Math 5600 6/23/14

- Gaussian Quadrature.

- $\int_a^b w(x) f(x) dx = \sum_{i=1}^{n} w_i f(x_i)$

want it exact for f being a polynomial of degree as high as possible

- w_i : weights ($w_i \neq w(x_i)$)
  x_i : knots

- we pick both the weights and the knots

- note: f is considered polynomial, not the integrand wf

- How high is possible?

  $2n$ parameters $\rightarrow$ degree $2n-1$

- we saw that we get a nonlinear system

- Gauss solved it

- Carl Friedrich Gauss 1777–1855

- pretty remarkable!

- Let p be a polynomial of degree $2n-1$.

- we have two types of parameters, the w_i and the x_i

- can we break the polynomial into two pieces?

- yes! long division with remainder

$$P_n(x) = Q_n(x)\, q(x) + r(x)$$

Q_n : degree n, divisor

q : degree $n-1$, quotient

r : degree $n-1$, remainder

- same concept as for numbers

$$45 = 7 \cdot 6 + 3$$

$$45 \div 7 = 6, \text{ remainder } 3$$

- Example (for $n=2$)

$$Q_n = x^2 - \frac{1}{3}, \text{ say}$$

$$p(x) = x^3 + 2x^2 + 3x + 4, \text{ say}$$

$$\begin{array}{r|l}
 & x + 2 \\
\hline
x^2 - \frac{1}{3} & x^3 + 2x^2 + 3x + 4 \\
 & x^3 \qquad\quad - \frac{x}{3} \\
\hline
 & \qquad 2x^2 + \frac{10}{3}x + 4 \\
 & \qquad 2x^2 \qquad\quad - \frac{2}{3} \\
\hline
 & \qquad\qquad \frac{10}{3}x + \frac{14}{3}
\end{array}$$

- easy to check:

$$x^3 + 2x^2 + 3x + 4 = \left(x^2 - \frac{1}{3}\right)(x+2) + \frac{10}{3}x + \frac{14}{3}$$

- How do we pick Q_n?

$$I = \int_a^b w(x)p(x)dx$$

$$= \int_a^b w(x)\big(Q_n(x)q(x) + r(x)\big)dx$$

$$= \int_a^b w(x)Q_n(x)q(x)dx + \int_a^b w(x)r(x)dx$$

how about picking Q_n so that the first integral vanishes for all polynomials q of degree up to $n-1$?

orthogonal polynomials with respect to

$$(u,v) = \int_a^b w(x)u(x)v(x)dx$$

old hat!

$$= \int_a^b w(x)r(x)dx$$

$$= \sum_{i=1}^n w_i\, p(x_i)$$

pick x_i roots of Q_n

$$= \sum_{i=1}^n w_i\big(Q_n(x_i)q(x_i) + r(x_i)\big)$$

$$= \sum_{i=1}^n w_i\, r(x_i) = \int_a^b w(x)r(x)dx$$

- we know how to pick the w_i

$$r(x) = \sum_{i=1}^{n} r(x_i) L_i(x)$$

- $$\int_a^b w(x) r(x) dx = \int_a^b w(x) \sum_{i=1}^{n} r(x_i) L_i(x) dx$$

Lagrange basis functions

$$= \sum_{i=1}^{n} r(x_i) \int_a^b w(x) L_i(x) dx$$

$$= \sum_{i=1}^{n} w_i r(x_i)$$

$$w_i = \int_a^b w(x) L_i(x) dx$$

$$L_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

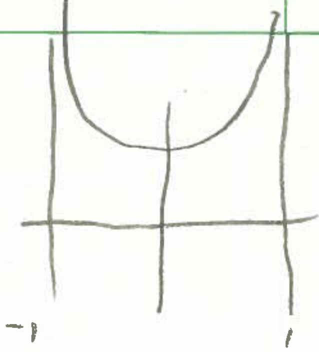
x_i : roots of Q_n

what if some roots of Q_n are outside the interval $[a,b]$, or if they are multiple?

- That won't happen!
- The roots of Q_n are real, distinct, and in the interval (a,b).
- suppose Q_n changes sign at the points
  $$z_1, z_2, \dots, z_k \in (a,b)$$
- we want to show that $k = n$
- consider
  $$\int_a^b w(x) Q_n(x)(x-z_1)(x-z_2)\cdots(x-z_k) > 0$$
- $\Rightarrow k \geq n$ since Q_n is orthogonal to all polynomials of degree $\leq n$

- Example.

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$



$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \sum_{i=1}^{n} w_i f(x_i) + R_n$$

$$x_i = \cos\frac{(2i-1)\pi}{2n} \qquad i = 1, \cdots, n$$

$$w_i = \frac{\pi}{n}$$

$$R_n = \frac{\pi}{(2n)!\,2^{2n-1}} f^{(2n)}(\xi)$$

- Assuming the derivative is 1

n	1	2	3	4	5
R_n	$\frac{\pi}{4}$	$\frac{\pi}{192}$	$\frac{\pi}{23{,}040}$	$\frac{\pi}{5{,}160{,}960}$,	$\frac{\pi}{1{,}857{,}445{,}600}$

5 points will suffice for most purpose

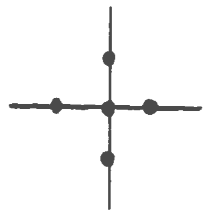
- If you are not impressed try approximating

$$\int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}}\,dx$$

with an open Newton–Cotes formulae.

Laplacian

25.3.30



$$\nabla^2 u_{0,0}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)_{0,0}$$

$$=\frac{1}{h^2}(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1}-4u_{0,0})+O(h^2)$$

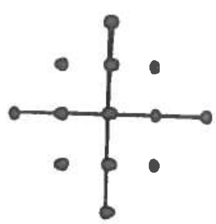
25.3.31

$$\nabla^2 u_{0,0}=\frac{1}{12h^2}[-60u_{0,0}+16(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1})$$

$$-(u_{2,0}+u_{0,2}+u_{-2,0}+u_{0,-2})]+O(h^4)$$

Biharmonic Operator

25.3.32



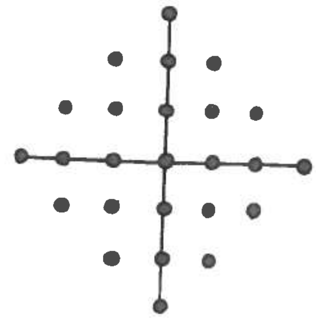
$$\nabla^4 u_{0,0}=\left(\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}\right)_{0,0}$$

$$=\frac{1}{h^4}[20u_{0,0}-8(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1})$$

$$+2(u_{1,1}+u_{1,-1}+u_{-1,1}+u_{-1,-1})$$

$$+(u_{0,2}+u_{2,0}+u_{-2,0}+u_{0,-2})]+O(h^2)$$

25.3.33



$$\nabla^4 u_{0,0}=\frac{1}{6h^4}[-(u_{0,3}+u_{0,-3}+u_{3,0}+u_{-3,0})$$

$$+14(u_{0,2}+u_{0,-2}+u_{2,0}+u_{-2,0})$$

$$-77(u_{0,1}+u_{0,-1}+u_{1,0}+u_{-1,0})$$

$$+184u_{0,0}+20(u_{1,1}+u_{1,-1}+u_{-1,1}+u_{-1,-1})$$

$$-(u_{1,2}+u_{2,1}+u_{1,-2}+u_{2,-1}+u_{-1,2}+u_{-2,1}$$

$$+u_{-1,-2}+u_{-2,-1})]+O(h^4)$$

25.4. Integration

Trapezoidal Rule

25.4.1

$$\int_{x_0}^{x_1} f(x)dx=\frac{h}{2}(f_0+f_1)-\frac{1}{2}\int_{x_0}^{x_1}(t-x_0)(x_1-t)f''(t)\,dt$$

$$=\frac{h}{2}(f_0+f_1)-\frac{h^3}{12}f''(\xi)\qquad (x_0<\xi<x_1)$$

Extended Trapezoidal Rule

25.4.2

$$\int_{x_0}^{x_m} f(x)dx=h\left[\frac{f_0}{2}+f_1+\dots+f_{m-1}+\frac{f_m}{2}\right]$$

$$-\frac{mh^3}{12}f''(\xi)$$

Error Term in Trapezoidal Formula for Periodic Functions

If $f(x)$ is periodic and has a continuous k^{th} derivative, and if the integral is taken over a period, then

25.4.3 $$|\text{Error}|\leq\frac{\text{constant}}{m^k}$$

Modified Trapezoidal Rule

25.4.4

$$\int_{x_0}^{x_m} f(x)dx=h\left[\frac{f_0}{2}+f_1+\dots+f_{m-1}+\frac{f_m}{2}\right]$$

$$+\frac{h}{24}[-f_{-1}+f_1+f_{m-1}-f_{m+1}]+\frac{11m}{720}h^5f^{(4)}(\xi)$$

25.4.5 **Simpson's Rule**

$$\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}[f_0+4f_1+f_2]+\frac{1}{6}\int_{x_0}^{x_1}(x_0-t)^2(x_1-t)f^{(3)}(t)dt+\frac{1}{6}\int_{x_1}^{x_2}(x_2-t)^2(x_1-t)f^{(3)}(t)dt$$

$$=\frac{h}{3}[f_0+4f_1+f_2]-\frac{h^5}{90}f^{(4)}(\xi)$$

25.4.6 **Extended Simpson's Rule**

$$\int_{x_0}^{x_{2n}} f(x)dx=\frac{h}{3}[f_0+4(f_1+f_3+\ldots+f_{2n-1})+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}]-\frac{nh^5}{90}f^{(4)}(\xi)$$

Euler-Maclaurin Summation Formula

25.4.7

$$\int_{x_0}^{x_n} f(x)dx=h\left[\frac{f_0}{2}+f_1+f_2+\ldots+f_{n-1}+\frac{f_n}{2}\right]-\frac{B_2}{2!}h^2(f'_n-f'_0)-\ldots-\frac{B_{2k}h^{2k}}{(2k)!}[f_n^{(2k-1)}-f_0^{(2k-1)}]+R_{2k}$$

$$R_{2k}=\frac{\theta n B_{2k+2}h^{2k+3}}{(2k+2)!}\max_{x_0\le x\le x_n}|f^{(2k+2)}(x)|, \qquad (-1\le\theta\le1)$$

(For B_{2k}, Bernoulli numbers, see chapter 23.)

If $f^{(2k+2)}(x)$ and $f^{(2k+4)}(x)$ do not change sign for $x_0<x<x_n$ then $|R_{2k}|$ is less than the first neglected term. If $f^{(2k+2)}(x)$ does not change sign for $x_0<x<x_n$, $|R_{2k}|$ is less than twice the first neglected term.

25.4.8 **Lagrange Formula**

$$\int_a^{b*} f(x)dx=\sum_{i=0}^{n}(L_i^{(n)}(b)-L_i^{(n)}(a))f_i+R_n$$

(See 25.2.1.)

25.4.9

$$L_i^{(n)}(x)=\frac{1}{\pi'_n(x_i)}\int_{x_0}^{x}\frac{\pi_n(t)}{t-x_i}dt=\int_{x_0}^{x}l_i(t)dt$$

25.4.10 $$R_n=\frac{1}{(n+1)!}\int_a^b \pi_n(x)f^{(n+1)}(\xi(x))dx$$

25.4.11 **Equally Spaced Abscissas**

$$\int_{x_0}^{x_k} f(x)dx=\frac{1}{h^n}\sum_{i=0}^{n}f_i\frac{(-1)^{n-i}}{i!(n-i)!}\int_{x_0}^{x_k}\frac{\pi_n(x)}{x-x_i}dx+R_n$$

25.4.12 $$\int_{x_m}^{x_{m+1}} f(x)dx=h\sum_{i=-\left[\frac{n-1}{2}\right]}^{\left[\frac{n}{2}\right]}A_i(m)f_i+R_n$$

(See **Table 25.3** for $A_i(m)$.)

Newton-Cotes Formulas (Closed Type)

(For Trapezoidal and Simpson's Rules see **25.4.1–25.4.6**.)

25.4.13 **(Simpson's $\frac{3}{8}$ rule)**

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{8}(f_0+3f_1+3f_2+f_3)-\frac{3f^{(4)}(\xi)h^5}{80}$$

25.4.14 **(Bode's rule)**

$$\int_{x_0}^{x_4} f(x)dx=\frac{2h}{45}(7f_0+32f_1+12f_2+32f_3+7f_4)-\frac{8f^{(6)}(\xi)h^7}{945}$$

25.4.15

$$\int_{x_0}^{x_5} f(x)dx=\frac{5h}{288}(19f_0+75f_1+50f_2+50f_3+75f_4+19f_5)-\frac{275f^{(6)}(\xi)h^7}{12096}$$

25.4.16

$$\int_{x_0}^{x_6} f(x)dx=\frac{h}{140}(41f_0+216f_1+27f_2+272f_3+27f_4+216f_5+41f_6)-\frac{9f^{(8)}(\xi)h^9}{1400}$$

25.4.17

$$\int_{x_0}^{x_7} f(x)dx=\frac{7h}{17280}(751f_0+3577f_1+1323f_2+2989f_3+2989f_4+1323f_5+3577f_6+751f_7)-\frac{8183f^{(8)}(\xi)h^9}{518400}$$

25.4.18

$$\int_{x_0}^{x_8} f(x)dx=\frac{4h}{14175}(989f_0+5888f_1-928f_2+10496f_3-4540f_4+10496f_5-928f_6+5888f_7+989f_8)-\frac{2368}{467775}f^{(10)}(\xi)h^{11}$$

25.4.19

$$\int_{x_0}^{x_9} f(x)dx=\frac{9h}{89600}\{2857(f_0+f_9)+15741(f_1+f_8)+1080(f_2+f_7)+19344(f_3+f_6)+5778(f_4+f_5)\}-\frac{173}{14620}f^{(10)}(\xi)h^{11}$$

*See page II.

25.4.20

$$\int_{x_0}^{x_{10}} f(x)dx=\frac{5h}{299376}\{16067(f_0+f_{10})$$
$$+106300(f_1+f_9)-48525(f_2+f_8)+272400(f_3+f_7)$$
$$-260550(f_4+f_6)+427368f_5\}$$
$$-\frac{1346350}{326918592}f^{(12)}(\xi)h^{13}$$

Newton-Cotes Formulas (Open Type)

25.4.21

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{2}(f_1+f_2)+\frac{f^{(2)}(\xi)h^3}{4}$$

25.4.22

$$\int_{x_0}^{x_4} f(x)dx=\frac{4h}{3}(2f_1-f_2+2f_3)+\frac{28f^{(4)}(\xi)h^5}{90}$$

25.4.23

$$\int_{x_0}^{x_5} f(x)dx=\frac{5h}{24}(11f_1+f_2+f_3+11f_4)+\frac{95f^{(4)}(\xi)h^5}{144}$$

25.4.24

$$\int_{x_0}^{x_6} f(x)dx=\frac{6h}{20}(11f_1-14f_2+26f_3-14f_4+11f_5)$$
$$+\frac{41f^{(6)}(\xi)h^7}{140}$$

25.4.25

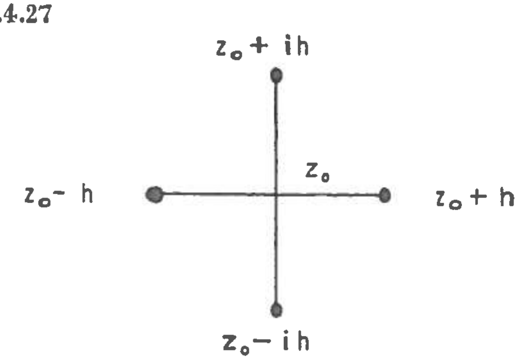
$$\int_{x_0}^{x_7} f(x)dx=\frac{7h}{1440}(611f_1-453f_2+562f_3+562f_4$$
$$-453f_5+611f_6)+\frac{5257}{8640}f^{(6)}(\xi)h^7$$

25.4.26

$$\int_{x_0}^{x_8} f(x)dx=\frac{8h}{945}(460f_1-954f_2+2196f_3-2459f_4$$
$$+2196f_5-954f_6+460f_7)+\frac{3956}{14175}f^{(8)}(\xi)h^9$$

Five Point Rule for Analytic Functions

25.4.27



$$\int_{z_0-h}^{z_0+h} f(z)dz=\frac{h}{15}\{24f(z_0)+4[f(z_0+h)+f(z_0-h)]$$
$$-[f(z_0+ih)+f(z_0-ih)]\}+R$$

$|R|\leq\frac{|h|^7}{1890}\max_{z\in S}|f^{(6)}(z)|$, S designates the square with vertices $z_0+i^kh\,(k=0,1,2,3)$; h can be complex.

Chebyshev's Equal Weight Integration Formula

25.4.28 $$\int_{-1}^{1} f(x)dx=\frac{2}{n}\sum_{i=1}^{n} f(x_i)+R_n$$

Abscissas: x_i is the i^{th} zero of the polynomial part of

$$x^n\exp\left[\frac{-n}{2\cdot 3x^2}-\frac{n}{4\cdot 5x^3}-\frac{n}{6\cdot 7x^4}-\ldots\right]$$

(See **Table 25.5** for x_i.)

For $n=8$ and $n\geq 10$ some of the zeros are complex.

Remainder:

$$R_n=\int_{-1}^{+1}\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)dx$$
$$-\frac{2}{n(n+1)!}\sum_{i=1}^{n}x_i^{n+1}f^{(n+1)}(\xi_i)$$

where $\xi=\xi(x)$ satisfies $0\leq\xi\leq x$ and $0\leq\xi_i\leq x_i$

$$(i=1,\ldots,n)$$

Integration Formulas of Gaussian Type

(For Orthogonal Polynomials see chapter 22)

Gauss' Formula

25.4.29 $$\int_{-1}^{1} f(x)dx=\sum_{i=1}^{n} w_if(x_i)+R_n$$

Related orthogonal polynomials: Legendre polynomials $P_n(x)$, $P_n(1)=1$

Abscissas: x_i is the i^{th} zero of $P_n(x)$

Weights: $w_i=2/(1-x_i^2)[P_n'(x_i)]^2$ *

(See **Table 25.4** for x_i and w_i.)

$$R_n=\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi)\qquad(-1<\xi<1)$$

Gauss' Formula, Arbitrary Interval

25.4.30 $$\int_a^b f(y)dy=\frac{b-a}{2}\sum_{i=1}^{n} w_if(y_i)+R_n$$

$$y_i=\left(\frac{b-a}{2}\right)x_i+\left(\frac{b+a}{2}\right)$$

*See page II.

25.4.36 $$\int_0^1 \frac{f(x)}{\sqrt{1-x}}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$P_{2n}\left(\sqrt{1-x}\right),\ P_{2n}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.
Weights: $w_i=2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.
Remainder:

$$R_n=\frac{2^{4n+1}}{4n+1}\frac{[(2n)!]^3}{[(4n)!]^2}f^{(2n)}(\xi)\qquad(0<\xi<1)$$

25.4.37 $$\int_a^b \frac{f(y)}{\sqrt{b-y}}\,dy=\sqrt{b-a}\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$P_{2n}\left(\sqrt{1-x}\right),P_{2n}(1)=1$$

Abscissas:
$x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.

Weights: $w_i=2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.

25.4.38 $$\int_{-1}^{+1} \frac{f(x)}{\sqrt{1-x^2}}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of First Kind

$$T_n(x),\ T_n(1)=\frac{1}{2^{n-1}}$$

Abscissas:

$$x_i=\cos\frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i=\frac{\pi}{n}$$

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{2n-1}}f^{(2n)}(\xi)\quad(-1<\xi<1)$$

25.4.39

$$\int_a^b \frac{f(y)dy}{\sqrt{(y-a)(b-y)}}=\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=\frac{b+a}{2}+\frac{b-a}{2}x_i$$

Related orthogonal polynomials:

$$T_n(x),\ T_n(1)=\frac{1}{2^{n-1}}$$

Abscissas:

$$x_i=\cos\frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i=\frac{\pi}{n}$$

25.4.40

$$\int_{-1}^{+1} f(x)\sqrt{1-x^2}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of Second Kind

$$U_n(x)=\frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)}$$ *

Abscissas:

$$x_i=\cos\frac{i}{n+1}\pi$$ *

Weights:

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i}{n+1}\pi$$ *

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{2n+1}}f^{(2n)}(\xi)\qquad(-1<\xi<1)$$

25.4.41

$$\int_a^b \sqrt{(y-a)(b-y)}\,f(y)dy=\left(\frac{b-a}{2}\right)^2\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=\frac{b+a}{2}+\frac{b-a}{2}x_i$$

Related orthogonal polynomials:

$$U_n(x)=\frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)}$$ *

Abscissas:

$$x_i=\cos\frac{i}{n+1}\pi$$ *

Weights:

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i}{n+1}\pi$$ *

25.4.42 $$\int_0^1 f(x)\sqrt{\frac{x}{1-x}}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}}T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i=\cos^2\frac{2i-1}{2n+1}\cdot\frac{\pi}{2}$$

Weights:

$$w_i=\frac{2\pi}{2n+1}x_i$$

*See page II.

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{4n+1}}f^{(2n)}(\xi) \qquad (0<\xi<1)$$

25.4.43

$$\int_a^b f(x)\sqrt{\frac{x-a}{b-x}}\,dx=(b-a)\sum_{i=1}^n w_i f(y_i)+R_n$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}}\,T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i=\cos^2\frac{2i-1}{2n+1}\cdot\frac{\pi}{2}$$

Weights:

$$w_i=\frac{2\pi}{2n+1}\,x_i$$

25.4.44

$$\int_0^1 \ln x\, f(x)dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: polynomials orthogonal with respect to the weight function $-\ln x$
Abscissas: See **Table 25.7**
Weights: See **Table 25.7**

25.4.45

$$\int_0^\infty e^{-x}f(x)dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: Laguerre polynomials $L_n(x)$.
Abscissas: x_i is the i^{th} zero of $L_n(x)$
Weights:

$$w_i=\frac{(n!)^2 x_i}{(n+1)^2[L_{n+1}(x_i)]^2}$$

(See **Table 25.9** for x_i and w_i.)
Remainder:

$$R_n=\frac{(n!)^2}{(2n)!}f^{(2n)}(\xi) \qquad (0<\xi<\infty)$$

25.4.46

$$\int_{-\infty}^\infty e^{-x^2}f(x)dx=\sum_{i=1}^n w_i f(x_i)+R_n$$

Related orthogonal polynomials: Hermite polynomials $H_n(x)$.
Abscissas: x_i is the i^{th} zero of $H_n(x)$
Weights:

$$\frac{2^{n-1}n!\sqrt{\pi}}{n^2[H_{n-1}(x_i)]^2}$$

(See **Table 25.10** for x_i and w_i.)

Remainder:

$$R_n=\frac{n!\sqrt{\pi}}{2^n(2n)!}f^{(2n)}(\xi) \qquad (-\infty<\xi<\infty)$$

Filon's Integration Formula [3]

25.4.47

$$\int_{x_0}^{x_{2n}} f(x)\cos tx\,dx=h\Big[\alpha(th)(f_{2n}\sin tx_{2n}-f_0\sin tx_0)+\beta(th)\cdot C_{2n}+\gamma(th)\cdot C_{2n-1}+\frac{2}{45}th^4S'_{2n-1}\Big]-R_n$$

25.4.48

$$C_{2n}=\sum_{i=0}^n f_{2i}\cos(tx_{2i})-\tfrac{1}{2}[f_{2n}\cos tx_{2n}+f_0\cos tx_0]$$

25.4.49

$$C_{2n-1}=\sum_{i=1}^n f_{2i-1}\cos tx_{2i-1}$$

25.4.50

$$S'_{2n-1}=\sum_{i=1}^n f^{(3)}_{2i-1}\sin tx_{2i-1}$$

25.4.51

$$R_n=\frac{1}{90}nh^5f^{(4)}(\xi)+O(th^7)$$

25.4.52

$$\alpha(\theta)=\frac{1}{\theta}+\frac{\sin 2\theta}{2\theta^2}-\frac{2\sin^2\theta}{\theta^3}$$

$$\beta(\theta)=2\left(\frac{1+\cos^2\theta}{\theta^2}-\frac{\sin 2\theta}{\theta^3}\right)$$

$$\gamma(\theta)=4\left(\frac{\sin\theta}{\theta^3}-\frac{\cos\theta}{\theta^2}\right)$$

For small θ we have

25.4.53

$$\alpha=\frac{2\theta^3}{45}-\frac{2\theta^5}{315}+\frac{2\theta^7}{4725}-\dots$$

$$\beta=\frac{2}{3}+\frac{2\theta^2}{15}-\frac{4\theta^4}{105}+\frac{2\theta^6}{567}-\dots$$

$$\gamma=\frac{4}{3}-\frac{2\theta^2}{15}+\frac{\theta^4}{210}-\frac{\theta^6}{11340}+\dots$$

25.4.54

$$\int_{x_0}^{x_{2n}} f(x)\sin tx\,dx=h\Big[\alpha(th)(f_0\cos tx_0-f_{2n}\cos tx_{2n})+\beta S_{2n}+\gamma S_{2n-1}+\frac{2}{45}th^4C'_{2n-1}\Big]-R_n$$

25.4.55

$$S_{2n}=\sum_{i=0}^n f_{2i}\sin(tx_{2i})-\frac{1}{2}[f_{2n}\sin(tx_{2n})+f_0\sin(tx_0)]$$

[3] For certain difficulties associated with this formula, see the article by J. W. Tukey, p. 400, "On Numerical Approximation," Ed. R. E. Langer, Madison, 1959.

Related orthogonal polynomials: $P_n(x)$, $P_n(1)=1$
Abscissas: x_i is the i^{th} zero of $P_n(x)$
* Weights: $w_i=2/(1-x_i^2)[P_n'(x_i)]^2$

$$R_n=\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}2^{2n+1}f^{(2n)}(\xi)$$

Radau's Integration Formula

25.4.31

$$\int_{-1}^{1} f(x)dx=\frac{2}{n^2}f_{-1}+\sum_{i=1}^{n-1} w_i f(x_i)+R_n$$

Related polynomials:

$$\frac{P_{n-1}(x)+P_n(x)}{x+1}$$

Abscissas: x_i is the i^{th} zero of

$$\frac{P_{n-1}(x)+P_n(x)}{x+1}$$

Weights:

$$w_i=\frac{1}{n^2}\frac{1-x_i}{[P_{n-1}(x_i)]^2}=\frac{1}{1-x_i}\frac{1}{[P'_{n-1}(x_i)]^2}$$

Remainder:

$$R_n=\frac{2^{2n-1}\cdot n}{[(2n-1)!]^3}[(n-1)!]^4 f^{(2n-1)}(\xi) \qquad (-1<\xi<1)$$

Lobatto's Integration Formula

25.4.32

$$\int_{-1}^{1} f(x)dx=\frac{2}{n(n-1)}[f(1)+f(-1)]+\sum_{i=2}^{n-1} w_i f(x_i)+R_n$$

Related polynomials: $P'_{n-1}(x)$

Abscissas: x_i is the $(i-1)^{st}$ zero of $P'_{n-1}(x)$

Weights:

$$w_i=\frac{2}{n(n-1)[P_{n-1}(x_i)]^2} \qquad (x_i\neq\pm1)$$

(See **Table 25.6** for x_i and w_i.)

Remainder:

$$R_n=\frac{-n(n-1)^3 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3}f^{(2n-2)}(\xi) \qquad (-1<\xi<1)$$

*See page II.

25.4.33

$$\int_0^1 x^k f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$q_n(x)=\sqrt{k+2n+1}P_n^{(k,0)}(1-2x)$$

(For the Jacobi polynomials $P_n^{(k,0)}$ see chapter

Abscissas:

x_i is the i^{th} zero of $q_n(x)$

Weights:

$$w_i=\left\{\sum_{j=0}^{n-1}[q_j(x_i)]^2\right\}^{-1}$$

(See **Table 25.8** for x_i and w_i.)

Remainder:

$$R_n=\frac{f^{(2n)}(\xi)}{(k+2n+1)(2n)!}\left[\frac{n!(k+n)!}{(k+2n)!}\right]^2 \qquad (0<\xi<1)$$

25.4.34

$$\int_0^1 f(x)\sqrt{1-x}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}}P_{2n+1}(\sqrt{1-x}),\ P_{2n+1}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.
Weights: $w_i=2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.
Remainder:

$$R_n=\frac{2^{4n+3}[(2n+1)!]^4}{(2n)!(4n+3)[(4n+2)!]^2}f^{(2n)}(\xi) \qquad (0<\xi<1)$$

25.4.35

$$\int_a^b f(y)\sqrt{b-y}\,dy=(b-a)^{3/2}\sum_{i=1}^{n} w_i f(y_i)$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}}P_{2n+1}(\sqrt{1-x}),\ P_{2n+1}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.
Weights: $w_i=2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.

22.2. Orthogonality Relations

	$f_n(x)$	Name of Polynomial	a	b	$w(x)$	Standardization	h_n	Remarks
22.2.1	$P_n^{(\alpha,\beta)}(x)$	Jacobi	-1	1	$(1-x)^\alpha(1+x)^\beta$	$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$	$\alpha>-1, \beta>-1$
22.2.2	$G_n(p, q, x)$	Jacobi	0	1	$(1-x)^{p-q}x^{q-1}$	$k_n=1$	$\frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}$	$p-q>-1, q>0$
22.2.3	$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer)	-1	1	$(1-x^2)^{\alpha-\frac{1}{2}}$	$C_n^{(\alpha)}(1)=\binom{n+2\alpha-1}{n}$ $(\alpha\neq 0)$	$\frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}$ $\alpha\neq 0$	$\alpha>-\frac{1}{2}$
						$C_n^{(0)}(1)=\frac{2}{n}$, $C_0^{(0)}(1)=1$	$\frac{2\pi}{n^2}$ $\alpha=0$	
22.2.4	$T_n(x)$	Chebyshev of the first kind	-1	1	$(1-x^2)^{-\frac{1}{2}}$	$T_n(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq 0\\ \pi & n=0\end{cases}$	
22.2.5	$U_n(x)$	Chebyshev of the second kind	-1	1	$(1-x^2)^{\frac{1}{2}}$	$U_n(1)=n+1$	$\frac{\pi}{2}$	
22.2.6	$S_n(x)$	Chebyshev of the first kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$	$S_n(2)=n+1$	$\begin{cases}4\pi & n\neq 0\\ 8\pi & n=0\end{cases}$	
22.2.7	$C_n(x)$	Chebyshev of the second kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}$	$C_n(2)=2$	4π	
22.2.8	$T_n^*(x)$	Shifted Chebyshev of the first kind	0	1	$(x-x^2)^{-\frac{1}{2}}$	$T_n^*(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq 0\\ \pi & n=0\end{cases}$	
22.2.9	$U_n^*(x)$	Shifted Chebyshev of the second kind	0	1	$(x-x^2)^{\frac{1}{2}}$	$U_n^*(1)=n+1$	$\frac{\pi}{8}$ *	
22.2.10	$P_n(x)$	Legendre (Spherical)	-1	1	1	$P_n(1)=1$	$\frac{2}{2n+1}$	
22.2.11	$P_n^*(x)$	Shifted Legendre	0	1	1		$\frac{1}{2n+1}$	

*See page II.

22.2. Orthogonality Relations—Continued

22.2.12	$L_n^{(\alpha)}(x)$	Generalized Laguerre	0	∞	$e^{-x}x^{\alpha}$	$k_n=\frac{(-1)^n}{n!}$	$\frac{\Gamma(\alpha+n+1)}{n!}$	$\alpha>-1$
22.2.13	$L_n(x)$	Laguerre	0	∞	e^{-x}	$k_n=\frac{(-1)^n}{n!}$	1	
22.2.14	$H_n(x)$	Hermite	$-\infty$	∞	e^{-x^2}	$a_n=(-1)^n$	$\sqrt{\pi}2^n n!$	
22.2.15	$He_n(x)$	Hermite	$-\infty$	∞	$e^{-\frac{x^2}{2}}$	$a_n=(-1)^n$	$\sqrt{2\pi}n!$	

22.3. Explicit Expressions

$$f_n(x)=d_n\sum_{m=0}^{N} c_m g_m(x)$$

	$f_n(x)$	N	d_n	c_m	$g_m(x)$	k_n	Remarks
22.3.1	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{1}{2^n}$	$\binom{n+\alpha}{m}\binom{n+\beta}{n-m}$	$(x-1)^{n-m}(x+1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.2	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}$	$\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)}$	$(x-1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.3	$G_n(p,q,x)$	n	$\frac{\Gamma(q+n)}{\Gamma(p+2n)}$	$(-1)^m\binom{n}{m}\frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)}$	x^{n-m}	1	$p-q>-1,\ q>0$
22.3.4	$C_n^{(\alpha)}(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{\Gamma(\alpha)}$	$(-1)^m\frac{\Gamma(\alpha+n-m)}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$	$\alpha>-\frac{1}{2},\ \alpha\neq 0$
22.3.5	$C_n^{(0)}(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n}\quad n\neq 0$	$n\neq 0,\ C_0^{(0)}(1)=1$
22.3.6	$T_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{n}{2}$	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^{n-1}	
22.3.7	$U_n(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	
22.3.8	$P_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{2^n}$	$(-1)^m\binom{n}{m}\binom{2n-2m}{n}$	x^{n-2m}	$\frac{(2n)!}{2^n(n!)^2}$	
22.3.9	$L_n^{(\alpha)}(x)$	n	1	$(-1)^m\binom{n+\alpha}{n-m}\frac{1}{m!}$	x^m	$\frac{(-1)^n}{n!}$	$\alpha>-1$
22.3.10	$H_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	see **22.11**
		$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!2^m(n-2m)!}$	x^{n-2m}	1	