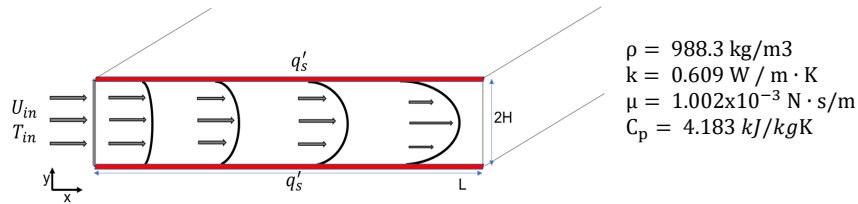


MEEN 644 - Homework 4

Logan Harbour

April 15, 2019

1 Problem statement



Consider an incompressible laminar flow between two infinite parallel plates. Flow enters at constant velocity u_{in} and a constant heat flux, $q' = 500 \text{ W/m}^2$ is applied to each wall. Height of channel $2H$ and length of channel L are 0.02 m and 2.0 m, respectively. Constant velocity u_{in} and temperature $T_{\text{in}} = 25^\circ\text{C}$ is set as the inlet condition. Reynolds number is defined as $Re = 2\rho u_{\text{in}} H / \mu$.

1. **(10 points)** Specify your boundary condition for u, v, P , and T .
2. **(50 points)** Write a finite volume method code to predict velocity, pressure, and temperature profiles for $Re = 100$. Employ the Power Law scheme to represent a solution to a 1-D convection-diffusion equation. Use the SIMPLE algorithm to link velocity and pressure fields. Use 10 uniformly sized CVs in the x -direction and 5 uniformly sized CVs in the y -direction. Declare convergence at R_u, R_v, R_P and $R_T < 10^{-6}$.
3. **(10 points)** Run your program with the 10×5 grid and tabulate $u, (P - P_0)$, and T to check for symmetry. P_0 is the pressure at the outlet.
4. **(30 points)** Use a 180×54 grid to solve for u, v, P , and T .
 - (a) Plot u/u_{in} along the centerline of the channel (as a function of x).
 - (b) Plot u, v , and T at $x = 0.8 \text{ m}$ (as a function of y).
 - (c) Plot the Nusselt number as a function of stream-wise distance x from the channel entrance.

2 Preliminaries

2.1 Boundary conditions

With two-dimensional diffusion and convection with constant material properties, we have the boundary conditions

$$\left\{ \begin{array}{l} u(x, L_y) = 0, \\ \frac{\partial u}{\partial x} \big|_{(L_x, y)} = 0, \\ u(x, 0) = 0, \\ u(0, y) = \frac{Re\mu}{\rho L_y}, \\ v(x, L_y) = 0, \\ v(L_x, y) = 0, \\ v(x, 0) = 0, \\ v(0, y) = 0, \\ -k \frac{\partial T}{\partial y} \big|_{(x, L_y)} = 500 \text{ W/m}^2, \\ T(0, y) = 25 \text{ }^\circ\text{C}, \\ -k \frac{\partial T}{\partial y} \big|_{(x, 0)} = 500 \text{ W/m}^2, \\ \frac{\partial T}{\partial x} \big|_{(L_x, y)} = \text{constant}, \end{array} \right. \quad (1)$$

Note that as velocity is prescribed on the boundary, we do not have boundary conditions for P . However, special care is taken on the pressure boundary in the algorithm in that we set P' on the boundary to be zero.

2.2 Solving methodology

The wide majority of the solving methodology (utilizing the SIMPLE algorithm) remains the same from the previous homework (homework 4), therefore said description will not be repeated in this report. The primary addition is the solve of the auxiliary equation (for temperature) once the primary solutions have been obtained (u, v , and P). The discussion of solving the temperature equation follows.

Recall again that the the domain of size $L_x \times L_y$ is discretized into $N_x \times N_y$ uniformly sized control volumes with $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$. The numbering for all variables begins at the origin at $(i, j) = (0, 0)$. The maximum index for each variable, ϕ , is defined as (M_x^ϕ, M_y^ϕ) .

2.3 Temperature solve

The domain of size $L_x \times L_y$ is discretized into $N_x \times N_y$ uniformly sized control volumes with $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$. The numbering for all variables begins at the origin at $(i, j) = (0, 0)$. The maximum index for each variable, ϕ , is defined as (M_x^ϕ, M_y^ϕ) . The temperature control volume is similar to a pressure control volume, as follows in Figure 1.

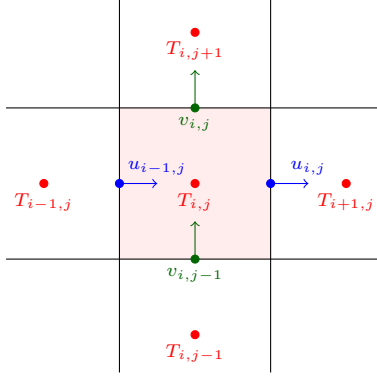


Figure 1: An internal temperature control volume.

The integration of the energy equation equations using the power-law scheme results in the equation (for $i = 1, \dots, M_x^T - 1$, $j = 1, \dots, M_y^T - 1$)

$$a_p^{T_{i,j}} T_{i,j} = a_n^{\phi_{i,j}} T_{i,j+1} + a_e^{\phi_{i,j}} T_{i,j-1} + a_s^{T_{i,j}} T_{i,j}^* + a_w^{T_{i,j}} T_{i-1,j}, \quad (2a)$$

$$a_n^{T_{i,j}} = D_n^{T_{i,j}} \max [0, (1 - 0.1 |F_n^{T_{i,j}} / D_n^{T_{i,j}}|)^5] + \max [-F_n^{T_{i,j}}, 0], \quad (2b)$$

$$a_e^{T_{i,j}} = D_e^{T_{i,j}} \max [0, (1 - 0.1 |F_e^{T_{i,j}} / D_e^{T_{i,j}}|)^5] + \max [-F_e^{T_{i,j}}, 0], \quad (2c)$$

$$a_s^{T_{i,j}} = D_s^{T_{i,j}} \max [0, (1 - 0.1 |F_s^{T_{i,j}} / D_s^{T_{i,j}}|)^5] + \max [F_s^{T_{i,j}}, 0], \quad (2d)$$

$$a_w^{T_{i,j}} = D_w^{T_{i,j}} \max [0, (1 - 0.1 |F_w^{T_{i,j}} / D_w^{T_{i,j}}|)^5] + \max [F_w^{T_{i,j}}, 0], \quad (2e)$$

$$a_p^{T_{i,j}} = a_n^{T_{i,j}} + a_e^{T_{i,j}} + a_s^{T_{i,j}} + a_w^{T_{i,j}}. \quad (2f)$$

The flow rates are defined as

$$F_n^{T_{i,j}} = \rho C_p \Delta x v_{i,j}, \quad (3a)$$

$$F_e^{T_{i,j}} = \rho C_p \Delta x u_{i,j}, \quad (3b)$$

$$F_s^{T_{i,j}} = \rho C_p \Delta x v_{i,j-1}, \quad (3c)$$

$$F_w^{T_{i,j}} = \rho C_p \Delta x u_{i-1,j}, \quad (3d)$$

and the “diffusion” coefficients are defined as

$$D_n^{T_{i,j}} = k \begin{cases} \frac{2\Delta x}{\Delta y}, & j = M_y^T - 1 \\ \frac{\Delta x}{\Delta y} & \text{otherwise} \end{cases}, \quad (4a)$$

$$D_e^{T_{i,j}} = k \begin{cases} \frac{2\Delta y}{\Delta x}, & i = M_x^T - 1 \\ \frac{\Delta y}{\Delta x} & \text{otherwise} \end{cases}, \quad (4b)$$

$$D_s^{T_{i,j}} = k \begin{cases} \frac{2\Delta x}{\Delta y}, & j = 1 \\ \frac{\Delta x}{\Delta y} & \text{otherwise} \end{cases}, \quad (4c)$$

$$D_w^{T_{i,j}} = k \begin{cases} \frac{2\Delta y}{\Delta x}, & i = 1 \\ \frac{\Delta y}{\Delta x} & \text{otherwise} \end{cases}. \quad (4d)$$

3 Results

3.1 Problem 3: 10×5 grid

The results requested for problem 3 follow in Tables 1, 2, and 3.

Table 1: The u -velocity solution with the 10×5 grid. A row corresponds to a x -position and a column corresponds to an y -position.

	1	2	3	4	5	6	7
1	5.06931E-3	5.06931E-3	5.06931E-3	5.06931E-3	5.06931E-3	5.06931E-3	5.06931E-3
2	0.00000E0	3.02156E-3	6.10805E-3	7.08733E-3	6.10805E-3	3.02156E-3	0.00000E0
3	0.00000E0	2.84164E-3	6.18930E-3	7.28469E-3	6.18930E-3	2.84164E-3	0.00000E0
4	0.00000E0	2.81967E-3	6.19551E-3	7.31618E-3	6.19551E-3	2.81967E-3	0.00000E0
5	0.00000E0	2.81677E-3	6.19585E-3	7.32132E-3	6.19585E-3	2.81677E-3	0.00000E0
6	0.00000E0	2.81636E-3	6.19583E-3	7.32217E-3	6.19583E-3	2.81636E-3	0.00000E0
7	0.00000E0	2.81630E-3	6.19583E-3	7.32231E-3	6.19583E-3	2.81630E-3	0.00000E0
8	0.00000E0	2.81629E-3	6.19582E-3	7.32233E-3	6.19582E-3	2.81629E-3	0.00000E0
9	0.00000E0	2.81628E-3	6.19582E-3	7.32234E-3	6.19582E-3	2.81628E-3	0.00000E0
10	0.00000E0	2.81628E-3	6.19582E-3	7.32234E-3	6.19582E-3	2.81628E-3	0.00000E0
11	0.00000E0	2.81628E-3	6.19582E-3	7.32234E-3	6.19582E-3	2.81628E-3	0.00000E0

Table 2: The $(P - P_0)$ solution with the 10×5 grid. A row corresponds to a x -position and a column corresponds to an y -position.

	1	2	3	4	5	6	7
1	2.88374E-1	2.88374E-1	2.88353E-1	2.88346E-1	2.88353E-1	2.88374E-1	2.88374E-1
2	2.88374E-1	2.88374E-1	2.88353E-1	2.88346E-1	2.88353E-1	2.88374E-1	2.88374E-1
3	2.40648E-1	2.40648E-1	2.40651E-1	2.40653E-1	2.40651E-1	2.40648E-1	2.40648E-1
4	2.11749E-1	2.11749E-1	2.11750E-1	2.11750E-1	2.11750E-1	2.11749E-1	2.11749E-1
5	1.83440E-1	1.83440E-1	1.83440E-1	1.83440E-1	1.83440E-1	1.83440E-1	1.83440E-1
6	1.55208E-1	1.55208E-1	1.55208E-1	1.55208E-1	1.55208E-1	1.55208E-1	1.55208E-1
7	1.26987E-1	1.26987E-1	1.26987E-1	1.26987E-1	1.26987E-1	1.26987E-1	1.26987E-1
8	9.87671E-2	9.87671E-2	9.87671E-2	9.87671E-2	9.87671E-2	9.87671E-2	9.87671E-2
9	7.05479E-2	7.05479E-2	7.05479E-2	7.05479E-2	7.05479E-2	7.05479E-2	7.05479E-2
10	4.23287E-2	4.23287E-2	4.23287E-2	4.23287E-2	4.23287E-2	4.23287E-2	4.23287E-2
11	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0
12	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0

Table 3: The T solution with the 10×5 grid. A row corresponds to a x -position and a column corresponds to an y -position.

	1	2	3	4	5	6	7
1	26.64204	25.00000	25.00000	25.00000	25.00000	25.00000	26.64204
2	27.69621	26.05418	25.36548	25.17769	25.36548	26.05418	27.69621
3	28.84253	27.20050	25.72681	25.36877	25.72681	27.20050	28.84253
4	29.72586	28.08382	26.15140	25.63232	26.15140	28.08382	29.72586
5	30.44158	28.79955	26.60972	25.95974	26.60972	28.79955	30.44158
6	31.06105	29.41902	27.08232	26.33547	27.08232	29.41902	31.06105
7	31.62445	29.98241	27.56009	26.74529	27.56009	29.98241	31.62445
8	32.15448	30.51244	28.03926	27.17831	28.03926	30.51244	32.15448
9	32.66429	31.02226	28.51841	27.62677	28.51841	31.02226	32.66429
10	33.16165	31.51962	28.99712	28.08535	28.99712	31.51962	33.16165
11	33.65119	32.00915	29.47532	28.55050	29.47532	32.00915	33.65119
12	33.89596	32.25392	29.71442	28.78307	29.71442	32.25392	33.89596

3.2 Problem 4: 180×54 grid

The requirements for problem (b) part i and ii were combined into Figure ?? as seen below. With increasing grid refinement, both centerline velocity profiles approached towards the reference solution obtained from Roy et. al. In addition, a once-more-refined run is compared with 256x256 CVs with good agreement.

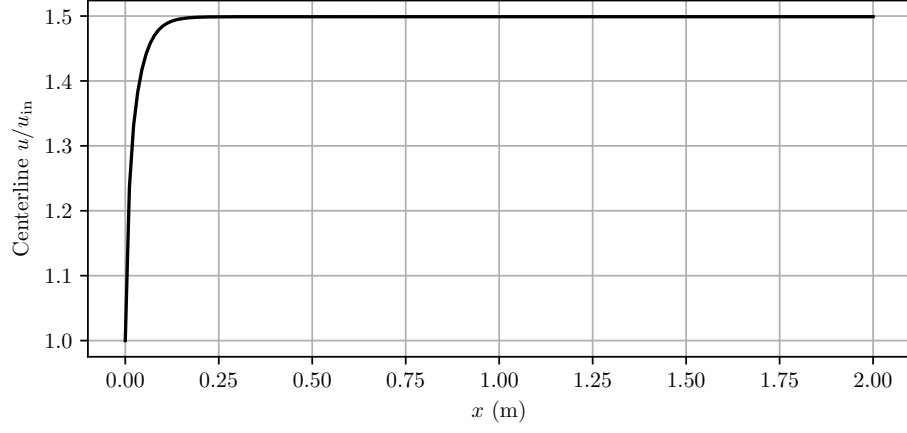


Figure 2: u/u_{in} plotted along the centerline of the channel for the 180×54 grid.

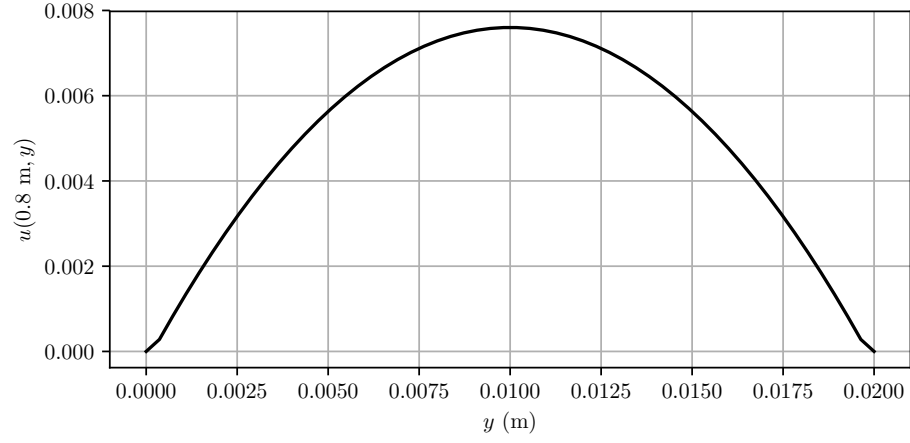


Figure 3: u plotted at $x = 0.8$ m for the 180×54 grid.

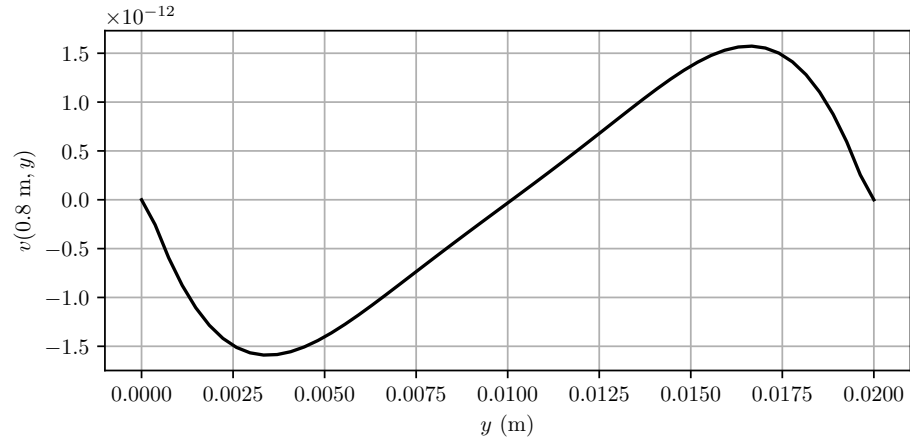


Figure 4: v plotted at $x = 0.8$ m for the 180×54 grid.

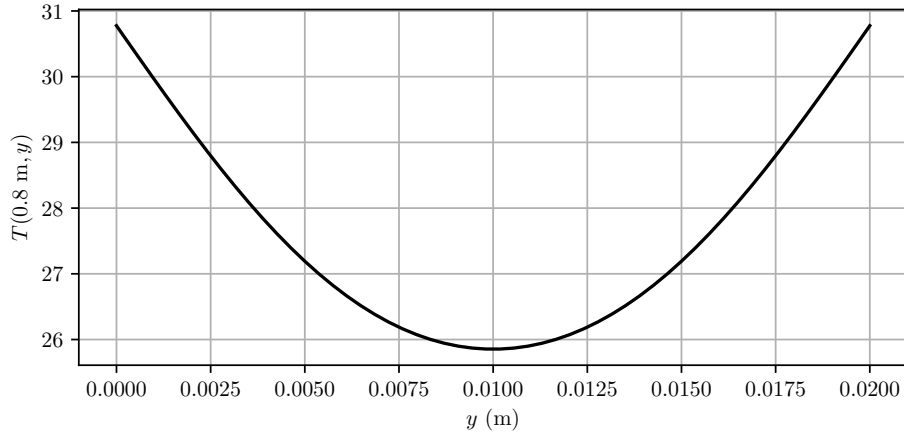


Figure 5: T plotted at $x = 0.8$ m for the 180×54 grid.

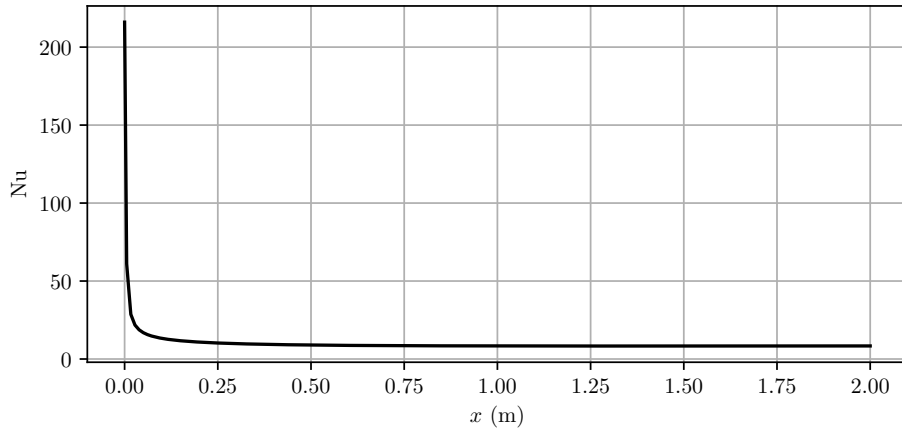


Figure 6: The Nusselt number plotted as a function of stream-wise distance for the 180×54 grid.

Code listing

For the implementation, we have the following files:

- **Makefile** – Allows for compiling the `c++` project with **make**.
- **hwk5.cpp** – Contains the `main()` function that is required by C that runs the cases requested in this problem set.
- **Problem.h** – Contains the header for the **Problem** class which is the main driver for a **Flow2D::Problem**.
- **Variable.h** – Contains the **Flow2D::Variable** class, which is a storage container for a single variable (i.e., u).
- **Problem.cpp** – Contains the `run()` functions that executes a **Problem**.
- **Problem_coefficients.cpp** – Contains the functions for solving coefficients in a **Problem**.

- `Problem_corrections.cpp` – Contains the functions for correcting solutions in a `Problem`.
- `Problem_residuals.cpp` – Contains the functions for computing residuals in a `Problem`.
- `Problem_solvers.cpp` – Contains the functions for sweeping and solving in a `Problem`.
- `Matrix.h` – Contains the `Matrix` class which provides storage for a matrix with various standard matrix operations.
- `TriDiagonal.h` – Contains the `TriDiagonal` class which provides storage for a tri-diagonal matrix including the TDMA solver found in the member function `solveTDMA()`.
- `Vector.h` – Contains the `Vector` class for one-dimensional vector storage.
- `postprocess.py` - Produces the plots and tables in this report.

Makefile

```
src = $(wildcard *.cpp)
obj = $(src:.cpp=.o)
CXXFLAGS = -std=c++14
CXXFLAGS = $(CXXFLAGS)

hwk-opt: $(obj)
        clang++ -o $@ $^

.PHONY: clean
clean:
        rm -f $(obj) hwk-opt
```

hwk5.cpp

```
#include "Problem.h"

using namespace Flow2D;

int
main()
{
    // Problem wide constants
    double Lx = 2;
    double Ly = 0.02;
    double cp = 4183;
    double k = 0.609;
    double rho = 988.3;
    double mu = 0.001002;
    double Re = 100;
    double u_bc_val = Re * mu / (rho * Ly);
    double T_bc_val = 25;
    double q_bc_val = 500;

    // Standard inputs
    InputArguments input;
    input.Lx = Lx;
    input.Ly = Ly;
    input.cp = cp;
    input.k = k;
    input.mu = mu;
    input.rho = rho;
    input.u_ref = u_bc_val;
    input.L_ref = Lx;
    input.T_left = T_bc_val;
    input.u_bc = BoundaryCondition(0, 0, 0, u_bc_val);
    input.v_bc = BoundaryCondition(0, 0, 0, 0);
    input.q_top_bot = q_bc_val;

    // Problem 3: check symmetry
    std::cout << "Problem 3: check symmetry" << std::endl;
    Problem problem3(10, 5, input);
```

```

problem3.run();
problem3.save(Variables::u, "results/coarse_u.csv");
problem3.save(Variables::p, "results/coarse_p.csv");
problem3.save(Variables::T, "results/coarse_T.csv");

// Problem 4: 180x54 grid
std::cout << "Problem 4: 180 x 54 grid" << std::endl;
Problem problem4(180, 54, input);
problem4.run();
problem4.save(Variables::u, "results/fine_u.csv");
problem4.save(Variables::v, "results/fine_v.csv");
problem4.save(Variables::T, "results/fine_T.csv");
}

```


Problem.h

```
#ifndef PROBLEM_H
#define PROBLEM_H

#include <cmath>
#include <ctime>
#include <iomanip>
#include <iostream>
#include <map>

#include "Variable.h"

namespace Flow2D
{
    using namespace std;

    struct InputArguments
    {
        double Lx, Ly;
        BoundaryCondition u_bc, v_bc;
        double T_left, q_top_bot;
        double L_ref, u_ref;
        double cp, k, mu, rho;
        bool debug = false;
        double alpha_p = 0.7;
        double alpha_uv = 0.5;
        unsigned int max_main_its = 50000;
        unsigned int max_aux_its = 20000;
        double tol = 1.0e-6;
    };

    class Problem
    {
    public:
        Problem(const unsigned int Nx, const unsigned int Ny, const InputArguments & input);

        void run();

        // Public access to printing and saving variable results
        void print(const Variables var,
                  const string prefix = "",
                  const bool newline = false,
                  const unsigned int pr = 5) const
        {
            variables.at(var).print(prefix, newline, pr);
        }
        void save(const Variables var, const string filename) const { variables.at(var).save(filename); }

    private:
        // Problem_corrections.cpp
        void correctMain();
        void correctAux();
        void pCorrect();
        void pBCCorrect();
        void TBCCorrect();
        void uCorrect();
        void uBCCorrect();
        void vCorrect();

        // Problem_coefficients.cpp
        void fillCoefficients(const Variable & var);
        void pcCoefficients();
        void TCoefficients();
        void uCoefficients();
        void vCoefficients();
        void fillPowerLaw(Coefficients & a,
                         const Coefficients & D,
                         const Coefficients & F,
                         const double & b = 0);

        // Problem_residuals.cpp
        void computeMainResiduals();
        void computeAuxResiduals();
        double pResidual() const;
        double TResidual() const;
        double velocityResidual(const Variable & var) const;
    };
}
```

```

// Problem_solvers.cpp
void solveMain();
void solveAux();
void solve(Variable & var);
void sweepColumns(Variable & var, const bool west_east = true);
void sweepRows(Variable & var, const bool south_north = true);
void sweepColumn(const unsigned int i, Variable & var);
void sweepRow(const unsigned int j, Variable & var);
void solveVelocities();

// Quicker v^5 for velocityCoefficients() (yes, it's actually much faster...)
static const double pow5(const double & v) { return v * v * v * v * v; }

protected:
// Number of pressure CVs
const unsigned int Nx, Ny;

// Geometry [m]
const double Lx, Ly, dx, dy;
// Material properties
const double cp, k, mu, rho;
// Residual references
const double L_ref, u_ref;
// Other boundary conditions
const double q_top_bot;
// Mass inflow
const double m_in;

// Enable debug mode (printing extra output)
const bool debug;

// Maximum iterations
const unsigned int max_main_its, max_aux_its;
// Iteration tolerance
const double tol;
// Pressure relaxation
const double alpha_p;
// Number of iterations completed
unsigned int main_iterations = 0;
unsigned int aux_iterations = 0;

// Variables
Variable u, v, pc, p, T;
// Variable map
map<const Variables, const Variable &> variables;

// Whether or not we converged
bool main_converged = false;
bool converged = false;
// Run start time
clock_t start;
};

} // namespace Flow2D
#endif /* PROBLEM_H */

```

Variable.h

```
#ifndef VARIABLE_H
#define VARIABLE_H

#include "Matrix.h"
#include "TriDiagonal.h"
#include "Vector.h"

namespace Flow2D
{
    using namespace std;

    // Storage for boundary conditions
    struct BoundaryCondition
    {
        BoundaryCondition() {}
        BoundaryCondition(const double top, const double right, const double bottom, const double left)
            : top(top), right(right), bottom(bottom), left(left)
        {
        }
    };
    double top = 0, right = 0, bottom = 0, left = 0;

    // Storage for coefficients for a single CV
    struct Coefficients
    {
        double p = 0, n = 0, e = 0, s = 0, w = 0, b = 0;
        void print(const unsigned int pr = 5) const
        {
            cout << setprecision(pr) << scientific << "n = " << n << ", e = " << e << ", s = " << s
                << ", w = " << w << ", p = " << p << ", b = " << b << endl;
        }
    };

    // Enum for variable types
    enum Variables
    {
        u,
        v,
        pc,
        p,
        T
    };

    // Conversion from variable type to its string
    static string
    VariableString(Variables var)
    {
        switch (var)
        {
            case Variables::u:
                return "u";
            case Variables::v:
                return "v";
            case Variables::pc:
                return "pc";
            case Variables::p:
                return "p";
            case Variables::T:
                return "T";
        }
    }

    // General storage structure for primary and auxiliary variables
    struct Variable
    {
        // Constructor for a primary variable
        Variable(const Variables name,
                 const unsigned int Nx,
                 const unsigned int Ny,
                 const double alpha,
                 const BoundaryCondition bc = BoundaryCondition())
            : name(name),
              string(VariableString(name)),
              Nx(Nx),
              Ny(Ny),
              Mx(Nx - 1),
    
```

```

        My(Ny - 1),
        w(1 / alpha),
        bc(bc),
        a(Nx, Ny),
        phi(Nx, Ny),
        Ax(Nx - 2),
        Ay(Ny - 2),
        bx(Nx - 2),
        by(Ny - 2)
    {
        // Apply initial boundary conditions
        if (bc.left != 0)
            phi.setColumn(0, bc.left);
        if (bc.right != 0)
            phi.setColumn(Mx, bc.right);
        if (bc.bottom != 0)
            phi.setRow(0, bc.bottom);
        if (bc.top != 0)
            phi.setRow(My, bc.top);
    }

    // Constructor for an auxiliary variable (no solver storage)
    Variable(const Variables name, const unsigned int Nx, const unsigned int Ny)
        : name(name), string(VariableString(name)), Nx(Nx), Ny(Ny), Mx(Nx - 1), My(Ny - 1), phi(Nx, Ny)
    {
    }

    // Solution matrix operations
    void operator=(const double v) { phi = v; }
    const double & operator()(const unsigned int i, const unsigned int j) const { return phi(i, j); }
    double & operator()(const unsigned int i, const unsigned int j) { return phi(i, j); }
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 5) const
    {
        phi.print(prefix, newline, pr);
    }
    void save(const string filename) const { phi.save(filename); }
    void reset() { phi = 0; }

    // Coefficient debug
    void printCoefficients(const string prefix = "",
                          const bool newline = false,
                          const unsigned int pr = 5) const
    {
        for (unsigned int i = 1; i < Nx - 1; ++i)
            for (unsigned int j = 1; j < Ny - 1; ++j)
            {
                cout << prefix << "(" << i << ", " << j << "): ";
                a(i, j).print(pr);
            }
        if (newline)
            cout << endl;
    }

    // Variable enum name
    const Variables name;
    // Variable string
    const string string;
    // Variable size
    const unsigned int Nx, Ny;
    // Maximum variable index that is being solved
    const unsigned int Mx, My;
    // Relaxation coefficient used in solving linear systems
    const double w = 0;
    // Boundary conditions
    const BoundaryCondition bc = BoundaryCondition();
    // Matrix coefficients
    Matrix<Coefficients> a;
    // Variable solution
    Matrix<double> phi;
    // Linear system LHS for both sweep directions
    TriDiagonal<double> Ax, Ay;
    // Linear system RHS for both sweep directions
    Vector<double> bx, by;
};

} // namespace Flow2D
#endif /* VARIABLE_H */

```

Problem.cpp

```
#include "Problem.h"

namespace Flow2D
{
    Problem::Problem(const unsigned int Nx, const unsigned int Ny, const InputArguments & input)
        : // Number of pressure CVs
          Nx(Nx),
          Ny(Ny),
          // Domain sizes
          Lx(input.Lx),
          Ly(input.Ly),
          dx(Lx / Nx),
          dy(Ly / Ny),
          // Residual references
          L_ref(input.L_ref),
          u_ref(input.u_ref),
          // Other boundary conditions
          q_top_bot(input.q_top_bot),
          // Mass inflow
          m_in(input.u_bc.left * input.rho * Ly),
          // Material properties
          cp(input.cp),
          k(input.k),
          mu(input.mu),
          rho(input.rho),
          // Enable debug
          debug(input.debug),
          // Solver properties
          max_main_its(input.max_main_its),
          max_aux_its(input.max_aux_its),
          tol(input.tol),
          alpha_p(input.alpha_p),
          // Initialize variables for u, v, pc (solved variables)
          u(Variables::u, Nx + 1, Ny + 2, input.alpha_uv, input.u_bc),
          v(Variables::v, Nx + 2, Ny + 1, input.alpha_uv, input.v_bc),
          pc(Variables::pc, Nx + 2, Ny + 2, 1),
          T(Variables::T, Nx + 2, Ny + 2, 1),
          // Initialize aux variables
          p(Variables::p, Nx + 2, Ny + 2)
    {
        // Add into variable map for access outside of class
        variables.emplace(Variables::u, u);
        variables.emplace(Variables::v, v);
        variables.emplace(Variables::pc, pc);
        variables.emplace(Variables::p, p);
        variables.emplace(Variables::T, T);

        // T initial condition
        T = input.T_left;
    }

    void
    Problem::run()
    {
        // Store start time
        start = clock();

        // Solve main variables
        for (unsigned int l = 0; l < max_main_its; ++l)
        {
            solveMain();
            correctMain();
            computeMainResiduals();

            // Break out if we've converged
            if (main_converged)
                break;
        }

        // Ensure main variables converged
        if (!main_converged)
            cout << "Main variables did not converge after " << max_main_its << " iterations!" << endl;

        // Solve aux variables
        for (unsigned int l = 0; l < max_aux_its; ++l)
        {

```

```

    solveAux();
    correctAux();
    computeAuxResiduals();

    // Exit if everything is converged
    if (converged)
        return;
}

// Oops. Didn't converge
cout << "Aux variables did not converge after " << max_aux_its << " iterations!" << endl;
}
} // namespace Flow2D

```

Problem_coefficients.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::fillCoefficients(const Variable & var)
    {
        if (var.name == Variables::pc)
            pcCoefficients();
        else if (var.name == Variables::T)
            TCoefficients();
        else if (var.name == Variables::u)
            uCoefficients();
        else if (var.name == Variables::v)
            vCoefficients();

        if (debug)
        {
            cout << var.string << " coefficients: " << endl;
            var.printCoefficients(var.string, true);
        }
    }

    void
    Problem::pcCoefficients()
    {
        for (unsigned int i = 1; i < pc.Mx; ++i)
            for (unsigned int j = 1; j < pc.My; ++j)
            {
                Coefficients & a = pc.a(i, j);

                if (i != 1)
                    a.w = rho * dy * dy / u.a(i - 1, j).p;
                if (i != pc.Mx - 1)
                    a.e = rho * dy * dy / u.a(i, j).p;
                if (j != 1)
                    a.s = rho * dx * dx / v.a(i, j - 1).p;
                if (j != pc.My - 1)
                    a.n = rho * dx * dx / v.a(i, j).p;
                a.p = a.n + a.e + a.s + a.w;
                a.b = rho * (dy * (u(i - 1, j) - u(i, j)) + dx * (v(i, j - 1) - v(i, j)));
            }
    }

    void
    Problem::TCoefficients()
    {
        Coefficients D, F;

        for (unsigned int i = 1; i < T.Mx; ++i)
            for (unsigned int j = 1; j < T.My; ++j)
            {
                // Diffusion coefficient
                D.n = (j == T.My - 1 ? 2 * dx * k / dy : dx * k / dy);
                D.e = (i == T.Mx - 1 ? 2 * dy * k / dx : dy * k / dx);
                D.s = (j == 1 ? 2 * dx * k / dy : dx * k / dy);
                D.w = (i == 1 ? 2 * dy * k / dx : dy * k / dx);

                // Heat flows
                F.n = dx * cp * rho * v(i, j);
                F.e = dy * cp * rho * u(i, j);
                F.s = dx * cp * rho * v(i, j - 1);
                F.w = dy * cp * rho * u(i - 1, j);

                // Compute and store power law coefficients
                fillPowerLaw(T.a(i, j), D, F);
            }
    }

    void
    Problem::uCoefficients()
    {
        Coefficients D, F;
        double W, dy_pn, dy_ps, b;

        for (unsigned int i = 1; i < u.Mx; ++i)
```

```

for (unsigned int j = 1; j < u.My; ++j)
{
    // Width of the cell
    W = (i == 1 || i == u.Mx - 1 ? 3 * dx / 2 : dx);
    // North/south distances to pressure nodes
    dy_pn = (j == u.My - 1 ? dy / 2 : dy);
    dy_ps = (j == 1 ? dy / 2 : dy);

    // Diffusion coefficients
    D.n = mu * W / dy_pn;
    D.e = mu * dy / dx;
    D.s = mu * W / dy_ps;
    D.w = mu * dy / dx;

    // East and west flows
    F.e = (i == u.Mx - 1 ? rho * dy * u(u.Mx, j) : rho * dy * (u(i + 1, j) + u(i, j)) / 2);
    F.w = (i == 1 ? rho * dy * u(0, j) : rho * dy * (u(i - 1, j) + u(i, j)) / 2);
    // North and south flows
    if (i == 1) // Left boundary
    {
        F.n = rho * W * (v(0, j) + 3 * v(1, j) + 2 * v(2, j)) / 6;
        F.s = rho * W * (v(0, j - 1) + 3 * v(1, j - 1) + 2 * v(2, j - 1)) / 6;
    }
    else if (i == u.Mx - 1) // Right boundary
    {
        F.n = rho * W * (2 * v(i, j) + 3 * v(i + 1, j) + v(i + 2, j)) / 6;
        F.s = rho * W * (2 * v(i, j - 1) + 3 * v(i + 1, j - 1) + v(i + 2, j - 1)) / 6;
    }
    else // Interior (not left or right boundary)
    {
        F.n = rho * W * (v(i, j) + v(i + 1, j)) / 2;
        F.s = rho * W * (v(i, j - 1) + v(i + 1, j - 1)) / 2;
    }

    // Pressure RHS
    b = dy * (p(i, j) - p(i + 1, j));

    // Compute and store power law coefficients
    fillPowerLaw(u.a(i, j), D, F, b);
}
}

void
Problem::vCoefficients()
{
    Coefficients D, F;
    double H, dx_pe, dx_pw, b;

    for (unsigned int i = 1; i < v.Mx; ++i)
        for (unsigned int j = 1; j < v.My; ++j)
        {
            // Height of the cell
            H = (j == 1 || j == v.My - 1 ? 3 * dy / 2 : dy);
            // East/west distances to pressure nodes
            dx_pe = (i == v.Mx - 1 ? dx / 2 : dx);
            dx_pw = (i == 1 ? dx / 2 : dx);

            // Diffusion coefficient
            D.n = mu * dx / dy;
            D.e = mu * H / dx_pe;
            D.s = mu * dx / dy;
            D.w = mu * H / dx_pw;

            // North and east flows
            F.n = (j == v.My - 1 ? rho * dx * v(i, v.My) : rho * dx * (v(i, j + 1) + v(i, j)) / 2);
            F.s = (j == 1 ? rho * dx * v(i, 0) : rho * dx * (v(i, j - 1) + v(i, j)) / 2);
            // East and west flows
            if (j == 1) // Bottom boundary
            {
                F.e = rho * H * (u(i, 0) + 3 * u(i, 1) + 2 * u(i, 2)) / 6;
                F.w = rho * H * (u(i - 1, 0) + 3 * u(i - 1, 1) + 2 * u(i - 1, 2)) / 6;
            }
            else if (j == v.My - 1) // Top boundary
            {
                F.e = rho * H * (2 * u(i, j) + 3 * u(i, j + 1) + u(i, j + 2)) / 6;
                F.w = rho * H * (2 * u(i - 1, j) + 3 * u(i - 1, j + 1) + u(i - 1, j + 2)) / 6;
            }
            else // Interior (not top or bottom boundary)
            {
                F.e = rho * H * (u(i, j) + u(i, j + 1)) / 2;
                F.w = rho * H * (u(i - 1, j) + u(i - 1, j + 1)) / 2;
            }
        }
}

```



```

    }

    // Pressure RHS
    b = dx * (p(i, j) - p(i, j + 1));

    // Compute and store power law coefficients
    fillPowerLaw(v.a(i, j), D, F, b);
}

void
Problem::fillPowerLaw(Coefficients & a,
                     const Coefficients & D,
                     const Coefficients & F,
                     const double & b)
{
    a.n = D.n * fmax(0, pow5(1 - 0.1 * fabs(F.n / D.n))) + fmax(-F.n, 0);
    a.e = D.e * fmax(0, pow5(1 - 0.1 * fabs(F.e / D.e))) + fmax(-F.e, 0);
    a.s = D.s * fmax(0, pow5(1 - 0.1 * fabs(F.s / D.s))) + fmax(F.s, 0);
    a.w = D.w * fmax(0, pow5(1 - 0.1 * fabs(F.w / D.w))) + fmax(F.w, 0);
    a.p = a.n + a.e + a.s + a.w;
    a.b = b;
}

} // namespace Flow2D

```

Problem_corrections.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::correctMain()
    {
        uCorrect();
        vCorrect();
        pCorrect();
        pBCCorrect();
        uBCCorrect();
    }

    void
    Problem::correctAux()
    {
        TBCCorrect();
    }

    void
    Problem::pCorrect()
    {
        for (unsigned int i = 1; i < pc.Mx; ++i)
            for (unsigned int j = 1; j < pc.My; ++j)
                p(i, j) += alpha_p * pc(i, j);

        // Set pressure correction back to zero
        pc.reset();

        if (debug)
            p.print("p corrected = ", true);
    }

    void
    Problem::pBCCorrect()
    {
        // Apply the edge values as velocity is set
        for (unsigned int i = 0; i <= pc.Mx; ++i)
        {
            p(i, 0) = p(i, 1);
            p(i, pc.My) = p(i, pc.My - 1);
        }
        for (unsigned int j = 0; j <= pc.My; ++j)
        {
            p(0, j) = p(1, j);
            p(pc.Mx, j) = p(pc.Mx - 1, j);
        }

        if (debug)
            p.print("p boundary condition corrected = ", true);
    }

    void
    Problem::TBCCorrect()
    {
        for (unsigned int j = 0; j <= T.My; ++j)
            T(T.Mx, j) = (3 * T(T.Mx - 1, j) - T(T.Mx - 2, j)) / 2;
        for (unsigned int i = 0; i <= T.Mx; ++i)
        {
            T(i, 0) = T(i, 1) + q_top_bot * dy / (2 * k);
            T(i, T.My) = T(i, T.My - 1) + q_top_bot * dy / (2 * k);
        }
    }

    void
    Problem::uCorrect()
    {
        for (unsigned int i = 1; i < u.Mx; ++i)
            for (unsigned int j = 1; j < u.My; ++j)
                u(i, j) += dy * (pc(i, j) - pc(i + 1, j)) / u.a(i, j).p;

        if (debug)
            u.print("u corrected = ", true);
    }

    void
```

```

Problem::uBCCorrect()
{
    double m_out = 0;
    for (unsigned int j = 0; j <= u.My; ++j)
        m_out += rho * dy * u(u.Mx - 1, j);
    for (unsigned int j = 0; j <= u.My; ++j)
        u(u.Mx, j) = m_in * u(u.Mx - 1, j) / m_out;

    if (debug)
        u.print("u boundary condition corrected = ", true);
}

void
Problem::vCorrect()
{
    for (unsigned int i = 1; i < v.Mx; ++i)
        for (unsigned int j = 1; j < v.My; ++j)
            v(i, j) += dx * (pc(i, j) - pc(i, j + 1)) / v.a(i, j).p;

    if (debug)
        v.print("v corrected = ", true);
}

} // namespace Flow2D

```

Problem_residuals.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::computeMainResiduals()
    {
        const double Rp = pResidual();
        const double Ru = velocityResidual(u);
        const double Rv = velocityResidual(v);

        if (Ru < tol && Rv < tol && Rp < tol)
        {
            if (debug)
                cout << "Main variables converged in " << main_iterations << " iterations" << endl;
            main_converged = true;
        }
    }

    void
    Problem::computeAuxResiduals()
    {
        double RT = TResidual();

        // Still not converged
        if (RT > tol)
            return;

        // Converged, finish up
        converged = true;

        // Print the result
        const double Rp = pResidual();
        const double Ru = velocityResidual(u);
        const double Rv = velocityResidual(v);
        cout << "Converged in " << noshowpos << fixed << setprecision(3)
            << 1.0 * (clock() - start) / CLOCKS_PER_SEC << " sec in " << main_iterations << " main and "
            << aux_iterations << " aux iterations: ";
        cout << noshowpos << setprecision(1) << scientific;
        cout << "p = " << Rp;
        cout << ", v = " << Rv;
        cout << ", p = " << Rp;
        cout << ", T = " << RT << endl;
    }

    double
    Problem::pResidual() const
    {
        double numer = 0;
        for (unsigned int i = 1; i < pc.Mx; ++i)
            for (unsigned int j = 1; j < pc.My; ++j)
                numer += abs(dy * (u(i - 1, j) - u(i, j)) + dx * (v(i, j - 1) - v(i, j)));
        return numer / (u_ref * L_ref);
    }

    double
    Problem::TResidual() const
    {
        double numer, numer_temp, denom = 0;
        for (unsigned int i = 1; i < T.Mx; ++i)
            for (unsigned int j = 1; j < T.My; ++j)
            {
                const Coefficients & a = T.a(i, j);
                numer_temp = a.p * T(i, j);
                denom += abs(numer_temp);
                numer_temp -= a.n * T(i, j + 1) + a.e * T(i + 1, j);
                numer_temp -= a.s * T(i, j - 1) + a.w * T(i - 1, j) + a.b;
                numer += abs(numer_temp);
            }
        return numer / denom;
    }

    double
    Problem::velocityResidual(const Variable & var) const
    {
        double numer, numer_temp, denom = 0;
    }
}
```

```

for (unsigned int i = 1; i < var.Mx; ++i)
  for (unsigned int j = 1; j < var.My; ++j)
  {
    const Coefficients & a = var.a(i, j);
    numer_temp = a.p * var(i, j);
    denom += abs(numer_temp);
    numer_temp -= a.n * var(i, j + 1) + a.e * var(i + 1, j);
    numer_temp -= a.s * var(i, j - 1) + a.w * var(i - 1, j) + a.b;
    numer += abs(numer_temp);
  }
return numer / denom;
}

} // namespace Flow2D

```

Problem_solvers.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::solveMain()
    {
        ++main_iterations;
        if (debug)
            cout << endl << "Main iteration " << main_iterations << endl << endl;

        solve(u);
        solve(v);
        solve(pc);
    }

    void
    Problem::solveAux()
    {
        ++aux_iterations;
        if (debug)
            cout << endl << "Aux iteration " << aux_iterations << endl << endl;

        solve(T);
    }

    void
    Problem::solve(Variable & var)
    {
        if (debug)
            cout << "Solving variable " << var.string << endl << endl;

        // Fill the coefficients
        fillCoefficients(var);

        // Solve west to east
        sweepColumns(var);
        // Solve south to north
        sweepRows(var);
        // Solve east to west
        sweepColumns(var, false);

        if (debug)
            var.print(var.string + " sweep solution = ", true);
    }

    void
    Problem::sweepRows(Variable & var, const bool south_north)
    {
        if (debug)
            cout << "Sweeping " << var.string << (south_north ? " south to north" : " north to south")
                << endl;

        // Sweep south to north
        if (south_north)
            for (int j = 1; j < var.My; ++j)
                sweepRow(j, var);
        // Sweep north to south
        else
            for (int j = var.My - 1; j > 0; --j)
                sweepRow(j, var);
    }

    void
    Problem::sweepColumns(Variable & var, const bool west_east)
    {
        if (debug)
            cout << "Sweeping " << var.string << (west_east ? " east to west" : " west to east") << endl;

        // Sweep west to east
        if (west_east)
            for (int i = 1; i < var.Mx; ++i)
                sweepColumn(i, var);
        // Sweep east to west
        else
            for (int i = var.Mx - 1; i > 0; --i)
```

```

        sweepColumn(i, var);
    }

void
Problem::sweepColumn(const unsigned int i, Variable & var)
{
    if (debug)
        cout << "Solving " << var.string << " column " << i << endl;

    auto & A = var.Ay;
    auto & b = var.by;

    // Fill for each cell
    for (unsigned int j = 1; j < var.My; ++j)
    {
        const Coefficients & a = var.a(i, j);
        b[j - 1] = a.b + a.w * var(i - 1, j) + a.e * var(i + 1, j);
        if (var.w != 1)
            b[j - 1] += a.p * var(i, j) * (var.w - 1);
        if (j == 1)
        {
            A.setTopRow(a.p * var.w, -a.n);
            if (var.name != Variables::pc)
                b[j - 1] += a.s * var(i, j - 1);
        }
        else if (j == var.My - 1)
        {
            A.setBottomRow(-a.s, a.p * var.w);
            if (var.name != Variables::pc)
                b[j - 1] += a.n * var(i, j + 1);
        }
        else
            A.setMiddleRow(j - 1, -a.s, a.p * var.w, -a.n);
    }

    if (debug)
    {
        A.print("A =");
        b.print("b =");
    }

    // Solve
    A.solveTDMA(b);

    if (debug)
        b.print("sol =", true);

    // Store solution
    for (unsigned int j = 1; j < var.My; ++j)
        var(i, j) = b[j - 1];
}

void
Problem::sweepRow(const unsigned int j, Variable & var)
{
    if (debug)
        cout << "Solving " << var.string << " row " << j << endl;

    auto & A = var.Ax;
    auto & b = var.bx;

    // Fill for each cell
    for (unsigned int i = 1; i < var.Mx; ++i)
    {
        const Coefficients & a = var.a(i, j);
        b[i - 1] = a.b + a.s * var(i, j - 1) + a.n * var(i, j + 1);
        if (var.w != 1)
            b[i - 1] += a.p * var(i, j) * (var.w - 1);
        if (i == 1)
        {
            A.setTopRow(a.p * var.w, -a.e);
            if (var.name != Variables::pc)
                b[i - 1] += a.w * var(i - 1, j);
        }
        else if (i == var.Mx - 1)
        {
            A.setBottomRow(-a.w, a.p * var.w);
            if (var.name != Variables::pc)
                b[i - 1] += a.e * var(i + 1, j);
        }
        else

```

```

        A.setMiddleRow(i - 1, -a.w, a.p * var.w, -a.e);
    }

    if (debug)
    {
        A.print("A =");
        b.print("b =");
    }

    // Solve
    A.solveTDMA(b);

    if (debug)
        b.print("sol =", true);

    // Store solution
    for (unsigned int i = 1; i < var.Mx; ++i)
        var(i, j) = b[i - 1];
}

} // namespace Flow2D

```


Matrix.h

```
#ifndef MATRIX_H
#define MATRIX_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include <vector>

using namespace std;

/**
 * Class that holds a N x M matrix with common matrix operations.
 */
template <typename T>
class Matrix
{
public:
    Matrix() {}
    Matrix(const unsigned int N, const unsigned int M) : N(N), M(M), A(N, vector<T>(M)) {}

    // Const operator for getting the (i, j) element
    const T & operator()(const unsigned int i, const unsigned int j) const
    {
        assert(i < N && j < M);
        return A[i][j];
    }
    // Operator for getting the (i, j) element
    T & operator()(const unsigned int i, const unsigned int j)
    {
        assert(i < N && j < M);
        return A[i][j];
    }
    // Operator for setting the entire matrix to a value
    void operator=(const T v)
    {
        for (unsigned int j = 0; j < M; ++j)
            setRow(j, v);
    }

    // Prints the matrix
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 5) const
    {
        if (prefix.length() != 0)
            cout << prefix << endl;
        for (unsigned int j = 0; j < M; ++j)
        {
            for (unsigned int i = 0; i < N; ++i)
                cout << showpos << scientific << setprecision(pr) << A[i][j] << " ";
            cout << endl;
        }
        if (newline)
            cout << endl;
    }
    // Saves the matrix in csv format
    void save(const string filename, const unsigned int pr = 12) const
    {
        ofstream f;
        f.open(filename);
        for (unsigned int j = 0; j < M; ++j)
        {
            for (unsigned int i = 0; i < N; ++i)
            {
                if (i > 0)
                    f << ",";
                f << setprecision(pr) << A[i][j];
            }
            f << endl;
        }
        f.close();
    }

    // Set the j-th row to v
    void setRow(const unsigned int j, const T v)
    {
        assert(j < M);
        for (unsigned int i = 0; i < N; ++i)
            A[i][j] = v;
    }
};
```

```

    }
    // Set the i-th column to v
    void setColumn(const unsigned int i, const T v)
    {
        assert(i < N);
        for (unsigned int j = 0; j < M; ++j)
            A[i][j] = v;
    }

private:
    // The size of this matrix
    const unsigned int N = 0, M = 0;

    // Matrix storage
    vector<vector<T>> A;
};

#endif /* MATRIX_H */

```

TriDiagonal.h

```
#ifndef TRIDIAGONAL_H
#define TRIDIAGONAL_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include "Vector.h"

using namespace std;

/**
 * Class that holds a tri-diagonal matrix and is able to perform TDMA in place
 * with a given RHS.
 */
template <typename T>
class TriDiagonal
{
public:
    TriDiagonal() {}
    TriDiagonal(const unsigned int N, const T v = 0) : N(N), A(N, v), B(N, v), C(N - 1, v) {}

    // Setters for the top, middle, and bottom rows
    void setTopRow(const T b, const T c)
    {
        B[0] = b;
        C[0] = c;
    }
    void setMiddleRow(const unsigned int i, const T a, const T b, const T c)
    {
        assert(i < N - 1 && i != 0);
        A[i] = a;
        B[i] = b;
        C[i] = c;
    }
    void setBottomRow(const T a, const T b)
    {
        A[N - 1] = a;
        B[N - 1] = b;
    }

    // Prints the matrix
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 6) const
    {
        if (prefix.length() != 0)
            cout << prefix << endl;
        for (unsigned int i = 0; i < N; ++i)
            cout << showpos << scientific << setprecision(pr) << (i > 0 ? A[i] : 0) << " " << B[i] << " "
                << (i < N - 1 ? C[i] : 0) << endl;
        if (newline)
            cout << endl;
    }

    // Saves the matrix in csv format
    void save(const string filename, const unsigned int pr = 12) const
    {
        ofstream f;
        f.open(filename);
        for (unsigned int i = 0; i < N; ++i)
        {
            if (i > 0)
                f << setprecision(pr) << A[i] << ",";
            else
                f << "0"
                    << ",";
            f << setprecision(pr) << B[i] << ",";
            if (i != N - 1)
                f << setprecision(pr) << C[i] << endl;
            else
                f << 0 << endl;
        }
        f.close();
    }

    // Solves the system Ax = d in place where d eventually stores the solution
    void solveTDMA(Vector<T> & d)
    {
        // Forward sweep
        T tmp = 0;
    }
};
```

```

    for (unsigned int i = 1; i < N; ++i)
    {
        tmp = A[i] / B[i - 1];
        B[i] -= tmp * C[i - 1];
        d[i] -= tmp * d[i - 1];
    }

    // Backward sweep
    d[N - 1] /= B[N - 1];
    for (unsigned int i = N - 2; i != numeric_limits<unsigned int>::max(); --i)
    {
        d[i] -= C[i] * d[i + 1];
        d[i] /= B[i];
    }
}

protected:
    // Matrix size (N x N)
    unsigned int N = 0;

    // Left/main/right diagonal storage
    vector<T> A, B, C;
};

#endif /* TRIDIAGONAL_H */

```

Vector.h

```
#ifndef VECTOR_H
#define VECTOR_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include <vector>

using namespace std;

/**
 * Class that stores a 1D vector and enables printing and saving.
 */
template <typename T>
class Vector
{
public:
    Vector(const unsigned int N) : v(N), N(N) {}
    Vector() {}

    const T & operator()(const unsigned int i) const
    {
        assert(i < N);
        return v[i];
    }
    T & operator()(const unsigned int i)
    {
        assert(i < N);
        return v[i];
    }
    const T & operator[](const unsigned int i) const
    {
        assert(i < N);
        return v[i];
    }
    T & operator[](const unsigned int i)
    {
        assert(i < N);
        return v[i];
    }

    // Prints the vector
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 6) const
    {
        if (prefix.length() != 0)
            cout << prefix << endl;
        for (unsigned int i = 0; i < v.size(); ++i)
            cout << showpos << scientific << setprecision(pr) << v[i] << " ";
        cout << endl;
        if (newline)
            cout << endl;
    }

    // Saves the vector
    void save(const string filename, const unsigned int pr = 12) const
    {
        ofstream f;
        f.open(filename);
        for (unsigned int i = 0; i < v.size(); ++i)
            f << scientific << v[i] << endl;
        f.close();
    }

private:
    vector<T> v;
    const unsigned int N = 0;
};

#endif /* VECTOR_H */
```

postprocess.py

```
import numpy as np
import matplotlib.pyplot as plt

plt.rc('text', usetex=True)
plt.rc('font', family='serif')

Lx = 2
Ly = 0.02
cp = 4183
rho = 998.3
q = 500
k = 0.609

#####
# Problem 3 prints

# Load coarse results
u = np.loadtxt('results/coarse_u.csv', delimiter=',').T
p = np.loadtxt('results/coarse_p.csv', delimiter=',').T
T = np.loadtxt('results/coarse_T.csv', delimiter=',').T
print(p.shape)

# Normalize P
p -= p[p.shape[0] - 1, :]

for var in [u, p, T]:
    for i in range(var.shape[0]):
        line = '{} & {}'.format(i + 1)
        for j in range(var.shape[1]):
            if var is T:
                line += '{:.5f}'.format(var[i, j])
            else:
                line += '{:.5e}'.format(var[i, j])
            if j == var.shape[1] - 1:
                line += " \\\\"
            else:
                line += " & "
        print(line)
    print()

#####
# Load refined results

u = np.loadtxt('results/fine_u.csv', delimiter=',').T
v = np.loadtxt('results/fine_v.csv', delimiter=',').T
T = np.loadtxt('results/fine_T.csv', delimiter=',').T

dx = Lx / (T.shape[0] - 2)
dy = Ly / (T.shape[1] - 2)

#####
# Plot problem 4a

x_center = np.linspace(0, Lx, num = u.shape[0])
u_center = np.copy(u[:, int(u.shape[1] / 2)])
u_center /= u_center[0]
fig, ax = plt.subplots(1)
fig.set_figwidth(6)
fig.set_figheight(3)
ax.plot(x_center, u_center, 'k')
ax.set_xlabel(r'$x$ (m)')
ax.set_ylabel('Centerline $u / u_{\{\mathrm{in}\}}$')
ax.grid()
fig.tight_layout()
fig.savefig('results/u_centerline.pdf', bbox_inches='tight')

#####
# Plot problem 4b

for var, name in [(u, 'u'), (v, 'v'), (T, 'T')]:
    if name == 'T':
        y_8 = np.hstack((0, (np.linspace(dy / 2, Ly - dy / 2, num = var.shape[1] - 2)), Ly))
    else:
        y_8 = np.linspace(0, Ly, num = var.shape[1])
    var_8 = var[int(0.8 * var.shape[0] / Lx), :]

    fig, ax = plt.subplots(1)
```

```

fig.set_figwidth(6)
fig.set_figheight(3)
if name == 'y':
    ax.semilogy(y_8, var_8, 'k')
else:
    ax.plot(y_8, var_8, 'k')
ax.set_xlabel(r'$y$ (m)')
ax.set_ylabel('${0.8$ m$, y}$'.format(name))
ax.grid()
fig.tight_layout()
fig.savefig('results/{_0p8.pdf'.format(name), bbox_inches='tight')

#####
# Plot problem 4b

# Sample u at temperature nodes
uT = np.zeros(T.shape)
for i in range(1, T.shape[0] - 1):
    for j in range(u.shape[1]):
        uT[0, j] = u[0, j]
        uT[u.shape[0], j] = u[u.shape[0] - 1, j]
        uT[i, j] = (u[i - 1, j] + u[i, j]) / 2

# Compute Nusselt numbers
Nu = np.zeros(T.shape[0])
u_left = u[0, 0]
for i in range(T.shape[0]):
    Tw = T[i, 0]
    Tb = 0
    for j in range(1, T.shape[1] - 1):
        Tb += uT[i, j] * T[i, j]
    Tb *= rho * cp * dy / (u_left * cp * Ly * rho)
    Nu[i] = 2 * Ly * q / (k * (Tw - Tb))

# Plot
x = np.hstack((0, (np.linspace(dx / 2, Lx - dx / 2, num = len(Nu) - 2)), Lx))
fig, ax = plt.subplots(1)
fig.set_figwidth(6)
fig.set_figheight(3)
ax.plot(x, Nu, 'k')
ax.set_xlabel(r'$x$ (m)')
ax.set_ylabel('Nu')
ax.grid()
fig.tight_layout()
fig.savefig('results/Nu.pdf', bbox_inches='tight')

```