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Section 301

Application of Matrix Methods for Ranking Incomplete Tournaments on Seeded Tournaments

1. Abstract.

In 1983, Thomas Jech published a paper outlining mathematical methods to rank the teams that have participated in incomplete sports tournaments. By incomplete, Jech meant tournaments where certain teams may have played an unequal number of games against the other competitors, or if certain teams have differing playing schedules. The goal of his application was to see if there was some mathematical way to rank the teams even though they all had different schedules – and it made me wonder if this could be applied to a more extreme case. I will be applying Jech's methods to a seeded tournament, the 2019 Wimbledon Men's Singles tournament, to see if they apply to cases where a majority of competitors do not play one another.

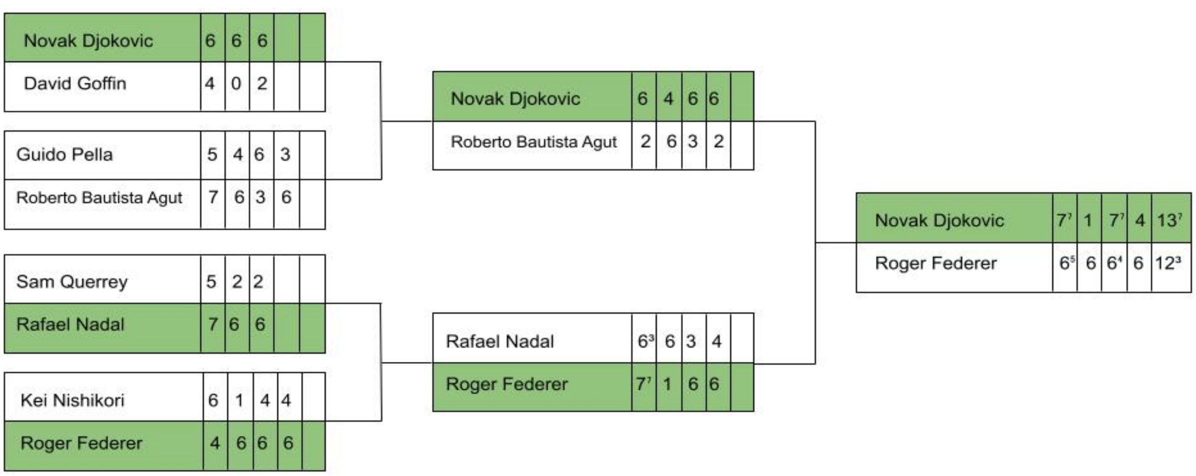
During the application there were several obstacles that were not outlined in Jech's paper that I had to consider and alter certain methods to reach a reasonable ranking of the players. Jech spoke very briefly on *Comparability*, whether one team can be compared to another - his remarks were very vague, only giving several specific examples of what would not be comparable, but never disclosing any rules to see if certain teams are comparable or not. We had to see for ourselves whether the seeded tournaments were comparable or not through our own calculations and judgement. Throughout our calculations and using Jech's method, we noticed a trait of our ranking that went against one of Jech's proofs: he proved that there can only be <u>one</u> unique ranking, but in our calculations, we found that there were infinite rankings if we solved for our rank in the same manner that Jech did. This led us to the conclusion that seeded tournaments are not in fact completely comparable using his methods. Once we learned this, we formed partial rankings, and then used a specific method to estimate a ranking of all the players.

2. Introduction.

Tennis was first played in the 1870's, and shortly after, the first tennis tournament was held in 1877, the Wimbledon tournament. The 133rd Wimbledon Gentlemen's Singles Championship just concluded Sunday, July 11, so I thought it was only right to analyze the 2019 gentlemen's singles championship (The 2020 one did not occur due to COVID-19). Wimbledon is a seeded tournament with 128 players, we are to work with several square matrices, so I would need a 128x128 matrix, or over 16,000 values if I am to track every single player. Since this paper is not about ranking each player in the 2019 Wimbledon, but primarily about whether Jech's methods apply to seeded tournaments, we will be working with only the quarterfinals (the final 8 players) and the semifinals (the final 4 players). These samples should be enough to see the behavior of this ranking method on a seeded tournament.

The scoring system for tennis is quite important in the conclusions of the ranks themselves, so I will briefly explain it now. Each match decides a winner, XX vs YY is a match where either XX or YY wins. Each match is comprised of several sets depending on the tournament, for Wimbledon, it is the best of 5 sets, or the first player to win 3 sets wins the match. Similarly, each set is decided by several games, where the first to 6 games wins the set, unless both players both win 5 games, then the winner must win by 2 games; if the players both win 6 games, then the set is finally decided by a single tiebreaker game where the first to 7 points wins. Finally, each game is comprised of points where the first one to 4 points wins, unless both players have 3 points each then you must win by 2. Although it is very complicated it was quite important when deciding how to score each player in the rankings and the importance of total score over an actual players skill since one player may end up with fewer total sets played (and perhaps fewer sets won) but a higher placement than others in the competition bracket. To rank the players, we will look at both the number of sets and games played instead of looking at matches. If we look at matches, there is only one data point to look at: did a player win the entire match or lose the entire match? Looking at the sets and games though allows us to look more closely at the skill of each player, "sure XX may have lost the match against ZZ, but he won 2 of the 5 sets, whereas YY didn't win a single set against ZZ. XX appears to be slightly better than YY."

In Jech's paper, he described a ranking matrix and a partial ranking matrix; The goal in this ranking method is to create these matrices that contain probabilities for a certain player beating another. Some teams are not comparable so they can only have a partial ranking matrix. Upon following through with the calculations, we will see that only partial ranking matrices are possible to create for a seeded tournament but using some properties that Jech outlined in his paper, we will roughly estimate the chance for one player to beat another to form our final rankings.



(Wimbledon 2019 Gentlemen's Singles Quarterfinals Bracket and Outcomes)

Above is the bracket for the quarterfinals of the tournament, it is comprised of the top 8 players. The green boxes represent the winner between a specific match, if a score of a set has a superscript, it means the set went to a tie game, the superscripted number is the score of that tie game. The very final set between Djokovic and Federer has a score of 13^7 - 12^3 because the two players were playing a set with no tie break until they reached 12 – 12, so the final set was 24 games with a final tiebreak game (Fun fact, the entire match between Djokovic and Federer is the longest final match ever at Wimbledon, it took 4 hours and 57 minutes!). Since we will be looking calculating the rank for both the sets and the games, I will be counting each as follows: the number of sets played is simply the total number of sets not including tie games, for example, the match between Nadal and Federer was 4 sets. The number of games played will be each game in the set, and each tie break game will count as a single game, in the same example of the Nadal and Federer match, the two played

6 + 6 + 3 + 4 + 7 + 1 + 6 + 6 + 1 (tie game) = 40 games.

One final thing to note which you will see in the paper, when we change between looking at the sets vs. the games played, we will be looking at a likelihood for a certain player to win a game vs a certain player to win a set, which is simply something to keep in mind when observing the final ranking.

3. Mathematical Formulation.

(Please note that all the following equations, definitions, and properties are from *The Ranking of Incomplete Tournaments: A Mathematician's Guide to Popular Sports*, by T, Jech, The American Mathematical Monthly (1983).)

Consider a tournament with n teams:

$$T_1, T_2, \dots T_n$$

In the ranking method, there is a **schedule matrix**, M, to order the playing schedules for each team. Since there are n teams M is an $n \ x \ n$ symmetric matrix, where each i^{th} row and column represent a single player for $1 \leq i \leq n$. Each entry in M represents the number of times T_i will play T_j and so the ij entries equal the ji entries, or

$$m_{ij} = m_{ji}$$

The outcome of the tournament is represented by the **result matrix**, R. R is also an $n \ x \ n$ matrix where each column and row still represent a different team, but the entries along the i^{th} row will show the score that T_i got against the player in the j^{th} column, T_j, for $1 \leq i, j \leq n$. Since r_{ij} and r_{ji} is the score of both teams, the sum must equal the number of games played, m_{ij}.

$$r_{ij} + r_{ji} = m_{ij} = m_{ji}$$

There is also a **score vector**, s, which is represented by the sums of each row in R.

$$s_i = r_{i1} + r_{i2} + \cdots + r_{in}$$

$$\boldsymbol{s} = \begin{pmatrix} s_1 \\ s_2 \\ \ldots \\ s_n \end{pmatrix}.$$

As I said earlier, Jech's goal was to form a **ranking matrix** or a **partial ranking matrix**, *P*. This is an *n x n* matrix that holds the probabilities of the team in the i^{th} row beating the team in the j^{th} column for $i, j = 1, 2, \ldots, n.$ a probability is usually shown as

$$p = \frac{r}{m}$$

Where p is the probability, r is the number of games won by a team, and m is the number of games played by a team. Since the probability of two teams is directly related, the following must be true,

$$p_{ij} + p_{ji} = 1.$$

Another way of showing these probabilities is by representing them as a *chance* for a team to beat another team. The *chance* to win is given by

$$x_{ij} = \frac{p_{ij}}{1 - p_{ij}}$$

Which must follow the laws of probability, meaning:

$$x_{ik} = x_{ij} * x_{jk}$$

Or in terms of p,

$$(1) \quad p_{ij} * p_{jk} * p_{ki} = p_{kj} * p_{ji} * p_{ik}.$$

And the values in the ranking matrix had to accurately predict the score of a team when multiplied with its number of games played, in mathematical terms:

$$(2) \quad \sum_{j=1}^{n} m_{ij} * p_{ij} = s_i.$$

If the ranking matrix met both conditions (1) and (2) then it was, according to Jech, an accurate ranking matrix.

An important note about the ranking matrices is that in the paper, Jech proved that for every tournament in which all teams are comparable, there is a *unique* ranking matrix, and for every tournament in which some teams are incomparable, there is a *unique* partial ranking. So, if we find one ranking that satisfies the two conditions above, then there are no other solutions.

In the next section, we are going to look at the results and application of Jech's methods, but I will say that after finding the partial rankings, we will estimate the chances for incompatible teams using some of the relationships above. We will be assuming that if we know the partial ranking or the chance for one team to beat another, we can use the known values and the laws of probability,

$$x_{ik} = x_{ij} * x_{jk}$$

to *estimate* another team's chance of beating an incomparable team.

There are several methods to solve for these p_{ij}, but one way that Jech proved for a tournament **with comparable teams** is the following formulas:

$$\sum_{j=1}^{n} \frac{m_{ij}}{1 + e^{v_j - v_i}} = s_i$$

And if there were values of v that satisfied that, then

$$p_{ij} = \frac{1}{1 + e^{v_j - v_i}} = \frac{e^{v_i}}{e^{v_i} + e^{v_j}}$$

Would obey the conditions Jech had set for a successful ranking matrix.

A final definition that was not in Jech's paper was the **chance matrix**, *X.* The chance matrix is quite self-explanatory, instead of holding the probability, *p*, of a player beating another, it would hold the chance.

4. Examples and Numerical Results.

We will first be looking at the semifinals to apply the techniques on a small scale.
As I said earlier, we will be looking at both the number of sets played, and the number of games played. First, we are going to rank the tournament according to the number of sets played, so the schedule matrix for the number of sets plays looks as follows:

$$M_s = \begin{bmatrix} \blacksquare & 4 & 0 & 5 \\ 4 & \blacksquare & 0 & 0 \\ 0 & 0 & \blacksquare & 4 \\ 5 & 0 & 4 & \blacksquare \end{bmatrix}$$

Where column 1 and row 1 represent Novak Djokovic, column 2 and row 2 represent Roberto Bautista Agut, column 3 and row 3 represent Rafael Nadal, and column 4 and row 4 represent Roger Federer. The matrix says that Djokovic and Agut played 4 sets against each other, Nadal and Federer played 4 sets, and Djokovic and Federer played 5 sets against each other.

Next, we formulate the result matrix by filling the matrix, R, with the values of player i's score against player j:

$$R_s = \begin{bmatrix} \blacksquare & 3 & 0 & 3 \\ 1 & \blacksquare & 0 & 0 \\ 0 & 0 & \blacksquare & 1 \\ 2 & 0 & 3 & \blacksquare \end{bmatrix}$$

And so, our score vector looks like:

$$\boldsymbol{s}_s = \begin{pmatrix} 6 \\ 1 \\ 1 \\ 5 \end{pmatrix}$$

Note that if the rank were based solely upon the score, Djokovic would be ranked 1st, Federer 2nd and Agut and Nadal tie for 4th.

When we try to use the summation equation near the end of section 3 to solve for the p_{ij} values we get the following:

$$\sum_{j=1}^{n} \frac{m_{ij}}{1 + e^{v_j - v_i}} = s_i$$

So,

$$\text{Row 1: } \frac{4}{1 + e^{v_2 - v_0}} + \frac{0}{1 + e^{v_3 - v_0}} + \frac{5}{1 + e^{v_4 - v_0}} = 6$$

$$\text{Row 2: } \frac{4}{1 + e^{v_1 - v_0}} + \frac{0}{1 + e^{v_3 - v_0}} + \frac{0}{1 + e^{v_4 - v_0}} = 1$$

$$\text{Row 3: } \frac{0}{1 + e^{v_1 - v_0}} + \frac{0}{1 + e^{v_2 - v_0}} + \frac{4}{1 + e^{v_4 - v_0}} = 1$$

$$\text{Row 4: } \frac{5}{1 + e^{v_1 - v_0}} + \frac{0}{1 + e^{v_2 - v_0}} + \frac{4}{1 + e^{v_3 - v_0}} = 5$$

But this is quite complicated to solve and will give us the same results as solving the set of equations below

$$(1) \quad 4p_{12} + 0p_{13} + 5p_{14} = 6$$

$$(2) \quad 4p_{21} + 0p_{23} + 0p_{24} = 1$$

$$(3) \quad 0p_{31} + 0p_{32} + 4p_{34} = 1$$

$$(4) \quad 5p_{41} + 0p_{42} + 4p_{43} = 5$$

From (2) and (3) we know that

$$p_{21} = .25 = p_{34}$$

And since

$$p_{12} = 1 - p_{21} \text{ and } p_{43} = 1 - p_{34}$$

Then

$$p_{12} = .75 = p_{43}$$

And finally, we can plug in to (1) and (4) to solve and see that

$$p_{14} = .6 \text{ and } p_{41} = .4$$

But neither of the methods above give us the have several probabilities that are still unknown (p_{13}, p_{23}, p_{24}, p_{31}, p_{32}, and p_{42}) since no games directly were played between them. I pointed out earlier though that if it is a solution to the summation method, then it is a viable probability, **BUT** I also pointed out earlier that there are only <u>unique</u> rank matrices. If we look back at the summations:

$$\text{Row 1: } \frac{4}{1+e^{v_2-v_0}} + \frac{0}{1+e^{v_3-v_0}} + \frac{5}{1+e^{v_4-v_0}} = 6$$

$$\text{Row 2: } \frac{4}{1+e^{v_1-v_0}} + \frac{0}{1+e^{v_3-v_0}} + \frac{0}{1+e^{v_4-v_0}} = 1$$

$$\text{Row 3: } \frac{0}{1+e^{v_1-v_0}} + \frac{0}{1+e^{v_2-v_0}} + \frac{4}{1+e^{v_4-v_0}} = 1$$

$$\text{Row 4: } \frac{5}{1+e^{v_1-v_0}} + \frac{0}{1+e^{v_2-v_0}} + \frac{4}{1+e^{v_3-v_0}} = 5$$

We can see that **any** denominator for the values with a 0 in the numerator will solve this set of equations. If there are infinite solutions to this set of equations, then there are infinite rank matrices. Since there should only be a unique matrix we can conclude that the seeded tournaments are **not comparable.**

This means our solutions result in a partial rank matrix:

$$P_s = \begin{bmatrix} \blacksquare & .75 & * & .6 \\ .25 & \blacksquare & * & * \\ * & * & \blacksquare & .25 \\ .4 & * & .75 & \blacksquare \end{bmatrix}$$

We still wanted some estimate of a player's ability against someone they had not played, so we used one of the laws of probability that Jech had pointed out earlier:

$$x_{ik} = x_{ij} * x_{jk}$$

This calculation was much more convenient to work with since it only required 3 values, so we formed our chance matrix:

$$X_s = \begin{bmatrix} \blacksquare & 3 & * & 1.5 \\ .333 & \blacksquare & * & * \\ * & * & \blacksquare & .333 \\ .667 & * & 3 & \blacksquare \end{bmatrix}$$

Then, one by one we filled in the empty spaces by using the equation above in the following order:

$$x_{13} = x_{14} * x_{43}$$
$$x_{24} = x_{21} * x_{14}$$
$$x_{23} = x_{21} * x_{13}$$

And the property that

$$x_{ji} = \frac{1}{x_{ij}}$$

For the other 3 values. This resulted in

$$X_s^* = \begin{bmatrix} \blacksquare & 3 & 4.5 & 1.5 \\ .333 & \blacksquare & 1.5 & 0.5 \\ .222 & .667 & \blacksquare & .333 \\ .667 & 2 & 3 & \blacksquare \end{bmatrix}$$

To get an estimated ranking from this, we used a method that Jech used, in which we put the players on a linear scale where we assign the "worst" player a value of 1, and each other team will have a value such that

$$x_{ij} = \frac{a_i}{a_j}$$

So, our estimated ranking of the seeded tournament by set was:

1. Djokovic = 4.5
2. Federer = 3
3. Agut = 1.5
4. Nadal = 1

Meaning that Agut was 1.5 times better at winning sets than Nadal, Federer was 2 times better then Agut and 3 times better than Nadal, and Djokovic was 1.5 times better than Federer, 3 times better than Agut, and 4.5 times better than Nadal.

Now we will look at the final 8 players in the quarterfinals by the **games** played and won instead of the sets. We will see both the difference in a smaller seed vs. a larger seed and the prediction between games vs. sets. When we look at the games, it gives us a significantly larger data pool and more accurate estimate for how much "better" one player may be than another. The same conclusion will form that those did not play are not comparable, so we will form the partial ranking matrix, then make the same assumption and fill in the chance matrix with estimations.

Our schedule matrix for the quarterfinals is:

$$M_g = \begin{bmatrix} \blacksquare & 24 & 0 & 35 & 0 & 0 & 0 & 71 \\ 24 & \blacksquare & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \blacksquare & 40 & 0 & 0 & 0 & 0 \\ 35 & 0 & 40 & \blacksquare & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \blacksquare & 28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28 & \blacksquare & 0 & 49 \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & 37 \\ 71 & 0 & 0 & 0 & 0 & 49 & 37 & \blacksquare \end{bmatrix}$$

Where row 1 and column 1 (r1c1) is Novak Djokovic, r2c2 is David Goffin, r3c3 is Guido Pella, r4c4 is Roberto Bautista Agut, r5c5 is Sam Querrey, r6c6 is Rafael Nadal, r7c7 is Kei Nishikori, and r8c8 is Roger Federer.

The result matrix is

$$R_g = \begin{bmatrix} \blacksquare & 18 & 0 & 22 & 0 & 0 & 0 & 35 \\ 6 & \blacksquare & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \blacksquare & 18 & 0 & 0 & 0 & 0 \\ 13 & 0 & 22 & \blacksquare & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \blacksquare & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 19 & \blacksquare & 0 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & 15 \\ 36 & 0 & 0 & 0 & 0 & 27 & 22 & \blacksquare \end{bmatrix}$$

Again, to find the *p* values, solving a system of linear equations is the easiest way to do this. Doing so results in the games won divided by the games played, just like above in the semifinals. So, our partial rank matrix is:

$$P_g = \begin{bmatrix} \blacksquare & .75 & * & .629 & * & * & * & .493 \\ .25 & \blacksquare & * & * & * & * & * & * \\ * & * & \blacksquare & .45 & * & * & * & * \\ .371 & * & .55 & \blacksquare & * & * & * & * \\ * & * & * & * & \blacksquare & .321 & * & * \\ * & * & * & * & .679 & \blacksquare & * & .449 \\ * & * & * & * & * & * & \blacksquare & .405 \\ .507 & * & * & * & * & .551 & .595 & \blacksquare \end{bmatrix}$$

And a chance matrix of

$$X_g = \begin{bmatrix} \blacksquare & 3 & * & 1.692 & * & * & * & .972 \\ .333 & \blacksquare & * & * & * & * & * & * \\ * & * & \blacksquare & .818 & * & * & * & * \\ .591 & * & 1.222 & \blacksquare & * & * & * & * \\ * & * & * & * & \blacksquare & .474 & * & * \\ * & * & * & * & 2.111 & \blacksquare & * & .815 \\ * & * & * & * & * & * & \blacksquare & .682 \\ 1.029 & * & * & * & * & 1.227 & 1.467 & \blacksquare \end{bmatrix}$$

We then made the same assumption and estimated the rest of the values one by one using

$$x_{ik} = x_{ij} * x_{jk}$$

And found a final estimated chance matrix of

$$X_g^* = \begin{bmatrix} \blacksquare & 3 & 2.068 & 1.692 & 2.519 & 1.193 & 1.426 & .972 \\ .333 & \blacksquare & .689 & .564 & .840 & .398 & .475 & .324 \\ .483 & 1.450 & \blacksquare & .818 & 1.218 & .577 & .689 & .470 \\ .591 & 1.773 & 1.222 & \blacksquare & 1.488 & .705 & .843 & .574 \\ .397 & 1.191 & .821 & .672 & \blacksquare & .474 & .566 & .386 \\ .838 & 2.514 & 1.733 & 1.418 & 2.111 & \blacksquare & 1.195 & .815 \\ .701 & 2.104 & 1.451 & 1.187 & 1.767 & .837 & \blacksquare & .682 \\ 1.029 & 3.086 & 2.127 & 1.741 & 2.591 & 1.227 & 1.467 & \blacksquare \end{bmatrix}$$

We will rank this using the same method as above. The lowest chance to win is Goffin vs. Federer with a .324 chance, so he will be our bottom rank.

1. Roger Federer = 3.086
2. Novak Djokovic = 3.003
3. Rafael Nadal = 2.513
4. Kei Nishikori = 2.105
5. Roberto Bautista Agut = 1.773

6. Guido Pella = 1.451
7. Sam Querrey =1.19
8. David Goffin = 1

5. Discussion and Conclusions.

As we can see, we have two very different rankings compared to the sets and the games, but both *do* make sense. It makes sense that Federer, who won more games against Djokovic, would win more games against everyone else in the other bracket. But it also makes sense that if Djokovic won more sets than Federer, he would win more sets than the others in Federer's half of the bracket. It does make sense that Kei Nishikori, who won 15 games against Federer, could win more games against Roberto Bautista Agut, who only won 13 games against Djokovic, who won fewer games against Federer. It *does* seem like it makes sense although it might not be the most accurate ranking. Unfortunately, we cannot get an absolutely accurate ranking with seeded tournaments, although we are able to estimate them.

In his paper, Jech said "it is clear that the tournament has to satisfy some conditions if we want to get a meaningful ranking." (250) When talking about comparability, but he never defined those conditions. Later in his proofs, Jech proved that "… comparability depends only on *M* and ***s***." (256) but never elaborated on *how* it depended on *M* and ***s***. So, in my calculations, I never was quite sure if I came to the correct conclusion about certain teams being comparable or not. My guess is when Jech said comparability depends on *M* and ***s***, he was talking about his method for finding the *p* values. If there are empty spaces in the schedule matrix, then there can be more than one solution that solves the summation. When I first read Jech's paper, before looking through the proofs, he said that this could be applied to situations where you need to compare several objects over a limited number of comparisons, and I originally thought that even seeded tournaments could be successfully ranked, but that is not the case. However, we did see that with partial rankings seemingly reasonable estimates could be made about a player's ranking compared to another – as the tournament size gets larger however, they might be less statistically accurate because you rely on more assumptions and multiplied probabilities between several players who have never played. As for future work in this topic, it seems that we would need a new method to rank seeded tournaments completely because an estimation likely won't be enough for professional applications. This might require a completely new method of ranking that is unlike Jech's or simply a new way to calculate *p* values.

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