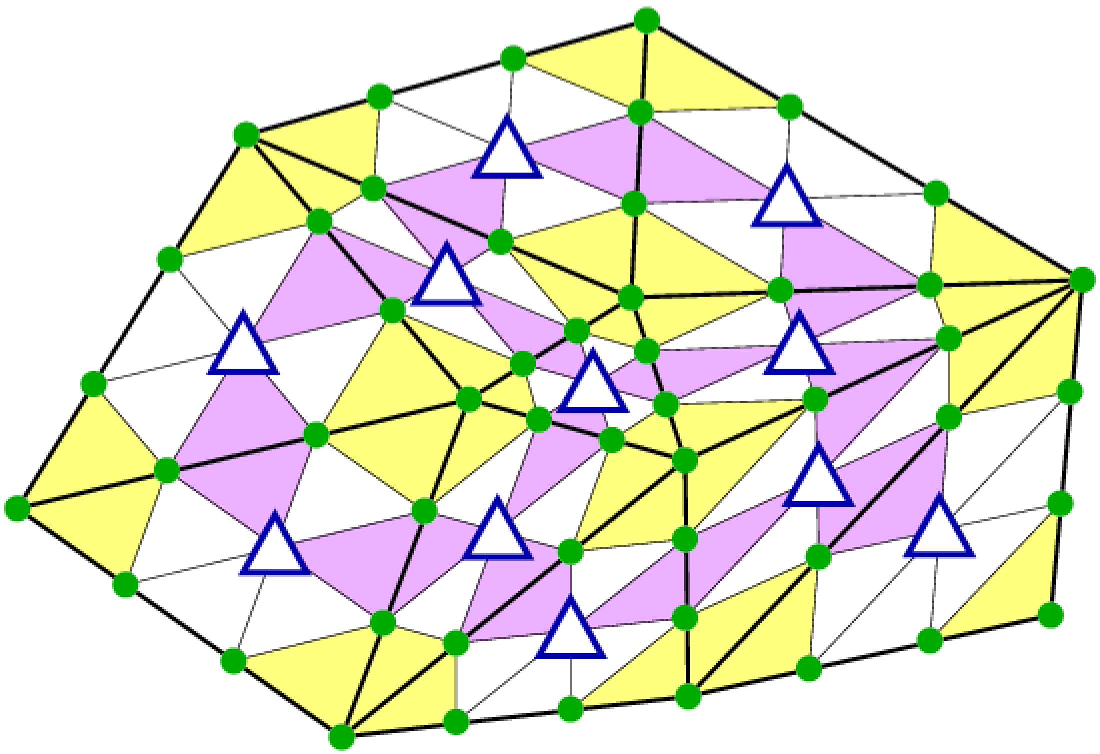
  Problem first mentioned in a conjecture by Gil Strang in the early seventies. What's the dimension of S_3^1?

  $$\dim S_3^1 \geq 3V_B + 2V_I + 1 + \sigma$$

  Does equality hold? Conjecture: yes.

  If you solve this problem I want to know about it!

Why is this so hard?

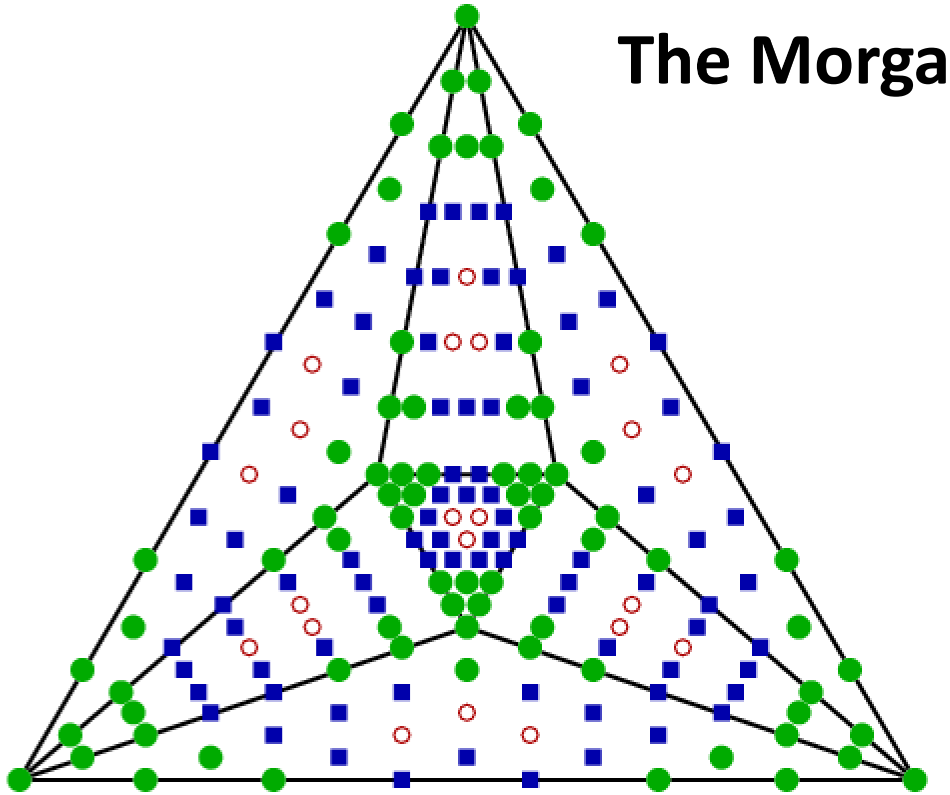


You can't localize things.

$$\dim S_3^1 \geq 3V + N - E_I + \sigma = 3V_B + 2V_I + 1 + \sigma.$$

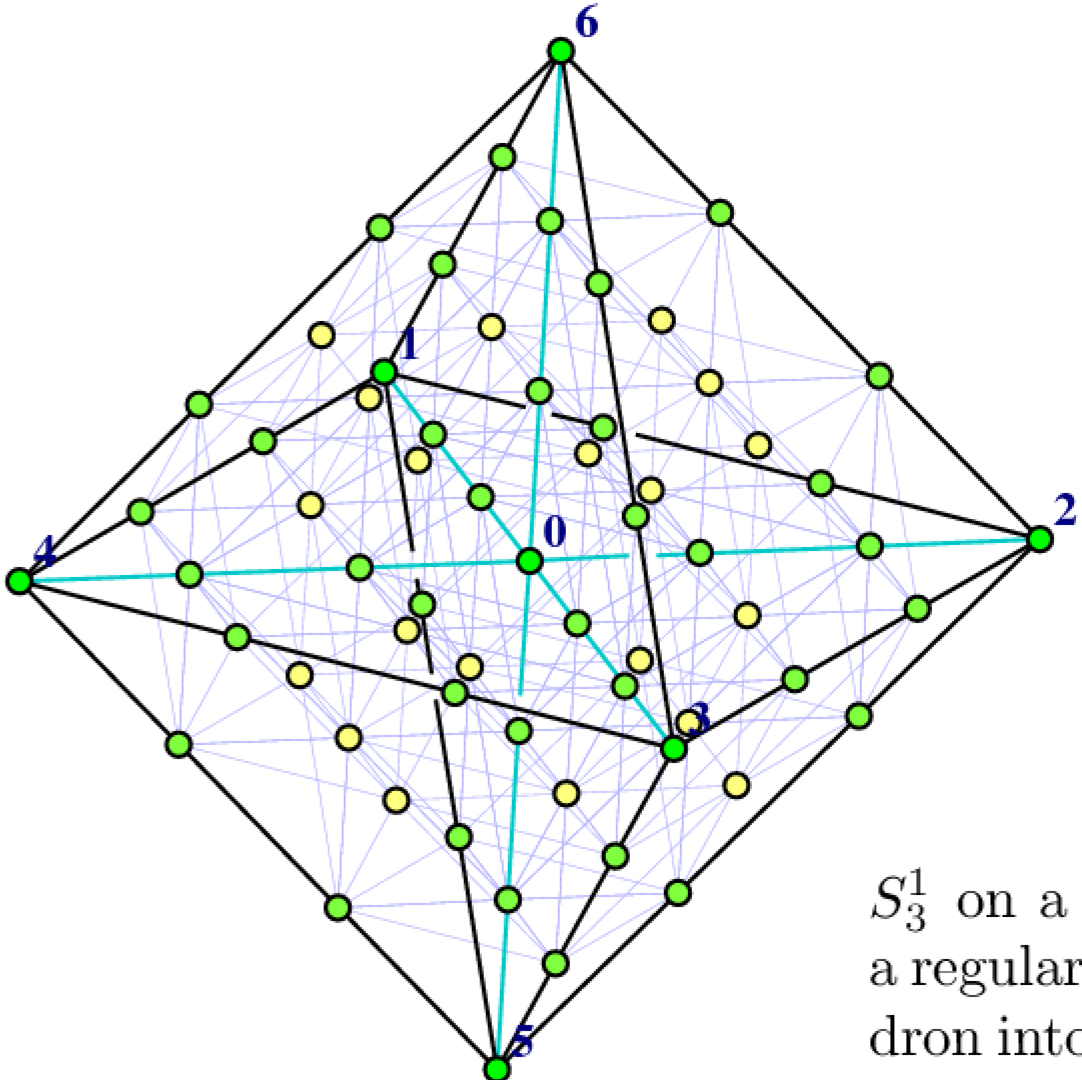
What works for large values of *d*?

The Morgan-Scott Idea



Use vertex globs (green), edge globs (blue) , and inactive points (red). Requires $d > 4r$. Smoothness conditions decouple.

Trivariate Splines



Consider a triangulation of a polyhedral domain by tetrahedra.

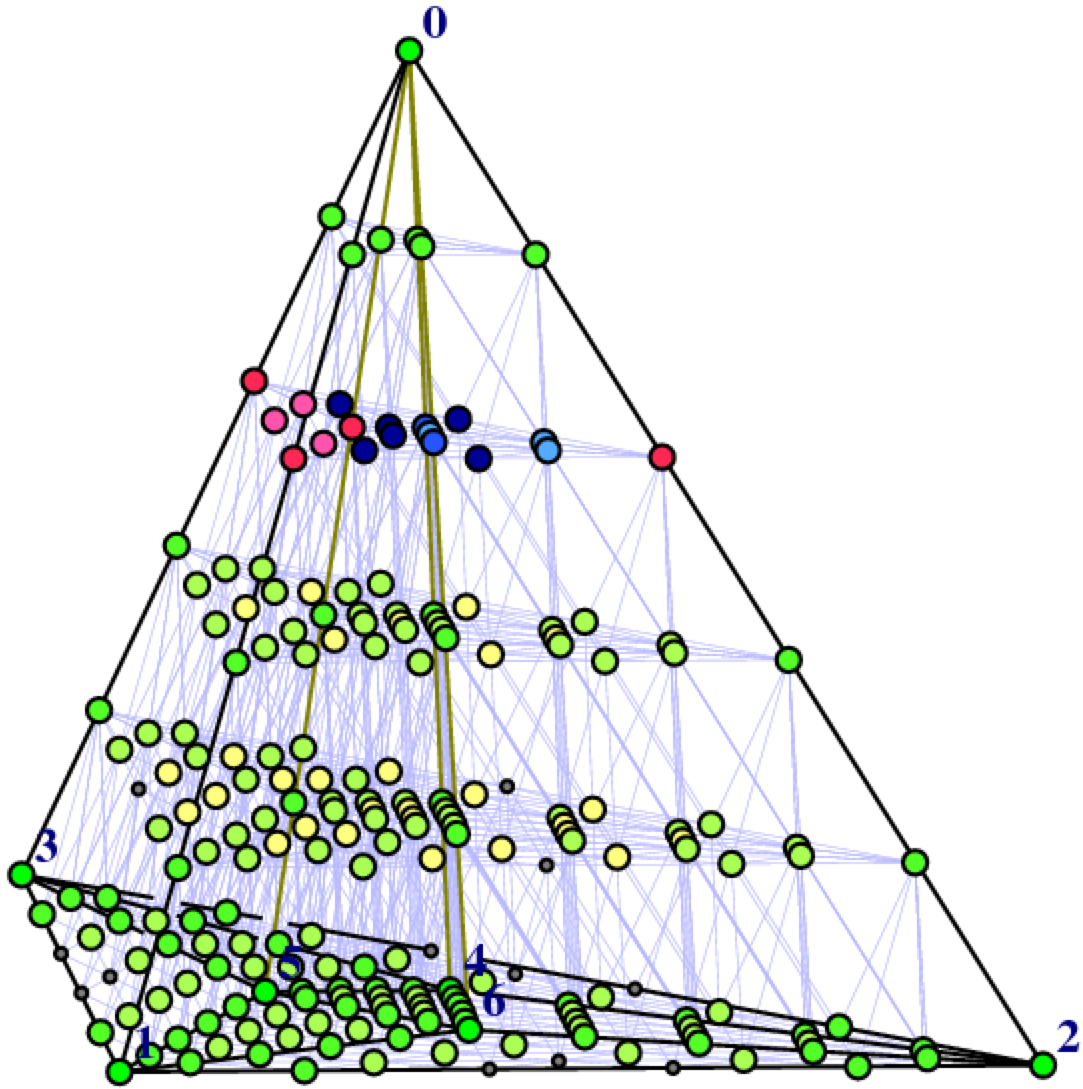
Similar definition as before.

S_3^1 on a split of a regular octahedron into 8 tetrahedra.

- Similar problems. Dimension depends on the geometry.
- Morgan-Scott idea requires $d > 8r$.
- Generic dimension is known for $r = 1$ and $d \geq 8$.
- One new problem:

— Knowing the dimension of the trivariate space S_d^r for sufficiently large d means we know the dimension of the bivariate space **for all** d.

— To see this construct a three dimensional triangulation by starting with a planar triangulation T and then connecting every vertex of T to a new vertex in $\mathbb{R}^3$ outside of the plane containing T.



d+1 **bivariate spline spaces**

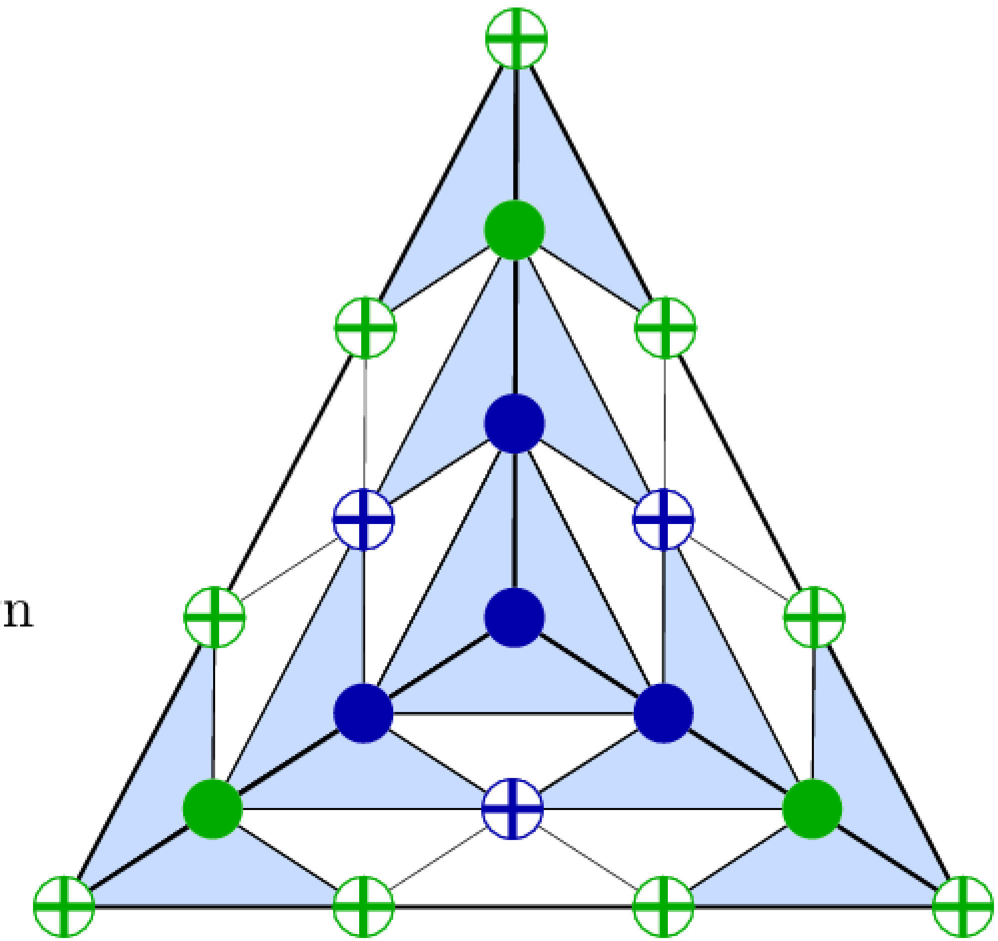
Lift planar Triangulation to R^3. Smoothness conditions de-couple.

Macro Schemes

Most applications of multivariate splines are based on so called **macro elements** (when approximating data) or **finite elements** (when solving differential equations).

- The interpolant is determined on each simplex by data on that simplex.
- Simplices may be subdivided.
- The overall spline space is a sub or superspace of the full space S_d^r
- Many macro schemes are known in 2, 3, or n variables.

Clough-Tocher, *r=1, d=3,*
3 micro-triangles



Some Recent Tetrahedral Macro-Schemes

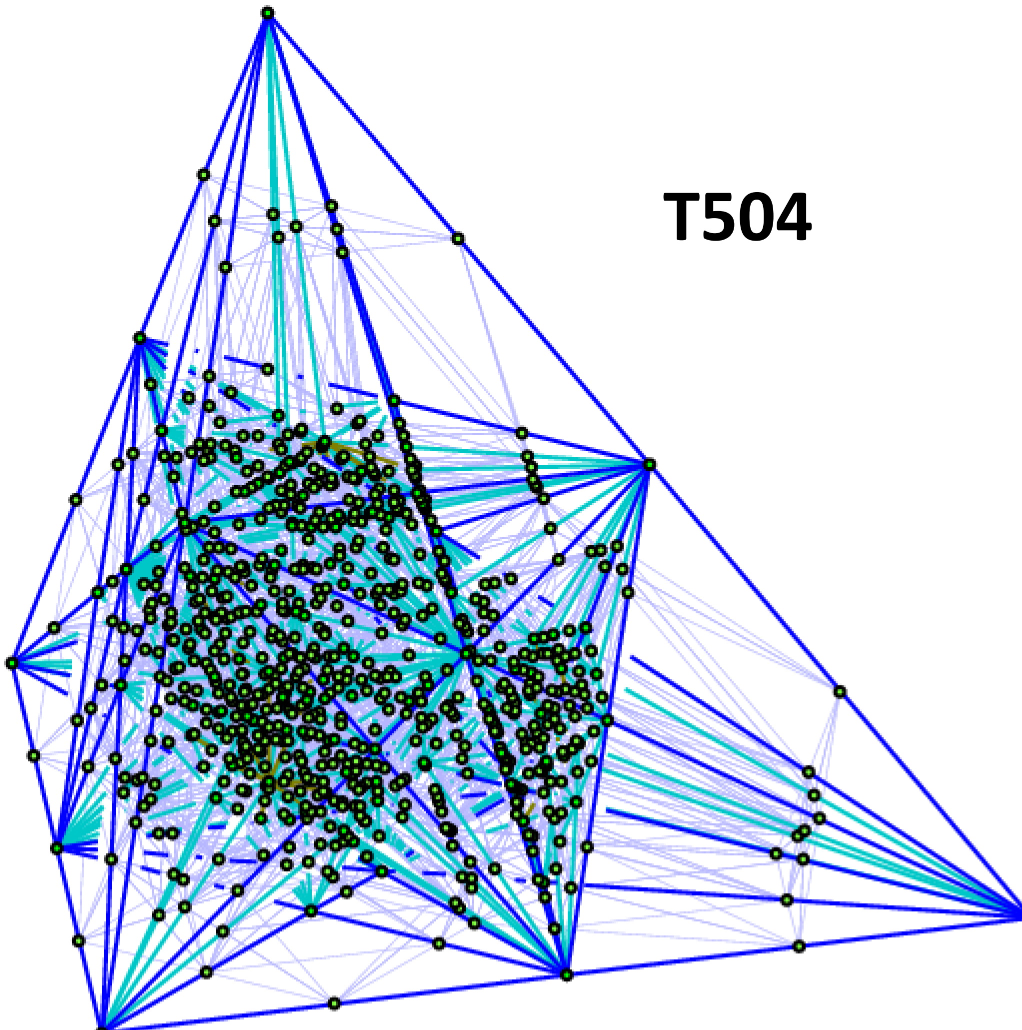
Schumaker, Sorokina, Worsey: Journal of Approximation Theory, 2009.

$r = 1$

$d = 2$

504 micro-tetrahedra

no geometric constraints



T60

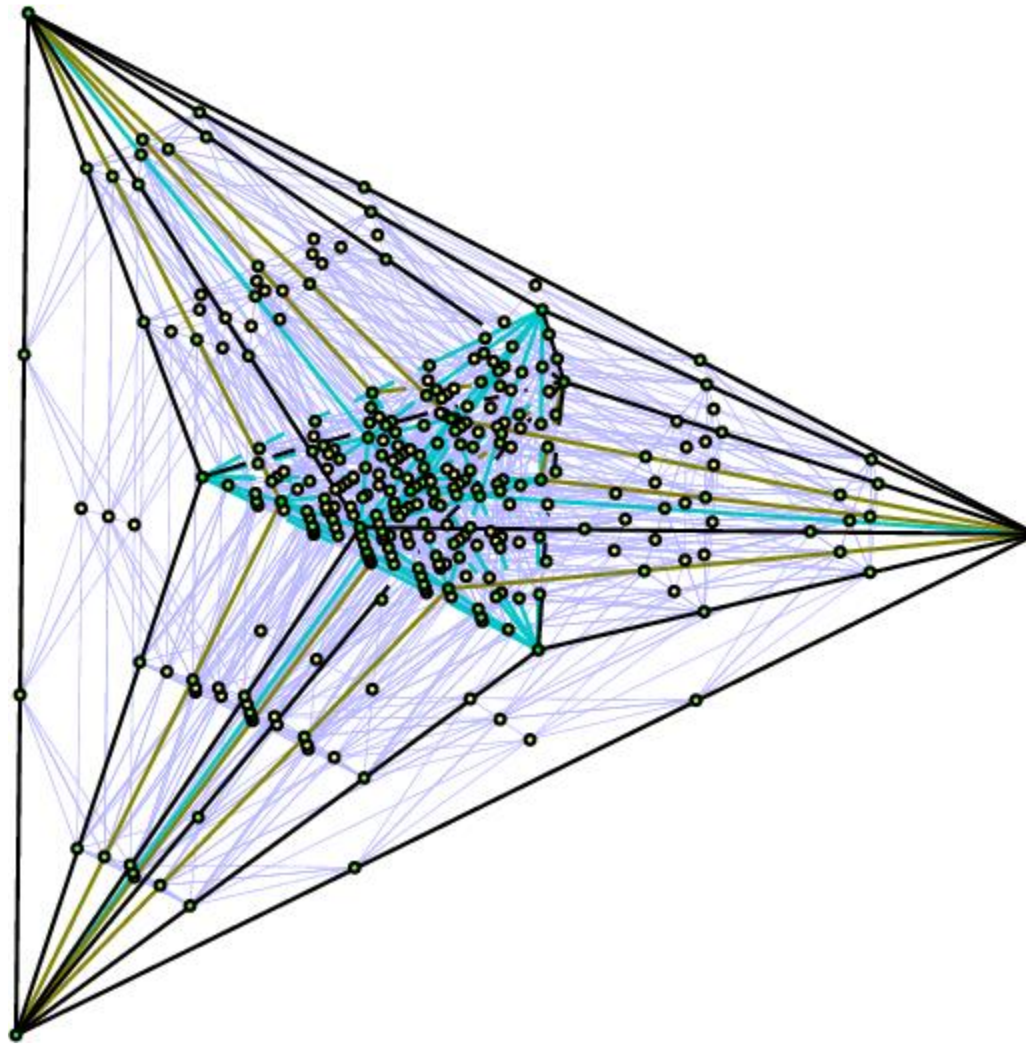
Alfeld and Sorokina, Journal of Approximation Theory, 2009.

$r = 1$

$d = 3$

60 micro-tetrahedra

no geometric constraints



Thank You

Multivariate Splines

Applied Mathematics Seminar

March 24, 2014

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More information and software on www.math.utah.edu/~pa/

Pdf version of slides on www.math.utah.edu/~pa/mvs.pdf