Discrete Ordinates Radiation Transport with Diffusion Acceleration

MATH 676 – Final Presentation

Logan H. Harbour

Department of Nuclear Engineering
Texas A&M University

Introduction

Background

- My Ph.D. work involves various acceleration techniques for method of characteristics (MOC) radiation transport
- Some acceleration methods for discrete ordinates (S_N) transport are similar to those in MOC transport
- In specific, the focus is diffusion synthetic acceleration (DSA), which attenuates the errors most poorly attenuated by source iteration

Goals

- Develop a one-group, DGFEM, S_N radiation transport code
- Develop a one-group, DGFEM, diffusion radiation transport code
- Accelerate the S_N source iterations with DSA using the diffusion code
- More... if time permits (which it did!)

One-group Linear Boltzmann Equation

Start with the one-group S_N transport equation for a single direction d (neglecting boundary conditions for simplicity), as

$$\boldsymbol{\Omega}_d \cdot \nabla \psi_d(\mathbf{x}) + \left(\sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})\right) \psi_d(\mathbf{x}) - \frac{\sigma_s(\mathbf{x})}{2\pi} \sum_{d=1}^{N_\Omega} \omega_d \psi_d(\mathbf{x}) = q(\mathbf{x}) \,, \qquad (1)$$

Let $\mathbb{T}_h$ be the set of all cells of the triangulation in a discontinuous approximation space. The DG weak form with test function v_d is

$$\sum_{K \in \mathbb{T}_h} \Big[\left(-\boldsymbol{\Omega}_d \cdot \nabla v_d, \psi_d\right)_K + \left(\psi_d^+ \boldsymbol{\Omega}_d \cdot \mathbf{n}, v_d\right)_{\delta K} + \left(\sigma_t \psi_d, v_d\right)_K - \left(\sigma_s \phi, v_d\right)_K = \left(q, v_d\right)_K \Big] \,, \qquad (2)$$

where ϕ is the *scalar flux*, $\phi = \sum_d^{N_\Omega} w_d \psi_d$, and ψ_d^+ is the upwind value of ψ_d (the value from the side of the face in which $\boldsymbol{\Omega} \cdot \mathbf{n} \geq 0$).

Source Iteration

To solve, cast Eq. (1) with iterative index ℓ as

$$\mathbf{\Omega}_d \cdot \nabla \psi_d^{(\ell+1)} + \sigma_t \psi_d^{(\ell+1)} = \sigma_s \phi^{(\ell)} + q\,, \tag{3}$$

where ℓ is the iterative index, $\psi_d^{(0)} = \phi^{(0)} = \vec{0}$. After solving each direction, d, for an iteration ℓ in Eq. (3), update the scalar flux with

$$\phi^{(\ell+1)} = \sum_{d=1}^{N_\Omega} w_d \psi_d^{(\ell+1)}\,.$$

As $\sigma_s/\sigma_t \to 1$, particles scatter more before they are absorbed $\to$ **the number of source iterations becomes significant!**

Diffusion Acceleration

Simple algebraic manipulations can show that the error in $\psi^{\ell+1}$ satisfies the transport equation with a source equal to:

$$R^{\ell+1} = \frac{\sigma_s}{2\pi}(\phi^{\ell+1} - \phi^{\ell})\,.$$

Fourier analysis shows that the angular flux error has a linearly anisotropic angular dependence. The diffusion approximation is exact for such a dependence $\rightarrow$ cast the diffusion problem to form an error equation with the diffusion approximation that will attenuate the errors most poorly attenuated by the transport solve.

Said approximation is cast as:

$$-\nabla \cdot D \nabla e^{\ell+1} + \sigma_a e^{\ell+1} = \sigma_s \left(\phi^{\ell+1} - \phi^{\ell}\right)\,, \tag{4}$$

where $e^{\ell+1}$ is the approximated error in $\phi^{\ell+1}$ and $D = 1/3\sigma_t$.

Diffusion Acceleration (cont.)

Casting Equation (4) in the same DG space with interior edges $\mathcal{E}_h^i$ and boundary edges $\mathcal{E}_h^b$ using a modified interior penalty method for the face terms we obtain

$$\int_{\mathbb{T}_h} (D\nabla e \cdot \nabla v + \sigma_a ev) + \int_{\mathcal{E}_h^i} (\{\!\{D\delta_n e\}\!\}[\![v]\!] + \{\!\{D\delta_n v\}\!\}[\![e]\!] + \kappa[\![e]\!][\![v]\!]) \\ + \int_{\mathcal{E}_h^b} (\kappa ev - Dv\delta_n e - De\delta_n v) = \int_{\mathbb{T}_h} (\phi^{\ell+1} - \phi^{\ell})v\,, \quad (5)$$

where

$$\{\!\{u\}\!\} \equiv \frac{u^+ + u^-}{2} \quad \text{and} \quad [\![u]\!] \equiv u^+ - u^-\,,$$

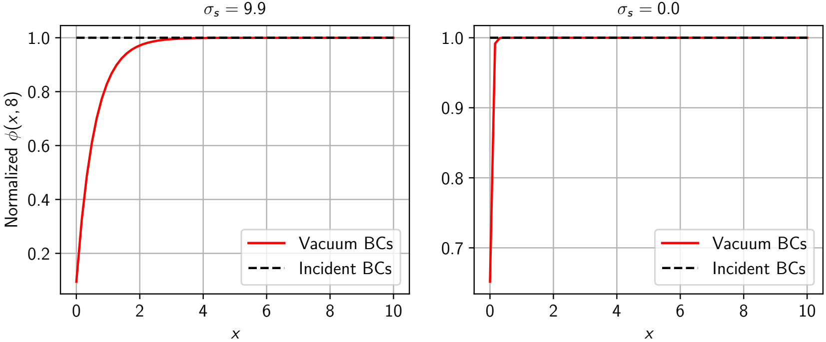
in which the penalty coefficient is

$$\kappa = \begin{cases} 2\left(\frac{D^+}{h_\perp^+} + \frac{D^-}{h_\perp^-}\right) & \text{for interior edges}\,, \\ 8\frac{D^-}{h_\perp^-} & \text{for boundary edges}\,, \end{cases}$$

and $h_\perp^\pm$ is a characteristic length of the cell in the direction orthogonal to the edge.

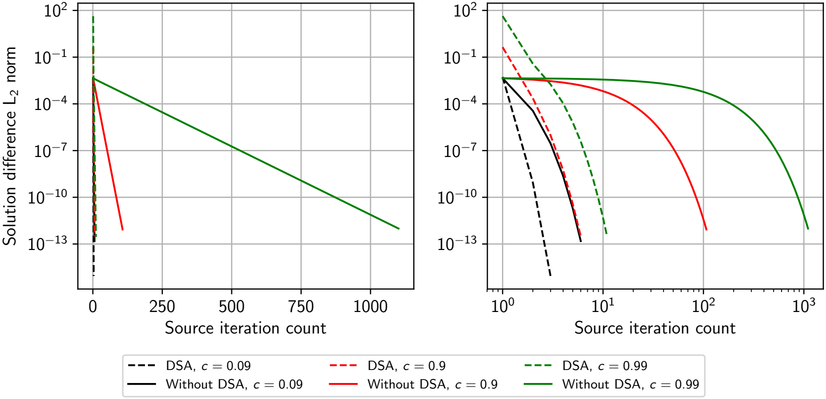
Constant Solution Verification

Consider $\mathcal{D} = [0, 10]^2$, $N_\Omega = 4$, $q = 1$, $\sigma_t = 10$, and 64^2 elements. Top and right boundary conditions are reflective. Bottom and left boundary conditions are either vacuum or incident isotropic flux of q/σ_a.



Diffusion Acceleration Results

Consider $\mathcal{D} = [0, 10]^2$, $N_\Omega = 20$, $q = 1$, $\sigma_a + \sigma_s = \sigma_t = 100$, and 16^2 elements. Increase the scattering ratio, $c = \sigma_s / \sigma_t$, with and without diffusion acceleration and observe the results:



Additional Goals Completed

The primarily implementation (transport with diffusion acceleration) was completed earlier than expected. Therefore, additional goals were added to round out the project:

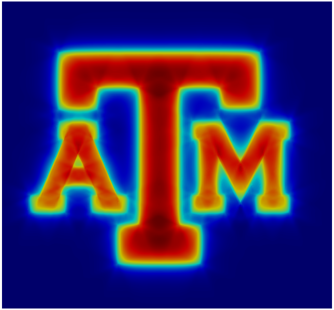
Parallel support

- Supports parallel solves using MPI and Trillinos wrappers, completed primarily by following step-40
- Transport is solved with GMRES and the AMG preconditioner
- Diffusion is solved with CG and the AMG preconditioner

Reflecting boundary conditions

- Reflecting boundaries require storing the outgoing flux on the boundaries and then reflecting on the incoming boundaries (bit of a pain, but it works)
- Also supported in the diffusion acceleration scheme through adding an additional source term for boundary flux error

Conclusions



- S_N and diffusion codes completed as desired
- Diffusion acceleration was completed
- Both were verified using simple test cases
- Additional goals were met: parallelism, reflecting boundary conditions
- Had a lot of fun digging through Deal.ii tutorials and Doxygen
- Looking forward to using this code as a simple test bed in the future

Thank you for your time ☺