Contents

1 Linear Algebra

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using *Linear Algebra with Applications*, by Otto Bretsch

1.1 Linear Equations

Linear algebra begins as the study of solving linear equations. For example, consider a system of three equations given by

$$\begin{aligned} a_1x + a_2y + a_3z = d_1 \qquad (A)\\ b_1x + b_2y + b_2z = d_2 \qquad (B)\\ c_1x + c_2y + c_3z = d_3 \qquad (C) \end{aligned}$$

for $a_1, \dots d_3 \in \mathbb{R}$. Each equation defines some plane in $\mathbb{R}^3$, depending on the coefficients of x, y, and z. Geometrically, solutions to this system of equations are equivalent to the intersections of these planes; then, solutions may take one of three forms for a linear system in $\mathbb{R}^3$:

1. No solutions. In this case, there are no points on which all three planes intersect. The system of equations defining these planes is said to be **inconsistent**.
2. One solution. In this case, all three planes intersect at a single point in space. The system of equations defining these planes is said to be **consistent** with a **unique** solution.
3. Infinitely many solutions. In this case, all three planes intersect along a line (or plane) in space. The system of equations defining these planes is said to be **consistent**, but solutions are **not unique**.

In higher dimensions, a more general definition of the solution to a linear system involves the intersection of sets. In particular, let A, B, and C be sets containing all points defined by the equations (A), (B), and (C) respectively. Then, $A \cap B \cap C$ is the set of solutions. A system of equations is inconsistent only if the solution set is equal to the empty set, $\varnothing$; otherwise, the system is said to be consistent and at least one solution is contained in the intersection of the sets comprising the system.

For convenience, linear systems are often represented as a **matrix** with m rows and n columns (and therefore dimension $m \times n$). For a linear system in the form previously shown, the **augmented matrix** representing the system is given by

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

A series of row operations may be used to reduce any matrix to a simpler form. The following operations are the same as those allowed on a system of linear equations in standard form, and are also allowed when dealing with augmented matrices:

1. Multiply a row by a scalar.
2. Subtract a multiple of one row from another.
3. Swap the position of any two rows.

When a matrix A is in its simplest form, it is said to be in **reduced row echelon form**, denoted $\text{rref}(A)$. $\text{rref}(A)$ the above augmented matrix may be similar to the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{array}\right]$$

Sometimes, it may be impossible to reduce a matrix directly to this form; in this case, there may be some constants located above the diagonal.

The **rank** of a matrix is defined loosely to be the number of leading 1s in the reduced row echelon form of the coefficient matrix. A square coefficient matrix which is of **full rank** (as many 1s as variables as equations) has one unique solution.

Sometimes, equations may "drop out" of a system. This is the case when two rows (equations) are **linearly dependant**; that is to say, they may be written as a **linear combination** of each other. For vectors (any object which may be multiplied by a scalar and added to another vector of equal size) to be **linearly independent**, $\vec{v_1} \neq k\vec{v_2},\ k \in \mathbb{R}$. Then, computing whether a vector $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ is a linear combination of $\vec{v_1}, \dots \vec{v_n} = \begin{bmatrix} v_{11} \\ \vdots \\ v_{1n} \end{bmatrix}, \dots, \begin{bmatrix} v_{n1} \\ \vdots \\ v_{nn} \end{bmatrix}$, the question is simply whether $\vec{w} = x_1\vec{v_1} + \cdots + x_n\vec{v_n}$ for some $x_1, \dots, x_n$. But this is simply a linear system given by

$$\begin{cases} w_1 &= x_1 v_{11} + \cdots + x_n v_{n1} \\ &\vdots \\ w_n &= x_1 v_{1n} + \cdots + x_n v_{nn} \end{cases}$$

and therefore by the augmented matrix

$$\left[\begin{array}{ccc|c} v_{11} & \dots & v_{n1} & w_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{1n} & \dots & v_{nn} & w_n \end{array}\right]$$

Then, the columns of a matrix are simply vectors. In particular, a matrix A with m rows and n columns may be written with n vectors of length m:

$$A = \left[\begin{array}{ccc} | & & | \\ \vec{v_1} & \dots & \vec{v_n} \\ | & & | \end{array}\right]$$

The product $A\vec{x}$ is given by the product of $\vec{x}$ with the rows of A:

$$A\vec{x} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

This is the special case $A\vec{x} = \vec{b}$, which is equivalent to the augmented matrix formulation which merely suppresses $\vec{x}$ and places $\vec{b}$ at the rightmost column of the matrix. The same operation is allowed on matrices under certain dimensional restrictions. Namely, for AB to be defined for matrices A and B, A must have the same number of columns as B has rows, and the product will have as many rows as A and as many columns as B. In this case,

$$AB = \left[\begin{array}{ccc} | & & | \\ A\vec{v_1} & \dots & A\vec{v_n} \\ | & & | \end{array}\right]$$

where $A\vec{v_k}$ is the product of A with the kth column of B. Alternatively (and more simply for computations), AB may be computed by dotting the rows of A by the columns of B.

1.2 Linear Transformations

A **linear transformation**, T, is any mapping which satisfies certain properties of linearity:

A **linear transformation** $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is a map (function) satisfying both of the following:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})\ \forall\ \vec{x}, \vec{y} \in \mathbb{R}^m$.
2. $T(a\vec{x}) = aT(\vec{x})\ \forall\ a \in \mathbb{R},\ \vec{x} \in \mathbb{R}^m$.

A *matrix* is one such linear transformation. Concurrent linear transformations are also linear (i.e. for matrix A, B, if AB is defined, AB is also a linear transformation).

A matrix is **invertible** only if it is square (i.e. A is $n \times n$) and $\text{rref}(A) = I$ (or equivalently, if $\text{rank}(A) = n$). This is to say, if $A\vec{x} = \vec{b}$ has a unique solution, A is an invertible matrix. This inverse matrix A^{-1} satisfies the property $A^{-1}A = AA^{-1} = I$.

Finding the inverse of a matrix essentially involves computing $\vec{x} = A^{-1}\vec{b}$. This involves solving the $n \times 2n$ augmented matrix $[\ A \mid I\ ]$ until the left hand side is the identity matrix, resulting in $[\ I \mid A^{-1}\ ]$. If this process is not possible, A is not invertible. Since any matrix A is unique (i.e. no two matrices produce the exact same transformation; a matrix product or sum may be written as a new, unique matrix), every inverse matrix is also unique. There exists only one A^{-1} such that $AA^{-1} = A^{-1}A = I$.

Given unit vectors $\vec{e_1}, \dots \vec{e_n} = \begin{bmatrix}1\\ \vdots \\ 0\end{bmatrix}, \dots, \begin{bmatrix}0\\ \vdots \\ 1\end{bmatrix}$ and $A = \begin{bmatrix} & & \\ \vec{v_1} & \dots & \vec{v_n} \\ & & \end{bmatrix}$, multiplication shows that $A\vec{e_1} = \vec{v_1}, \dots, A\vec{e_n} = \vec{e_n}$ such that each unit vector simply returns its associated column of A. This gives an *unreasonably useful lemma*:

Unreasonably Useful Lemma. For a linear transformation matrix $A \in \mathbb{R}^{m \times n}$ and vector $\vec{e_j} \in \mathbb{R}^m$ being the jth canonical unit vector in $\mathbb{R}^m$, the jth column of A is given by $A\vec{e_j}$.

Then, it becomes clear that a linear transformation A essentially transforms each coordinate axis into each column of A. It is useful to geometrically interpret some common transformation matrices, using the knowledge that each coordinate axis takes on the form of the columns of the matrix. Common examples are given in $\mathbb{R}^2$:

Scaling. The matrix which scales a vector by c is given by cI where I is the identity matrix. In $\mathbb{R}^2$, this is simply

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

Orthogonal Projection. Any vector $\vec{x}$ may be written in terms of its components, $\vec{x} = \vec{x_{||}} + \vec{x_{\perp}}$. Projection onto the x-axis is simply given by

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and projection onto the y-axis is simply given by

$$P_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Considering the general projection onto a line L, given a unit vector $\vec{u}$ in the direction of L, the projection onto L is given by

$$\text{proj}_L(\vec{x}) = \boxed{(\vec{x} \cdot \vec{u})\vec{u}}$$

By multiplying the dot product through and then factoring out x_1, x_2 into a separate vector, this leaves the projection matrix:

$$P_L = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$$

Reflection. For any vector $\vec{x}$ reflected over a line L, $\vec{x_{||}} - \vec{x_{\perp}}$. By some addition, the reflection matrix M is given by $2P_L - I$, or

$$M = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Rotation. The rotation matrix uses trigonometric functions to map any vector $\vec{x}$ to itself rotated by θ. This matrix is given by

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Shear. A shear matrix causes a shift on vector $\vec{x}$ in either the horizontal or vertical direction. The horizontal shear matrix is given by

$$S_h = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and the vertical shear matrix is given by

$$S_v = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

1.3 Vector Spaces and Subspaces

The **image** of a function f refers to the set of all outputs which may be reached from a particular set of inputs. In particular,

$$\begin{aligned}\text{image}(f) &= \{f(x) \mid x \in X\} \\ &= \{y \in Y \mid y = f(x) \text{ for } x \in X\}\end{aligned}$$

The image of a matrix $A \in \mathbb{R}^{m \times n}$ is then given by

$$\text{image}(A) = \{\vec{y} \in \mathbb{R}^m \mid \vec{y} = A\vec{x} \text{ for } \vec{x} \in \mathbb{R}^n\}$$

Alternatively, the image of any linear transformation $T(\vec{x}) = A\vec{x}$ is the **span** of the column vectors of A, where *span* refers to the set of linear combinations which may be produced from A. For vectors $\vec{v_1}, \dots \vec{v_m} \in \mathbb{R}^n$, $\text{span}(\vec{v_1}, \dots, \vec{v_m}) = \{c_1\vec{v_1} + \cdots + c_m\vec{v_m} | c_1, \dots c_m \in \mathbb{R}\}$. Then, image$(A)$ is given by

$$\text{image}(A) = \text{span}(A\vec{x}) = \begin{bmatrix} | & & | \\ \vec{v_1} & \dots & \vec{v_m} \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v_1} + \cdots + x_m\vec{v_m}$$

Properties of the image. For a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$:

1. The zero vector $\vec{0} \in \mathbb{R}^n$ is in the image of T.
2. If $\vec{v_1}, \vec{v_2}$ are in the image of T, $\vec{v_1} + \vec{v_2}$ is in the image of T (the image is **closed under addition**).
3. If $\vec{v}$ is in the image of T and $k \in \mathbb{R}$, then $k\vec{v}$ is in the image of T (the image is **closed under scalar multiplication**).

The **kernel** of a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is defined to be the set of all vectors $\vec{x} \in \mathbb{R}^m$ satisfying $T(\vec{x}) = \vec{0}$. Alternatively, the kernel of a matrix equivalent to T, $\ker(A)$, is simply the set of $\vec{x}$ which satisfy the equation $A\vec{x} = \vec{0}$. $\ker(A)$ is also known as the **null space** of A.

Properties of the kernel. For a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$:

1. The zero vector $\vec{0} \in \mathbb{R}^m$ is in the kernel of T.
2. The kernel is closed under addition.
3. The kernel is closed under scalar multiplication.

In general, $\ker(A) = \{\vec{0}\}$ under certain conditions:

1. If, for $m \times n$ matrix A, $\text{rank}(A) = m$ and $m \leq n$ (for $m > n$, there are nonzero vectors in the set $\ker(A)$)
2. If and only if, for $n \times n$ (square) matrix A, A is invertible.

The image and kernel of a linear transformations are each special **subspaces**. A subspace has the following properties:

A subset W of the vector space $\mathbb{R}^n$ is called a (linear) subspace of $\mathbb{R}^n$ if is satisfies the following properties:

1. W contains the zero vector in $\mathbb{R}^n$.
2. W is closed under addition (for all $w_1, w_2 \in W$, $w_1 + w_2 \in W$).
3. W is closed under scalar multiplication (for all $k \in \mathbb{R}, w \in W$, $kw \in W$)

From this, it follows that a subspace is closed under any linear combination of its elements.

The **basis** for a subspace is a set of vectors which spans the space. For example, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis for $\mathbb{R}^3$, but any other set of three linearly independent vectors in $\mathbb{R}^3$ also forms a basis for $\mathbb{R}^3$. If a set of vectors is not linearly independent, at least one of the vectors is **redundant** and is not included in the basis. A basis for a space always contains the same amount of vectors. For example, any basis for $\mathbb{R}^n$ must contain n vectors. The number of vectors in the basis for a space is known as the **dimension** of the space. For example, $\dim(\mathbb{R}^n) = n$, because n linearly independent basis vectors are needed to span $\mathbb{R}^n$. A subspace V of $\mathbb{R}^n$ may have a dimension less than or equal to, but not greater than, n.

It is possible to check for linear independence of a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ by considering the linear combination

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

If this relation holds only when $c_1 = \cdots = c_n = 0$, then all vectors in the set are linearly independent. Notice that this equation may be converted into an equivalent augmented matrix, given by

$$\left[\begin{array}{ccc|c} | & & | & 0 \\ \vec{v}_1 & \cdots & \vec{v}_n & \vdots \\ | & & | & 0 \end{array}\right]$$

If this matrix reduces to $[\ I \mid \vec{0}\ ]$, then all $c_1, \ldots, c_n$ must be 0, meaning the vectors are linearly independent. In general, for a matrix $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, the columns of A are linearly independent if and only if $\ker(A) = \{\vec{0}\}$, or equivalently, $\text{rank}(A) = n$.

By reducing A to reduced row echelon form, it is possible to pick a basis for $\text{im}(A)$. In particular, each column which may be reduced to have a pivot is a non-redundant column, and is therefore necessary to span the image. The set of these columns forms the basis for $\text{im}(A)$.

The rank-nullity theorem relates the dimension of the kernel and image to the number of columns in a matrix:

Rank-Nullity Theorem. Generally, for a linear transformation $T : V \mapsto W$,

$$\dim(\text{im}(T)) + \dim(\ker(T)) = \dim(V)$$

and equivalently (but with different terminology), for an $m \times n$ matrix such that $T(\vec{x}) = A\vec{x}$,

$$\text{rank}(A) + \text{null}(A) = n$$

In other words, the number of columns with pivots (rank) and free variables (nullity) must sum to the total number of columns in a matrix.

It is now possible to think of a matrix as a function, with all the properties of a function applying to a matrix as well:

For a function $f : A \mapsto B$,

f is **surjective** (onto) if, for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

f is **injective** (one-to-one) if, for all $x, y \in A$, $f(x) = f(y)$ only when $x = y$.

f is **bijective** (invertible) if it is both injective and surjective.

For an $m \times n$ matrix A, surjectivity requires $m \leq n$ (at least as many columns as rows, at least as many many variables as equations) and injectivity requires $m \geq n$ (at least as many rows as columns, at least as many equations as variables), so only when $m = n$ (so A is square) can A be bijective.

These definitions and properties extend to all vector spaces, not merely $\mathbb{R}^n$. For example, the complex numbers, polynomials, differentiable functions, and any other set of object with well-defined zero which is closed under addition and scalar

multiplication may constitute a vector space. For any vector space, a linear transformation $T : V \mapsto W$ is an **isomorphism** if T is bijective, and V is **isomorphic** to W is there exists an isomorphism T from V to W.

1.4 Change of Coordinates

Coordinates are essential for any geometric interpretation of a vector space. In particular, Cartesian coordinates are generally used to describe the x-y plane or the x-y-z space.

For an ordered basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ for a vector space V, all elements $\vec{v} \in V$ may be written as a linear combination

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

and the $\mathcal{B}$-coordinates are defined as

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Notice that V is now related to $\mathbb{R}^n$ by $[\vec{v}]_{\mathcal{B}}$. This makes bases particularly useful when working with vector spaces besides $\mathbb{R}^n$ (i.e. differentiable functions, polynomials, etc).

For example, for the set $\mathcal{P}_2$ of polynomials of at most degree two, a basis $\mathcal{B} = (1, x, x^2)$ spans $\mathcal{P}_2$ such that for any $p \in \mathcal{P}_2$, $p(x) = a + bx + cx^2$, so $[p]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ relates $p \in \mathcal{P}_2$ to $\mathbb{R}^2$.

Formally, this process is an isomorphic mapping from V to $\mathbb{R}^n$:

The mapping

$$V \to \mathbb{R}^n, \text{ given by } \vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$$

is an isomorphism of V and $\mathbb{R}^n$, so

$$[\vec{v} + \vec{w}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$$

and

$$[c\vec{v}]_{\mathcal{B}} = c[\vec{v}]_{\mathcal{B}}$$

For a linear transformation T given by a square matrix such that $T(\vec{x}) = A\vec{x}$, it is relatively simple to compute the basis for the transformation. In particular, for $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ where $\vec{v}_1, \ldots \vec{v}_n$ form a basis,

$$[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\vec{v}_1]_{\mathcal{B}} & \ldots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = B[\vec{x}]_{\mathcal{B}}$$

where B is the $\mathcal{B}$-coordinate matrix given by

$$B = \begin{bmatrix} | & & | \\ [T(\vec{v}_1]_{\mathcal{B}} & \ldots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

Alternatively, B may be computed by

$$B = S^{-1}AS$$

where S is the matrix of the bases of A, given by

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}$$

Then, the $\mathcal{B}$-matrix for a linear transformation may be constructed two ways:

For a linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $T(\vec{x}) = A\vec{x}$ with basis $\vec{v}_1, \ldots, \vec{v}_n$ for $\mathbb{R}^n$, the $\mathcal{B}$-matrix $B = [T]_{\mathcal{B}}$ may be computed either column-wise, as

$$B = \begin{bmatrix} | & & | \\ [T(\vec{v}_1]_{\mathcal{B}} & \ldots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

or with the basis, as

$$B = S^{-1}AS, \qquad S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}$$

And since S is invertible, the relationship

$$SB = AS$$

holds generally.

In general, *any isomorphic transformation $T : V \mapsto V$ may be represented with a B-matrix*:

For any isomorphic transformation $T : V \mapsto V$ with basis $\mathcal{B}$ for V given by $\mathcal{B} = \{f_1, \ldots, f_n\}$, the change of basis matrix B is given by

$$B = \begin{bmatrix} | & & | \\ [T(f_1)]_{\mathcal{B}} & \ldots & [T(f_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

The entire change of basis process may be best understood with a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \downarrow{\scriptstyle L_{\mathcal{B}}} & & \downarrow{\scriptstyle L_{\mathcal{B}}} \\ \mathbb{R}^n & \xrightarrow{[T]_{\mathcal{B}}} & \mathbb{R}^n \end{array}$$

$L_{\mathcal{B}}$ is the coordinate isomorphism associating an arbitrary vector space V with some basis $\mathcal{B}$ in $\mathbb{R}^n$. Direct paths (i.e. by T or $[T]_{\mathcal{B}}$) are equivalent to the change-of-coordinate path (i.e. by $L_{\mathcal{B}}[T]_{\mathcal{B}}L_{\mathcal{B}}^{-1}$ or $L_{\mathcal{B}}^{-1} \circ T \circ L_{\mathcal{B}}$ respectively).

1.5 Orthogonality in $\mathbb{R}^n$, Inner Product Spaces, and Least Squares

In higher dimensions, the concept of orthogonality (perpendicularity) and length of vectors becomes difficult to visualize. Instead, a rigorous definition replaces intuition:

Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are **orthogonal** if their dot product $\vec{v} \cdot \vec{w} = v_1w_1 + \cdots + v_nw_n = 0$.

The **length** of a vector is given by $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$. A vector is a **unit vector** if its length is equal to 1. For a vector $\vec{v}$ which is not a unit vector, the corresponding unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$.

The concept of the dot product generalizes for any vector space as an **inner product**. In fact, the dot product is one example of an inner product in the *inner product space* $\mathbb{R}^n$. An inner product is defined as any transformation $\langle \cdot, \cdot \rangle : V \to \mathbb{R}$ satisfying the following properties:

For all $f, g, h \in V$ and all $c \in \mathbb{R}$, an inner product $\langle \cdot, \cdot \rangle$ satisfies the following:

1. $\langle f, g \rangle = \langle g, f \rangle$
2. $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
3. $\langle cf, g \rangle = c\langle f, g \rangle$
4. $\langle f, f \rangle > 0$ for all nonzero $f \in V$

An inner product space is a vector space which has an assigned inner product. In a particular inner product space, the magnitude of an element f is defined as $\sqrt{\langle f, f \rangle}$, and elements f, g are orthogonal if $\langle f, g \rangle = 0$.

Vectors are **orthonormal** if they are of unit length and orthogonal. Orthonormal vectors must be linearly independent. Then, a set of n orthonormal vectors in $\mathbb{R}^n$ have special properties. In particular, for orthonormal vectors $\vec{u}_1, \ldots, \vec{u}_n \in \mathbb{R}^n$, $\vec{u}_1, \ldots, \vec{u}_n$ must form a basis for $\mathbb{R}^n$.

The projection may be generalized in $\mathbb{R}^n$. In particular:

For a subspace V of $\mathbb{R}^n$ with orthonormal basis $\vec{u}_1, \ldots, \vec{u}_n$, the projection of any $\vec{x} \in \mathbb{R}^n$ onto the subspace V is given by

$$\text{proj}_V \vec{x} = \vec{x}_{||} = (\vec{u}_1 \cdot \vec{x}_1)\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x}_n)\vec{x}_n$$

The **orthogonal compliment** of a vector space V refers to the set of all vectors $\vec{x}$ which are perpendicular to every vector $\vec{v}$ in V. In set notation, $V_\perp$ is given by

$$V_\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{x} = 0 \ \forall \ \vec{v} \in V\}$$

$V_\perp$ has several unique properties:

Properties of the orthogonal complement of a subspace V of $\mathbb{R}^n$:

1. $V_\perp$ is a subspace of V.
2. $V \cap V_\perp = \{\vec{0}\}$
3. $\dim(V) + \dim(V_\perp) = n$
4. $(V_\perp)_\perp = V$

Using this information, any basis may be converted into an **orthonormal basis** by the **Gram-Schmidt process**, a recursive process outlined as follows:

Gram-Schmidt Process. For a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$, take

$$\begin{aligned} u_1 &= v_1 \\ u_i &= v_i - v_i^\perp, \text{ where } v_i^\perp = \text{proj}_{u_1}(v_i) + \cdots + \text{proj}_{u_{i-1}} v_i, \qquad \text{for } i > 1 \end{aligned}$$

which yields the new orthonormal basis

$$\mathcal{B}' = \left\{ \frac{u_1}{||u_1||}, \ldots, \frac{u_n}{||u_n||} \right\}$$

A linear transformation is called an **orthogonal transformation** if it preserves length and orthogonality. In particular, a linear transformation has the following properties:

Properties of an Orthogonal Transformation. A linear transformation $T : V \to V$ is orthogonal if and only if any of the following hold:

1. $||T(\vec{x})|| = ||\vec{x}||$ for all $\vec{x} \in \mathbb{R}^n$
2. for orthogonal $\vec{v}, \vec{w} \in V$, $T(\vec{v})$ and $T(\vec{w})$ are orthogonal
3. The $n \times n$ standard matrix A of T is an **orthogonal matrix**, such that the columns of A form an orthonormal basis for $\mathbb{R}^n$
4. $A^T A = AA^T = I_n$, where A^T is the transpose of A such that entry a_{ij} of A becomes entry a_{ji} of A^T. Equivalently, A is orthogonal if $A^T = A^{-1}$.

The transpose has several other useful properties:

Properties of the Transpose. For any $A, B \in \mathbb{R}^{m \times n}$, the following properties of the transpose apply:

1. $(A + B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $\text{rank}(A^T) = \text{rank}(A)$
5. $\ker(A) = \ker(A^T A)$
6. $(\text{im}A)^\perp = \ker(A^T)$

The transpose has several applications, particularly in data fitting. In particular, for any *inconsistent system* $A\vec{x} = \vec{b}$ (for which there is no solution), the **least-squares solution** approximates a solution.

Least Squares Solutions. Let $A \in \mathbb{R}^{m \times n}$. The least-squares solution of a linear system $A\vec{x} = \vec{b}$ is the vector $\vec{x}^*$ such that $||\vec{b} - A\vec{x}^*|| \leq ||\vec{b} - A\vec{x}||$ for all $\vec{x} \in \mathbb{R}^n$. In other words, $\vec{x}^*$ minimizes the error between $\vec{b}$ and $A\vec{x}$.

The least-squares solutions of a system $A\vec{x} = \vec{b}$ are the exact solutions of the consistent system

$$A^T A\vec{x} = A^T \vec{b}$$

If A is invertible, then $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$. For a consistent linear system, $\vec{X}^* = \vec{x}$ (the least squares solution is the solution to $A\vec{x} = \vec{b}$).

Least-squares solutions are frequently applied in data fitting to fit vectors (generally polynomials or some other differentiable function) to a set of data–for example, in fitting a trend line, a quadratic function, or an exponential curve to an over-constrained dataset (which cannot fit exactly to the function in question).

1.6 Determinants

The **determinant** of a matrix, $\det A$, is a real number associated with a square matrix. In general, the determinant is computed recursively by *Laplace expansion*. Several properties of the determinant allow its computation to be simplified at times:

For any square $n \times n$ matrix, the following properties hold:

1. $\det(AB) = (\det A)(\det B)$
2. $\det(A^n) = (\det A)^n$
3. $\det A^T = \det A$
4. $\det(A^{-1}) = \frac{1}{\det A}$

For special matrices, the following properties hold:

1. If a matrix has linearly dependent rows or columns, $\det A = 0$. Alternatively, a matrix is invertible if and only if $\det A \neq 0$.
2. The determinant of a triangular matrix is given by $\det A = \mathrm{tr}A$, where $\mathrm{tr}A$ is the sum of the diagonal entries of A.
3. If A and B are similar matrices, then $\det A = \det B$

1.7 Eigenvalues and Eigenvectors

For a linear transformation $T : V \to V$, an **eigenvalue** of T is a real number λ such that $T(\vec{v}) = \lambda\vec{v}$, where (nonzero) $\vec{v} \in V$ is an **eigenvector** corresponding to λ.

If there exist linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$, with corresponding $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $A\vec{v}_i = \lambda_i\vec{v}_i$, then $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ forms an **eigenbasis** for $\mathbb{R}^n$.

If an eigenbasis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ exists for some matrix A, then A is **diagonalizable** (similar to a diagonal matrix) by the diagonalization

$$D = S^{-1}AS, \qquad S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

A method to calculate eigenvalues emerges from the definition of an eigenvalue-eigenvector pair. In particular, if λ is an arbitrary real number and $\vec{v}$ is any nonzero vector, then the condition for an eigenvalue is:

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0} \iff A\vec{v} - \lambda I\vec{v} = \vec{0} \iff (A - \lambda I)\vec{v} = \vec{0}$$

Thus, all vectors in $\ker(A - \lambda I)$ are eigenvectors to corresponding eigenvalues λ. If there are eigenvectors (in the kernel), then $\det(A - \lambda I) = 0$ as $A - \lambda I$ is not invertible in this case. Thus, (real or complex) eigenvalues and eigenvectors may be computed as follows:

Eigenvalues and eigenvectors satisfy the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

For each λ_i found from the roots of the characteristic polynomial, eigenvectors consist of all vectors in $\ker(A - \lambda_i I)$.

Each eigenvalue has an **algebraic multiplicity** $\text{almu}(\lambda_i)$ and a **geometric multiplicity** $\text{gemo}(\lambda_i)$. Algebraic multiplicity refers to the number of times λ_i appears as a root in the characteristic equation, while geometric multiplicity refers to $\dim(\ker(A - \lambda_i I)$. If $\text{almu}(\lambda_i) \neq \text{gemu}(\lambda_i)$ for any eigenvector, then A is not diagonalizable.

In some cases, there exists an *orthonormal eigenbasis* for a transformation. This is only the case when S is orthogonal such that $S^{-1}AS = S^TAS = D$ for some diagonalization D. The **spectral theorem** outlines the condition for an orthonormal eigenbasis:

Spectal Theorem. A matrix A is orthogonally diagonalizable if and only if A is symmetric, where a symmetric matrix satisfies $A^T = A$.

2 Real Analysis

Notes from **Math 351: Principles of Analysis**

Taken at University of Michigan, Winter 2025

using *Elementary Analysis: The Theory of Calculus*, by Kenneth Ross

2.1 Real Numbers

This material is essentially a development of the two Fundamental Theorems of Calculus, beginning from sequences. To begin, certain axioms about real numbers and sets are taken for granted. These axioms are outlined:

Archimedean Property. If $a > 0$ and $b > 0$, then there exists a natural number $n \in \mathbb{N}$ such that $na > b$, or $n > b/a$.

Density of Rational Numbers. If $a, b \in \mathbb{R}$ and $a < b$, then there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$.

Density of Irrational Numbers. If $a, b \in \mathbb{R}$ and $a < b$, then there exists an irrational number $x \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < x < b$.

For the final axiom about the real numbers, a definition is necessary:

Definition. Let $S \subseteq \mathbb{R}$. If there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$, then M is an **upper bound** for S. Similarly, if there exists $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$, then m is a **lower bound** for S. S is said to be **bounded** if it has both an upper and lower bound. Equivalently, S is bounded if and only if there exists $M_0 \in \mathbb{R}$ such that if $s \in S$, then $|s| \leq M_0$.

Completeness Axiom: If S is a non-empty subset of $\mathbb{R}$ which has an upper bound, then S has a least upper bound $\sup S$. Similarly, if S has a lower bound, then S has a greatest lower bound $\inf S$.

2.2 Sequences

A sequence $s : \mathbb{N} \to \mathbb{R}$ is a mapping of natural numbers to real numbers. Generally, the nth term of a sequence $s(n)$ is denoted s_n.

Definition. A sequence $\{s_n\}$ is said to **converge** to $s \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that if $n > N$, then $|s_n - s| < \epsilon$. Here, s is equivalently denoted as $s = \lim s_n = \lim_{n\to\infty} s_n$.

Essentially, if the values of a sequence become arbitrarily close to some number after a given point, then the sequence is said to converge to that number. It can be shown that this limit s is *unique*.

Example. Prove that $\lim \frac{1}{n} = 0$.

Proof. Fix $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Then, if $n > N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon$$

Thus, $\left|\frac{1}{n} - 0\right| < \epsilon$, and $\lim \frac{1}{n} = 0$. ■

From the definition of the limit, various facts may be proved which aid in computing limits. Let $\lim s_n = s$ and $\lim t_n = t$. Then, the following limits hold:

Scalar Multiplication. If $c \in \mathbb{R}$, then $\lim cs_n = cs$

Addition. $\lim(s_n + t_n) = \lim s_n + \lim t_n = s + t$

Subtraction. $\lim(s_n - t_n) = \lim s_n - \lim t_n = s - t$

Multiplication. $\lim(s_n t_n) = st$

Division. If $t \neq 0$ and $t_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim(s_n/t_n) = \lim(s_n)/\lim(t_n) = s/t$

Note that these limit laws *only apply to sequences known to converge.* A particularly important limit law, the *squeeze limit law*, is given:

Theorem. (Squeeze Principle) Let $a_n \leq s_n \leq b_n$ for all but finitely many values of n and $\lim a_n = \lim b_n = s$. Then, $\lim a_n = \lim s_n = \lim b_n = s$.

Proof. Fix $\epsilon > 0$. Since $a_n \leq s_n \leq b_n$ for all but finitely many values of n, there exists N_0 such that if $n > N_0$, then $a_n \leq s_n \leq b_n$. Similarly, since $\lim a_n = \lim b_n = s$, there exists N_1, N_2 such that if $n > N_1$, then $|a_n - s| < \epsilon$ and if $n > N_2$, then $|b_n - s| < \epsilon$. Let $N = \max(N_0, N_1, N_2)$. Then, if $n > N$,

$$|a_n - s| < \epsilon, \text{ so } s - \epsilon < a_n < s + \epsilon$$

and

$$|b_n - s| < \epsilon, \text{ so } s - \epsilon < b_n < s + \epsilon$$

Thus,

$$s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon$$

and thus $|s_n - s| < \epsilon$, so $\lim s_n = s$. ■

It is possible to derive other *comparison principles*, i.e. if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$, and if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

Definition. A sequence $\{s_n\}$ is said to **diverge to** ∞ if, for all $M \in \mathbb{R}$, there exists a real number N such that if $n > N$, then $s_n > M$. This is denoted $\lim s_n = +\infty$. A sequence $\{s_n\}$ is said to **diverge to** $-\infty$ if, for all $M \in \mathbb{R}$, there exists a real number N such that if $n > N$, then $s_n < M$. This is denoted $\lim s_n = -\infty$.

A sequence will either *converge* to a finite limit, *diverge* to $+\infty$ or $-\infty$, or *not converge* if neither convergence nor divergence are met. For example, the sequence $s_n = (-1)^n$ does not converge.

Definition. A sequence $\{s_n\}$ is **monotone increasing** if, for all $n \in \mathbb{N}$, $s_{n+1} \geq s_n$. $\{s_n\}$ is **monotone decreasing** if, for all $n \in \mathbb{N}$, $s_n \geq s_{n+1}$. $\{s_n\}$ is **monotone** if it is monotone increasing and/or monotone decreasing.

Monotone sequences are special sequences for two reasons: (1) a bounded monotone sequence is guaranteed to converge, and it converges to its infimum if it is monotone decreasing or its supremum if it is monotone increasing; and (2) a *monotone subsequence* can be found in any sequence. The meaning of subsequence is an infinite sampling of terms of an original sequence, ordered in the same manner:

Definition. A sequence $\{s_{n_k}\}$ is a **subsequence** of $\{s_n\}$ if each $\{n_k\}$ is a strictly increasing sequence of natural numbers (i.e. $n_{k-1} < n_k$), indexed by $k \in \mathbb{N}$.

Using these two facts, it is possible to prove a key result in analysis:

Bolzano-Weierstrass Theorem. Every bounded sequence has a convergent subsequence.

Proof. Let $\{s_n\}$ be a bounded sequence. Then, there exists a subsequence $\{s_{n_k}\}$ which is monotone. Since $\{s_n\}$ is bounded, $\{s_{n_k}\}$ must also be bounded. Since $\{s_{n_k}\}$ is monotone and bounded, $\{s_{n_k}\}$ is convergent. Thus, $\{s_n\}$ has a convergent subsequence. ■

Subsequences of convergent or divergent sequences have special properties. If a sequence $\{s_n\}$ is convergent to s, then every subsequence s_{n_k} converges to s. Similarly, if $\{s_n\}$ diverges to $+\infty$, then every $\{s_{n_k}\}$ also diverges to $+\infty$, and if $\{s_n\}$ diverges to $-\infty$, then every $\{s_{n_k}\}$ also diverges to $-\infty$. Therefore, if the convergence (or divergence) of any two subsequences disagree, then the sequence is not convergent.

Example. Prove that $\{s_n\} = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$ does not converge by using subsequences.

Proof. Observe that

$$\lim s_{2k} = \lim 1 = 1$$

and

$$\lim s_{2k+1} = \lim 0 = 0$$

and thus $\lim s_{2k} \neq \lim s_{2k+1}$, so $\{s_n\}$ does not converge. ■

A final method to evaluate the convergence of a sequence is to determine whether the sequence is *Cauchy*.

Definition. A sequence $\{s_n\}$ is a **Cauchy sequence** if, for any $\epsilon > 0$, there exists a real number N such that if $m, n > N$, then $|s_m - s_n| < \epsilon$.

In other words, if the terms in a sequence become arbitrarily close together for sufficiently large index values, the sequence is Cauchy. An important equivalence between convergent sequences and Cauchy sequences can be shown:

Theorem. A sequence $\{s_n\}$ is a Cauchy sequence if and only if it is convergent.

This theorem provides a criteria for convergence even when the value to which a sequence converges is not obvious; it is sufficient to shown that the difference between consecutive terms becomes arbitrarily small for large enough indexing terms.

A special type of sequence is the **infinite series**. A series $\sum_{n=m}^{\infty} a_n$ is said to converge if its **partial sums**, a sequence with nth term $s_n = a_m + a_{m+1} + \cdots + a_n$, is convergent.

Theorem. If $\sum_{n=m}^{\infty} a_n$ is an infinite series, then $\lim a_n = 0$.

Discussion of closedness and compactness.

Definition. A subset $S \subseteq \mathbb{R}$ is **closed** if every convergent sequence with values in S has limit in S. A subset $S \subseteq \mathbb{R}$ is **compact** if every sequence with values in S has a convergent subsequence with limit in S.

An important theorem in analysis, the Heine-Borel theorem, relates compactness and closedness of a set:

Heine-Borel Theorem. A set $S \subseteq \mathbb{R}$ is compact if and only if S is closed and bounded.

The Heine-Borel theorem is readily applied to an interval $[a, b]$ of $\mathbb{R}$. The set $[a, b]$ clearly is bounded below by a and above by b, and this set is closed. Therefore, any closed interval of $\mathbb{R}$ is compact.

2.3 Continuity

Continuity of a function may be defined immediately from sequences.

Definition. A function $f : D \to \mathbb{R}$ is **continuous** at $x \in D$ if for every sequence $\{x_n\}$ converging to x with values in D, $\lim f(x_n) = f(x)$.

This is the *sequence definition of continuity*. The criteria here is essentially that the limit distributes inside the function, as this definition requires $\lim f(x_n) = f(\lim x_n)$.

Example. Prove that $f(x) = x^2$ is continuous on $\mathbb{R}$.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Let $\{x_n\}$ converge to x. Then,

$$\lim f(x_n) = \lim(x_n^2) = \lim(x_n) \cdot \lim(x_n) = (\lim x_n)^2 = f(\lim x_n) = f(x)$$

Our choice of x is arbitrary, so for any $x \in \mathbb{R}$, $\lim f(x_n) = f(x)$, so x^2 is continuous on $\mathbb{R}$. ■

$f(x_n)$ is simply a sequence, and if f is continuous, this is a convergent sequence. Thus, limit laws for sequences essentially carry through to continuous functions, with these operations preserving continuity. In particular, for functions f, g, h continuous at x, $(f+g)(x)$ is continuous, $cf(x)$ is continuous for any $c \in \mathbb{R}$, and $\left(\frac{f}{g}\right)(x)$ is continuous if g is nonzero on the domain D. Additionally, a composition of continuous functions $f(g(x))$ is continuous at x. Finally, the squeeze principle carries through; if $f(x) \leq g(x) \leq h(x)$ for all $x \in D$ and $f(x_0) = h(x_0)$, then g must be continuous at x_0.

Two key theorems may be proven immediately from these facts:

Extreme Value Theorem. If $f : D \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded and it achieves its maximum and its minimum on $[a, b]$. Equivalently, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

Using the extreme value theorem, another key theorem may be proven:

Intermediate Value Theorem. If f is continuous on $[a, b]$ and y is between $f(a)$ and $f(b)$, then there exists $x_0 \in [a, b]$ such that $f(x_0) = y$.

Continuity may also be defined without reference to sequences. This definition sometimes simplifies the process of proving or disproving continuity. This definition, which is equivalent to the sequence definition of continuity, is given below:

Definition. A function $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if, for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in D$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

In the epsilon-delta definition of continuity, a function f is continuous at x_0 if for any inputs δ-close to x_0, the outputs are ϵ-close to $f(x_0)$.

Example. Prove that $f(x) = x^3$ is continuous at $x = 1$.

Scratch Work. We often make the assumption that δ is small, i.e. $\delta < 1$. Making this assumption, we have

$$|x - 1| < 1 \implies -1 < x - 1 < 1 \implies 0 < x < 2$$

And we also have

$$|f(x) - f(1)| = |x^3 - 1| = |x - 1||x^2 + x + 1| < \delta \underbrace{|2^2 + 2 + 1|}_{\text{given } x<2} < 7\delta$$

If $\epsilon \geq 7\delta$, we are done. Thus, we choose δ to be the minimum $\epsilon/7$ (as found by assumption) and 1 (as an upper bound).

Proof. Fix $\epsilon > 0$. Choose $\delta = \min\left(\frac{\epsilon}{7}, 1\right)$. If $|x - 1| < \delta$, then

$$|f(x) - f(1)| = |x^3 - 1| = |x - 1||x^2 + x + 1| < 7\delta \leq 7\frac{\epsilon}{7} = \epsilon$$

For our chosen δ, we are guaranteed that if $|x - 1| < \delta$, then $|f(x) - f(1)| < \epsilon$. Thus, f is continuous at $x = 1$. ■

The $\epsilon - \delta$ definition lends itself to the definition of a stronger form of continuity, *uniform continuity*:

Definition. A function $f : D \to \mathbb{R}$ is **uniformly continuous** on $S \subseteq D$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Functions which grow at increasing rate are generally *not uniformly continuous* on an unbounded interval. For example, $f(x) = x^2$ can be shown to be continuous for any value in $\mathbb{R}$, but x^2 is *not uniformly continuous* on $\mathbb{R}$ because δ cannot take on a form to account for arbitrary ϵ. By contrast, $f(x) = x$ is both continuous for every point in $\mathbb{R}$ and uniformly continuous on $\mathbb{R}$, with $\delta = \epsilon$. The most important result of uniform continuity is the following theorem:

Theorem. If f is continuous on a closed interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

2.4 Differentiation

Before defining the derivative, it is necessary to define the limit at a point.

Definition. Suppose $f : D \to \mathbb{R}$ is a function, $x_0 \in \mathbb{R}$, and that there exists $\beta > 0$ such that D contains $(x_0 - \beta, x_0) \cup (x_0, x_0 + \beta)$. The *limit* as $x \to x_0$, denoted

$$\lim_{x \to x_0} f(x) = L$$

if for any sequence $\{x_n\}$ with values in $D \backslash \{x_0\}$ such that $\lim x_n = x_0$, $\lim f(x_n) = L$.

Notice that x_0 is not considered in evaluating the limit. Furthermore, the limit is only defined if the function can be evaluated within β of x_0 (in either direction). Since $\lim_{x \to x_0} f(x)$ is defined in terms of sequences, the limit laws derived for sequences still apply. Furthermore, the limit as $x \to x_0$ may be equivalently defined in terms of the $\epsilon - \delta$ definition:

Theorem. $\lim_{x \to x_0} f(x) = L$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \in D$, then $|f(x) - L| < \epsilon$.

An important result relates limits to continuity:

Theorem. A function f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

With the limit definition, it is possible to define the derivative:

Definition. Suppose $f : D \to \mathbb{R}$, $x_0 \in \mathbb{R}$ and D contains an open interval around x_0. f is called *differentiable* at x_0 with derivative $f'(x_0)$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

If this limit does not exist, then f is said to be *not differentiable* at x_0 and $f'(x_0)$ does not exist. Notice that x_0 is *not* in the domain of $\frac{f(x) - f(x_0)}{x - x_0}$ to avoid division by zero. Differentiability and continuity are related by the following theorem:

Theorem. Suppose f is differentiable at x_0. Then, f is continuous at x_0.

Notice that the opposite statement is clearly *false*. As a counterexample, the function $|x|$ may be shown to be continuous but not differentiable at $x = 0$.

The common differentiation rules may be derived:

Linearity of derivatives. If f, g are differentiable at x_0 and $c \in \mathbb{R}$, then $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ and $(cf)'(x_0) = cf'(x_0)$

Product Rule. If f, g are differentiable at x_0, then $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

Quotient Rule. If f, g are differentiable at x_0 and $g(x_0) \neq 0$, then $(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$

Chain Rule. If g is differentiable at x_0 and f is differentiable at $g(x_0)$, then $(f(g(x_0))' = f'(g(x_0))g'(x_0)$

Two important theorems involving the existence of derivatives may be derived directly from the definition:

Rolle's Theorem. If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Rolle's Theorem is used in the proof of a stronger existence theorem, the *mean value theorem*:

Mean Value Theorem. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$

In other words, for any interval where f is continuous and differentiable, the average slope over the interval must equal the instantaneous slope, the derivative, for at least one point on that interval. Observe that Rolle's Theorem is the Mean Value Theorem under the special case $f(b) - f(a) = 0$.

Following immediately from the Mean Value Theorem, derivatives can be shown to relate to the behavior of their original function.

Function behavior and derivatives. Suppose f is differentiable on (a, b).

If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x) = c$ on (a, b) for some constant $c \in \mathbb{R}$.

If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b).

If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).

Each result follows from Mean Value Theorem. The case of strictly decreasing is illustrated as example:

Proof. Suppose $f'(x) < 0$ for all $x \in (a, b)$. Let $x_1, x_2 \in (a, b)$ such that $x_2 > x_1$. Then, f is differentiable on (x_1, x_2), and there exists some value x_0 where $x_1 < x_0 < x_2$ and

$$f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

Thus, $f(x_2) < f(x_1)$, so f is strictly increasing on (a, b). ■

As a final result, the intermediate value theorem may be extended to derivatives.

Intermediate Value Theorem for Derivatives. If f is differentiable on (a, b) and $x_1, x_2 \in (a, b)$ where $x_1 < x_2$ and c is between $f'(x_1)$ and $f'(x_2)$, then there exists $x_0 \in (x_1, x_2)$ such that $f'(x_0) = c$.

If a function is at least twice differentiable, then this result is trivial as f' is continuous. However, this theorem applies more generally, including to functions which are only once differentiable, making this result particularly useful.

2.5 Integration

Definition. A **partition** P of $[a, b]$ is a finite collection of points $P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$

Notably, each point $t_k \in [a, b]$ is distinct, and the endpoints are *always included* in the partition. Thus, the smallest partition contains two points, i.e. the two endpoints.

Integration is then defined by summation over a partition:

Let f be a bounded function on $[a, b]$. Let P be a partition of $[a, b]$.

The **upper Darboux sum** of f with partition P is defined by

$$U(f, P) = \sum_{k=1}^{n} \sup\{f[t_{k-1}, t_k]\}(t_k - t_{k-1})$$

and the **upper Darboux integral** of f on $[a, b]$ is defined as

$$U(f) = \inf\{U(f, P) | P \text{ partition of } [a, b]\}$$

Similarly, the **lower Darboux sum** of f with partition P is defined by

$$L(f, P) = \sum_{k=1}^{n} \inf\{f[t_{k-1}, t_k]\}(t_k - t_{k-1})$$

and the **lower Darboux integral** of f on $[a, b]$ is defined as

$$L(f) = \sup\{L(f, P) | P \text{ partition of } [a, b]\}$$

f is said to be **integrable** on $[a, b]$ if $U(f) = L(f)$, and thus

$$L(f) = \int_a^b f(x)dx = U(f)$$

In other words, integration is defined by a special type of summation, where a *partition* splits up an interval $[a, b]$ into subintervals. On each subinterval defined by the partition, the greatest or least value is multiplied by the length of that subinterval, thus computing the area of a rectangle defined by f. When the greatest possible lower Darboux sums and the smallest possible upper Darboux sums are equal, this value is said to be the *integral* of f on $[a, b]$.

The following inequality can be shown:

$$\inf\{f[a, b]\}(b - a) \leq L(f, P) \leq L(f) \leq U(f) \leq U(f, P) \leq \sup\{f[a, b]\}(b - a)$$

Informally, this means that for any partition P, the lower and upper Darboux sums are bounded respectively by rectangles with width $b - a$ and the least and greatest value of f on the interval respectively. Additionally, the greatest lower Darboux sum is naturally bounded from below by the lower Darboux sum on any partition, and the least upper Darboux sum is bounded from above by the upper Darboux sum on any partition.

Theorem (Cauchy Integrability Criterion). Let f be bounded on $[a, b]$. f is integrable on $[a, b]$ if and only if for all $\epsilon > 0$, there exists a partition P over $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Basic facts of integrability can be shown from the definition:

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Let f be integrable on $[a,b]$. If $c,d \in (a,b)$ and $c<d$, then f is integrable on $[c,d]$.

Integration is linear. In particular, if f,g are integrable on $[a,b]$ and $c \in \mathbb{R}$, then $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

It is now possible to prove the fundamental theorems of calculus. These theorems essentially relate differentiation and integration as, in essence, "inverse" operations. The 2nd Fundamental Theorem is simpler to prove, and is therefore outlined first:

2nd Fundamental Theorem of Calculus. Suppose f is continuous on $[a,b]$ and differentiable on (a,b). If f' is integrable on $[a,b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Proof. Fix $\epsilon > 0$. Since f' is integrable, f' meets the Cauchy criterion, so there exists a partition P such that $U(f',P) - L(f',P) < \epsilon$. By the Mean Value Theorem, for each interval $[t_k, t_{k+1}]$ there exists some x_k such that

$$f'(x_k) = \frac{(f(t_{k+1}) - f(t_k))}{(t_{k+1} - t_k)}$$

and thus $f(t_{k+1}) - f(t_k) = f'(x_k)(t_{k+1} - t_k)$. Thus,

$$f(b) - f(a) = \sum_{k=1}^{n} [f(t_{k+1}) - f(t_k)] = \sum_{k=1}^{n} f'(x_k)(t_{k+1} - t_k)$$

Therefore,

$$L(f',P) \leq f(b) - f(a) \leq U(f',P)$$

Since $U(f',P) - L(f',P) < \epsilon$, and by definition of integration $L(f',P) \leq \int_a^b f'(x)dx \leq U(f',P)$, then

$$0 \leq \int_a^b f'(x)dx - L(f',P) \leq U(f',P) - L(f',P) < \epsilon$$

and $L(f',P) \leq f(b) - f(a)$, so

$$\int_a^b f'(x)dx - L(f',P) \leq \left| \int_a^b f'(x)dx - [f(b) - f(a)] \right| < \epsilon$$

■

In other words, if a function f' is known to be the derivative of another function f, computing $\int_a^b f'(x)dx$ simply involves computing the difference between f at the endpoints of the interval.

1st Fundamental Theorem of Calculus. Let f be integrable on $[a,b]$. Define $F(x) = \int_a^x f(t)dt$, then F is uniformly continuous on $[a,b]$. If f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 with $F'(x_0) = f(x_0)$.

Proof. Uniform Continuity. Fix $\epsilon > 0$. f is integrable and therefore bounded by the Cauchy criterion for integration, so there exists some $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. If $x, y \in [a,b]$, and $|x-y| < \epsilon/M$, then (assuming without loss of generality that $x < y$):

$$|F(y) - F(x)| = \left| \int_x^y f(t)dt \right| \leq \int_x^y |f(t)|dt \leq \int_x^y M dt = M(y-x) = M|x-y| < \epsilon$$

Thus, $|F(y) - F(x)| < \epsilon$ and F is uniformly continuous on $[a,b]$.

Differentiability. Suppose f is continuous at $x_0 \in (a,b)$. For any $x \neq x_0$,

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(t)dt$$

and

$$f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f(x_0)dt$$

by simple evaluation of the constant integral. Thus,

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \int_{x_0}^x [f(t) - f(x_0)]dt$$

Fix $\epsilon > 0$. Because f is continuous at x_0, there exists $\delta > 0$ such that if $t \in (a,b)$ and $|t - x_0| < \delta$, then $|f(t) - f(x_0)| < \epsilon$. Thus,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \frac{1}{x - x_0} \int_{x_0}^x \epsilon dt = \epsilon$$

Therefore,

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

and thus $F'(x_0) = f(x_0)$. ■

The fact that $\left|\int_x^y f(t)dt\right| \leq \int_x^y |f(t)|dt$ can be shown by evaluating the Darboux sums of each respectively. Intuitively, $|f(t)|$ is always nonnegative, so summing multiples of $|f(t)|$ must be at least as large as the absolute value of the original integral (which may come out positive, negative, or zero before taking the absolute value, i.e. a different number and one necessarily smaller than $\int_x^y |f(t)|dt$ if f is negative for any value on $[x,y]$).

This fundamental theorem gives two facts: first, if f is integrable, its integral function F is uniformly continuous and therefore continuous; second, if the original function f is continuous, then F is differentiable and its derivative at a point is the value of f at that point.