SAGI 2024 Lesson: Sampling and Aliasing

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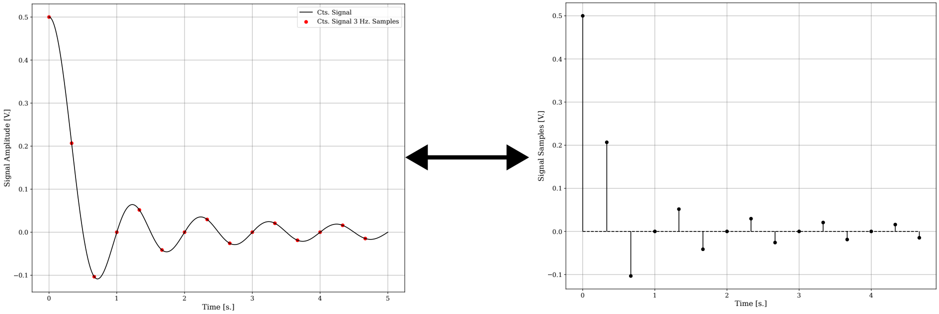
Sampling

1. **Sampling** refers to the process in which we record a series of *samples* from a signal at points equally spaced in time.
2. Under certain conditions, a *continuous-time* signal can be *uniquely represented* by this set of samples. This extremely important and useful property of sampling is known as the **Nyquist Sampling Theorem**.



Sampling

1. Intuitive examples of the **Nyquist Sampling Theorem** come from images and movies.
2. Movies consist of a set of individual images (*samples*) that when viewed in sequence fast enough, accurately resemble a continuously changing scene.
3. Digital images hold a set of finely spaced pixels each of which are *samples* within a spatially continuous scene.

1. In general, we would not expect that a continuous signal could be uniquely represented by a sequence of equally spaced samples. See the following figure from [1]

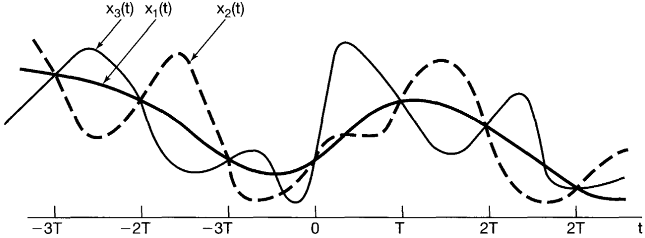


Figure: Three independent signals $x_1(t)$, $x_2(t)$, $x_3(t)$ sampled at evenly spaced intervals of time T produce the same values. Hence, these signals *are not* uniquely represented by this sampling process.

[1] A. Oppenheim, A. Willsky, and S. Nawab, Signals and Systems, 2nd edition. Upper Saddle River, N.J: Pearson, 1996.

Sampling

1. Under which conditions does the sampling theorem hold? To answer this question we must formalize our discussion of sampling.
2. We begin by defining our sampled signal $x_p(t)$ in the following manner:

$$x_p(t) = x(t) \cdot p(t) \tag{1}$$

Where $p(t)$, the *sampling function*, is a periodic impulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2}$$

T is referred to as the *sampling period* and $f_s = 1/T$, the *sampling frequency*.

Sampling

1. Multiplying $x(t)$ by a unit-impulse samples the value of x at that point. I.e. $x(t) \cdot \delta(t - t_o) = x(t_o)$
2. We then have:

$$x_p(t) = x(t) \cdot p(t) \tag{3}$$

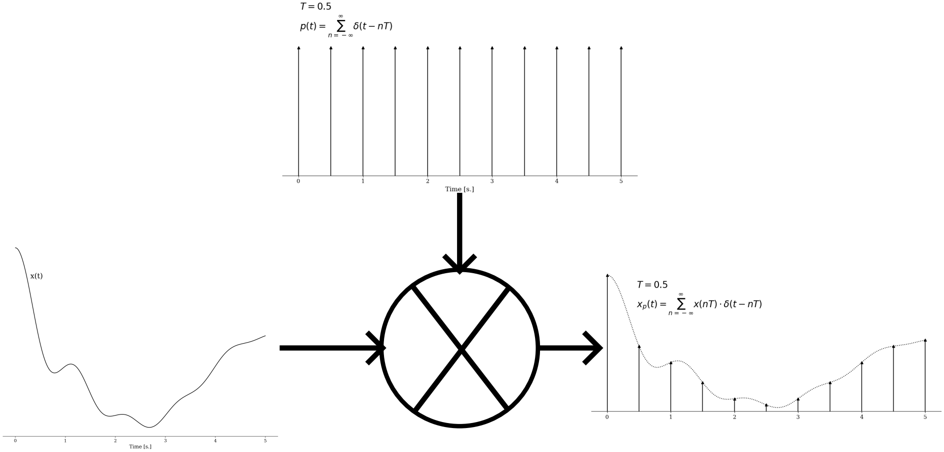
$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{4}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \tag{5}$$

3. $x_p(t)$ is an impulse train with the amplitudes of the impulses equal to the samples of $x(t)$ at intervals spaced by T.

Sampling

1. This process, referred to as **Impulse Train Sampling**, is illustrated below for a sampling frequency of $f_s = 2$ Hz ($T = 1/2$ s.):



Sampling

1. Now let's consider our sampled signal in the frequency domain. From the convolution/multiplication property of the Fourier Transform we have:

$$x_p(t) = x(t) \cdot p(t) \xrightarrow[F.T.]{} X_p(f) = \int_{-\infty}^{\infty} X(f - f_o)P(f_o)df_o \tag{6}$$

2. Further, it can be shown that the Fourier Transform of our sampling function is given by:

$$p(t) \xrightarrow[F.T.]{} P(f) = f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s) \tag{7}$$

3. Convolution with an impulse shifts a signal. I.e. $X(f) * \delta(f - f_o) = X(f - f_o)$, thus:

$$\boxed{X_p(f) = f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)} \tag{8}$$

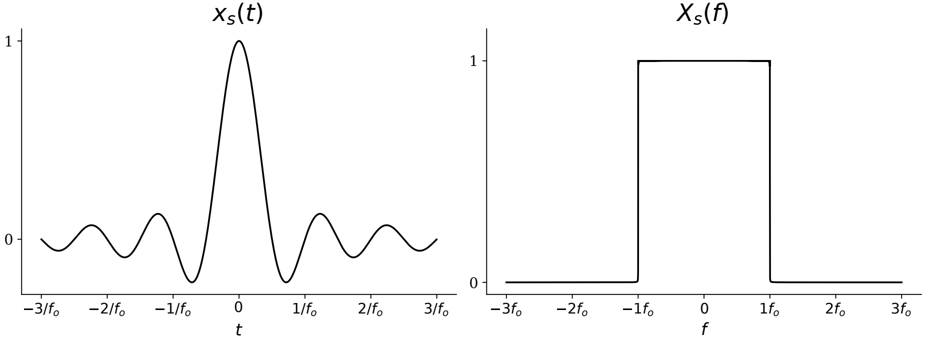
Aliasing and Nyquist Sampling

1. In words, the Fourier Transform of our sampled signal $X_p(f)$ is an infinite set of copies of the Fourier Transform of the original signal $X(f)$, where each copy is shifted by the sampling frequency f_s.
2. We are getting closer to the **Nyquist Sampling Theorem**. We now must think about a distortion effect known as **aliasing**. Let's begin by considering two special signals– the sinc and sinc squared signal.

Aliasing and Nyquist Sampling

1. **Sinc** and it's Fourier Transform:

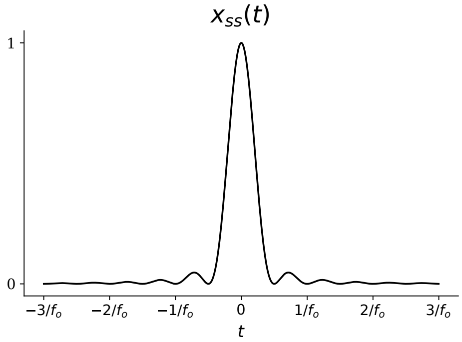
$$x_s(t) = \frac{\sin(2\pi f_o t)}{2\pi f_o t} \xrightarrow[F.T.]{} X_s(f) = \begin{cases} 1, & |f| \leq f_o \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

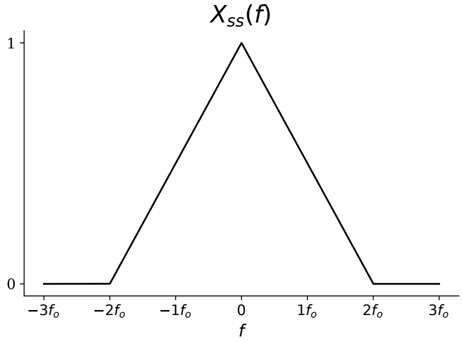


Aliasing and Nyquist Sampling

Sinc squared and it's Fourier Transform:

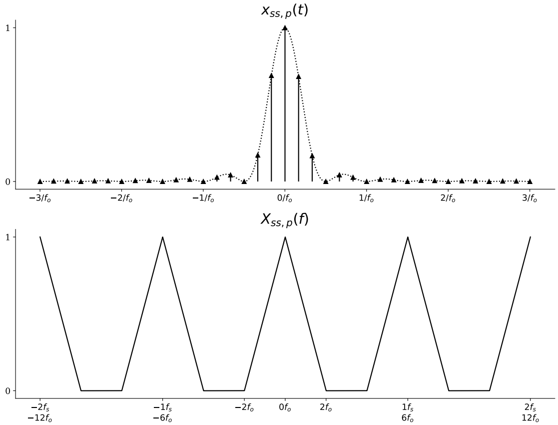
$$x_{ss}(t) = \left[\frac{\sin(2\pi f_o t)}{2\pi f_o t}\right]^2 \xrightarrow[F.T.]{} X_{ss}(f) = \begin{cases} \text{sign}(f)\cdot\left(\frac{-f}{2f_o}\right)+1, & |f| \leq 2f_o \\ 0, & \text{elsewhere} \end{cases} \tag{10}$$





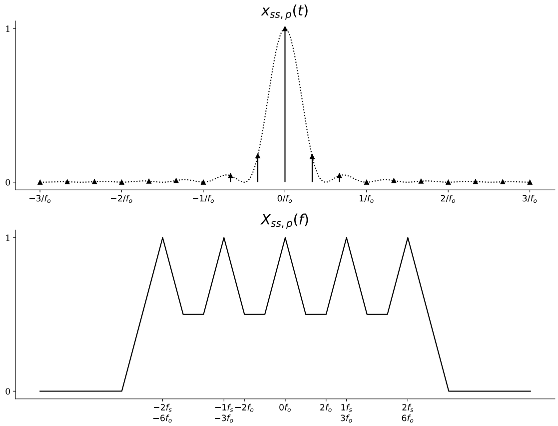
Aliasing and Nyquist Sampling

1. Now, let's sample the sinc squared signal with a sampling frequency of $f_s = 6f_o$
2. Using equations 3 and 8 we obtain the following results in the time and frequency domains:



Aliasing and Nyquist Sampling

1. Let's say that for practical reasons, we can only sample at $f_s = 3f_o$.
2. Our sampled signal and it's Fourier Transform then looks like:



Aliasing and Nyquist Sampling

1. Notice that we can no longer recover our original signal by applying a sinc filter. We have introduced *higher frequency* components into our sampled signal's Fourier transform and thus we do not have a reliable representation of the original signal!
2. This effect is known as **aliasing**.
3. From the intuition gained in this demonstration we can now state (without rigorous proof) the **Nyquist Sampling Theorem:**

> Let $x(t)$ be a band-limited signal with $X(f) = 0$ for $|f| > f_m$.
> Then, $x(t)$ is reliably (uniquely) determined by it's samples if: $f_s > 2f_m$