SOME NECESSARY CONCEPTS TO SOLVE PARTIAL DIFFERENTIAL EQUATIONS USING FINITE DIFFERENCE TECHNIQUES

Physical Classification:

1. **Equilibrium Problems**

These are problems in which a solution of a given partial differential equation (PDE) is desired in a closed domain subject to a prescribed set of boundary conditions. e.g. (1) Steady State temperature distribution in solids (2) Incompressible invicid flow and (3) equilibrium stress distribution in solids.

Sometimes equilibrium problems are referred to as Jury Problems. The solution of the PDE at every point in the domain (D) depends upon the prescribed boundary condition at every point on the boundary (B). Mathematically, the equilibrium problems are governed by elliptic partial differential equations.

2. **Marching Problems**

Marching or propagation problems are transient-like problems where the solution of a PDE is required in an open domain subject to a set of initial conditions and set of boundary conditions. Problems in this category are initial value or initial value boundary value problems. The solution should be obtained by marching outward from an initial data surface while satisfying the boundary condition. Mathematically these problems are governed by either hyperbolic or parabolic partial differential equations. e.g. Unsteady heat conduction and wave equation.

Mathematical Classification:

A large number of engineering problems are governed by the second order partial differential equations and some engineering problems are governed by first order partial differential equations. However, very few problems are governed by fourth order partial differential equations and mathematical classification of PDE's higher than second order is considerably more complicated. Hence, our discussion will be confined to the mathematical classification of second order quasi-linear and first order quasi-linear PDEs. The term, quasi-linear, implies that the equation is linear in its highest order derivative. Consider a second order quasi-linear PDE of the form

$$AU_{XX} + BU_{XY} + CU_{YY} + DU_X + EU_Y + FU = G$$

Now, the classification of the PDE depends on the sign of the discriminant $B^2 - 4AC$. Note that coefficients such as A, B and C may be functions of X and Y. Then, the given PDE can be classified as follows:

$$B^2 - 4AC < 0 \quad \text{Elliptic}$$
$$B^2 - 4AC = 0 \quad \text{Parabolic}$$
$$B^2 - 4AC > 0 \quad \text{Hyperbolic}$$

The terminology, elliptic, parabolic and hyperbolic, is chosen to classify PDEs and reflects the analogy between the form of discriminant $B^2 - 4AC$ for PDEs and the form of discriminant $B^2 - 4AC$ that classifies conic sections. Conic sections are described by the general second order algebraic equation of the form:

$$AX^2 + BXY + CY^2 + DX + EY + F = 0$$

From an engineering point, we should ask the question, what impact does this classification have on the numerical procedure? The classification is directly linked to the way information propagates in the domain, D(X,Y) and the pattern of information propagation dictates to the type of numerical procedure to be used.

The concept of the characteristics of a PDE is strongly linked to the classification of a PDE. The characteristics are (N - 1) dimensional hypersurfaces in N dimensional hyperspace that has some very special features. Here, N is the number of independent variables. The prefix, hyper, is used to denote spaces that can be of more than three spaces, such as X, Y, Z, and t spaces.

As stated earlier, the discussion here will be confined to second order quasi-linear equations and the corresponding domain D(X,Y) will be two-dimensional. In a two-dimensional domain, characteristics will be lines which are curves in D(X,Y) along which information or signals propagate. Discontinuities in the derivatives of the dependent variable, if they exist, propagate along the characteristic curves. If a PDE possesses real characteristics, then information propagates along these characteristics. If there are no real characteristics (i.e., complex), then there are no preferred paths of information propagation.

In this section we will determine characteristics of a second order quasi-linear equation. If discontinuities in the derivatives of the solution exist, then they must propagate along the characteristics. Alternatively, one can ask if there are any paths in D(X,Y) passing through a general point, P, along which the second derivatives, U_{XX}, U_{XY} and U_{YY}, are multivalued or discontinuous. If such paths do exist, then they are the paths of information propagation. For a second order equation there are three second derivatives and one needs three equations. One equations is the given PDE i.e.,

$$AU_{XX} + BU_{XY} + CU_{YY} + DU_X + EU_Y + F = G$$

the other two required equations are obtained by applying the chain rule to determine the total derivatives of U_X and U_Y, thus

$$d(U_X) = U_{XX}dX + U_{XY}dY$$
$$d(U_Y) = U_{YX}dX + u_{YY}dY$$

Therefore, the three equations can be written in matrix form as:

$$\begin{bmatrix} A \ \ B \ \ C \\ dX dY \ 0 \\ 0 \ \ dX dY \end{bmatrix} \begin{Bmatrix} U_{XX} \\ U_{XY} \\ U_{YY} \end{Bmatrix} = \begin{Bmatrix} -DU_X - EU_Y - F + G \\ d(U_X) \\ d(U_Y) \end{Bmatrix}$$

By applying Cramer's rule, one can determine unique values of U_{XX}, U_{XY}, and U_{YY}, unless the determinant of the coefficient matrix (Δ) is zero. If $\Delta = 0$, then the second derivatives are discontinuous or multivalued. i.e.:

$$AdY^2 - BdXdY + CdX^2 = 0$$

This is the characteristic equation and can be solved to yield:

$$\frac{dY}{dX} = \frac{B +- \sqrt{B^2 - 4AC}}{2A}$$

These two families of curves, if they exist, are the characteristic curves along which the second derivatives of U are multivalued or discontinuous. These curves may be complex (imaginary), real and repeated, or real and distinct depending on $B^2 - 4AC$.

$B^2 - 4AC$	Characteristic Curves	Classification
< 0	complex	elliptic
$= 0$	real and repeated	parabolic
> 0	real and distinct	hyperbolic

An example: As an exercise, show that a first order PDE of the form $aU_t + bU_X = c$8is hyperbolic for all values of a, b, and c.

Elliptic: Information propagates at infinite speed in all directions (in directions it displays two-way street behavior).

Hyperbolic: Information propagates at finite speed along special lines called characteristic lines. Along these lines PDE_$\rightarrow$ODE and resulting ODE can be solved by one of the many methods known a Priori (e.g. RK Methods).

Parabolic: Information propagates at finite speed in the direction that displays one way street behavior.

Types of Boundary Equations

Consider $\_^2 U = 0 \quad D9$

1.**Dirichlet boundary condition:** in this case the dependent variable is specified on the boundary i.e. $U = g(X,Y)$.

2.**Neumann boundary condition:** the normal derivative is specified on the boundary i.e. $\frac{\partial U}{\partial \eta} = g(X,Y). \quad 10$

3.The combination of Dirichlet and Neumann problems will lead to mixed boundary conditions. This is often referred to as **Robin's problems**. i.e. $a_1(X)\frac{\partial U}{\partial \eta} + a_2(X)U = N(X).11$

A Model Equation

Consider a one-dimensional transient heat conduction equation

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial X^2}, \quad U = U(X,t)$$

BCS

$$X = 0, \quad U = U_1, \quad \Big| \ I.C.\ U = U_0$$

$$X = 1, \quad U = U_2, \quad \Big| \ i.e.\ \ U(X,0) = U_0$$

1. This is a parabolic equation.
2. Displays parabolic behavior in time co-ordinate (one-way).
3. Displays elliptic behavior in X co-ordinate (two-way).
4. Marching type problem.

Consistency

Consistency deals with the extent to which the finite difference equations approximate the partial differential equation. Remember that we call the technique as finite difference because we include only a finite number of terms to represent a mathematical operator. We know that T.E. = PDE - FDE. We say that the method is consistent if lim (PDE - FDE) = 0 as mesh $\_\rightarrow 0$ or $\rightarrow 0 \quad T.E. = 014$ This should be the case when

$$T.E. = 0[\Delta t, \quad \Delta X].$$

However, if

$$T.E. = 0(\Delta t/\Delta X)$$

then for a method to be consistent the mesh should be refined such that T.E. $\rightarrow 0$ as $(\Delta t/\Delta X) \rightarrow 0$

Stability

Numerical stability is a concept applicable only to marching problems. A stable numerical scheme is one for which errors from any source (R.E. or T.E.) are not permitted to grow in sequence of numerical procedures, as calculations proceed from one step to another.

Convergence

Convergence means the solution to the finite-difference equation approaches the true solution to the PDE having the same initial and boundary conditions as the mesh is refined. In general, consistent stable scheme is also convergent.

Lax's Equivalence Theorem

Given a properly posed initial value problem and a finite difference to it that satisfies the consistency conditions, stability is the necessary and sufficient condition for convergence.

A Note on equilibrium Problems

If an iterative procedure is used then we say that the solutions has converged if:

$$\left| U_{I,J}^{K+1} - U_{I,J}^{K} \right| < \varepsilon$$

This is known as iteration convergence and ε is a very small number, e.g. 10^{-6}. When a non-iterative technique is used, make sure that the round-off errors do not get out of hand.

Conservative Variable:

The variable which is conserved. For example, energy, mass and momentum.

Primitive Variable:

The variable may not be conserved. For example, pressure, temperature and specific volume.

References:

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