Solving for orbital elements given observations in 6-d phase space

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1. Introduction

This document describes the method and equations I used for adopting the methodology of OFTI to make use of 3-d velocities for fitting for orbital elements.

If the two components of a binary system are resolved in Gaia, we can make use of Gaia's astrometric observations of RA/Dec and proper motion in RA/Dec as constraints for fitting orbital elements. If Gaia also obtained radial velocities for both bodies (or if they're been attained independently), then that gives a 5th measurement, leaving only one unconstrained phase space measurement of Z position. If accelerations have been measured for any coordinate, then this provides even more constraints, and allows Z to be removed as a free parameter.

The orbital elements are: a (semi-major axis) [and thus T (period - derived from Kepler's 3rd law)], t_0 (time of periastron passage), e (eccentricity), i (inclination), ω (argument of periastron - angle from ascending node to periapse), and Ω (longitude of periastron - angle of location of ascending node from reference direction). (Note - other sybmols are often used from these variables [for example, P rather than T for period]. I will be using the symbols as they are defined in Seager (2010) for consistency).

These formulae are taken from the book *Exoplanets* (Seager (2010)), Part 1: Keplerian Orbits and Dynamics of Exoplanets, by C.D. Murray and A.C.M. Correia (Murray & Correia 2010).

1.1. Positions and Velocities from orbital elements

Taking the central body of a 2-body Keplerian orbit to be at the origin of the 3-d cartesian coordinate system, the position of the orbiting body is given by the coordinates (X,Y,Z), where +X is the reference direction and is +Declination in the on-sky coordinates. +Y is the negative RA direction, and +Z is the line of sight direction towards the observer.

Murray & Correia (2010) eqns 53, 54, and 55 gives the following formulae for projecting orbital elements onto the plane of the sky:

$$X = r(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) \tag{1}$$

$$Y = r(\sin\Omega\cos(\omega+f) - \cos\Omega\sin(\omega+f)\cos i) \tag{2}$$

$$Z = r\sin(\omega+f)\sin i \tag{3}$$

Where f is the true anomaly, the angle from periapse to the current position of the orbiting body in the plane of the orbit, and r is the separation distance of the orbiting body from the focus in orbital plane. X and Y correspond to the observed ΔDec ($\Delta\delta$) and -ΔRA (-$\Delta\alpha$) respectively between the orbiting body and host. (-ΔRA because RA is defined as increasing in the -Y direction)

Murray & Correia (2010) derives in eqn 63 the formula for velocity in the Z direction as:

$$\dot{Z} = \dot{r}\sin(\omega + f)\sin i + r\dot{f}\cos(\omega + f)\sin i \tag{4}$$

which is the time derivative of Z. In the equations above, only r and f vary with time. This corresponds to the observed radial velocity.

Taking the time derivative of X and Y we obtain the velocity in the X and Y direction, which corresponds to proper motion in the Dec and RA directions respectively (μ_δ and μ_α).

$$\dot{X} = \dot{r}(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) + r\dot{f}(-\cos\Omega\sin(\omega+f) - \sin\Omega\cos(\omega+f)\cos i) \tag{5}$$

$$\dot{Y} = \dot{r}(\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i) + r\dot{f}(-\sin\Omega\sin(\omega+f) + \cos\Omega\cos(\omega+f)\cos i) \tag{6}$$

Thus equations (1)-(6) form the 6-dimensional phase space of position and velocity in three dimensions.

r is the radius of the orbiting body in the orbital plane, and is given as:

$$r = \frac{a(1-e^2)}{1+e\cos f} \tag{7}$$

f is the true anomaly and is given by solving Kepler's equation at the observation date (for Gaia DR2 this is 2015.5):

$$M = \frac{2\pi}{T}\left(Date - t_o\right) \tag{8}$$

$$g(E) = E - e\sin E - M \tag{9}$$

which is a transcendental equation which must be solved numerically (such as Newton-Raphson method). $\left[T\right.$ is derived from Kepler's 3rd law as $T = \sqrt{\frac{4\pi^2 a^3}{\mu}}$, where $\mu = G(m_1 + m_2)$.$\left.\right]$ The true anomaly then is given by:

$$f = 2\ \arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) \tag{10}$$

$\dot{r}$ and $r\dot{f}$ are the time rate of change of separation and angular distance from the focus of the ellipse (the central body). Eqns (31) and (32) in Murray & Correia (2010) define $\dot{r}$ and $r\dot{f}$ in terms of a and e and f:

$$\dot{r} = \frac{na}{\sqrt{1-e^2}}e\sin f \tag{11}$$

$$r\dot{f} = \frac{na}{\sqrt{1-e^2}}\left(1 + e\cos f\right) \tag{12}$$

Where $n = \frac{2\pi}{T}$.

And equations (1)-(6) become:

$$X = \frac{a(1-e^2)}{1+e\cos f}(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) = \Delta\delta \tag{13}$$

$$Y = \frac{a(1-e^2)}{1+e\cos f}(\sin\Omega\cos(\omega+f) - \cos\Omega\sin(\omega+f)\cos i) = -\Delta\alpha \tag{14}$$

$$Z = \frac{a(1-e^2)}{1+e\cos f}\sin(\omega+f)\sin i \tag{15}$$

$$\begin{aligned}\dot{X} = \frac{na}{\sqrt{1-e^2}}\,[\, & e\sin f(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) + \\ & \big(1+e\cos f\big)(-\cos\Omega\sin(\omega+f) - \sin\Omega\cos(\omega+f)\cos i)\,] = \mu_\delta\end{aligned} \tag{16}$$

$$\begin{aligned}\dot{Y} = \frac{na}{\sqrt{1-e^2}}\,[\, & e\sin f(\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i) + \\ & \big(1+e\cos f\big)(-\sin\Omega\sin(\omega+f) + \cos\Omega\cos(\omega+f)\cos i)\,] = -\mu_\alpha\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{Z} = \frac{na}{\sqrt{1-e^2}}\left[e\sin f\sin(\omega+f)\sin i + \big(1+e\cos f\big)\cos(\omega+f)\sin i\right] = \\ \textit{radial velocity}\end{aligned} \tag{18}$$

With RA/Dec, proper motion in RA/Dec, and radial velocity, only Z position is unconstrained in the fit.

1.2. Accelerations from orbital elements

Beginning with eqns (4)-(6), we derive the second time derivative for X,Y, and Z position as

$$\begin{aligned}\ddot{X} = {} & (\ddot{r} - r\dot{f}^2)\,(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) + \\ & (-2\dot{r}\dot{f} - r\ddot{f})\,(\cos\Omega\sin(\omega+f) + \sin\Omega\cos(\omega+f)\cos i)\end{aligned} \tag{19}$$

$$\begin{aligned}\ddot{Y} = {} & (\ddot{r} - r\dot{f}^2)\,(\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i) + \\ & (2\dot{r}\dot{f} + r\ddot{f})\,(\sin\Omega\sin(\omega+f) + \cos\Omega\cos(\omega+f)\cos i)\end{aligned} \tag{20}$$

$$\ddot{Z} = \sin i\,\Big[(\ddot{r} - r\dot{f}^2)\,\sin(\omega+f) + (2\dot{r}\dot{f} + r\ddot{f})\,\cos(\omega+f)\Big] \tag{21}$$

Now here is where trusty Murray & Correia (2010) fails us, as they do not derive expressions for $\ddot{r}$ or $\ddot{f}$. As f depends on E, which varies with time, and is a transcendental, this gets messy quickly. Fortunately Klioner (2016) has a lovely discussion of the two-body problem, and while they do not derive what we need directly, we can make use of their work to assemble needed expressions.

Klioner (2016) gives two helpful expressions for $\dot{E}$:

$$\dot{E} = \frac{n}{1-e\cos E} \tag{22}$$

$$\dot{E} = \frac{an}{r} = \frac{n\,(1+e\cos f)}{1-e^2} \tag{23}$$

Thus we derive from (22):

$$\ddot{E} = \frac{-\,n\;e\;\sin E}{(1-e\cos E)^2}\;\dot{E} = \frac{n^2\;e}{(1-e\cos E)^2}\;\frac{\sin f}{\sqrt{1-e^2}} \tag{24}$$

Or from (23):

$$\ddot{E} = \frac{-n\;e\;\sin f}{1-e^2}\;\dot{f} \tag{25}$$

Reverse-engineering (12), we get that

$$\dot{f} = \frac{n\sqrt{1-e^2}}{(1-e\cos E)^2} = \dot{E}\frac{\sqrt{1-e^2}}{1-e\cos E} = \dot{E}\;\frac{\sin f}{\sin E} \tag{26}$$

where $\sin f = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}$. Which is kind of a nice result I think. Anyway.

We can write r in a more derivative friendly way as $r = a\,(1-e\cos E)$, and thus:

$$\dot{r} = a\;e\;\sin E\;\dot{E} \tag{27}$$

$$\ddot{r} = a\,e\;\cos E\,\dot{E}^2\;+\;a\,e\;\sin E\;\ddot{E} \tag{28}$$

And from (26) we get:

$$\ddot{f} = \ddot{E}\;\frac{\sqrt{1-e^2}}{1-e\cos E} + \dot{E}^2\,\frac{e\sqrt{1-e^2}\sin E}{(1-e\cos E)^2} \tag{29}$$

which reduces to:

$$\ddot{f} = \ddot{E}\;\frac{\sin f}{\sin E} + \dot{E}^2\;\frac{e\sin f}{1-e\cos E} \tag{30}$$

And now we have all the pieces for computing $\ddot{X}$, $\ddot{Y}$, and $\ddot{Z}$.

1.3. Fitting method

The fitter is based on the Orbits for the Impatient (OFTI) rejection sampling methodology of Blunt et al. (2017). In brief, OFTI works by generating a set of random orbits from probability distributions of the orbital parameters, with the semi-major axis initially fixed at some distance and the longitude of periastron initially fixed at zero. OFTI computes astrometry at the observation date, and then scales semi-major axis and rotates longitude of periastron to match observation. It then computes astrometry and radial velocity (if applicable) of the scaled orbits at the observation date, computes the χ^2 goodness of fit for each orbit, and accepts or rejects based on the probability of that orbit producing observations compared to a random "dice roll". (Note: Users can now perform their own OFTI fits to astrometry with the python package Orbitize! (`https://orbitize.readthedocs.io/en/latest/`). Version 1 allows fitting one orbiting body using multiple astrometric measurements (either RA/Dec or separation/position angle) with either OFTI (for poorly constrained orbits) or MCMC (for better constrained orbits). Later versions will allow incorporating radial velocity and fitting multiple orbiting bodies).

This method is a variation on OFTI by replacing multiple astrometric measurements with measurements of position and velocity at one point in time. The scale-and-rotate and accept/reject step remain from traditional OFTI. This makes this method well suited to working with Gaia astrometry. Many wide binaries have good quality Gaia solutions for both objects, including radial velocity (For a discussion on good quality Gaia solutions, see my tutorial here: https://github.com/logan-pearce/breakthroughlisten/blob/master/Gaia_parallaxes/Data%20Quality%20and%20Gaia%20Archive%20tutorial.ipynb).

In this github repository I have an example of code for fitting one wide binary in Gaia using this method, and for generating nice plots from the fit.

REFERENCES

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