Investigating the Period Dependance on the Amplitude of the Simple Pendulum

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***Abstract*—The period *T* and angular amplitude θ_0 can be obtained by recording a video of an oscillating pendulum mass and extracting the displacement of the centre of mass with time whilst plotting a damped harmonic function. The displacement was measured at each frame of the video by converting the position of the mass in the image to binary and taking an average of the position of each pixel. The dependence of the period on amplitude can be described by a Legendre polynomial of order *n* that is a polynomial of even orders of θ_0. By plotting the Legendre function for the known values of θ_0 with their respective *T* the relationship can be tested for different values of *n*. The closer the gradient is to 1, the better the relationship. For the range of amplitudes, A, used, (0<A<16) cm, the Legendre approximation of the period was found to be most accurate at *n*=2 with an error (to the true value of *T)* of 4.2 ± 0.1 %. The error of the approximation was also found to be 5.2 ± 0.1% for a quadratic and 4.4 ± 0.1% for the Legendre orders greater than or equal to *n*=3.**

I. INTRODUCTION

THE simple pendulum arises from the most elementary principle in physics – the conservation of energy. A pendulum constantly converts kinetic energy to potential energy and vice versa producing oscillations (harmonic motion). These oscillations have typical properties such as a period (time of one complete oscillation) and amplitude (the greatest displacement from the equilibrium position). The relationship between these two quantities can be approximated to be proportional in a range where $sin\theta \approx \theta$. However more complex models and approximations can be formed outside this range and can be tested for larger amplitudes.

II. THEORETICAL BACKGROUND

A. *Displacement as a Function of Time*

Systems such as the simple pendulum are described by the differential equation for simple harmonic motion. However, we must consider the damping factor as the pendulum is very lightly damped by air by a force $F_d = -bv$, the negative showing the force acting opposite to motion. The second-order differential equation that describes damped harmonic motion for a mass m is [1]:

$$\frac{d^2\psi}{dt^2} + \gamma\frac{d\psi}{dt} + \sqrt{\frac{g}{l}}\psi = 0 \tag{1}$$

Where $\gamma = b/m$ and is the damping parameter of ψ. l is the natural length of the pendulum which we will assume to be constant (non-extensible); and g is the acceleration due to gravity. The solution to the differential equation yields the function:

$$\psi(t) = Ae^{-\gamma t/2}\cos(\omega t + \phi) \tag{2}$$

$\psi(t)$ describes the displacement of the centre of mass at any time t where A is the amplitude of oscillations and ω is the angular frequency of the oscillator which is also equal to $2\pi/T$ – this will be used to calculate the period, T, for known parameters of $\psi(t)$.

B. *Relationship between Amplitude and Period*

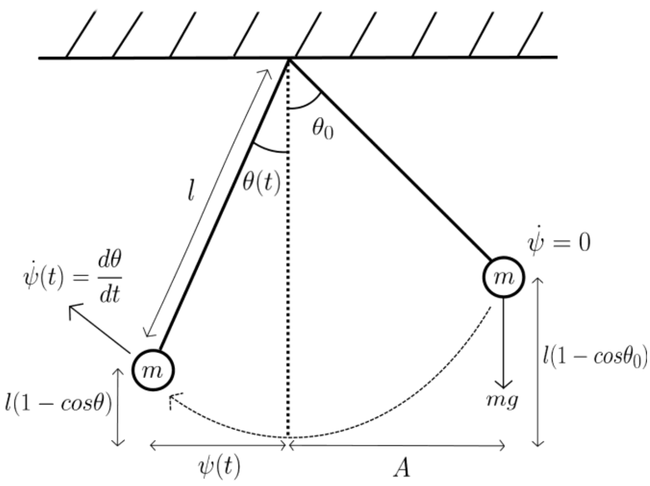

Figure 1. Shows the motion of a pendulum from a maximum displacement where $\theta = \theta_0$, to some arbitrary angle $\theta(t)$, as well as the mass's velocity $\dot{\psi}$.

Another way of mathematically describing a simple pendulum is in terms of energies and using the angle from the equilibrium position instead of a cartesian displacement. [2] Using conservation of momentum, we can state that the energy at a maximum angle (maximum displacement), θ_0, is equal to the energy at some arbitrary angle $\theta(t)$ – this is shown in figure. (1).

$$mgl(1-\cos\theta_0) = \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta) \tag{3}$$

Which we can rearrange in terms of $dt/d\theta$ as

$$\frac{dt}{d\theta} = \sqrt{\frac{l}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}} \tag{4}$$

We can solve this simple differential equation using the boundary conditions of $\theta_{max} = \theta_0$ and $\dot{\theta}_0 = 0$ by integrating in the interval $[0, \theta_0]$ four times to find the period T.

$$T(\theta) = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}d\theta \qquad (5)$$

We can simplify our integral by using the $cos\theta = 1 - sin\,(\theta/2)$, and using the substitutions $sin\varphi = sin\,(\theta/2)/sin\,(\theta_0/2)$ as well as $\kappa = sin\,(\theta_0/2)$. This results in our integral to be rearranged to:

$$T(\varphi) = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1 - \kappa^2\sin\varphi}}d\varphi \qquad (6)$$

The resulting integral is an incomplete elliptic integral of the first kind. The solution to an elliptic integral is a Legendre polynomial [3].

$$T(\theta_0) = 2\pi\sqrt{\frac{l}{g}}\cdot\sum_{n=0}^{\infty}\left(\left(\frac{2n!}{(2^n\cdot n!)^2}\right)^2\sin^{2n}\left(\frac{\theta_0}{2}\right)\right) \qquad (7)$$

This relationship shows a polynomial of even powers of θ and although it may seem rather complex, when $\theta_0 < 1/4$ we can see the summation reduces to 1. In other words, for a small amplitude, we can approximate the polynomial to *n*=0, we will call this period T_0.

$$T(\theta_0 < \frac{1}{4}) \approx T_0 = 2\pi\sqrt{\frac{l}{g}} \qquad (8)$$

However, we can test the validity of the polynomial by testing relationships of quadratic or quartic order by comparing the coefficients. For larger values we can expand the maclurin series for $sin(\theta/2)$ to produce a polynomial:

$$T(\theta_0) = T_0\cdot K_n(\theta_0) \qquad (9)$$

$$T(\theta_0) = T_0\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \dots\right) \qquad (10)$$

Where,

$$K_n(\theta_0) = \sum_{n=0}^{\infty}\left(\left(\frac{2n!}{(2^n\cdot n!)^2}\right)^2\sin^{2n}\left(\frac{\theta_0}{2}\right)\right) \qquad (11)$$

III. METHODOLOGY

The pendulum is set up by hanging a mass of approximately 50g by a string – this mass allows the damping factor to remain small (as *γ=b/m*) while still being small enough to not exert a large tension on the string as that would discount our assumption that the string is inextensible. The largest possible string length was also used to ensure that the period is large enough to get approximately five oscillations in a 5-second video – the calculated value of the length will be described in at a later stage.

A camera was clamped down onto a stand such that it looked down upon the mass onto a black mat, this allows the camera to see the pendulum mass as significantly brighter than the background such that python can distinguish the two. The relative position of the camera does not matter.

A 5-second-long video of the pendulum oscillating would be taken after being displaced at some arbitrary amplitude. The amplitude chosen for each video depended on the previous values such that a range of amplitudes was chosen from almost 0 to the maximum width of the camera frame. 25 videos were taken ranging from 1cm to 15cm of amplitude were then processed by python.

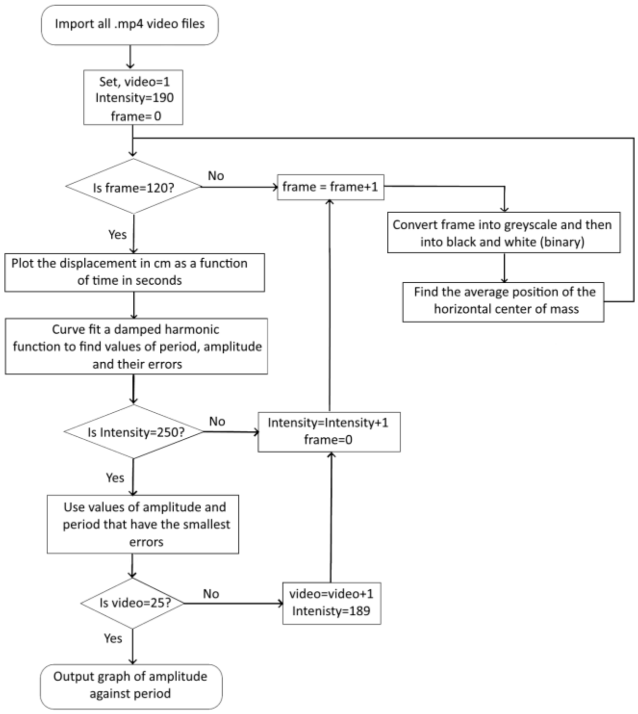

Figure 2. The algorithm used to analyze the video footage and minimize errors to produce a graph of period against amplitude.

As shown in fig. (2), the data analysis occurs in 3 loops. The first loop goes through a single video and selects each frame one at a time. Each frame comprises a 3-dimensional array containing the red-green-blue (RGB) values of each pixel that give the frame it's colour. The array size is (1280,720,3) in accordance with the resolution of the camera and RGB values. The frame is first converted to greyscale by taking the dot product of the RGB vector with the vector (0.3,0.6,0.1) [4] – the relative intensities of light for a red, green or blue pixel. The result is a greyscale image showing the intensity of light for each pixel (or a 2-dimensional array) shown in fig. (3a). The greyscale frame is then converted into binary (black or white) to find the position of each pixel in the frame where the mass is located. The intensity of a greyscale image has values in the range 0 to 255 (0 being a completely black pixel and 255 a completely white pixel). To convert to binary, we can tell python that any pixel with an intensity between 190 and 255 is a white pixel (255) and all pixels out of that range are assigned a value of 0. The position of the centre of mass can simply be calculated by taking an average position of the white pixels in the array to get a single value of position for each frame in units of pixels. For a 120-frame video, 120 positions of the mass will be measured. An example of a frame converted to black and white is shown in fig. (3b).

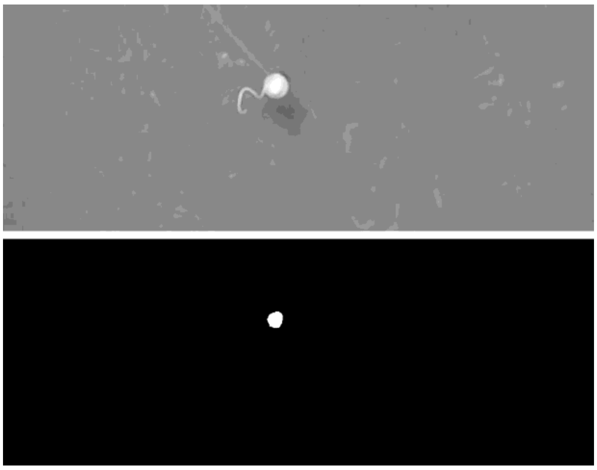

Figure 3. A single frame that has been converted to (3a) greyscale (top) and then to (3b) binary (bottom).

Once, the position of the centre of mass has been found for each frame, the displacements are converted to centimetres. By placing a meter ruler under the camera (at the same height as the pendulum to avoid parallax errors) we measured that 28 .0 ± 0.1 *cm* results in 944 ± 1 pixels; this means that the cm per pixels value is 0.0297 ±0.0001 cm/pixel. Multiplying the pixel positions converts the values to cm. To plot time on the x-axis in seconds the time interval between each frame must be calculated. The video records 120 frames at 23.976 ± 0.001 fps resulting in a frame every 0.041708 ± 0.000001 s. Once the data is plotted, a curve is fitted, namely the damped harmonic function from (2). In the preliminary test video, the damping parameter γ has a value of 0.010 ± 0.001 s^{-1} which results in an approximate difference in 1.2% difference in amplitude which justifies using a damped curve fit over a simple cosine (non-damped) function.

When the curve is first fitted, the point of 0 displacement is not on the *x*-axis. To ensure this is a case an offset parameter is added to the function, which when fitted to the data is calculated and then subtracted from every value of position to produce a graph like the one shown in fig. (4)

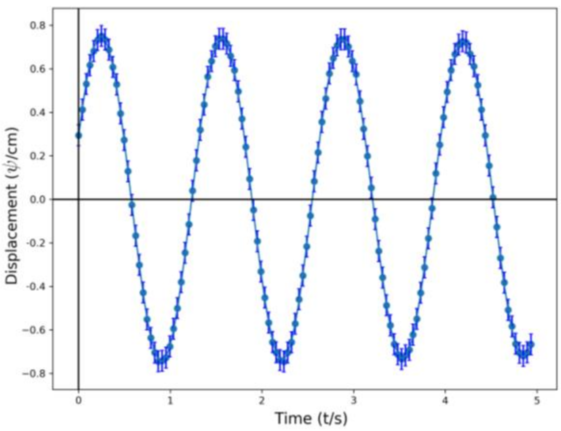

Figure 4. An example of a displacement-time graph for the pendulum with a damped harmonic function fitted.

The graph above also included errors in the positions calculated from the standard deviation of the white pixel positions in the binary frame. When the curve from (2) with parameters *A* and ω is fitted, the values of the amplitude and period can be calculated since $T=2\pi/\omega$ along with their errors.

The major flaw in using this method is that when converting from greyscale to black and white using a different minimum intensity, I_{min}, that is a different range of greyscale intensities that converts a pixel to a white pixel, the centre of mass position changes. By looping through different values of I_{min}, the errors of amplitude and period can be calculated for different intensities and the optimal value of I_{min} for that particular video finds the most accurate center of mass position. For each video, python looped through the range $190 \leq I_{min} \leq 250$ and plotted a graph of the percentage errors of amplitude and period against I_{min} and quadratic curve. The assumption that the error increases quadratically either side of the optimum I_{min} can be explained by the idea that the number of white pixels increases quadratically as I_{min} increases. An example of this graph is shown in fig. (5). The minimum point of the quadratic is found and the values of amplitude and period for this optimum value is used as the values of *A* and *T* for that video.

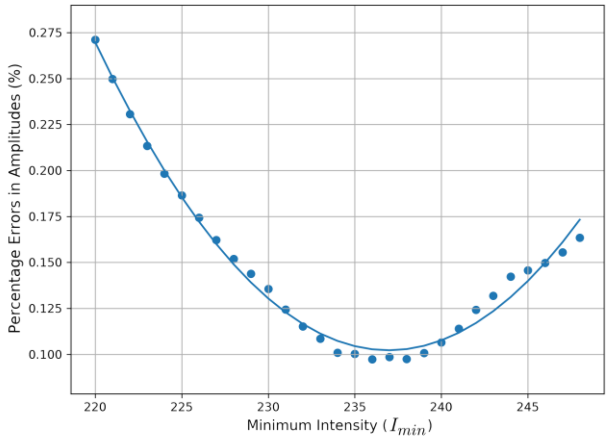

Figure 5. Graph showing how the error in amplitude varies with the minimum intensity used in the range 220 to 250.

This is then done for all video files of the pendulum via a third and final loop. It is looped through a total of 25 times, one for each video. The following step was finding the length of the pendulum used. Instead of using a meter rule that has a large error, (8) can be rearranged to find *l*.

$$l = g\left(\frac{T_0}{2\pi}\right)^2 \tag{12}$$

However, this equation only holds for very small amplitudes; we can find *l* by using the calculated value of *T* that the amplitude was the smallest – this value of *A* was ~0.75 cm with a corresponding period, T_0, of 1.3159 ± 0.0002 s. Using the value of g in London to be 9.8364 ms^{-2} [5] we can calculate the length of the pendulum to be 0.43144 ± 0.00009 m. Using this length and elementary trigonometry, we can convert all values of the amplitude to the angle θ_0 from the vertical such that:

$$\theta_0 = \arctan\left(\frac{A}{l}\right) \tag{13}$$

Once every value of θ_0 is calculated, a graph of the period against $T_0 \cdot K_n(\theta_0)$ (9) for different values of *n* is produced.

IV. Results and Discussion

The graphs of $T_0 \cdot K_n(\theta_0)$ against T has created plots showing 20 periods with their value approximated by $K_1(\theta_0)$. Three of the 25 videos were unfit for the data as it was calculated they had negative damping values γ; this is of course impossible as it would suggest the amplitude increases with no driving force. Another two data points were ignored as they were outside of 1.5σ from the best fit line – they had periods that were much lower than the expected value and is likely due to the fact that the centre of mass position calculated was not its true position, this can occur when the plot of percentage errors against I_{min} (fig. (5)) is not parabolic.

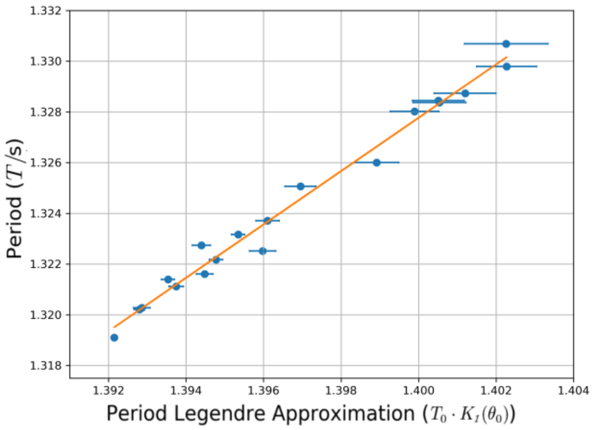

Figure 6. A plot of period T against the second-order polynomial function $T_0 \cdot K_1(\theta_0)$.

The gradient of the line of best fit for the $K_1(\theta_0)$ graph (shown in fig. (6)) has a calculated value 1.052 ± 0.001 resulting in a 5.2% error with the desired theoretical value of 1 with the second-order approximation model. The accuracy of $2n$ = 2, 4, 6 and 8 order were calculated from their respective K_n graph, the results of each model is shown in table I.

TABLE I
Accuracy of Legendre Approximation of T

n	Error in $T_0 \cdot K_n(\theta_0)$ Approximation (%)
1	5.2 ± 0.1
2	4.2 ± 0.1
3	4.4 ± 0.1
4	4.4 ± 0.1

Table I shows how accurate the approximated value of T is compared to the actual value for more terms of the Legendre approximation. We can see that the model is most accurate at fourth order θ_0 showing the terms of order greater than four has little effect on the accuracy of model, it is likely that the maximum angles (amplitudes) used were too small to have an effect on higher-order terms. Despite this accuracy, the model over approximates for any number of terms n and the desired value of a gradient of 1 is not in the error range. This could come down to the idea that the camera sees the horizontal motion of the mass, but not the vertical – as the pendulum is closer to the table the closer to the equilibrium position it is, the meters per pixel value is smallest and largest when its displacement is at a maximum. This parallax error would cause an overestimation in the amplitude and hence the value θ_0. Although we considered the damping force in the fitting of the displacement-time graph, the derivation of the Legendre approximation derived from (3) does not take into account the energy lost due to damping meaning the energy at a time $\theta(t)$ will be smaller than when the angular displacement is θ_0.

It is worth noting that the errors throughout the experiment remain small due to the accurate fitting of the damped harmonic curve yielding values of period and amplitude accurate to 10^{-3} % which is why error bars are not seen in fig. (6). It is the conversion of amplitude from pixels to cm that propagates the greatest error as the measured value of meters per pixel is accurate to ~1 %. This error is largest for larger values of θ_0 seen in larger error bars for greater values of T in fig. (6).

V. Conclusion

To conclude, the experimental method was very effective in showing how the Legendre approximation is a suitable way of showing that the period of harmonic oscillation does indeed depend on its amplitude. The polynomial function is proven to be effective at showing this dependence up to order 2 which produces a quartic function of θ_0 which is understandable for the range of values of amplitudes used (maximum of ~15cm). However, to show that the model becomes more accurate for more terms larger amplitudes need to be used. To accurately measure these larger amplitudes the video should have been recorded from an additional side view as the larger amplitudes will have greater heights. Finally, the algorithm used to minimize the errors and find the centre of mass position by finding the optimal minimum intensity was very efficient as the errors in the final values of T and θ_0 was <1 %.

VI. References

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[5] B. Adams, 'Local Acceleration due to Gravity Calculator', Wolfram Alpha, Physics, Added: 6 February 2014