Solving for orbital elements given observations in 6-d phase space

Logan Pearce

1. Introduction

Given the observations of a 2 body system of RA/Dec, proper motion in RA/Dec, and radial velocity, determine the 6 orbital elements describing a 3-d orbit. These elements are: a (semi-major axis) [and thus T (period - derived from Kepler's 3rd law)], t_0 (time of periastron passage), e (eccentricity), i (inclination), ω (argument of periastron - angle from ascending node to periapse), and Ω (longitude of periastron - angle of location of ascending node from reference direction). (Note - other sybmols are often used from these variables [for example, P rather than T for period]. I will be using the symbols as they are defined in Seager (2010) for consistency).

These formulae are taken from the book *Exoplanets* (Seager (2010)), Part 1: Keplerian Orbits and Dynamics of Exoplanets, by C.D. Murray and A.C.M. Correia (Murray & Correia 2010).

2. Method

Taking the central body of a 2-body Keplerian orbit to be at the origin of the 3-d cartesian coordinate system, the position of the orbiting body is given by the coordinates (X,Y,Z), where +X is the reference direction and is +Declination in the on-sky coordinates. +Y is the negative RA direction, and +Z is the line of sight direction towards the observer.

Murray & Correia (2010) eqns 53, 54, and 55 gives the following formulae for projecting orbital elements onto the plane of the sky:

$$X = r(\cos\Omega\cos(\omega + f) - \sin\Omega\sin(\omega + f)\cos i) \tag{1}$$

$$Y = r(\sin\Omega\cos(\omega + f) - \cos\Omega\sin(\omega + f)\cos i) \tag{2}$$

$$Z = r\sin(\omega + f)\sin i \tag{3}$$

Where f is the true anomaly, the angle from periapse to the current position of the orbiting body in the plane of the orbit, and r is the separation distance of the orbiting body from the focus in orbital plane. X and Y correspond to the observed ΔDec ($\Delta\delta$) and -ΔRA (-$\Delta\alpha$) respectively between the orbiting body and host. (-ΔRA because RA is defined as increasing in the -Y direction)

Murray & Correia (2010) derives in eqn 63 the formula for velocity in the Z direction as:

$$\dot{Z} = \dot{r}\sin(\omega + f)\sin i + r\dot{f}\cos(\omega + f)\sin i \tag{4}$$

which is the time derivative of Z. In the equations above, only r and f vary with time. This corresponds to the observed radial velocity.

Taking the time derivative of X and Y we obtain the velocity in the X and Y direction, which corresponds to proper motion in the Dec and RA directions respectively (μ_δ and μ_α).

(In the case of DS Tuc, which is a binary star system of nearly equal masses, we reduce the system to a reduced mass "planet" (m$_{red} = \frac{mA\, mB}{mA+mB}$) orbiting the total system mass located at the origin (m$_{tot} = mA + mB$), and take the relative separations in RA and Dec, and difference in proper motions in the two directions as our measurements)

$$\dot{X} = \dot{r}(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) + r\dot{f}(-\cos\Omega\sin(\omega+f) - \sin\Omega\cos(\omega+f)\cos i) \tag{5}$$

$$\dot{Y} = \dot{r}(\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i) + r\dot{f}(-\sin\Omega\sin(\omega+f) + \cos\Omega\cos(\omega+f)\cos i) \tag{6}$$

Thus equations (1)-(6) form the 6-dimensional phase space of position and velocity in three dimensions.

But both r and f are defined in the plane of the orbit, rather than the plane of the sky, so they must be eliminated in order to solve for the orbital elements.

Astrometry gives X and Y, but we don't have a measurement for Z position. We are free to define Z = 0, and thus that the ascending node is located in the reference plane. Now we have from (3):

$$0 = r\sin(\omega + f)\sin i \tag{7}$$

$r = 0$ is not physical, and we have no reason to restrict inclination to either 0 or π (in fact it is a parameter of our solution), so to make (7) true we set $\omega + f = 0$ or π. Setting $\omega + f = 0$ makes $f = -\omega$. (This also makes sense physically. f is defined as the angle from periapse to planet in the orbital plane; ω is angle from asc node to periapse. If Z = 0, then the planet is at the ascending node, so $f = -\omega$).

Now all $\cos(\omega + f)$ terms become 1, and all $\sin(\omega + f)$ terms become zero, and (1) - (6) simplify to:

$$X = r\cos\Omega \tag{8}$$

$$Y = r\sin\Omega \tag{9}$$

$$\dot{X} = \dot{r}\cos\Omega - r\dot{f}\sin\Omega\cos i \tag{10}$$

$$\dot{Y} = \dot{r}\sin\Omega + r\dot{f}\cos\Omega\cos i \tag{11}$$

$$\dot{Z} = r\dot{f}\sin i \tag{12}$$

(Again, this makes sense physically. Ω is angle from reference direction +X to planet in the reference plane, so X and Y are trig functions of that angle)

Eqns (31) and (32) in Murray & Correia (2010) define $\dot{r}$ and $r\dot{f}$ in terms of a and e and f:

$$\dot{r} = \frac{na}{\sqrt{1-e^2}} e\sin f = -\frac{na}{\sqrt{1-e^2}} e\sin\omega \tag{13}$$

$$r\dot{f} = \frac{na}{\sqrt{1-e^2}}(1 + e\cos f) = \frac{na}{\sqrt{1-e^2}}(1 + e\cos\omega) \tag{14}$$

n is the mean motion of the planet, defined as $n = \frac{2\pi}{T}$ (Eqn (24) in Murray & Correia (2010)). T is derived from Kepler's 3rd law as $T = \sqrt{\frac{4\pi^2 a^3}{\mu}}$, where $\mu = G(m_1 + m_2)$. Therefore na reduces to $na = \sqrt{\frac{\mu}{a}}$ and (13) and (14) become:

$$\dot{r} = \sqrt{\frac{\mu}{a(1-e^2)}} e \sin f = -\sqrt{\frac{\mu}{a(1-e^2)}} e \sin\omega \tag{15}$$

$$r\dot{f} = \sqrt{\frac{\mu}{a(1-e^2)}}(1 + e\cos f) = \sqrt{\frac{\mu}{a(1-e^2)}}(1 + e\cos\omega) \tag{16}$$

r is also defined in Murray & Correia (2010) eqn (20) as:

$$r = \frac{a(1-e^2)}{1 + e\cos\omega} \tag{17}$$

Now our system of equations become:

$$X = \frac{a(1-e^2)}{1 + e\cos\omega}\cos\Omega = \Delta\delta \tag{18}$$

$$Y = \frac{a(1-e^2)}{1 + e\cos\omega}\sin\Omega = -\Delta\alpha \tag{19}$$

$$\dot{X} = -\sqrt{\frac{\mu}{a(1-e^2)}}\left[e\sin\omega\cos\Omega + (1 + e\cos\omega)\sin\omega\cos i\right] = \mu_\delta \tag{20}$$

$$\dot{Y} = \sqrt{\frac{\mu}{a(1-e^2)}}\left[e\sin\omega\sin\Omega + (1 + e\cos\omega)\cos\omega\cos i\right] = \mu_\alpha \tag{21}$$

$$\dot{Z} = \sqrt{\frac{\mu}{a(1-e^2)}}(1 + e\cos\omega)\sin i = radial\ velocity \tag{22}$$

Equations (18) - (22) form a system of 5 equations defined in terms of 5 of the six orbital elements — a, e, i, ω, Ω — and constrained by five observations — $\Delta\delta, \Delta\alpha, \mu_\delta, \mu_\alpha$, and radial velocity.

The final element needed is the time, which in Murray & Correia (2010) is defined from the epoch of periastron passage, t_0. Eqn (37) in Murray & Correia (2010) defines r in terms of the eccentric anomaly (E), which we can set equal to (17) to obtain E in terms of e and ω

$$r = \frac{a(1-e^2)}{1 + e\cos\omega} = a(1 - e\cos E) \tag{23}$$

which we can solve for E. The mean anomaly M is defined as $M = n(t - t_0)$, where t is the time of the observation. M is also defined from the eccentric anomaly as $M = E - e\ sinE$, thus we can solve for t_0.

Using these equations in reverse allows us to predict the position and proper motions that would be observed from orbits described by orbital elements.

REFERENCES

Murray, C. D., & Correia, A. C. M. 2010, Keplerian Orbits and Dynamics of Exoplanets, ed. S. Seager, 15–23

Seager, S. 2010, Exoplanets