SAGI 2024: Fourier Series, Transform, and Transform Properties

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Overview

1. This lesson develops some of the machinery and intuition around Fourier analysis– an essential tool for work in any experimental science!
2. The material here largely follows the development in [1]. We refer you here for more details.
3. Outline:
    1. Representation of a square wave by *Fourier series coefficients*.
    2. Fourier series representation of general periodic signals.
    3. Continous time Fourier Transform.
    4. Fourier Transform of rectangle signal.
    5. Fourier Transform properties.

[1] A. Oppenheim, A. Willsky, and S. Nawab, Signals and Systems, 2nd edition. Upper Saddle River, N.J: Pearson, 1996.

Periodic Signals

1. A signal $x(t)$ is *periodic* if for all real values of t and some positive value T we have:

$$x(t) = x(t \pm T) \tag{1}$$

2. The *fundamental period* of $x(t)$ is the minimum nonzero value of T for which equation 1 is satisfied.
3. The *fundamental frequency* of $x(t)$ is $w_o = \frac{2\pi}{T}$ when T is the fundamental period.
4. Two basic periodic signals are the sinusoidal signal:

$$x(t) = \cos(w_o t) \tag{2}$$

and the complex exponential:

$$x(t) = e^{iw_o t} \tag{3}$$

Periodic Signals

1. Each of these signals are periodic with *fundamental frequency* w_o and *fundamental period* $T = \frac{2\pi}{w_o}$
2. Associated with these signals are the set of *harmonically related* complex exponentials, which are also periodic with T:

$$\phi_k(t) = e^{ikw_o t};\ k = 0, \pm 1, \pm 2, \ldots \tag{4}$$

3. Any linear combination of these *harmonically related* complex exponentials is also periodic with T. Therefore, we can reasonably expect that for some periodic signal $x(t)$ with fundamental period T, there exist a set of coefficients $\{a_k\}$ for which:

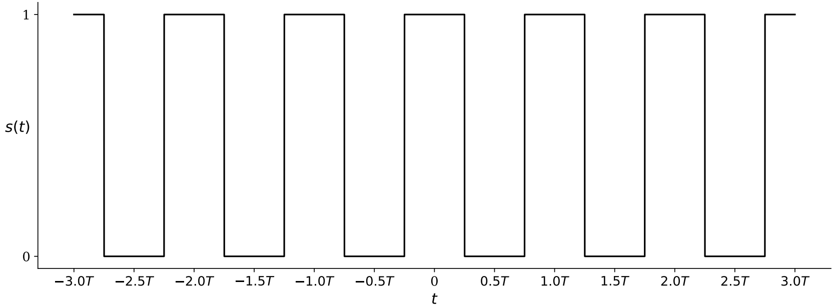
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikw_o t} = \sum_{k=-\infty}^{\infty} a_k \left[\cos(kw_o t) + i\sin(kw_o t)\right] \tag{5}$$

Square Wave Signal

1. Let's explore this idea with the periodic square wave signal. Over one period T, we define the square wave signal as:

$$s(t) = \begin{cases} 1, & |t| \leq \frac{T}{4} \\ 0, & \frac{T}{4} < |t| \leq \frac{3T}{4} \\ 1, & \frac{3T}{4} < |t| \end{cases} \tag{6}$$

2. Several periods of the square wave are plotted here:



Square Wave Signal

1. Can the square wave be represented by a sum of harmonically related complex exponentials as in equation 5? Yes! We claim that with the following values of $\{a_k\}$ we recover the square wave exactly!

$$a_0 = \frac{1}{2} \tag{7}$$

$$a_k = \frac{2\sin\left(\frac{k\pi}{2}\right)}{k\pi}, \quad k \neq 0 \tag{8}$$

2. Notice that for all even values of k we have $a_k = 0$, thus equation 5 can be written as follows for the square wave:

$$s(t) = \frac{1}{2} + 2\sum_{k=1}^{\infty}\left[\frac{2\sin\left(\frac{k\pi}{2}\right)}{k\pi}\right]\cos\left(\frac{2\pi k}{T}t\right) \tag{9}$$

Square Wave Signal

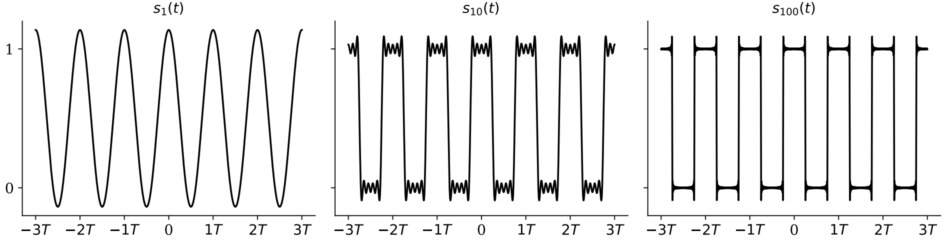
1. Equation 9 is referred to as the *Fourier Series representation* of the square wave signal. Notice that in the summation we have neglected the $k < 0$ terms. We can do this because the square wave is completely real-valued.
2. To illustrate that this works, let's define an *n term* representation of the full infinite series as:

$$s_0(t) = \frac{1}{2} \tag{10}$$

$$s_n(t) = \frac{1}{2} + 2\sum_{k=1}^{n}\left[\frac{2\sin\left(\frac{k\pi}{2}\right)}{k\pi}\right]\cos\left(\frac{2\pi k}{T}t\right), \quad n = 1, 2, 3, \ldots \tag{11}$$

Square Wave Signal

1. Plotting s_1, s_{10}, s_{100}, we see that as n increases we obtain a signal that more closely resembles the true square wave:



2. To explore this further, and see some basics of the numpy and matplotlib python packages, please open the following notebook: *sagi/lessons/instrument_control/lesson4/sqr_wave_fourier_series.ipynb*

Fourier Series of General Periodic Signals

1. How were the coefficients in equations 7 and 8 determined? We will now develop this, taking a general approach that works for any periodic signal $x(t)$. We start with equation 5, repeated here:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikw_o t} \tag{12}$$

2. Multiplying both sides by $e^{-inw_o t}$ and integrating over one fundamental period T, we obtain:

$$\int_t^{t+T} x(t) e^{-inw_o t} dt = \int_t^{t+T} \sum_{k=-\infty}^{\infty} a_k e^{ikw_o t} e^{-inw_o t} dt \tag{13}$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_t^{t+T} e^{i(k-n)w_o t} dt \tag{14}$$

Fourier Series of General Periodic Signals

1. Harmonically related complex exponentials are *orthogonal* when integrated over one fundamental period. Therefore:

$$\int_{t}^{t+T} e^{i(k-n)w_o t} dt = \begin{cases} 0, & k \neq n \\ T, & k = n \end{cases} \tag{15}$$

2. So, equation 14 reduces to:

$$\boxed{a_n = \frac{1}{T}\int_{t}^{t+T} x(t) e^{-inw_o t} dt} \tag{16}$$

Fourier Series of General Periodic Signals

1. In summary, a periodic signal $x(t)$ with fundamental period T can be represented as a linear combination of harmonically related complex exponentials in the following summation. The coefficients of the linear combination $\{a_n\}$ are given by equation 16 above.

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{inw_o t}} \tag{17}$$

Fourier Transform

1. We have illustrated (but not proven) that a periodic signal can be represented by an infinite summation of complex exponentials. Can the same be done for aperiodic signals? Yes, the *Fourier Transform* (distinct from the Fourier Series) allows us to do this.
2. Here, we simply state the *Fourier Transform* equations and again refer to [2] for more details on where the equations come from.
3. Given an aperiodic signal $x(t)$, we can represent this signal as a linear combination of infinitesimally spaced complex exponentials. This linear combination is similar to equation 17, but now the summation becomes an integral and $nw_o \rightarrow w$, with w being a continous variable.

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{X}(w)e^{iwt}dw} \tag{18}$$

[2] A. Oppenheim, A. Willsky, and S. Nawab, Signals and Systems, 2nd edition. Upper Saddle River, N.J: Pearson, 1996.

Fourier Transform

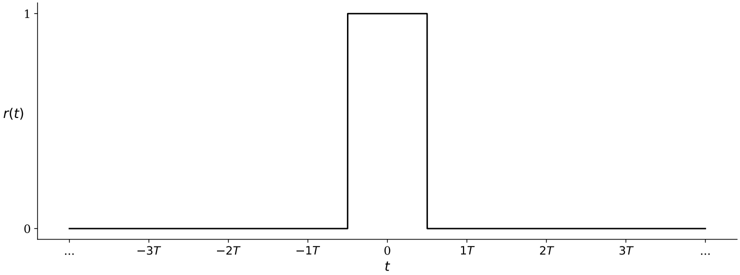
1. The coefficients $\{a_n\}$ become a continuous function $\tilde{X}(w)$ known as the *Fourier Transform* of $x(t)$.

$$\boxed{\tilde{X}(w) = \int_{-\infty}^{+\infty} x(t)e^{iwt}dt} \quad (19)$$

Rectangle Signal

1. Let's consider an example– the Fourier Transform of the rectangle signal.
2. We define the rectangle signal as a non-repeating aperiodic version of the square wave:

$$r(t) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & \frac{T}{2} < |t| \end{cases} \tag{20}$$



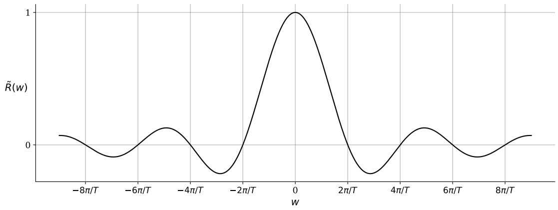
Rectangle Signal

1. Apply equation 19 to the rectangle signal:

$$\tilde{R}(w) = \int_{-T/2}^{+T/2} e^{iwt} dt = \frac{1}{w}\left[\frac{e^{iwT/2} - e^{-iwT/2}}{i}\right] \tag{21}$$

2. We recognize the bracketed term as $2\sin(wT/2)$, so the Fourier Transform becomes:

$$\tilde{R}(w) = 2\frac{\sin\left(\frac{wT}{2}\right)}{w} \tag{22}$$



Fourier Transform Properties: Linearity

1. We now turn to some important properties of the Fourier Transform, starting with Linearity.
2. Linearity of the Fourier Transform means that if $\tilde{X}(w) = \mathbb{F}\{x(t)\}$ and $\tilde{Y}(w) = \mathbb{F}\{y(t)\}$, then:

$$\mathbb{F}\{ax(t) + by(t)\} = a\tilde{X}(w) + b\tilde{Y}(w) \tag{23}$$

3. Proof of this property follows easily from direct application of equation 19 to $ax(t) + by(t)$

Fourier Transform Properties: Time Shift

1. If $\tilde{X}(w) = \mathbb{F}\{x(t)\}$, then:

$$\mathbb{F}\{x(t - t_o)\} = e^{-iwt_o}\tilde{X}(w) \tag{24}$$

2. Proof: Start with equation 18 and substitute $t \rightarrow (t - t_o)$:

$$x(t - t_o) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{X}(w)e^{iw(t-t_o)}dw \tag{25}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[e^{-iwt_o}\tilde{X}(w)\right]e^{iwt}dw \tag{26}$$

3. We identify the bracketed term above as the Fourier Transform of $x(t - t_o)$ according to equation 18

Fourier Transform Properties: Frequency Shift

1. If $x(t) = \mathbb{F}^{-1}\{\tilde{X}(w)\}$, then:

$$\mathbb{F}^{-1}\{\tilde{X}(w - w_o)\} = e^{iw_o t}x(t) \tag{27}$$

2. Proof: Start with equation 19 and substitute $w \rightarrow (w - w_o)$:

$$\tilde{X}(w - w_o) = \int_{-\infty}^{+\infty} x(t)e^{-i(w-w_o)t}dt \tag{28}$$

$$= \int_{-\infty}^{+\infty} \left[e^{iw_o t}x(t)\right] e^{-iwt}dt \tag{29}$$

3. We identify the bracketed term above as the inverse transform of $\tilde{X}(w - w_o)$ according to equation 19.

Fourier Transform Properties: Convolution

1. The Fourier Transform convolution property is very important. We will revisit this in the next lecture and in the IC/DAQ + electronics lab project next week.
2. The *convolution* $y(t)$ between two signals $x(t)$ and $h(t)$ is defined as:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \tag{30}$$

3. Now, let's find $\tilde{Y}(w) = \mathbb{F}\{y(t)\}$:

$$\tilde{Y}(w) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right] e^{-iwt}dt \tag{31}$$

Fourier Transform Properties: Convolution

1. Interchange the order of integration and note that $x(\tau)$ is not a function of t:

$$\tilde{Y}(w) = \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} h(t-\tau) e^{-iwt} dt \right] d\tau \tag{32}$$

2. From the *time shift* property (equation 24), we recognize the bracketed term as $\mathbb{F}\{h(t-\tau)\} = e^{-iw\tau}\tilde{H}(w)$:

$$\tilde{Y}(w) = \int_{-\infty}^{+\infty} x(\tau) e^{-iw\tau} \tilde{H}(w) d\tau \tag{33}$$

$$= \tilde{H}(w) \int_{-\infty}^{+\infty} x(\tau) e^{-iw\tau} d\tau \tag{34}$$

$$= \tilde{H}(w)\tilde{X}(w) \tag{35}$$

Fourier Transform Properties: Convolution

1. We summarize this result, denoting the convolution integral of equation 30 by $*$:

$$\boxed{\mathbb{F}\{y(t) = h(t) * x(t)\} = \tilde{Y}(w) = \tilde{H}(w)\tilde{X}(w)} \tag{36}$$

2. As we see, convolution in the time-domain corresponds to multiplication in the frequency domain. Is there an analogous property for multiplication in the time domain? Yes! Multiplication in time corresponds to convolution in frequency. We show this next.

Fourier Transform Properties: Multiplication

1. The multiplication property states that given a signal $r(t) = s(t)p(t)$ we have:

$$\boxed{\mathbb{F}\{r(t)\} = \tilde{R}(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{S}(\theta)\tilde{P}(w-\theta)d\theta = \tilde{S}(w) * \tilde{P}(w)} \tag{37}$$

2. To prove this, we perform an inverse transform on $\tilde{R}(w)$ and show that we obtain $r(t) = s(t)p(t)$:

$$r(t) = \mathbb{F}^{-1}\{\tilde{R}(w)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{R}(w)e^{iwt}dw \tag{38}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{S}(\theta)\tilde{P}(w-\theta)d\theta\right] e^{iwt}dw \tag{39}$$

Fourier Transform Properties: Multiplication

1. Interchange the order of integration and note that $\tilde{S}(\theta)$ is not a function of w:

$$r(t) = \mathbb{F}^{-1}\{\tilde{R}(w)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{S}(\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{P}(w-\theta)e^{iwt}dw\right]d\theta \quad (40)$$

2. From the *frequency shift* property we recognize the bracketed term as $\mathbb{F}^{-1}\{\tilde{P}(w-\theta)\} = e^{i\theta t}p(t)$:

$$r(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{S}(\theta)e^{i\theta t}p(t)d\theta \quad (41)$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{S}(\theta)e^{i\theta t}d\theta\right]p(t) \quad (42)$$

$$= s(t)p(t) \quad (43)$$