

SAGI 2024: Fourier Series, Fourier Transform, and Fourier Transform Properties

Sam Condon

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Fourier
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- ① This lesson develops some of the machinery and intuition around Fourier analysis– an essential tool for work in any experimental science!
- ② The material here largely follows the development in ¹. We refer you here for more details.
- ③ Outline:
 - ① Fourier transform motivation.
 - ② Representation of a square wave by *Fourier series coefficients*.
 - ③ Fourier series representation of general periodic signals.
 - ④ Continuous time Fourier Transform.
 - ⑤ Fourier Transform of rectangle signal.
 - ⑥ Fourier Transform properties.

¹A. Oppenheim, A. Willsky, and S. Nawab, Signals and Systems, 2nd edition. Upper Saddle River, N.J: Pearson, 1996.

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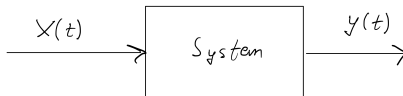
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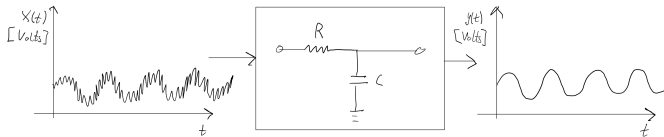
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- ① Why do we want to study and understand the Fourier Transform?
- ② I will motivate the Fourier Transform from a signal processing perspective, starting with some definitions.
- ③ **System:** any physical process that transforms an input signal $x(t)$ to an output $y(t)$:



- ④ Ex: low-pass filter circuit:



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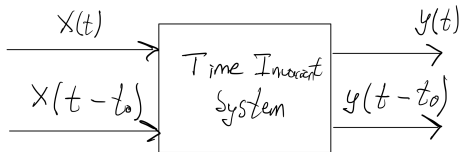
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- 1 **Time-invariant system:** A system whose behavior has no time independence. That is, if $y(t)$ is the output corresponding to $x(t)$, then $y(t - t_o)$ is the output corresponding to $x(t - t_o)$
- 2 If R and C remain constant in the low-pass filter, we will always get the same output for a given input, no matter when we run the experiment. Hence, the low-pass filter is a *time-invariant system*.



- 3 Non time-invariant example:

$$y(t) = x(2t) \quad (1)$$

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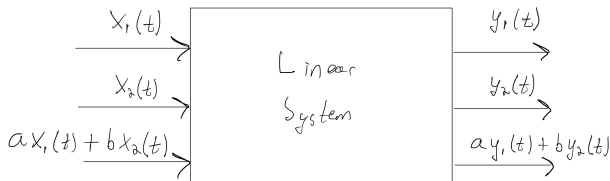
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- 1 **Linear system:** A system which obeys superposition— if $y_1(t)$ is the output corresponding to $x_1(t)$ and $y_2(t)$ is the output corresponding to $x_2(t)$, then $ay_1(t) + by_2(t)$ is the output corresponding to $ax_1(t) + bx_2(t)$, where a and b are any constant scalar values.



- 2 Systems which possess both linearity and time-invariance are called **linear time-invariant systems** or LTI systems for short.

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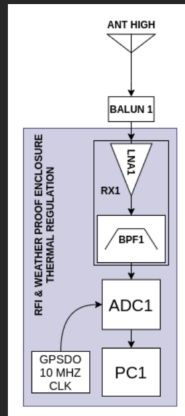
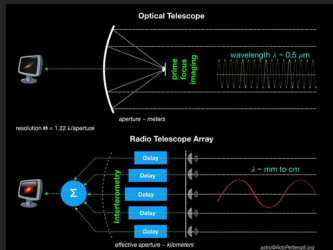
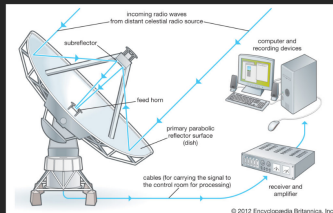
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1 From Bang's radio astronomy lecture:

Radio Telescope



Antenna
(Optics)

Balun

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amplifier

Bandpass
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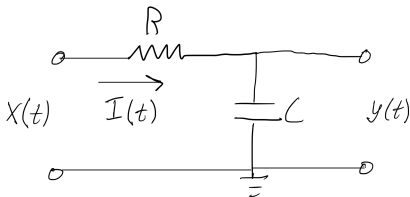
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- 1 Let's take a look at the low-pass filter circuit again:



- 2 Which obeys the differential equation:

$$y(t) + RC \frac{dy(t)}{dt} = x(t) \quad (2)$$

- 3 This is a first-order linear differential equation \rightarrow **the low-pass filter is an LTI system.**

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- 1 Take $x(t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$, which has the solution:

$$y(t) = \left(\frac{1}{1 + i\omega RC} \right) e^{i\omega t} = \tilde{H}(\omega)x(t) \quad (3)$$

- 2 With a complex exponential input, the low-pass filter has a scaled complex exponential output \rightarrow complex exponentials are **eigenfunctions** of the low-pass filter, and in fact of **all LTI systems**.
- 3 Further, if the input to an LTI system can be represented as a linear combination of complex exponentials:

$$x(t) = \sum_k a_{\omega_k} e^{i\omega_k t} \quad (4)$$

- 4 Then the superposition property gives us:

$$y(t) = \sum_k a_{\omega_k} H(\omega_k) e^{i\omega_k t} \quad (5)$$

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- 1 $H(w)$ is called the frequency response. When we design filters in a telescope receiver, it is this function that we want to optimize for our specific application!
- 2 Complex exponentials are *complete orthonormal functions*. This means that we can represent any signal $x(t)$ as a summation of complex exponentials.
- 3 If $x(t)$ is periodic:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{inw_0 t} \quad (6)$$

- 4 If $x(t)$ is aperiodic:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(w) e^{iwt} dw \quad (7)$$

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- 1 A signal $x(t)$ is *periodic* if for all real values of t and some positive value T we have:

$$x(t) = x(t \pm T) \quad (8)$$

- 2 The *fundamental period* of $x(t)$ is the minimum nonzero value of T for which equation 8 is satisfied.
- 3 The *fundamental frequency* of $x(t)$ is $\omega_o = \frac{2\pi}{T}$ when T is the fundamental period.
- 4 Two basic periodic signals are the sinusoidal signal:

$$x(t) = \cos(\omega_o t) \quad (9)$$

and the complex exponential:

$$x(t) = e^{i\omega_o t} \quad (10)$$

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- 1 Each of these signals are periodic with *fundamental frequency* ω_0 and *fundamental period* $T = \frac{2\pi}{\omega_0}$
- 2 Associated with these signals are the set of *harmonically related* complex exponentials, which are also periodic with T :

$$\phi_k(t) = e^{ik\omega_0 t}; \quad k = 0, \pm 1, \pm 2, \dots \quad (11)$$

- 3 Any linear combination of these *harmonically related* complex exponentials is also periodic with T . Therefore, we can reasonably expect that for some periodic signal $x(t)$ with fundamental period T , there exist a set of coefficients $\{a_k\}$ for which:

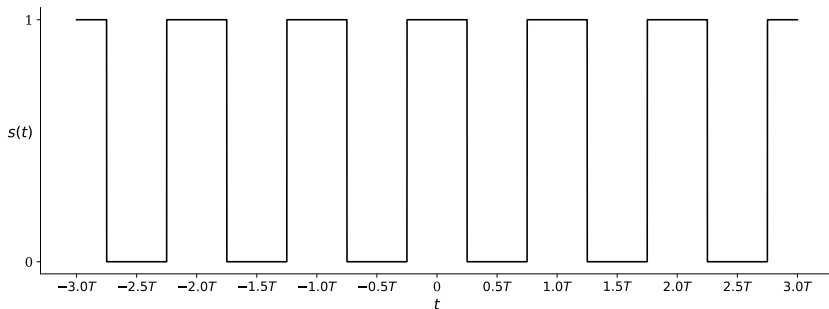
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k [\cos(k\omega_0 t) + i \sin(k\omega_0 t)] \quad (12)$$

Square Wave Signal

- 1 Let's explore this idea with the periodic square wave signal. Over one period T , we define the square wave signal as:

$$s(t) = \begin{cases} 1, & |t| \leq \frac{T}{4} \\ 0, & \frac{T}{4} < |t| \leq \frac{T}{2} \end{cases} \quad (13)$$

- 2 Several periods of the square wave are plotted here:



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- 1 Can the square wave be represented by a sum of harmonically related complex exponentials as in equation 12? Yes! We claim that with the following values of $\{a_k\}$ we recover the square wave exactly!

$$a_0 = \frac{1}{2} \quad (14)$$

$$a_k = \frac{2 \sin\left(\frac{k\pi}{2}\right)}{k\pi}, \quad k \neq 0 \quad (15)$$

- 2 Notice that for all even values of k we have $a_k = 0$, thus equation 12 can be written as follows for the square wave:

$$s(t) = \frac{1}{2} + 2 \sum_{k=1}^{\infty} \left[\frac{2 \sin\left(\frac{k\pi}{2}\right)}{k\pi} \right] \cos\left(\frac{2\pi k}{T} t\right) \quad (16)$$

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- 1 Equation 16 is referred to as the *Fourier Series representation* of the square wave signal. Notice that in the summation we have neglected the $k < 0$ terms. We can do this because the square wave is completely real-valued.
- 2 To illustrate that this works, let's define an n term representation of the full infinite series as:

$$s_0(t) = \frac{1}{2} \quad (17)$$

$$s_n(t) = \frac{1}{2} + 2 \sum_{k=1}^n \left[\frac{2 \sin\left(\frac{k\pi}{2}\right)}{k\pi} \right] \cos\left(\frac{2\pi k}{T} t\right), \quad n = 1, 2, 3, \dots \quad (18)$$

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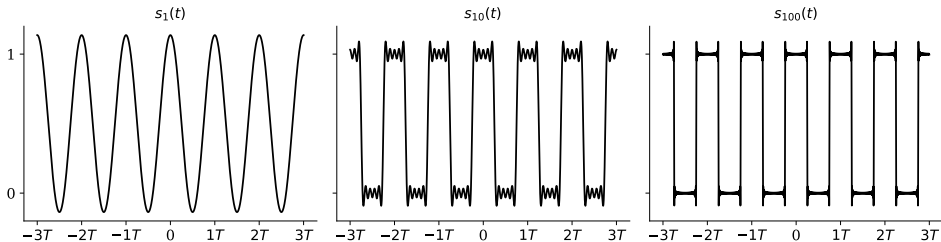
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- 1 Plotting s_1 , s_{10} , s_{100} , we see that as n increases we obtain a signal that more closely resembles the true square wave:



- 2 To explore this further, and see some basics of the numpy and matplotlib python packages, please open the following notebook:
[*sagi/lessons/instrument_control/lesson4/sqr_wave_fourier_series.ipynb*](#)

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- 1 How were the coefficients in equations 14 and 15 determined? We will now develop this, taking a general approach that works for any periodic signal $x(t)$. We start with equation 12, repeated here:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikw_0 t} \quad (19)$$

- 2 Multiplying both sides by $e^{-inw_0 t}$ and integrating over one fundamental period T , we obtain:

$$\int_t^{t+T} x(t) e^{-inw_0 t} dt = \int_t^{t+T} \sum_{k=-\infty}^{\infty} a_k e^{ikw_0 t} e^{-inw_0 t} dt \quad (20)$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_t^{t+T} e^{i(k-n)w_0 t} dt \quad (21)$$

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- ① Harmonically related complex exponentials are *orthogonal* when integrated over one fundamental period. Therefore:

$$\int_t^{t+T} e^{i(k-n)\omega_0 t} dt = \begin{cases} 0, & k \neq n \\ T, & k = n \end{cases} \quad (22)$$

- ② So, equation 21 reduces to:

$$a_n = \frac{1}{T} \int_t^{t+T} x(t) e^{-in\omega_0 t} dt \quad (23)$$

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- 1 In summary, a periodic signal $x(t)$ with fundamental period T can be represented as a linear combination of harmonically related complex exponentials in the following summation. The coefficients of the linear combination $\{a_n\}$ are given by equation 23 above.

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega_0 t} \quad (24)$$

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- ① We have illustrated (but not proven) that a periodic signal can be represented by an infinite summation of complex exponentials. Can the same be done for aperiodic signals? Yes, the *Fourier Transform* (distinct from the Fourier Series) allows us to do this.
- ② Here, we simply state the *Fourier Transform* equations and again refer to ² for more details on where the equations come from.
- ③ Given an aperiodic signal $x(t)$, we can represent this signal as a linear combination of infinitesimally spaced complex exponentials. This linear combination is similar to equation 24, but now the summation becomes an integral and $nw_o \rightarrow w$, with w being a continuous variable.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(w) e^{iwt} dw \quad (25)$$

²A. Oppenheim, A. Willsky, and S. Nawab, Signals and Systems, 2nd edition. Upper Saddle River, N.J: Pearson, 1996.

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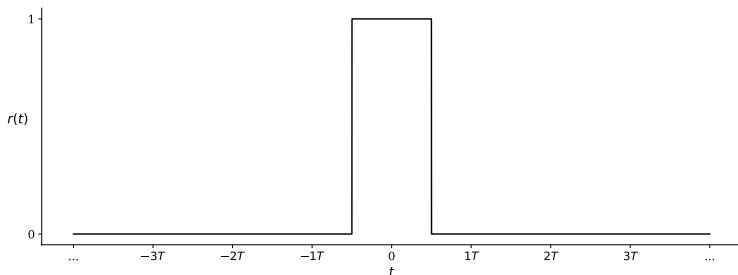
- 1 The coefficients $\{a_n\}$ become a continuous function $\tilde{X}(w)$ known as the *Fourier Transform* of $x(t)$.

$$\tilde{X}(w) = \int_{-\infty}^{+\infty} x(t)e^{-iwt} dt \quad (26)$$

Rectangle Signal

- ① Let's consider an example— the Fourier Transform of the rectangle signal.
- ② We define the rectangle signal as a non-repeating aperiodic version of the square wave:

$$r(t) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & \frac{T}{2} < |t| \end{cases} \quad (27)$$



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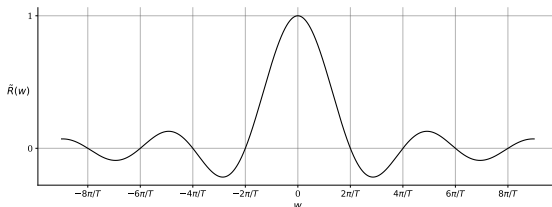
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- 1 Apply equation 26 to the rectangle signal:

$$\tilde{R}(w) = \int_{-T/2}^{+T/2} e^{iwt} dt = \frac{1}{w} \left[\frac{e^{iwT/2} - e^{-iwT/2}}{i} \right] \quad (28)$$

- 2 We recognize the bracketed term as $2 \sin(wT/2)$, so the Fourier Transform becomes:

$$\tilde{R}(w) = 2 \frac{\sin\left(\frac{wT}{2}\right)}{w} \quad (29)$$



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- 1 We now turn to some important properties of the Fourier Transform, starting with Linearity.
- 2 Linearity of the Fourier Transform means that if $\tilde{X}(w) = \mathbb{F}\{x(t)\}$ and $\tilde{Y}(w) = \mathbb{F}\{y(t)\}$, then:

$$\mathbb{F}\{ax(t) + by(t)\} = a\tilde{X}(w) + b\tilde{Y}(w) \quad (30)$$

- 3 Proof of this property follows easily from direct application of equation 26 to $ax(t) + by(t)$

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- ① If $\tilde{X}(w) = \mathbb{F}\{x(t)\}$, then:

$$\mathbb{F}\{x(t - t_o)\} = e^{-iwt_o} \tilde{X}(w) \quad (31)$$

- ② Proof: Start with equation 25 and substitute $t \rightarrow (t - t_o)$:

$$x(t - t_o) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(w) e^{iw(t-t_o)} dw \quad (32)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[e^{-iwt_o} \tilde{X}(w) \right] e^{iwt} dw \quad (33)$$

- ③ We identify the bracketed term above as the Fourier Transform of $x(t - t_o)$ according to equation 25

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- ① If $x(t) = \mathbb{F}^{-1}\{\tilde{X}(w)\}$, then:

$$\mathbb{F}^{-1}\{\tilde{X}(w - w_o)\} = e^{iw_o t} x(t) \quad (34)$$

- ② Proof: Start with equation 26 and substitute $w \rightarrow (w - w_o)$:

$$\tilde{X}(w - w_o) = \int_{-\infty}^{+\infty} x(t) e^{-i(w - w_o)t} dt \quad (35)$$

$$= \int_{-\infty}^{+\infty} [e^{iw_o t} x(t)] e^{-iwt} dt \quad (36)$$

- ③ We identify the bracketed term above as the inverse transform of $\tilde{X}(w - w_o)$ according to equation 26.

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- 1 The Fourier Transform convolution property is very important. We will revisit this in the next lecture and in the IC/DAQ + electronics lab project next week.
- 2 The *convolution* $y(t)$ between two signals $x(t)$ and $h(t)$ is defined as:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau \quad (37)$$

- 3 Now, let's find $\tilde{Y}(w) = \mathbb{F}\{y(t)\}$:

$$\tilde{Y}(w) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau \right] e^{-iwt} dt \quad (38)$$

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- 1 Interchange the order of integration and note that $x(\tau)$ is not a function of t :

$$\tilde{Y}(w) = \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} h(t - \tau) e^{-iwt} dt \right] d\tau \quad (39)$$

- 2 From the *time shift* property (equation 31), we recognize the bracketed term as $\mathbb{F}\{h(t - \tau)\} = e^{-i\omega\tau} \tilde{H}(w)$:

$$\tilde{Y}(w) = \int_{-\infty}^{+\infty} x(\tau) e^{-i\omega\tau} \tilde{H}(w) d\tau \quad (40)$$

$$= \tilde{H}(w) \int_{-\infty}^{+\infty} x(\tau) e^{-i\omega\tau} d\tau \quad (41)$$

$$= \tilde{H}(w) \tilde{X}(w) \quad (42)$$

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- 1 We summarize this result, denoting the convolution integral of equation 37 by $*$:

$$\mathbb{F}\{y(t) = h(t) * x(t)\} = \tilde{Y}(w) = \tilde{H}(w)\tilde{X}(w) \quad (43)$$

- 2 As we see, convolution in the time-domain corresponds to multiplication in the frequency domain. Is there an analogous property for multiplication in the time domain? Yes! Multiplication in time corresponds to convolution in frequency. We show this next.

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- ① The multiplication property states that given a signal $r(t) = s(t)p(t)$ we have:

$$\mathbb{F}\{r(t)\} = \tilde{R}(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{S}(\theta) \tilde{P}(w - \theta) d\theta = \frac{1}{2\pi} \tilde{S}(w) * \tilde{P}(w) \quad (44)$$

- ② To prove this, we perform an inverse transform on $\tilde{R}(w)$ and show that we obtain $r(t) = s(t)p(t)$:

$$r(t) = \mathbb{F}^{-1}\{\tilde{R}(w)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{R}(w) e^{iwt} dw \quad (45)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{S}(\theta) \tilde{P}(w - \theta) d\theta \right] e^{iwt} dw \quad (46)$$

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Fourier
Transform
Properties

- 1 Interchange the order of integration and note that $\tilde{S}(\theta)$ is not a function of w :

$$r(t) = \mathbb{F}^{-1}\{\tilde{R}(w)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{S}(\theta) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{P}(w - \theta) e^{iwt} dw \right] d\theta \quad (47)$$

- 2 From the *frequency shift* property we recognize the bracketed term as $\mathbb{F}^{-1}\{\tilde{P}(w - \theta)\} = e^{i\theta t} p(t)$:

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{S}(\theta) e^{i\theta t} p(t) d\theta \quad (48)$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{S}(\theta) e^{i\theta t} d\theta \right] p(t) \quad (49)$$

$$= s(t) p(t) \quad (50)$$