Comprehensive Exam Question, Fall 2023

Logan Hall

I. Prompt

In this problem, we implement the discrete elastic beam algorithm to simulate the deformation of the cantilevered beam shown in Figure 1. The beam has the following parameters: $\mathbf{L = 0.1m}, \mathbf{b = 0.01m}$, and $\mathbf{h = 0.002m}$, and Young's modulus is $\mathbf{E = 200 * 10^9}$ Pa. The applied load is P = 10 N.

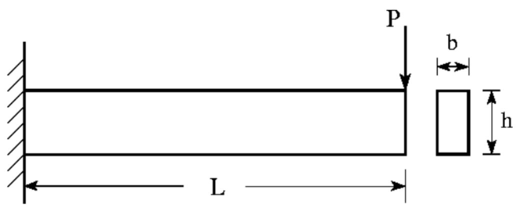

Fig. 1. Problem Statement

II. Approach

To solve this problem, I implemented the Discrete Elastic Beam algorithm. To start, I imported the problem parameters and discretized the beam into N = 200 nodes. In doing this, each node is evenly spaced across the length, L. The geometry is well defined since it begins as a straight beam, and thus the natural curvature is equal to 0. Additionally, the given beam is assumed to be massless, so the weight of the beam will not factor into the solving of this problem.

We then begin to calculate the positions and velocities of each successive step. We do this by implementing a time marching scheme, as described in Algorithm 1 [3].

For this cantilevered beam, the right side is unconstrained. However, the left side is restricted in x displacement, y displacement, and rotation. To constrain the x and y displacement of the left end of the beam, we fix the first node by fixing the first two degrees of freedom. To constrain the rotation, we fix the second node by fixing the third and fourth degrees of freedom. This fixes the slope. So in total, the first two nodes (first four degrees of freedom) are fixed. We fix these degrees of freedom by setting their values equal to 0, and only updating the free degrees of freedom in our time marching scheme. The remaining N-4 degrees of freedom are free.

III. Simulation

A. Deflection vs Point Load

The results of this simulation are shown in Figure 2. As observed, the left end remains fixed and the deformation ramps up as we look from left to right. The maximum deflection is at the right end of the beam, where the load P is applied. For the simulation, the steady state value of

Algorithm 1 Discrete Beam Algorithm Time Marching

```
% Define time marching steps
t = 0 : Δt : maxTime
% Compute f using bending energy and stretching energy
Compute f = f(q(t_{k+1}))
% Compute Jacobian using bending energy and stretching
% energy
Compute 𝕁 = f(q(t_{k+1}))
% Define free indices of f
f_free = f(free index)
% Define free indices of 𝕁
𝕁_free = 𝕁(free index, free index)
% Compute the changes in dof for free indices
Δq_free = 𝕁_free \ f_free
% Update the dof for free indices
New estimate q_free = Δq_free + q_free
New estimate q(free index) = q_free
% Update the dof for fixed indices
New estimate q(fixed index) = q_fixed
```

the maximum deflection is $y_{max} = -0.002480m$ For the theoretical prediction from Euler beam theory, $y_{max} = -0.0025m$. The percent error between these two values is 0.82%, thus showing the accuracy between both the theoretical and simulated methods for an applied load of this size to this beam.

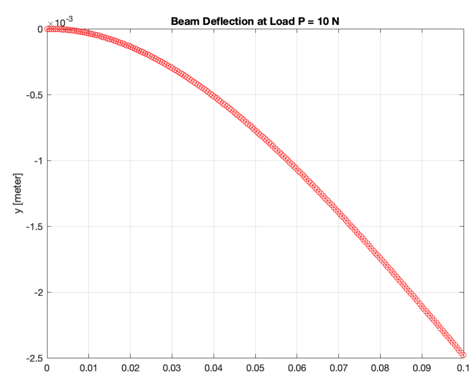

Fig. 2. Beam Deflection with Load P = 10N

IV. Discrete Elastic Beams vs Euler-Bernoulli Beam Theory

This section performs and discusses the simulation of the deformation of elastic beams and compares it with Euler-Bernoulli beam theory.

Figure 3 shows a plot comparing the maximum vertical displacements between the Discrete Elastic Beam method (simulation) and the Euler-Bernoulli beam theory (theoretical). This plot shows the maximum beam deflections after the simulation has reached a steady state value, ignoring the transient dynamics. From this plot, it is apparent that the two approaches begin diverging as the load increases.

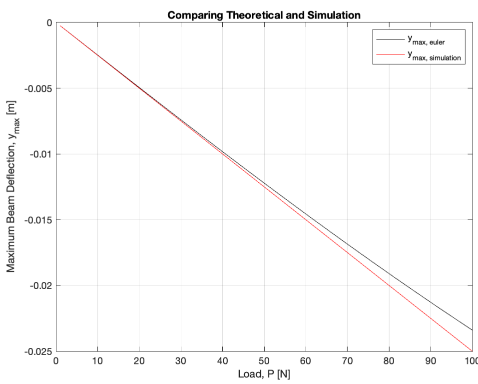

Fig. 3. Comparing Theoretical vs Simulated Maximum Vertical Displacements

Figures 4 and 5 show the percent error in calculating the maximum vertical displacements between these two methods, at different levels of zoom. We can observe that once the load reaches P = 124.7 N, the percent error grows to be greater than 10 %. It should be noted, that the exact value of the load where the percent error reaches 10 % is dependent on the number of nodes (N) defined in the simulation.

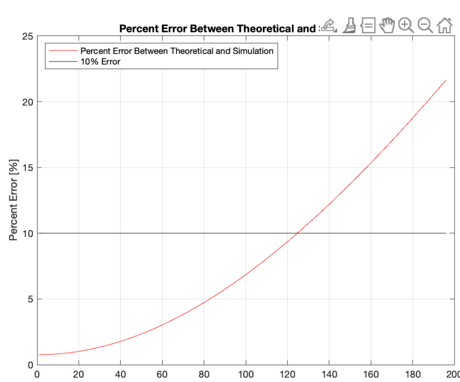

Fig. 4. Percent Error Between Theoretical vs Simulated Maximum Vertical Displacements

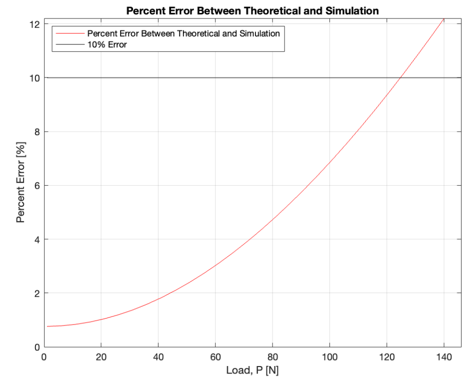

Fig. 5. Percent Error Between Theoretical vs Simulated Maximum Vertical Displacements - Zoomed In

As the value of P continues to increase past P = 124.7 N, the two methods continue to diverge at an increasing rate. This is a visualization of how the Euler-Bernoulli beam theory method cannot accurately support larger deformations. Euler-Bernoulli beam theory is only applicable when the slope is small. In the derivation of the Euler-Bernoulli beam theory, the following approximation is made

$$\kappa = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} \approx y'' \tag{1}$$

This approximation assumes that the length of the beam remains equal to the horizontal length of the beam as it undergoes deformation [3]. For small deformations, this is acceptable. However, once the value of P is large enough that it results in visible deformation, the slope is no longer small enough and the Euler-Bernoulli beam theory method is no longer accurate. However, this discrete elastic beam simulation is able to support much larger deformations and remain accurate.

REFERENCES

[1] Jawed, Khalid. "Lecture 3", 9 Oct 2023,
[2] Jawed, Khalid. "Lecture 4", 16 Oct 2023,
[3] Jawed, K., Lim, S. "Discrete Simulation of Slender Structures