Homework 2

Logan Hall

I. Problem 1 - Chapter 6 - Discrete Twist

This section pertains to the optional prompt from the homework assignment, relating to chapter 6: discrete twist. Note: My code is not provided for this problem, as instructed. The code for this section is inspired by UCLA MAE 263F lectures by Professor Khalid Jawed [1], [2], [3] and the course notes [4].

A. Problem 1.1 - Parallel Transport

This problem asks to calculate the integrated twist in the given rod. From the given parameters, we can calculate the initial twist by completing the following sequence. First, we compute the tangent vectors from the given degrees of freedom. Next, we perform parallel transport of $\mathbf{m}^{k-1}$ from $\mathbf{t}^{k-1}$ to $\mathbf{t}^k$. Then at edge $\mathbf{t}^k$, we calculate the signed angle from the parallel transported $\mathbf{m}^{k-1}$ to $\mathbf{m}^k$

The integrated twist is thus: $\tau_2 = 0$ at $\mathbf{x_2}$ and $\tau_3 = 0.3398$ at $\mathbf{x_3}$.

B. Problem 1.2 - Space-Parallel Reference Frame

This problem asks to construct a space-parallel reference frame for the given rod, and calculate the angle ν^k at each edge and to calculate the twist. To start, we once again compute the tangent vectors $\mathbf{t}^k$ from the given degrees of freedom. Then we perform parallel transport of frame director $\mathbf{u}^{k-1}$ from $\mathbf{t}^{k-1}$ to $\mathbf{t}^k$. Next, we calculate the signed angle between $\mathbf{u}^k$ and the material director $\mathbf{m}_1^k$ to get ν_k. Lastly, we calculate the difference between ν_k and ν_{k-1} to get τ_k.

The angles at each edge for this problem are $\nu^1 = 0$, $\nu^2 = 0$ and $\nu^3 = 0.3398$. The twist is $\tau_2 = 0$ at $\mathbf{x_2}$ and $\tau_3 = 0.3398$ at $\mathbf{x_3}$. This agrees with the results from problem 1.1.

C. Problem 1.3 - Calculation of Twist Using Time-Parallel Reference Frame

This problem provides the nodal coordinates and twist angles at three different times t = 0.00 sec, t = 0.05 sec, and t = 0.10 sec. First it asks to solve for the space parallel reference frame and the material frame for t = 0 s. Then, it asks to solve for the time-parallel reference frames and material frames at t = 0.05 and t = 0.10 s. It also asks to solve for the reference twist at t = 0, 0.05, and 0.10 sec and the integrated discrete twist.

To calculate the space parallel reference frame, we start by calculating the tangent vectors from the degrees of freedom. Then we calculate the reference frame by using space parallel transport. From the space parallel transport, we can then calculate the material directors from $\mathbf{a}_1$ and $\mathbf{a}_2$. Then we compute the time-parallel reference frame from the space parallel director $\mathbf{a}_1$ and both degree of freedom vectors (at t = 0.00 and t = 0.05). The material directors can then be calculated from the time-parallel reference frame directors a1, and a2, and the twist, θ. The time-parallel reference frame is identical to the initial reference frame for t = 0 s. At t = 0.05 s and t = 0.10 s, the time-parallel reference frame is calculated from the old reference director $\mathbf{a_1}$, the old dof vector $\mathbf{q_{j-1}}$ and the new dof vector $\mathbf{q_j}$ The reference twist is calculated as a function of

$$\Delta m_{\text{k,ref}} = signedAngle(P_{k-1}^k(\mathbf{a}_1^{k-1}(t_{j+1})), \mathbf{a}_1^k(t_{j+1}), \mathbf{t}^k(t_{j+1}))$$

The space-parallel reference frame and the time parallel reference frame at t = 0.00 sec:
$\mathbf{a}_1^1$ = [-0.8110 -0.5851 0.0000]
$\mathbf{a}_1^2$ = [0.2996 -0.9541 0.0000]
$\mathbf{a}_1^3$ = [0.9999 -0.0167 0.0000]
$\mathbf{a}_1^4$ = [0.3311 0.9436 0.0000]

The time parallel reference frame at t = 0.05 sec:
$\mathbf{a}_1^1$ = [-0.8110 -0.5851 0.0000]
$\mathbf{a}_1^2$ = [0.2838 -0.9589 -0.0031]
$\mathbf{a}_1^3$ = [1.0000 0.0079 0.0054]
$\mathbf{a}_1^4$ = [0.3708 0.9287 0.000281]

The time parallel reference frame at t = 0.10 sec:
$\mathbf{a}_1^1$ = [-0.8110 -0.5851 0.0000]
$\mathbf{a}_1^2$ = [0.2531 -0.9672 -0.0218]
$\mathbf{a}_1^3$ = [0.9982 0.0480 0.0371]
$\mathbf{a}_1^4$ = [0.4213 0.9069 -0.0035]

The material frame at t = 0.00 sec:
$\mathbf{m}_1^1$ = [-0.8110 -0.5851 0.0000]
$\mathbf{m}_2^1$ = [0.0000 0.0000 1.0000]
$\mathbf{m}_1^2$ = [0.2996 -0.9541 0.0000]
$\mathbf{m}_2^2$ = [0.0000 0.0000 1.0000]
$\mathbf{m}_1^3$ = [0.9999 -0.0167 0.0000]
$\mathbf{m}_2^3$ = [0.0000 0.0000 1.0000]
$\mathbf{m}_1^4$ = [0.3311 0.9436 0.0000]
$\mathbf{m}_2^4$ = [0.0000 0.0000 1.0000]

The material frame at t = 0.05 sec:
$\mathbf{m}_1^1$ = [-0.8110 -0.5851 0.0000]
$\mathbf{m}_2^1$ = [0.0000 0.0000 1.0000]
$\mathbf{m}_1^2$ = [0.3470 -0.9186 -.1888]
$\mathbf{m}_2^2$ = [-0.2823 -0.2943 0.9131]
$\mathbf{m}_1^3$ = [0.9757 -0.0834 0.2026]
$\mathbf{m}_2^3$ = [-0.2189 -0.4107 0.8851]
$\mathbf{m}_1^4$ = [0.3245 0.8290 0.4554]

$\mathbf{m}_2^4$ = [-0.1798 -0.4186 0.8902]

The material frame at t = 0.10 sec:
$\mathbf{m}_1^1$ = [-0.8110 -0.5851 0.0000]
$\mathbf{m}_2^1$ = [0.0000 0.0000 1.0000]
$\mathbf{m}_1^2$ = [0.4681 -0.8264 -0.3130]
$\mathbf{m}_2^2$ = [-0.4717 -0.5331 0.7023]
$\mathbf{m}_1^3$ = [0.9404 -0.2302 0.2505]
$\mathbf{m}_2^3$ = [-0.3351 -0.7540 0.5650]
$\mathbf{m}_1^4$ = [0.3538 0.4697 0.8088]
$\mathbf{m}_2^4$ = [-0.2668 -0.7781 0.5686]

The reference twist at t = 0.00 is:
$\Delta m_{2,\text{ref}} = 0.0000$
$\Delta m_{3,\text{ref}} = 0.0000$
$\Delta m_{4,\text{ref}} = 0.0000$

The reference twist at t = 0.05 sec is:
$\Delta m_{2,\text{ref}} = -0.2667$
$\Delta m_{3,\text{ref}} = -0.5650$
$\Delta m_{4,\text{ref}} = -0.2895$

The reference twist at t = 0.10 sec is:
$\Delta m_{2,\text{ref}} = -0.5050$
$\Delta m_{3,\text{ref}} = -1.02830$
$\Delta m_{4,\text{ref}} = -0.6649$

The discrete integrated twist is given by:

$$\tau_k = \Delta\theta_k + \Delta m_{k,ref}$$

The discrete integrated twist at t = 0.00 sec is:
$\tau_2 = 0.0000$
$\tau_3 = 0.0000$
$\tau_4 = 0.0000$

The discrete integrated twist at t = 0.05 sec is:
$\tau_2 = -0.4673$
$\tau_3 = -0.1453$
$\tau_4 = -0.0360$

The discrete integrated twist at t = 0.10 sec is:
$\tau_2 = -0.8958$
$\tau_3 = -0.2803$
$\tau_4 = -0.0605$

II. Problem 2 - Chapter 7 - Discrete Elastic Rods Algorithm

The section performs and discusses the simulation of the deformation of a rod under gravity. The code for this section is heavily inspired by and partly taken directly from UCLA MAE 263F lectures by Professor Khalid Jawed [1], [2], [3] and the UCLA MAE 263F course notes [4].

A. Problem 2.1 - Simulation

The simulation included in my github repository simulates the deformation of the rod under gravity that is described in the problem statement, from t = 0 to t = 5 seconds.

The simulation is performed by implementing the DER (Discrete Elastic Rods) algorithm. Page 41 of the course notes describes the psuedocode for the DER algorithm [4]. That psuedocode is appended to the end of this report. The implementation begins by defining inputs, such as the number of nodes, edges, degrees of freedom, rod geometry, material parameters and gravitational acceleration. Then it defines the stiffness variables, and the mass vector. Next, it defines the geometry based on the initial parameters and defines the initial degrees of freedom vector, where the twist is initially set to 0. Next, we calculate the edge lengths and from the edge lengths, we define the Voronoi lengths. From the gravity and mass vectors already defined, we now calculate the force due to gravity.

For the reference frames, we start at time t = 0. From the degree of freedom vector, we can calculate the tangent vectors, $\mathbf{t^k}$. We then pick an arbitrary frame $\mathbf{a_1^1}$ at the first edge, orthonormal to $\mathbf{t^1}$. Then the second director is calculated by taking the cross product between $\mathbf{t^1}$ and $\mathbf{a_1^1}$. Once the first directors are chosen, we than do space parallel transport to construct the reference frame. We do this by taking the vector $\mathbf{a_1^{k-1}}$ and transporting it from $\mathbf{t^{k-1}}$ to $\mathbf{t^k}$.

Next, we can calculate the material directors from $\mathbf{a_1}$, $\mathbf{a_2}$, and the twist angle θ. After next defining the boundary conditions by fixing the first 7 degrees of freedom, we begin the time marching scheme.

Here is where we implement the core of the DER algorithm. The goal for each time step is to calculate the new degree of freedom vector ($\mathbf{q}$), the velocity between time steps ($\dot{\mathbf{q}}$), and the new time-parallel reference directors ($\mathbf{a_1}$ and $\mathbf{a_2}$) at the new time step. We start start the iteration of Newton's method like usual, with an initial guess of $\mathbf{q}$ and by defining our tolerance to be sufficiently small, such that our approximation is close enough. In the time marching scheme, we first calculate the time-parallel reference directors $\mathbf{a_1}$ and $\mathbf{a_2}$ from the old reference director $\mathbf{a_1}$, and the old and current degree of freedom vectors. We then calculate the reference twist $\Delta m_{k,ref}^{(n)}$. Next, we calculate the material frame by defining the material directors $\mathbf{m}_1^k(t_{j+1})$ and $\mathbf{m}_2^k(t_{j+1})$. After this, we need to calculate the forces from the equations of motion and Jacobians associated with bending, twisting, stretching. Next we calculate our values of $\mathbf{f}$ and J and isolate the indices associated with the free degrees of freedom. We then calculate our $\Delta\mathbf{q}_{free} = J_{free}\backslash\mathbf{f}_{free}$. Then we update the values of our free degrees of freedom, while leaving our fixed degrees of freedom as they were. Then we check the error of $\mathbf{f}_{free}$ and if it is above our tolerance, we reimpliment this implicit method. We continue reimplimenting this method until the error is below our tolerance. Once the error is below that tolerance, we will have the degree of freedom vector at time t_{k+1}. Knowing this and our Δt, we can calculate the velocities ($\dot{\mathbf{q}}$) at t_{k+1}. Through our final iteration at each time step, we have calculated the time-parallel reference directors $\mathbf{a_1}$ and $\mathbf{a_2}$. Thus, we can now return $\mathbf{q}, \dot{\mathbf{q}}, \mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})$, as required. Once these values are calculated, my code will plot the data at t = 0.01 seconds, t = 0.05 seconds, and then every t = 0.05 seconds until t = 5.00 seconds.

The below figures (figures 1-21) show the plots of the simulation at time t = 0.01 seconds, then every 0.05 seconds for the first 0.75 seconds, and then at every whole second interval (t = 1.00, 2.00, 3.00, 4.00, 5.00 seconds). All of these figures are set with the same scale x, y, and z axis and are shown from the same isometric view for ease of comparison. These figures show the oscillatory nature of the deformation of the rod. The boundary conditions set the first two nodes and the first twist to be fixed. When released from it's initial configuration, the rod accelerates downward, with the free end accelerating the fastest. Then it begins to decelerate until it reaches a maximum deformation. It then continues this accelerating/decelerating cycle, as it oscillates up and down in the z-direction. As time progresses, the magnitudes of each oscillation decrease until it approaches a steady state. This deformation of the rod is shown in the figures below.

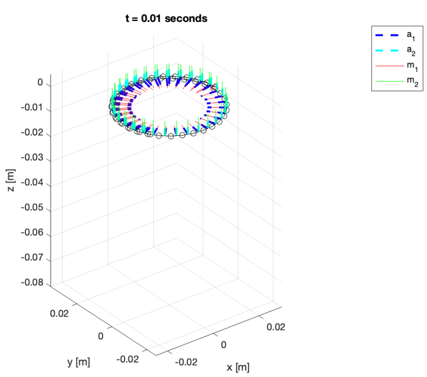

Fig. 1. Plot of Given Elastic Rod at Time t = 0.01 Seconds

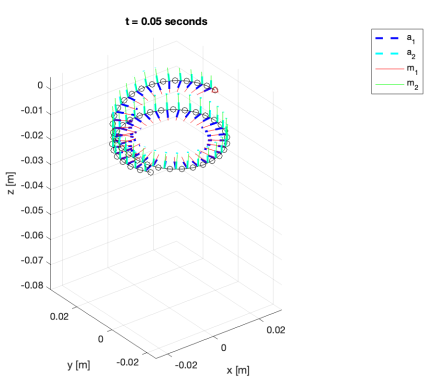

Fig. 2. Plot of Given Elastic Rod at Time t = 0.05 Seconds

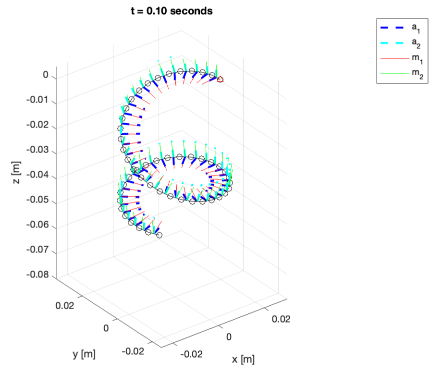

Fig. 3. Plot of Given Elastic Rod at Time t = 0.10 Seconds

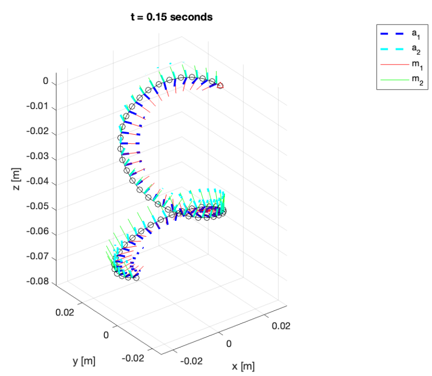

Fig. 4. Plot of Given Elastic Rod at Time t = 0.15 Seconds

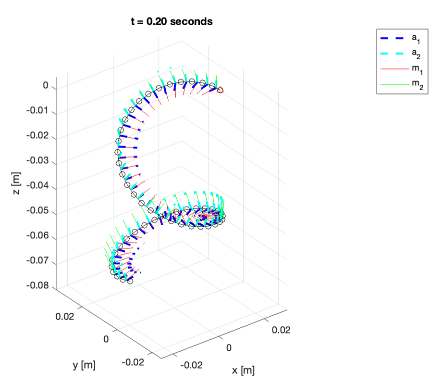

Fig. 5. Plot of Given Elastic Rod at Time t = 0.20 Seconds

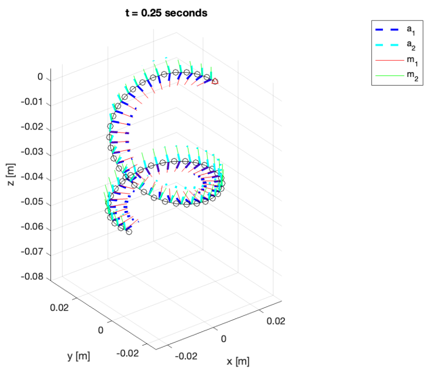

Fig. 6. Plot of Given Elastic Rod at Time t = 0.25 Seconds

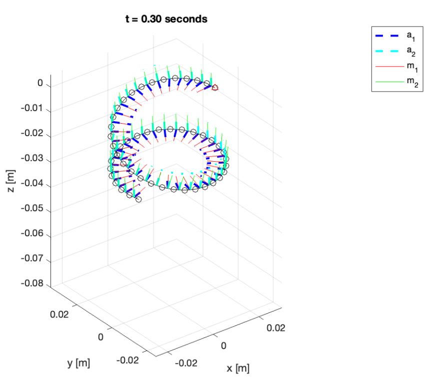

Fig. 7. Plot of Given Elastic Rod at Time t = 0.30 Seconds

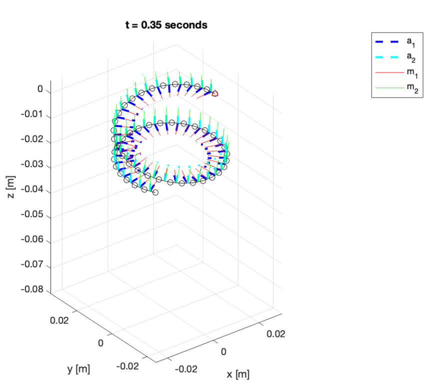

Fig. 8. Plot of Given Elastic Rod at Time t = 0.35 Seconds

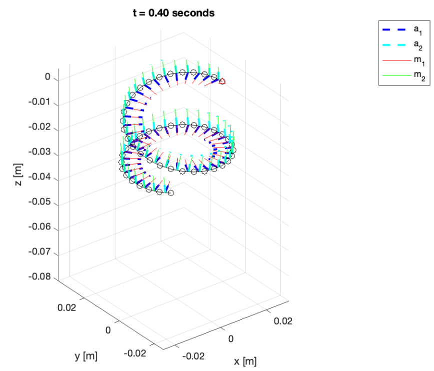

Fig. 9. Plot of Given Elastic Rod at Time t = 0.40 Seconds

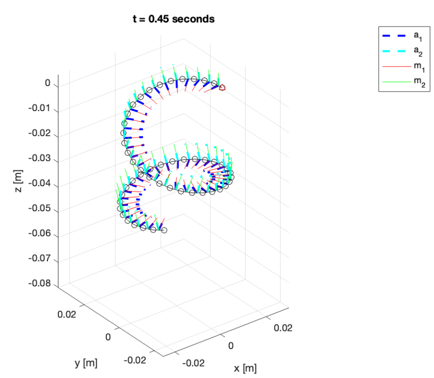

Fig. 10. Plot of Given Elastic Rod at Time t = 0.45 Seconds

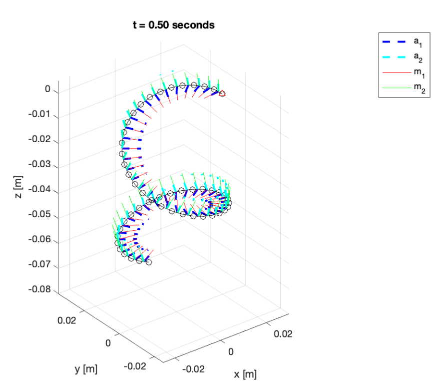

Fig. 11. Plot of Given Elastic Rod at Time t = 0.50 Seconds

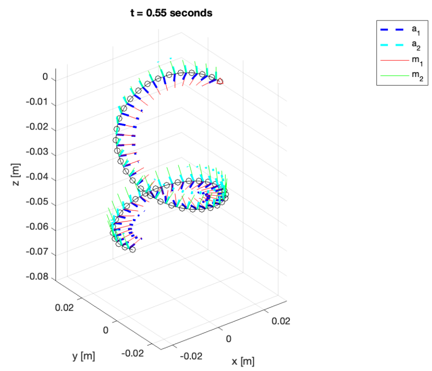

Fig. 12. Plot of Given Elastic Rod at Time t = 0.55 Seconds

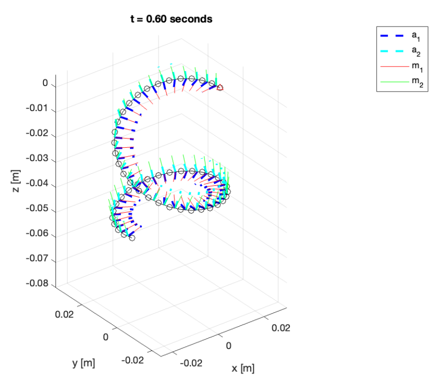

Fig. 13. Plot of Given Elastic Rod at Time t = 0.60 Seconds

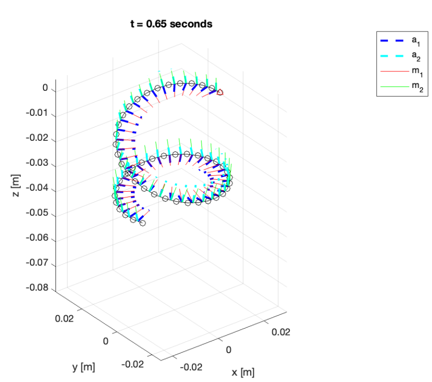

Fig. 14. Plot of Given Elastic Rod at Time t = 0.65 Seconds

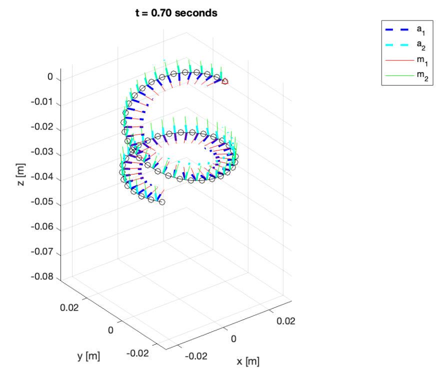

Fig. 15. Plot of Given Elastic Rod at Time t = 0.70 Seconds

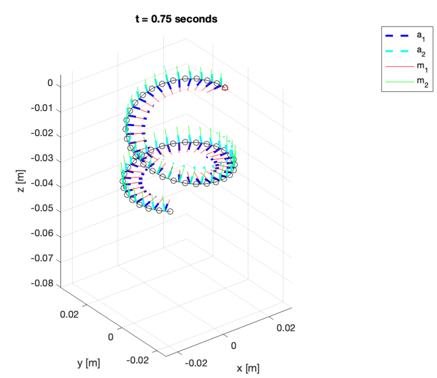

Fig. 16. Plot of Given Elastic Rod at Time t = 0.75 Seconds

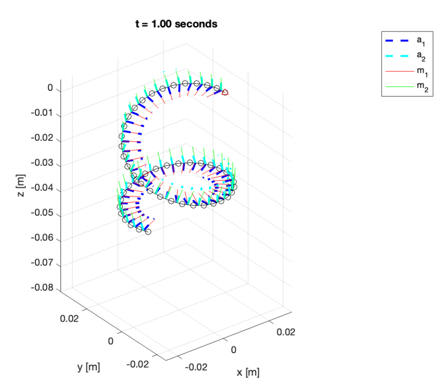

Fig. 17. Plot of Given Elastic Rod at Time t = 1.00 Seconds

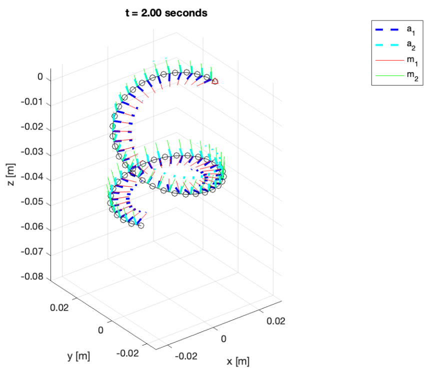

Fig. 18. Plot of Given Elastic Rod at Time t = 2.00 Seconds

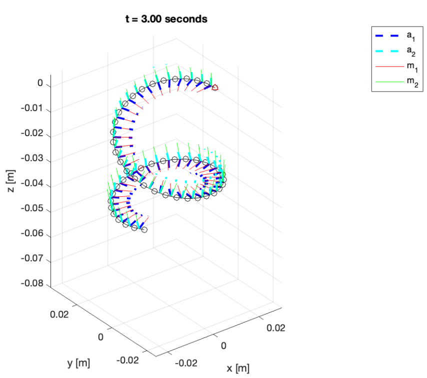

Fig. 19. Plot of Given Elastic Rod at Time t = 3.00 Seconds

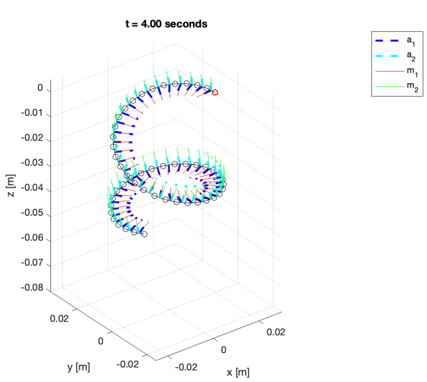

Fig. 20. Plot of Given Elastic Rod at Time t = 4.00 Seconds

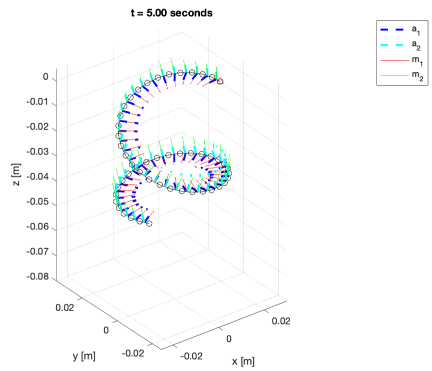

Fig. 21. Plot of Given Elastic Rod at Time t = 5.00 Seconds

B. Problem 2.2 - z-coordinate

The below figure (figure 22) shows how the z-coordinate of the last node changes with time. This plot is an excellent visualization of the oscillatory nature of how the rod deforms. The z coordinate of the last node undergoes the most change of any coordinate in this rod during the simulation. The speed can be seen on the graph by looking at the distance between adjacent points. Since the time step (Δt) is constant, the larger the gap there is between two points, the higher the speed. Thus, it is seen that it accelerates as it moves away from the peaks, and decelerates as it approaches each new peak. In the early stages of this simulation, the position of this last node has a larger oscillation and is moving at a greater speed when compared to the end (t=5 seconds), where the points are approaching linearity with time and are very close together. This indicates that it is approaching steady-state. By the end, it has approached a steady state value of $\delta_z \approx$-0.04 m. This simulation was able to converge to that value because it had a sufficiently large enough N and a sufficiently small enough Δt.

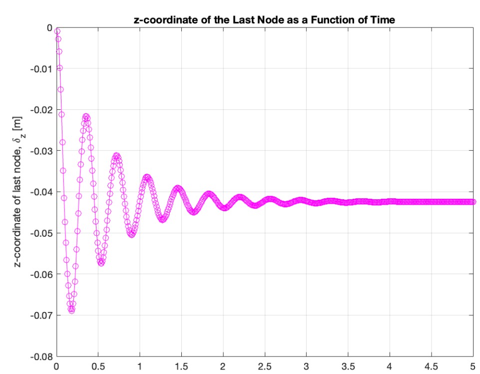

Fig. 22. Plot of Given Elastic Rod After Time t = 5 Seconds

REFERENCES

[1] Jawed, Khalid. "Lecture 4", 16 Oct 2023
[2] Jawed, Khalid. "Lecture 5", 18 Oct 2023
[3] Jawed, Khalid. "Lecture 6", 23 Oct 2023
[4] Jawed, K., Lim, S. "Discrete Simulation of Slender Structures