SAGI 2024: Sampling, Aliasing, and the Discrete Fourier Transform

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Overview

1. Review Fourier Transform equations and linear time-invariant systems.
2. Look in more depth at the convolution integral and the multiplication and convolution properties of the Fourier Transform.
3. Sampling: what this means, and a mathematical formalism to represent it.
4. Aliasing and Nyquist sampling $\rightarrow$ Nyquist sampling theorem.
5. Discrete signals and the Discrete Fourier Transform.

Fourier Transform

1. Any signal $x(t)$ can be represented as a summation of infinitesimally spaced weighted complex exponentials:

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{X}(w)e^{iwt}dw} \tag{1}$$

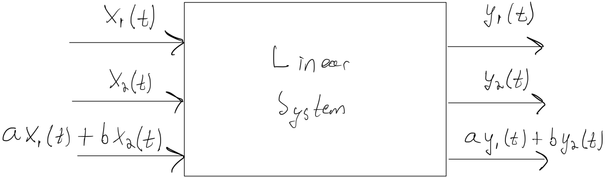
2. The weightings $\tilde{X}(w)$, called the **Fourier Transform** of $x(t)$, provide a measure of how the signal $x(t)$ is distributed in the frequency domain.
3. We find the Fourier Transform with:

$$\boxed{\tilde{X}(w) = \int_{-\infty}^{+\infty} x(t)e^{-iwt}dt} \tag{2}$$

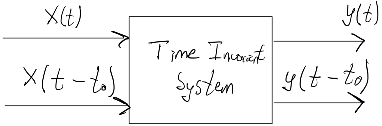
4. **Shorthand notation:**
    1. $\tilde{X}(w) = \mathbb{F}\{x(t)\}$ is the Fourier Transform of $x(t)$.
    2. $x(t) = \mathbb{F}^{-1}\{\tilde{X}(w)\}$ is the Inverse Fourier Transform of $\tilde{X}(w)$.

Linear Time Invariant Systems

1. A **System** is any physical process which maps an input signal $x(t)$ to an output signal $y(t)$.
2. A linear time-invariant (LTI) system obeys both superposition (linearity):



As well as time-invariance:

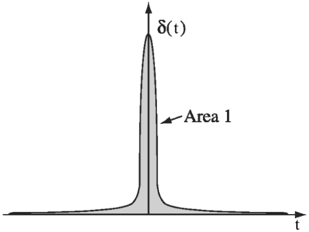


Dirac Delta Function

1. Formally, the *Dirac Delta* is actually a generalization of a function called a *distribution*. We won't worry about that and define it loosely here in the following manner:

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \infty, & t = 0 \end{cases} \tag{3}$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1 \tag{4}$$



Convolution

1. The convolution integral between two signals $x_1(t)$ and $x_2(t)$ is defined as:

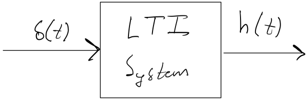
$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)x_2(t-\tau)d\tau \tag{5}$$

2. Because of equation 4, we can represent a signal $x(t)$ as a linear combination of delayed delta functions: $x(t)$ convolved with $\delta(t)$:

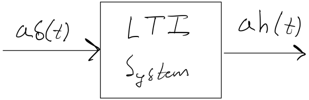
$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \tag{6}$$

Convolution

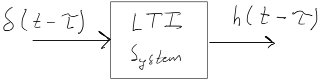
1. Now suppose that we have an LTI system with output $h(t)$ corresponding to the input $\delta(t)$:



2. Because of linearity, the output $ah(t)$ then corresponds to the input $a\delta(t)$:



3. Because of time-invariance:



Convolution

1. We know that $\delta(t) \rightarrow h(t)$ and that our system is LTI. So, for any input function $x(t)$:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \tag{7}$$

2. We obtain the output function:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)} \tag{8}$$

3. In signal processing literature, the function $h(t)$ is called the *impulse response* of the system. In mathematics and physics this type of function is referred to as a **Green's Function**.

Convolution

1. Recall from last lecture the correspondence between multiplication and convolution in the time and frequency domain.
2. Convolution in time corresponds to multiplication in frequency:

$$\boxed{\mathbb{F}\{y(t) = x(t) * h(t)\} = \tilde{Y}(w) = \tilde{X}(w)\tilde{H}(w)} \tag{9}$$

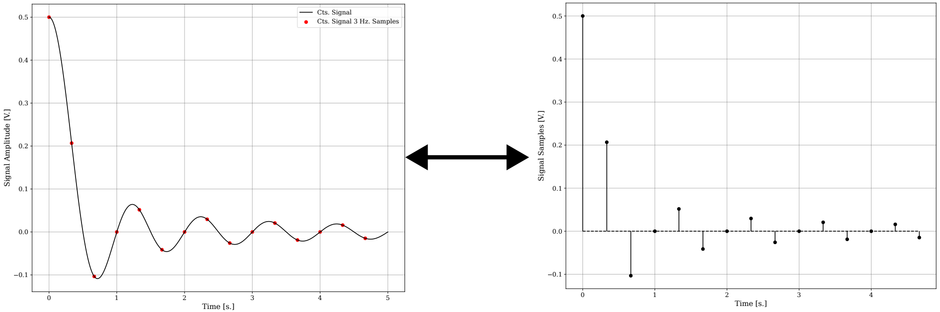
3. Multiplication in time corresponds to convolution in frequency:

$$\boxed{\mathbb{F}\{x_p(t) = x(t) \cdot p(t)\} = \tilde{X}_p(w) = \frac{1}{2\pi}\tilde{X}_p(w) * \tilde{P}(w)} \tag{10}$$

4. We will use the latter of these two properties to develop the **Nyquist sampling theorem.**

Sampling

1. **Sampling** refers to the process in which we record a series of *samples* from a signal at points equally spaced in time.



Sampling

1. Example from astronomy: Sample the brightness of stars versus time to detect exoplanets.

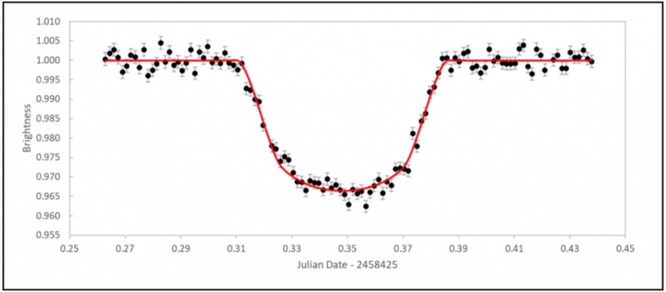


Figure: Transit light curve of WASP-52b, Mark Salisbury

Sampling

1. Intuitive examples of sampling come from images and movies.
2. Movies consist of a set of individual images (*samples*) that when viewed in sequence fast enough, accurately resemble a continuously changing scene.
3. Digital images hold a set of finely spaced pixels each of which are *samples* within a spatially continuous scene.

Sampling

1. When sampling a signal to do science, we need to be confident that we can **uniquely** represent a continuous signal of interest.
2. If we do not sample star brightness fast enough, we can not claim to accurately detect exoplanets via the transit method!
3. In general, we would not expect that a continuous signal could be uniquely represented by a sequence of equally spaced samples. See the following figure from [1]

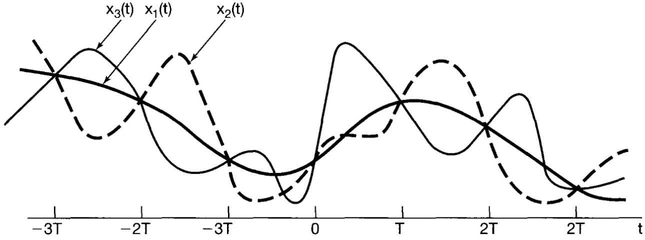


Figure: Three independent signals $x_1(t)$, $x_2(t)$, $x_3(t)$ sampled at evenly spaced intervals of time T produce the same values.

Sampling

1. **Nyquist Sampling Theorem:** provides us with the conditions under which we can *uniquely* represent a continuous signal with a discrete set of samples.

Sampling

1. Consider a continuous signal of time $x(t)$ containing some scientific information of interest.
2. We define our sampled signal $x_p(t)$ as $x(t)$ multiplied by a sampling function $p(t)$:

$$x_p(t) = x(t) \cdot p(t) \tag{11}$$

3. Where the sampling function $p(t)$ is:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{12}$$

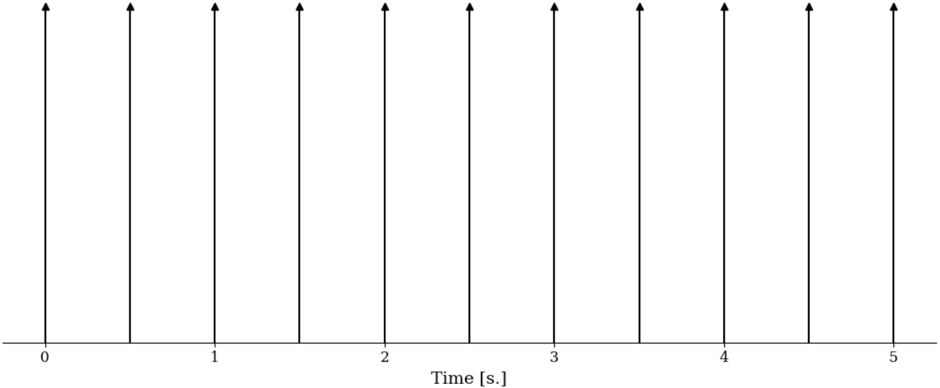
T is referred to as the *sampling period* and $w_s = 2\pi/T$, the *sampling frequency*.

Sampling

1. The plot below shows an example sampling function for $T = 0.5$:

$$T = 0.5$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Sampling

1. Since the Dirac delta $\delta(t - nT)$ is zero everywhere except for at $t = nT$, our sampled signal $x_p(t)$ becomes:

$$x_p(t) = x(t) \cdot p(t) \tag{13}$$

$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{14}$$

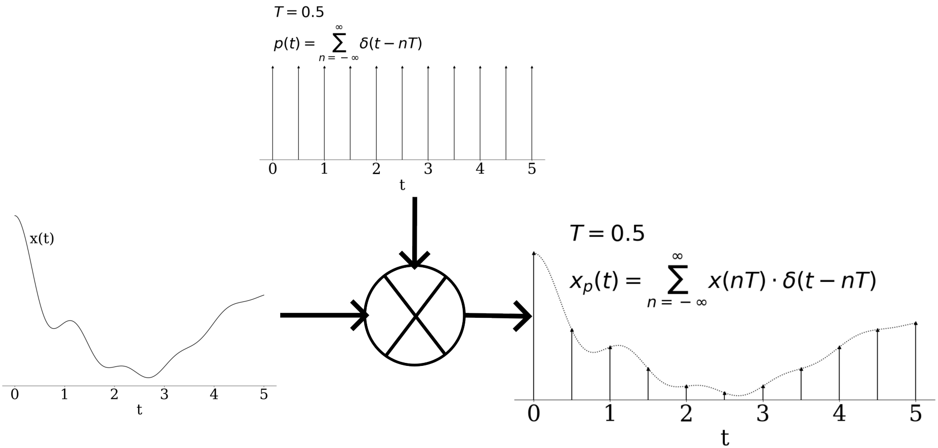
$$= \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \tag{15}$$

2. Defining the *sampling frequency* $w_s = \frac{2\pi}{T}$, this expression becomes:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x\left(n2\pi/w_s\right) \cdot \delta\left(t - n2\pi/w_s\right) \tag{16}$$

Sampling

1. The sampling process is illustrated graphically here:



Sampling

1. Now let's consider our sampled signal in the frequency domain by taking it's Fourier Transform. Applying the convolution/multiplication property:

$$\boxed{\mathbb{F}\{x_p(t) = x(t) \cdot p(t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{X}(\theta)\tilde{P}(w-\theta)d\theta = \frac{1}{2\pi}\tilde{X}(w) * \tilde{P}(w)} \tag{17}$$

   Where $\tilde{X}(w) \equiv \mathbb{F}\{x(t)\}$, the Fourier Transform of our original signal, and $\tilde{P}(w) \equiv \mathbb{F}\{p(t)\}$, the Fourier Transform of our sampling function.

2. It can be shown that the Fourier Transform of our sampling function is as follows:

$$\mathbb{F}\{p(t)\} = \tilde{P}(w) = w_s \sum_{n=-\infty}^{+\infty} \delta(w - nw_s) \tag{18}$$

Sampling

1. So continuing:

$$\mathbb{F}\{x_p(t)\} = \tilde{X}_p(w) = \frac{1}{2\pi}\tilde{X}(w) * \left[w_s \sum_{n=-\infty}^{+\infty} \delta(w - nw_s) \right] \tag{19}$$

2. Convolution with the Dirac delta shifts the signal, i.e. $\tilde{X}(w) * \delta(w - kw_s) = \tilde{X}(w - kw_s)$ so we are left with:

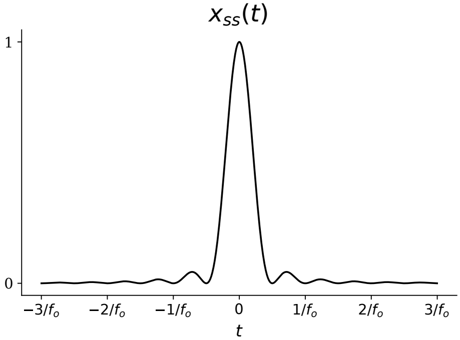
$$\tilde{X}(w) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \tilde{X}(w - kw_s) \tag{20}$$

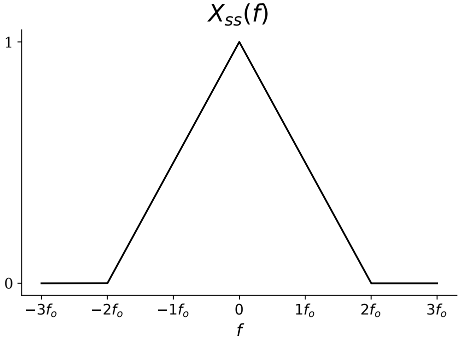
3. In words, this result says that the Fourier Transform of a sampled function is an infinite sum of copies of the Fourier Transform of the original signal, with each copy shifted by an integer multiple of the sampling frequency w_s.

Sampling

1. **Sinc squared** and it's Fourier Transform:

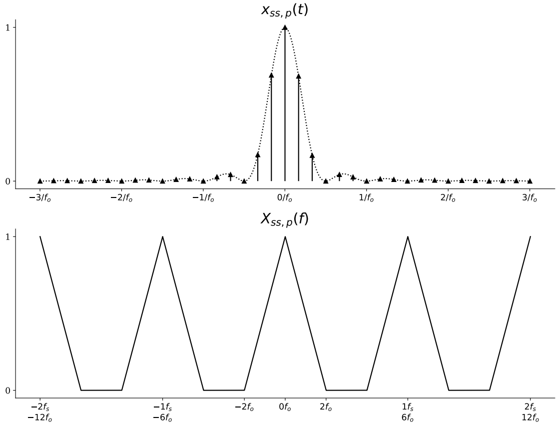
$$x_{ss}(t) = \left[\frac{\sin(2\pi f_o t)}{2\pi f_o t}\right]^2 \xrightarrow[F.T.]{} \tilde{X}_{ss}(f) = \begin{cases} \frac{-f}{2f_o} + 1, & 0 \leq f \leq 2f_o \\ \frac{f}{2f_o} + 1, & -2f_o \leq f \leq 0 \\ 0, & \text{elsewhere} \end{cases} \tag{21}$$



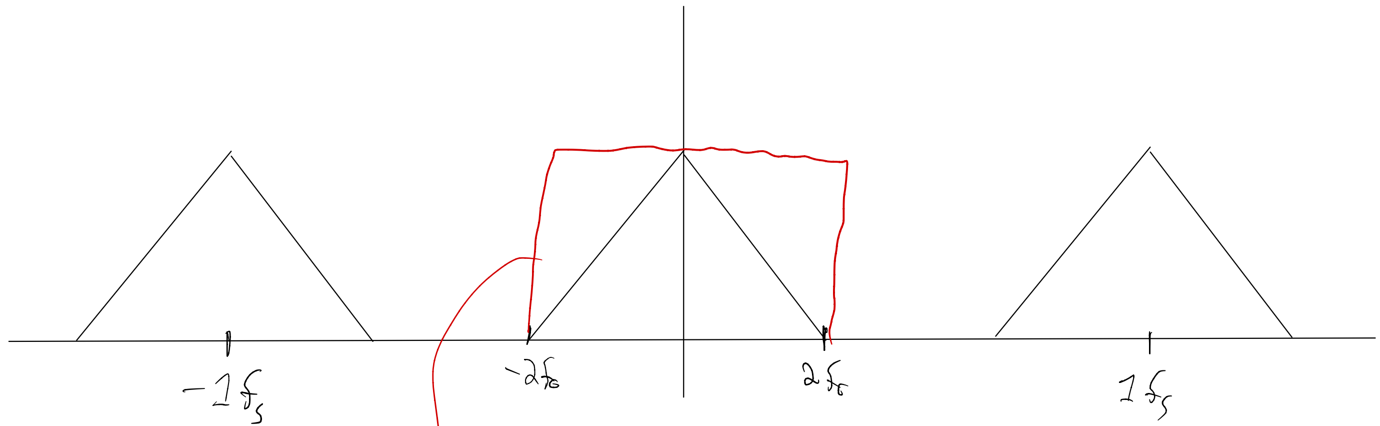


Aliasing and Nyquist Sampling

1. Now, let's sample the sinc squared signal with a sampling frequency of $f_s = 6f_o$
2. We obtain the following results in the time and frequency domains:



How can we recover our original frequency spectrum?



Apply a low-pass filter!

Ideal low-pass filter:

Frequency domain:

$$\tilde{H}(f) = \begin{cases} 1, & |f| < 2f_0 \\ 0, & \text{otherwise} \end{cases} \xrightarrow{\mathcal{F}^{-1}}$$

Time domain:

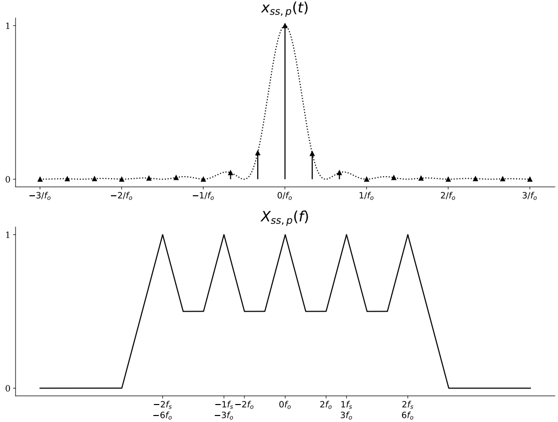
$$h(t) = \frac{2\sin(2\pi f t)}{2\pi t}$$

$$\tilde{X}_{ss}(f) = \tilde{H}(f)\,\tilde{X}_{ss,p}(f) \xrightarrow{\text{Conv. prop}} x_{ss}(t) = \int_{-\infty}^{+\infty} x_{ss,p}(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} x_{ss,p}(\tau) \left[\frac{\sin(2\pi f(t-\tau))}{\pi t} \right] d\tau$$

Aliasing and Nyquist Sampling

1. Let's say that for practical reasons, we can only sample at $f_s = 3f_o$.
2. Our sampled signal and it's Fourier Transform then looks like:



1. In the second example, we can no longer recover the frequency representation of our original signal! We have introduced higher frequency components into the signal by not sampling fast enough. This effect is called **aliasing**.

Aliasing and Nyquist Sampling

1. **Nyquist Sampling Theorem:**

   Let $x(t)$ be a band-limited signal with $\tilde{X}(f) = 0$ for $|f| > f_m$.
   Then, $x(t)$ is reliably (uniquely) represented by it's samples if: $f_s > 2f_m$

2. $2f_m$ is called the Nyquist frequency.

Discrete Signals

1. We have seen the conditions under which sampling a continuous signal into a discrete set of points gives us a unique representation of the continuous signal.
2. After sampling a continuous signal $x(t)$, we have the sampled signal $x_p(t)$ given by:

$$\begin{aligned} x_p(t) &= x(t) \cdot p(t) && (22) \\ &= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) && (23) \\ &= \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) && (24) \end{aligned}$$

3. This representation of a sampled signal provides a mathematical formalism to see how aliasing happens, and to develop the Nyquist sampling theorem but it is not a very convenient expression to work with.

Discrete Signals

1. In practice, when we sample a signal, we are simply left with a sequence of discrete points. Sampling a signal $x(t)$ with a frequency $f_s = 1/T$, we can represent this sequence with the following notation:

$$x[n] = x(nT) \tag{25}$$

2. Take $x(t) = \cos(2\pi f_o t)$. Sampling this signal with sampling rate f_s we get:

$$x[n] = \cos\left(\frac{2\pi f_o n}{f_s}\right) \tag{26}$$

3. **Exercise:** if $f_o = 4$ Hz. what does f_s need to be in order to reliably represent the above signal?

Discrete Fourier Transform

1. Now we will define the Discrete Fourier Transform, which performs the analog of the continuous time Fourier Transform, but for discrete signals.
2. Take a discrete signal $x[n]$ with N points. The Discrete Fourier Transform maps this sequence of N complex numbers into another sequence of N complex numbers given by:

$$\boxed{\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i2\pi \frac{kn}{N}}} \tag{27}$$

3. And the inverse transform:

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \cdot e^{i2\pi \frac{kn}{N}}} \tag{28}$$

Discrete Fourier Transform

1. The Discrete Fourier Transform (DFT) is the mathematical operation we perform on our computers.
2. The **Fast Fourier Transform** (FFT) is the most efficient algorithm to perform the DFT operation.
3. Tomorrow, we will see some practical examples using the FFT with Python.
4. The electronics + IC/DAQ joint lab project next week will explore many of the topics covered in the past few lectures, but from a much more practical hands-on perspective.