2D S_N Radiation Transport with Diffusion Acceleration

MATH 676 – Final Presentation

Logan H. Harbour

Department of Nuclear Engineering
Texas A&M University

NUCLEAR ENGINEERING
TEXAS A&M UNIVERSITY

One-group Linear Boltzmann Equation

Start with the one-group S_N transport equation for a single direction d (neglecting boundary conditions for simplicity), as

$$\mathbf{\Omega}_d \cdot \nabla \psi_d(\mathbf{x}) + (\sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x}))\,\psi_d(\mathbf{x}) - \frac{\sigma_s(\mathbf{x})}{2\pi} \sum_{d=1}^{N_\Omega} \omega_d \psi_d(\mathbf{x}) = q(\mathbf{x})\,, \qquad (1)$$

Let $\mathbb{T}_h$ be the set of all cells of the triangulation in a discontinuous approximation space. The DG weak form with test function v_d is

$$\sum_{K \in \mathbb{T}_h} \Big[(-\mathbf{\Omega}_d \cdot \nabla v_d, \psi_d)_K + \left(\psi_d^+ \mathbf{\Omega}_d \cdot \mathbf{n}, v_d\right)_{\delta K} + (\sigma_t \psi_d, v_d)_K - (\sigma_s \phi, v_d)_K = (q, v_d)_K \Big]\,, \qquad (2)$$

where ϕ is the *scalar flux*, $\phi = \frac{1}{2\pi} \sum_d^{N_\Omega} \omega_d \psi_d$, and ψ_d^+ is the upwind value of ψ_d (the value from the side of the face in which $\mathbf{\Omega} \cdot \mathbf{n} \geq 0$).

Source Iteration

To solve, cast Eq. (1) with iterative index ℓ as

$$\mathbf{\Omega}_d \cdot \nabla \psi_d^{(\ell+1)} + \sigma_t \psi_d^{(\ell+1)} = \sigma_s \phi^{(\ell)} + q \,, \tag{3}$$

where ℓ is the iterative index, $\psi_d^{(0)} = \phi^{(0)} = \vec{0}$. After solving each direction, d, for an iteration ℓ in Eq. (3), update the scalar flux with

$$\phi^{(\ell+1)} = \frac{1}{2\pi} \sum_{d=1}^{N_\Omega} w_d \psi_d^{(\ell+1)} \,.$$

As $\sigma_s/\sigma_t \to 1$, particles scatter more before they are absorbed $\to$ **the number of source iterations becomes significant!**

Diffusion Acceleration

Simple algebraic manipulations can show that the error in $\psi^{\ell+1}$ satisfies the transport equation with a source equal to:

$$R^{\ell+1} = \frac{\sigma_s}{2\pi}(\phi^{\ell+1} - \phi^{\ell})\,.$$

Fourier analysis shows that the angular flux error has a linearly anisotropic angular dependence. The diffusion approximation is exact for such a dependence, therefore we can cast the diffusion problem with the source above to form an error equation with the diffusion approximation that will attenuate the errors most poorly attenuated by the transport solve. The approximation is cast as:

$$-\nabla \cdot D\nabla\delta e^{\ell+1} + \sigma_a\delta e^{\ell+1} = \sigma_s\left(\phi^{\ell+1} - \phi^{\ell}\right)\,, \tag{4}$$

where $e^{\ell+1}$ is the approximated error in $\phi^{\ell+1}$ and $D = 1/3\sigma_t$.

Diffusion Acceleration (cont.)

Casting Equation (4) in the same DG space with interior edges $\mathcal{E}_h^i$ and boundary edges $\mathcal{E}_h^b$ using a modified interior penalty method for the face terms we obtain

$$\begin{aligned}\int_{\mathbb{T}_h} (D\nabla e \cdot \nabla v + \sigma_a e v) + \int_{\mathcal{E}_h^i} (\{\!\{D\delta_n e\}\!\}[\![v]\!] + \{\!\{D\delta_n v\}\!\}[\![e]\!] + \kappa[\![e]\!][\![v]\!]) \\ + \int_{\mathcal{E}_h^b} (\kappa e v - D v \delta_n e - D e \delta_n v) = \int_{\mathbb{T}_h} (\phi^{\ell+1} - \phi^{\ell}) v\,, \quad (5)\end{aligned}$$

where

$$\{\!\{u\}\!\} \equiv \frac{u^+ + u^-}{2} \quad \text{and} \quad [\![u]\!] \equiv u^+ - u^-\,,$$

in which the penalty coefficient is

$$\kappa = \begin{cases} 2\left(\frac{D^+}{h_\perp^+} + \frac{D^-}{h_\perp^-}\right) & \text{for interior edges}\,, \\ 8\frac{D^-}{h_\perp^-} & \text{for boundary edges}\,, \end{cases}$$

and $h_\perp^\pm$ is a characteristic length of the cell in the direction orthogonal to the edge.

Diffusion Acceleration Results

Consider $\mathcal{D} = [0, 10]^2$, $N_\Omega = 20$, $q = 1$, $\sigma_a + \sigma_s = \sigma_t = 100$, and 16^2 elements. Increase the scattering ratio, σ_s/σ_t with and without diffusion acceleration and observe the results: