MATH 676 PROJECT: S$_N$ SUMMARY

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1 Problem definition

Begin with the spatial domain $\mathcal{D} \in \mathbb{R}^2$ in which $\delta\mathcal{D}$ is on the boundary of $\mathcal{D}$. The set of propagation directions $\mathcal{S}$ is the unit disk.

The linear Boltzmann equation for one-group transport is

$$\mathbf{\Omega} \cdot \nabla\Psi(\mathbf{\Omega}, \mathbf{x}) + \sigma_t(\mathbf{x})\Psi(\mathbf{\Omega}, \mathbf{x}) - \sigma_s(\mathbf{x})\Phi(\mathbf{x}) = q(\mathbf{x})\,, \qquad \forall(\mathbf{\Omega}, \mathbf{x}) \in \mathcal{S} \times \mathcal{D}\,, \tag{1.1a}$$

$$\Phi(\mathbf{\Omega}, \mathbf{x}) = \Phi^{\text{inc}}(\mathbf{\Omega}, \mathbf{x})\,, \qquad \forall(\mathbf{\Omega}, \mathbf{x}) \in \mathcal{S} \times \delta\mathcal{D}\,,\ \mathbf{\Omega} \cdot \mathbf{n}(\mathbf{x}) < 0\,, \tag{1.1b}$$

where Φ is the scalar flux, defined by

$$\Phi = \frac{1}{2\pi} \int_{\mathcal{S}} \Phi(\mathbf{\Omega}, \mathbf{x}) d\Omega\,.$$

2 S$_N$ discretization

Introduce the S$_N$ discretization, which replaces the angular flux with a discrete angular flux, as

$$\psi(\mathbf{x}) = [\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \dots \psi_{N_\Omega}(\mathbf{x})]^T\,. \tag{2.1}$$

We then introduce the quadrature rule $\{(\mathbf{\Omega}_d, \omega_d), d = 1, \dots, N_\Omega\}$ where $\sum_d \omega_d = 2\pi$. With said quadrature rule, we have

$$\int_{\mathcal{S}} f(\mathbf{\Omega}, \mathbf{x}) d\Omega \approx \sum_{d=1}^{N_\Omega} w_d f(\mathbf{\Omega}_d, \mathbf{x})\,.$$

This discretization allows us to write the system in Equation (1.1) as

$$\mathbf{\Omega}_d \cdot \nabla\psi_d(\mathbf{x}) + \sigma_t(\mathbf{x})\psi_d(\mathbf{x}) - \sigma_s(\mathbf{x})\phi(\mathbf{x}) = q(\mathbf{x})\,, \qquad \forall \mathbf{x} \in \mathcal{D} \tag{2.2a}$$

$$\psi_d(\mathbf{x}) = \Psi_j^{\text{inc}}(\mathbf{x})\,, \qquad \forall \mathbf{x} \in \delta\mathcal{D}\,,\ \mathbf{\Omega}_d \cdot \mathbf{n}(\mathbf{x}) < 0\,, \tag{2.2b}$$

where the discrete scalar flux, ϕ, is

$$\phi(\mathbf{x}) = \frac{1}{2\pi} \sum_{d=1}^{N_\Omega} w_j \psi_j(\mathbf{x})\,.$$

3 Discontinuous Galerkin discretization

Define $\mathbb{T}_h$ as the set of all active cells of the triangulation for $\mathcal{D}$ and $\mathbb{F}_h$ as the set of all active interior faces. Define a discontinuous approximation space for the scalar flux based on the mesh $\mathbb{T}_h$ as

$$V_h \in \{v \in L^2(\mathcal{D}) \mid \forall K \in \mathbb{T}_h\,, v|_K \in P_K\}\,, \tag{3.1}$$

where the finite-dimensional space P_K is assumed to contain $\mathbb{P}_k$, the set of polynomials of degree at most k. The discrete space for the angular flux consists of copies of V_h for each of the discrete ordinates, as

$$W_h = (V_h)^{N_\Omega}\,. \tag{3.2}$$

Multiply Equation (2.2b) by the test function $v_d \in V_h$ and integrate as

$$\sum_{K \in \mathbb{T}_h} \left[(\mathbf{\Omega}_d \cdot \nabla \psi_d, v_d)_K + (\sigma_t \psi_d, v_d)_K - (\sigma_s \phi, v_d)_K = (q, v_d)_K \right] , \tag{3.3}$$

and integrate the first term by parts to obtain

$$\sum_{K \in \mathbb{T}_h} \left[(-\mathbf{\Omega}_d \cdot \nabla v_d, \psi_d)_K + (\psi_d \mathbf{\Omega}_d \cdot \mathbf{n}, v_d)_{\delta K} + (\sigma_t \psi_d, v_d)_K - (\sigma_s \phi, v_d)_K = (q, v_d)_K \right] , \tag{3.4}$$

where $\mathbf{n}$ is the outward normal. Note that the surface integration in Equation (3.4) is double-valued due to the discontinuous approximation. We introduce the upwind approximation

$$\psi_d \mathbf{\Omega}_d \cdot \mathbf{n} = \psi_d^+ \mathbf{\Omega}_d \cdot \mathbf{n} \, , \tag{3.5}$$

where ψ_d^+ is the upwind value of ψ_d, that is, the value from the side of the face in which $\mathbf{\Omega} \cdot \mathbf{n} \geq 0$. The weak form is then defined as

$$\sum_{K \in \mathbb{T}_h} \left[(-\mathbf{\Omega}_d \cdot \nabla v_d, \psi_d)_K + \left(\psi_d^+ \mathbf{\Omega}_d \cdot \mathbf{n}, v_d \right)_{\delta K} + (\sigma_t \psi_d, v_d)_K - (\sigma_s \phi, v_d)_K = (q, v_d)_K \right] . \tag{3.6}$$