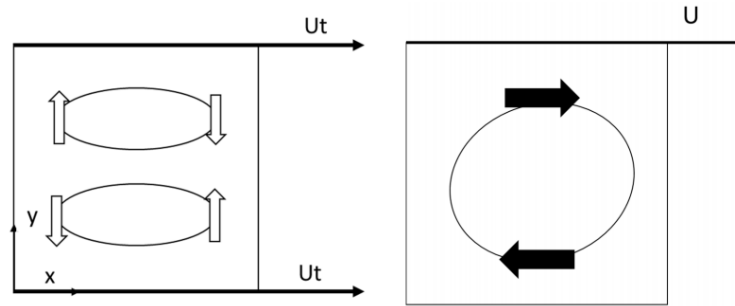


MEEN 644 - Homework 4

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1 Problem statement

A viscous fluid (water at 20°C: $\rho = 998.3 \text{ kg/m}^3$ and $\mu = 1.002 \times 10^{-3} \text{ N}\cdot\text{s/m}^2$) is trapped in a square 2-D cavity of dimension 0.2 m by 0.2 m. Either top or bottom walls are pulled to the right at a uniform velocity on purpose.



Left: flow for problem (a) to verify symmetry; right: flow for problem (b) to compare with Roy et al. (2015).

Write a finite-volume base computer program to predict the 2-D steady laminar flow field for $Re = 400$. Solve the velocity and pressure fields by linking them through the SIMPLE algorithm in a staggered grid. Represent the solution to the one-dimensional convection-diffusion problem using the power law scheme.

- (a) **(60 points)** In order to verify your code for symmetry, make calculations using 5×5 uniformly sized control volumes (CVs). Declare convergence when R_u and $R_v < 10^{-6}$ and $R_p < 10^{-5}$. Print your velocity and pressure fields up to 5 decimal places.
- (b) With the top plate pulled right at a constant velocity at $Re = 400$, calculate velocity and pressure fields using 8×8 , 16×16 , 32×32 , 64×64 , and 128×128 CVs.
 - i) **(20 points)** Plot the centerline u and v velocities for each case (for centerline u plot at $x = 0.1$ m, while for centerline v plot at $y = 0.1$ m).
 - ii) **(20 points)** Compare your solutions of the 128×128 CV case with the benchmark solution of Roy et al (2015) on Tables 4 and 5.

2 Preliminaries

2.1 Two-dimensional diffusion-convection

With two-dimensional diffusion and convection with constant material properties, we have the PDE

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}, \\ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}. \end{cases} \quad (1)$$

where Dirichlet boundary conditions are applied on the boundary for u and v .

The boundary conditions for problem (a) are:

$$\begin{cases} u(x, L_y) = \frac{\mu \text{Re}}{\rho L_x}, \\ u(L_x, y) = 0, \\ u(x, 0) = \frac{\mu \text{Re}}{\rho L_x}, \\ u(0, y) = 0, \\ v(x, L_y) = 0, \\ v(L_x, y) = 0, \\ v(x, 0) = 0, \\ v(0, y) = 0, \end{cases} \quad (2)$$

and the boundary conditions for problem (b) are:

$$\begin{cases} u(x, L_y) = \frac{\mu \text{Re}}{\rho L_x}, \\ u(L_x, y) = 0, \\ u(x, 0) = 0, \\ u(0, y) = 0, \\ v(x, L_y) = 0, \\ v(L_x, y) = 0, \\ v(x, 0) = 0, \\ v(0, y) = 0. \end{cases} \quad (3)$$

2.2 Solving methodology

Using the SIMPLE algorithm, the problem is solved in the following order:

1. Explicitly fill the boundary conditions into the u and v solution vector in order to enforce them in all of the integrations that follow.
2. Guess a pressure field, p^* .
3. Use the guessed (or previously iterated) pressure field to obtain the velocity guesses, u^* and v^* , as discussed in 2.4.
4. Solve the pressure correction, p' , as discussed in 2.5.
5. Compute the velocity corrections, u' and v' , and correct the velocity and pressure field, as discussed in 2.6.
6. Check for convergence. If not converged, return to 2.

2.3 Domain discretization

The domain of size $L_x \times L_y$ is discretized into $N_x \times N_y$ uniformly sized control volumes with $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$. The numbering for all variables begins at the origin at $(i, j) = (0, 0)$. The maximum index for each variable, ϕ , is defined as (M_x^ϕ, M_y^ϕ) .

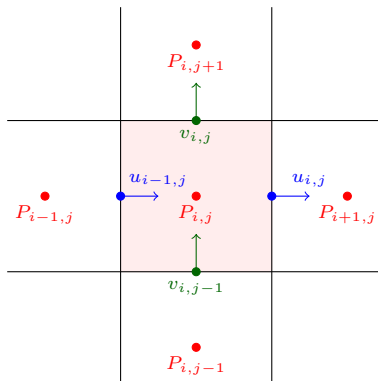


Figure 1: An internal pressure control volume.

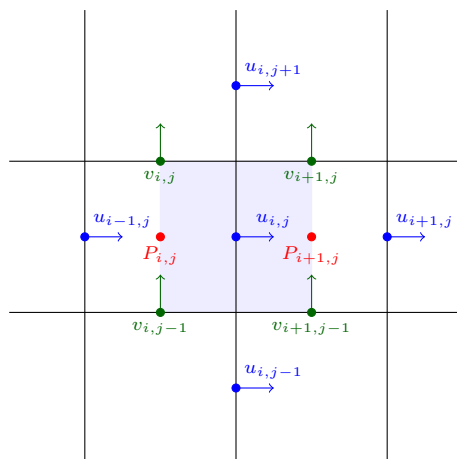


Figure 2: An internal u -velocity control volume.

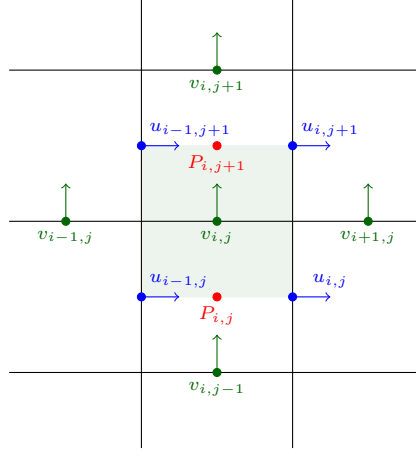


Figure 3: An internal v -velocity control volume.

2.4 Velocity guess

Define the Pechlet number on each boundary of a CV for variable ϕ centered at node $\phi_{i,j}$ as

$$P_{\text{bd}}^{\phi_{i,j}} = \frac{F_{\text{bd}}^{\phi_{i,j}}}{D_{\text{bd}}^{\phi_{i,j}}}, \quad \text{where } \text{bd} = [n, e, s, w] \quad \text{and} \quad \phi = [u, v]. \quad (4)$$

The integration of the x and y -momentum equations (generalizing again with $\phi = [u, v]$) using the power-law scheme results in the equation (for $i = 1, \dots, M_x^\phi - 1$, $j = 1, \dots, M_y^\phi - 1$)

$$a_p^{\phi_{i,j}} \phi_{i,j}^* = a_n^{\phi_{i,j}} \phi_{i,j+1}^* + a_e^{\phi_{i,j}} \phi_{i+1,j}^* + a_s^{\phi_{i,j}} \phi_{i,j-1}^* + a_w^{\phi_{i,j}} \phi_{i-1,j}^* + a_b^{\phi_{i,j}}, \quad (5a)$$

$$a_n^{\phi_{i,j}} = D_n^{\phi_{i,j}} \max [0, (1 - 0.1 |P_n^{\phi_{i,j}}|)^5] + \max [-F_n^{\phi_{i,j}}, 0], \quad (5b)$$

$$a_e^{\phi_{i,j}} = D_e^{\phi_{i,j}} \max [0, (1 - 0.1 |P_e^{\phi_{i,j}}|)^5] + \max [-F_e^{\phi_{i,j}}, 0], \quad (5c)$$

$$a_s^{\phi_{i,j}} = D_s^{\phi_{i,j}} \max [0, (1 - 0.1 |P_s^{\phi_{i,j}}|)^5] + \max [F_s^{\phi_{i,j}}, 0], \quad (5d)$$

$$a_w^{\phi_{i,j}} = D_w^{\phi_{i,j}} \max [0, (1 - 0.1 |P_w^{\phi_{i,j}}|)^5] + \max [F_w^{\phi_{i,j}}, 0], \quad (5e)$$

$$a_p^{\phi_{i,j}} = a_n^{\phi_{i,j}} + a_e^{\phi_{i,j}} + a_s^{\phi_{i,j}} + a_w^{\phi_{i,j}}, \quad (5f)$$

$$a_b^{\phi_{i,j}} = \begin{cases} \Delta y (p_{i,j}^* - p_{i+1,j}^*), & \phi = u \\ \Delta x (p_{i,j}^* - p_{i,j+1}^*), & \phi = v \end{cases}. \quad (5g)$$

2.4.1 u -velocity guess update

In all discussion that follow, we are considering a u -CV defined by the central node $u_{i,j}$. For simplicity we will define the width of each u -CV as

$$\Delta x^{u_{i,j}} = \begin{cases} \Delta x, & 1 < i < M_x^u - 1 \\ \frac{3}{2} \Delta x, & \text{otherwise} \end{cases}, \quad (6)$$

and we will also define the y -distance from $u_{i,j}$ to the north and south pressure interfaces, respectively, as

$$\delta y_{p_n}^{u_{i,j}} = \begin{cases} \frac{1}{2}\Delta y, & j = M_y^u - 1, \\ \Delta y, & \text{otherwise} \end{cases}, \quad (7a)$$

$$\delta y_{p_s}^{u_{i,j}} = \begin{cases} \frac{1}{2}\Delta y, & j = 1, \\ \Delta y, & \text{otherwise} \end{cases}. \quad (7b)$$

The diffusion coefficients are then defined as

$$D_n^{u_{i,j}} = \frac{\mu \Delta x^{u_{i,j}}}{\delta y_{p_n}^{u_{i,j}}}, \quad (8a)$$

$$D_e^{u_{i,j}} = \frac{\mu \Delta y}{\Delta x}, \quad (8b)$$

$$D_s^{u_{i,j}} = \frac{\mu \Delta x^{u_{i,j}}}{\delta y_{p_s}^{u_{i,j}}}, \quad (8c)$$

$$D_w^{u_{i,j}} = \frac{\mu \Delta y}{\Delta x}. \quad (8d)$$

Lastly, the flow rates are defined as

$$F_n^{u_{i,j}} = \rho \Delta x^{u_{i,j}} \begin{cases} \frac{1}{6} (v_{0,j}^* + 3v_{1,j}^* + 2v_{2,j}^*), & i = 1 \\ \frac{1}{6} (2v_{i,j}^* + 3v_{i+1,j}^* + v_{i+2,j}^*), & i = M_x^u - 1, \\ \frac{1}{2} (v_{i,j}^* + v_{i+1,j}^*), & \text{otherwise} \end{cases} \quad (9a)$$

$$F_e^{u_{i,j}} = \rho \Delta y \begin{cases} u_{M_x^u,j}^*, & i = M_x^u - 1 \\ \frac{1}{2} (u_{i+1,j}^* + u_{i,j}^*), & \text{otherwise} \end{cases}, \quad (9b)$$

$$F_s^{u_{i,j}} = \rho \Delta x^{u_{i,j}} \begin{cases} \frac{1}{6} (v_{0,j-1}^* + 3v_{1,j-1}^* + 2v_{2,j-1}^*), & i = 1 \\ \frac{1}{6} (2v_{i,j-1}^* + 3v_{i+1,j-1}^* + v_{i+2,j-1}^*), & i = M_x^u - 1 \\ \frac{1}{2} (v_{i,j-1}^* + v_{i+1,j-1}^*), & \text{otherwise} \end{cases} \quad (9c)$$

$$F_w^{u_{i,j}} = \rho \Delta y \begin{cases} u_{0,j}^*, & i = 1 \\ \frac{1}{2} (u_{i-1,j}^* + u_{i,j}^*), & \text{otherwise} \end{cases}. \quad (9d)$$

2.4.2 v -velocity guess update

Similarly, we will consider a v -CV defined by the central node $v_{i,j}$. The width of each v -CV is defined as

$$\Delta y^{v_{i,j}} = \begin{cases} \Delta y, & 1 < j < M_y^v - 1, \\ \frac{3}{2}\Delta y, & \text{otherwise} \end{cases}, \quad (10)$$

and we will also define the x -distance from $v_{i,j}$ to the east and west pressure interfaces, respectively, as

$$\delta x_{p_e}^{v_{i,j}} = \begin{cases} \frac{1}{2}\Delta x, & i = M_x^v - 1, \\ \Delta x, & \text{otherwise} \end{cases}, \quad (11a)$$

$$\delta x_{p_w}^{v_{i,j}} = \begin{cases} \frac{1}{2}\Delta x, & i = 1, \\ \Delta x, & \text{otherwise} \end{cases}. \quad (11b)$$

The diffusion coefficients are then defined as

$$D_n^{v_{i,j}} = \frac{\mu \Delta x}{\Delta y} , \quad (12a)$$

$$D_e^{v_{i,j}} = \frac{\mu \Delta y^{v_{i,j}}}{\delta x_{p_e}^{v_{i,j}}} , \quad (12b)$$

$$D_s^{v_{i,j}} = \frac{\mu \Delta x}{\Delta y} , \quad (12c)$$

$$D_w^{v_{i,j}} = \frac{\mu \Delta y^{v_{i,j}}}{\delta x_{p_w}^{v_{i,j}}} . \quad (12d)$$

Lastly, the flow rates are defined as

$$F_n^{v_{i,j}} = \rho \Delta x \begin{cases} v_{i,M_y^v}^* , & j = M_y^v - 1 \\ \frac{1}{2} (v_{i,j+1}^* + v_{i,j}^*) , & \text{otherwise} \end{cases} , \quad (13a)$$

$$F_e^{v_{i,j}} = \rho \Delta y^{v_{i,j}} \begin{cases} \frac{1}{6} (u_{i,0}^* + 3u_{i,1}^* + 2u_{i,2}^*) , & j = 1 \\ \frac{1}{6} (2u_{i,j}^* + 3u_{i,j+1}^* + 2u_{i,j+2}^*) , & j = M_y^v - 1 \\ \frac{1}{2} (u_{i,j}^* + u_{i,j+1}^*) , & \text{otherwise} \end{cases} , \quad (13b)$$

$$F_s^{v_{i,j}} = \rho \Delta x \begin{cases} v_{i,0}^* , & j = 1 \\ \frac{1}{2} (v_{i,j+1}^* + v_{i,j}^*) , & \text{otherwise} \end{cases} , \quad (13c)$$

$$F_w^{v_{i,j}} = \rho \Delta y^{v_{i,j}} \begin{cases} \frac{1}{6} (u_{i-1,0}^* + 3u_{i-1,1}^* + 2u_{i-1,2}^*) , & j = 1 \\ \frac{1}{6} (2u_{i-1,j}^* + 3u_{i-1,j+1}^* + 2u_{i-1,j+2}^*) , & j = M_y^v - 1 \\ \frac{1}{2} (u_{i-1,j}^* + u_{i-1,j+1}^*) , & \text{otherwise} \end{cases} . \quad (13d)$$

2.5 Pressure correction solve

At convergence, $u' = v' = p' = 0$, therefore it is irrelevant as to how we find them. Subtracting (exact - guessed) forms of Equation (5) one obtains

$$a_p^{u_{i,j}} u'_{i,j} = a_n^{u_{i,j}} u'_{i,j+1} + a_e^{u_{i,j}} u'_{i+1,j} + a_s^{u_{i,j}} u'_{i,j-1} + a_w^{u_{i,j}} u'_{i-1,j} + \Delta y (p'_{i,j} - p'_{i+1,j}) , \quad (14a)$$

$$a_p^{v_{i,j}} v'_{i,j} = a_n^{v_{i,j}} v'_{i,j+1} + a_e^{v_{i,j}} v'_{i+1,j} + a_s^{v_{i,j}} v'_{i,j-1} + a_w^{v_{i,j}} v'_{i-1,j} + \Delta x (p'_{i,j} - p'_{i,j+1}) . \quad (14b)$$

Drop the neighbor terms in the equations above (implying that velocity corrections are local) to obtain

$$u'_{i,j} = \frac{\Delta y}{a_p^{u_{i,j}}} (p'_{i,j} - p'_{i+1,j}) , \quad (15a)$$

$$v'_{i,j} = \frac{\Delta x}{a_p^{v_{i,j}}} (p'_{i,j} - p'_{i,j+1}) . \quad (15b)$$

Integrate the continuity equation over a p -CV defined by the central node $p_{i,j}$ and substitute $u = u^* + u'$, $v = v^* + v'$, and the above Equations to obtain

$$a_p^{p'_{i,j}} p'_{i,j} = a_n^{p'_{i,j}} p'_{i,j+1} + a_e^{p'_{i,j}} p'_{i+1,j} + a_s^{p'_{i,j}} p'_{i,j-1} + a_w^{p'_{i,j}} p'_{i-1,j} + a_b^{p'_{i,j}}, \quad (16a)$$

$$a_n^{p'_{i,j}} = \begin{cases} \rho \Delta x^2 / a_p^{v_{i,j}}, & j < M_y^p - 1 \\ 0, & \text{otherwise} \end{cases}, \quad (16b)$$

$$a_e^{p'_{i,j}} = \begin{cases} \rho \Delta y^2 / a_p^{u_{i,j}}, & i < M_x^p - 1 \\ 0, & \text{otherwise} \end{cases}, \quad (16c)$$

$$a_s^{p'_{i,j}} = \begin{cases} \rho \Delta x^2 / a_p^{v_{i,j-1}}, & j > 1 \\ 0, & \text{otherwise} \end{cases}, \quad (16d)$$

$$a_w^{p'_{i,j}} = \begin{cases} \rho \Delta y^2 / a_p^{u_{i-1,j}}, & i > 1 \\ 0, & \text{otherwise} \end{cases}, \quad (16e)$$

$$a_p^{p'_{i,j}} = a_n^{p'_{i,j}} + a_e^{p'_{i,j}} + a_s^{p'_{i,j}} + a_w^{p'_{i,j}}, \quad (16f)$$

$$a_b^{p'_{i,j}} = \rho (\Delta y (u_{i-1,j}^* - u_{i,j}^*) + \Delta x (v_{i,j-1}^* - v_{i,j}^*)). \quad (16g)$$

2.6 Velocity and pressure correction

Lastly, the velocities are then updated using Equation (15) with

$$u_{i,j} = u_{i,j} + \frac{\Delta y}{a_p^{u_{i,j}}} (p'_{i,j} - p'_{i+1,j}), \quad i = 1, \dots, M_x^u - 1, \quad j = 1, \dots, M_y^u - 1, \quad (17a)$$

$$v_{i,j} = v_{i,j} + \frac{\Delta x}{a_p^{v_{i,j}}} (p'_{i,j} - p'_{i,j+1}), \quad i = 1, \dots, M_x^v - 1, \quad j = 1, \dots, M_y^v - 1, \quad (17b)$$

and the pressures (take note of the relaxation parameter α_p) with

$$p_{i,j} = p_{i,j} + \alpha_p p'_{i,j}, \quad i = 1, \dots, M_x^p - 1, \quad j = 1, \dots, M_y^p - 1. \quad (18)$$

2.7 System solver

The systems in Equations (5) and (16) are solved using the line-by-line method with TDMA as the matrix solver. In this method, a tri-diagonal system is formed as the terms from one of the dimensions are lagged. Consider the simple system

$$a_p^{i,j} \phi_{i,j} = a_n^{i,j} \phi_{i,j+1} + a_e^{i,j} \phi_{i+1,j} + a_s^{i,j} \phi_{i,j-1} + a_w^{i,j} \phi_{i-1,j} + a_b^{i,j}, \quad i = 1, \dots, N_x, \quad j = 1, \dots, N_y. \quad (19)$$

Now, consider ϕ^* to be a *lagged* value of ϕ , i.e., it is known and is moved to the right hand side of each equation. In solving a single physical column i using the line-by-line method, the following system is solved:

$$a_p^{i,j} \phi_{i,j} = a_n^{i,j} \phi_{i,j+1} + a_e^{i,j} \phi_{i+1,j}^* + a_s^{i,j} \phi_{i,j-1} + a_w^{i,j} \phi_{i-1,j}^* + a_b^{i,j}, \quad j = 1, \dots, N_y. \quad (20)$$

In solving a single physical row j using the line-by-line method, the following system is solved:

$$a_p^{i,j} \phi_{i,j} = a_n^{i,j} \phi_{i,j+1}^* + a_e^{i,j} \phi_{i+1,j} + a_s^{i,j} \phi_{i,j-1}^* + a_w^{i,j} \phi_{i-1,j} + a_b^{i,j}, \quad i = 1, \dots, N_x. \quad (21)$$

3 Results

3.1 Problem a: Lid driven cavity, symmetry check

Symmetric solutions were identified for problem (a), as presented in Tables 1, 2, and 3 below.

Table 1: The u -velocity solution with 5x5 pressure CVs and symmetric BCs.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|------------|-------------|-------------|-------------|-------------|------------|
| 1 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 |
| 2 | 0.00000E0 | 1.69059E-4 | 2.96268E-4 | 3.47982E-4 | 2.74870E-4 | 0.00000E0 |
| 3 | 0.00000E0 | -6.05705E-5 | -1.24904E-4 | -1.54163E-4 | -1.09384E-4 | 0.00000E0 |
| 4 | 0.00000E0 | -2.16978E-4 | -3.42727E-4 | -3.87638E-4 | -3.30974E-4 | 0.00000E0 |
| 5 | 0.00000E0 | -6.05705E-5 | -1.24904E-4 | -1.54163E-4 | -1.09384E-4 | 0.00000E0 |
| 6 | 0.00000E0 | 1.69059E-4 | 2.96268E-4 | 3.47982E-4 | 2.74870E-4 | 0.00000E0 |
| 7 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 | 2.00741E-3 |

Table 2: The v -velocity solution with 5x5 pressure CVs and symmetric BCs.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|-----------|-------------|-------------|-------------|-------------|-------------|-----------|
| 1 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 |
| 2 | 0.00000E0 | -1.69059E-4 | -1.27208E-4 | -5.17144E-5 | 7.31116E-5 | 2.74870E-4 | 0.00000E0 |
| 3 | 0.00000E0 | -1.08489E-4 | -6.28747E-5 | -2.24555E-5 | 2.83322E-5 | 1.65487E-4 | 0.00000E0 |
| 4 | 0.00000E0 | 1.08489E-4 | 6.28746E-5 | 2.24555E-5 | -2.83322E-5 | -1.65487E-4 | 0.00000E0 |
| 5 | 0.00000E0 | 1.69059E-4 | 1.27208E-4 | 5.17144E-5 | -7.31116E-5 | -2.74870E-4 | 0.00000E0 |
| 6 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 | 0.00000E0 |

Table 3: The p solution with 5x5 pressure CVs and symmetric BCs.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|-------------|-------------|-------------|-------------|-------------|------------|------------|
| 1 | -9.59831E-5 | -9.59831E-5 | -5.96700E-6 | 1.14365E-5 | 6.78274E-5 | 2.13791E-4 | 2.13791E-4 |
| 2 | -9.59831E-5 | -9.59831E-5 | -5.96700E-6 | 1.14365E-5 | 6.78274E-5 | 2.13791E-4 | 2.13791E-4 |
| 3 | 1.42128E-6 | 1.42128E-6 | -2.15270E-5 | -3.57174E-5 | -1.98216E-5 | 7.52326E-5 | 7.52326E-5 |
| 4 | 1.86045E-5 | 1.86045E-5 | -9.11188E-6 | -2.11868E-5 | 1.15363E-5 | 8.26938E-5 | 8.26938E-5 |
| 5 | 1.42128E-6 | 1.42128E-6 | -2.15270E-5 | -3.57174E-5 | -1.98216E-5 | 7.52326E-5 | 7.52326E-5 |
| 6 | -9.59831E-5 | -9.59831E-5 | -5.96700E-6 | 1.14365E-5 | 6.78274E-5 | 2.13791E-4 | 2.13791E-4 |
| 7 | -9.59831E-5 | -9.59831E-5 | -5.96700E-6 | 1.14365E-5 | 6.78274E-5 | 2.13791E-4 | 2.13791E-4 |

3.2 Problem b: Lid driven cavity, top right BC

The requirements for problem (b) part i and ii were combined into Figure 4 as seen below. With increasing grid refinement, both centerline velocity profiles approached towards the reference solution obtained from Roy et. al. In addition, a once-more-refined run is compared with 256x256 CVs with good agreement.

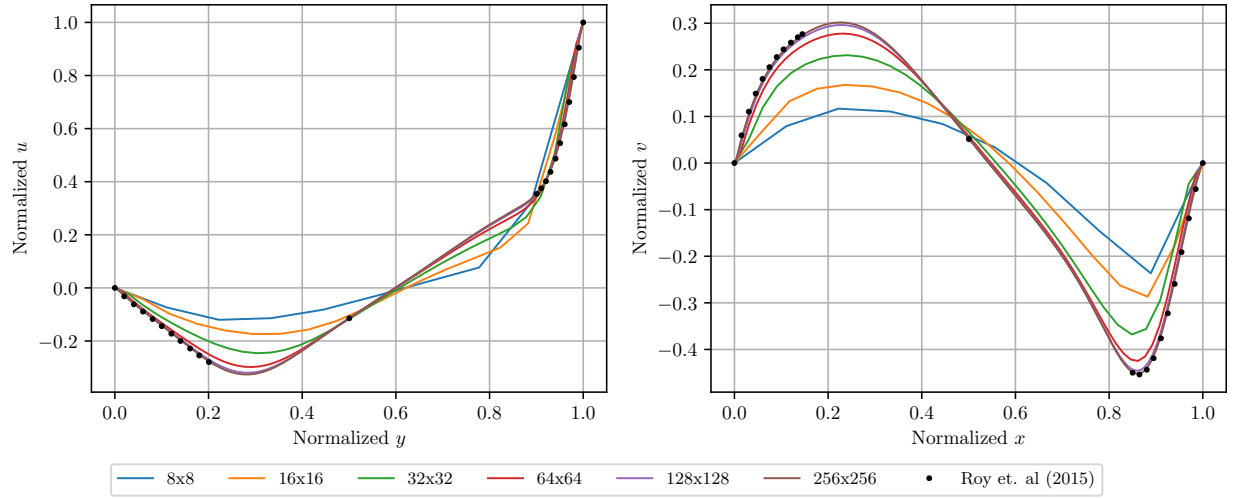


Figure 4: Centerline u and v -velocity fields with the top plate pulled right at a constant velocity.

Code listing

For the implementation, we have the following files:

- **Makefile** – Allows for compiling the c++ project with **make**.
- **hwk4.cpp** – Contains the **main()** function that is required by C that runs the cases requested in this problem set.
- **Problem.h** – Contains the header for the **Problem** class which is the main driver for a **Flow2D::Problem**.
- **Variable.h** – Contains the **Flow2D::Variable** class, which is a storage container for a single variable (i.e., u).
- **Problem.cpp** – Contains the **run()** functions that executes a **Problem**.
- **Problem_coefficients.cpp** – Contains the functions for solving coefficients in a **Problem**.
- **Problem_corrections.cpp** – Contains the functions for correcting solutions in a **Problem**.
- **Problem_residuals.cpp** – Contains the functions for computing residuals in a **Problem**.
- **Problem_solvers.cpp** – Contains the functions for sweeping and solving in a **Problem**.
- **Matrix.h** – Contains the **Matrix** class which provides storage for a matrix with various standard matrix operations.
- **TriDiagonal.h** – Contains the **TriDiagonal** class which provides storage for a tri-diagonal matrix including the TDMA solver found in the member function **solveTDMA()**.
- **Vector.h** – Contains the **Vector** class for one-dimensional vector storage.
- **plots.py** - Produces the plots in this report.

Makefile

```
src = $(wildcard *.cpp)
obj = $(src:.cpp=.o)
CXXFLAGS = -std=c++14
CCLFLAGS = $(CXXFLAGS)

hwk-opt: $(obj)
        clang++ -o $@ $^

.PHONY: clean
clean:
        rm -f $(obj) hwk-opt
```

hwk4.cpp

```
#include "Problem.h"

using namespace Flow2D;

int
main()
{
    // Problem wide constants
    double L = 0.2;
    double Re = 400;
    double rho = 998.3;
    double mu = 1.002e-3;
    double bc_val = Re * mu / (rho * L);

    // Standard inputs
    InputArguments input;
    input.Lx = L;
    input.Ly = L;
    input.mu = mu;
    input.rho = rho;
    input.u_ref = bc_val;
    input.L_ref = L;

    // Problem 1: check symmetry
    input.u_bc = BoundaryCondition(bc_val, 0, bc_val, 0);
    input.v_bc = BoundaryCondition(0, 0, 0, 0);
    std::cout << "Problem 1, check symmetry" << std::endl;
    {
        Problem problem(5, 5, input);
        problem.run();
        problem.print(Variables::u, "u =");
        problem.print(Variables::v, "v =");
        problem.print(Variables::p, "p =", true);
    }

    // Problem 2: change to top plate BC
    input.u_bc = BoundaryCondition(bc_val, 0, 0, 0);
    input.v_bc = BoundaryCondition(0, 0, 0, 0);
    std::cout << "Problem 2, top plate BC" << std::endl;
    for (unsigned int N : {8, 16, 32, 64, 128, 256})
    {
        std::cout << "N = " << N << "x" << N << " - ";
        Problem problem(N, N, input);
        problem.run();
        problem.save(Variables::u, "results/" + to_string(N) + "_u.csv");
        problem.save(Variables::v, "results/" + to_string(N) + "_v.csv");
    }
}
```

Problem.h

```
#ifndef PROBLEM_H
#define PROBLEM_H

#include <cmath>
#include <ctime>
#include <iomanip>
#include <iostream>
#include <map>

#include "Variable.h"

namespace Flow2D
{
    using namespace std;

    struct InputArguments
    {
        double Lx, Ly;
        BoundaryCondition u_bc, v_bc;
        double L_ref, u_ref;
        double rho, mu;
        bool debug = false;
        double alpha_p = 0.7;
        double alpha_uv = 0.5;
        unsigned int max_its = 100000;
        double tol = 1.0e-6;
    };

    class Problem
    {
    public:
        Problem(const unsigned int Nx, const unsigned int Ny, const InputArguments & input);

        void run();

        // Public access to printing and saving variable results
        void print(const Variables var,
                  const string prefix = "",
                  const bool newline = false,
                  const unsigned int pr = 5) const
        {
            variables.at(var).print(prefix, newline, pr);
        }
        void save(const Variables var, const string filename) const { variables.at(var).save(filename); }

    private:
        // Problem_corrections.cpp
        void correct();
        void pCorrect();
        void uCorrect();
        void vCorrect();

        // Problem_coefficients.cpp
        void fillCoefficients(const Variable & var);
        void pcCoefficients();
        void uCoefficients();
        void vCoefficients();
        void velocityCoefficients(Coefficients & a,
                                  const Coefficients & D,
                                  const Coefficients & F,
                                  const double & b);

        // Problem_residuals.cpp
        void computeResiduals();
        double pResidual() const;
        double velocityResidual(const Variable & var) const;

        // Problem_solvers.cpp
        void solve();
        void solve(Variable & var);
        void sweepColumns(Variable & var, const bool west_east = true);
        void sweepRows(Variable & var, const bool south_north = true);
        void sweepColumn(const unsigned int i, Variable & var);
        void sweepRow(const unsigned int j, Variable & var);
        void solveVelocities();
    };
}
```

```

protected:
    // Number of pressure CVs
    const unsigned int Nx, Ny;

    // Geometry [m]
    const double Lx, Ly, dx, dy;
    // Material properties
    const double rho, mu;
    // Residual references
    const double L_ref, u_ref;

    // Enable debug mode (printing extra output)
    const bool debug;

    // Maximum iterations
    const unsigned int max_its;
    // Iteration tolerance
    const double tol;
    // Pressure relaxation
    const double alpha_p;
    // Number of iterations completed
    unsigned int iterations = 0;

    // Variables
    Variable u, v, pc, p;
    // Variable map
    map<const Variables, const Variable &> variables;

    // Whether or not we converged
    bool converged = false;
    // Run start time
    clock_t start;
};

} // namespace Flow2D
#endif /* PROBLEM_H */

```

Variable.h

```
#ifndef VARIABLE_H
#define VARIABLE_H

#include "Matrix.h"
#include "TriDiagonal.h"
#include "Vector.h"

namespace Flow2D
{
    using namespace std;

    // Storage for boundary conditions
    struct BoundaryCondition
    {
        BoundaryCondition() {}
        BoundaryCondition(const double top, const double right, const double bottom, const double left)
            : top(top), right(right), bottom(bottom), left(left)
        {
        }
        bool nonzero() const { return top != 0 || right != 0 || bottom != 0 || left != 0; }
        double top = 0, right = 0, bottom = 0, left = 0;
    };

    // Storage for coefficients for a single CV
    struct Coefficients
    {
        double p = 0, n = 0, e = 0, s = 0, w = 0, b = 0;
        void print(const unsigned int pr = 5) const
        {
            cout << setprecision(pr) << scientific << "n = " << n << ", e = " << e << ", s = " << s
                << ", w = " << w << ", p = " << p << ", b = " << b << endl;
        }
    };

    // Enum for variable types
    enum Variables
    {
        u,
        v,
        pc,
        p
    };

    // Conversion from variable type to its string
    static string
    VariableString(Variables var)
    {
        switch (var)
        {
            case Variables::u:
                return "u";
            case Variables::v:
                return "v";
            case Variables::pc:
                return "pc";
            case Variables::p:
                return "p";
        }
    }

    // General storage structure for primary and auxiliary variables
    struct Variable
    {
        // Constructor for a primary variable
        Variable(const Variables name,
                 const unsigned int Nx,
                 const unsigned int Ny,
                 const double alpha,
                 const BoundaryCondition bc = BoundaryCondition())
            : name(name),
              string(VariableString(name)),
              Nx(Nx),
              Ny(Ny),
              Mx(Nx - 1),
              My(Ny - 1),
              w(1 / alpha),
    
```

```

        bc(bc),
        a(Nx, Ny),
        phi(Nx, Ny),
        Ax(Nx - 2),
        Ay(Ny - 2),
        bx(Nx - 2),
        by(Ny - 2)
    {
        // Apply initial boundary conditions
        if (bc.left != 0)
            phi.setColumn(0, bc.left);
        if (bc.right != 0)
            phi.setColumn(Mx, bc.right);
        if (bc.bottom != 0)
            phi.setRow(0, bc.bottom);
        if (bc.top != 0)
            phi.setRow(My, bc.top);
    }

    // Constructor for an auxiliary variable (no solver storage)
    Variable(const Variables name, const unsigned int Nx, const unsigned int Ny)
        : name(name), string(VariableString(name)), Nx(Nx), Ny(Ny), Mx(Nx - 1), My(Ny - 1), phi(Nx, Ny)
    {
    }

    // Solution matrix operations
    const double & operator()(const unsigned int i, const unsigned int j) const { return phi(i, j); }
    double & operator()(const unsigned int i, const unsigned int j) { return phi(i, j); }
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 5) const
    {
        phi.print(prefix, newline, pr);
    }
    void save(const string filename) const { phi.save(filename); }
    void reset() { phi = 0; }

    // Coefficient debug
    void printCoefficients(const string prefix = "",
                          const bool newline = false,
                          const unsigned int pr = 5) const
    {
        for (unsigned int i = 1; i < Nx - 1; ++i)
            for (unsigned int j = 1; j < Ny - 1; ++j)
            {
                cout << prefix << "(" << i << ", " << j << "): ";
                a(i, j).print(pr);
            }
        if (newline)
            cout << endl;
    }

    // Variable enum name
    const Variables name;
    // Variable string
    const string string;
    // Variable size
    const unsigned int Nx, Ny;
    // Maximum variable index that is being solved
    const unsigned int Mx, My;
    // Relaxation coefficient used in solving linear systems
    const double w = 0;
    // Boundary conditions
    const BoundaryCondition bc = BoundaryCondition();
    // Matrix coefficients
    Matrix<Coefficients> a;
    // Variable solution
    Matrix<double> phi;
    // Linear system LHS for both sweep directions
    TriDiagonal<double> Ax, Ay;
    // Linear system RHS for both sweep directions
    Vector<double> bx, by;
};

} // namespace Flow2D
#endif /* VARIABLE_H */

```

Problem.cpp

```
#include "Problem.h"

namespace Flow2D
{
    Problem::Problem(const unsigned int Nx, const unsigned int Ny, const InputArguments & input)
        : // Number of pressure CVs
          Nx(Nx),
          Ny(Ny),
          // Domain sizes
          Lx(input.Lx),
          Ly(input.Ly),
          dx(Lx / Nx),
          dy(Ly / Ny),
          // Residual references
          L_ref(input.L_ref),
          u_ref(input.u_ref),
          // Material properties
          rho(input.rho),
          mu(input.mu),
          // Enable debug
          debug(input.debug),
          // Solver properties
          max_its(input.max_its),
          tol(input.tol),
          alpha_p(input.alpha_p),
          // Initialize variables for u, v, pc (solved variables)
          u(Variables::u, Nx + 1, Ny + 2, input.alpha_uv, input.u_bc),
          v(Variables::v, Nx + 2, Ny + 1, input.alpha_uv, input.v_bc),
          pc(Variables::pc, Nx + 2, Ny + 2, 1),
          // Initialize aux variables
          p(Variables::p, Nx + 2, Ny + 2)
    {
        // Add into variable map for access outside of class
        variables.emplace(Variables::u, u);
        variables.emplace(Variables::v, v);
        variables.emplace(Variables::pc, pc);
        variables.emplace(Variables::p, p);
    }

    void
    Problem::run()
    {
        // Store start time
        start = clock();

        for (unsigned int l = 0; l < max_its; ++l)
        {
            ++iterations;
            if (debug)
                cout << "Iteration " << setw(3) << left << l << ": " << endl;

            // Solve for all variables
            solve();

            // Apply corrections
            correct();

            // Compute residuals and exit if converged
            computeResiduals();
            if (converged)
                break;
        }

        // Oops. Didn't converge
        if (!converged)
            cout << "Did not converge after " << max_its << " iterations!" << endl;
    }
}
```

Problem_coefficients.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::fillCoefficients(const Variable & var)
    {
        if (var.name == Variables::u)
            uCoefficients();
        else if (var.name == Variables::v)
            vCoefficients();
        else if (var.name == Variables::pc)
            pcCoefficients();

        if (debug)
            var.printCoefficients(var.string, true);
    }

    void
    Problem::pcCoefficients()
    {
        for (unsigned int i = 1; i < pc.Mx; ++i)
            for (unsigned int j = 1; j < pc.My; ++j)
            {
                Coefficients & a = pc.a(i, j);

                if (i != 1)
                    a.w = rho * dy * dy / u.a(i - 1, j).p;
                if (i != pc.Mx - 1)
                    a.e = rho * dy * dy / u.a(i, j).p;
                if (j != 1)
                    a.s = rho * dx * dx / v.a(i, j - 1).p;
                if (j != pc.My - 1)
                    a.n = rho * dx * dx / v.a(i, j).p;
                a.p = a.n + a.e + a.s + a.w;
                a.b = rho * (dy * (u(i - 1, j) - u(i, j)) + dx * (v(i, j - 1) - v(i, j)));
            }
    }

    void
    Problem::uCoefficients()
    {
        Coefficients D, F;
        double W, dy_pn, dy_ps, b;

        for (unsigned int i = 1; i < u.Mx; ++i)
            for (unsigned int j = 1; j < u.My; ++j)
            {
                // Width of the cell
                W = (i == 1 || i == u.Mx - 1 ? 3 * dx / 2 : dx);
                // North/south distances to pressure nodes
                dy_pn = (j == u.My - 1 ? dy / 2 : dy);
                dy_ps = (j == 1 ? dy / 2 : dy);

                // Diffusion coefficients
                D.n = mu * W / dy_pn;
                D.e = mu * dy / dx;
                D.s = mu * W / dy_ps;
                D.w = mu * dy / dx;

                // East and west flows
                F.e = (i == u.Mx - 1 ? rho * dy * u(u.Mx, j) : rho * dy * (u(i + 1, j) + u(i, j)) / 2);
                F.w = (i == 1 ? rho * dy * u(0, j) : rho * dy * (u(i - 1, j) + u(i, j)) / 2);
                // North and south flows
                if (i == 1) // Left boundary
                {
                    F.n = rho * W * (v(0, j) + 3 * v(1, j) + 2 * v(2, j)) / 6;
                    F.s = rho * W * (v(0, j - 1) + 3 * v(1, j - 1) + 2 * v(2, j - 1)) / 6;
                }
                else if (i == u.Mx - 1) // Right boundary
                {
                    F.n = rho * W * (2 * v(i, j) + 3 * v(i + 1, j) + v(i + 2, j)) / 6;
                    F.s = rho * W * (2 * v(i, j - 1) + 3 * v(i + 1, j - 1) + v(i + 2, j - 1)) / 6;
                }
                else // Interior (not left or right boundary)
                {

```



```

        F.n = rho * W * (v(i, j) + v(i + 1, j)) / 2;
        F.s = rho * W * (v(i, j - 1) + v(i + 1, j - 1)) / 2;
    }

    // Pressure RHS
    b = dy * (p(i, j) - p(i + 1, j));

    // Compute and store power law coefficients
    velocityCoefficients(u.a(i, j), D, F, b);
}

void
Problem::vCoefficients()
{
    Coefficients D, F;
    double H, dx_pe, dx_pw, b;

    for (unsigned int i = 1; i < v.Mx; ++i)
        for (unsigned int j = 1; j < v.My; ++j)
        {
            // Height of the cell
            H = (j == 1 || j == v.My - 1 ? 3 * dy / 2 : dy);
            // East/west distances to pressure nodes
            dx_pe = (i == v.Mx - 1 ? dx / 2 : dx);
            dx_pw = (i == 1 ? dx / 2 : dx);

            // Diffusion coefficient
            D.n = mu * dx / dy;
            D.e = mu * H / dx_pe;
            D.s = mu * dx / dy;
            D.w = mu * H / dx_pw;

            // North and east flows
            F.n = (j == v.My - 1 ? rho * dx * v(i, v.My) : rho * dx * (v(i, j + 1) + v(i, j)) / 2);
            F.s = (j == 1 ? rho * dx * v(i, 0) : rho * dx * (v(i, j - 1) + v(i, j)) / 2);
            // East and west flows
            if (j == 1) // Bottom boundary
            {
                F.e = rho * H * (u(i, 0) + 3 * u(i, 1) + 2 * u(i, 2)) / 6;
                F.w = rho * H * (u(i - 1, 0) + 3 * u(i - 1, 1) + 2 * u(i - 1, 2)) / 6;
            }
            else if (j == v.My - 1) // Top boundary
            {
                F.e = rho * H * (2 * u(i, j) + 3 * u(i, j + 1) + u(i, j + 2)) / 6;
                F.w = rho * H * (2 * u(i - 1, j) + 3 * u(i - 1, j + 1) + u(i - 1, j + 2)) / 6;
            }
            else // Interior (not top or bottom boundary)
            {
                F.e = rho * H * (u(i, j) + u(i, j + 1)) / 2;
                F.w = rho * H * (u(i - 1, j) + u(i - 1, j + 1)) / 2;
            }

            // Pressure RHS
            b = dx * (p(i, j) - p(i, j + 1));

            // Compute and store power law coefficients
            velocityCoefficients(v.a(i, j), D, F, b);
        }
}

void
Problem::velocityCoefficients(Coefficients & a,
                             const Coefficients & D,
                             const Coefficients & F,
                             const double & b)
{
    a.n = D.n * fmax(0, pow(1 - 0.1 * fabs(F.n / D.n), 5)) + fmax(-F.n, 0);
    a.e = D.e * fmax(0, pow(1 - 0.1 * fabs(F.e / D.e), 5)) + fmax(-F.e, 0);
    a.s = D.s * fmax(0, pow(1 - 0.1 * fabs(F.s / D.s), 5)) + fmax(F.s, 0);
    a.w = D.w * fmax(0, pow(1 - 0.1 * fabs(F.w / D.w), 5)) + fmax(F.w, 0);
    a.p = a.n + a.e + a.s + a.w;
    a.b = b;
}

} // namespace Flow2D

```

Problem_corrections.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::correct()
    {
        uCorrect();
        vCorrect();
        pCorrect();
    }

    void
    Problem::pCorrect()
    {
        for (unsigned int i = 1; i < pc.Mx; ++i)
            for (unsigned int j = 1; j < pc.My; ++j)
                p(i, j) += alpha_p * pc(i, j);

        // Set pressure correction back to zero
        pc.reset();

        // Apply the edge values as velocity is set
        for (unsigned int i = 0; i <= pc.Mx; ++i)
        {
            p(i, 0) = p(i, 1);
            p(i, pc.My) = p(i, pc.My - 1);
        }
        for (unsigned int j = 0; j <= pc.My; ++j)
        {
            p(0, j) = p(1, j);
            p(pc.Mx, j) = p(pc.Mx - 1, j);
        }

        if (debug)
            p.print("p corrected = ", true);
    }

    void
    Problem::uCorrect()
    {
        for (unsigned int i = 1; i < u.Mx; ++i)
            for (unsigned int j = 1; j < u.My; ++j)
                u(i, j) += dy * (pc(i, j) - pc(i + 1, j)) / u.a(i, j).p;

        if (debug)
            u.print("u corrected = ", true);
    }

    void
    Problem::vCorrect()
    {
        for (unsigned int i = 1; i < v.Mx; ++i)
            for (unsigned int j = 1; j < v.My; ++j)
                v(i, j) += dx * (pc(i, j) - pc(i, j + 1)) / v.a(i, j).p;

        if (debug)
            v.print("v corrected = ", true);
    }
}
```

Problem_residuals.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::computeResiduals()
    {
        double Ru = velocityResidual(u);
        double Rv = velocityResidual(v);
        double Rp = pResidual();

        // Check for convergence
        if (Ru < tol && Rv < tol && Rp < tol)
            converged = true;

        // Print residuals
        if (converged)
            cout << "Converged in " << noshowpos << fixed << setprecision(3)
                << 1.0 * (clock() - start) / CLOCKS_PER_SEC << " sec in " << iterations << " iterations: ";
        if (converged || debug)
        {
            cout << noshowpos << setprecision(2) << scientific;
            cout << "u = " << Ru;
            cout << ", v = " << Rv;
            cout << ", p = " << Rp << endl;
        }
    }

    double
    Problem::pResidual() const
    {
        double numer = 0;
        for (unsigned int i = 1; i < pc.Mx; ++i)
            for (unsigned int j = 1; j < pc.My; ++j)
                numer += abs(dy * (u(i - 1, j) - u(i, j)) + dx * (v(i, j - 1) - v(i, j)));
        return numer / (u_ref * L_ref);
    }

    double
    Problem::velocityResidual(const Variable & var) const
    {
        double numer, numer_temp, denom = 0;
        for (unsigned int i = 1; i < var.Mx; ++i)
            for (unsigned int j = 1; j < var.My; ++j)
            {
                const Coefficients & a = var.a(i, j);
                numer_temp = a.p * var(i, j);
                denom += abs(numer_temp);
                numer_temp -= a.n * var(i, j + 1);
                numer_temp -= a.e * var(i + 1, j);
                numer_temp -= a.s * var(i, j - 1);
                numer_temp -= a.w * var(i - 1, j);
                numer_temp -= a.b;
                numer += abs(numer_temp);
            }
        return numer / denom;
    }
} // namespace Flow2D
```

Problem_solvers.cpp

```
#include "Problem.h"

namespace Flow2D
{
    void
    Problem::solve()
    {
        solve(u);
        solve(v);
        solve(pc);
    }

    void
    Problem::solve(Variable & var)
    {
        // Fill the coefficients
        fillCoefficients(var);

        // BC is in the x-direction, sweep left to right
        if (u.bc.nonzero()) {
            sweepColumns(var);
            sweepRows(var);
            sweepColumns(var, false);
        }
        // BC is in the y-direction, sweep south to north
        else {
            sweepRows(var);
            sweepColumns(var);
            sweepRows(var, false);
        }

        if (debug)
            var.print(var.string + " sweep solution = ", true);
    }

    void
    Problem::sweepRows(Variable & var, const bool south_north)
    {
        // Sweep south to north
        if (south_north)
            for (int j = 1; j < var.My; ++j)
                sweepRow(j, var);
        // Sweep north to south
        else
            for (int j = var.My - 1; j > 0; --j)
                sweepRow(j, var);
    }

    void
    Problem::sweepColumns(Variable & var, const bool west_east)
    {
        // Sweep west to east
        if (west_east)
            for (int i = 1; i < var.Mx; ++i)
                sweepColumn(i, var);
        // Sweep east to west
        else
            for (int i = var.Mx - 1; i > 0; --i)
                sweepColumn(i, var);
    }

    void
    Problem::sweepColumn(const unsigned int i, Variable & var)
    {
        if (debug)
            cout << "Solving " << var.string << " column " << i << endl;

        auto & A = var.Ay;
        auto & b = var.by;

        // Fill for each cell
        for (unsigned int j = 1; j < var.My; ++j)
        {
            const Coefficients & a = var.a(i, j);
            b[j - 1] = a.b + a.w * var(i - 1, j) + a.e * var(i + 1, j) + a.p * var(i, j) * (var.w - 1);
            if (j == 1)

```

```

    {
        A.setTopRow(a.p * var.w, -a.n);
        if (var.name != Variables::pc)
            b[j - 1] += a.s * var(i, j - 1);
    }
    else if (j == var.My - 1)
    {
        A.setBottomRow(-a.s, a.p * var.w);
        if (var.name != Variables::pc)
            b[j - 1] += a.n * var(i, j + 1);
    }
    else
        A.setMiddleRow(j - 1, -a.s, a.p * var.w, -a.n);
}

if (debug)
{
    A.print("A =");
    b.print("b =");
}

// Solve
A.solveTDMA(b);

if (debug)
    b.print("sol =", true);

// Store solution
for (unsigned int j = 1; j < var.My; ++j)
    var(i, j) = b[j - 1];
}

void
Problem::sweepRow(const unsigned int j, Variable & var)
{
    if (debug)
        cout << "Solving " << var.string << " row " << j << endl;

    auto & A = var.Ax;
    auto & b = var.bx;

    // Fill for each cell
    for (unsigned int i = 1; i < var.Mx; ++i)
    {
        const Coefficients & a = var.a(i, j);
        b[i - 1] = a.b + a.s * var(i, j - 1) + a.n * var(i, j + 1) + a.p * var(i, j) * (var.w - 1);
        if (i == 1)
        {
            A.setTopRow(a.p * var.w, -a.e);
            if (var.name != Variables::pc)
                b[i - 1] += a.w * var(i - 1, j);
        }
        else if (i == var.Mx - 1)
        {
            A.setBottomRow(-a.w, a.p * var.w);
            if (var.name != Variables::pc)
                b[i - 1] += a.e * var(i + 1, j);
        }
        else
            A.setMiddleRow(i - 1, -a.w, a.p * var.w, -a.e);
    }

    if (debug)
    {
        A.print("A =");
        b.print("b =");
    }

    // Solve
    A.solveTDMA(b);

    if (debug)
        b.print("sol =", true);

    // Store solution
    for (unsigned int i = 1; i < var.Mx; ++i)
        var(i, j) = b[i - 1];
}

} // namespace Flow2D

```

Matrix.h

```
#ifndef MATRIX_H
#define MATRIX_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include <vector>

using namespace std;

/**
 * Class that holds a N x M matrix with common matrix operations.
 */
template <typename T>
class Matrix
{
public:
    Matrix() {}
    Matrix(const unsigned int N, const unsigned int M) : N(N), M(M), A(N, vector<T>(M)) {}

    // Const operator for getting the (i, j) element
    const T & operator()(const unsigned int i, const unsigned int j) const
    {
        assert(i < N && j < M);
        return A[i][j];
    }
    // Operator for getting the (i, j) element
    T & operator()(const unsigned int i, const unsigned int j)
    {
        assert(i < N && j < M);
        return A[i][j];
    }
    // Operator for setting the entire matrix to a value
    void operator=(const T v)
    {
        for (unsigned int j = 0; j < M; ++j)
            setRow(j, v);
    }

    // Prints the matrix
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 5) const
    {
        if (prefix.length() != 0)
            cout << prefix << endl;
        for (unsigned int j = 0; j < M; ++j)
        {
            for (unsigned int i = 0; i < N; ++i)
                cout << showpos << scientific << setprecision(pr) << A[i][j] << " ";
            cout << endl;
        }
        if (newline)
            cout << endl;
    }
    // Saves the matrix in csv format
    void save(const string filename, const unsigned int pr = 12) const
    {
        ofstream f;
        f.open(filename);
        for (unsigned int j = 0; j < M; ++j)
        {
            for (unsigned int i = 0; i < N; ++i)
            {
                if (i > 0)
                    f << ",";
                f << setprecision(pr) << A[i][j];
            }
            f << endl;
        }
        f.close();
    }

    // Set the j-th row to v
    void setRow(const unsigned int j, const T v)
    {
        assert(j < M);
        for (unsigned int i = 0; i < N; ++i)
            A[i][j] = v;
    }
};
```

```

    }
    // Set the i-th column to v
    void setColumn(const unsigned int i, const T v)
    {
        assert(i < N);
        for (unsigned int j = 0; j < M; ++j)
            A[i][j] = v;
    }

private:
    // The size of this matrix
    const unsigned int N = 0, M = 0;

    // Matrix storage
    vector<vector<T>> A;
};

#endif /* MATRIX_H */

```

TriDiagonal.h

```
#ifndef TRIDIAGONAL_H
#define TRIDIAGONAL_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include "Vector.h"

using namespace std;

/**
 * Class that holds a tri-diagonal matrix and is able to perform TDMA in place
 * with a given RHS.
 */
template <typename T>
class TriDiagonal
{
public:
    TriDiagonal() {}
    TriDiagonal(const unsigned int N, const T v = 0) : N(N), A(N, v), B(N, v), C(N - 1, v) {}

    // Setters for the top, middle, and bottom rows
    void setTopRow(const T b, const T c)
    {
        B[0] = b;
        C[0] = c;
    }
    void setMiddleRow(const unsigned int i, const T a, const T b, const T c)
    {
        assert(i < N - 1 && i != 0);
        A[i] = a;
        B[i] = b;
        C[i] = c;
    }
    void setBottomRow(const T a, const T b)
    {
        A[N - 1] = a;
        B[N - 1] = b;
    }

    // Prints the matrix
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 6) const
    {
        if (prefix.length() != 0)
            cout << prefix << endl;
        for (unsigned int i = 0; i < N; ++i)
            cout << showpos << scientific << setprecision(pr) << (i > 0 ? A[i] : 0) << " " << B[i] << " "
                << (i < N - 1 ? C[i] : 0) << endl;
        if (newline)
            cout << endl;
    }

    // Saves the matrix in csv format
    void save(const string filename, const unsigned int pr = 12) const
    {
        ofstream f;
        f.open(filename);
        for (unsigned int i = 0; i < N; ++i)
        {
            if (i > 0)
                f << setprecision(pr) << A[i] << ",";
            else
                f << "0"
                    << ",";
            f << setprecision(pr) << B[i] << ",";
            if (i != N - 1)
                f << setprecision(pr) << C[i] << endl;
            else
                f << 0 << endl;
        }
        f.close();
    }

    // Solves the system Ax = d in place where d eventually stores the solution
    void solveTDMA(Vector<T> & d)
    {
        // Forward sweep
        T tmp = 0;
    }
};
```



```

    for (unsigned int i = 1; i < N; ++i)
    {
        tmp = A[i] / B[i - 1];
        B[i] -= tmp * C[i - 1];
        d[i] -= tmp * d[i - 1];
    }

    // Backward sweep
    d[N - 1] /= B[N - 1];
    for (unsigned int i = N - 2; i != numeric_limits<unsigned int>::max(); --i)
    {
        d[i] -= C[i] * d[i + 1];
        d[i] /= B[i];
    }
}

protected:
    // Matrix size (N x N)
    unsigned int N = 0;

    // Left/main/right diagonal storage
    vector<T> A, B, C;
};

#endif /* TRIDIAGONAL_H */

```

Vector.h

```
#ifndef VECTOR_H
#define VECTOR_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include <vector>

using namespace std;

/**
 * Class that stores a 1D vector and enables printing and saving.
 */
template <typename T>
class Vector
{
public:
    Vector(const unsigned int N) : v(N), N(N) {}
    Vector() {}

    const T & operator()(const unsigned int i) const
    {
        assert(i < N);
        return v[i];
    }
    T & operator()(const unsigned int i)
    {
        assert(i < N);
        return v[i];
    }
    const T & operator[](const unsigned int i) const
    {
        assert(i < N);
        return v[i];
    }
    T & operator[](const unsigned int i)
    {
        assert(i < N);
        return v[i];
    }

    // Prints the vector
    void print(const string prefix = "", const bool newline = false, const unsigned int pr = 6) const
    {
        if (prefix.length() != 0)
            cout << prefix << endl;
        for (unsigned int i = 0; i < v.size(); ++i)
            cout << showpos << scientific << setprecision(pr) << v[i] << " ";
        cout << endl;
        if (newline)
            cout << endl;
    }

    // Saves the vector
    void save(const string filename, const unsigned int pr = 12) const
    {
        ofstream f;
        f.open(filename);
        for (unsigned int i = 0; i < v.size(); ++i)
            f << scientific << v[i] << endl;
        f.close();
    }

private:
    vector<T> v;
    const unsigned int N = 0;
};

#endif /* VECTOR_H */
```

plots.py

```
import numpy as np
import matplotlib.pyplot as plt

plt.rc('text', usetex=True)
plt.rc('font', family='serif')

# Load problem 2 results
L = 0.2
Re = 400
rho = 998.3
mu = 1.002e-3
u_ref = Re * mu / (rho * L)
Ns = [8, 16, 32, 64, 128, 256]
x, y, u, v = {}, {}, {}, {}
for N in Ns:
    valu = np.loadtxt('results/{}_u.csv'.format(N), delimiter=',')
    u[N] = valu[:, int(N / 2)] / u_ref
    y[N] = np.linspace(0, 1, num=len(u[N]))
    valv = np.loadtxt('results/{}_v.csv'.format(N), delimiter=',')
    v[N] = valv[int(N / 2), :] / u_ref
    x[N] = np.linspace(0, 1, num=len(v[N]))

# Problem 2 reference
ref_u = [0, -0.03177, -0.06189, -0.08923, -0.11732, -0.14410, -0.17257, -0.19996,
         -0.22849, -0.25458, -0.27956, -0.11446, 0.35427, 0.37502, 0.40187,
         0.43679, 0.48695, 0.54512, 0.61626, 0.70001, 0.79419, 0.90472, 1.0]
ref_y = [0, 0.02, 0.0405, 0.0601, 0.0806, 0.1001, 0.1206, 0.1401, 0.1606, 0.1802,
         0.2007, 0.5005, 0.9009, 0.9106, 0.9204, 0.9302, 0.9409, 0.9507, 0.9604,
         0.9702, 0.9800, 0.9907, 1]
ref_v = [0, 0.05951, 0.11028, 0.14906, 0.18047, 0.20578, 0.22746, 0.24397,
         0.25854, 0.27001, 0.27667, 0.05146, -0.44994, -0.45381, -0.44362,
         -0.41888, -0.37613, -0.32251, -0.25931, -0.19118, -0.11873, -0.05590, 0]
ref_x = [0, 0.0151, 0.0308, 0.0454, 0.0600, 0.0747, 0.0902, 0.1049, 0.1206,
         0.1352, 0.1450, 0.5005, 0.8501, 0.8647, 0.8804, 0.8950, 0.9106, 0.9253,
         0.9399, 0.9546, 0.9702, 0.9849, 1]

# Problem 2 plot
fig, ax = plt.subplots(1, 2)
fig.set_figwidth(9)
fig.set_figheight(3.5)
for N in Ns:
    ax[0].plot(y[N], u[N], label='{}x{}'.format(N, N), linewidth=1)
    ax[1].plot(x[N], v[N], linewidth=1)
ax[0].plot(ref_y, ref_u, '.k', label='Roy et. al (2015)', markersize=4)
ax[1].plot(ref_x, ref_v, '.k', markersize=4)
ax[0].set_xlabel(r'Normalized $y$')
ax[0].set_ylabel(r'Normalized $u$')
ax[1].set_xlabel(r'Normalized $x$')
ax[1].set_ylabel(r'Normalized $v$')
ax[0].grid()
ax[1].grid()
handles, labels = ax[0].get_legend_handles_labels()
lgd = ax[0].legend(handles, labels, loc='lower center', bbox_to_anchor=(1.0, -0.29),
                  ncol=7, fontsize=9)
fig.tight_layout()
fig.savefig('results/p2.pdf', bbox_inches='tight', bbox_extra_artists=(lgd,))
```