1-Introduction

Natural language abounds with vague and imprecise concepts, such as “Ali is tall”, or “It is very hot today”. Such statements are difficult to translate into more precise language without losing some of their semantic value: for example, the statement “Ali’s height is 155 cm.” does not explicitly state that he is tall, and the statement “ Ali’s height is 1.2 standard deviations about the mean height for men of his age in his culture” is fraught with difficulties : would mean 1.999999 standard deviations above the mean be tall? Which culture does Ali belong to, and how is membership in it defined?

While it might be argued that such vagueness is an obstacle to clarity of meaning, only the most staunch traditionalists would hold that there is no loss of richness of meaning when statements such as "Ali is tall" are discarded from a language. Yet this is just what happens when one tries to translate human language into classic logic. Such a loss is not noticed in the development of a payroll program, perhaps, but when one wants to allow for natural language queries, or "knowledge representation" in expert systems, the meanings lost are often those being searched for.

For example, when one is designing an expert system to mimic the diagnostic powers of a physician, one of the major tasks is to codify the physician's decision-making process. The designer soon learns that the physician's view of the world, despite her dependence upon precise, scientific tests and measurements, incorporates evaluations of symptoms, and relationships between them, in a "fuzzy," intuitive manner: deciding how much of a particular medication to administer will have as much to do with the physician's sense of the relative "strength" of the patient's symptoms as it will their height/weight ratio. While some of the decisions and calculations could be done using traditional logic w will see how fuzzy logic affords a broader, richer field of data and the manipulation of that data than do more traditional methods.

2.0-History

The precision of mathematics owes its success in large part to the efforts of Aristotle and the philosophers who preceded him. In their efforts to devise a concise theory of logic, and later mathematics, the so-called "Laws of Thought" were posited[1] . One of these, the

"Law of the Excluded Middle," states that every proposition must either be True or False. Even when Parminedes proposed the first version of this law (around 400 B.C.) there were strong and immediate objections: for example, Heraclitus proposed that things could be simultaneously True and not True.

It was Plato who laid the foundation for what would become fuzzy logic, indicating that there was a third region (beyond True and False) where these opposites "tumbled about." Other, more modern philosophers echoed his sentiments, notably Hegel, Marx, and Engels. But it was Lukasiewicz who first proposed a systematic alternative to the bi-valued logic of Aristotle . [2]

In the early 1900's, Lukasiewicz described a three-valued logic, along with the mathematics to accompany it. The third value he proposed can best be translated as the term "possible," and he assigned it a numeric value between True and False. Eventually, he proposed an entire notation and axiomatic system from which he hoped to derive modern mathematics.

Later, he explored four-valued logics, five-valued logics, and then declared that in principle there was nothing to prevent the derivation of an infinite-valued logic. Lukasiewicz felt that three- and infinite-valued logics were the most intriguing, but he ultimately settled on a four-valued logic because it seemed to be the most easily adaptable to Aristotelian logic.

Knuth proposed a three-valued logic similar to Lukasiewicz's, from which he speculated that mathematics would become even more elegant than in traditional bi-valued logic. His insight, apparently missed by Lukasiewicz, was to use the integral range [-1, 0 +1] rather than [0, 1, 2]. Nonetheless, this alternative failed to gain acceptance, and has passed into relative obscurity.

It was not until relatively recently that the notion of an infinite-valued logic took hold. In 1965 Lotfi A. Zadeh published his seminal work "Fuzzy Sets" [3]which described the mathematics of fuzzy set theory, and by extension fuzzy logic. This theory proposed making the membership function (or the values False and True) operate over the range of real numbers [0.0, 1.0]. New operations for the calculus of logic were proposed, and showed to be in principal.

3.0-Basic definitions

Before illustrating the mechanisms which make fuzzy logic to work, it is important to realize what fuzzy logic actually is? Fuzzy logic is a superset of conventional(Boolean) logic that has been extended to handle the concept of partial truth- truth values between "completely true" and "completely false". As its name suggests, it is the logic underlying modes of reasoning which are approximate rather than exact. The importance of fuzzy logic derives from the fact that most modes of human reasoning and especially common sense reasoning are approximate in nature. According to Zadeh Lotfi following are the essential characteristics of fuzzy logic:

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
- In fuzzy logic everything is a matter of degree.
- Any logical system can be fuzzified
- In fuzzy logic, knowledge is interpreted as a collection of elastic or, equivalently , fuzzy constraint on a collection of variables
- Inference is viewed as a process of propagation of elastic constraints.

The third statement hence, define Boolean logic as a subset of Fuzzy logic.

3.1-Classical or Crisp Sets

In this section we shall briefly over view the concept of classical sets in order to facilitate the concept of fuzzy sets. The concept of collection of objects is common in every days experience. The objects in a sets are called elements or members of that set. We denote all elements by small letters a,b,c……….,x,y,x and the sets by capital letters A,B,C……….X,Y,Z.

The fundamental notion in set theory is that of belonging or membership. If an object y belongs to set A we write $y \in A$; if y is not member of A we write $y \notin A$. The set of all objects under consideration in a particular situation is called universal set and denoted by U.A set without elements is called empty set and denoted by ϕ .X is subset of y if every element of X is also an element of Y. It is denoted by $X \subseteq Y$.X is proper subset of Y, denoted by $X \subset Y$, if $X \subseteq Y$ and there is at least one element in Y which does not belong to X.

Intersection of set X and Y denoted as $X \cap Y$, is defined by

$X \cap Y = \{ x| \; x \in X \text{ and } x \in Y \}$ $X \cap Y$ is a set whose elements are common to X and Y.

Union set is a set whose elements are in sets X or Y. It is denoted by $X \cup Y$ and defined as follow

$X \cup Y = \{ x| \; x \in X \text{ or } x \in Y \}$

The complement of set consists of all elements in universal set that are not in given set denoted by $\overline{X}$

An example will help us to understand the classical set theory in more effective way.

Illustration

X= ={*3,4,5,6,*} , Y={ *2,4,6,8* }, U={ 1,*2,3,5,4,6,7,8* },

$X \cap Y=\{4,6\}$, $X \cup Y=\{2,3.4,5,6,8\}$, $\overline{X}=\{1,2,7,8\}$ $\overline{Y}=\{1,3,5,7\}$

3.2-Functions

A function f is relation $\Re$ (the concept of relation is based on the concept of ordered pair and cartesian product)such that for every element of x in the domain of f there corresponds a unique element y in the range of f. i.e.

y= f(x)

3.3-Characteristic function

The concept of characteristic function will help us to understand the concept of fuzzy sets .The membership rule that characterizes the elements of a set $X \subset U$ can be established by the concept of characteristic function $\mu_X(x)$ taking only two values 1 and 0, indicating whether or not $x \in U$ is member of X :

$$\mu_X(x) = \begin{cases} 1 & \text{for} \quad x \in X, \\ 0 & \text{for} \quad x \notin X. \end{cases} \quad \text{.............} [1]$$

Hence $\mu_X(x) \in \{0, 1\}$. Inversely, if a function $\mu_X(x)$ is defined by 1, then it is characteristic function for a set $X \subset U$ in the sense that X consists of the values of $x \in U$ for which $\mu_X(x) = 1$. In other words every set is defined by its characteristic function.

Illustration

Let U={1,2,3,4,5,6,}and its sub set X ={1,3,5}.

Only three out of six elements in U belong X.Using the notation in [1] gives

$\mu_X(x_1) = 1$, $\mu_X(x_2) = 0$, $\mu_X(x_3) = 1$, $\mu_X(x_4) = 0$,

$\mu_X(x_5) =1$, $\mu_X(x_6) =0$.

Hence the characteristic function of a set X is

$$\mu_X(x) = \begin{cases} \mathbf{1} & \boldsymbol{for} \quad x = 1,3,5, \\ \mathbf{0} & \boldsymbol{for} \quad \boldsymbol{x = 2,4,6}; \end{cases}$$

The set X can be presented as

X = {(1,1),(2,0),(3,1),(4,0),(5,1),(6,0)}.

3.4-Fuzzy Sets

We have seen in classical set that membership of an object to a set is precise concept; the object is either member or it is not, hence membership function can take two values 1 or 0.But a Fuzzy set is a set containing elements that have varying degree of membership in set. This idea is in contrast with classical set because member of crisp set would not be member unless its membership was full. Formally a fuzzy set Ã is defined by a set or ordered pair, a binary relation,

$\tilde{A}=\{(x, \mu_{\tilde{A}}(x)) \mid x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$, OR

$\tilde{A}=\{(x, \mu_{\tilde{A}}(x)) / x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$,

Where $\mu_{\tilde{A}}(x)$ is a function called membership function, $\mu_{\tilde{A}}(x)$ specifies the degree to which any element *x* in A a universe set belongs to the fuzzy set Ã and symbol / is not division sign but shows that the top number $\mu_{\tilde{A}}(x)$ is membership value of the *x* in the bottom. Above definition associates with each element *x* in a real number $\mu_{\tilde{A}}(x)$ in the interval [0,1] which is assigned to *x*. Larger the value of $\mu_{\tilde{A}}(x)$, higher will be the degree of membership.

A fuzzy set is called normalized when at least one $x \in A$ attains the maximum membership degree 1; otherwise the set is called non-normalized. Let set Ã is non-normalized, then max $\mu_{\tilde{A}}(x)<1$. To normalize the set Ã means to normalize its membership function $\mu_{\tilde{A}}(x)$ i.e. to divide it by max $\mu_{\tilde{A}}(x)$, gives $\frac{\mu\tilde{A}\,(x)}{\max \mu\tilde{A}\,(x).}$

Illustration

Let X be the universe of war aircrafts as defined as follow:

X= {b 52, b117,c130 , f4.f14,f15,f16}

and Ã be the fuzzy set of fighter class of aircrafts

Ã ={(b52,1.0),(b117,0.95),(c130,0.1),(f4,0.6),(f14,0.7),(f15,0.8),(f16,0.9)} which can also be represented as

$\tilde{A}$={1.0/b52 + 0.95/b117 +0.1/c130 +f4/0.6 +f14/0.7 +f15/0.8+f16/0.9}

It is a discrete fuzzy set consisting of seven ordered pairs. The membership function $\mu_{\tilde{A}}(x)$ of

$\tilde{A}$ takes the following values on [0,1]:

$\mu_{\tilde{A}}$ (b52) = 1.0 $\mu_{\tilde{A}}$ (b117) =0.95 $\mu_{\tilde{A}}$ (c130) = 0.1

$\mu_{\tilde{A}}(f4)$ =0.6 $\mu_{\tilde{A}}(f14)$ =0.7 $\mu_{\tilde{A}}(f15)$ =0.8 $\mu_{\tilde{A}}$ (f16) =0.9

Interpretation is as follow:

B52 is full member of $\tilde{A}$, while b117 and f 16 are almost full members of $\tilde{A}$. C130 is very little member of $\tilde{A}$ and f4 are little more member of $\tilde{A}$, while f14 and f15 are more members of fuzzy set $\tilde{A}$.

3.5-Basic Operations on Fuzzy Sets

Consider the fuzzy sets $\tilde{A}$ and $\tilde{B}$ in the universe U ,

$\tilde{A}=\{(x, \mu_{\tilde{A}}(x))\}$, $\mu_{\tilde{A}}(x) \in [0,1]$,

$\tilde{B}=\{(x, \mu_{\tilde{B}}(x))\}$, $\mu_{\tilde{B}}(x) \in [0,1]$.

The operations with $\tilde{A}$ and $\tilde{B}$ are introduced via operations on their membership function

$\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$.

3.5.1-Equality

The fuzzy sets $\tilde{A}$ and $\tilde{B}$ $\tilde{A} = \tilde{B}$ if and only if for every $x \in U$ $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$.

3.5.2-Inclusion

The fuzzy set $\tilde{A}$ is included in the fuzzy set $\tilde{B}$ denoted by $\tilde{A} \subseteq \tilde{B}$ if for every $x \in U$

$\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ then $\tilde{A}$ is called a sub set of $\tilde{B}$.

3.5.3-Proper subset

The fuzzy set $\tilde{A}$ is called a proper subset of the fuzzy set $\tilde{B}$ denoted by $\tilde{A} \subset \tilde{B}$ when $\tilde{A}$ is the subset of $\tilde{B}$ and $\tilde{A} \neq \tilde{B}$ i.e.

$\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ for every $x \in$ U

$\mu_{\tilde{A}}(x)$ c $\mu_{\tilde{B}}(x)$ for at least one $x \in$ U

3.5.4-Intersection

The intersection of the fuzzy sets Ã and $\tilde{B}$ denoted by Ã ∩ $\tilde{B}$ is defined by

$\mu_{\tilde{A}} \cap \mu_{\tilde{B}(x)}) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

If $a_1 < a_2$, min $(a_1, a_2) = a_1$. For instance min(5 , 7) = 5

3.5.5-Union

The union of the fuzzy sets Ã and $\tilde{B}$ denoted by Ã ∪ $\tilde{B}$ is defined by

$\mu_{\tilde{A}} \cup \mu_{\tilde{B}(x)}) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

If $a_1 < a_2$, max $(a_1, a_2) =$. For instance max (5 , 7) = 7

4.0-Fuzzy Relations

Consider the Cartesian product

A×B = {(x , y)| x ∈ A ,y ∈ B},

Where A × B are subset of the universal sets U_1 and U_2 respectively. A fuzzy relation

on A × B denoted by $\Re$ or $\Re$ (x.y) is defined as the set

$\Re$ ={(x , y), $\mu_{\Re}$ (x, y)|(x, y) ∈ A × B, $\mu_{\Re}$ (x , y) ∈ [0, 1]})

Where $\mu_{\Re}$ (x , y) is function in two variables called membership function. It gives the

degree of membership of the ordered pair(x , y) in $\Re$ associating with each pair (x , y)

in A × B , a real number in the interval [0 , 1]. The degree of membership indicates the degree to which x is in relation with y.

4.1-Operations on Fuzzy Relations

Let $\Re_1$ and $\Re_2$ be two relations on A × B

$\Re_1$ ={(x , y), $\mu_{\Re 1}$ (x, y)} , (x ,y) ∈ A × B,

$\Re_2 = \{(x, y), \mu_{\Re 2}(x, y)\}$, $(x, y) \in A \times B$.

4.1.1-Equality

$\Re_1 = \Re_2$ if and only if for every pair $(x, y) \in A \times B$

$\mu\Re_1(x, y) = \mu\Re_2(x, y)$

4.1.2-Inclusion

If for every pair $(x, y) \in A \times B$

$\mu\Re_1(x, y) \le \mu\Re_2(x, y)$

the relation $\Re_1$ included in $\Re_2$or $\Re_2$ is larger than $\Re_1$, denoted by $\Re_1 \subseteq \Re_2$

if $\Re_1 \subseteq \Re_2$is in addition if for at least on pair (x, y)

$\mu\Re_1(x, y) < \mu\Re_2(x, y)$

then we have proper inclusion $\Re_1 \subset \Re_2$

4.1.3-Intersection

The intersection $\mu\Re_1$ and $\mu\Re_2$ denoted by $\mu\Re_1 \cap \mu\Re_2$ is defined by

$\mu\Re_1 \cap_\mu \Re_2(x, y) = \min\{\mu\Re_1(x, y), \mu\Re_2(x, y)\}$ $(x, y) \in A \times B$

4.1.4-Union

The union $\mu\Re_1$ and $\mu\Re_2$ denoted by $\mu\Re_1 \cup \mu\Re_2$ is defined by

$\mu\Re_1 \cup_\mu \Re_2(x, y) = \max\{\mu\Re_1(x, y), \mu\Re_1(x, y)\}$ $(x, y) \in A \times B$

The operations of intersection and union on fuzzy relations are illustrated in following Example.

Illustration

Let fuzzy set $\tilde{A} = \{x_1, x_2, x_3\}$ and $\tilde{B} = \{y_1, y_2, y_3\}$ and also suppose that Cartesian product

between $\tilde{A}$ and $\tilde{B}$ is given as follow:

$$\mathfrak{R}_1 = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.5 & 0.7 \\ x_2 & 0.3 & 0.6 & 0.7 \\ x_3 & 0.4 & 0.8 & 0.9 \end{array}$$

$$\mathfrak{R}_2 = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline y_1 & 1 & 0.8 & 0.6 \\ y_2 & 0.5 & 0.6 & 0.7 \\ y_3 & 0.4 & 0.3 & 0.1 \end{array}$$

$$\mathfrak{R}_1 \cap \mathfrak{R}_2 = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.5 & 0.6 \\ x_2 & 0.3 & 0.6 & 0.7 \\ x_3 & 0.4 & 0.3 & 0.1 \end{array}$$

$$\mathfrak{R}_1 \cup \mathfrak{R}_2 = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0.8 & 0.7 \\ x_2 & 0.5 & 0.6 & 0.7 \\ x_3 & 0.4 & 0.8 & 0.9 \end{array}$$

5.0-Max-Min Composition of fuzzy Relations

Fuzzy relation in different product space can be combined with each other by the operation called "Composition". There are many composition methods in use , e.g. max-product method, max-average method and max-min method. But max-min composition method is best known in fuzzy logic applications.

Definition:

Max –min composition

Let $\mathfrak{R}_1(x, y), (x, y \in A \times B)$ and $\mathfrak{R}_2(y, z), (y, z) \in B \times C)$ be the two relations.

The max- min composition is then the fuzzy set

$\Re_1 \circ \Re_2$ ={[(x,y), $\max_y${min { $\mu\Re_1$(x,y), $\mu\Re_2$(y.z)}}] x∈A,y∈B,c∈C}

Here $\mu\Re_1$ and $\mu\Re_2$ is membership function of a fuzzy relation on fuzzy sets.

Illustration

Let $\Re_1$(x, y), and $\Re_2$ (y, z) be defined by the following relational matrices

$$\Re_1 = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.5 & 0.7 \\ x_2 & 0.3 & 0.6 & 0.7 \\ x_3 & 0.4 & 0.8 & 0.9 \end{array}$$

$$\Re_2 = \begin{array}{c|cc} & z_1 & z_2 \\ \hline y_1 & 1 & 0.8 \\ y_2 & 0.5 & 0.6 \\ y_3 & 0.4 & 0.3 \end{array}$$

Now computing the max –min composition as per definition
Calculations are as follow:

min { $\mu\Re_1(x_1,y_1)$, $\mu\Re_2(y_1.z_1)$}=min (0.2,1)= 0.2

min { $\mu\Re_1(x_1,y_2)$, $\mu\Re_2(y_2.z_1)$}=min (0.5,0.5)=0.5

min { $\mu\Re_1(x_1,y_3)$, $\mu\Re_2(y_3.z_1)$}=min (0.7,0.4)=0.4

min { $\mu\Re_1(x_2,y_1)$, $\mu\Re_2(y_1.z_1)$}=min (0.3,1)= 0.3

min { $\mu\Re_1(x_2,y_2)$, $\mu\Re_2(y_2.z_1)$}=min (0.6,0.5)= 0.5

min { $\mu\Re_1(x_2,y_3)$, $\mu\Re_2(y_3.z_1)$}=min (0.7, 0.4)= 0.4

min { $\mu\Re_1(x_3,y_1)$, $\mu\Re_2(y_1.z_1)$}=min (0.4 0.1)= 0.4

min { $\mu\Re_1(x_3,y_2)$, $\mu\Re_2(y_2.z_1)$}=min (0.8 0.5)= 0.5

min { $\mu\Re_1(x_3,y_3)$, $\mu\Re_2(y_3.z_1)$}=min (0.9,0.4)= 0.4

min { $\mu\Re_1(x_1,y_1)$, $\mu\Re_2(y_1.z_2)$}=min (0.2,0.8)= 0.2

$\min\{\mu\Re_1(x_1,y_2), \mu\Re_2(y_2.z_2)\}=\min(0.5,0.6)=0.5$

$\min\{\mu\Re_1(x_1,y_3), \mu\Re_2(y_3.z_2)\}=\min(0.7,0.3)=0.3$

$\min\{\mu\Re_1(x_2,y_1), \mu\Re_2(y_1.z_2)\}=\min(0.3,0.8)= 0.3$

$\min\{\mu\Re_1(x_2,y_2), \mu\Re_2(y_2.z_2)\}=\min(0.6,0.6)= 0.6$

$\min\{\mu\Re_1(x_2,y_3), \mu\Re_2(y_3.z_2)\}=\min(0.7,0.3)= 0.3$

$\min\{\mu\Re_1(x_3,y_1), \mu\Re_2(y_1.z_2)\}=\min(0.4, 0.8)= 0.4$

$\min\{\mu\Re_1(x_3,y_2), \mu\Re_2(y_2.z_2)\}=\min(0.8 , 0.6)= 0.6$

$\min\{\mu\Re_1(x_3,y_3), \mu\Re_2(y_3.z_1)\}=\min(0.9,0.3)= 0.3$

$\Re_1\circ\Re_2 \ (x_1,z_1) = ((x_1,z_1), \mu\Re_{1,}\ \Re_2(x_1.z_1))$

$=(x_1,z_1),\max(0.2,0.5,0.3) =0.5$

$\Re_1\circ\Re_2 \ (x_2,z_1) = ((x_2,z_1), \mu\Re_{1,}\ \Re_2(x_2.z_1))$

$=(x_2,z_1),\max(0.3 ,0.5,0.4) =0.5$

$\Re_1\circ\Re_2 \ (x_3,z_1) = ((x_3,z_1), \mu\Re_{1,}\ \Re_2(x_3.z_1))$

$=(x_3,z_1),\max(0.4, 0.5, 0.4) =0.5$

$\Re_1\circ\Re_2 \ (x_1,z_2) = ((x_1,z_2), \mu\Re_{1,}\ \Re_2(x_1.z_2))$

$=(x_1,z_2),\max(0.2, 0.5, 0.3) =0.5$

$\Re_1\circ\Re_2 \ (x_2,z_2) = ((x_2,z_2), \mu\Re_{1,}\ \Re_2(x_2.z_2))$

$=(x_2,z_2),\max(0.3,0.6\ 0.3) =0.6$

$\Re_1\circ\Re_2 \ (x_3,z_2) = ((x_3,z_2), \mu\Re_{1,}\ \Re_2(x_3.z_2))$

$=(x_3,z_2),\max(0.4,0.6, 0.3) =0.6$

So the resultant matrix of the relation between (x ,z) is

$$\Re_1\circ\Re_2 \ (x, z) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0.5 & 0.5 \\ x_2 & 0.5 & 0.6 \end{array}$$

x_3	0.5	0.6

5.1-Properties of max-min composition

5.1.1-Associativity:

Max-min composition is associative that is

$$(\mathfrak{R}_1 \circ \mathfrak{R}_2) \circ \mathfrak{R}_3 = \mathfrak{R}_1 \circ (\mathfrak{R}_2 \circ \mathfrak{R}_3)$$

5.1.2-Reflexivity: (Zadeh 1971)

Let $\mathfrak{R}$ be a fuzzy relation in X× X then $\mathfrak{R}$ is called reflexive if

$\mu_{\mathfrak{R}}(x, x) = 1 \quad \forall\, x \in X$

Illustarion

Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$

The following relation "y is close to x" is reflexive:

$\mathfrak{R}$ =

	y_1	y_2	y_3	y_4
x_1	1	0	.2	.3
x_2	0	1	.1	1
x_3	.2	.7	1	.4
x_4	0	1	.4	1

if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are reflexive fuzzy relation then the max-min composition $\mathfrak{R}_1 \circ \mathfrak{R}_2$ is also reflexive.

5.1.3-Symmetry

A fuzzy relation $\mathfrak{R}$ is symmetric if

$\mathfrak{R}(x, y) = \mathfrak{R}(y, x) \forall x, y \in X$

So if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are symmetric, then $\mathfrak{R}_1 \circ \mathfrak{R}_2$ is symmetric if

$\mathfrak{R}_1 \circ \mathfrak{R}_2 = \mathfrak{R}_2 \circ \mathfrak{R}_1$

5.1.4-Transitivity

A fuzzy relation $\Re$ is transitive if

$$\Re \circ \Re \subseteq \Re$$

So if $\Re_1$ and $\Re_2$ are transitive and $\Re_1 \circ \Re_2 = \Re_2 \circ \Re_1$ then $\Re_1 \circ \Re_2$ is transitive.

6.0-Fuzzy decision making

Decision making is defined as making choices between future, uncertain alternatives. It is a choice between various ways of getting an end accomplished. Decision making plays an important role in business, finance, and economics as well as in engineering ,social, physical and medical sciences. It must be emphasized that all decision making relates to the future. Where no alternatives exist, no decision making can be made.

It is a difficult process due to factors like incomplete and imprecise information vagueness and uncertainty of situation. These factors show that decisions take place in fuzzy logic environment. Decision making is characterized by choice from alternatives which are available .In this process specified goals have to be fulfilled keeping constraints in mind.

Consider a simple decision making model which consists of goals and constraints. Let $\tilde{G}$ be fuzzy set defined as goal with membership function $\mu_{\tilde{G}}(x)$

And constraint described by $\tilde{C}$ with membership function $\mu_{\tilde{C}}(x)$

Where x is an element of the crisp set of alternatives $A_{alt.}$

By definition (Bellman and Zadeh 1970) the decision is a fuzzy set $\tilde{D}$ with membership function $\mu_{\tilde{D}}(x)$ expressed as intersection of $\tilde{G}$ and $\tilde{C}$ as follow:

$$\tilde{D} = \tilde{G} \cap \tilde{C} = \{(x, \mu_{\tilde{D}}(x))\ x \in [d_1, d_2], \mu_{\tilde{D}}(x) \in [0, h \leq 1]\}$$

It is a multiple decision making resulting in selection the crisp set $[d_1,d_2]$, from the set of alternatives A_{alt}; $\mu_{\tilde{D}}(x)$ indicates the degree to which any $x \in [d_1,d_2]$ belongs to decision $\tilde{D}$. using the membership function and intersection operation formula we get

$$\mu_{\tilde{D}}(x) = \min(\mu_{\tilde{G}}(x), \mu_{\tilde{C}}(x)), \quad x \in A_{alt}$$

The intersection operation is commutative. Hence $\tilde{G}$ and $\tilde{C}$ can be interchanged.