Kinetic Inductance Detector resonance model and fitting

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1 Resonance circuit derivation

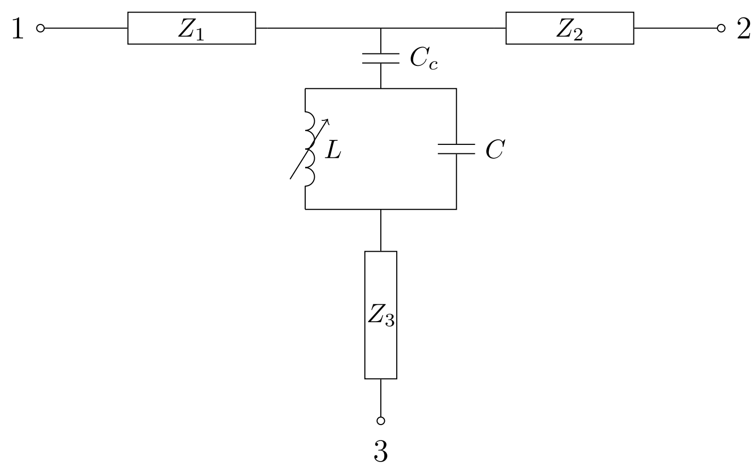

Figure 1: Schematic of KID circuit.

A schematic of the KID resonance circuit is shown in figure 1. The KID is represented by a variable inductor L, the parallel capacitor is C, and the coupling capacitor is C_c. The transmission S_{21} is measured from port 1 to port 2. $Z_1 = Z_2$ in an ideal circuit, but we will see in section 1.2 that differences in the input and output impedance must be accounted for in the resonance model.

1.1 Basic resonance model

The basic formula for the transmission S_{21} through the resonance circuit is derived in many sources, including Jiansong Gao's thesis (Gao, "The Physics of Superconducting Microwave Resonators."), Omid Noroozian's thesis (Noroozian, "Superconducting Microwave Resonator Arrays for Submillimeter/Far-Infrared Imaging."), and Ben Mazin's thesis (Mazin, "Microwave Kinetic Inductance Detectors."). An example of the basic resonance model curve is plotted in figure 2. The transmission is

$$\boxed{S_{21} = 1 - \frac{Q_r}{Q_c}\frac{1}{1+2jy}} \tag{1}$$

where Q_r is the total resonator quality factor and Q_c is the coupling quality factor. y is related to the fraction frequency shift x through

$$\boxed{y = Q_r x = Q_r \frac{f - f_r}{f_r}.} \tag{2}$$

The total resonator quality factor is related to the coupling quality factor and the internal quality factor Q_i through

$$\boxed{\frac{1}{Q_r} = \frac{1}{Q_c} + \frac{1}{Q_i}.} \tag{3}$$

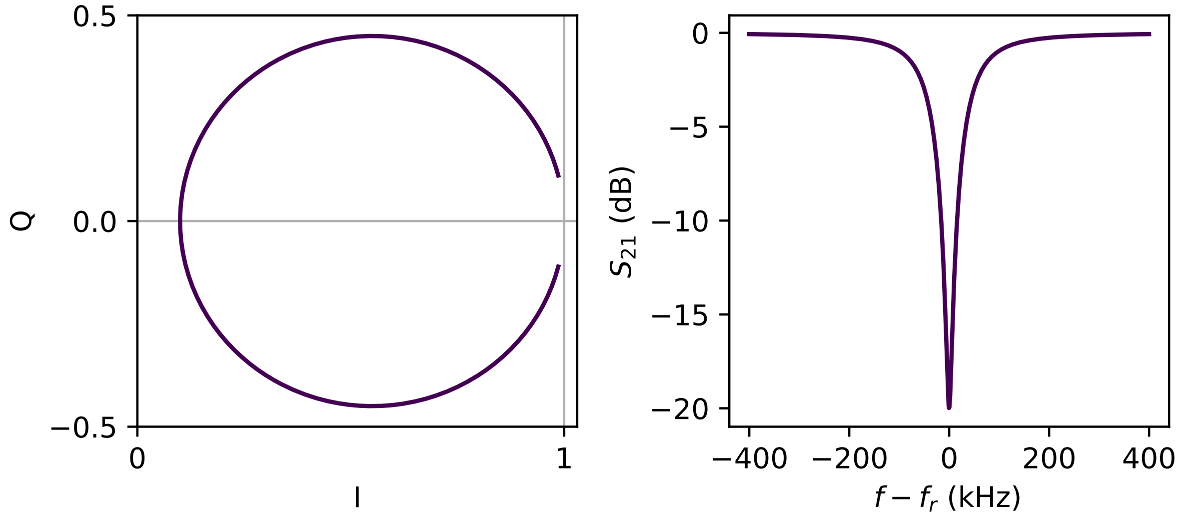

Figure 2: Example of the basic resonance curve. Left: imaginary (Q) versus real (I) component of the transmission. The resonance traces a circle with diameter Q_r/Q_c. Right: transmission magnitude versus frequency. The resonance dip is centered on f_r and the FWHM is $\sqrt{3}f_r/Q_r$.

1.2 Impedance mismatch

A mismatch in the impedance of the input and output lines can be modelled by introducing an imaginary component to the coupling quality factor through the angular parameter ϕ. This behavior is explained in detail in (Khalil et al., "An Analysis Method for Asymmetric Resonator Transmission Applied to Superconducting Devices."). An example of the shift in the resonance cause by the impedance mismatch is plotted in figure 3. The modification to Q_c is

$$\frac{1}{\hat{Q}_c} = \frac{1}{|\hat{Q}_c|} e^{j\phi}. \tag{4}$$

The resonator transmission becomes

$$\boxed{S_{21} = 1 - \frac{Q_r}{Q_c \cos\phi} e^{j\phi} \frac{1}{1 + 2jy}.} \tag{5}$$

The cosine term arises from the fact that the coupling quality factor is $Q_c = \mathrm{Re}\left\{\hat{Q}_c\right\}$.

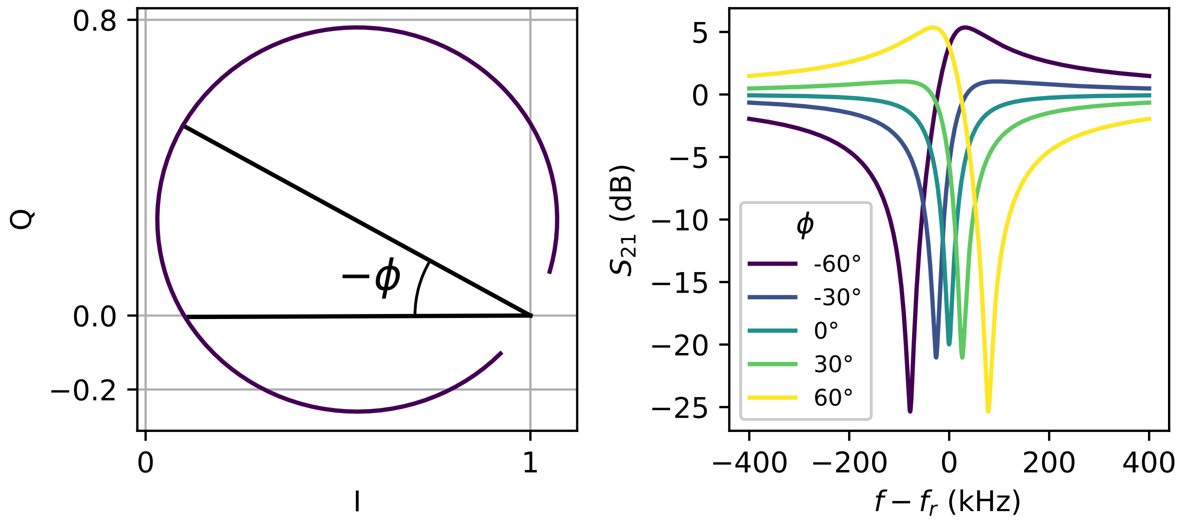

Figure 3: Example of the shift in the resonance shape cause by an impedance mismatch. Left: Q versus I. The angle ϕ determines the angle between the real axis and the point at the resonance frequency. Right: transmission magnitude versus frequency for several values of ϕ.

1.3 Nonlinear inductance

KIDs exhibit nonlinear inductance. The nonlinear inductance can be modelled as the series

$$L(I) = L(I = 0)\left(1 + \frac{I^2}{I_\star^2} + \cdots\right), \tag{6}$$

where odd powers of I must be dropped to ensure that the inductance is the same in both directions. The modification to S_{21} due to the nonlinear inductance is derived in (Swenson et al. 2013: "Operation of a titanium nitride superconducting microresonator detector in the nonlinear regime"). The nonlinearity manifests as a modification to y in equation 5. y becomes the solution to

$$\boxed{y - \frac{a}{1 + 4y^2} = y_0 = Q_r \frac{f - f_{r,0}}{f_{r,0}},} \tag{7}$$

where the nonlinearity parameter a is given by

$$\boxed{a = \frac{2Q_r^3}{Q_c} \frac{P}{2\pi f_r E_\star}.} \tag{8}$$

This equation can be solved for y, and it reaches a critical value at $a_{\mathrm{bif}} = 4\sqrt{3}/9 \approx 0.77$. For $a \leq a_{\mathrm{bif}}$, there is only one real solution for all values of y_0. For $a > a_{\mathrm{bif}}$, there is a range of y_0 values with three real solutions. Two of these three solutions are stable (the largest and smallest stored energy states), the resonator is said to have undergone bifurcation. Since a is proportional to power, there exists an upper limit on the power at which the KID can be read out, above which bifurcation leads to excess noise. An example of the effect of the nonlinear kinetic inductance on the resonance shape is plotted in figure 4.

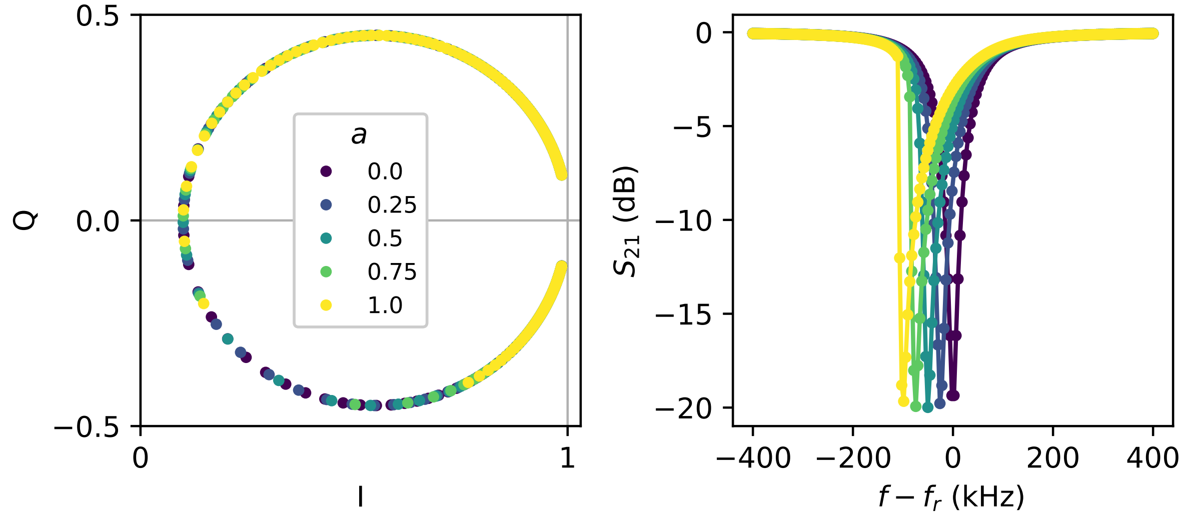

Figure 4: Effect of the nonlinear kinetic inductance on resonance shape for a range of values of the nonlinearity parameter a. Left: Q versus I. Right: transmission magnitude versus frequency. Note that the frequency spacing becomes more dense one one side of the resonance and less dense on the other as a is increased.

1.4 Readout circuit

The readout circuit contributes an additional gain that can be expressed as

$$S_{21}^{\text{readout}} = A(f)e^{-j2\pi f\tau_{\text{cable}}+j\delta} \tag{9}$$

where $A(f)$ is the gain amplitude, τ_{cable} is the cable delay, and δ_0 is the phase offset (at $f = 0$). An example of the gain phase and amplitude is plotted in figure 5.

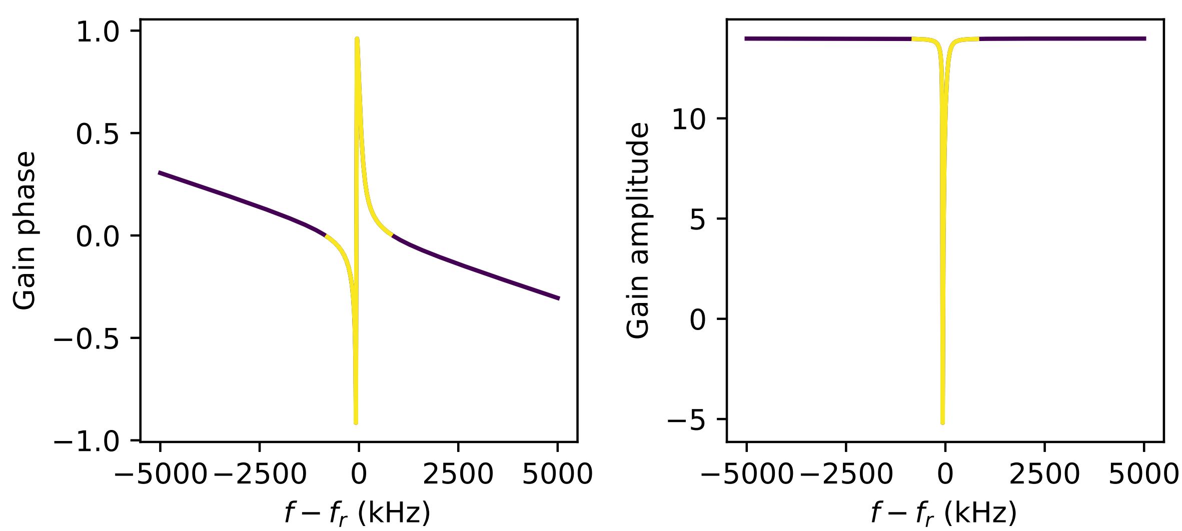

Figure 5: Example of the gain phase and amplitude. The resonance is yellow and the gain portion of the data is purple. Left: Phase versus frequency. The slope of the phase is $-2\pi\tau_{\text{cable}}$ and the offset is δ. Right: Amplitude versus frequency.

2 Fitting

The equation to fit is

$$S_{21} = A(f)e^{-j2\pi f\tau + j\delta}\left[1 - \frac{1}{1+2jy}\frac{Q_r}{Q_c\cos\phi}e^{j\phi}\right]. \tag{10}$$

In practice, the gain amplitude and phase are first removed from the data through polynomial fits to the data off-resonance on either side of the resonance. A second- or third-order polynomial fit is used for the gain amplitude, and a first-order fit is used for the gain phase (represented the phase offset and cable delay). After removing the gain amplitude and phase, it is advised to keep a multiplicative complex parameter $z_0 = i_0 + jq_0$ and the cable delay in the fitting equation for fine-tuning. The fitting is more robust to errors in the gain fit if these parameters are kept. The gain-removed data can be fit to

$$\boxed{S_{21} = (i_0 + jq_0)e^{-j2\pi f\tau}\left[1 - \frac{1}{1+2jy}\frac{Q_r}{Q_c\cos\phi}e^{j\phi}\right].} \tag{11}$$

The fitting procedure uses the following eight parameters:

$$\begin{aligned}
f_r &: \text{resonance frequency} \\
Q_r &: \text{total resonator quality factor} \\
\text{amp} &: Q_r/Q_c \\
\phi &: \text{impedance mismatch angle} \\
a &: \text{nonlinearity parameter} \\
i_0 &: \text{real component of gain} \\
q_0 &: \text{imaginary component of gain} \\
\tau &: \text{cable delay.}
\end{aligned}$$

2.1 Bounds

The following suggestions are reasonable bounds for the fitter.

$$\begin{aligned}
f_r &: \text{lower and upper range of the frequency data} \\
Q_r &: \text{factor of 10 from the initial guess} \\
\text{amp} &: \text{0 to 1} \\
\phi &: -\pi \text{ to } \pi \\
a &: \text{0 to 2.} \\
i_0,\ q_0 &: \text{-10 to 10, assuming the gain has been removed} \\
\tau &: \text{-1 us to 1 us}
\end{aligned}$$

2.2 Initial guess

The initial guess can be tricky, because many of the features of the data can depend on multiple parameters.

2.2.1 Gain and cable delay

These parameters should be $i_0 = 1$, $q_0 = 0$, and $\tau = 0$ if the gain removal is done correctly. In practice, i_0 and q_0 can be fine-tuned in the initial guess by using the average of the data on both edges.

2.2.2 Impedance mismatch

The impedance mismatch angle is the angle between the off-resonance data and the center of the circular resonance. The IQ data can be fit to a circle to extract the center and radius to calculate the angle.

2.2.3 amp

Q_r/Q_c is given by $d|\cos\phi|/|z_0|$, where d is the diameter of the circle fit from the impedance mismatch and $z_0 = i_0 + jq_0$.

2.2.4 Total quality factor

Q_r is guessed through the width of the data with the gain and impedance mismatch removed. In practice, a Guassian filter is first applied to this data, and Q_r is calculated from a first-order polynomial fit with parameters determined from many sets of simulated resonances.

2.2.5 Nonlinearity parameter

The nonlinearity parameter is related to the ratio of the maximum to minimum spacing between adjacent points in IQ space multiplied by Q_r/f_r, where f_r is guessed using a simple maximum spacing method. A third-order polynomial fit is used to determine a, and the fit parameters were determined from many sets of simulated resonances.

2.2.6 Resonance frequency

The resonance frequency turns out to be the hardest parameter to predict, since it depends on many other parameters. As a first guess, the frequency of maximum spacing in IQ space of the data with gain and impedance mismatch rotation removed is used. This guess is modified using a polynomial fit to a/Q_r, where the fit parameters were determined from many sets of simulated resonances.