APPM 4350 Project

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Abstract

We consider a mathematical model of Chladni plates. We solve this model by separation of variables for two different boundary value problems. For each boundary value problem we find some allowable values and then use those values to estimate the nodes and frequencies expected of the system. We then compare these estimates to a real physical Chladni plate system to gauge the predictive ability of our model.

1 Introduction

1.1 What are Chladni Plates

Chladni Plates are flat plates that Ernst Chladni first used to visualize sound waves through physical surfaces. They have significantly contributed to the science of acoustics and improving musical instruments.[1] Since his applications, they have been used to advance quantum mechanics when Schrödinger used them to understand single electron orbitals.[2] Today they are used to visualize sound in instruments with violin and guitar shaped Chladni plates.

1.2 PDE to model Chladni plates

One might think the wave equation would be a good model for Chladni plates, but because we want a 2D model that works for *rigid* surfaces we will have to use the following modified model.

$$\Delta\Delta u = -\beta\frac{\partial^2 u}{\partial t^2}, \quad 0 < r < R, \quad t > 0,$$

where β is a physical parameter specific to the system.

We will solve this system for the Fourier modes and then use these modes to find the zeros of the systems and the frequencies at which we expect these zeros to occur.

We will then compare our solved values to real world experimental data to see how well our model stacks up compared to real physical findings.

For reference, our physical system has these properties:

Parameter	Value
Inner Radius (R_i)	$0.21cm$
Outer Radius (R)	$12cm$
Height/Thickness (h)	$0.0095cm$
Density (ρ)	$2700\frac{kg}{m^3}$
Young's Modulus (E)	$68.9 * 10^9\frac{kg}{ms^2}$
Poisson Ratio (v)	0.33
$D = \frac{Eh^2}{12(1-v^2)}$	$58.15121\frac{kgm}{s^2}$
$\beta = \frac{\rho h}{D}$	$0.0044109\frac{s^2}{m^3}$

Table 1 - Physical properties of the Chladni Plate.

2 Analysis

2.1 Solving the system

We begin our analysis with the given system:

$$\Delta\Delta u = -\beta \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < R, \quad t > 0, \tag{1}$$

There are two sets of boundary conditions for this system. One set of boundary conditions assumes the Chladni plate is a circular disk with no hole in the middle - this is an idealized system and is not reflective of the reality of the physical system. The physical system has a hole through the middle of the disk where the wave driver is, making an annulus a more accurate representation.

The boundary conditions for the disk:

$$|u(0,t)| < \infty, \quad t > 0, \tag{2a}$$

$$|\frac{1}{r}\frac{\partial}{\partial r}u(0,t)| < \infty, \quad t > 0, \tag{2b}$$

$$\frac{\partial^2}{\partial r^2}u(R,t) = 0, \quad t > 0, \tag{2c}$$

$$\frac{\partial^3}{\partial r^3}u(R,t) = 0, \quad t > 0, \tag{2d}$$

The boundary conditions for the annulus:

$$u(R_i,t) = 0, \quad t > 0, \tag{3a}$$

$$\frac{\partial}{\partial r}u(R_i,t) = 0, \quad t > 0, \tag{3b}$$

$$\frac{\partial^2}{\partial r^2}u(R,t) = 0, \quad t > 0, \tag{3c}$$

$$\frac{\partial^3}{\partial r^3}u(R,t) = 0, \quad t > 0, \tag{3c}$$

It is important to understand the physical implications of our mathematical model, particularly the boundary conditions. Let us first consider equation (2). Boundary condition (2a) means that at $r = 0$ the amplitude of vibration of the plate is bounded for any $t > 0$. Boundary condition (2b) means that the change in the amplitude of vibration at and as we approach $r = 0$ is bounded. Boundary condition (2c) means that at $r = R$ the change in the amplitude of vibration is constant, and as this is only defined for $r = R$, this means that the amplitude of vibration for $r = R$ is constant. Boundary condition (2d) follows from (2c) mathematically and a physical interpretation does little to inform.

Now lets consider equation (3). Boundary condition (3a) means that the plate is not moving at $r = R_i$. Boundary condition (3b) is a consequence of boundary condition (3a), as a plate that is not moving will have no change in its amplitude with respect to any variable. Recall from the whole disk case, boundary condition (3c) means that at $r = R$ the change in the amplitude of vibration is constant, and as this is only defined for $r = R$, this means that the amplitude of vibration for $r = R$ is constant. Boundary condition (3d) follows from (3c) mathematically and a physical interpretation does little to inform.

2.1.1 Separation of Variables

Since the Chladni plates are circular, our solution is going to be of the form $u(\theta, r, t)$, but we know that from the boundary conditions there is no angular dependence in the system, thus u is a function of r and t only. From separation of variables using the assumption $u(r,t) = F(r)G(t) \neq 0$ we have:

$$\begin{aligned}
\Delta\Delta F(r)G(t) &= -\beta F(r)G''(t) \\
\implies \frac{\Delta\Delta F(r)}{F(r)} &= -\beta\frac{G''(t)}{G(t)} \\
\frac{\Delta\Delta F(r)}{F(r)} &= -\beta\frac{G''(t)}{G(t)} = k^4
\end{aligned} \tag{4}$$

Thus we have our F (spatial) equation in the form:

$$\Delta\Delta F(r) = k^4 F(r) \tag{5}$$

The boundary conditions for $F(r)$ follow from above:
Whole Disk:

$$|F(0)| < \infty, \tag{6a}$$
$$|\frac{1}{r}F'(0)| < \infty, \tag{6b}$$
$$F''(R) = 0, \tag{6c}$$
$$F'''(R) = 0. \tag{6d}$$

Annulus:

$$F(R_i) = 0, \tag{7a}$$
$$F'(R_i) = 0, \tag{7b}$$
$$F''(R) = 0, \tag{7c}$$
$$F'''(R) = 0. \tag{7d}$$

And the G (time) equation in the form:

$$G''(t) = \frac{k^4 G(t)}{-\beta} \tag{8}$$

First, we will solve the G equation. We will look at the case that $k^4 > 0$ as other values of k will produce trivial solutions for this system. We then let $G(t) = e^{lt}$ be our guess for the system and get the resulting characteristic equation $l^2 = \frac{-k^4}{\beta}$. With the assumption that $k^4 > 0$ we get a general solution of the form:

$$\begin{aligned}
G(t) &= c_1 e^{i\frac{k^2}{\beta}t} + c_2 e^{-i\frac{k^2}{\beta}t} \\
G(t) &= c_1\cos(\frac{k^2}{\sqrt{\beta}}t) + c_2\sin(\frac{k^2}{\sqrt{\beta}}t)
\end{aligned} \tag{9}$$

Now we will solve the spatial problem. From equation (5) and using properties of operators, it follows that:

$$\begin{aligned}
\Delta\Delta F(r) - F(r)k^4 &= 0 \\
F(r)(\Delta\Delta - k^4) &= 0 \\
F(r)(\Delta - k^2)(\Delta + k^2) &= 0
\end{aligned} \tag{10}$$

There are two equations to solve, $(\Delta F(r) - k^2 F(r)) = 0$ and $(\Delta F(r) + k^2 F(r)) = 0$. Using the definition of the laplacian in polar coordinates, we get:

$$
\begin{aligned}
\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} \pm k^2 F(r) = 0 \\
r^2 \frac{\partial^2 F}{\partial r^2} + r\frac{\partial F}{\partial r} \pm k^2 r^2 F(r) = 0
\end{aligned} \tag{11}
$$

If we use a change of variables, letting $z = kr$ and letting $f(z) = F(r)$, we have

$$\frac{\partial f}{\partial z} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial z} = \frac{\partial F}{\partial r}\frac{1}{k}$$

$$\frac{\partial F}{\partial r} = k\frac{\partial f}{\partial z}$$

$$\frac{\partial^2 F}{\partial r^2} = k^2 \frac{\partial^2 f}{\partial z^2}$$

So, equation (11) becomes:

$$z^2 \frac{\partial^2 f}{\partial z^2} + z\frac{\partial f}{\partial z} \pm z^2 f(z) = 0 \tag{12}$$

We now have our spatial problem in a form that is solved by a linear combination of Bessel functions. There is a set of solutions for the $+z^2$ case and for the $-z^2$ case. The solutions for the $+z^2$ case are the Bessel functions, $J_0(z)$ and $Y_0(z)$ and for the $-z^2$ case are the modified Bessel functions, $I_0(z)$ and $K_0(z)$. Thus by superposition, our general solution of the spatial equation will have the form:

$$F(r) = AJ_0(kr) + BY_0(kr) + CI_0(kr) + DK_0(kr) \tag{13}$$

Where A, B, C, D are constants.

We can apply the boundary conditions to $F(r)$ using the following identities and derivatives of Bessel and modified Bessel functions. The derivative of normal Bessel functions is defined as

$$C'_v(z) = \frac{C_{v-1}(z) - C_{v+1}(z)}{2}[3] \tag{14a}$$

For modified Bessel functions, the derivative is defined as

$$C'_v(z) = \frac{C_{v-1}(z) + C_{v+1}(z)}{2}[3] \tag{14b}$$

And note these useful identities:

$$J'_0(z) = -J_1(z) \tag{15a}$$
$$Y_0`(z) = -Y_1(z) \tag{15b}$$
$$I_0`(z) = I_1(z) \tag{15c}$$
$$K_0`(z) = -K_1(z)[3] \tag{15d}$$

From this we can find the derivatives with respect to r of each Bessel function in our solution:

$$\frac{d^2}{dr^2}(J_0(kr)) = \frac{k^2}{2}(-J_0(kr) + J_2(kr)) \tag{16a}$$
$$\frac{d^3}{dr^3}(J_0(kr)) = \frac{k^3}{4}(3J_1(kr) - J_3(kr)) \tag{16b}$$
$$\frac{d^2}{dr^2}(Y_0(kr)) = \frac{k^2}{2}(-Y_0(kr) + Y_2(kr)) \tag{16c}$$
$$\frac{d^3}{dr^3}(Y_0(kr)) = \frac{k^3}{4}(3Y_1(kr) - Y_3(kr)) \tag{16d}$$
$$\frac{d^2}{dr^2}(I_0(kr)) = \frac{k^2}{2}(I_0(kr) + I_2(kr)) \tag{16e}$$
$$\frac{d^3}{dr^3}(I_0(kr)) = \frac{k^3}{4}(3I_1(kr) + I_3(kr)) \tag{16f}$$
$$\frac{d^2}{dr^2}(K_0(kr)) = \frac{k^2}{2}(-K_0(kr) - K_2(kr)) \tag{16g}$$
$$\frac{d^3}{dr^3}(K_0(kr)) = \frac{k^3}{4}(K_1(kr) - K_3(kr)) \tag{16h}$$

And now we can apply the boundary conditions for each case.

Whole Disk:

Since $F(r)$ must be bounded at $r = 0$ from equation (6a), the two unbounded Bessel functions, $Y_0(kr)$ and $K_0(kr)$, drop out of the solution, leaving the equation in the form:

$$F(r) = AJ_0(kr) + CI_0(kr) \tag{17}$$

Now applying the other boundary conditions in equation (6) yields the following system of equations:

$$F''(R) = 0 = A\frac{k^2}{2}(-J_0(kR) + J_2(kR)) + C\frac{k^2}{2}(I_0(kR) + I_2(kR)) \tag{18a}$$
$$F'''(R) = 0 = A\frac{k^3}{4}(3J_1(kR) - J_3(kR)) + C\frac{k^3}{4}(3I_1(kR) + I_3(kR)) \tag{18b}$$

Annulus:

Since $R_i < r < R$ for the annulus model, we have no boundedness condition at $r = 0$, thus all four of the Bessel and modified Bessel functions are a part of our solution. Applying the boundary conditions from equation (7) yields the system:

$$F(R_i) = 0 = AJ_0(kR_i) + BY_0(kR_i) + CI_0(kR_i) + DK_0(kR_i) \tag{19a}$$

$$F'(R_i) = 0 = -AkJ_1(kR_i) - BkY_1(kR_i) + CkI_1(kR_i) - DkK_1(kR_i) \tag{19b}$$

$$\begin{aligned} F''(R) = 0 = {} & A\frac{k^2}{2}(-J_0(kR) + J_2(kR)) + B\frac{k^2}{2}(-Y_0(kr) + Y_2(kr)) \\ & + C\frac{k^2}{2}(I_0(kR) + I_2(kR)) + D\frac{k^2}{2}(-K_0(kr) - K_2(kr)) \end{aligned} \tag{19c}$$

$$
\begin{aligned}
F'''(R) = 0 = A\frac{k^3}{4}(3J_1(kR) - J_3(kR)) + B\frac{k^3}{4}(3Y_1(kR) - Y_3(kR)) \\
+C\frac{k^3}{4}(3I_1(kR) + I_3(kR)) + D\frac{k^3}{4}(K_1(kR) - K_3(kR))
\end{aligned} \tag{19d}
$$

Both of these systems can be made into a matrix system. For the disk:

$$
\begin{bmatrix} (-J_0(kR) + J_2(kR)) & (I_0(kR) + I_2(kR) \\ (3J_1(kR) - J_3(kR)) & (3I_1(kR) + I_3(kR)) \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The annulus:

$$
\begin{bmatrix} J_0(kR_i) & Y_0(kR_i) & I_0(kR_i) & K_0(kR_i) \\ -J_1(kR_i) & -Y_1(kR_i) & I_1(kR_i) & -K_1(kR_i) \\ (-J_0(kR) + J_2(kR)) & (-Y_0(kr) + Y_2(kr)) & (I_0(kR) + I_2(kR)) & (-K_0(kr) - K_2(kr)) \\ (3J_1(kR) - J_3(kR)) & (3Y_1(kR) - Y_3(kR)) & (3I_1(kR) + I_3(kR)) & (K_1(kR) - K_3(kR)) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

We then set the determinant of each matrix equal to 0 and solve for the possible k values numerically.

Whole Disk Case:

$$
(-J_0(kR) + J_2(kR))(3I_1(kR) + I_3(kR) - (3J_1(kR) - J_3(kR))(I_0(kR) + I_2(kR) = 0
$$

Resulting k values:

k_n	Value
k_1	0.2305
k_2	0.5229
k_3	0.7935
k_4	1.0598
k_5	1.3245
k_6	1.588

Table 2 - k values for the disk case.

Annulus Case:

$$
\begin{aligned}
&\mathrm{J}_0(kR_i)(Y_1(kR_i)[(I_0(kR) + I_2(kR))(K_1(kR) - K_3(kR)) - (-K_0(kr) - K_2(kr))(3I_1(kR) + I_3(kR))] \\
&+ I_1(kR_i)[(-Y_0(kr) + Y_2(kr))(K_1(kR) - K_3(kR)) - (-K_0(kr) - K_2(kr))(3Y_1(kR) - Y_3(kR))] \\
&+ K_1(kR_i)[(-Y_0(kr) + Y_2(kr))(3I_1(kR) + I_3(kR)) - (I_0(kR) + I_2(kR))(3Y_1(kR) - Y_3(kR))]) \\
&+ \mathrm{Y}_0(kR_i)(J_1(kR_i)[(I_0(kR) + I_2(kR))(K_1(kR) - K_3(kR)) - (-K_0(kr) - K_2(kr))(3I_1(kR) + I_3(kR))] \\
&+ I_1(kR_i)[(-J_0(kr) + J_2(kr))(K_1(kR) - K_3(kR)) - (-K_0(kr) - K_2(kr))(3J_1(kR) - J_3(kR))] \\
&+ K_1(kR_i)[(-J_0(kr) + J_2(kr))(3I_1(kR) + I_3(kR)) - (I_0(kR) + I_2(kR))(3J_1(kR) - J_3(kR))]) \\
&+ \mathrm{I}_0(kR_i)(-J_1(kR_i)[(-Y_0(kr) + Y_2(kr))(K_1(kR) - K_3(kR)) - (-K_0(kr) - K_2(kr))(3Y_1(kR) - Y_3(kR))] \\
&+ Y_1(kR_i)[(-J_0(kr) + J_2(kr))(K_1(kR) - K_3(kR)) - (-K_0(kr) - K_2(kr))(3J_1(kR) - J_3(kR))] \\
&- K_1(kR_i)[(-J_0(kr) + J_2(kr))(3Y_1(kR) - Y_3(kR)) - (-Y_0(kR) + Y_2(kR))(3J_1(kR) - J_3(kR))]) \\
&- \mathrm{K}_0(kR_i)(-J_1(kR_i)[(-Y_0(kr) + Y_2(kr))(3I_1(kR) + I_3(kR)) - (I_0(kr) + I_2(kr))(3Y_1(kR) - Y_3(kR))] \\
&+ Y_1(kR_i)[(-J_0(kr) + J_2(kr))(3I_1(kR) + I_3(kR)) - (I_0(kr) + I_2(kr))(3J_1(kR) - J_3(kR))] \\
&- I_1(kR_i)[(-J_0(kr) + J_2(kr))(3Y_1(kR) - Y_3(kR)) - (-Y_0(kR) + Y_2(kR))(3J_1(kR) - J_3(kR))]) = 0
\end{aligned}
$$

Resulting k values

k_n	Value
k_1	0.0835
k_2	0.3880
k_3	0.6515
k_4	0.9216
k_5	1.1901
k_6	1.4581

Table 3 - k values for the annulus case.

Next, we wanted to substitute the respective k values back into equation (17) for the whole disk case and equation (13) for the annulus case to find the zeroes of our spatial equation to predict where the nodes would form on the plate. We can solve for B, C, D in terms of A for both the whole disk and annulus case. To demonstrate this consider the whole disk case in equation (18). It is readily apparent that one can solve for C in terms of A as follows:

$$F''(R) = 0 = A\frac{k^2}{2}(-J_0(kR) + J_2(kR)) + C\frac{k^2}{2}(I_0(kR) + I_2(kR))$$

$$C(I_0(kR) + I_2(kR)) = -A(-J_0(kR) + J_2(kR))$$

$$C = \frac{-A(-J_0(kR) + J_2(kR))}{(I_0(kR) + I_2(kR))}$$

Then, we substituted this into equation (17) and set equation (17) equal to zero. Upon substitution, we factored an A out and divided it over to the side with zero leaving an equation with all known values except for r which we can then solve for numerically. This procedure was repeated for the annulus case in a similar fashion however, we omit the details here as the algebra for the annulus case is exceptionally long and leads to no new conclusions other than the roots. We will present the r values for each case in the Discussion section.

The next area of interest is which frequencies we should expect to produce these zero points on the plate. We can determine this from our G equation, which has a frequency term of $\frac{k^2}{\sqrt{\beta}}$. We can try each k value with this equation and thus find a frequency for each k value. We will present these frequency values in the Discussion section.

3 Discussion

3.1 Spatial Equation

Following from the end of the Analysis section, here are the predicted radii of the nodes from both models for each k.

The whole disk:

k_n	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
k_1	8.498	-	-	-	-	-
k_2	4.638	9.939	-	-	-	-
k_3	3.03	6.978	10.538	-	-	-
k_4	2.269	5.208	8.181	10.864	-	-
k_5	1.816	4.168	6.533	8.914	11.07	-
k_6	1.514	3.476	5.45	7.425	9.412	11.214

Table 4 - Predicted radii (cm) of nodes using the disk model.

The annulus:

k_n	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
k_1	-	-	-	-	-	-
k_2	2.695	8.056	-	-	-	-
k_3	-	7.242	9.786	-	-	-
k_4	0.867	3.974	8.617	10.367	-	-
k_5	-	3.874	5.647	9.378	10.705	-
k_6	0.575	2.648	5.397	6.76	9.857	10.924

Table 5 - Predicted radii (cm) of nodes using the annulus model.

Here is the experimental data:

k_n	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
k_1	7.5	-	-	-	-	-
k_2	3.25	9.5	-	-	-	-
k_3	1.5	6.75	10.75	-	-	-
k_4	0.75	4.75	8	11	-	-
k_5	0.25	3.75	6	8.75	11.25	-
k_6	0.1	3	5	7.5	9.5	11.5

Table 6 - Radii (cm) of nodes produced experimentally.

And we have the following percent errors for each case:
The whole disk:

k_n	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
k_1	13.31	-	-	-	-	-
k_2	42.707	4.621	-	-	-	-
k_3	102	3.378	-1.97	-	-	-
k_4	202.53	9.642	2.26	-1.24	-	-
k_5	626.4	11.15	8.88	1.87	-1.6	-
k_6	1414	15.87	9	-1	-0.92	-2.49

Table 7 - Percent error of each node for disk model.

The annulus:

k_n	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
k_1	-	-	-	-	-	-
k_2	-17.08	-15.2	-	-	-	-
k_3	-	7.29	-8.97	-	-	-
k_4	15.6	-16.34	7.71	-5.75	-	-
k_5	-	3.304	-5.88	7.177	-4.84	-
k_6	475	-11.73	7.94	-9.86	3.76	-5.01

Table 8 - Percent error of each node for annulus model.

Both models are decent predictors of the location of the nodes with some caveats. Both nodes struggle for nodes closer to center of the plate, but particularly the whole disk model has a lot of problems here. As both models go farther out their estimates are fairly close to the real ones, giving acceptable to good results. This is really promising, while there are clear flaws in the model, we do see a lot of predictive ability and suggests that this model is well suited for the system. Some edge cases, particularly when r is close to zero, may need to be worked on. But the overall model is promising. One piece of odd behavior seen is that the model for the annulus does not produce zeros for some odd k's first nodes. We don't really know why this occurs and with how generally good the data is besides that it seems improbable to be a calculation or algebra error, but we do not exclude this possibility due to the length of the algebra.

3.2 Frequency Analysis

From our G equation we expect our frequency values to be equal to $\frac{k^2}{\sqrt{\beta}}$. For our k values we expect these frequencies (with proper unit conversion) which we compare to the experimental frequencies:

The whole disk case:

k_n	Expected (Hz)	Experimental (Hz)	Percent Error
k_1	78.00	105	-25.71
k_2	411.69	359	14.48
k_3	948.05	877	8.10
k_4	1691.16	1683	0.4848
k_5	2641.44	2773	-4.74
k_6	3796.97	4097	-7.32

Table 9 - Expected node producing freq. vs. experimental freq. for disk case.

The annulus case:

k_n	Expected (Hz)	Experimental (Hz)	Percent Error
k_1	10.50	105	-90
k_2	226.67	359	-36.86
k_3	639.09	877	-27.13
k_4	1278.85	1683	-24.01
k_5	2132.57	2773	-23.10
k_6	3201.19	4097	-21.87

Table 10 - Expected node producing freq. vs. experimental freq. for annulus case.

Some observations regarding these results. For smaller k's and thus smaller frequencies larger percent errors are observed. The whole disk case produces far more accurate frequency estimates than the annulus, with the annulus consistently under predicting the true frequency for every single k value and no percent error in magnitude less than 20 percent. Compared to the whole disk case, where only one estimate has a percent error higher than 20 percent. To highlight this, the annulus k_1 estimate has a percent error of 90 percent which is decidedly not close to a reasonable estimate. Reasons why the annulus is markedly worse than the whole disk are not obvious as we expect the annulus to produce better data as the boundary condition assumptions for the annulus are more accurate to our system than for the whole disk case. A potential reason for this would be we treat the annulus as separate from the wave driver its attached to in our boundary conditions, but it is not without merit to suggest that the additional mass of the wave driver in the hole in the center of the annulus could have an effect on this model. This mass is missing in the annulus case, however in the whole disk case we approximate this mass with simply more mass from the Chladni plate where the wave driver would be. This may lead to more accurate predictions.

Next we will change our β value such that the expected frequency matches the experimental frequency for some k's and then determine the expected frequencies with this new β. We will avoid k's with small percent errors as our expected frequencies will change little and not much will be gained. Thus we will choose k_1 as it has most percent error for both the whole disk case and the annulus. Our results are as follows:

k	New β
Disk:k_1	$0.00256\frac{s^2}{m^3}$
Annulus:k_1	$0.0000441\frac{s^2}{m^3}$

Table 11 - new β values for disk and annulus based on k_1.

The whole disk k_1 case:

k_n	Expected (Hz)	Experimental (Hz)	Percent Error
k_1	105	105	0
k_2	540.41	359	50.53
k_3	1244.45	877	41.90
k_4	2219.89	1683	31.90
k_5	3467.27	2773	25.04
k_6	4984.01	4097	21.65

Table 12 - Expected node producing freq. vs. experimental freq. for disk based on new β.

The annulus k_1 case:

k_n	Expected (Hz)	Experimental (Hz)	Percent Error
k_1	105	105	0
k_2	2267.23	359	531.
k_3	6392.35	877	628.89
k_4	12791.36	1683	660.01
k_5	21330.39	2773	669.22
k_6	32018.91	4097	681.52

Table 13 - Expected node producing freq. vs. experimental freq. for annulus based on new β.

For the whole disk case we see that percent errors rose across the board in magnitude except for the lowest frequency, but the percent error appears to get lower as the frequency increases. This matches what we would expect using our standard β, as our percent errors generally decreased and followed the same pattern. For the annulus case, every single percent error is absurdly high. To a degree of losing any even remote semblance of credibility of predictive power for this new β. Our percent errors do increase, as they do using our standard β. However it is obvious using k_1 in this model to fix β is unacceptable due to how far off k_1 is from the experimental expectation. This suggests we should try k's that have lower percent errors for their expected frequencies, but k's with too low error will just return a β nearly identical to our standard β and thus may seem trivial. Further study seems warranted. We will focus on the whole disk case as that case much better represents the data and we will examine k_5 as a sort of middle ground of percent errors in the data set.

k	New β
Disk:k_5	$0.0040023\frac{s^2}{m^3}$

Table 14 - new β value for disk based on k_5.

The whole disk k_5 case :

k_n	Expected (Hz)	Experimental (Hz)	Percent Error
k_1	83.99	105	-20.01
k_2	432.22	359	20.40
k_3	995.32	877	13.49
k_4	1775.49	1683	5.496
k_5	2773	2773	0
k_6	3986.32	4097	-2.701

Table 15 - Expected node producing freq. vs. experimental freq. for disk based on new β from k_5.

This model has a very similar spread of percent errors in magnitude as our original with the standard β and simply shifted the percent errors to be relative around k_5. From this it appears that redefining β is only good for k's with small error values. For those that is true, it simply shifts the relative estimates of the frequencies from the standard β. For k's with small percent errors this is not really a positive or a negative, just a choice.

4 Conclusion

4.1 Spatial Equation

The two models for the spatial equation both had their pros and cons. The disk seemed to be poorly suited for estimating the inner-most nodes, but it did produce an estimate for all nodes at all modes of vibration. The annulus model had notably smaller error for the innermost nodes, *but* appears to only estimate the first node for even modes of vibration (k_2, k_4, k_6), but odd modes (k_1, k_3, k_5) were always missing only the first node. Interestingly enough, the annulus' predictions for the odd modes seemed to be more accurate on average for the nodes it did predict, but the fact that it completely missed the first node was a major drawback. The solutions reminded us of how the wave equation's fundamental harmonic, second harmonic, third harmonic, etc. correlated to $n = 1, 2, 3$ respectively, and how the solutions to what we might call the harmonics for the Chladni plate correlate to k_1, k_2, k_3, etc. The only difference being k was not an integer but rather a real number that ensured a nontrivial solution for the spatial equation.

As for which model might be better, it depends on what you might want to do. The k values from the disk model seemed to more accurately predict the expected frequencies that would produce nodes, but the annulus model seemed to more accurately predict the radius of said nodes. The annulus model also becomes more useful for k_{odd} if we notice the trend that the radius of the first node seems to shrink and shrink to $r = R_i$ as the mode of vibration increases. This would lead us to believe that one could manually estimate the first node's radius as $< .1$ for odd vibrational modes above 6, and use the produced prediction for other nodes.

4.2 Frequency

It's quite odd that the disk equation was actually a much better model for the expected frequencies than the annulus method, but it might lead to more evidence that there may have been an error when calculating the k values. When redefining β we did get a slightly better estimate for the expected frequencies (provided we changed β for the correct k), so the model could be influenced by the mass or some other physical property of the plate which we weren't taking into account.

4.3 Future Work

There is some work that could be done in the future to improve these models. Solving for the large determinants by hand was extremely tedious and led to the increased potential of human error. While we did find k values that lead to close solutions, perhaps due to algebraic errors there are better k values that would lead to better models. Potentially there is another way to solve for the k values numerically by comparing experimental data with the $F(r)$ plotted along a reasonable set of k values and finding the best matches for each mode of vibration. This isn't unrealistic especially since a range of the k values that work for each mode is known after our analysis (i.e. $.1 < k_1 < .4$ for the disk just estimating from our k's that produced nodes).

There is also another way to solve for a model using variation of parameters that models the wave equation with known forcing, but it makes many assumptions and uses functions and solvability values we didn't have access to. Future research on this solution method could lead to more accurate models in the future.

5 Appendix

Editor - /Users/woahgan/Documents/MATLAB/homework1/disk.m

```matlab
f=@(x) (besselj(2,x) - besselj(0,x)).*(3*besseli(1,x)-besseli(3,x))-(3*besselj(1,x)-besselj(3,x)).*(besseli(0,x)+besseli(2,x)); %expanded determinant

xx= 0:.1:3; %domain of k values(k*R, technically) to search for root.

t = f(xx) > 0;
i0 = find(diff(t(:))~=0);
i0 = [i0(:)';i0(:)'+1];
n = size(i0,2);
xout = zeros(n,1);
for jj = 1:n
    xout(jj) = fzero(f,xx(i0(:,jj))); %stores all k values (again, k*R technically) on domain in xout. only need first 6.
end
```

Matlab function to find k values for disk.

Editor - /Users/woahgan/Documents/MATLAB/homework1/diskSol.m

```matlab
k = 1.588; %manually changing k values
R = 12;
f = @(r) besselj(0,k*r)-((besselj(2,k*R)-besselj(0,k*R))./(besseli(0,k*R)+besseli(2,k*R))).*besseli(0,k*r); %F in terms of A for disk.
range = 0:.1:12;
plot(range,f(range)); %plotting to estimate zeroes, using fzero in command window to find precise values.
```

Matlab function to find nodes for disk.

Editor - /Users/woahgan/Documents/MATLAB/homework1/annulus.m

```matlab
xi = .21; %inner radius
x = 12;     %outer radius
f = @(k) -besselj(0,k*xi).*(bessely(1,k*xi).*((besseli(0,k*x)+besseli(2,k*x)).*(besselk(1,k*x)-besselk(3,k*x))+(besselk(0,k*x)+besselk(2,k*x)).*(3*besseli(1,k*x)+besseli(3,k*x)))...
    +besseli(1,k*xi).*((bessely(2,k*x)-bessely(0,k*x)).*(besselk(1,k*x)-besselk(3,k*x))+(besselk(0,k*x)+besselk(2,k*x)).*(3*bessely(1,k*x)-bessely(3,k*x)))+besselk(1,k*xi).*...
    ((bessely(2,k*x)-bessely(0,k*x)).*(3*besseli(1,k*x)+besseli(3,k*x))-(besseli(0,k*x)+besseli(2,k*x)).*(3*bessely(1,k*x)-bessely(3,k*x)))) + bessely(0,k*xi).*(besselj(1,k*xi).*((besseli(0,k*x)...
    +besseli(2,k*x)).*(besselk(1,k*x)-besselk(3,k*x))+(besselk(0,k*x)+besselk(2,k*x)).*(3*besseli(1,k*x)+besseli(3,k*x)))+besseli(1,k*xi).*((besselj(2,k*x)-besselj(0,k*x)).*(besselk(1,k*x)-besselk(3,k*x))...
    +(besselk(0,k*x)+besselk(2,k*x)).*(3*besselj(1,k*x)-besselj(3,k*x)))+besselk(1,k*xi).*((besselj(2,k*x)-besselj(0,k*x)).*(3*besseli(1,k*x)+besseli(3,k*x))-(besseli(0,k*x)+besseli(2,k*x)).*...
    (3*besselj(1,k*x)-besselj(3,k*x)))) + besseli(0,k*xi).*(-besselj(1,k*xi).*((bessely(2,k*x)-bessely(0,k*x)).*(besselk(1,k*x)-besselk(3,k*x))+(besselk(0,k*x)+besselk(2,k*x)).*(3*bessely(1,k*x)-...
    bessely(3,k*x)))+bessely(1,k*xi).*((besselj(2,k*x)-besselj(0,k*x).*(besselk(1,k*x)-besselk(3,k*x))+(besselk(0,k*x)+besselk(2,k*x)).*(3*besselj(1,k*x)-besselj(3,k*x)))-besselk(1,k*xi).*...
    ((besselj(2,k*x)-besselj(0,k*x)).*(3*bessely(1,k*x)-bessely(3,k*x))-(bessely(2,k*x)-bessely(0,k*x)).*(3*besselj(1,k*x)-besselj(3,k*x)))) - besselk(0,k*xi).*(-besselj(1,k*xi).*((bessely(2,k*x)-...
    bessely(0,k*x)).*(3*besseli(1,k*x)+besseli(3,k*x))-(besseli(0,k*x)+besseli(2,k*x)).*(3*bessely(1,k*x)-bessely(3,k*x)))+bessely(1,k*xi).*((besselj(2,k*x)-besselj(0,k*x)).*...
    (3*besseli(1,k*x)+besseli(3,k*x))-(besseli(0,k*x)+besseli(2,k*x)).*(3*besselj(1,k*x)-besselj(3,k*x)))-besseli(1,k*xi).*((besselj(2,k*x)-besselj(0,k*x)).*(3*bessely(1,k*x)-bessely(3,k*x))...
    -(bessely(2,k*x)-bessely(0,k*x)).*(3*besselj(1,k*x)-besselj(3,k*x))))); %determinant for annulus problem expanded manually.

xx= .001:.1:2; % range of k values

t = f(xx) > 0;
i0 = find(diff(t(:))~=0);
i0 = [i0(:)';i0(:)'+1];
n = size(i0,2);
xout = zeros(n,1);
for jj = 1:n
    xout(jj) = fzero(f,xx(i0(:,jj))); %stores all k values that are a root on the domain, (.001, 2) in this case. Only looking for first 6.
end
```

Matlab function to find k values for annulus.

Editor - /Users/woahgan/Documents/MATLAB/homework1/annulusSol.m

```matlab
k = 1.1901; %manually changing k
Ri = .21;
R = 12;

i = (besseli(1,k*Ri).*besselk(0,k*Ri)+besseli(0,k*Ri).*besselk(1,k*Ri))./besselk(0,k*Ri);
j = (besselj(1,k*Ri).*besselk(0,k*Ri)-besselj(0,k*Ri).*besselk(1,k*Ri))./besselk(0,k*Ri);
y = (bessely(1,k*Ri).*besselk(0,k*Ri)-bessely(0,k*Ri).*besselk(1,k*Ri))./besselk(0,k*Ri); %i,j,y,b all used for simplification when writing F in terms of A. just reallly long constants.
b = ((besselj(2,k*R)-besselj(0,k*R))+j/i*(besseli(0,k*R)+besseli(2,k*R))+besselj(0,k*Ri).*(-besselk(0,k*R)-besselk(2,k*R))./besselk(0,k*Ri)...
    + j/i*besseli(0,k*Ri).*(-besselk(0,k*R)-besselk(2,k*R))./besselk(0,k*Ri)) ./((bessely(2,k*R)-bessely(0,k*R))+y/i*(besseli(0,k*R)+besseli(2,k*R))...
    +bessely(0,k*Ri).*(-besselk(0,k*R)-besselk(2,k*R))./besselk(0,k*Ri) + y/i*besseli(0,k*Ri).*(-besselk(0,k*R)-besselk(2,k*R))./besselk(0,k*Ri));

F = @(r) besselj(0,k*r) - b*bessely(0,k*r) + (j/i - b*y/i)*besseli(0,k*R) - (besselj(0,k*Ri)./besselk(0,k*Ri)-b*bessely(0,k*Ri)./besselk(0,k*Ri)...
    +(j/i - b*y/i)*besseli(0,k*Ri)./besselk(0,k*Ri))*besselk(0,k*r); %F in terms of A for annulus.
range = 0:.0001:12;
plot(range,F(range)); %plotting F along 0<r<12 to estimate roots and use fzero in command window to solve for roots, or estimated nodes.
```

Matlab function to find nodes for disk

References

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