ITERATIVE METHODS

Solution to Simultaneous Linear Algebraic Equations

Consider:

$$a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 = C_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 = C_2$$

$$a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 = C_3$$

Diagonal dominance is sufficient but not a necessary condition for convergence.

An ith equation:

$$\sum_{j=1}^{N} a_{ij}\,x_j = C_i$$

is said to be diagonally dominant if

$$|\,a_{ii}\,| > \sum_{\substack{j=1\\ j\neq i}}^{N} |\,a_{ij}\,|$$

1. Jacobi Iteration:

$$X_i^{(K+1)} = \frac{1}{a_{ii}}\left[\, C_i - \sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij}\,X_j^{(K)} \right]$$

2. Gauss-Seidel:

$$X_i^{(K+1)} = \frac{1}{a_{ii}} \left[C_i - \sum_{j=1}^{i-1} a_{ij} X_j^{(K+1)} - \sum_{j=i+1}^{N} a_{ij} X_j^{(k)} \right]$$

$$\Delta X_i = X_i^{(K+1)} - X_i^{(K)}$$

3. Successive over Relaxation (SOR):

Consider:

$$a_{i1} X_1 + a_{i2} X_2 + \text{- - - - - -} + a_i X_N = C_i$$

$$a_{ii} X_i^{K+1} = \left[C_i - \sum_{j=1}^{i-1} a_{ij} X_j^{K+1} - \sum_{j=i+1}^{N} a_{ij} X_j^{K} \right]$$

$$X_i^{K+1} = \frac{1}{a_{ii}} \left[C_i - \sum_{j=1}^{i-1} a_{ij} X_j^{K+1} - \sum_{j=i+1}^{N} a_{ij} X_j^{K} \right]$$

Add and subtract X_i^K on the right hand side of the above equation.

$$X_i^{K+1} = X_i^K + \frac{1}{a_{ii}} \left[C_i - \sum_{j=1}^{i-1} a_{ij} X_j^{K+1} - \sum_{j=i+1}^{N} a_{ij} X_j^{K} - a_{ii} X_i^K \right]$$

$$X_i^{K+1} = X_i^K + \frac{1}{a_{ii}}\left[C_i - \sum_{j=1}^{i-1} a_{ij} X_j^{K+1} - \sum_{j=i}^{N} a_{ij} X_j^K \right]$$

Note: The second term on the right hand side represents the change (ΔX_i) in the variable produced by the current iteration. This change can be modified by introducing a relaxation factor W. The reason for introducing W is to speed up or to slow down the process of convergence.

$$X_i^{K+1} = X_i^K + W \Delta X_i$$

W is the **Over Relaxation Factor**. For convergence

$$1 \le W \le 2,\ e.g.\ W = 1.03$$

To satisfy the condition of diagonal dominance the value of W has to be less than 2. If W < 1, then W is termed **successive under the relaxation factor**. The successive under relaxation method is used to stabilize and slow down the process of convergence. This method is adapted in solving linearized versions of highly non-linear systems of equations.

***General Steps:**

1. Assume $X_i^{(0)}$ (superscript indicates iteration number)

2. Substitute and solve for $X_i^{(1)}$

3. Repeat until $\left| X_i^{(K+1)} - X_i^{(K)} \right| \le \varepsilon$ 13

Note: Iterative methods are ideal for sparse matrices.

An example: (SOR Method):

Solve:

$$10X_1 + X_2 + 2X_3 = 44$$

$$2X_1 + 10X_2 + X_3 = 51$$

$$X_1 + 2X_2 + 10X_3 = 61$$

Initial guess: $X_1 = 1.0$, $X_2 = 2.0$, $X_3 = 3.0$

Use W = 1.03

$$X_1^{K+1} = X_1^K + \frac{1.03}{10}\left[44 - 10X_1^K - X_2^K - 2X_3^K\right]$$

$$X_2^{K+1} = X_2^K + \frac{1.03}{10}\left[51 - 2X_1^{K+1} - 10X_2^K - X_3^K\right]$$

$$X_3^{K+1} = X_3^K + \frac{1.03}{10}\left[61 - X_1^{K+1} - 2X_2^{K+1} - 10X_3^K\right]$$

K	X_1	X_2	X_3
0	1.0	2.0	3.0
1	3.678	4.126	4.964
2	2.974	4.005	5.002
3	2.999	3.999	5.000