A Quantitative Analysis of One-Dimensional *N*-slit Fraunhofer Diffraction

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***Abstract*—Fraunhofer diffraction is a type of diffraction that occurs at small apertures on a diffracting object which produces a diffraction pattern [1]. This pattern is visible on a focal plane when focused by a convex lens. When modelling the pattern mathematically, we can see that the pattern produced depends highly on the number of apertures on the diffracting object - in one dimension these apertures are slits. We can measure changes in the diffraction pattern from an image of the pattern by recording the focal plane and reducing the high frequency noise of the image by means of Fourier analysis. In my experiment, I investigated how the peak widths Δ*p* of the pattern change with the number of slits *N* and found that the relationship is a reciprocal relationship accurate to 2.7%. The errors primarily due to impurities in the camera and the natural diffraction effect of a lens. The widely used application of such a relationship can be used to understand diffraction gratings and how the number of slits *N* is directly linked to the resolution of a spectrometer.**

I. Introduction

THE theory of diffraction roots from the Fresnel-Huygens principle that states that a propagating wave has spherical wavelets that are produced at each point on the wave front [2]. These wavelets interfere with each other through superposition to produce a new wave front. When passing through an aperture, the wave is seen to have a dispersion effect – this effect can be measured as a variation of the intensity of the wave on a screen behind the aperture. Combining multiple apertures increases the amount of Fraunhofer diffraction which produces a different diffraction pattern. This again is a result of the interference of increasing number of diffracted waves.

II. Theoretical Background

[3] The mathematical interpretation of the Fresnel-Huygens principle is an integral that sums the complex amplitudes of each wavelet over the aperture $\mathcal{A}$ in the plane of wave propagation (the $\xi\eta$-plane). Our interest is the diffraction pattern in one-dimension meaning our apertures are slits of infinite height, so we are only interested in the ξ-direction. The complex amplitude ψ at a very distant point P, is defined as:

$$\psi(P) = \int_{\mathcal{A}} G(\chi) e^{if(\xi)} d\xi \tag{1}$$

Where the function $G(\chi)$ contains the directional parameters of the incoming light through the aperture – different directions of incoming light will produce diffracted rays with different optical paths to the point P and hence will interfere differently. However, we can set $G(\chi)$ to be a constant, C, by putting a well-correlated lens behind the aperture. We can do this as the optical path to a point in the focal plane P' will be the same for rays diffracted in the same direction and hence their interference will be the same – this is illustrated in figure. (1). We can now define the function $f(\xi)$ to be a function of p where p is a distance on the focal plane in the ξ-direction. The integral for Fraunhofer diffraction becomes:

$$\psi(p) = C \int_{\mathcal{A}} e^{-ikp\xi} d\xi \tag{2}$$

Where k is a constant inversely related to the wavelength by $k = 2\pi/\lambda$ where λ denotes the wavelength of incoming light.

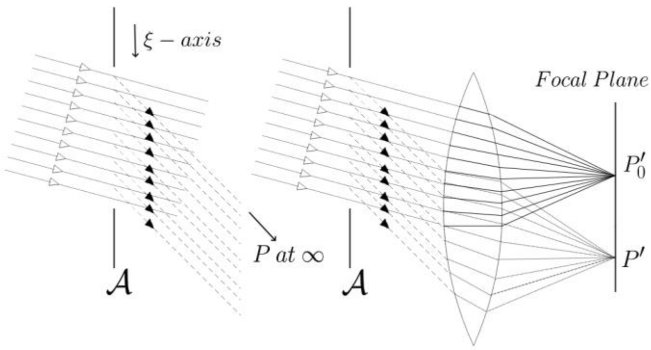

Figure 1. Shows how diffracted rays have the same optical path to a point on the focal plane, whereas without the lens, the rays meet at ∞.

We can calculate the amplitude $\psi^{(0)}(p)$ for a single slit by integrating $\mathcal{A}$ for a slit width s in the interval $[-\frac{1}{2}s, \frac{1}{2}s]$. When integrating the exponential, the result is a *sinc* function.

$$\psi^{(0)}(p) = C \int_{-s/2}^{s/2} e^{-ikp\xi} d\xi = C\,\mathrm{sinc}\left(\frac{ksp}{2}\right) \tag{3}$$

We now consider the multi-slit case for N apertures $\mathcal{A}_n$ with coordinates $(\xi_1, \dots, \xi_N)$. We start by taking a sum over n of the contribution of each wave front from an aperture at $\xi_n + \xi'$, integrating with respect to ξ'.

$$\psi_n(p) = C \sum_n \int_{\mathcal{A}_n} e^{-ikp(\xi_n+\xi')} d\xi' \tag{4}$$

We can split the exponential and take the ξ_n term out of the integral as it does not have dependence on ξ'.

$$\psi_n(p) = C\sum_n e^{-ikp\xi_n}\int_{\mathcal{A}} e^{-ikp\xi'}d\xi' \quad (5)$$

The integral also has no dependence on n and is in fact the integral we evaluated for a single slit, $\psi^{(0)}$, in (3). We can also solve the sum in the range where $n = (0,1,\dots,N-1)$ such that the coordinate $\xi_n = nd$ where d is the distance between the slits.

$$\psi_n(p) = \psi^{(0)}\sum_{n=0}^{N-1} e^{-ikndp} = \psi^{(0)}\frac{1-e^{-ikndpN}}{1-e^{-ikdp}} \quad (6)$$

This result gives us a complex amplitude at a point *P'* in the focal plane at a distance p. the intensity can be calculated using the identity $I = \psi^*\psi$. Applying this identity to (6) and substituting our result for $\psi^{(0)}(p)$ from (3) yields the intensity at *P*'.

$$I(p) = \frac{1-\cos(kdpN)}{1-\cos kpd}\,|\psi^{(0)}(p)|^2 \quad (7)$$

$$I(p) = \left(\frac{\sin(kdpN/2)}{\sin(kdp/2)}\right)^2\left(\frac{\sin(ksp/2)}{ksp/2}\right)^2 \quad (8)$$

This equation for intensity shows how the interference of diffracted waves from *N*-slits gives an oscillating function with two frequencies, that is modulated by the $1/p^2$ term – meaning intensity goes to 0 as $p \to \infty$. $I(p)$ is plotted in figure. (2) for various *N*.

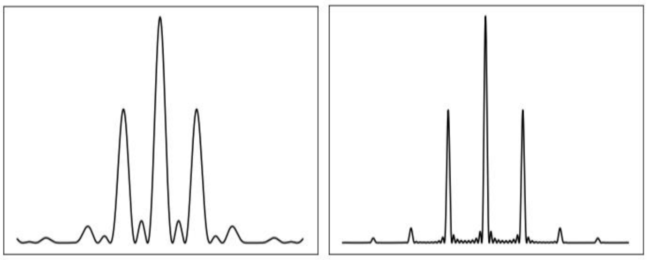

Figure 2. A theoretical plot of intensity against distance at a screen on the focal plane for (a) *N*=3 and (b) *N*=10 slits

Figure. (2) shows a plot of $I(p)$ using the function that was derived in (10). The main observation in increasing the value of *N* is that the width of each peak decreases. We can see from our expression for $I(p)$ that the distance Δp between minima for *N* slits can be given by [4]:

$$\Delta p = \frac{2\pi}{Nkd} = \frac{\lambda}{Nd} \quad (11)$$

III. Methodology

To test whether the intensity pattern does follow our derived $I(p)$ by measuring values of Δp for various number of slits and testing the relationship. For my experiment, 6 slits were used from *N=1* to *N=6* and was set up as follows. A 670 ± 1 nm [5] laser provides are light source (parallel rays) which goes onto our diffracting object that contains the slits. Behind the diffracting object (~1 cm after), a biconvex lens was placed to conform to our mathematical derivation in (2) and produce a measurable diffraction pattern in the focal plane. The focal length of the lens was *f=500* mm as it produced a diffraction pattern where peaks are wide enough to be measurable whilst giving many visible peaks. A longer focal length lens is ideal for such an experiment as smaller lenses have a greater diffraction effect of their own which I wanted to minimise. A larger *f=1000* mm lens produced an image too wide. Finally, a CMOS camera was placed in the focal plane at a distance of 50.0 ± 0.1 cm from the centre of the lens to observe the pattern. A CMOS camera is preferred over a CCD camera as CMOS cameras have a higher dynamic range [6] meaning it can see a greater contrast between bright and dark light, so peaks have a higher resolution. The live image is then sent into a computer for analysis. The experimental set up is shown in figure. (3).

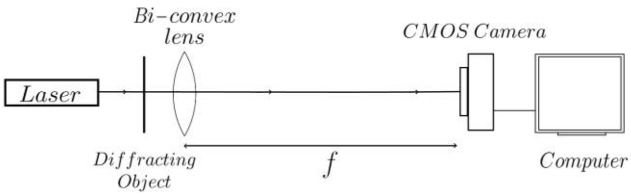

Figure 3. The experimental set-up used to produce a live image of the diffraction pattern on the focal plane.

For each slit, the camera was displaced approximately 5 cm to one side of the central maxima and moved in the focal plane to the other side of the diffraction pattern. The image sent to the computer was recorded using the *ThorCam* software which produces a 120-150 frame video of the pattern. The video had to be turned into a single image of the complete diffraction pattern in order to be analysed. To do this the video was decomposed into individual frames using *ImageJ*. Each frame then had be stitched together using *PanoramaStudio* which created a wide image stitch of the entire video, however due to the orientation of the camera, the diffraction pattern wasn't always perfectly horizontal, hence it had to the rotated such that the fringes were exactly vertical. A straight line was fitted down the fringes and the angle the line made to the vertical was the angle the image was rotated by. The final image was about 3000 pixels in width covering the entire visible pattern; in general, the larger the value of *N*, the wider the image of the diffraction pattern. Some examples of the diffraction patterns captured are shown in figure. (4).

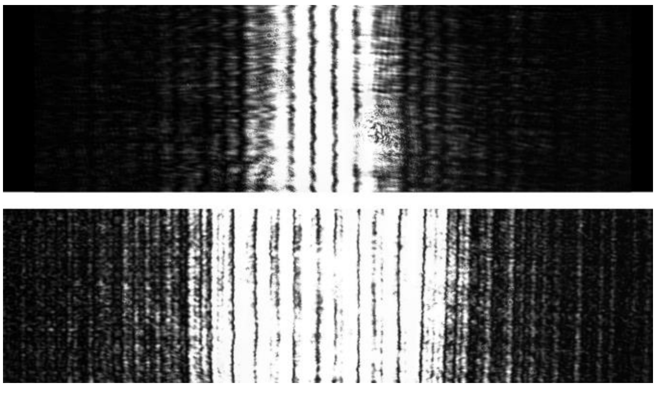

Figure 3. The diffraction patterns observed using the CMOS camera for (a) *N=2* (top) and (b) *N=6* (bottom).

Each image could then be analysed using python. Although the image was an image of intensity, each pixel had a red, green, blue (RGB) value that corresponds to the colour of the pixel. To covert to a greyscale image, each pixel must be multiplied by the vector (0.6,0.3,0.1) [5], the vector being the relative intensity of each RGB value. Once this is done, each pixel possesses an intensity value in the range 0 to 255. The image as seen in figure. 3 is 2 dimensional, whereas our analysis is only interested in the variation in the horizontal change in intensity. To do so, each column of pixels was averaged into a single intensity value. However, the error in doing this operation varied from 0.4% to 9.9%. The larger errors are due to the poor resolution of the CMOS camera as well as the idea that bright fringes will not be perfectly parallel resulting in some darker pixels included in the fringes. To avoid these affects, all horizontal rows outside of 1.5σ were ignored as this resulted in the smallest error of 1.5%. The average intensities were plotted against the distance in pixels; the plot is shown in figure. 4

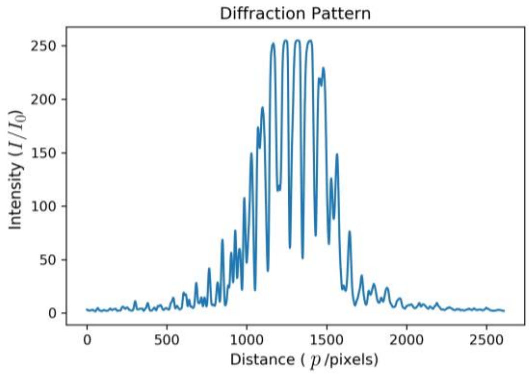

Figure 4. The average intensity of each column of pixels I/I_0 against the distance in pixels for the N=6 case.

We can see from the plot above that there is a large amount of noise from the image which must be removed in order to extract the diffraction pattern. This means the high frequency data must be discarded from the plot as it is primarily due to limits in the camera's ability to optimise the image at a certain frequency combined with impurities in the camera lens. We do this by doing a Fourier transform of the data and removing all frequencies greater than a value F, followed by an inverse Fourier transform to produce a clean pattern. We know from (8), that intensity varies with a high a low frequency, hence a frequency must be found that minimises the noise whilst still containing the two frequencies of the diffraction pattern. By looping through intensities 0-100 in python, the value of F that provided the high frequency was found to be 39 pixels^{-1}. By doing this process on each graph, the peaks become clearer. Additionally, we are only interested in peaks of higher intensity as they are of greater resolution and yield better data, hence the pattern is cropped for high intensity maxima. Both graphs are shown in figure. (5).

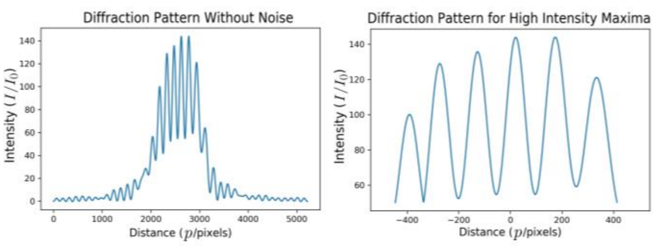

Figure 4. Intensity pattern after removing high frequency noise (left) and cropped for high intensity maxima (right) for $N=6$.

IV. Results and Discussion

Using the clean data acquired, the peak widths (minima to minima distances) Δp can be calculated from the respective graph of each N-slit. This done in python by finding the differences in the local minima and taking an average. Some outliers needed to be removed in the process as inflection points would also arise from the data due to the cleaning of the data. In order to test the relationship between intensity I and number of slits N, the values of Δp were plotted against $1/N$ as we see from (11), this would theoretically produce a straight line. The graph is shown in figure. (6).

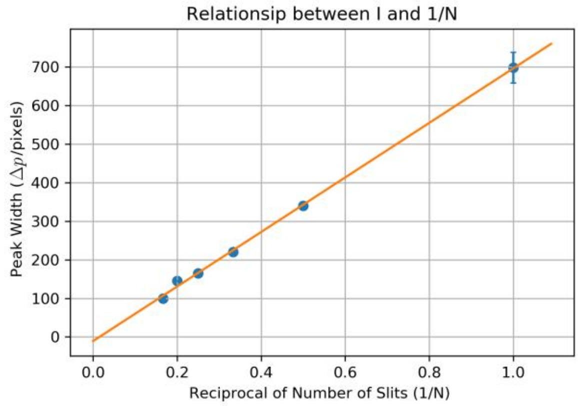

Figure 6. The plot of peak width against the reciprocal of the number of slits N

As shown in figure. (6) there is indeed a straight-line relationship between Δp and $1/N$ with a fit error of 2.7%. Furthermore, the straight line intercepts the y-axis at -6 ± 10 pixels which is in range of the origin and is what was expected from the relationship. It is worth noticing that the value of Δp is higher than expected for the $N=5$ case (i.e. $1/N=0.2$) that is possibly a result of the slits not being of equal width due to poor manufacturing. If slit widths were not equal, the width of peaks would vary as the contribution of interference from some slits would be greater than others.

The error in the fit is largely propagated by the single slit peak (where the slit width Δp=700 ± 30 pixels, an error of 2.8%) is approximately twice the width of the $N=2$ case. The error is much greater than the multi-slit cases as there are multiple high intensity peaks and more data to analyse for $N>1$ – for example, the value of Δp for N=3 was 223 ± 1 pixel a significant decrease in error of just 0.4%. The primary sources of error in the experiment for the single slit case and that was suppressed for the multi-slit cases could possibly be the small amount of diffraction caused by the lens, this makes sense as it would cause a variation in the peak width proportional to the peak width itself as light travelling in a particular direction would focus to a small range rather than to a single point on the focal plane. This is seen to be true for $N=1$ as it has the largest peak width. Finally, impurities in the camera lens would slightly distort the image and cause very noticeable systematic errors in the image. For example, figure 3(a) shows the image for the $N=2$ case where there is a noticeable distortion in the lower centre. Although such noise is reduced by the Fourier process, it is likely these large impurities propagate as lower frequency data.

V. Conclusion

There is clear evidence to a low degree of error, that the increase of number of slits on a diffracting object causes the peak widths of the diffraction pattern on the focal plane to decreases by a reciprocal relationship. Despite the initial image looking very noisy, very accurate values of Δp were extracted by reducing noise via a Fourier band pass of the data and removing noticeable outliers. The primary use of such a relationship has important implications in terms of spectrometry and diffraction gratings. A higher resolution grating will have very thin bright fringes that is produced by a very high number of slits N. Hence the more slits, the better the resolution of the spectrometer.

VI. References

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