References

Thermodynamics

Ideal Gas Relationships

Ideal Gas Law

$$PV = mRT$$
$$Pv = RT$$
$$PV = nR_uT$$

Compressibility Factor

$$Pv = ZRT$$

Internal Energy of Ideal Gas

$$\Delta u = \int_{T_1}^{T_2} c_v(T)dT \approx c_{v,avg}\Delta T$$

Enthalpy of Ideal Gas

$$\Delta h = \int_{T_1}^{T_2} c_p(T)dT \approx c_{p,avg}\Delta T$$

Entropy of Ideal Gas

$$\Delta S_{\text{ideal gas}} = \underbrace{c_v \ln(T_2/T_1) + R\ln(V_2/V_1)}_{c_v \text{ at average temperature}}$$

$$= \underbrace{c_p \ln(T_2/T_1) - R\ln(P_2/P_1)}_{c_p \text{ at average temperature}}$$

Isentropic Ideal Gas Relationships

$$\left(\frac{T_2}{T_1}\right)_{s=\text{const}} = \left(\frac{v_1}{v_2}\right)^{k-1}$$

$$\left(\frac{T_2}{T_1}\right)_{s=\text{const}} = \left(\frac{P_2}{P_1}\right)^{(k-1)/k}$$

$$\left(\frac{P_2}{P_1}\right)_{s=\text{const}} = \left(\frac{v_1}{v_2}\right)^{k}$$

Energy Balances

Nozzle and Diffuser

$$\dot{m}\left(h_1 + \frac{v_1^2}{2}\right) = \dot{m}\left(h_2 + \frac{v_2^2}{2}\right)$$

Compressor

$$\dot{m}h_1 + W_{\text{in}} = \dot{m}h_2$$

Pump (liquids only)

$$\dot{m}h_1 + V(P_2 - P_1) = \dot{m}h_2$$

Turbine

$$\dot{m}h_1 = \dot{m}h_2 + W_{\text{out}}$$

Carnot Efficiency

Heat Engine

$$\eta = \frac{T_H - T_L}{T_H}$$

Heat Pump Coefficient of Performance

$$\text{COP}_{\text{HP}} = \frac{T_H}{T_H - T_L}$$

Refrigerator Coefficient of Performance

$$\text{COP}_{\text{R}} = \frac{T_L}{T_H - T_L}$$

Heat-Entropy Relationship

$$\Delta S = \frac{Q}{T_0}$$

$$Q = \int_{T_1}^{T_2} TdS$$

Isentropic Efficiency

Turbine

$$\eta = \frac{W_a}{W_s} = \frac{h_1 - h_{2a}}{h_1 - h_{2s}}$$

Compressor

$$\eta = \frac{W_s}{W_a} = \frac{h_{2s} - h_1}{h_{2a} - h_1} = \underbrace{\frac{V(P_2 - P_1)}{h_{2a} - h_1}}_{\text{Pumps}}$$

Cycle Efficiency

Otto Cycle

$$\eta = 1 - \frac{1}{r^{k-1}}$$

Diesel Cycle

$$\eta = 1 - \frac{1}{r^{k-1}}\left[\frac{r_c^k - 1}{k(r_c - 1)}\right]$$

Brayton Cycle

$$\eta = 1 - \frac{1}{r_p^{(k-1)/k}}$$

Rankine Cycle

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}$$

Backwork Ratio

$$\text{BWR} = \frac{W_{\text{in}}}{W_{\text{out}}}$$

Statics and Mechanics of Materials

Stress

Normal Stress

$$\sigma_{zz} = \frac{N}{A} + \frac{M_x y}{I_x} - \frac{M_y x}{I_y}$$

Shear Stress (average)

$$\tau_{zx} = \frac{V_x}{A}, \ \tau_{zy} = \frac{V_y}{A}$$

Shear Stress due to Torsion

$$\tau_{z\theta} = \frac{Tr}{J}$$

Strain

Extensional Strain

$$\epsilon = \frac{\sigma}{E} = \frac{\delta}{L_0}$$

Shear Strain

$$\gamma = \frac{\tau}{G} = \phi_1 + \phi_2$$

Multiaxial Hooke's Law

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E} + \alpha \Delta T$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu \sigma_x}{E} - \frac{\nu \sigma_z}{E} + \alpha \Delta T$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu \sigma_x}{E} - \frac{\nu \sigma_y}{E} + \alpha \Delta T$$

Poisson's Ratio

$$E = 2G(1 + \nu)$$

Extension

Normal Extension

$$\delta = \frac{F L_0}{AE}$$

Angle of Torsion

$$\phi = \frac{TL}{JG}$$

Centroid Locations

$$Ac_x = \sum A_i \bar{x}_i \iff c_x = \frac{\sum A_i \bar{x}_i}{A}$$

$$Ac_y = \sum A_i \bar{y}_i \iff c_y = \frac{\sum A_i \bar{y}_i}{A}$$

Area Moments of Inertia

Parallel Axis Theorem

$$I_{x'} = I_x + Ad^2$$

Rectangular Section

$$I = \frac{1}{12} bh^3$$

Circular Section

$$I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

Circular Section, Normal Axis

$$J = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

Principle Stress and Stress Transformations

Stress Transformations

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Principle Stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \underbrace{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}_{\tau_{\max}}$$

Principle Strain and Strain Transformations

Strain Transformations

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta)$$

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos(2\theta) - \frac{\gamma_{xy}}{2} \sin(2\theta)$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\epsilon_x - \epsilon_y}{2} \sin(2\theta) + \frac{\gamma_{xy}}{2} \cos(2\theta)$$

Principle Strains

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \underbrace{\sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}}_{\gamma_{\max}/2}$$

Strain Gauge Rosette Relationship

$$\epsilon_a = \epsilon_x \cos^2 \theta_a + \epsilon_y \sin^2 \theta_a + \gamma_{xy} \sin \theta_a \cos \theta_a$$

Bending Deflections

$$\frac{M}{EI} = \frac{d^2 u}{dz^2} = \frac{d\theta}{dz}$$

Dynamics and Vibrations

Rectilinear Motion

$$a = v\frac{dv}{ds} \iff a\,ds = v\,dv$$

Coordinate Systems

Polar Coordinates
$$\vec{r} = r\hat{e}_r + z\hat{k}$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$$

Normal-Tangential Coordinates
$$\vec{v} = v\hat{e}_t$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

Particle Kinetics

Newton's 2nd Law

$$\sum \vec{F} = m\vec{a_c}$$

Work-Energy Theorem

$$\underbrace{\int_{\vec{R}_1}^{\vec{R}_2} \vec{F} \cdot d\vec{R}}_{\text{Work}} = \underbrace{T_2 - T_1}_{\text{Kinetic energy}}$$

Work-Energy Theorem, Conservative Forces

$$T_1 + V_1 + W_{\text{nonconservative}} = T_2 + V_2$$

Impulse-Momentum Equation

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2 - m\vec{v}_1$$

Coefficient of Restitution, rectilinear speeds

$$e := \frac{v'_B - v'_A}{v_A - v_B}$$

Particle Vibrations

Undamped natural frequency, spring-mass system

$$\omega_n = \sqrt{\frac{k}{m}}$$

Equation of Motion, Undamped Vibrations

$$\ddot{x} + \omega_n^2 x = 0$$

Solution, Undamped Vibrations

$$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n}\sin(\omega_n t)$$

Damping ratio, spring-mass damper

$$\zeta = \frac{c}{2m\omega_n}$$

Damped frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Equation of Motion, Damped Vibrations

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Solution, Damped Vibrations

$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cos(\omega_d t) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\right)$$

Time Constant ($t = 4\tau \implies 98\%$ decay)

$$\tau = \frac{1}{\zeta\omega_n}$$

Planar Rigid Body Kinetics

General Velocity for Rigid Body (or bodies, points o and p)
$$\vec{v}_p = \vec{v}_o + \vec{\omega_o} \times \vec{r}_{p/o} + \vec{v}_{p,\text{rel } o}$$

General Acceleration for Rigid Body (or bodies, points o and p)

$$\vec{a}_p = \vec{a}_o + \vec{\alpha} \times \vec{r}_{p/o} + \vec{\omega}_o \times (\vec{\omega_o} \times \vec{r}_{p/o}) + 2\vec{\omega}_o \times \vec{v}_{p,\text{rel } o} + \vec{a}_{p,\text{rel } o}$$

Parallel Axis Theorem (from center of mass c to point p at distance d)
$$I_p = I_c + md^2$$

Euler's Equation

$$\sum M_p = I_p\dot{\vec{\omega}} + \vec{r}_{c/p} \times m\vec{a}_p$$

Planar Rigid Body Vibrations

Equation of Motion, Forced Vibrations (with cosine forcing term)

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega_t)$$

Tuning Ratio

$$\eta = \frac{\omega}{\omega_n}, \text{ where } \eta \approx 1 \text{ at resonance}$$

Steady-State Amplitude of Response, Forced Vibrations

$$X = \frac{F_0/k_{\text{effective}}}{\sqrt{(2\zeta\eta)^2 + (1 - \eta^2)^2}}$$

Steady-State Amplitude of Response, Base Excitations

$$X = x_0\sqrt{\frac{1 + (2\zeta\eta)^2}{(2\zeta\eta)^2 + (1 - \eta^2)^2}}$$

Fluid Mechanics

Fluid Properties

Shear

$$\tau = \mu \frac{du}{dy}$$

Kinematic Viscosity

$$\nu = \frac{\mu}{\rho}$$

Fluid Statics

Hydrostatic Equation

$$P = P_{\text{atm}} + \rho g h$$

Resultant Force of Hydrostatics

$$F_R = \rho g h_c A$$

Center of Pressure

$$(x_R, \ y_R) = \left(\frac{I_{xy,c}}{A y_c} + x_c, \ \frac{I_{x,c}}{y_c A} + y_c \right)$$

Buoyant Force

$$F_b = \rho_{\text{fluid}} g V_{\text{displaced}}$$

Fluid Kinematics

Acceleration of a Fluid

$$\vec{a} = \frac{D}{Dt}(\vec{v}) = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}$$

Streamline Equation

$$\frac{dy}{dx} = \frac{v}{u}$$

Pathline Equations

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v$$

Stream Function

$$\frac{\partial \psi}{\partial y} = u, \qquad \frac{\partial \psi}{\partial x} = -v$$

Bernoulli Equation

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 + \underbrace{\int_{s_1}^{s_2} \frac{\partial v}{\partial t} ds}_{\text{neglect if steady}} \qquad \text{(Unsteady Bernoulli Equation)}$$

Control Volume Analysis

$$\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho (\vec{v} - \vec{v}_{CV}) \cdot \hat{n} \, dA = 0 \qquad \text{(Conservation of Mass)}$$

$$\frac{\partial}{\partial t} \int_{CV} \rho \vec{v} \, dV + \int_{CS} \rho \vec{v} [(\vec{v} - \vec{v}_{CV}) \cdot \hat{n}] dA = -\int P \hat{n} \, dA + \int_{CV} \rho \vec{g} \, dV + \vec{F}_{\text{viscous}} + \vec{F}_{\text{ext}} \qquad \text{(Conservation of Momentum)}$$

$$\frac{\partial}{\partial t} \int_{CV} \rho e \, dV + \int_{CS} \rho e (\vec{v} - \vec{v}_{CV}) \cdot \hat{n} \, dA = \dot{Q}_{\text{net,in}} + \dot{W}_{\text{shaft,in}} - \int P(\vec{v} - \vec{v}_{CV}) \cdot \hat{n} \, dA \qquad \text{(Conservation of Energy)}$$

$$\frac{P_{\text{out}}}{\rho} + \frac{1}{2} v_{\text{out}}^2 + g z_{\text{out}} = \frac{P_{\text{in}}}{\rho} + \frac{1}{2} v_{\text{in}}^2 + g z_{\text{in}} + w_{\text{shaft,in}} - \underbrace{(\hat{u}_{\text{out}} - \hat{u}_{\text{in}} - q_{\text{net,in}})}_{\text{losses}} \qquad \text{(Extended Bernoulli Equation)}$$

Differential Analysis, Rectangular Coordinates

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \xrightarrow{\text{if incompressible}} \nabla \cdot \vec{v} := \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(Continuity)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + g_x \qquad (x)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + g_y \qquad (y)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + g_z \qquad (z)$$

Differential Analysis, Polar Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(Incompressible Continuity)}$$

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r \qquad (r)$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] + \rho g_\theta \qquad (\theta)$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial \theta} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \qquad (z)$$

Contents

1 Thermodynamics

Notes from **MECHENG 235: Thermodynamics I**

Taken at University of Michigan, Fall 2024

using *Thermodynamics: an Engineering Approach*, by Boles and Cengel

1.1 Basic Concepts of Thermodynamics

Thermodynamics is the study of energy and its interaction with matter. In particular, *classical thermodynamics* is the study of energy and energy transfer at a macroscopic level. Thermodynamics is concerned with the **equilibrium states** of systems.

A system has properties which may be **extensive** (varying with system size, i.e. total volume and total momentum) or **intensive** (not variable with size, i.e. temperature and pressure). A system is in *equilibrium* when there are no forces driving change to the system. This equilibrium may be *thermal* (temperature is constant with time), *mechanical* (pressure is constant with time), or *chemical* (chemical composition does not progress with time, no net reaction). A system may be **closed**, with a control mass and potentially variable volume, or **open**, with a control volume and potentially variable mass. A system is **adiabatic** if no heat is transferred (i.e. the system is well-insulated, or system is the same temperature as surroundings).

The State Postulate. The state of a simple compressible system is completely specified by *two independent intensive properties*.

Other intensive properties of a system include **internal energy** u, and **enthalpy** h. Enthalpy is defined in terms of internal energy:
$$H = U + PV \qquad (kJ)$$
and specific enthalpy is similarly defined by

$$h = u + Pv \qquad (kJ/kg)$$

Internal energy is usually used in evaluating closed systems, while enthalpy may be used in evaluating open (flowing) systems.

Zeroth Law of Thermodynamics. If two bodies are in thermal equilibrium with a third body, they are also in thermal equilibrium with each other. If the third body is a thermometer, then the two bodies are in thermal equilibrium if they have the same temperature.

1.2 Energy and the First Law of Thermodynamics

Energy exists in several forms. There is **macroscopic energy** (kinetic, potential), **microscopic energy** (related to molecular structure and activity) and **internal energy,** U (sum of microscopic energies). In general, energy exists in either **static** or **dynamic** form:

1. **Static energy**: stored in the system (ex. energy in chemical bonds).

2. **Dynamic energy**: recognized only as it crosses a boundary (ex. heat, the flow of energy due to temperature difference at the boundaries; work, force applied to move a boundary)

The **First Law of Thermodynamics** is the conservation law for energy:

First Law of Thermodynamics. Energy is not created or destroyed during a process and may only change forms.

$$\underbrace{E_{in} - E_{out}}_{\text{macroscopic}} = \underbrace{\Delta E_{system}}_{\text{microscopic}}.$$

For a cycle ($\Delta E_{system} = 0$), the first law reduces to

$$Q_{net} - W_{net} = 0$$

1.3   Properties of Pure Substances

A **pure substance** is any substance with a fixed, uniform chemical composition. A pure substance may be comprised of one element (i.e. nitrogen gas) or more (i.e. water), or a mixture of several substances which is homogeneous (i.e. air). The **phase** of a substance is dependent on the average movement of its molecules (temperature). During a **phase change**, some portion of a sample of substance exists in either phase, depending on the properties of substance.

During a phase change from the liquid state to the gaseous state, a pure substance may be in any of the following phases:

1. **Compressed liquid**. The substance is a liquid which is not about to vaporize.

2. **Saturated liquid**. The substance is liquid, but any increase in temperature or decrease in pressure will cause it to partially vaporize.

3. **Saturated liquid-vapor mixture**. The substance is a mix of gas and liquid. Without change to pressure, the temperature of a saturated vapor is constant, with only the volume changing during the process.

4. **Saturated vapor**. The substance is a gas, but any decrease in temperature or increase in pressure will cause it to partially condense.

5. **Superheated vapor**. The substance is a gas which is not about to vaporize.

Pressure and temperature are dependent on each other during phase change, so another intensive property (generally specific volume, v) is required to describe the system. Plotting temperature or pressure against specific volume during a phase change gives the following:

(a) T-v diagram of a pure substance   (b) P-v diagram of a pure substance
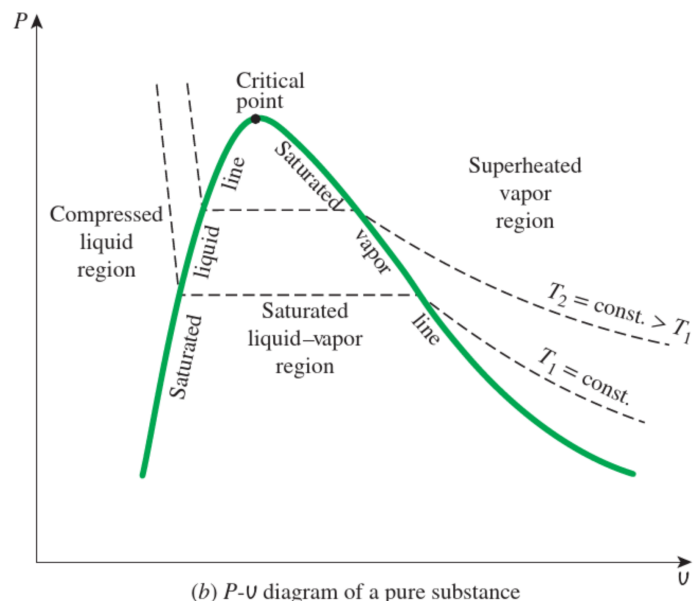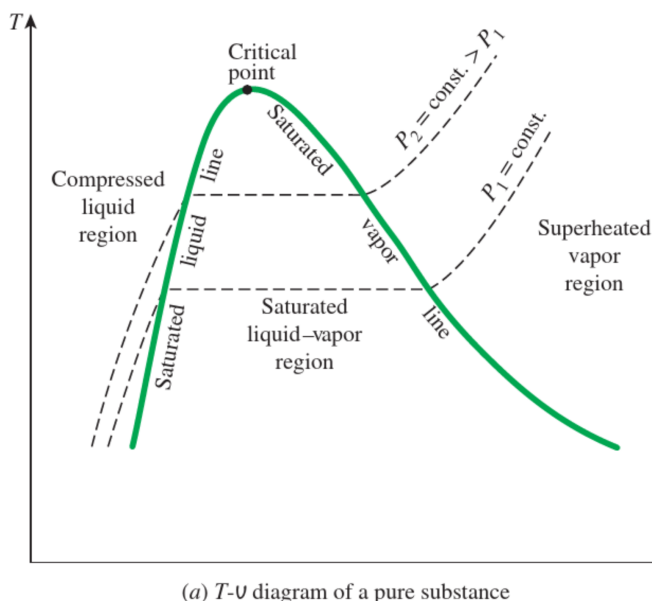
The isolines show constant pressures or temperatures respectively. Importantly, holding pressure or temperature constant under the dome necessarily holds the other quantity constant.

In general, an intensive property p (standing in for volume, enthalpy, internal energy, etc) of a system are related to pressure and temperature by:

Compressed liquid: $p \approx p_f$ at T. Properties of a liquid at fixed temperature vary negligibly.

Saturated liquid: $p = p_f$ from a property table.

Liquid-vapor mixture: $p = p_f + x(v_g - v_f)$, where x is the **quality** of the mixture, defined by the mass ratio of vapor to total mass:

$$x = \frac{m_{\text{vapor}}}{m_{\text{liquid}} + m_{\text{vapor}}}$$

Quality directly corresponds to a position along the horizontal portion of an isoline under the dome.

Saturated vapor: $p = p_g$ from a property table.

Superheated vapor: read from a property table for most values.

In absence of property tables, the **ideal gas law** sometimes provides a decent approximation for the behavior of gases. In particular, the ideal gas law is given by the equivalent relationships

$$Pv = RT \qquad \text{or} \qquad PV = mRT \qquad \text{or} \qquad PV = nR_uT$$

where R is the gas constant for a specific substance and R_u is the universal gas constant. Under certain conditions, a gas deviates from ideal behaviors. In this case, a **compressibility factor** $Z = v_{\text{real}}/v_{\text{ideal}}$ adjusts the ideal gas law:

$$Pv = ZRT$$

Z is generally read from a compressibility chart (where P_{cr} and T_{cr} are particular to a substance and read from a table):

NELSON — OBERT
GENERALIZED
COMPRESSIBILITY CHARTS

REDUCED PRESSURE, $P_R = \dfrac{P}{P_{\text{cr}}}$

REDUCED TEMPERATURE, $T_R = \dfrac{T}{T_{\text{cr}}}$

PSEUDO REDUCED VOLUME, $v_R = \dfrac{v}{RT_{\text{cr}}/P_{\text{cr}}}$

CHART No. 2

1953
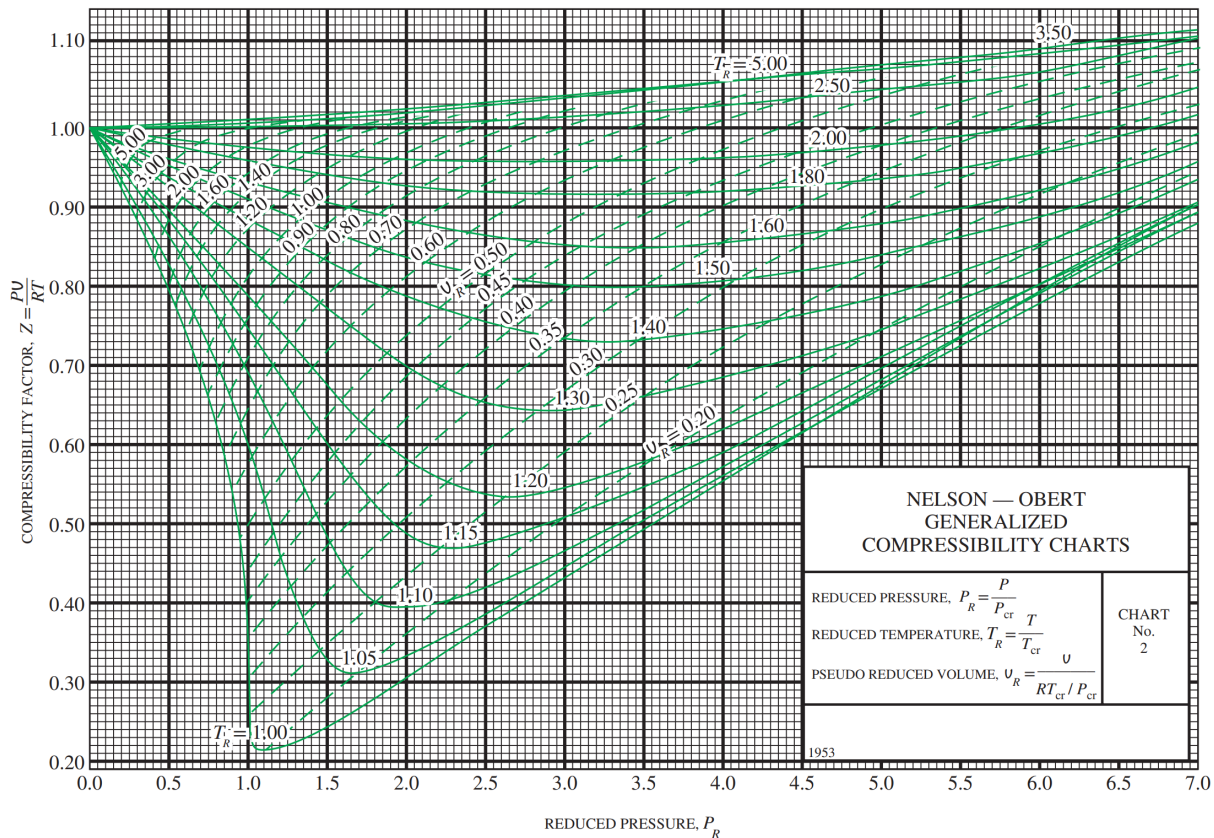
A gas is generally "ideal" (the ideal gas law is a reasonable approximation) under any of the following conditions:

1. Pressure is low: $P_R << 1$ (irrespective of temperature).

2. Temperature is high: $T_R > 2$ (irrespective of pressure).

3. $Z > 0.9$

1.4 Energy Analysis of Closed Systems

In a closed system, mass does not transfer across the system boundary. For these systems, energy analysis simply involves energy balance and an analysis of the temperature, pressure, and volume changes in the system.

Boundary work (also known as *moving boundary work*) involves the *compression* or *expansion* of a system boundary, such as in a piston. In the case of a piston, pressure remains constant because the weight of the piston head pushes on the gas in the piston with constant force and area, so only volume changes between states. From the definition of mechanical work, the boundary work done moving a piston (or any other boundary) is given by

$$\Delta W_B = F\Delta x = PA\Delta x = P\Delta V$$

Most generally, total boundary work W_B may be found by integration:

$$W_B = \int_{V_1}^{V_2} PdV$$

Using this definition, it is possible to derive an expression for the boundary work of various common processes:

1. **Boundary Work for Constant Volume (Isochoric) Process.** $\Delta V = 0$, so

$$W_{B,\ \text{const. } V} = 0$$

2. **Boundary Work for Constant Volume (Isobaric) Process.** P is a constant and not a function of V, so it may be pulled out of the integral:

$$W_{B,\ \text{const. } P} = P\int_{V_1}^{V_2} dV = P(V_2 - V_1)$$

3. **Boundary Work for Constant Temperature (Isothermal) Process.** Assuming ideal gas, PV is a constant. So, let $PV = C \implies W_B = \int_{V_1}^{V_2} \frac{C}{V}dV = C\ln\left(\frac{V_2}{V_1}\right)$. But $C = P_1V_1$ or $C = P_2V_2$ or $C = mRT$ because all three are equivalent given ideal gas with constant temperature, so

$$W_{B,\ \text{const. } T} = P_1V_1\ln\left(\frac{V_2}{V_1}\right) = P_2V_2\ln\left(\frac{V_2}{V_1}\right) = mRT\ln\left(\frac{V_2}{V_1}\right)$$

4. **Boundary Work for Polytropic Process.** A polytropic process has $PV^k = C$ for some constant k and C. Given $k = 1$ (ideal gas), this is simply the isothermal case. For $k \neq 1$, integration gives

$$W_{B,\ \text{polytropic}} = \frac{P_2V_2 - P_1V_1}{1-k} = \frac{mR(T_2 - T_1)}{1-k}$$

For a process, work is given visually by the area under the curve on a P-v diagram. For a process which is a cycle, net work W_{net} is given by the area enclosed by the process curves. Boundary work should be *negative for compression* (work is an input) and *positive for expansion* (work is an output).

Specific heat is a measure of the energy required to raise the temperature of a unit mass of a substance by one degree. In general, thermodynamics considers specific heat at *constant volume* c_v and specific heat at *constant pressure* c_p. Formally, specific heats are given by the following differentials:

$$c_v = \left(\frac{\partial u}{\partial T}\right)_{v=\text{const}} \qquad\qquad c_p = \left(\frac{\partial h}{\partial T}\right)_{P=\text{const}}$$

For ideal gases, both u and h are functions of temperature alone, so these differentials become exact:

$$\frac{du}{dT} = c_v(t) \qquad\qquad \frac{dh}{dT} = c_p(T)$$

Thus,

$$\Delta u = u_2 - u_1 = \int_{T_1}^{T_2} c_v(T)dT \qquad\qquad \Delta h = h_2 - h_1 = \int_{T_1}^{T_2} c_p(T)dT$$

where c_v and c_p are generally polynomials with exact equations available in a property table. For some temperature ranges, Δu and Δh may be approximated by

$$\Delta u \approx c_{v,avg}\Delta T, \qquad\qquad \Delta h \approx c_{p,avg}\Delta T$$

For incompressible substances (liquids and solids), pressures and specific volumes are constant, so $c_p = c_v = c$. Thus, change in internal energy for a liquid or solid is given by

$$\Delta u \approx c_{avg}\Delta T$$

1.5   Energy Analysis of Open Systems (Control Volumes)

In an open system, a fixed control volume is considered where both mass and energy may be transferred across the system boundary. Like energy, mass is a conserved quantity with a conservation law.

Conservation of mass. Given a constant volume,

$$m_{\text{in,net}} - m_{\text{out,net}} = \Delta m \qquad\qquad \text{kg}$$

In (differential) rate form,

$$\dot{m}_{\text{in,net}} - \dot{m}_{\text{out,net}} = \frac{dm}{dt} \qquad\qquad \text{kg/s}$$

If mass changes with time, there is some **mass flow rate**, given by

$$\dot{m} = \rho\vec{v}_{avg}A = \rho\dot{V} = \frac{\dot{V}}{v}$$

Here, $\vec{v}_{avg}$ is the magnitude of the average velocity, v is specific volume, and A is the cross-sectional area through which the mass is flowing. The **volume flow rate** is then given by

$$\dot{V} = \vec{v}_{avg}A$$

Many open systems, such as power plants, operate in **steady flow** (i.e. mass flow rate does not vary considerably with time). In this case, it is assumed that $\Delta\dot{m} = 0$, yielding $\dot{m}_{\text{in,net}} = \dot{m}_{\text{out,net}}$

For an open system, the general energy balance is given by

$$\dot{Q}_{\text{in}} + \dot{W}_{\text{in}} + \sum_{\text{in}} \dot{m}(h + \vec{v}^2/2 + gz) = \dot{Q}_{\text{out}} + \dot{W}_{\text{out}} + \sum_{\text{out}} \dot{m}(h + \vec{v}^2/2 + gz)$$

For a particular system, many of these quantities are negligible. In particular, kinetic and potential energies are neglected in many systems, and insulated systems may take $\dot{Q}_{\text{in}} = \dot{Q}_{\text{out}} = 0$. Similar analysis can be used to reach an energy balance for any system.

1.6 Entropy

The Second Law of Thermodynamics describes the direction of spontaneous change for a system. In particular, the Second Law relates to processes operating between two different **thermal reservoirs** (external bodies, large enough that their temperature is roughly constant; i.e. the atmosphere, rivers, etc). Some cyclic devices of interest (and their ideal, reversible (Carnot) counterparts) are outlined:

Heat engine (i.e. a steam engine): cyclic process where heat is taken from a high temperature source, converted partially into work, and the rest rejected into a low-temperature sink. Thermal efficiency given by

$$\eta_{\text{thermal}} = \frac{W_{\text{net,out}}}{Q_{\text{out}}} = \frac{Q_{\text{in}} - Q_{\text{out}}}{Q_{\text{in}}} \longrightarrow \eta_{\text{Carnot}} = \frac{T_H - T_L}{T_H}$$

Construction of a heat engine is mechanically limited by the Kelvin-Planck Statement:

Kelvin-Planck Statement:

It is impossible for any device that operates on a cycle to receive heat from a single reservoir and produce a net amount of work.

Refrigerators: cyclic process where work is added to a system to remove heat. Thermal efficiency is given by a coefficient of performance:

$$\text{COP}_R = \frac{Q_{\text{Low}}}{W_{\text{net,in}}} = \frac{Q_{\text{Low}}}{Q_{\text{High}} - Q_{\text{Low}}} \longrightarrow \eta_{\text{Carnot}} = \frac{T_L}{T_H - T_L}$$

Construction of a refrigerator is limited by the Clausius Statement:

Clausius Statement: Is it impossible to construct a device that operates in a cycle and produces no effect other than the transfer of heat from a lower-temperature body to a higher-temperature body.

Heat Pumps: cyclic process where work is added to a system to generate heat. Thermal efficiency is given by a coefficient of performance:

$$\text{COP}_{HP} = \frac{Q_{\text{High}}}{W_{\text{net,in}}} = \frac{Q_{\text{High}}}{Q_{\text{High}} - Q_{\text{Low}}} \longrightarrow \eta_{\text{Carnot}} = \frac{T_H}{T_H - T_L}$$

A process is **irreversible** if the process cannot be spontaneously undone. Irreversibility may be introduced in real processes by friction, electrical resistance, mixing of fluids, etc.

Carnot Principle. The efficiency of an irreversible process is always less than the efficiency of a reversible one (Carnot) operating between the same two reservoirs. Carnot efficiency is upper bound.

Entropy is introduced as a measure of irreversibility. Change in entropy for a (cyclic) process is given by

$$\Delta S = S_2 - S_1 = \oint \frac{dQ}{T}$$

where

$$\oint \frac{dQ}{T} \leq 0 \text{ always } (\textbf{Clausius Inequality})$$

The Second Law of Thermodynamics emerges from this definition of entropy:

Second Law of Thermodynamics. Entropy cannot decrease during a process; entropy generated internally in a process is either zero or positive.

$$S_{gen} \geq 0$$

where

$$S_{\text{gen}} \begin{cases} > 0 \implies \text{irreversible process} \\ = 0 \implies \text{reversible process} \\ < 0 \implies \text{impossible process} \end{cases}$$

For a process, entropy balance is given by

$$S_{\text{in}} - S_{\text{out}} + S_{\text{gen}} = \Delta S_{\text{system}}$$

Entropy change is related to heat generation. Rearranging the differential form of the Clausius Inequality gives:

$$\frac{dQ}{T} = dS \implies dQ = TdS$$

$$\implies Q = \int_{T_1}^{T_2} TdS \quad \text{(area under } T\text{-}s \text{ graph)}$$

For mechanical processes, entropy is generally changed through the addition of heat. At a microscopic level (in statistical thermodynamics), entropy increases as the "disorder" of a system increases. The following may increase disorder in a system:

Change of volume: molecules have more space to move, causing more disorder (i.e. expansion of a gas).

Change of composition: Some compositions are more "chaotic" than others (i.e. a crystalline structure is less chaotic than a gaseous state)

Change of temperature: temperature is a measure of average molecule velocity. Higher temperatures mean higher velocities and therefore greater disorder.

The Boltzmann Postulate gives an exact value for entropy relying on statistical mechanics:

Boltzmann Postulate (statistical thermodynamics).

$$S = k \ln W$$

where W is the number of microstates and k is the Boltzmann constant.

Because entropy is defined in terms of microstates, entropy is an **intensive property** of a system. The limiting case of the Boltzmann Postulate, with $W = 1$, sets the reference point for zero entropy and comprises the Third Law of Thermodynamics:

Third Law of Thermodynamics A pure crystalline substance at absolute zero temperature is in perfect order, and its entropy is zero.

For an ideal gas, change in entropy is given by

$$\Delta S_{\text{ideal gas}} = \underbrace{c_v \ln(T_2/T_1) + R \ln(V_2/V_1)}_{c_v \text{ at average temperature}} = \underbrace{c_p \ln(T_2/T_1) - R \ln(P_2/P_1)}_{c_p \text{ at average temperature}}$$

And for a solid or liquid, change in entropy is given by

$$\Delta S = c_{avg} \ln(T_2/T_1)$$

For an ideal gas, isentropic processes are polytropic with $k = c_p/c_v$. The equations for an isentropic process are given:

$$\left(\frac{T_2}{T_1}\right)_{s=\text{const}} = \left(\frac{v_1}{v_2}\right)^{k-1}$$

$$\left(\frac{T_2}{T_1}\right)_{s=\text{const}} = \left(\frac{P_2}{P_1}\right)^{(k-1)/k}$$

$$\left(\frac{P_2}{P_1}\right)_{s=\text{const}} = \left(\frac{v_1}{v_2}\right)^{k}$$

Isentropic efficiency is a measure of how far a real process deviates from its ideal efficiency. Isentropic efficiencies for various common devices are given outlined:

$$\eta_{\text{turbine}} = \frac{W, \text{ actual turbine}}{W, \text{ isentropic turbine}} = \frac{W_a}{W_s} \approx \frac{h_1 - h_{2a}}{h_1 - h_{2s}}$$

$$\eta_{compressor} = \frac{\text{isentropic compressor work } W_s}{\text{actual compressor work } W_a} = \frac{h_{2s} - h_1}{h_{2a} - h_1} = \underbrace{\frac{V(P_2 - P_1)}{h_{2a} - h_1}}_{\text{Pumps only (liquids)}}$$

1.7   Power Cycles

Many important power-producing devices operate on a cycle. Two distinct types of cycles, relevant to everyday energy production, are of interest: gas power cycles, as is typical in engines, and vapor power cycles, as is typical in power plants.

For every cycle, the energy balance reduces to $Q_{\text{net}} - W_{\text{net}} = 0$, or $Q_{\text{net}} = W_{\text{net}}$. Thus, it is possible to analyze the net output of a cycle either using heat (on a T-s diagram) or work (on a P-v diagram). In each cycle analyzed, thermal efficiency is given as a ratio of *net work* to *heat in*:

$$\eta = \frac{W_{\text{net}}}{Q_{\text{in}}}$$

The **gas power cycles** listed here operate on the **air-standard assumptions**, which allow the processes to be simplified for analysis:

Air-Standard Assumptions.

1. The working fluid is air, which circulates through the system in a closed loop as an ideal gas.

2. All processes in the cycle are internally reversible.

3. Combustion processes are replaced by heat-addition from an external source.

4. Exhaust processes are replaced by heat-rejection which returns the air to its initial state.

Air-standard assumptions are used in analysis of engines. Several cycles are listed below:

Otto Cycle. Gasoline engines operate on the Otto cycle using an air-fuel mixture, generally approximated as air for the purposes of analysis. The Otto cycle has four distinct steps:

(a) Compression Stroke (replaced by isentropic compression)
(b) Power (expansion) Stroke (replaced by constant volume heat addition)
(c) Exhaust Stroke (replaced by isentropic expansion)
(d) Intake Stroke (replaced by constant volume heat rejection)
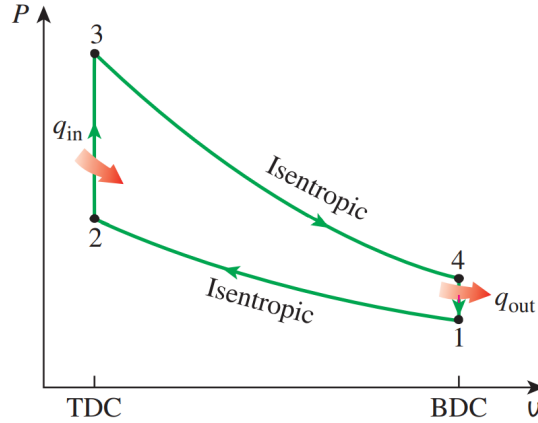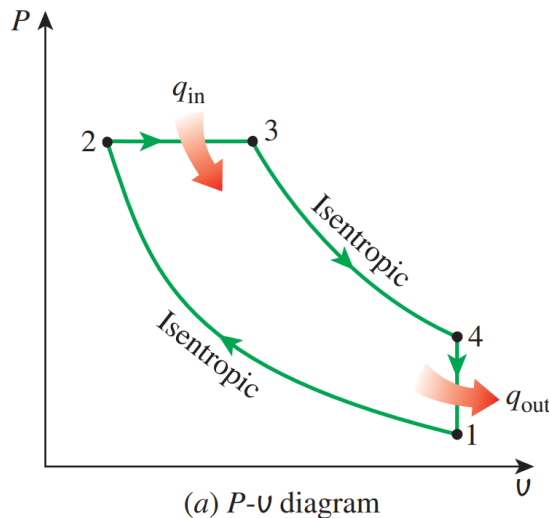

Efficiency for an Otto cycle is given by

$$\eta = 1 - \frac{1}{r^{k-1}}, \qquad r := \frac{V_{\max}}{V_{\min}}, \quad k := c_p/c_v$$

The Otto cycle is limited by *autoignition*, or *engine knock*. For large compression ratios r (around 12), pockets of fuel-air mixture have a tendency to ignite before the power stroke, causing an effective reduction in volume ratio and reducing efficiency.

Diesel Cycle. Diesel engines operate on the Diesel cycle, where air is compressed to high temperatures before pure Diesel fuel is injected and combusted due to the high pressure and high temperature. The Diesel cycle also have four distinct steps:

(a) Compression Stroke (replaced by isentropic compression)
(b) Combustion (fuel injection) Stroke (replaced by constant pressure heat addition)
(c) Exhaust Stroke (replaced by isentropic expansion)
(d) Intake Stroke (replaced by constant volume heat rejection)


(a) P-υ diagram

Efficiency for a Diesel cycle is given by

$$\eta = 1 - \frac{1}{r^{k-1}} \left[\frac{r_c^k - 1}{k(r_c - 1)} \right], \qquad r_c := \frac{V_3}{V_2}$$

The Diesel cycle always has a lower efficiency as the same compression ratio, but because combustion is initiated by fuel injection, autoignition is not a concern in Diesel engines and Diesel engines can reach higher raw efficiencies than an engine operating on the Otto cycle.

Brayton Cycle. Gas-turbine engines, such as those used in propulsion aircraft, generally operate on a Brayton cycle. A Brayton Cycle uses four devices, and the cycle has four distinct steps:

(a) Compressor (isentropic compression)

(b) Heat Exchanger (constant-pressure heat addition)

(c) Turbine (isentropic expansion)

(d) Heat Exchanger (constant-pressure heat rejection)

(a) T-s diagram
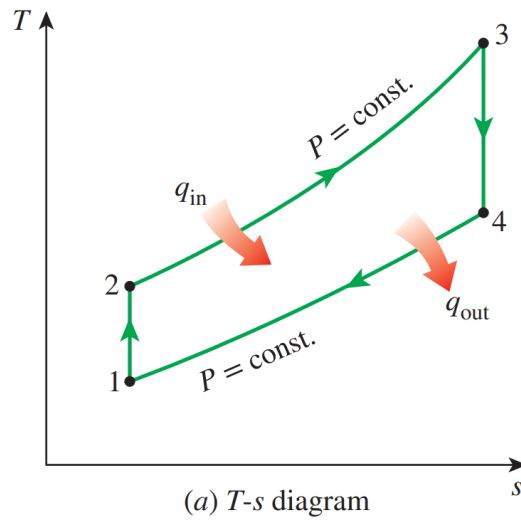
Efficiency for a Brayton Cycle is given by

$$\eta = 1 - \frac{1}{r_p^{(k-1)/k}}, \qquad r_p = \frac{P_2}{P_1}$$

Typical pressure ratios r_p for a Brayton cycle are between 5 and 20.

Many power plants operate on some form of vapor power cycle, utilizing the phase change of water into steam to produce useful work. The Rankine Cycle is a simple, common cycle utilizing water.

Rankine Cycle. An ideal Rankine cycle utilizes four devices and water in both liquid and vapor form to produce power:

(a) Pump (isentropic compression of liquid, initially saturated liquid)

(b) Boiler (constant pressure heat addition to superheated vapor)

(c) Turbine (isentropic expansion)

(d) Condenser (constant-pressure heat rejection)
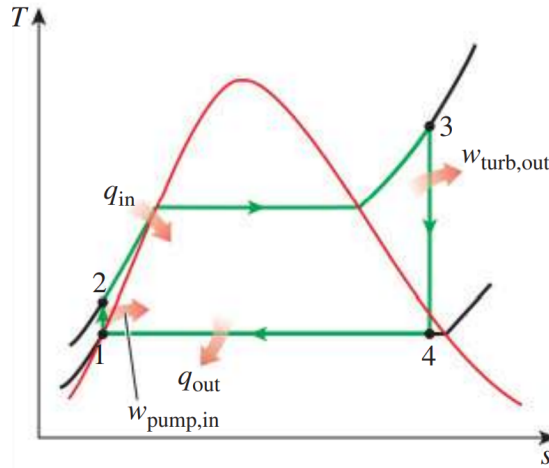
The efficiency of a Rankine Cycle is given by

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}$$

The efficiency of a Rankine cycle may be improved by adding regeneration, diverting some warm water to effectively reduce Q_{in}.

For power cycles, the **backwork ratio** is another measure related to efficiency. Backwork ratio is given by

$$\text{BWR} = \frac{W_{\text{in}}}{W_{\text{out}}}$$

Backwork ratio measures what fraction of the output work must be used to power the cycle (in raising the piston, powering the compressor, etc).

2   Statics and Mechanics of Materials

Notes from **MECHENG 211: Introduction to Solid Mechanics**

Taken at University of Michigan, Winter 2025

2.1   Static Equilibrium

A force $\vec{F}$ creates a **moment** about a point if it is located a distance $\vec{r}$ from the line of action of the $\vec{F}$. The moment $\vec{M}$ of a force is perpendicular to both $\vec{F}$ and $\vec{r}$, so in two dimensions, $\vec{M}$ necessarily points into or out of the page and is given by

$$M = Fr$$

In three dimensions, moments are computed by a cross product. For a moment about a point A produced by a force acting at point B, the moment $\vec{M}_A$ is given by

$$\vec{M}_A = \vec{r}_{AB} \times \vec{F} = (\vec{r}_B - \vec{r}_A) \times \vec{F}$$

A body is in **static equilibrium** if it does not translate or rotate. For this to be the case, the vector sum of the forces acting on the body must be zero, and the vector sum of the moments about any point on the body must also be zero.

At each *support* contacting a body, the body produces a *reaction force*. For a body to be fully constrained, there must be as many reaction forces as *degrees of freedom*. In two dimensions, there are *three degrees of freedom* (x and y translation, and a moment). In three dimensions, there are *six degrees of freedom* (translation in x y, z, and moments in x, y, and z).

Most supports fall into one of the following categories:

Fixed support. Sometimes called a built-in support. Constrains motion and rotation in all directions and therefore produces reaction forces and reaction moments in all directions.

Pin support. Constrains translational motion while allowing free rotation, therefore producing reaction forces in the horizontal and vertical directions. Pin supports may often be used to approximate joints in structures like beams and trusses.

Simple support. Prevents motion normal to the point of contact, producing a single reaction force. This joint is mathematically equivalent to a roller.

When bodies come into contact through *compressive forces*, a frictional force F opposes motion. The magnitude of frictional force is generally given by the inequality $|F| < \mu N$. *Incipient slip* refers to the force required to begin motion of a body, occurring when the frictional forces have just barely been overcome.

2.2   Composite Structures

It is possible to analyze a complicated, multi-element structure by breaking it into its components and supports. Such analysis involves drawing a free body diagram for each part, with external forces and reaction forces (and their 3rd Law reaction) listed at their appropriate positions and supports. Because the entire structure is in static equilibrium, each component must also be in static equilibrium and the forces on each component must thus satisfy the static equilibrium conditions.

A *two-force member* is any component loaded at only two points, such as a support member in a truss. In this case,

$$\vec{F}_A + \vec{F}_B = \vec{0}, \qquad \vec{r}_{AB} \times \vec{F}_B = \vec{0}$$

so $\vec{F}_A = -\vec{F}_B$ and thus forces act *along a single line* through the component. If the forces point outwards, the member is in **tension**; if the forces point inwards, the member is in **compression**.

There are two main ways to analyze the internal forces in a determinate structure, such as a truss:

Method of Pins. Most useful for solving systems where many or all internal forces are to be calculated. Steps:

1. Separate structure into separate two-force members.
2. Draw free-body diagrams for each two-force member, approximating joints as pins with reaction forces pointing away (in tension, which is positive by convention).
3. Solve the force balance for the desired internal forces. This involves (at most) an x and y force balance for each member.

Method of Sections. Most useful for solving systems where only some internal forces need to be known. Steps:

1. Cut a section through the body, cutting through at most three elements with unknown internal forces. The internal forces from cut members become external forces for the remaining section.
2. Draw a free-body diagram for the section in question, with unknown forces pointing away from cut members (in tension, which is positive by convention).
3. Solve the force and moment balance for the section. Since the entire body is in equilibrium, the section must also be in equilibrium. This involves setting net forces in x and y and net moment equal to zero.

Each section of a loaded body must experience internal reactions which balance the external forces and moments. If one were to make a cut through any section of the body, these reactions would be exactly what is required to support either section of the cut.

By convention, the internal reactions include a normal force N acting perpendicular to the cut, a shear force V acting parallel to the cut, and a moment M. The internal reactions have opposite orientations on opposite sides of the cut.

In three dimensions, there is a single normal vector $\vec{N}$, two directional components to $\vec{V}$ (i.e. x and y), and three rotational components (a bending moment $\vec{M}$ about the x and y axis, and a twisting torque T about the z axis).

2.3 Stress and Strain

In general, **stress** refers to the force acting over an area on a body (i.e. a "pressure" over a surface), and **strain** refers to the deformation of the body. Normal stress is denoted σ and shear stress is denoted τ.

Given a normal force N acting uniformly on an area A, the normal stress at the section is given by

$$\sigma = \frac{N}{A}$$

Moments may also impart a normal stress. In particular, for a bar with cross-sectional area A and area moment of inertia I_x, there is an additional component of stress given by

$$\sigma_{zz} = \frac{M_x y}{I_x} - \frac{M_y x}{I_y}$$

where x and y are taken from the centroid $c = (\bar{x}, \bar{y})$. Similarly, given a uniform shear force V, the (average) shear stress is given by

$$\tau = \frac{V}{A}$$

and torque on a bar can also produce additional shear stress:

$$\tau = \frac{Tr}{J}$$

Stress acting on a body produces a **strain**, or change in the dimensions of an object. When a bar is loaded with a tensile force, it may experience some extension from its initial length L_0, given by $\delta = L - L_0$. *Normal strain ε, or extensional strain*, is defined as the ratio of extension and initial length, given by

$$\varepsilon = \frac{\delta}{L_0}$$

Shear strain acting on a body is simply computed as the total change of angle ϕ_1 with respect to the horizontal and ϕ_2 with respect to the vertical:

$$\gamma = \phi_1 + \phi_2$$

Hooke's Law may be generalized to any material, and directly relates stress and strain.

Multiaxial Hooke's Law. For a material which is elastically (linearly) deformed, the strains on a material relate directly to the stresses acting on the material by Hooke's Law:

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E} + \alpha \Delta T$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu \sigma_x}{E} - \frac{\nu \sigma_z}{E} + \alpha \Delta T$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu \sigma_x}{E} - \frac{\nu \sigma_y}{E} + \alpha \Delta T$$

Here, E is Young's Modulus, and α is the coefficient of thermal expansion. Both are specific to a material. ν is **Poisson's Ratio**, the ratio by which compression in one direction results in extension in another, and is related to E and G by the relationship

$$\nu = -\frac{\epsilon_{\text{lateral}}}{\epsilon_{\text{axial}}} = \frac{E}{2G} - 1$$

More concisely,

$$E = 2G(1 + \nu)$$

If a problem is *statically indeterminate* (more unknowns than equations), another equation may be found by considering the extension or torsion in the body. In particular, for a beam built in at both ends, extension of the beam must be zero (both ends are fixed), giving a relation between forces. The same applies for a bar in torsion, with a relation between torques.

Extension is associated with normal strain. For a bar with area A, Young's Modulus E, and initial length L_0, the extension is given by:

$$\delta = \frac{FL_0}{AE}$$

A bar loaded with a torque T has an analogous relationship to the extension of a bar. The angle of torsion, ϕ, is given by

$$\phi = \frac{TL}{JG}$$

where G is the shear modulus for a material and J is the *second moment of inertia*. For a cylindrical bar in torsion, the relation between torque, shear, and angle of torsion is as follows:

$$\frac{T}{J} = \frac{\tau_z}{r} = G\frac{\phi_B - \phi_A}{L_{AB}}$$

Thin-walled pressure vessels have an internal pressure P (i.e. due to an internal gas or fluid) and a wall of thickness t, where t is negligible compared to radius r.

Cylindrical pressure vessels. Internal pressure P with a cap on either end. Stresses are given by

$$\underbrace{\sigma_\theta = \frac{Pr}{t}}_{\text{hoop stress, tangent to walls}} \quad , \qquad \underbrace{\sigma_{\text{axial}} = \frac{Pr}{2t}}_{\text{longitudinal stress, normal to cap}}$$

Spherical pressure vessels. Internal pressure P with symmetric hemispheres. Stresses are given by

$$\underbrace{\sigma_\theta = \sigma_\phi = \frac{Pr}{2t}}_{\text{tangent to walls}}$$

Superposition allows the addition of normal and shear stresses. Stresses emerge from loads applied to a body. For a coordinate system oriented (facing the surface) with z normal to the surface, y up, and x left:

Normal stress. May be imparted due to a moment or normal force.

$$\sigma_{zz} = \frac{N}{A} + \frac{M_x y}{I_x} - \frac{M_y x}{I_y}$$

Shear stress. May be imparted due to a torque or a shear force.

$$\tau_{zy} = \frac{V_y}{A}, \ \tau_{zx} = \frac{V_x}{A}, \qquad \underbrace{\tau_{z\theta} = \frac{Tr}{J}}_{\text{may be in } x \text{ or } y}$$

Note that other forces may be present, i.e. hoop stress or longitudinal stress if a pressure vessel is contributing support. It is necessary in such cases to identify the coordinate directions of any such stresses.

2.4   Transformation of Stress and Strain

For any element experiencing stresses, there exists a coordinate transformation to a *principal axis* where all stress is normal stress and another transformation where all stress is shear stress. These axes are normal to each other. Fracture is most likely to occur along one of these axes, depending on the nature of the material.

In general, if an element experiences coordinate stresses σ_x, σ_y, and τ_{xy}, then the stress in a new axis an orientation θ away is given by

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

The stresses along the *principal axis* may be found from these equations:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

where $\sigma_1 > \sigma_2$. The maximum shear stress in the plane is given by
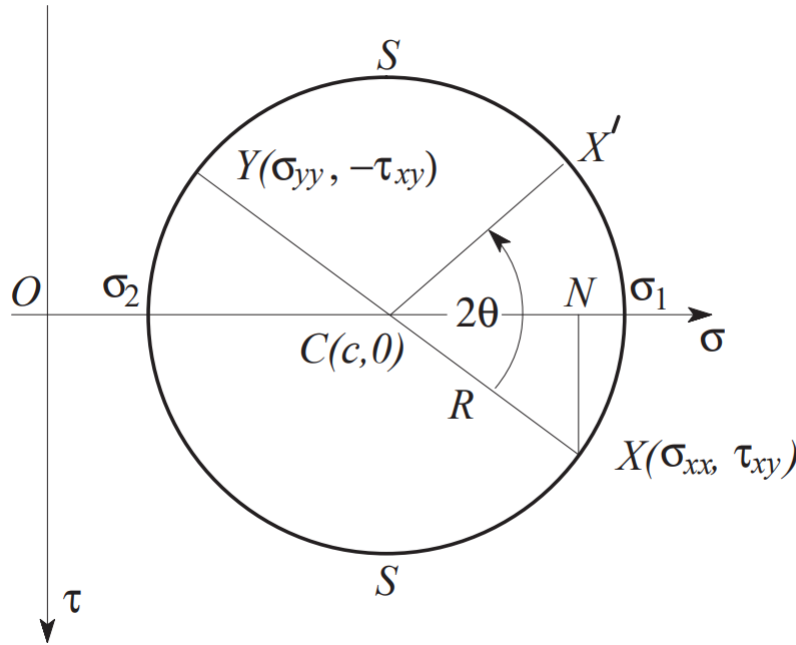
$$\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Mohr's circle is a visual representation which may be constructed to visualize planar stress.

Construction of Mohr's Circle. Mohr's circle may be constructed by the following steps:

1. Create coordinate axes, by convention positive stress to the right and positive shear to the bottom.

2. Plot the center of the circle at $\frac{\sigma_x + \sigma_y}{2}$.

3. Draw the circle with radius $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

4. Optionally, plot the reference stresses (σ_x, τ_{xy}) as a reference point $\theta = 0$. The angle θ_p to the principal axes may be determined by trigonometry.

Mohr's circle is drawn graphically as so:

Since stress and strain are related by Hooke's Law, a Mohr's Circle may also be constructed for strain. The transformation equations are analogous to the stress transformations:

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos(2\theta) + \frac{\gamma_{xy}}{2}\sin(2\theta)$$

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos(2\theta) - \frac{\gamma_{xy}}{2}\sin(2\theta)$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\epsilon_x - \epsilon_y}{2}\sin(2\theta) + \frac{\gamma_{xy}}{2}\cos(2\theta)$$

Principle strains also have an analogous transformations, and thus Mohr's circle is constructed in effectively the same manner for strains as with stresses:

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \underbrace{\sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}}_{\gamma_{max}/2}$$

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It is important to note that, for planar stresses, $\sigma_3 = \sigma_z = 0$, but this is *not the case* with planar strains, as a strain in one direction also causes strain in the others.

By using a *strain gauge*, which measures electrical resistance (and thus strain, as electrical resistance increases with strain), the strain in a particular direction is related as follows:

$$\epsilon_a = \epsilon_x \cos^2 \theta_a + \epsilon_y \sin^2 \theta_a + \gamma_{xy} \sin \theta_a \cos \theta_a$$

Here, ϵ_a is read by the gauge and is thus known, and θ_a is known by the position of the strain gauge, so there are three unknowns. Thus, a strain gauge *rosette*, with at least three strain gauges in unique directions, is required to find all the principal strains.

2.5   Beam Deflection

A loaded beam experiences some deflection, $u(z)$, as a function of the distance z along the beam. For relatively small deflections, the following relationship holds:

$$\frac{M(z)}{EI} = \frac{d\theta}{dz} = \frac{d^2 u}{dz^2}$$

EI is the rigidity of the beam, and M may vary as a function of position z. Integration gives $\theta(z)$ and $u(z)$ with constants of integration. These constants may be determined by evaluating θ or u at supports and considering the boundary conditions of each support. Common boundary conditions, for support at arbitrary position $z = L_0$, are outlined:

Built-in support. $u(L_0) = 0, \ \theta(L_0) = 0$

Simple support. $u(L_0) = 0, \ M(L_0) = 0$

Free end. $V(L_0) = 0, \ M(L_0) = 0$

3 Dynamics and Vibrations

Notes from **MECHENG 240: Introduction to Dynamics and Vibrations**

Taken at University of Michigan, Winter 2025

using *Dynamics*, by Meriam and Kraige

3.1 Particle Kinematics

Kinematics is the description of motion without reference to forces acting on an object. Particle kinematics is concerned with the movement of bodies which are physically small compared to their paths.

For a particle a distance $s(t)$ from a fixed point in direction of unit vector $\hat{e}$, position may be defined by

$$\vec{r}(t) = s(t)\hat{e}$$

Differentiation gives the general relationship for velocity:

$$\vec{v}(t) = \vec{r}'(t) = s'(t)\hat{e} + s(t)\frac{d\hat{e}}{dt}$$

In the case of *rectilinear motion* (motion along a line), $\hat{e}$ is constant with respect to time and thus its derivative is zero. Further, there is only one vector direction to consider, so rectilinear motion may be generally reduced to scalar quantities with a positive and negative direction, giving the following relationships:

$$v = \frac{ds}{dt} = \dot{s} \qquad a = \frac{dv}{dt} = \dot{v} \qquad a = \frac{d^2s}{dt^2} = \ddot{s}$$

Additionally, in the case of rectilinear motion, time may be eliminated by applying the chain rule:

$$a = v\frac{dv}{ds} \iff v\,dv = a\,ds$$

For the special case of *constant acceleration*, integration of these equations always yields the following direct relationships between displacement, velocity, and acceleration:

Uniformly Accelerated Motion. For a particle experiencing *constant acceleration*, the following relationships hold:

$$v = v_0 + at$$

$$v^2 = v_0^2 + 2a\Delta s$$

$$s = s_0 + v_0 t + \frac{1}{2}at^2$$

For motion in higher dimensions (i.e. in a plane or a space), the differential relationship for vectors continues to hold. In particular, for a particle a distance r from the origin,

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

Most dynamics problems may be greatly simplified when approached in an appropriate coordinate system. Three useful *orthonormal coordinate systems* are outlined:

Rectangular coordinates. Uses fixed unit vectors in each coordinate direction, generally $\hat{i}$, $\hat{j}$, $\hat{k}$. Requires a fixed (non-inertial) reference point for an origin. Position is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

By differentiation, velocity is given by

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

And by differentiation again, acceleration is given by

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

Normal and Tangential coordinates. Uses two moving vectors to describe motion of a particle: $\hat{e}_t$ tangential to travel, and $\hat{e}_n$ in the direction of the circle which best fits the path. Generally most applicable when the problem may be simplified at each instant to a planar problem. If necessary, a basis vector $\hat{k}$ may be defined perpendicular to both $\hat{e}_t$ and $\hat{e}_n$. Velocity is given by

$$\vec{v}(t) = v\hat{e}_t$$

and acceleration is given by

$$\vec{a}(t) = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

where ρ is the radius of curvature of the path at that moment in time.

Polar and Cylindrical coordinates. Uses a unit vector $\hat{e}_r$ pointing radially from the origin, a vector $\hat{e}_\theta$ pointing in the direction of rotation, and a fixed vector $\hat{k}$ about which the particle rotates. In two dimensions, $\hat{k}$ is always zero and thus can be ignored. Position is given by

$$\vec{r} = r\hat{e}_r + z\hat{k}$$

By differentiation, velocity is given by

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

And by differentiation again, acceleration is given by

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$$

The vectors $\vec{r}$, $\vec{v}$, and $\vec{a}$, and any other vectors, are *equivalent* in any coordinate system. By using a change of basis, any basis vector for a particular coordinate system may be decomposed into components and converted into another coordinate system, i.e. by a rotation matrix or trigonometric analysis.

3.2 Particle Kinetics

Kinetics is the study of motion caused by unbalanced forces. Newton's Second Law is a description of motion (acceleration), weighted by a property of the object in motion (inertia; or, mass):

$$\vec{F} = m\vec{a} \qquad \text{(Newton's 2nd Law)}$$

For a body acted upon by several forces, the vector sum of the forces acting on an object gives the net force and thus net acceleration:

$$\sum \vec{F} = \vec{F}_{\text{net}} = m\vec{a}$$

Some common forces are as follows:

Combined with a kinematic analysis, kinetics relates the forces acting on an object to its path of travel, where either one may be found from the other. For Newton's 2nd Law to apply, the frame of reference *must be inertial (non-accelerating, and/or stationary)*.

For some analyses, it may be convenient to consider only the initial and final states of some object. This analysis is thus time-independent, and may be carried out with an energy analysis. In particular, integrating Newton's 2nd Law with respect to position gives a useful result:

$$\underbrace{\int_{\vec{R}_1}^{\vec{R}_2} \vec{F} \cdot d\vec{R}}_{\text{Work, } W_{1-2}} = \int_{\vec{R}_1}^{\vec{R}_2} m\vec{a} \cdot d\vec{R}$$

$$= \int_{\vec{R}_1}^{\vec{R}_2} m(\dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n) \cdot ds\hat{e}_t$$

$$= m\int_{\vec{R}_1}^{\vec{R}_2} \dot{v}\hat{e}_t \cdot ds\hat{e}_t$$

$$= m\int_{s_1}^{s_2} a\,ds$$

$$= m\int_{v_1}^{v_2} v\,dv$$

$$= \underbrace{\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}_{\text{Kinetic energy}}$$

This result is known as the **work-energy theorem**:

Work-Energy Theorem. The net work on an object, defined by

$$U_{1-2} := \int_{\vec{R}_1}^{\vec{R}_2} \vec{F} \cdot d\vec{R}$$

is equal to the difference in kinetic energy between positions $\vec{R}_1$ and $\vec{R}_2$:

$$U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Often, $d\vec{R}$ is defined with respect to a Cartesian coordinate system, thus giving $d\vec{R} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, but $d\vec{R}$ may also be defined in another coordinate system, such as polar-cylindrical.

Energy analysis is often simplified when the forces acting on a body are **conservative forces**. A conservative force is any force where work done on a body depends only on the initial and final states and is *independent of the path of travel*, as is the case with spring and gravitational forces. If a force $\vec{F}$ is conservative, it may be defined to have a *potential energy*, where the work done by said force is defined as $U_F = -(V_2 - V_1)$. Potential energies for common forces are outlined:

1. **Gravitational Potential Energy**. For a gravitational force $\vec{F}_g = mg$ at the Earth's surface, potential energy is given by

$$V = mgz$$

2. **Spring Potential Energy**. For a Hookean spring force $\vec{F}_s = -k\Delta s$, potential energy is given by

$$V = \frac{1}{2}ks^2$$

Given this, the work-energy theorem may be broken into work by conservative and nonconservative forces:

$$\underbrace{-(V_2 - V_1)}_{U_{\text{conservative}}} + U_{\text{nonconservative}} = T_2 - T_1 \iff U_{\text{nonconservative}} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 + V_2 - V_1$$

Momentum methods also involve analysis of initial and final states, in this case integrating with respect to time. In particular,

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\implies \underbrace{\int_{t_1}^{t_2} \vec{F}dt}_{\text{impulse}} = \int_{\vec{v}_1}^{\vec{v}_2} md\vec{v}$$

$$= \underbrace{m\vec{v}_2 - m\vec{v}_1}_{\text{linear momentum}}$$

As opposed to work, $m\vec{v}$ is a vector quantity which relates to velocity rather than speed.

If a system of two particles is considered in a momentum analysis, then any forces exerted from particle A onto particle B and vice versa are internal, and thus cancel in integration. This makes momentum analysis particularly suited to collision problems, as internal forces at the moment of collision can be neglected. A collision between two particles can be further simplified if the collision is assumed to take place over an instant. Given this, consider two particles A and B which undergo a collision. Their momentum balance is given by:

$$\underbrace{\int_0^{\Delta t} (\vec{F}_A + \vec{F}_B)dt}_{\longrightarrow 0 \text{ as } \Delta t \to 0} = \underbrace{(m_A\vec{v}'_A + m_B\vec{v}'_B)}_{\text{after collision}} - \underbrace{(m_A\vec{v}_A + m_B\vec{v}_B)}_{\text{before collision}}$$

$$\implies (m_A\vec{v}'_A + m_B\vec{v}'_B) = (m_A\vec{v}_A + m_B\vec{v}_B)$$

If a collision is a *direct central impact*, the particles collide in a line, and their motion is rectilinear. This simplifies the momentum balance to one dimension:

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B$$

Note that, even in this simple case, there are generally at least two unknown values–the velocities of each body after collision. In this case, a known *coefficient of restitution* may be used to relate speeds along the axis of impact. The coefficient of restitution for two bodies is given by

$$e = \frac{v'_B - v'_A}{v_A - v_B}, \qquad 0 \le e \le 1$$

If $e = 0$, the objects move at the same final speed and therefore stick together in a *perfectly inelastic collision*. If $e = 1$, then one body transfers all its energy to the second body in a *perfectly elastic collision*.

For an *oblique impact*, objects make contact along a normal axis. Velocity in the normal direction changes according to the coefficient of restitution, while velocity along the tangential axis is not changed during collision.

3.3   Vibration of Particles

Many real-world systems may be modeled as particles undergoing vibration–in particular, as spring-mass systems.

A particle undergoing vibrations is either *free*, with no external forces besides the restoring force causing oscillations, or *forced*, with some kind of external force acting on the system.

Undamped, Free Vibrations

In the absence of external forces, such a system is said to undergo *free vibrations*. Typical applications involve a Hookean spring, such that the restoring force caused by the spring varies linearly with displacement. The standard equation of motion for such a system is as follows:

$$
\begin{aligned}
m\ddot{x} &= -kx \\
m\ddot{x} + kx &= 0 \\
\ddot{x} + \frac{k}{m}x &= 0 \\
\text{let } \omega_n^2 = \frac{k}{m} &\longrightarrow \boxed{\ddot{x} + \omega_n^2 x = 0}
\end{aligned}
$$

Here, ω_n is the *undamped natural frequency* for the vibrations. Using standard methods (i.e. characteristic equations), this differential equation can be shown to have the following solution:

$$
\ddot{x} + \omega_n^2 x = 0 \implies \boxed{x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t)}
$$

This may alternatively be written as a single sinusoidal function:

$$
x(t) = C \cos(\omega_n t + \phi), \qquad C := \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}, \qquad \phi := \arctan\left(\frac{\dot{x}_0}{\omega_n x_0}\right)
$$

Damped, Free Vibrations

In many applications, a spring is paired with a *damper*, which contains some fluid and thus produces a force proportional to velocity which resists motion. The standard equation of motion for such a system is as follows:

$$
\begin{aligned}
m\ddot{x} &= -c\dot{x} - kx \\
m\ddot{x} + c\dot{x} + kx &= 0 \\
\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x &= 0 \\
\text{let } \zeta = \frac{c}{2m\omega_n} &\longrightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}
\end{aligned}
$$

Here, ζ is the *damping ratio* for the system. If a damper is present in a system, $\zeta > 0$. Using method of characteristics, $\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$, and the form of the solution to the differential equation depends on ζ:

Underdamped, $\zeta < 1$: system oscillates and gradually decays to equilibrium. Frequency is reduced to $\omega_d = \omega_n\sqrt{1 - \zeta^2}$:

$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cos(\omega_d t) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\right)$$

This equation is also valid for systems without a damper, where $\zeta = 0$.

Critically damped, $\zeta = 1$: system returns to equilibrium as fast as possible. Solution is of the form

$$x(t) = A_1 e^{\lambda t} + A_2 t e^{\lambda t}$$

where A_1 and A_2 are found by solving for initial conditions.

Overdamped, $\zeta > 1$: system decays exponentially to equilibrium, but at a slower rate than critical damping. Solution is of the form

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

The *time constant* for a system is a measure of a system's decay rate. In general, for a decaying exponential $x_0 e^{-at}$, the time constant is defined as the time τ by which $x_0 e^{-a\tau} = e^{-1}$, so $\tau = \frac{1}{a}$. For the systems under consideration here, the time constant is given by

$$\tau = \frac{1}{\zeta w_n}$$

At a time 4τ, a system has decayed by $e^{-4} \approx 98\%$, at which point a system may generally be approximated as having returned to rest.

3.4   Rigid Body Kinematics

A **rigid body** is a collection of particles for which the distance between particles does not change or may otherwise be approximated as constant. A body may move with both *translation* and *rotation*. For translational motion, a body moves along its center of mass, and thus relationships from particle kinematics apply directly. New kinematic quantities are required to describe the rotation of a body.

Similar to particle kinematics, angular velocity and acceleration are given by a differential equation on angle θ:

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \qquad \alpha = \frac{d\omega}{dt} = \dot{\omega} \qquad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Again, time may be eliminated, yielding a differential equation in terms of differential quantities:

$$\omega d\omega = \alpha d\theta$$

For a general rigid body, given two points o and p on a rigid body, absolute motion is given by the following relationship:

General kinematics of a rotating body. Given two points o and p on a rigid body, where the distance from o in the direction of p is given by $\vec{r}_{p/o}$, absolute velocities and accelerations for the body are given by:

$$\vec{v}_p = \vec{v}_o + \vec{\omega} \times \vec{r}_{p/o}$$

$$\vec{a}_p = \vec{a}_o + \vec{\alpha} \times \vec{r}_{p/o} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{p/o})$$

An important application is *rolling without slipping*. For a wheel which rolls without slipping, the velocity as a point on the exterior rim of the wheel approaches the contacting surface shrinks to zero and is therefore momentarily at rest, experiencing *static friction*. A velocity or acceleration analysis shows that

$$v = -\omega r, \qquad a = -\alpha r$$

It is also possible to analyze motion between two separate rigid bodies directly:

Kinematics between two rotating bodies. Given a point o on one body and a point p on another (or the same) body, the motion of point p in terms of o is given by:

$$\vec{v}_p = \vec{v}_o + \vec{\omega} \times \vec{r}_{p/o} + \vec{v}_{p,\text{rel } o}$$

$$\vec{a}_p = \vec{a}_o + \vec{\alpha} \times \vec{r}_{p/o} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{p/o}) + 2\vec{\omega} \times \vec{v}_{p,\text{rel } o} + \vec{a}_{p,\text{rel } o}$$

3.5   Rigid Body Kinetics

Most generally, a rigid body has six degrees of freedom: three positional, and three rotational. Newton's equation describes positional acceleration when applied to the center of mass, and Euler's equation describes rotation.

Center of mass $\vec{r}_c$ of an object may be computed for a rigid body comprised of either a continuous or discrete distribution of points:

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n} = \int_0^M \frac{\vec{r}dm}{M}$$

Euler's equation relates the moments on a body about an axis to the body's angular acceleration. Moments may be computed as the cross product between the distance to a force and the force itself:

$$\vec{M}_c = \vec{r}_{p/c} \times \vec{F}_p$$

For planar motion, moments must act in or out of the plane, so a single Euler's equation describes motion of a body as there is only one axis of rotation. Euler's equation may be written equivalently in several ways:

Euler's equation. Euler's equation may be equivalently expressed as:

1. Euler's equation about the center of mass, c:
$$\sum \vec{M}_c = I_c\vec{\alpha}$$

2. Euler's equation about a fixed point, o:
$$\sum \vec{M}_o = I_o\vec{\alpha}$$

3. Euler's equation about an arbitrary point p:
$$\sum \vec{M}_p = I_p\vec{\alpha} + \vec{r}_{c/p} \times m\vec{a}_p$$

The *mass moment of inertia* about the center of mass of a rigid body is computed by an integral:

$$I_c = \int_m r^2 dm$$

The *parallel axis theorem* allows the calculation of a moment of inertia about any point on a body if the moment of inertia about the center of mass c is known. In particular, the moment of inertia about a point p on a body is given by

$$I_p = I_c + md^2$$

where d^2 is the scalar distance between c and p. Therefore, I_c is necessarily the smallest moment of inertia for a body.

For a rigid body, the kinetic energy may be expressed as a combination of the energy caused by linear and rotational velocity:

$$T = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 = \frac{1}{2}I_p\omega^2 \text{ if } \vec{v}_p = \vec{0}$$

Potential energy is calculated at the center of mass of a rigid body. With this, the *work-energy theorem* derived for particles may be applied to rigid bodies.

3.6 Vibration of Rigid Bodies

Rigid body vibrations may take advantage of both Newton's and Euler's equations to describe the motion of a body.

Harmonically Forced Vibrations

The system under consideration is a mass attached to a spring of constant k, a damper with damping ratio c, and a forcing term $F(t)$. The equation of motion is therefore

$$
\begin{aligned}
m\ddot{x} &= -kx - c\dot{x} + F(t) \\
\implies \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x &= \frac{F(t)}{m} \\
\implies \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x &= \frac{F_0}{m}\cos(\omega t) \text{ for forcing term with frequency } \omega
\end{aligned}
$$

The solution to this differential equation is a sum of the homogeneous and particular solution $x_c(t) + x_p(t)$. The homogeneous solution has already been found; the particular solution may be found by making a guess of the same form as the forcing term.

$$
\begin{aligned}
x_p(t) &= A_1\cos(\omega t) + A_2\sin(\omega t) \\
&= X\cos(\phi)\cos(\omega t) + X\sin(\phi)\sin(\omega t) \\
&= X\cos(\omega t + \phi), \text{ where } \phi = \arctan(-A_2/A_1) \text{ and } X = \sqrt{A_1^2 + A_2^2}
\end{aligned}
$$

Solving the differential equation by standard methods, this gives the following:

$$
X = \frac{F_0/m}{\sqrt{(2\zeta\omega_n\omega)^2 + (\omega_n^2 - \omega^2)^2}} \text{ and } \phi = \arctan\left(\frac{-2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)
$$

Defining the tuning ratio $\eta = \omega/\omega_n$, these constants may be alternatively defined in terms of the ratio of forcing frequency to natural frequency of the undamped system:

$$
X = \frac{F_0/k}{\sqrt{(2\zeta\eta)^2 + (1 - \eta^2)^2}} \text{ and } \phi = \arctan\left(\frac{-2\zeta\eta}{1 - \eta^2}\right)
$$

The *dynamic magnification factor* M is a measure of amplitude under forcing and can be used to determine resonance (i.e. as $M \to \infty$). M is defined as

$$
M := X/(F_0/k) = \frac{1}{\sqrt{(2\zeta\eta)^2 + (1 - \eta^2)^2}}
$$

Base Excitations

If the base on which a spring-mass system is attached is free to move in a known manner, the spring-mass system has a different equation of motion with a forcing term from the base:

$$
m\ddot{x} + c(\dot{x} + \dot{z}) + k(x + z) = 0 \iff m\ddot{x} + c\dot{x} + kx = \underbrace{c\dot{z} + kz}_{\text{forcing term}}
$$

Given a periodic $z(t)$ as input, this becomes a harmonic forcing term. In this case, the magnification factor is given by the ratio

$$
M = \frac{X}{z_0} = \sqrt{\frac{1 + (2\zeta\eta)^2}{(2\zeta\eta)^2 + (1 - \eta^2)^2}}
$$

4 Design and Manufacturing I

Notes from **MECHENG 250: Design and Manufacturing I**

Taken at University of Michigan, Winter 2025

4.1 Product Design

In general, a product undergoes various phases of design:

1. Specifying *functional requirements*, i.e. design-neutral, measurable goals which the product must accomplish

2. Product *ideation*, including research in existing solutions and generation of initial designs

3. *Analysis*, using first-principles models

4. *Performance evaluation*, where empirical data is analyzed by some means, i.e. by a regression model

These steps may repeat any number of times until a design is satisfactory.

A simplified *first principles model* is often useful for evaluating feasibility of a design. A first-principles model typically involves drawing a free-body diagram, then using Newton's and Euler's equations to evaluate forces and moments acting on the body. Such analysis often gives a desired output as a function of *design variables*–properties which can be changed to influence the ability of the product to perform, such as length, material, and so on.

An *empirical model* is produced by extrapolating from real-world data, typically through some kind of regression model which relates a desired output to a variable, tested input. A regression model works by minimizing the square of the error between extrapolated values and observed values.

4.2 Machine Elements

Transmissions

A transmission is a system which transfers power. Common transmissions involves *gears*, *belts*, or *chains*. There are various tradeoffs between these components:

Gears. Gears come in many forms and essentially transfer rotational motion through meshing teeth.

Belts. Belts are less expensive than gears, but require a tensioner in order to prevent slipping.

Chains. Chains are connected to two sprockets. Chains are often less expensive than gears and, unlike belts, do not require a tensioner.

These transmission systems are mathematically identical, and the following relationships hold (in magnitude):

$$\frac{\omega_2}{\omega_1} = \frac{T_1}{T_2} = \frac{d_1}{d_2} = \frac{N_1}{N_2}$$

For two gears to mesh, the *diametral pitch P* and pressure angle. Diametral pitch is given by

$$P = \frac{N}{d}$$

A *gear train* is a system consisting of several gears. The *gear speed ratio e* for a gear train is given by

$$e = \frac{\omega_{\text{output}}}{\omega_{\text{input}}} = -\frac{d_1}{d_2} = \underbrace{\frac{n_{\text{output}}}{n_{\text{input}}}}_{\text{speed (RPM)}}$$

From a conservation of energy, torque is given by

$$T_2 = \eta \frac{T_1}{e} = \eta T_1 \frac{N_2}{N_1}$$

where η is the gear efficiency, where η is typically between 0.9 and 0.95 *per contact on a gear train*.

Electric Motors

A *motor* is a common device which converts electrical power to mechanical power. A motor is defined by a no-load speed n_0 (with no output torque) and stall torque T_s (with zero speed). A *torque-speed curve* is an approximately linear relationship between output torque T and speed n, where

$$T = T_s - kn$$

A torque-speed curve has slope $1/k$ and shows speed n as a function of torque T. If a motor is too weak, it is often useful to use a gear train to increase maximum effective torque at the expense of speed, according to the relationship

$$\frac{T_{s2}}{\gamma T_{s1}} = \frac{n_{01}}{n_{02}} = M$$

where γ is the efficiency of the geartrain and M is the gear ration M:1 for a gearbox. Alternatively, increasing the voltage of a gearbox shifts the torque-speed curve up, according to the relationship:

$$\frac{T_{s2}}{T_{s1}} = \frac{n_{02}}{n_{01}} = \frac{V_2}{V_1}$$

Springs

A *spring* is a common device which is meant to apply a controlled force or torque and/or store potential energy. Linear springs are of potential interest, where the number of turns N_t on the spring is given by

$$N_t = \frac{L_s}{d}$$

where L_s is spring length when fully compressed and d is the diameter of the coil wire. There are two primary types of linear springs:

Extension Spring. Number of active coils $N_a = N_t$. Contains an initial preload F_{initial}, so $F_{s,\text{ext}} = F_{\text{initial}} + k\delta$.

Compression Spring. Number of active coils $N_a = N_t - 2$ because the ends are inactive. Compressive force given by $F_{s,\text{ext}} = k\delta$.

If the mean coil diameter D_m and the material's shear modulus G is known, the spring constant k may be calculated as follows:

$$k = \frac{F}{\delta} = \frac{Gd^4}{8D_m^3 N_a} = \frac{Gd}{8C^3 N_a}$$

where C is the *spring index*, defined by $C = D_m/d$.

Lead Screws

A *lead screw* is a screw which rotates, causing a nut to translate. These systems are simple, but low efficiency and typically quite slow. To raise a load of weight F given a lead screw with pitch diameter D_p, screw lead l, pitch p (distance between two threads), and coefficient of friction μ, the torque to raise and lower a load respectively is given by

$$T_{\text{raise}} = \frac{FD_p}{2}\left(\frac{l + \mu\pi D_p}{\pi D_p - \mu l}\right), \qquad T_{\text{lower}} = \frac{FD_p}{2}\left(\frac{\mu\pi D_p - l}{\pi D_p + \mu l}\right)$$

Mechanisms

A *mechanism* is any device which transfers motion and/or force from an input to an output. A *linkage* is a specific type of mechanism which consists of links connected at joints. A group of linkages forms a *kinematic chain*, which may be *open* (free at one end) or *closed* (fixed at both ends).

A *four-bar linkage* is a common type of linkage consisting of four links and four joints, where a single motor may be used to control the linkage. Given a four-bar linkage with shortest link of length s, longest link of length l, and the other links of length p, q, by **Grashof's Criterion**, the linkage may complete a 360-degree rotation relative to the ground if and only

$$s + l < p + q$$

4.3 Dimensioning and Tolerancing

Any two of the same part will have differing dimensions, introduced through variation in the manufacturing process (tool wear, machine precision, measurement inaccuracy, etc). *Tolerance* is the total size variation allowed in a part, given by

$$\text{tolerance} = \text{upper limit} - \text{lower limit}$$

The **basic size** of a part is the size around which tolerances are calculated. The **maximum material condition (MMC)** is the size of the part where it has the most material (i.e. smallest hole or largest shaft), and **least material condition (LMC)** is the size of the part where it has the least material (i.e. largest hole or smallest shaft). Measurement of MMC and LMC are useful for evaluating whether components can fit together at the limits of tolerance. For example, for a shaft and hole,

$$\text{max clearance} = \text{LMC}_{\text{hole}} - \text{LMC}_{\text{shaft}}, \qquad \text{min clearance} = \text{MMC}_{\text{hole}} - \text{MMC}_{\text{shaft}}$$

Standard dimensioning and tolerancing is often insufficient for ensuring a part will function. *Geometric Dimensioning and Tolerancing* (GD&T) allows parts to be more accurately defined, often with larger tolerances and fewer opportunities for

misinterpretation. A *feature control frame* is used to place a tolerance on a geometric feature with respect to a datum point on a part, as shown:
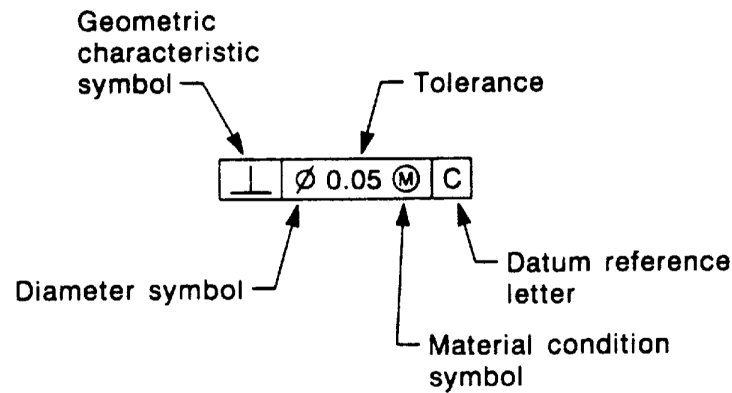
where M and L refer to maximum and least material conditions, respectively. Characteristics may be any of the following:

CHARACTERISTIC	SYMBOL
STRAIGHTNESS	—
FLATNESS	▱
CIRCULARITY (ROUNDNESS)	○
CYLINDRICITY	⌭
PROFILE OF A LINE	⌒
PROFILE OF A SURFACE	⌓
ANGULARITY	∠
PERPENDICULARITY	⊥
PARALLELISM	//
POSITION	⊕
CIRCULAR RUNOUT	↗
TOTAL RUNOUT	↗↗

4.4   Manufacturing Systems

Discrete Manufacturing

Discrete manufacturing refers to the production of distinct items (as opposed to *process manufacturing*, where production is by weight, i.e. chemical or pharmaceutical production). Discrete manufacturing systems can generally be analyzed as a production line, consisting of a series of separate manufacturing stations.

The production rate of an entire line is limited to the production rate of the slowest machine, where the slowest machine is the *bottleneck*. Optimizing the flow rate of a production line therefore involves optimizing the output of the bottleneck. Common ways to improve a process bottleneck include:

1. Add another bottleneck process in parallel, thus increasing the effective production rate

2. Place a *buffer* station (accumulating parts from a previous station) before the bottleneck to prevent the bottleneck being *starved*.

Discrete manufacturing occurring over a long period of time may be approximated as a steady-state flow, with constant flow rate λ parts per hour. **Little's Law** simply relates the production rate and the time of production W:

$$L = \lambda W$$

Importantly, Little's Law applies to both a single process and a group of processes (i.e. an entire factory). Therefore, L may be either the number of units in a system or the number of manufacturing stations in a factory, depending on the units of the variable λ.

Takt time is a measure of the average time between production of units, given by

$$\text{takt time} = \frac{\text{production time}}{\text{production quantity}}$$

If $1/\lambda >$ takt time, then the production system is too slow and/or the production numbers are unfeasible in the given time.

Process Capability

A *process capability index* is a standard way to measure the variability of a process output within a specification or tolerance. Given an *upper specification limit* USL and a *lower specification limit* LSL, the process capability index C_p is defined by

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

where σ is the standard deviation of measurements on output parts. C_p should ideally be as large as possible, with $C_p > 1.33$ being the typical industry-standard minimum. C_{pk} measures both the accuracy and the precision by comparing to the mean value, where C_{pk} is defined by

$$C_{pk} = \min\left\{\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right\}$$

C_{pk} is generally smaller than (or sometimes equal to) C_p and should also be maximized.

Manufacturing Cost Modeling

Total costs involved in the production of n parts of mass m are given by

$$C_{\text{total}} = C_{\text{material}} + C_{\text{capital}} + C_{\text{energy}} + C_{\text{overhead}}$$

Many processes are only viable given large batch size, effectively reducing per-part capital costs.

Whether an investment is likely to be profitable is determined by the related quantities *net present value* (NPV) and *internal rate of return* (IRR). Given an initial investment C_{init}, cash flow C_t in year t, and discount rate (annual return that could be earned with a similar investment, typically between 5% and 15%) i, the net present value of an investment is computed as

$$\text{NPV} = \sum_{t=1}^{T} \underbrace{\left(\frac{C_t}{(1+i)^t}\right)}_{\text{Yearly NPV}} - \underbrace{C_{\text{init}}}$$

Similarly, the internal rate of return is the value for which NPV $= 0$, or the solution to

$$\sum_{t=0}^{T} \left(\frac{C_t}{(1+\text{IRR})^t}\right) - C_{\text{init}} = 0$$

If IRR $< i$, an investment is a bad investment as it does not meet the minimum rate of return a firm could achieve with a similar investment. A higher IRR indicates a more desirable investment, and the best investment among a series of options is that with the highest IRR.

4.5 Manufacturing Processes

Manufacturing most fundamentally involves bringing stock material to a desired shape. Each manufacturing process has specific capabilities and associated processing times, making a process more or less feasible depending on the material properties, desired geometry, and the scale of production.

Subtractive Processes

Subtractive processes involve removing material from a stock piece. Common examples include:

Blanking, a process by which sheet-metal components are produced by shearing. This process is suitable for mass production because of its low processing time and the recyclability of leftover material "skeleton".

Machining, where the five basic machining processes are *turning* (on a lathe), *drilling*, *milling*, *grinding*, and *sawing*.

CNC Processes, including water jetting, laser cutting, CNC milling, CNC lathing, and other automated processes. CNC machines are programmed with G-Code and generally allow more complex geometries to be produced (in particular with CNC mills and CNC lathes).

Machining processes are relatively slow compared to other methods, and are therefore not suitable for mass manufacturing (i.e. >100,000 parts). As an example, for a lathe operation with feed rate f inches/rev, spindle speed N revs/min, initial diameter D_0 and a depth of removal d, the material removal rate is given by

$$\text{material removal rate} = \underbrace{\pi D_0 N}_{v} \, df$$

where v is the *cutting speed*, the tangential speed of the workpiece. The machining time is then given by

$$T_m = \frac{L}{Nf}$$

Metal Forming

Because machining is relatively slow, parts are generally brought near their final shape so that machining becomes a finishing process, rather than a shaping process. The processes here are *bulk deformation processes*, which plastically deform a material, causing *work hardening* and producing a tougher final part:

Extrusion. Involves forcing a metal through a die to form a long shape. Metals may be extruded *hot* (aluminum, copper, magnesium) or *cold* (aluminum or steel, with steel requiring high forces).

Forging. Produces a part near net shape of very high strength. Types include *open die forging* (suitable for large parts, generally requiring lots of machining) and *closed die forging* (suitable for producing smaller parts near net shape, where only the excess "flash" which squeezes out of the mold must be machined away).

The forces involved in forging can be quite large. The required forces can be evaluated by considering first the necessary deformation, then the strain in the material, then the stress in the material, and finally the surface area of the part.

Strain of work is given by

$$\epsilon = \ln \frac{h_0}{h}$$

where h_0 is the initial height of the stock and h is the final height. The stress is then given by

$$Y_f = K\epsilon^n$$

where K is the strength coefficient and n is the strain hardening exponent, both of which are material properties. Finally, the compressive force is found by

$$F = K_f Y_f A$$

where

$$K_f = 1 + \frac{.4\mu D}{h} \text{ (shape factor)}$$

K_f is a function of part height and diameter, and also friction between the part and the die. Shape factor K_f is not related the strength coefficient K.

There are different metal forming operations available for forming sheet metal parts:

Simple Bending. Involves bending straight lines into a sheet metal part using a punch and die. Both cheaper and slower than stamping operations.

Sheet Metal Stamping. Involves pressing a sheet metal part between a die and a punch to form a desired shape. Often used in mass production because the process is relatively fast and maintains sheet thickness.

Sheet Metal Drawing. Essentially the same as stamping, but with a specialized (more expensive) die allowing part to be drawn to a depth larger than the punch diameter.

Additive Manufacturing

Additive manufacturing processes are varied and involve forming a part from feedstock, rather than forming or removing from existing stock. There are seven categories of additive manufacturing process:

Process	Material Used	Tool Used
Material Jetting	Melted Plastic	Laser or Multi-Nozzle Print Head
Vat Photopolymerization / Stereolithography (SLA)	Melted Plastic	Laser
Material Extrusion	Melted Plastic	Single-Nozzle Print Head
Sheet Lamination	Material Sheets	Solid-State Welding or Adhesive
Directed Energy Deposition	Powdered Metal	Laser
Powder Bed Fusion	Plastic Powder (SLS) or Metal Powder (SLM/DMLS)	Laser
Binder Jetting	Powdered Metal	Multi-Nozzle Print Head, then Solid State Welding

Liquid to Shape.

As opposed to other metal forming processes, metals can also be *cast* into a nearly complete shape. This is accomplished by pouring a molten metal into some form of mold; the hardened metal then takes the desired shape. This process generally requires some amount of machining to process the part, particularly for threads and smooth surfaces, both of which casting is incapable of producing. There are two types of casting:

Die Casting. Metal is forced into a permanent die cavity under pressure. Allows for high production rate (suitable for mass manufacturing) and good tolerances and surface finish. Can occur in a *hot chamber* (where low melting point metals, i.e. zinc or magnesium, are under high pressure until they solidify in the mold) or in a *cold chamber* (where metal is melted in a separate furnace and driven into the die).

Sand Casting. Most common casting process, involving a mold made of sand which is broken after the part solidifies. Produces a rough surface finish and has low upfront cost, but is also a slower process as molds are not reused.

A mold used in sand casting consists of a *cope* (top half) and *drag* (bottom half), and often a *core* which produces the internal surfaces on a cast part. The core is supported by *chaplets* both above and below, which are placed below to support the core weight and above to counteract bouyant forces from the molten metal. For core of density ρ_c and volume V_c and molten metal of density ρ_m, the forces acting on the core are

$$\underbrace{F_b = (\rho_m - \rho_c)V_c g}_{\text{bouyant force}}, \qquad \underbrace{W = \rho_c V_c g}_{\text{weight}}$$

If a chaplet is able to support F_{chap}, then the number of chaplets necessary is given by

$$N_{\text{above}} = \left\lceil \frac{F_b}{F_{\text{chap}}} \right\rceil$$

Then, enough chaplets must be placed below to support the weight of both the core and the chaplets above. Thus, the number of chaplets required below is

$$N_{\text{below}} = \left\lceil \frac{W + N_{\text{above}} \cdot W_{\text{chap}}}{F_{\text{chap}}} \right\rceil$$

Even more diverse processes exist for forming liquid plastic:

Injection Molding. Produces open shapes by forcing high-pressure liquid plastic into a closed mold. Plastic is subject to shrinkage as it cools.

Blow Molding. Produces thin, bottle-like shapes with small openings by "blowing" heated plastic to walls of a closed mold.

Rotational Molding. Produces thin-walled, seamless, hollow shapes by rotating a mold, forcing liquid plastic to conform to the mold shape.

Thermoforming. Produces flat, thin-walled shapes by "sucking" heated plastic sheet onto an open mold.

Parts created in injection molding are particularly vulnerable to shrinkage as they cool within the mold. If a cavity has dimension D_c and a plastic has a shrinkage value S (typically in mm/mm or inch/inch, i.e. unitless), then the dimension of the molded part is given by

$$D_c = \frac{D_p}{1 - S}$$

5 Fluid Mechanics

Notes from **MECHENG 320: Introduction to Fluid Mechanics**

Taken at University of Michigan, Fall 2025

using *Fundamentals of Fluid Mechanics*, by Munson, Okiishi, and Young

5.1 Fluid Properties

A **fluid** is a substance which deforms *continuously* when acted upon by a shearing force of any magnitude. As opposed to a solid, which deforms to a finite extent when a force is applied, a fluid is easily deformed and this deformation does not stop; in other words, a fluid can be made to *flow*. The study of fluids tends to involve a **continuum approximation**, where the quantities associated with the fluid vary continuously, as opposed to an atomistic model.

Viscosity is a property of a fluid which measures its ability to resist flow. **Dynamic viscosity** is the proportionality constant relating fluid shearing stress and the rate of shearing strain,

$$\tau = \mu \frac{du}{dy}$$

Note that $\frac{du}{dy}$ is speed per distance, but can be shown to be equivalent to the rate of shearing strain. Dynamic viscosity has SI units $\frac{\text{kg}}{m \cdot \text{s}}$. **Kinematic viscosity** is found by normalizing dynamic viscosity to the density of a fluid, given by

$$\nu = \frac{\mu}{\rho}$$

Viscosity is dependent on the temperature of a fluid. If μ is linear, i.e. $\tau/(du/dy)$ is constant, then a fluid is called *Newtonian*; if this relationship is nonlinear, a fluid is called *Non-Newtonian*.

Pressure is another property of a fluid. Pressure is a measure of the normal stress in a fluid. Pressure is a *scalar* quantity, and forces arising due to pressure act *normal* to the interaction surface. The **compressibility** of a fluid is given by the *bulk modulus*, which quantifies the pressure required to compress a fluid:

$$E_v = -\frac{dp}{dV/V} = \frac{dp}{d\rho/\rho}$$

As $E_v \to \infty$, the quantity $\frac{d\rho}{\rho} \to 0$, which is the *incompressibility* condition. As opposed to viscosity, bulk modulus is a property of the fluid. Bulk modulus is related to speed of sound in a medium by

$$c = \sqrt{\frac{E_v}{\rho}}$$

Surface tension occurs when there is a pressure difference between the fluid and surrounding medium. Surface tension is denoted by σ, and is a force per length. In general, for a length L of an interface (i.e. $2\pi R$ for the arclength of a circular droplet) and an area enclosed by the interface A (i.e. πR^2 in this case), the pressure difference arising within a droplet from surface tension is given by force balance:

$$\sigma L = \Delta p A; \text{ or, for circular droplets, } \sigma \cdot 2\pi R = \Delta p \cdot \pi R^2$$

5.2 Fluid Statics

Fluid statics, sometimes called *hydrostatics*, is a subset of fluid mechanics studying forces acting on and caused by fluids at rest, and therefore without shearing stresses. In absence of shearing stress, pressure acts equally in each direction and therefore can be treated as a scalar.

Pascal's Law. Pressure at a point in a fluid at rest, or in motion, is independent of direction *so long as there are no shearing stresses present.*

In a static fluid, pressure distribution can be shown to be given by:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \ \frac{\partial p}{\partial z} = -\rho g$$

In other words, at a given depth z, movement in the x or y direction (i.e. movement at the same depth) does not affect pressure. Pressure can only change by a change in depth z. Given constant ρg, as would occur in an incompressible fluid where $\rho(z)$ is constant, integration yields the **hydrostatic equation**:

$$\frac{\partial p}{\partial z} = -\rho g \implies \boxed{p = p_{\text{atm}} + \rho g h} \qquad \text{(Hydrostatic Equation)}$$

In the hydrostatic equation, p_{atm} is the atmospheric pressure at the fluid surface, and h is the depth of immersion. Therefore, static pressure is independent of vessel shape and only depends on depth. Pressure may be measured as an *absolute pressure*, measured with respect to a perfect vacuum (i.e. from 0), or as a *gage pressure*, measured with respect to the local atmospheric pressure (i.e. $p_{\text{gage}} = p - p_{\text{atm}}$. The hydrostatic equation gives an *absolute pressure.*

Force on Submerged Objects. Hydrostatics enables the study of forces acting on submerged objects. Assuming a fluid is open (i.e. at $h = 0$ the fluid has pressure P_{atm}), the resultant forces due to the fluid may be calculated using a static balance. There are two cases in general: either a planar (flat) surface, or a curved surface, oriented to the free surface.

Flat Surface. Let θ be the angle between the surface and the free surface. From the hydrostatic equation, it can be shown that the resultant force has magnitude

$$F_R = \rho g h_c A$$

where h_c is the height of the centroid and A is the area of the shape. The location (x_R, y_R) of the resultant force is the **center of pressure**, which is given by

$$x_R = \frac{I_{xy,c}}{A y_c} + x_c \qquad y_R = \frac{I_{x,c}}{y_c A} + y_c$$

where $I_{x,c}$ is the second moment of area with respect to the centroid, and $I_{xy,c}$ is the *product of interia* with respect to the centroid. Note that the center of pressure is always below the centroid for any submerged shape.

Curved Surface. In general, net force acting on a submerged body may be found by balancing forces on a fluid volume. In particular, the horizontal forces on a volume must balance and the vertical forces on a volume must balance, as shown below:
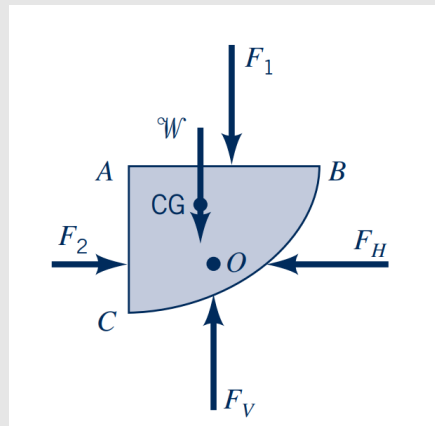
The procedure for computing forces on a general submerged body is as follows:

1. Compute $F_H = F_2$. The force acting horizontally on the curved surface must equal the force exerted on the flat, vertical plane through the fluid. Thus, F_H is given by

$$F_H = \rho g h_{c,\text{plane}} A_{\text{plane}}$$

where $h_{c,\text{plane}}$ is the centroid (center) of the plane, i.e. half the depth of the body, and A_{plane} is the area of the flat plane.

2. Compute $F_V = W + F_1$. If the surface of the fluid is free, $F_1 = 0$. Weight is computed by

$$F_V = \rho g V_{\text{fluid}}$$

3. Compute reaction force. The magnitude of the force is given by $F = \sqrt{F_H^2 + F_V^2}$ and the force acts in the unit direction $\hat{e} = \left(\frac{F_H}{F}, \ \frac{F_V}{F} \right)$ from the origin.

Buoyancy is the net vertical force acting on a submerged object due to fluid pressure. Buoyancy force F_b is given by

$$F_b = \rho_{\text{fluid}} g V_{\text{displaced}}$$

Archimedes Principle gives the condition for floating (where $W = \rho_{\text{solid}} g V$):

If $F_b > W$, the body will *rise* in the fluid.

If $F_b = W$, the body is *neutrally buoyant* and will not rise or sink in the fluid.

If $F_b < W$, the body will *sink* in the fluid.

A floating body is *stable* if the buoyancy force, acting at the center of buoyancy, produces a moment which restores the body to its upright position, a *restoring couple*. A body which produces an *overturning couple* is *unstable*. The *metacenter* is the intersection point between the line through the original center of buoyancy and the new center of buoyancy. If the metacenter is above the body, the body is stable; if the metacenter is on the body, the body is unstable.

5.3   Fluid Kinematics

Fluid kinematics is the description of fluid motion without regard to forces acting on and caused by the fluid. There are two primary kinematic frameworks for analyzing fluids:

Eulerian Description of Fluid Kinematics. Flow is observed with respect to a fixed position as time progresses. This framework provides a *lab-fixed reference frame.*

Lagrangian Description of Fluid Kinematics. Individual particles of fluid are tracked as they flow. This model is convenient for kinematics but will need to be modified when dealing with forces (i.e. fluid dynamics).

For either framework, the assumption is made that for any fluid particle (i.e. a small collection of fluid), mass is constant but shape and volume may change (depending on the compressibility of the fluid). Fluid velocity constitutes a vector field, where $\vec{v}(x, y, z, t) = u(x, y, z, t)\hat{i} + v(x, y, z, t)\hat{j} + w(x, y, z, t)\hat{k}$ denotes the velocity at any point (x, y, z) and any time t. Familiarly, if $\vec{r}$ is the position vector from the origin to a particle of fluid,

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}(\vec{r}(t), t) = \vec{v}(x, y, z, t)$$

As expected, the acceleration of a fluid is given by differentiation of $\vec{v}$:

$$
\begin{aligned}
\vec{a}(t) &= \frac{d\vec{v}}{dt} \\
&= \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \vec{v}}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial \vec{v}}{\partial z} \cdot \frac{\partial z}{\partial t} \\
&= \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}
\end{aligned}
$$

Then, for $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$, the components of acceleration are given by:

$$
\begin{cases}
a_x &= \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} \\[2ex]
a_y &= \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} \\[2ex]
a_z &= \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}
\end{cases}
$$

Acceleration is an example of a *material derivative*, also called a *substantial derivative*. The material derivative is an operator defined by:

$$
\frac{D}{Dt}(-) = \frac{\partial}{\partial t}(-) + \vec{v} \cdot \nabla(-)
$$

which is essentially the partial derivative of a property with respect to time plus the dot product of velocity with the gradient of the property. Physically the material derivative gives the total change in a property of a fluid flow as an observer moves with the flow. This is composed first of local temporal effects on the property, and the spatial *convective derivative*, which gives the rate of change of the property as fluid moves through locations. In steady-state flow, the temporal term is zero, but the total derivative may still be nonzero depending on the convective terms.

There are several ways to visualize fluid flow:

1. **Streamlines**. If a vector field is known or approximated analytically, the *streamlines* may be drawn across the flow field. Streamlines are drawn by definition as a line instantaneously tangent to the flow field. From this definition, the slope of a streamline in two dimensions is given by

$$
\frac{dy}{dx} = \frac{v}{u}
$$

where the slope of the streamline satisfies this differential equation, and the solution given by separation of variables is the slope of the streamline as a function of position.

2. **Pathlines** (Lagrangian concept). A given volume of fluid is somehow marked (i.e. by dye) and tracked. In a lab, this can be used to plot position of a particular particle as it progresses through time. Position is given by familiar kinematic relationships.

$$
\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v
$$

3. **Streaklines**. A dye or similarly visible material is inserted into a flow. The streaks as the dye moves are visible and allow for flow visualization. This method is common in laboratories due to its simplicity.

In steady flow, these lines are all equivalent; in this case, these lines are typically simply called the *streamlines* of the flow field.

5.4   Fluid Dynamics

Fluid dynamics is the study of forces causing and resulting from fluid motion. By applying Newton's second law to a fluid particle *moving along its streamline* and under a set of simplifying assumptions, Bernoulli's equation can be derived. Bernoulli's' equation essentially states conservation of energy along a streamline:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \qquad \text{(Bernoulli's Equation)}$$

P is the *static pressure*, where $\frac{1}{2}\rho v^2$ is the *dynamic pressure* which is the product of velocity (in magnitude) and density, and $\rho g h$ is the familiar hydrostatic pressure. Applied at two points along a streamline, Bernoulli's Equation is written as follows:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Bernoulli's Equation is typically written in one of three forms:

Bernoulli's Equation. Bernoulli's equation can be rearranged in several ways to solve for different quantities:

1. **Pressure Form**. Units are pressure.

$$P + \rho \frac{v^2}{2} + \rho g h = \text{const.}$$

2. **Energy Form**. Units are energy per mass.

$$\frac{P}{\rho} + \frac{v^2}{2} + gh = \text{const.}$$

3. **Head Form**. Units are length.

$$\frac{P}{\rho g} + \frac{v^2}{2g} + h = \text{const.}$$

   In the head form, the quantity $P/(\rho g)$ is referred to as the *pressure head*, and h as the *head length*.

It should be noted that Bernoulli's equation makes several simplifying assumptions and is therefore only valid for very specific fluid flows. In particular, Bernoulli's equation is valid only under *steady-state*, *inviscid* flow, with a *constant density* and evaluation occurs between two points *along the same streamline*.

Example. *Stagnation Pressure.* Stagnation pressure is the pressure resulting from a fluid coming to a complete stop at a point called a *stagnation point*, i.e. kinetic energy is converted completely into pressure. This pressure is given by:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \;\overset{0}{\nearrow}$$

$$\implies P_2 = P_1 + \frac{1}{2}\rho v_1^2$$

Bernoulli's equation can be modified to relax the assumption of steady-state. The result is the **unsteady Bernoulli equation**, which requires knowledge of the velocity at every point along a streamline.

$$\underbrace{p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2}_{\text{Bernoulli's equation}} + \underbrace{\rho \int_{s_1}^{s_2} \frac{\partial v}{\partial t}\,ds}_{\text{Unsteady effects}}$$

s_1 and s_2 are the initial and final points considered along the same streamline.

The description of a fluid given by Bernoulli's equation is limited. More generally, there is an interest in applying to laws of physics directly to the fluid. A *control volume* allows these laws to be applied, and the fluid may then be described in terms of integral equations or differential equations. In particular, the following relationships are of interest, and can be obtained through integral or differential analysis on the control volume:

1. Conservation of Mass.

2. Conservation of Momentum.

3. Conservation of Energy.

The coupled system given by (1) and (2) will be the *Navier-Stokes Equations*.

Control Volume Analysis. A control volume may be defined in either a Lagrangian or Eulerian framework. In a Lagrangian framework, the control volume is a collection of fluid particles; this volume moves with the flow and always contains the same particles. In an Eulerian framework, the control volume is an arbitrary region in space through which fluid may flow; the matter in this volume changes with time.

Suppose B is any extensive property of a system (momentum, energy, force, etc). Then, define $b = B/m$, where m is mass and b is therefore an intensive property. Reynold's Transport Theorem relates the time derivatives of the intensive properties of a system to a control volume.

Reynold's Transport Theorem. The rate of change of an extensive property is equal to the combination of the rate of change of the extensive property inside the control volume CV and the rate at which the extensive property is flowing across the surface of the control volume, the control surface CS.

$$\frac{d}{dt} B_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \rho b dV + \int_{CS} \rho b \vec{w} \cdot \hat{n} dA$$

The quantity $\vec{w} := \vec{v} - \vec{v}_{CV}$ is the difference between true velocity and surface velocity.

Derivation of the Continuity Equation. Applying Reynold's transport theorem to conservation of mass immediately gives the *continuity equation* (with $B = m \implies b = m/m = 1$):

$$\frac{dm}{dt} = 0 = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{w} \cdot \hat{n} dA \qquad \text{(Continuity Equation)}$$

Note that a positive flux term implies outflow, and a negative flux term implies inflow. The continuity equation can be simplified in several ways under different assumption sets. For *incompressible flow*, density is constant, so the continuity equation becomes:

$$\frac{\partial}{\partial t} \int_{CV} dV + \int_{CS} \vec{w} \cdot \hat{n} dA = 0$$

which is a statement on conservation of volume. For steady flow with constant control volume, the temporal variation disappears and $\vec{w} = \vec{v}$, the velocity of fluid flow. Thus, this equation reduces to:

$$\int_{CS} \vec{v} \cdot \hat{n} dA = 0 \iff \sum \dot{m}_{\text{in}} = \sum \dot{m}_{\text{out}}$$

Finally, if $\rho\vec{w}$ is a constant term over the entire surface, and if flow is steady, the flux term becomes:

$$\int_{CS} \rho \vec{w} \cdot \hat{n} dA = \sum_i \rho \vec{w} \cdot \hat{n} dA_i$$

Derivation of Conservation of Linear Momentum

Newton's 2nd law, in most general momentum form, is given by

$$\frac{d}{dt} (m\vec{v})_{\text{sys}} = \sum \vec{F}_{\text{sys}}$$

Observe that, if a control volume CV is coincident with the system, then $\sum \vec{F}_{\text{sys}} = \sum \vec{F}_{CV}$. Thus, Reynold's transport theorem may be applied:

$$\sum \vec{F}_{\text{sys}} = \frac{d}{dt}(m\vec{v})_{\text{sys}}$$

$$= \underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV}_{\text{change inside volume}} + \underbrace{\int_{CS}\rho\vec{v}\left[(\vec{v}-\vec{v}_{CV})\cdot\hat{n}\right]dA}_{\text{flux through surface}}$$

In general, three kinds of forces act on fluid in a control volume:

Surface forces. Surface forces act on the surface and are proportional to the surface area, such as pressure forces.

Body Forces. Body forces act on the entire volume and are proportional to volume, such as gravity.

External Forces. Force resulting from bodies within the control volume. These forces are unknown; common examples include lift and drag of a body within a control volume.

Therefore, a complete force balance is given by:

$$\frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV + \int_{CS}\rho\vec{v}[(\vec{v}-\vec{v}_{CV})\cdot\hat{n}]dA = \underbrace{-\int P\hat{n}dA}_{\text{surface forces}} + \underbrace{\int_{CV}\rho\vec{g}dV}_{\text{body forces}} + \vec{F}_{\text{viscous}} + \vec{F}_{\text{ext}}$$

where $\vec{F}_{\text{ext}}$ are external forces and $\vec{F}_{\text{viscous}}$ are the viscous forces, which can be ignored if a fluid is approximately inviscid.

Derivation of Conservation of Energy. Let $e = E/m$, or energy per unit mass. Note that $e = \hat{u} + \frac{1}{2}v^2 + gz$, for internal energy, kinetic energy, and potential energy respectively. Then, applying Reynold's transport theorem to the first law of thermodynamics gives conservation of energy for a fluid:

$$\dot{Q}_{\text{net,sys}} + \dot{W}_{\text{net,sys}} = \frac{d}{dt}E_{\text{sys}}$$

$$= \frac{\partial}{\partial t}\int_{CV}\rho e dV + \int_{CS}\rho e(\vec{v}-\vec{v}_{CV})\cdot\hat{n}dA$$

Note that this is a scalar equation. Note that in an *adiabatic* process, $\dot{Q}_{\text{net,in}} = 0$. In general, $\dot{Q}_{\text{net,in}} = \dot{Q}_{in} - \dot{Q}_{out}$. Similarly, $\dot{W}_{\text{net,in}} = \dot{W}_{in} - \dot{W}_{out}$, where work may be done by any of the following mechanisms:

- $\dot{W}_{\text{pressure,in}}$, work done by pressure forces on control surface.

- $\dot{W}_{\text{viscous}}$, work done on the control surface by viscous forces, which may often be neglected.

- $\dot{W}_{\text{shaft,in}}$, which is work done by a machine on the fluid in the control volume (i.e. pumps, pistons, fans, etc).

Neglecting viscous forces, a complete energy balance is given by:

$$\frac{\partial}{\partial t}\int_{CV}\rho e dV + \int_{CS}\rho e(\vec{v}-\vec{v}_{CV})\cdot\hat{n}dA = \dot{Q}_{\text{net,in}} + \dot{W}_{\text{shaft,in}} \underbrace{-\int P(\vec{v}-\vec{v}_{CV})\cdot\hat{n}dA}_{\text{work by pressure forces}}$$

The energy balance may be simplified under certain assumptions to produce the *mechanical energy equation*, which is also known as the *extended Bernoulli equation*. In particular, assuming steady flow, incompressibility, and a *single inlet and single outlet with uniform flow*, integration over the control volume and algebraic rearrangement gives the following expression:

$$\frac{P_{\text{out}}}{\rho} + \frac{1}{2}v_{\text{out}}^2 + gz_{\text{out}} = \frac{P_{\text{in}}}{\rho} + \frac{1}{2}v_{\text{in}}^2 + gz_{\text{in}} + w_{\text{shaft,in}} - \underbrace{(\hat{u}_{\text{out}} - \hat{u}_{\text{in}} - q_{\text{net,in}})}_{\text{losses}} \qquad \text{(Extended Bernoulli Equation)}$$

where that $q_{\text{net,in}}$ is defined by $\dot{Q}/\dot{m}$, and $w_{\text{shaft,in}}$ is defined by $\dot{W}/\dot{m}$. Note that this equation reduces to the Bernoulli equation if there is no shaft work or energy losses.

The integral form can be reduced to differential form. In particular, starting with the integral form of the conservation of mass, one can arrive at the **continuity equation**:

$$0 \;=\; \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho(\vec{v}-\vec{v}_{CV})\cdot \hat{n}\, dA$$

$$\text{assuming constant, fixed CV:} \quad = \int_{CV}\frac{\partial \rho}{\partial t}\, dV + \int_{CS}\rho\vec{v}\cdot \hat{n}\, dA$$

$$\text{applying divergence theorem:} \quad = \int_{C}V\frac{\partial \rho}{\partial t}\, dV + \int_{CV}\nabla\cdot(\rho\vec{v})\, dV$$

$$= \int_{CV}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right]dV$$

$$\text{as } V\to 0, \text{ i.e. at a point:} \quad = \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})$$

Thus, the **continuity equation**, which gives a condition which *every point* must satisfy to satisfy conservation of mass, is given by the differential equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad\qquad \text{(Continuity Equation)}$$

Assuming incompressibility, all derivatives involving ρ are zero; then, the *incompressible* continuity equation is given by:

$$\nabla\cdot\vec{v} = 0 \qquad\qquad \text{(Incompressible Continuity)}$$

Then, the continuity equation will hold on a fluid field *if and only if* the fluid is incompressible on that field. Note that a flow may be incompressible and still have variable density; in particular, an incompressible fluid may have variable density layers (i.e. water which is heated from the bottom is incompressible in the liquid phase, but the density increases further from the heat source).

Stream Functions. Consider the simple case of steady, constant density, two-dimensional flow. Define $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

Then, $\psi(x,y)$ allows a flow field to be defined with a single function, instead of two velocity functions (i.e. $(u(x,y), v(x,y))$. Note that ψ plus any constant defines the same flow field; by convention, $\psi(x,y)$ is typically written without any such constant. $\psi(x,y)$ has several useful properties:

The velocity field defined by ψ is incompressible. Applying the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies \underbrace{\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y}}_{\text{definition of }\psi} = 0$$

Lines of constant ψ are streamlines. Suppose $\psi(x,y) = c$. Then,

$$\underbrace{0 = d\psi}_{\text{on const. }\psi} = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -v\,dx + u\,dy \implies \underbrace{\frac{dy}{dx} = \frac{v}{u}}_{\text{streamline def'n}}$$

The volume flow rate between streamlines is the difference between streamlines. Specifically, if c_1 and c_2 are two streamlines given by ψ with $c_2 > c_1$, then $q = c_2 - c_1$ is the volumetric flow rate *per unit depth*.

Example. *Derive the stream function $\psi(x,y)$ for a flow field given by $(u,v) = (Ax, -Ay)$.*

There are two constrains ψ must satisfy:

$$\frac{\partial \psi}{\partial y} = u, \qquad \frac{\partial \psi}{\partial x} = -v$$

Integrating u gives:

$$\frac{\partial \psi}{\partial y} dy = Ax dy \implies d\psi = Ax dy \implies \psi(x,y) = Axy + f(y)$$

Then, integrating v gives:

$$\frac{\partial \psi}{\partial x} dx = Ay dx \implies d\psi = Ay dx \implies \psi(x,y) = Ayx + g(x)$$

Then, it must be the case that $g(x) = f(y)$; this can only be the case if $g(x) = f(y) = c$ for some constant c. This constant term may be neglected (by convention) to give:

$$\boxed{\psi(x,y) = Axy}$$

This stream function defines the given flow field. Note that the streamlines are given by

$$\psi(x,y) = k \implies y = \frac{k}{Ax}$$

where k is any arbitrary constant. For any k, this equation defines a unique streamline.

The integral form of the conservation of momentum can also be reduced to differential form by a similar process. The final result is the following equation:

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla P + \underbrace{\nu \nabla^2 \vec{u}}_{\nu = \mu/\rho} + \vec{g} \qquad \text{(Conservation of Momentum)}$$

Written in terms of its components, this is equivalently given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g_x \qquad (x)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + g_y \qquad (y)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + g_z \qquad (z)$$

The continuity equation and the conservation of momentum, when taken together, constitute the **Navier-Stokes equations**.

For many problems, it is convenient to write the Navier-Stokes equations in terms of cylindrical coordinates. Therefore, the Navier-Stokes equations are equivalently written as:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(Continuity)}$$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad (r)$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad (\theta)$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial \theta}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad (z)$$

Example: Plane Poiseuille Flow. Find an expression for the velocity of a fluid between two fixed, infinitely long plates separated by a distance b. Assume the flow is driven by a known pressure differential $\frac{\partial p}{\partial x}$, and that the fluid is incompressible.

Assumptions.

1. Planar flow $\implies$ no flow in z direction.

2. Infinitely long plates $\implies$ flow is fully developed $\implies$ no change in x direction, i.e. $\frac{\partial}{\partial x} \to 0$.

3. Incompressible fluid $\implies \nabla \cdot \vec{v} = 0$

4. Steady state $\implies$ no change with time $\implies \frac{\partial}{\partial t} = 0$

5. Neglect gravity (planar orientation not specified).

Boundary Conditions.

1. No-slip boundary condition at the walls: $u(0) = u(b) = 0$.

2. Impermeable walls: $v(0) = v(b) = 0$.

Since the fluid is incompressible, $\nabla \cdot \vec{v} = 0$:

$$\nabla \cdot \vec{v} = \underbrace{\frac{\partial u}{\partial x}}_{0,\,\text{fully developed}} + \frac{\partial v}{\partial y} = 0 \implies \frac{\partial v}{\partial y} = 0 \implies v = v_0$$

Applying boundary conditions, $v(0) = 0$, so $v_0 = 0$. Therefore, $v = 0$. Now, considering the momentum balance in the x direction:

$$\underset{(1)}{\frac{\partial u}{\partial t}} + u\underset{(2)}{\frac{\partial u}{\partial x}} + v\underset{(3)}{\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\underset{(4)}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

These terms are zero because the flow is assumed to be steady state (1), and fully developed (2,4). From the continuity equation, $v = 0$ (2). Therefore, the expression becomes:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}$$

Now, it must be checked whether p is a function of y; if not, this expression can be integrated with $\frac{\partial p}{\partial x}$ treated as a constant. Consider momentum in the y direction:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

v is constant, so all derivatives are 0. Therefore, $\frac{\partial p}{\partial y} = 0$ meaning p does not vary with y, i.e. p is not a function of y. Therefore, the original equation can be safely integrated:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2} \implies \frac{1}{\nu\rho}\frac{\partial p}{\partial x}y + c_1 = \frac{\partial u}{\partial y} \implies \frac{1}{2\nu\rho}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2 = u(y)$$

Applying boundary conditions:

$$u(0) = 0 \implies c_2 = 0; \quad u(b) = 0 \implies \frac{1}{2\nu\rho}\frac{\partial p}{\partial x}b^2 + c_1 b = 0 \implies c_1 = -\frac{b}{2\nu\rho}\frac{\partial p}{\partial x}$$

Substituting c_1 and $\mu = \nu\rho$ gives the (quadratic) velocity profile:

$$\boxed{u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - yb\right)}$$

5.5 Dimensional Analysis

The basic principle of *dimensional analysis* is that all physical laws must be independent of any particular system of measurement; in other words, any physical law can be expressed equivalently in any self-consistent unit system. Dimensional analysis involves formulating problems in terms of dimensionless variables and parameters.

The benefit of dimensional analysis is that it substantially reduces the number of necessary experiments to determine a relationship between parameters. In particular, if a physical phenomenon depends on n variables, and k data points are needed for each experiment, then n^k experiments must be conducted; dimensional analysis often reduces this into fundamental, non-dimensional parameters which define.

Buckingham Pi Theorem. Let $q = (q_1, \ldots, q_n)$ be n variables such that there exists a functional relation of the form $f(q) = 0$. Then, the n variables can be combined to form exactly $m = n - r$ independent, non-dimensional variables $\Pi = (\Pi_1, \ldots, \Pi_m)$, where r is the number of independent dimensions, which is the rank of the dimensional matrix. Equivalently, $f(q)$ can be written in the form $\phi(\Pi) = 0$.

The general steps for this dimensional analysis are as follows:

1. Determine relevant variables and parameters $x_1, \ldots, x_n$.

2. Form the *dimensional matrix*.

3. Determine the rank r of the dimensional matrix.

4. Determine the number of dimensionless groups $m = n - r$.

5. Construct the dimensionless Π-groups. That is, each group is of the form $\Pi_i = x_i x_{r_1}^{a_1} \ldots x_{r_r}^{a_r}$ where x_{r_i} are the repeated parameters which are the same for each Π group. The exponents $a_1, \ldots a_r$ are chosen so that Π_i is dimensionless.

6. State the dimensionless relationships,

7. Simplify the relationship, if possible; this generally requires physical insight, such as experiments which determine a proportionality coefficient.

One important application of dimensional analysis is in model testing. It is often difficult to test a full-scale prototype, both because of practical wind tunnel size restrictions and expenses in producing full-size models. Dimensional analysis allows one to design experiments where the scale model testing faithfully represents and predicts the full-scale behavior. This is the case when the model and prototype are **dynamically similar**. For a model and prototype to by dynamically similar, they must be **scale similar** (i.e. boundary conditions (geometry) and initial conditions are proportionally scaled) and their *dimensionless groups must match*. So long as flow is dynamically similar, the fluid used does not matter. In fact, it is often desirable to use water when testing aerodynamics, because this matches Reynolds numbers with considerably lower speeds.

Nondimensionalization of Governing Equations. If a governing equation is known, it can be non-dimensionalized to determine the dimensionless parameters characterizing its flow. For example, consider the conservation of linear momentum, given by

$$\rho \left(\frac{d\vec{u}}{dt} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{g}$$

Define the following dimensionless parameters:

$$\vec{u}^* = \vec{u}/V, \quad \vec{x}^* = \vec{x}/L, \quad t^* = t/\tau, \quad p^* = p/p_0, \quad \vec{g}^* = \vec{g}/g, \quad \nabla^* = L\nabla$$

where V, L, τ, p_0 are *characteristic* properties of the flow. By making substitutions, linear momentum becomes nondimensionalized as follows:

$$\left[\frac{L}{TV} \right] \frac{d\vec{u}^*}{dt^*} + \vec{u}^* \cdot \nabla^* \vec{u}^* = -\left[\frac{p_0}{\rho V^2} \right] \nabla^* p^* + \left[\frac{\mu}{\rho VL} \right] (\nabla^*) \vec{u}^* + \left[\frac{gL}{V^2} \right] \vec{g}^* \iff \text{St} \frac{d\vec{u}^*}{dt^*} + \vec{u}^* \cdot \nabla^* \vec{u}^* = -\text{Eu} \nabla^* p^* + \frac{1}{\text{Re}} (\nabla^*) \vec{u}^* + \frac{1}{\text{Fr}^2} \vec{g}^*$$

Therefore, the flow is characterized by the Strouhal number, the Euler number, the Reynolds number, and the Froude number.

5.6   Special Cases: Internal and External Flows

External and internal flows are special cases which often arise in engineering applications. Some general techniques are derived to simplify and evaluate simple systems. In both external and internal flows, behavior varies substantially between turbulent and laminar regimes; Reynolds number dependence is discussed. In general, a flow regime is **laminar** for $Re < 2000$, and a flow regime is **turbulent** for $Re > 4000$. Between these is the **transition** regime, characterized by $Re \in (2000, 4000)$.

Flow bounded by walls is called **internal flow**. Common examples include flow through pipes and ducts, which is of interest for many engineering applications. Key problems here involve energy loss as fluid flows through the pipes.

Characteristics of Laminar and Turbulent Pipe Flow. The Reynolds number characterizing pipe flow is related to the pipe diameter, and for a circular pipe is typically given by:

$$Re = \frac{\rho v D}{\mu}$$

The *entrance length*, L_e, is the distance from an entrance or geometry change after which flow is fully developed. L_e varies differently for different regimes, and is given by:

$$\frac{L_e}{D} \approx \begin{cases} 0.06 Re, & \text{laminar flow} \\ 4.4 Re^{1/6}, & \text{turbulent flow} \end{cases}$$

When flow is fully developed, it has a defined velocity profile. In laminar flow, this profile can be derived from Navier Stokes, and is a parabolic profile with $U_{\max} = 2 U_{\text{avg}}$.

In a turbulent pipe flow, there is a different velocity profile at each position z and time t, and the average profile is a flatter parabola, with $U_{\max} \approx U_{\text{avg}}$. The flatter parabolic profile means there is a larger velocity gradient, and therefore greater shear, at the walls. This means that, in turbulent flows, there is a large frictional drag penalty compared to laminar flows.

The extended Bernoulli equation can be adapted for general pipe flows, including turbulent flows. In particular, by dividing the mechanical energy equation by g and scaling kinetic energy terms by a coefficient α to scale for the non-uniform profiles, the mechanical energy equation becomes:

$$\frac{P_2}{\rho g} + \alpha_2 \frac{v_2^2}{2g} + z_2 = \frac{P_1}{\rho g} + \alpha_1 \frac{v_1^2}{2g} + z_1 + h_S - h_L$$

where h_S is the head shaft, given by $h_s = \dot{W}/(\dot{m}g)$, and h_L is the head loss. In general, the coefficients α depend on the flow regime:

$$\alpha = \begin{cases} 1, & \text{uniform flow} \\ 1.08, & \text{turbulent flow} \\ 2, & \text{laminar flow} \end{cases}$$

Head loss can be divided into major and minor losses. Major head losses are related to pressure drops which occur due to wall shear stress; minor head losses are related to pipe components, like bends and nozzles. Major head losses are given by:

$$h_{L,\text{major}} = \frac{\Delta P}{\rho g} = f \cdot \frac{L}{D} \cdot \frac{v^2}{2g}$$

where f is the **friction factor**; for laminar flow, f is only a function of Re, and for turbulent flow, wall friction dominates and f is only a function of the roughness ϵ/D, and f is only an empirical relationship; f can be computed or read from a table, such as a **Moody chart**, according to the following criterion:

$$f = \begin{cases} 64/Re, & \text{laminar flow} \\ f(\epsilon/D), & \text{turbulent flow} \end{cases}$$

where $f(\varepsilon/D)$ is the experimental relationship as a function of roughness. The minor losses are computed as follows:

$$h_{L,\text{minor}} = K_L \frac{v^2}{2g}$$

where K_L is a function of geometry. Total head loss can be found by summing all the major and minor head losses in a section of pipe.

This analysis can be extended from circular pipes to approximate flow in any arbitrary non-circular duct. In particular, the *hydraulic diameter* D_h can be introduced instead of circular diameter D, where:

$$D_h = \frac{4A}{P}$$

where A is the cross-sectional area of the duct, and P is the perimeter of the duct. Using D_h as the characteristic length in head loss calculations gives an approximation for non-circular geometries.

Flow over a body immersed in a fluid is called **external flow**. Flow past any object with a solid surface develops a thin *boundary layer* in which viscous effects are important, even if the flow far from the body is inviscid.

Consider a flat plate of infinite length, with uniform free-stream velocity U parallel to the plate. Let x be the direction of flow. As flow proceeds over the plate, fluid near the wall becomes stationary, resulting in a gradient from zero-velocity at the wall to the free-stream velocity U at some thickness δ from the wall. This thickness, δ, is known as the *boundary layer*.

Over sufficient length, the boundary layer tends to transition from a *laminar boundary layer* to a *turbulent boundary layer*. Note that, by choosing x as the characteristic length, the Reynolds number characterizing flow over the plate is given by:

$$Re_x = \frac{\rho U x}{\mu}$$

The *boundary layer* is laminar for $Re_x < 2 \times 10^5$, and is turbulent for $Re_x > 3 \times 10^6$.

Boundary Layer and Resulting Wall Shear. There are several metrics for thickness which relate some quantities:

> **Boundary Layer Thickness**. The boundary layer thickness, also called he *disturbance thickness*, δ_{99}, is the thickness at which 99% of the free-stream velocity is recovered:
>
> $$\delta_{99} = y|_{u=0.99U}$$
>
> **Displacement Thickness**. The *displacement thickness*, δ^*, is the displacement from the wall which results in an equivalent mass flow deficit. Displacement thickness is given by:
>
> $$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$
>
> **Momentum Thickness**. The *momentum thickness*, θ, is the displacement from the wall which results in an equivalent momentum deficit. Momentum thickness is given by:
>
> $$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Under the assumption of steady, two-dimensional flow with a thin boundary layer, the following relationship for wall shear stress τ can be derived:

$$\frac{\tau}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx} \qquad \text{(Momentum Integral Equation)}$$

Note that U is from the outer flow; if the outer flow is inviscid, relations can be found with Bernoulli's equation. In particular, by differentiating Bernoulli's equation:

$$P + \frac{1}{2}\rho U^2 = c \implies \frac{dP}{dx} = -\rho U \frac{dU}{dx}$$

For many geometries and flow conditions, a boundary layer may depart from the surface, resulting in a region of reversed flow near the surface; this phenomenon is called flow **separation**. A body is called *bluff*, or *blunt*, if large flow separations

occur (i.e. a sphere of a cube), and a body is called *streamlined* if flow remains mostly attached (i.e. a low-AoA airfoil). Quantitatively, flow separation occurs at a point on the surface where the velocity gradient cannot overcome the pressure difference over the body, meaning $\left.\dfrac{\partial u}{\partial y}\right|_{(x,0)} = 0$. Past this critical point, the velocity gradient reverses due to the pressure gradient.

Laminar boundary layers are generally more prone to separation than a turbulent boundary layer. Separation of a laminar boundary layer generally leads to transition flow, which energizes the flow and leads to reattachment further along the geometry, resulting in a separation bubble.

Force in the direction of the fluid stream in known as **drag**; flow opposite the direction of the fluid stream is known as **thrust**. Force perpendicular to the free stream is known as **lift** or **downforce**, depending on the direction. Dimensional analysis can be used to derive nondimensional relationships for lift and drag, written by convention as the *drag coefficient*, C_D, and the lift coefficient, C_L:

$$C_D = \frac{F_D}{\frac{1}{2}\rho v^2 A}, \qquad C_L = \frac{F_L}{\frac{1}{2}\rho v^2 A}$$

In both coefficients, A is any characteristic area, which can result in different coefficients. Therefore, by convention, for a bluff body, A is taken to be the cross-sectional area affected by the fluid, and for an airfoil the area is taken to be the chord length times span length, $A = cs$. Note that the quantities $C_L A$ and $C_D A$ are constant for a body at given conditions, regardless of the choice of characteristic area.

6 Mechanical Behavior of Materials

Notes from **MECHENG 382: Mechanical Behavior of Materials**

Taken at University of Michigan, Fall 2025

using *Engineering Materials*, by Ashby and Jones

6.1 Statics

The **stiffness** of an object is defined as the ratio of applied load to resultant deformation, i.e.

$$\text{stiffness} := \frac{F}{\Delta x}$$

The **compliance** of an object is the reciprocal of stiffness, i.e.

$$\text{compliance} := \frac{\Delta x}{F}$$

The **strength** of an object is its ability to resist failure. The **toughness** of an object is its ability to absorb and dissipate energy, where a *tough* object can dissipate energy effectively and a *brittle* object cannot.

The stiffness of an object, as well as its strength, depends on both geometry and material properties. For many materials, the *tensile test* is used to extrapolate fundamental material properties.

Tensile Test. The tensile test utilizes a (typically cylindrical) test piece of a material, from which a stress-strain curve for the material in general can be derived.

1. Measure initial length L_0 and cross-sectional area A_0.

2. Apply a tensile normal force F to the test piece and measure the change in length ΔL until fracture. This allows a plot of force against displacement.

3. Compute and plot stress and strain using F, A_0, L_0, and ΔL. This allows a plot of stress against strain, the engineering stress-strain curve.

4. Compute and plot true stress and true strain. This allows a plot of true stress against true strain.

The result of this procedure is a geometry-independent relationship between stress and strain for a given material. Note that the slope of the initially linear portion of the curve gives the Young's Modulus for the material.

Using this stress strain curve, it is possible to compute the energy absorption per unit volume of a material, given by

$$u = \frac{U}{V} = \int \sigma d\varepsilon$$

Note that the units are $[N \cdot m^{-2}]$ which is equivalent to $[N \cdot m \cdot m^{-3}]$, and thus the units are equivalently $[J \cdot m^{-3}]$, which is indeed energy per volume.

There are two definitions of stress and strain: *nominal* stress and strain, and *true* stress and strain. Nominal stress and strain is defined familiarly as

$$\sigma_n = \frac{F}{A_0}, \qquad \varepsilon_n = \frac{\Delta L}{L_0}$$

Nominal stress and strain are accurate for the (linear) elastic deformation portion of a stress-strain curve. However, for large

plastic deformations, these relations cease to hold. More generally, true strain is defined by differential relationship:

$$d\varepsilon_t = \frac{dL}{L}$$

$$\implies \varepsilon_t = \ln\left(\frac{L}{L_0}\right) = \ln\left(\frac{L_0 + \Delta L}{L_0}\right)$$

$$\implies \boxed{\varepsilon_t = \ln(1 + \varepsilon_n)}$$

True stress is still F/A, but because volume is conversed in plastic deformation, $A_0 L_0 = AL$, and therefore

$$A = \frac{A_0 L_0}{L} = \frac{A_0}{1 + \varepsilon_n}$$

$$\implies \sigma_t = \frac{F}{A_0}(1 + \varepsilon_n)$$

$$\implies \boxed{\sigma_t = \sigma_n(1 + \varepsilon_n)}$$
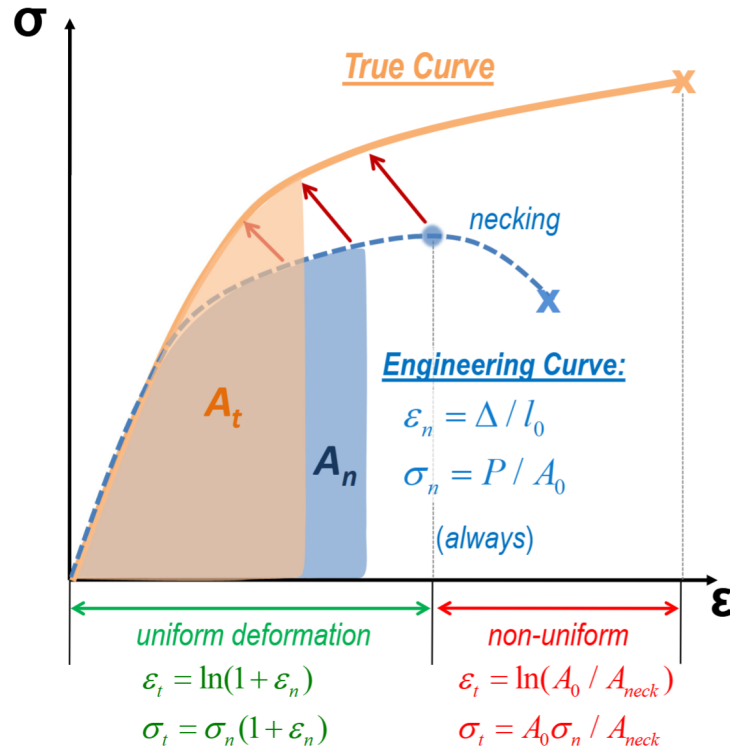
True stress may also be described by a power law, given by

$$\sigma_t = H\varepsilon_t^n$$

where H is the strength coefficient for a material, and n is the *strain-hardening exponent* which accounts for work hardening. n has a physical meaning, as necking occurs precisely at $\varepsilon_t = n$.

6.2   Structure

Material properties depend on the atomic structure of a material. In particular, material properties depend on: type of atomic bonds; the material's *crystal structure* (arrangement of atoms and molecules); defects in the crystal structure; and interactions of defects (the "microstructure"). Properties are either *structure-insensitive*, depending primarily on the crystal structure and atomic bond types (i.e. elastic modulus, density, and melting temperature), or *structure-sensitive*, depending primarily on the defects and microstructure (i.e. yield strength, ultimate tensile strength).

In general, the atomic structure of a material aims to *minimize energy* and *maximize entropy*. Recall that Gibbs free energy is given by $G = H - TS$ and force on a body is related to energy most generally by $F_x = -dU/dx$. Thus, the force acting on an atom or molecule is given by

$$F_x = -\frac{dG}{dx}$$

Attractive forces occur when atoms are distant, and repulsive forces occur when atoms are close (note that $F_x \to \infty$ as molecules approach zero separation). When $-dG/dx = 0$, the particle is in an energy well and no net force acts on the particle. Near $x = d_0$, a particle will oscillate with thermal energy $E_T = kT$, where k is Boltzmann's constant.
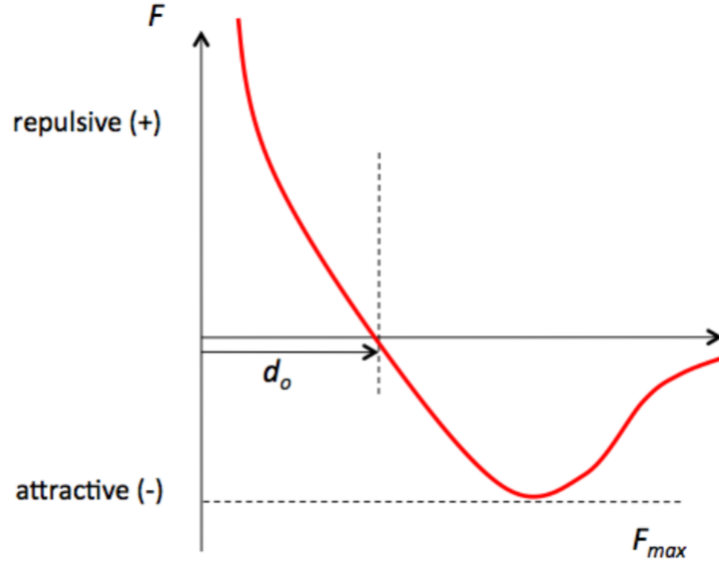
Physical origin on modulus. Consider atoms with separation distance d_0. Then, applied stress to maintain equilibrium is given by

$$\sigma = -\frac{F}{d_0^2} = \frac{1}{d_0^2}\frac{dG}{dx}$$

Then, Young's Modulus is given by

$$E = \frac{d\sigma}{d\varepsilon} = \frac{1}{d_0^2}\frac{dG/dx}{dx/d_0} = \frac{1}{d_0^3}\frac{dG}{d\varepsilon}$$

Substituting $G = H - TS$ and $\Omega := d_0^3$ (where Ω is the *atomic volume*) gives the following:

$$E = \frac{1}{\Omega}\underbrace{\left(\frac{d^2H}{d\varepsilon^2} - T\frac{d^2S}{d\varepsilon^2}\right)}_{d^2U/d\varepsilon^2}$$

The quantity $\frac{1}{\Omega}\frac{d^2H}{d\varepsilon^2}$ is **bonding strength**, and the quantity $\frac{T}{\Omega}\frac{d^2S}{d\varepsilon^2}$ is the entropy change for a material, which is negligible in many materials (with notable exceptions of polymers, glasses, and other disordered materials).

There are various types of bonds which occur between atoms. The main types are outlined below:

1. **Primary Bonds**. These types of bonds are very strong and of roughly equivalent strength to each other:

   (a) *Ionic bonds* (electron transfer). One atom donates an "excess" electron to another atom "lacking" an electron; the result is electrostatic attraction between atoms. Example: NaCl.

   (b) *Covalent bonds* (electron sharing). Electrons are shared by atoms to produce a stable configuration. Typical in organic molecules, i.e. ethane.

   (c) *Metallic bonds* (electron sea). Electrons are shared between atoms, giving ions freedom to move and capacity to conduct electricity and heat.

2. **Secondary Bonds**. These bonds are considerably weaker than primary bonds, i.e. they are broken with relative ease compared to primary bonds:

   (a) *Hydrogen bonds* (permanent dipole). Hydrogen atom attracts neighboring molecules, generating a permanent dipole (ex. H_2O). Can be induced by any individual atom bonding with hydrogen. Similar effects with other atoms than hydrogen, but the electronegativity of hydrogen has tendency to cause a more pronounced dipole moment.

   (b) *Van der Waals bond* (fluctuating dipole). Very weak bonds emerging from unequal pull of electrons as electron density fluctuates (ex. O_2).

The arrangement of these atoms results in different material properties. A material has, in general, one of the following structures:

1. **Crystalline Structure**. Ordered structure with a lattice which repeats periodically. Most metals and ceramics have this structure.

2. **Poly-Crystalline Structure**. Material is comprised of several small grains of different orientation, where each grain has an ordered lattice structure on the interior and a disordered structure on the boundary. Most engineering materials (i.e. real metals) have this structure.

3. **Amorphous Structure**. Structure is disordered; there is no periodic repetition of molecular structure. Glasses and polymers have this structure.

For a crystalline structure, it is possible to study the entire structure of a material by studying representative cells which tile the entire plane of atoms. A *primitive cell* is defined as the smallest unit for such a cell which tiles the plane by translation alone without missing area. A *conventional cell* is typically introduced instead to simplify the geometry involved, where a conventional cell only needs to tile the plane by translation alone.

Given a lattice structure, it is possible to approximate the density with which atoms are arranged. In particular, using a *hard sphere approximation*, i.e. that the atoms are replaced by hard spheres of radius r_0, the *packing efficiency* can be defined by:

$$\text{Packing Efficiency} = \frac{N_{\text{atoms}} \pi r_0^2}{A_{\text{unit cell}}}$$

where packing efficiency is a measure of space occupied by atoms per total space available in the plane. In three dimensions, the *packing factor* is the measure of volume occupied by atoms per total volume available in the space, given by:

$$\text{Packing Factor} = \frac{N_{\text{atoms}} \Omega}{V_{\text{unit cell}}}, \quad \text{where } \Omega = \frac{4\pi r_0^3}{3}$$

There are three main types of packing which occur in real materials:

1. **Body-Centered Cubic** (BCC). A central atom makes contact with its 8 nearest neighbor atoms along the diagonal. Packing factor of approximately 68%. This is the packing structure of lithium, chromium, potassium, etc.

2. **Face-Centered Cubic** (FCC). Atom packed on each of the six cubic faces between eight corner atoms, so each atom makes contact with 12 nearest neighbors. Packing factor of approximately 74%. This is the packing structure of aluminum, copper, lead, etc.

3. **Hexagonal Close Packing** (HCP). Sheets of triangular-packed atoms are pressed together, resulting in packing factor of 74% (the exact same as FCC). HCP also has 12 contacting nearest neighbors to each atom. This is the packing structure of magnesium, titanium, zinc, etc.

Real materials do not have perfect crystalline packing. Irregularities emerging in real structures, known as *defects*, have substantial effects on material properties. Defects are outlined below:

1. **Point Defects** (0D). A single missing atom on a lattice is called a *vacancy* defect. An extra atom on a lattice is called a *interstitial* defect. A different species of atom replacing an atom on the lattice is called a *substitutional* defect.

2. **Volume Defects** (3D). In three dimensions, a collection of vacancies produces a *void*. Interstitial defects become *precipitate* defects with addition of solute atoms and *compound* defects with addition of solutes and solvents.

3. **Area Defects** (2D). In two dimensions, defects emerge in the form of planar *cracks* through the material.

4. **Line Defects** (1D). Line defects have a substantial effect on material properties; these defects take the form of *dislocations*. An *edge dislocation* is an insertion of a line of atoms; these atoms must have a different number of nearest neighbors compared to the rest of the material, and their insertion deforms the neighboring crystalline structure. A *screw dislocation* involves shearing along a plane, resulting in a step of one lattice unit along the plane and a local spiral along the central axis.

Quantification of Line Defects. Line defects are quantified by *Burger's Vector*. The process is as follows:

1. Draw an arbitrary loop of size $m \times n$ containing line dislocation. If the loop is enclosed, there is no dislocation; if the loop is not enclosed, draw Burger's Vector $\vec{b}$ pointing from the starting point to the final point.

2. Let $\vec{\xi}$ be the direction of dislocation. If a defect is an edge dislocation, $\vec{b} \perp \vec{\xi}$. If a defect is a screw dislocation, $\vec{b} \parallel \vec{\xi}$.

3. If there are several dislocations, sum the Burger's vectors for each dislocation. The linear combination of these vectors is the net dislocation.

The resulting Burger's vector quantifies the dislocation felt by distant vectors from an ideal crystalline lattice structure.

All real solids are impure. Impurities, both interstitial and substitutional, can change the material properties of a solid. For example, a metal *alloys* is a deliberate mixture of different metals to produce a different metal, generally one which is stronger than either metal alone. A material containing a random dispersion of impurities is called a *solid solution*, where the *solvent* are the host atoms (the major component) and the *solute* are the impurities (the minor component). A *phase* of a solid solution is a region of the material with *uniform physical and chemical properties*.

Metal alloys may be described with a *binary phase diagram*. As opposed to traditional phase diagrams, i.e. phase diagrams for a pure substance, or for a dissolved mixture, a binary phase diagram does not have a well-defined melting temperature. Instead, there is an envelope within which there is a mixture of solid and liquid metals. In this case, the weight composition at a particular temperature may be found by reading off the concentration at the solid and liquid state and algebraically solving the resulting equation. In general, the *lever rule* gives the solution to this simple system of equations:

$$X_s = \frac{w_0 - w_l}{w_s - w_l}, \; X_l = \frac{w_s - w_0}{w_s - w_l}$$

The lever rule remains true in more complex cases as well, so long as one applies it within each distinct region.

The simplest binary phase diagram has one solid phase, one liquid phase, and one two-phase region. Qualitatively, the structure of this phase diagram explains the real-world development of microstructure: as the liquid mixture of metals cools, one solid precipitates before the other in small parts. These components are unable to fully align when cooling is completed, causing irregularities in the lattice.

A more realistic phase diagram is the *binary eutectic system*. In these systems, the *eutectic point* is a special point. In contrast to the simple binary phase system, a eutectic reaction moves immediately from liquid to solid as temperature drops below the eutectic point. A binary eutectic system typical for many metal alloys is illustrated below:
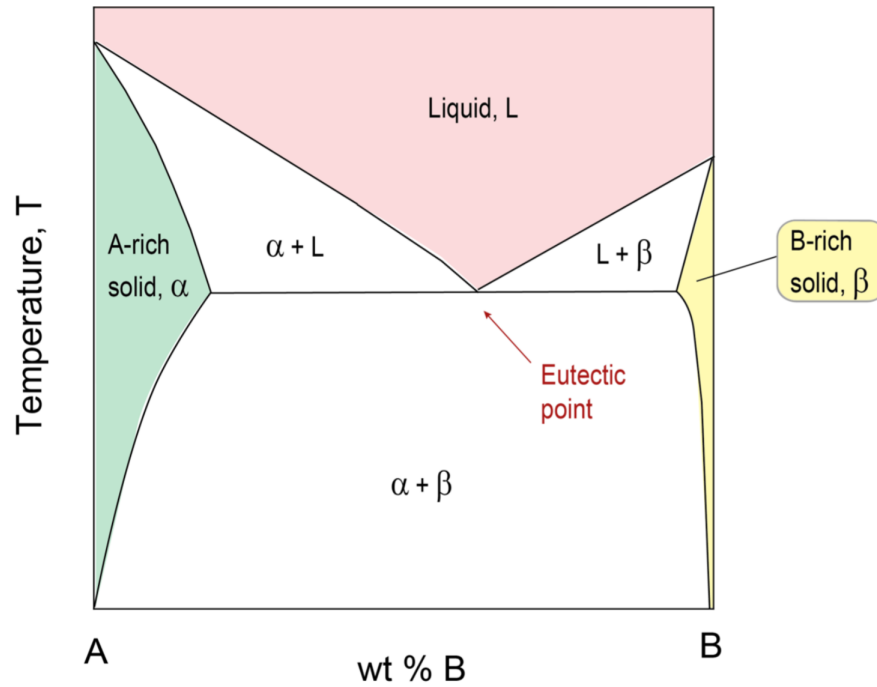
6.3   Yield in Ductile Materials

Real-world experiments show that ductile materials, under tensile loading as in the tensile test, typically fail during necking and at an angle of approximately 45°. From this, it is reasonable to assume that the material is failing under shear. In fact, if the yield stress is measured as σ_Y, a stress transformation gives the *yield criteria* as:

$$\tau_{\max} \geq k := \frac{\sigma_Y}{2} \tag{Yield Criterion}$$

Tresca Yield Criterion. Given the hypothesis that ductile material fails in shear, define $k := \dfrac{\sigma_Y}{2}$. Thus, material will fail if $\tau_{\max} \geq k = \dfrac{\sigma_Y}{2}$.

1. Define the loading parameter by:
$$\sigma_s = \max(|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|)$$
That is, find the diameter of the largest Mohr's circle constructed from the principle stresses.

2. By assumption, the material will yield at $\sigma_Y = 2k$.

3. By substitution, the material will yield if the Tresca stress exceeds yield stress:

$$\sigma_s \geq \sigma_Y$$

This result can be visualized with a **Tresca yield surface**. If stresses are contained within the yield surface, the material is safe; if the stresses are outside the yield surface, the material is susceptible to yield.

An alternative hypothesis states that the material will yield when *deviatoric strain energy* exceeds the deviatoric strain energy at the yield stress state. That is, given

$$U_{\text{dev}} = \frac{1+\nu}{3E} \cdot \frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]$$

and therefore deviatoric strain energy at yield given by

$$U_Y = \frac{1+\nu}{3E}\sigma_Y^2$$

then, yield is predicted to occur when

$$U_{\text{dev}} \geq U_Y$$

von Mises Yield Criteria. The von Mises yield criteria for failure emerges from the assumption that strain energy causes yield. The steps for evaluating yield in a von Mises framework is as follows:

1. Compute the equivalent von Mises stress as follows:

$$\sigma_H = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]}$$

2. Material property (conventially uniaxial tensile strength) given by σ_Y.

3. Yield is predicted to occur where

$$\sigma_H \geq \sigma_Y$$

This results in a yield surface which takes the shape of a smooth oval in two dimensions, and a cylinder in three dimensions.

The Tresca and von Mises criteria typically give different conditions for yield in the same material. Working from *tensile conditions*, the Tresca criteria is more conservative than the von Mises critera, but the two agree in uniaxial testing conditions. The two-dimensional projection of the two overlaid is shown below:
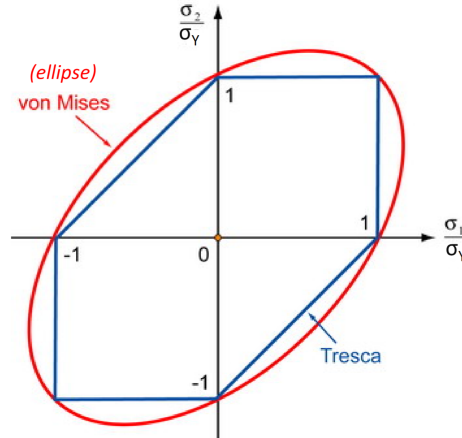
Note that, from another test, the von Mises criteria may be more conservative than the Tresca criteria, but the two tests will *always agree* at the initial testing points (in this case, yield stress; in another test, perhaps a shear stress).

Many geometries result in nonuniform stress concentration. The ratio of the maximum stress to the average stress defines the *stress concentration factor*:

$$K_T = \frac{\sigma_{\text{max}}}{\sigma_{\text{avg}}}$$ (Stress Concentration Factor)

6.4 Plastic Deformation and Metals

Recall that the mechanism of *plastic deformation*, which is irreversible deformation, is shear. Specifically, plastic deformation can be caused by any of the following shearing mechanisms which produce a new, stable state:

Martensitic Transformation. The structure of atoms is transformed, i.e. from FCC to BCC, by a transformation. This process will be discussed in-depth later.

Twinning. Some volume of the grain atoms are moved to mirror the original structure.

Slip (dislocation). Planes on the crystal lattice slide via dislocation motion, causing shape change.

The slip mechanism is the most important mechanism of plastic deformation. The other mechanisms require large amounts of energy, as many atoms must be rearranged; for slip, only local dislocation is necessary, thus making slip a more common process in plastic deformation.

Around an edge dislocation, a local dilation of approximately b^2 occurs, where b is the magnitude of Burger's vector. The dislocation energy is strain energy, which can be calculated by the same energy density as for macroscopic energy. The stress field around an edge dislocation is complicated; however, it can be shown that the **energy per unit length of an edge dislocation** is given by:

$$u_{\text{edge}} = \frac{Gb^2}{4\pi(1-\nu)} \ln\left(\frac{R}{r_0}\right) \propto Gb^2$$

Total energy can then be calculated by multiplying u_{edge} by the length of the edge dislocation. Similarly, one can derive the **energy per unit length of a screw dislocation**, which is given by

$$u_{\text{screw}} = \frac{Gb^2}{4\pi} \ln\left(\frac{R}{r_0}\right) \propto Gb^2$$

Note that, as expected, both energies are proportional to Gb^2. That is, Gb^2 is a sort of "minimum energy" carried by a dislocation, and therefore a sort of "minimum energy" required to cause dislocation.

To quantify dislocations, it is useful to define **dislocation density** ρ. Dislocation density is defined as the length l of dislocation per unit volume V_0, so

$$\rho := \frac{l}{V_0}$$

ρ has SI units m^{-2}. ρ may be equivalently defined as the number of dislocation lines crossing a unit area; the units are the same, and assuming random distribution of dislocations, the result is expected to be the same. To produce a dislocation, some force is required. Resistive (frictional) force prevents atoms from moving within a lattice; this resistive force, k, is intrinsic to a material and is minimized in the close-packing directions (i.e. in the direction of nearest neighbors). When enough stress is applied to a crystal to overcome internal friction, the dislocation will move, causing plastic dislocation. The **force per unit length** applied as an edge dislocation moves is given by:

$$f_x = \tau_{xy} b$$

More generally, assuming a glide plane in the xy plane, the local force exerted on the dislocation is given by projection:

$$\vec{f} = \left[\left(\tau_{xz}\hat{i} + \tau_{yz}\hat{j}\right) \cdot \vec{b}\right]\hat{n} \qquad \text{(Peach-Koehler Force)}$$

This force assumes dislocation is local, i.e. dislocations to not interact.

Work Hardening Effect. In practice, dense dislocations cause **work hardening** as dislocations interact and prevent neighboring dislocations, effectively "locking" dislocations in place, i.e.

$$\sigma_Y \propto \sqrt{\rho} \qquad \text{(Single Crystal)}$$

In a **polycrystal** substance, dislocations cannot move between grain boundaries, as these boundaries are not aligned. Therefore, increasing the grain size weakens yield strength as the grain boundaries are lengthened; this can be accomplished by decreasing the grain size D. An empirical relationship is given:

$$\sigma_Y = \sigma_0 + \frac{k}{\sqrt{D}} \implies \sigma_Y \propto \frac{1}{\sqrt{D}} \qquad \text{(Polycrystal)}$$

Note that, as expected, yield strength increases with smaller grain size.

Similar work-hardening effects occur for alloys, although the mechanism is different.

Work Hardening of Alloys. When multiple solid solutes are present, the dislocation line distorts to minimize energy. Attractive and repulsive forces balance to prevent motion of dislocations, which increases material strength. The **shear strength**, k, increases as follows:

$$k \propto \varepsilon_m^{3/2} c^{1/2}$$

where ε is the strain induced as misfitting atoms are inserted into the lattice, and c is the concentration of solutes. Note that these values are taken around a point defect on the lattice. From this, it is obvious that a large mismatch between atomic sizes or a large solute concentration increases yield strength.

Similarly, for small precipitate, hardening is also proportional to strain induced on the lattice and particle size. In fact, **small, coherent precipitate** is proportional as follows:

$$k \propto \left| \varepsilon_{coh}^{3/2} \right| (d_0 f / b)^{1/2}$$

where d_0 is precipitate particle size, and f is the volume fraction. Then, at a constant volume, a larger size results in a higher yield strength.

For large precipitate, incoherence tends to occur. In this case, a shearing force must overcome resistance from both the matrix and precipitate. It can be shown that, for **large, incoherent precipitate**, increased volume fraction results in a higher yield strength. Dislocations traveling through a lattice cannot penetrate the large, hard precipitate, and therefore dislocations bow around these points; additional (tensile) resistance is generated by this bowing, which is given by:

$$\tau_0 = \frac{\Gamma}{bR} = \frac{Gb}{2R}, \qquad\qquad \Gamma := \frac{dU}{dl} = \frac{1}{2} Gb^2$$

where τ_0 is the critical external stress required to bend the dislocation line. Therefore, **bowing shear strength** in a hard precipitate has the following proportionality:

$$k_b \approx \frac{Gb}{L} = \frac{Gb f^{1/3}}{d_0}$$

Therefore, at a constant volume fraction, increasing size results in lower yield strength.

Discussion of microstructure is incomplete without discussing **particle kinetics**. While a given microstructure might be favorable, there is often a necessary activation energy required to convert a material from one microstructure to a lower energy state. This process is temperature-dependent. For example, diffusivity (in liquids or solids) is given by the (approximate) relationship:

$$D = D_0 \exp(-q/k_B T) = D_0 \exp(-Q/RT)$$

where q is the activation barrier, and Q is the activation barrier per mol. The overall rate of a reaction is therefore a function of both thermodynamics and kinetics: the rate is proportional to the kinetic term *mobility*, related to diffusivity, and the thermodynamic term *driving force*, related to the difference between energy well depths $G_A - G_B$ and also determining the direction of the reactions. Assuming H and S are almost constant within the temperature range between states A and B, the relationship becomes

$$\text{rate}_{A \to B} \propto \underbrace{\exp(-q/k_B T)}_{\text{kinetics}} \cdot \underbrace{(T_e - T)}_{\text{prop. to } G}$$

where T_e is the temperature which results in thermodynamic equilibrium between A- and B-phase. This proportionality can be transformed into a *time-temperature-transformation* diagram:
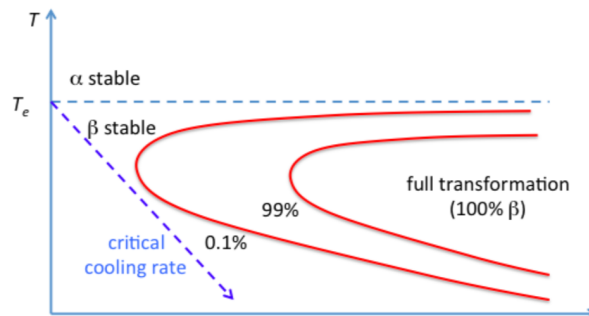
The implication of this graph is that there is a critical cooling rate which avoids transformation from α to β, limited by the kinetics. In general, the cooling rate determines the composition of a material.

During phase change from α- to β-phase, a certain energy barrier must be overcome in the formation of precipitate. It can be shown that, for a spherical precipitate droplet of radius r, there is an energy benefit proportional to $-r^3$, and an energy cost proportional to r^2; then, at low radii, the energy cost dominates and precipitate tends to collapse, while at large radii energy benefit dominates and precipitate will grow. The critical radius of this transformation is given by:

$$r^* = \frac{2\gamma_{\alpha\beta}}{\Delta G_{\alpha\beta}} = \frac{2\gamma_{\alpha\beta}T_e}{\Delta H_{\alpha\beta}(T_e - T_1)}$$

In other words, there is temperature sensitivity which determines whether random precipitate formations will be able to grow, i.e. initiate a *macroscopic* phase change. The tendency of large precipitate particles to grow is the physical origin of *grain coarsening*, as large precipitate will tend to grow at the expense of small precipitate; the result is, at a given temperature and fixed volume ratio, precipitate will combine to minimize volume, i.e. from several small particles into fewer large particles.

The tendency of individual particles to change phase is *homogeneous* nucleation; this process occurs randomly and without preference to any particular location in a material. In reality, *heterogeneous* nucleation also occurs, with a much lower every barrier and a reduced critical radius r^*, at certain preferential sites like grain boundaries or near impurities.

If diffusive transformations are not allowed to occur, i.e. by a fast-quenching process missing the "nose" of a TTT diagram, the result is an unstable material which results from *displacive* transformation. A **martensitic transformation** is one class, typical of steel, where large quantities of atoms are displaced in a coordinated manner by the fast-quenching process.

For iron materials, a key property is that martensite may be achieved by cooling exceeding the critical cooling rate (i.e. missing the nose); this is a transformation from FCC to a sheared BCC-like structure which is unstable. There is high amounts of shear energy stored in this structure, so an increase in temperature has a tendency to reverse plastic deformations.

Critical Cooling Rate in Steels. The critical cooling rate exists even in pure iron (and therefore martensite is technically achievable even in pure iron), but the critical cooling rate is very fast (i.e. $\mathcal{O}(10^5)°C$/sec)) and is not realistically achievable. The critical cooling rate is affected by two primary factors:

- **Material Composition**. Critical cooling rate decreases as impurity concentration increases (i.e. a reduction to $200°C$/sec for 0.8% C). This is the primary reason martensite is achievable in carbon steels (or in non-carbon steel alloys) but not in pure iron. The addition of impurities block diffusion and therefore impede kinetics.

- **Grain Size**. Critical cooling rate increases in fine-grained material. Fine grains have a tendency to block diffusion and therefore impede kinetics.

Note that cooling rates are not uniform in thick materials, so the critical cooling rate must be exceeded at the surface to achieve the critical cooling rate inside the material.

6.5 Modes of Failure

Fracture is the separation of a material into two or more parts. Fracture involves crack formation, followed by crack propagation. Ductile fracture occurs with most metals under standard conditions; fracture occurs after yielding and extensive plastic deformation. Brittle (cleavage) fracture occurs in ceramics and very cold metals; fracture occurs before macroscopic yield is observed, and there is very little plastic deformation. Yield occurs in ductile materials due to the formation of microscopic voids; the local stress concentration near these voids is high, so voids tend to grow (i.e. into macroscopic cracks).

Fracture occurs when maximum normal stress, σ_n, exceeds the *fracture strength* of a material, σ_f. Fracture strength is a material property. Comparing σ_f to σ_Y gives a criteria for whether a material behaves in a ductile or brittle manner.

- **Ductile Behavior**. A material exhibits ductile behavior if yielding occurs before fracture; that is, if $\sigma_H \geq \sigma_Y$ and/or $\tau_{\max} \geq k$ before $\sigma_n \geq \sigma_f$.

- **Brittle Behavior**. A material exhibits brittle behavior if fracture occurs before yielding; that is, if $\sigma_n \geq \sigma_f$ before $\sigma_H \geq \sigma_Y$ and $\tau_{\max} \geq k$.

In real materials, yield strength tends to *decrease* with temperature, while fracture strength is roughly *independent* of temperature. For this reason, at **low temperature**, a material tends to be more **brittle**; conversely, at **high temperature**, a material tends to be more **ductile**.

Crack propagation can be determined by an energy criterion. In particular, given a pre-existing crack, an external load P doing work ΔW, and a resulting change in strain energy ΔU, the crack will grow (without bound) if $\Delta W - \Delta U$ exceeds the energy cost to increase the crack surface area, approximately given by $\Gamma t \Delta a$ where Γ is a material property which describes energy cost, t is material thickness, and Δa is a small change in crack width. Under this condition, the material experiences fracture. At a given crack length a, it can be shown that the energy release per unit thickness is:

$$\Delta \Pi = \frac{\Delta W - \Delta U}{t} = \frac{P^2 \Delta C(a)}{2t}$$

where $C(a)$ is compliance, which is a function of crack size. The energy release rate, $\mathcal{G}$, is given by

$$\mathcal{G} = \frac{\partial \Pi}{\partial a} = \frac{1}{t}\frac{\partial}{\partial a}(W - U) = \frac{P^2}{2t}\frac{dC}{da} = \frac{P^2}{2\overline{E}t^2}\frac{d}{da}\left[g\left(\frac{a}{b}, \frac{L}{b}\right)\right]$$

where g is a function of the geometry relating change in surface area to crack size, and $\overline{E}$ is the corrected Young's modulus. In a free-expansion, plane-stress scenario, $\overline{E} = E$; in a plane-strain scenario, i.e. where expansion is constrained, $\overline{E} = \frac{E}{1-\nu^2}$. Critically, fracture occurs if $\mathcal{G} \geq \Gamma$. Note that $\mathcal{G}$ depends on $\overline{E}$, which is a material property; multiplying both sides by $\overline{E}$ separates loading-related terms from material-related terms, resulting in the following (equivalent) criterion for fracture:

$$\mathcal{G} \geq \Gamma \iff \frac{P^2}{t^2 b}f(a/b, L/b) \geq \overline{E} \cdot \Gamma \qquad \text{(Fracture Criterion)}$$

Define $K := \sqrt{\overline{E} \cdot \mathcal{G}} = \sqrt{\frac{P^2}{t^2 b}f(a/b, L/b)}$, which is the stress-intensity factor determined only by loading parameters. Define $K_c := \sqrt{\overline{E} \cdot \Gamma}$, which is called **fracture toughness** and is a measured quantity. Then failure alternatively occurs under the following condition:

$$K \geq K_c \qquad \text{(Alt. Fracture Criterion)}$$

This expression is useful, because because K may be rewritten as

$$K = \sigma_t^\infty \sqrt{a} F$$

where σ_t^∞ is the far-field stress, $\sqrt{a}$ is the crack-length dependent term, and F is a number which depends on the ratio a/b. This fracture criterion is the *Mode I* fracture mode, which is tensile fracture. For this reason, K is often denoted K_I, and K_C is often denoted K_{IC} to specify mode 1 fracture. Similarly, a factor of $\sqrt{\pi}$ is often common to F, so the expression is often written as $K_I = F\sigma_t^\infty \sqrt{\pi a}$.

Design Against Fracture in Pressure Vessel. Many engineering designs can be approximated as thin-walled pressure vessels (i.e. an aircraft cabin). A design must be sufficient to prevent both yield and fracture. Therefore, a design must meet the following coupled criteria:

$$\begin{cases} K_I < K_{IC} \\ \sigma_H < \sigma_Y \end{cases}$$

Suppose a crack forms along the length (i.e. in the axial direction). Then, $\sigma_{\theta\theta}$ could pull the crack apart and cause failure. Then, the criteria becomes:

$$\begin{cases} \frac{PR}{t} < \frac{K_{IC}}{\sqrt{\pi a}} \\ \frac{PR}{t} < \frac{2}{\sqrt{3}}\sigma_Y \end{cases}$$

The transition crack length, after which fracture will occur before yield, can be found by rearrangement:

$$a_T = \frac{3}{4\pi}\left(\frac{K_{IC}}{\sigma_Y}\right)^2$$

Cyclic loading can cause crack size to increase during operation; *assuming the same scenario with cracking in the axial direction*, a pressure vessel may be designed to leak before fracture to prevent catastrophic failure. Thus, the requirement is (for a semicircular crack) that crack size exceeds thickness and penetrates the wall *before* fracture occurs. This design requirement for thickness can be found by rearrangement *for fixed pressure P*:

$$t > \pi\left(\frac{PR}{K_{IC}}\right)^2$$

Note that t must *exceed* the right-hand size to ensure leak occurs, which can be shown from rearrangement.

Note that, for a crack perpendicular to the axial direction, the axial stress σ_{zz} is the tensile stress on the crack; this affects the failure criteria (by a factor of $1/2$) and slightly changes a_T, although irrespective of geometry $a_T \propto (K_{IC}/\sigma_Y)^2$. Similarly, the thickness criteria may have a different pre-factor depending on geometry. However, in a real material, cracks are randomly orientated throughout a material; an effective design against failure must prevent crack propagation in the worst-case scenario. Cracks in the axial direction experience the largest tensile force, and are therefore most susceptible to crack propagation; therefore, designing against propagation of axially-aligned cracks is a conservative design.

In many brittle materials, the critical crack length is so small that random, naturally occurring cracks in the material may cause material failure. Therefore, the question of whether a material (especially brittle material) survives is a question of probability. The survival probability for a given sample of volume V_0 under a stress σ is given by:

$$P_s(V) = \exp\left[-\frac{1}{V_0 S_0^m}\int_V \sigma^m dV\right]$$

The quantities m and S_0 are material properties, where S_0 is the *Weibull parameter* at V_0, and m is a material-dependent exponent. If surface flaws dominate, the integral is instead given by:

$$P_s(A) = \exp\left[-\frac{1}{A_0 S_0^m}\int_A \sigma^m dA\right]$$

In this expression, A_0 is the normalizing area. Note that, for large m, the probability of survival approaches a step function; ductile materials, such as steels, have large m (i.e. on $O\left(10^3\right)$), and therefore it is typically acceptable to treat these materials deterministically. Other materials, like cements or ceramics have lower m (i.e. between 5 and 10) and therefore must be treated statistically.

Often, *pre-existing cracks* which are subjected to some stress have a tendency to *grow* with time. The two main mechanisms for this are *stress-corrosion cracking* and *cyclic loading.*

Stress-Corrosion Cracking has the following main stages:

Stage 0. Below a certain stress-intensity threshold, no stress-corrosion cracking occurs.

Stage 1. Crack velocity depends on reaction rate at the crack tip:

$$\frac{da}{dt} = AK_I^n, \qquad A = A_0 \exp(-Q_I/RT)$$

Stage 2. Crack velocity is limited by reactant diffusion to the crack tip:

$$\frac{da}{dt} = B \qquad B = B_0 \exp(-Q_{II}/RT)$$

Stage 3. Fast fracture, which occurs irrespective of reactions; $\frac{da}{dt} \to \infty$ as $K_I \to K_{IC}$. This is the same physics as general fast fracture, which have already been discussed.

Stage 2 occurs if neighboring oxygen molecules (or another compound causing oxidation) are in excess; as humidity and/or temperature increases, the oxidation reaction happens more readily, and reactant diffusion may cease to be a bottleneck. In this case, stage 2 crack propagation may not be present. A representative crack propagation velocity curve for a stress-corrosion cracking scenario is shown below:
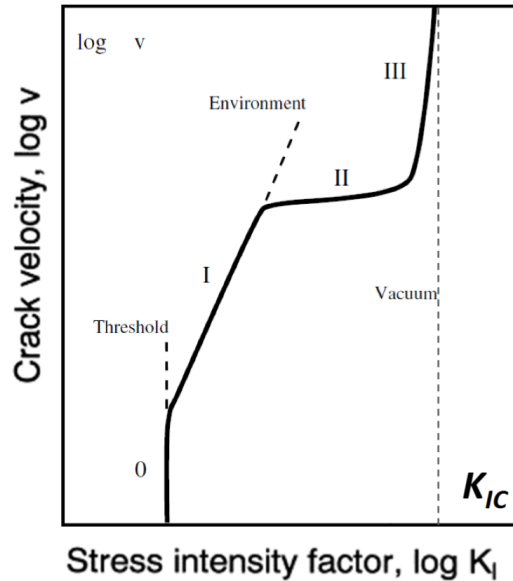
Cyclic loading occurs whenever stress is a function of time; for example, this occurs in automotive and aerospace applications, as well as in many static applications where loads are not constant (i.e. bridges, which are constantly loaded and unloaded by traffic). In general, the mechanism for crack propagation under cyclic loading is the time-dependent change in K_I. Of particular interest is the maximum change in K_I, given by the difference between maximum and minimum *tensile load* applied to the specimen:

$$\Delta K = F(\sigma_{\max} - \sigma_{\min})\sqrt{\pi a}$$

Note that $\sigma_{\min} \geq 0$ because compressive stress does not contribute to crack growth; therefore, if the stress $\sigma(t)$ is compressive for some time, $\sigma_{\min}$ is taken to be 0 and $\Delta\sigma = \sigma_{\max}$.

In general, the crack growth rate $\frac{da}{dN}$ for cyclic loading, which is the change in crack size per number of cycles, has three distinct phases:

Stage 1. Almost no growth below a certain threshold; that is, $\Delta K_I < \Delta K_{th} \implies \frac{da}{dN} \approx 0$.
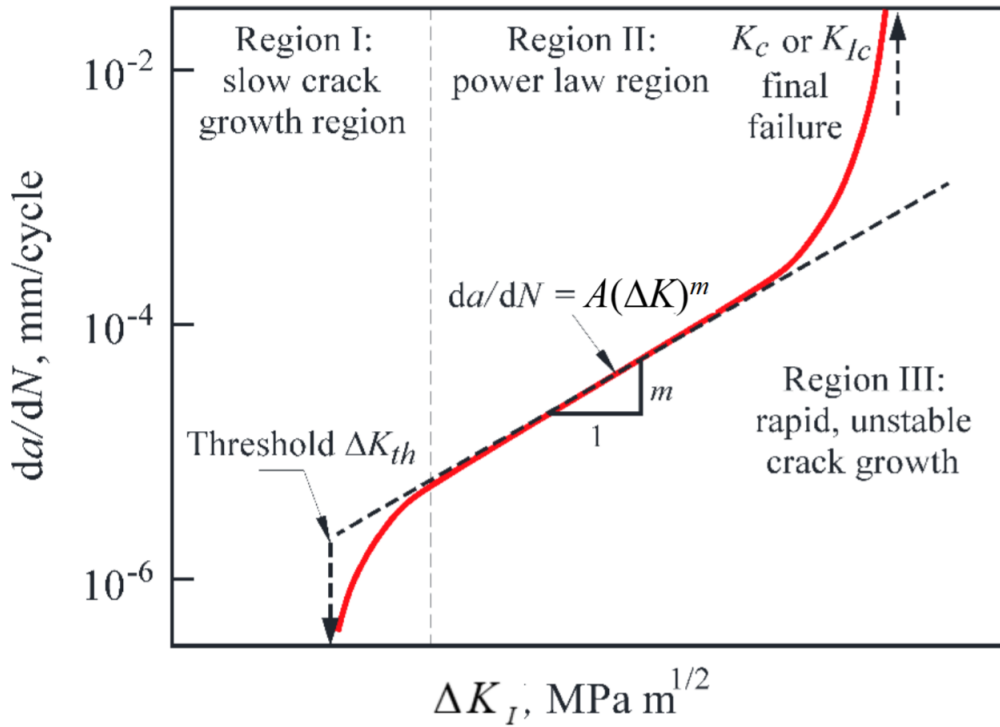
Stage 2. Crack growth follows a power-law relationship, known as **Paris' Law**:

$$\frac{da}{dN} = A(\Delta K_I)^m$$

where m and A are material parameters.

Stage 3. Fast fracture, which occurs if $K_{\max} > K_{IC}$, i.e. when $\sigma(t)$ is sufficiently large to cause fracture for some t.

A representative crack propagation curve for cyclic loading is shown below:

Even without a pre-existing crack, materials may still fail under cyclic loading. This phenomenon is called **fatigue**.

Crack initiation due to fatigue has two stages. First, in any material, there is some dislocation; cyclic loading can cause local dislocation motion, which causes nucleation of cracks. In the second stage, the crack nucleate will grow perpendicular to applied stresses according to Paris' law. Once a crack appears, lifetime is generally relatively short (i.e. on the order of 10000 cycles or fewer); on the other hand, initial crack nucleation may take many cycles. Therefore, the lifespan of a part with no pre-existing cracks is dominated by the time of crack initiation.

There are two modes of fatigue: *low-cycle*, and *high-cycle*. In low-cycle fatigue, nominal stress is sufficiently low that plastic deformation does not occur, i.e. nominal stress is less than σ_Y. In high-cycle fatigue, nominal stress exceeds yield strength σ_Y; the resulting plastic deformation drastically shortens the lifespan of a part. In general, the transition point between low-cycle and high-cycle fatigue occurs around 10^3 or 10^4 cycles. The *fatigue limit*, or *endurance limit*, denoted σ_e, is the stress amplitude below which fatigue in effect does not occur; for amplitudes below σ_e, one can expect at least 10^7 to 10^8 cycles before failure. Fatigue strength is often defined as the stress amplitude which gives a certain fatigue life.

High-cycle fatigue for a **fully-reversed signal** is described by **Basquin's Law**:

$$\sigma_{ar} N_f^a = C_1 \tag{Basquin's Law}$$

In Basquin's Law, $\sigma_{ar} < \sigma_Y$ is the **fully-reversed stress amplitude** (meaning mean stress of 0), N_f is fatigue life (cycles until failure), and a and C_1 are material properties.

Low-cycle fatigue for a **fully-reversed signal** is described by the **Coffin-Manson Law**:

$$\Delta\varepsilon_{pl}N_f^b = C_2 \qquad \text{(Coffin-Manson Law)}$$

The control variable here is the plastic deformation $\Delta\varepsilon_{pl}$ per cycle. This law quantifies the accumulation of plastic deformation when operating at a stress beyond σ_Y but below fracture.

Because a material experiencing low-cycle fatigue has already yielded, most systems are designed for high-cycle fatigue (i.e. the system is already considered failed if yield has occurred). This motivates an extension of Basquin's Law to predict failure given real stress signals, i.e. including signals which are *not* fully reversed. This generalization is done by calculating a *conservative*, equivalent "fully-reversed stress" by normalizing the stress amplitude and mean stress. This normalization is known as **Goodman's Law**:

$$\begin{cases} \dfrac{\sigma_a}{\sigma_{ar}} + \dfrac{\sigma_m}{\sigma_u} = 1 & \text{conservative if } \sigma_m > 0 \\[3ex] \dfrac{\sigma_a}{\sigma_{ar}} = 1 & \text{conservative if } \sigma_m < 0 \end{cases}$$

where σ_u is the ultimate tensile strength for a material. Goodman's Law is conservative in that it predicts a shorter lifetime than reality.

In reality, most systems experience *non-uniform* signals. In general, many different amplitudes may occur for different numbers of cycles. The **Palmgren-Miner Rule** gives a criterion for failure with non-uniform cycles:

$$\sum_i \frac{N_i}{N_{f,i}} = \frac{N_1}{N_{f,1}} + \frac{N_2}{N_{f,2}} + \cdots = 1 \qquad \text{(Palmgren-Miner Rule)}$$

In other words, each $\frac{N_i}{N_{f,i}}$ moves the part some fraction closer to failure, until 100% failure is achieved. Essentially, the Palmgren-Miner rule tracks damage to a part, and fracture occurs when this damage reaches 100%, or 1.

Another time-dependent mode of failure is **creep**. Creep is time-dependent, *shear-driven* deformation. A typical creep curve has three distinct stages:

Primary Stage. Nonlinear region involving complicated deformation behavior due to microstructure.

Secondary Stage (*Steady-State Creep*. Strain rate $d\varepsilon/dt$ is constant at given σ and T. This stage is very dependent on temperature, with an exponential increase in strain rate with temperature increase.

Tertiary Stage. Accumulated damage, such as cracking, rapidly increases strain nonlinearly with time and leads to creep rupture.

In general, if a material has a melting temperature T_M, creep becomes observable on short timescales when the material is deployed at 30% to 50% of the melting temperature.

Creep may be linear or nonlinear, depending on the material and loading conditions. Linear creep is given by the following relationship:

$$\frac{d\gamma_{xy}}{dt} = \frac{\tau_{xy}}{\eta}, \qquad \text{where } \eta = \frac{\text{shear stress}}{\text{strain rate}}$$

The viscosity η decreases at T increases, because $\eta(T) \propto \exp\left(\frac{Q}{RT}\right)$. Because creep is a shear-driven process, it is related to von Mises (or Tresca) stress. It can be shown that, for an effective stress σ_H, the effective strain is given by:

$$\varepsilon_H = \sqrt{\frac{2}{9}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_1 - \varepsilon_3)^2 + (\varepsilon_2 - \varepsilon_3)^2\right]}$$

Substituting values, it can be shown that the strain rate is given by:

$$\frac{d\varepsilon_H}{dt} = \frac{\sigma_H}{3\eta}$$

Note that, for a specimen experiencing uniaxial tension (in the x-direction), $\sigma_H = \sigma_{xx}$.

Nonlinear creep, also known as power-law creep, is also related to σ_H. The following relationship gives the strain rate for a specimen:

$$\frac{d\varepsilon_H}{dt} = A\sigma_H^n = \dot{\varepsilon}_0 \left(\frac{\sigma_H}{\sigma_0}\right)^n$$

In this equation, n and A (or σ_0) are material parameters for creep, and $\dot{\varepsilon}_0$ is a dimensionalizing parameter typically defined as $1 \cdot \text{s}^{-1}$. A is an energy barrier, of the form $A = A_0 \exp(-Q/RT)$. It can be shown that, in pure shear, creep is given by

$$\frac{\gamma_{xy}}{dt} = \dot{\varepsilon}_0 (\sqrt{3})^{n+1} \left(\frac{\tau_{xy}}{\sigma_0}\right)^n$$

In any case, a specimen which is loaded experiences an initial, instantaneous strain (or deflection) which can be determined by standard static analysis, as well as a time-dependent, viscous strain (or deflection), determined by the creep of a material. The summation of these two gives the total strain (or deflection) with time.

6.6 Polymers and Composite Materials

A **polymer** is distinguished by its long-chain atomic structure, formed from linking together monomers, resulting in a chain where each link has the form CH_2. Carbon atoms have a tendency to share electrons, resulting in a single bond between neighboring carbons and the formation of a monomer chain (a polymer). Polymers typically fall in one of three categories:

Thermoplastics. Polymer is formed by long chains, held together by only hydrogen and/or Van der Walls bonding between chains. Chains are disconnected, and can slide past each other at high temperatures. This structure can be melted and recast.

- Chains tend to fold to reach energy-efficient state, resulting in a crystalline structure. The crystalline structure can break down due to branching, resulting in a lower density amorphous region.
- Examples of thermoplastics include high-density polyethylene (HDPE) and low-density polyethylene (LDPE).

Thermosets. Polymer is formed by covalent cross-linked structure. Structure is very rigid and does not significantly deform. This structure cannot be recast after melting, as a melting process breaks the covalent bonds.

- Cross-links tend to form by introduction of an activator molecule, which allows two polymers to connect due to a chemical reaction at some site on the molecule.
- Examples include polyesters and some epoxies.

Elastomers. Polymer is formed by a few covalent cross-links. Stronger bonds (covalent) compared to thermoplastics, and fewer links compared to thermosets, so material may deform significantly with a "memory" of its original shape. Cannot be recast after melting because melting breaks the covalent bonds.

- Formed by same mechanism as thermosets, but with much lower concentration of activator. Results in nearly linear polymers with some occasional cross-links.

Recall that E is proportional to the Gibbs free energy for a material. While entropy is negligible for crystalline structures (i.e. ductile materials), it makes a significant contribution for polymer structures. In particular, polymer chains are naturally disordered; stretching of the chains, i.e. by applying a stress, reduces this disorder and is therefore resisted by the material's tendency to maximize entropy.

A **composite** material is a combination of two or more materials. Typically, composite materials are designed to give better material performance per weight. A key example in this category is carbon fiber, which is a resin-fiber composite. In this example, however, the strength of the material depends on the fiber direction, and the material is therefore *orthotropic*, meaning Young's modulus, shear modulus, and Poisson's ratio differ in each direction.

Derivation of Composite Modulus. Consider a composite material with cross-sectional area A_c consisting of fibers with isotropic modulus E_f and cross-section A_f, and a matrix with modulus E_m and cross-section A_m. Suppose the volume fraction of fiber is f.

Suppose loading by force on the composite F_c is carried out in the direction of the fibers. Then, the fiber and matrix experience the same strain, so $\varepsilon_c = \varepsilon_f = \varepsilon_m$. Then:

$$
\begin{aligned}
F_c &= \sigma_C A_c \\
&= \sigma_f A_f + \sigma_m A_m \\
&= f A_c \sigma_f + (1-f) A_c \sigma_m \\
\implies \sigma_c &= f \sigma_f + (1-f) \sigma_m \\
&= f \varepsilon_f E_f + (1-f) \varepsilon_m E_m \\
&= f \varepsilon_c E_f + (1-f) \varepsilon_c E_m \\
\implies \sigma_c / \varepsilon_c = E_c &= f E_f + (1-f) E_m
\end{aligned}
$$

Therefore, loading in the direction of the fibers is characterized by a Young's modulus of $E_c = f E_f + (1-f) E_m$. Suppose now that F_c is applied perpendicular to the direction of the fibers. Then, the stress experienced by the fibers and the matrix must be the same. Suppose l_c is the length of the composite, and each fiber has length l_f separated by a matrix length l_m. Then:

$$
\begin{aligned}
\Delta l_f &= l_f \varepsilon_f \\
&= l_f \frac{\sigma_f}{E_f} \\
&= l_f \frac{\sigma_c}{E_f}
\end{aligned}
$$

The exact same process shows that the matrix has:

$$
\Delta l_m = l_m \frac{\sigma_c}{E_m}
$$

Then:

$$
\begin{aligned}
\varepsilon_c &= \frac{\Delta l_c}{l_c} \\
&= \frac{\Delta l_f + \Delta l_m}{l_c} \\
&= \sigma_c \left(\frac{f}{E_f} + \frac{1-f}{E_m} \right) \\
\implies \sigma_c / \varepsilon_c = E_c &= \left(\frac{f}{E_f} + \frac{1-f}{E_m} \right)^{-1}
\end{aligned}
$$

The contribution to Young's modulus in this case is nonlinear. This is the origin of orthotropic behavior, which emerges even when combining two isotropic materials.

This derivation gives an upper-bound and lower-bound for Young's modulus, as a function of the material composition and the elastic modulus of the matrix and fiber separately:

$$
\left(\frac{f}{E_f} + \frac{1-f}{E_m} \right)^{-1} \leq E_c \leq f E_f + (1-f) E_m
$$

The lower-bound is attained when forces are applied perpendicular to the fibers, and the upper-bound is attained when forces are applied parallel to the fibers. For loads applied off-axis, the behavior of the composite can only be derived with the isotropic Hooke's law and stress and/or strain transformations.