MEEN 644 - Homework 4

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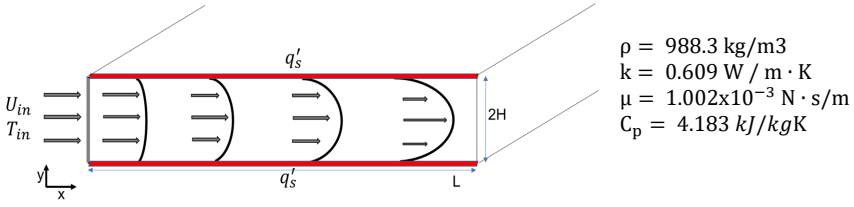
1 Problem statement



Consider an incompressible laminar flow between two infinite parallel plates. Flow enters at constant velocity u_{in} and aconstant heat flux, $q' = 500$ W/m^2 is applied to each wall. Height of channel $2H$ and length of channel L are 0.02 m and 2.0 m, respectively. Constant velocity u_{in} and temperature $T_{\text{in}} = 25°$C is set as the inlet condition. Reynolds number is defined as $Re = 2\rho u_{\text{in}} H/\mu$.

1. **(10 points)** Specify your boundary condition for u, v, P, and T.
2. **(50 points)** Write a finite volume method code to predict velocity, pressure, and temperature profiles for Re $= 100$. Employ the Power Law scheme to represent a solution to a 1-D convection-diffusion equation. Use the SIMPLE algorithm to link velocity and pressure fields. Use 10 uniformly sized CVs in the x-direction and 5 uniformly sized CVs in the y-direction. Declare convergence at R_u, R_v, R_P and $R_T < 10^{-6}$.
3. **(10 points)** Run your program with the 10×5 grid and tabulate $u, (P - P_0)$, and T to check for symmetry. P_0 is the pressure at the outlet.
4. **(30 points)** Use a 180×54 grid to solve for u, v, P, and T.
   (a) Plot u/u_{in} along the centerline of the channel (as a function of x).
   (b) Plot u, v, and T at $x = 0.8$m m (as a function of y).
   (c) Plot the Nusselt number as a function of stream-wise distance x from the channel entrance.

2 Preliminaries

2.1 Two-dimensional diffusion-convection

2.2 Solving methodology

2.3 Domain discretization

The domain of size $L_x \times L_y$ is discretized into $N_x \times N_y$ uniformly sized control volumes with $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$. The numbering for all variables begins at the origin at $(i, j) = (0, 0)$. The maximum index for each variable, ϕ, is defined as (M_x^ϕ, M_y^ϕ).

3 Results

3.1 Problem 3: 10 × 5 grid

The results requested for problem 3 follow in Tables 1, 2, and 3.

Table 1: The u-velocity solution with the 10×5 grid. A row corresponds to a x-position and a column corresponds to an y-position.

	1	2	3	4	5	6	7
1	2.50927E−3	2.50927E−3	2.50927E−3	2.50927E−3	2.50927E−3	2.50927E−3	2.50927E−3
2	0.00000E0	1.44730E−3	3.04425E−3	3.56324E−3	3.04425E−3	1.44730E−3	0.00000E0
3	0.00000E0	1.39756E−3	3.06604E−3	3.61913E−3	3.06604E−3	1.39756E−3	0.00000E0
4	0.00000E0	1.39429E−3	3.06686E−3	3.62401E−3	3.06686E−3	1.39429E−3	0.00000E0
5	0.00000E0	1.39406E−3	3.06688E−3	3.62445E−3	3.06688E−3	1.39406E−3	0.00000E0
6	0.00000E0	1.39404E−3	3.06688E−3	3.62449E−3	3.06688E−3	1.39404E−3	0.00000E0
7	0.00000E0	1.39404E−3	3.06688E−3	3.62449E−3	3.06688E−3	1.39404E−3	0.00000E0
8	0.00000E0	1.39404E−3	3.06688E−3	3.62449E−3	3.06688E−3	1.39404E−3	0.00000E0
9	0.00000E0	1.39404E−3	3.06688E−3	3.62449E−3	3.06688E−3	1.39404E−3	0.00000E0
10	0.00000E0	1.39404E−3	3.06688E−3	3.62449E−3	3.06688E−3	1.39404E−3	0.00000E0
11	0.00000E0	1.39404E−3	3.06688E−3	3.62449E−3	3.06688E−3	1.39404E−3	0.00000E0

Table 2: The $(P - P_0)$ solution with the 10×5 grid. A row corresponds to a x-position and a column corresponds to an y-position.

	1	2	3	4	5	6	7
1	1.41200E−1	1.41200E−1	1.41190E−1	1.41187E−1	1.41190E−1	1.41200E−1	1.41200E−1
2	1.41200E−1	1.41200E−1	1.41190E−1	1.41187E−1	1.41190E−1	1.41200E−1	1.41200E−1
3	1.18832E−1	1.18832E−1	1.18832E−1	1.18833E−1	1.18832E−1	1.18832E−1	1.18832E−1
4	1.04769E−1	1.04769E−1	1.04769E−1	1.04769E−1	1.04769E−1	1.04769E−1	1.04769E−1
5	9.07942E−2	9.07942E−2	9.07942E−2	9.07942E−2	9.07942E−2	9.07942E−2	9.07942E−2
6	7.68254E−2	7.68254E−2	7.68254E−2	7.68254E−2	7.68254E−2	7.68254E−2	7.68254E−2
7	6.28571E−2	6.28571E−2	6.28571E−2	6.28571E−2	6.28571E−2	6.28571E−2	6.28571E−2
8	4.88889E−2	4.88889E−2	4.88889E−2	4.88889E−2	4.88889E−2	4.88889E−2	4.88889E−2
9	3.49206E−2	3.49206E−2	3.49206E−2	3.49206E−2	3.49206E−2	3.49206E−2	3.49206E−2
10	2.09524E−2	2.09524E−2	2.09524E−2	2.09524E−2	2.09524E−2	2.09524E−2	2.09524E−2
11	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0
12	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0	0.00000E0

Table 3: The T solution with the 10×5 grid. A row corresponds to a x-position and a column corresponds to an y-position.

	1	2	3	4	5	6	7
1	26.64204	25.00000	25.00000	25.00000	25.00000	25.00000	26.64204
2	28.47372	26.83169	25.80190	25.50210	25.80190	26.83169	28.47372
3	30.21235	28.57032	26.64040	26.07996	26.64040	28.57032	30.21235
4	31.50432	29.86229	27.56664	26.82629	27.56664	29.86229	31.50432
5	32.60351	30.96148	28.51796	27.67470	28.51796	30.96148	32.60351
6	33.62161	31.97957	29.47406	28.57704	29.47406	31.97957	33.62161
7	34.60455	32.96252	30.43017	29.50634	30.43017	32.96252	34.60455
8	35.57185	33.92982	31.38560	30.44874	31.38560	33.92982	35.57185
9	36.53202	34.88999	32.34047	31.39739	32.34047	34.88999	36.53202
10	37.48883	35.84679	33.29498	32.34898	33.29498	35.84679	37.48883
11	38.44391	36.80187	34.24921	33.30189	34.24921	36.80187	38.44391
12	39.39900	37.75696	35.20344	34.25481	35.20344	37.75696	39.39900

3.2 Problem 4: 180×54 grid

The requirements for problem (b) part i and ii were combined into Figure **??** as seen below. With increasing grid refinement, both centerline velocity profiles approached towards the reference solution obtained from Roy et. al. In addition, a once-more-refined run is compared with 256x256 CVs with good agreement.

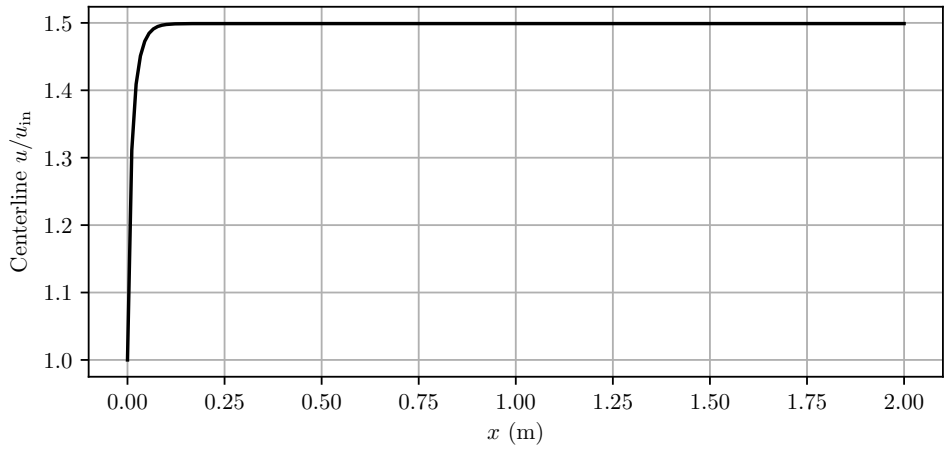


Figure 1: u/u_{in} plotted along the centerline of the channel for the 180×54 grid.

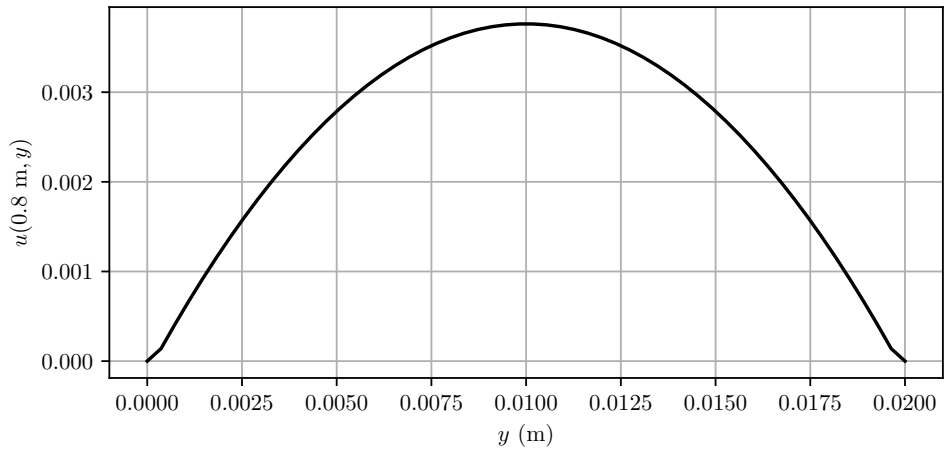


Figure 2: u plotted at $x = 0.8$ m for the 180×54 grid.

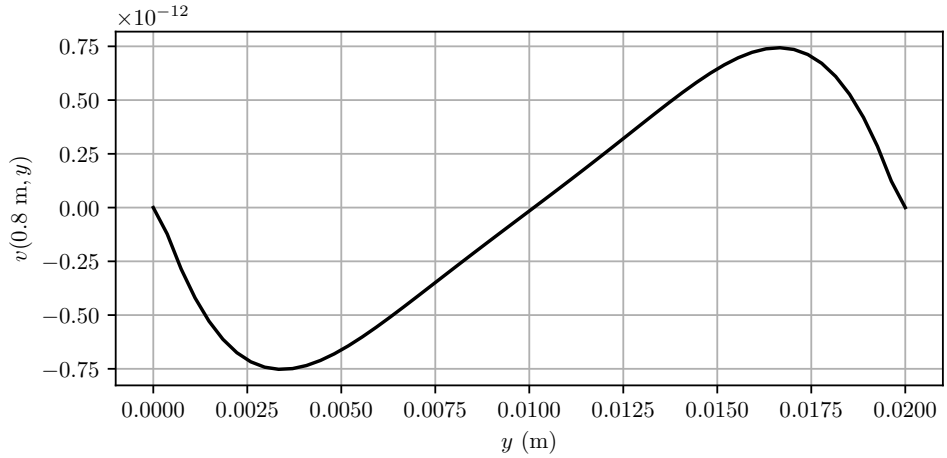


Figure 3: v plotted at $x = 0.8$ m for the 180×54 grid.

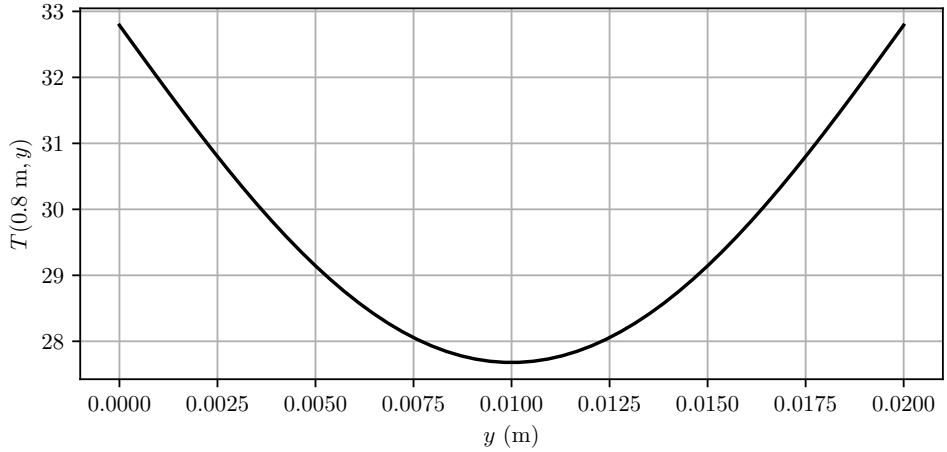


Figure 4: T plotted at $x = 0.8$ m for the 180×54 grid.

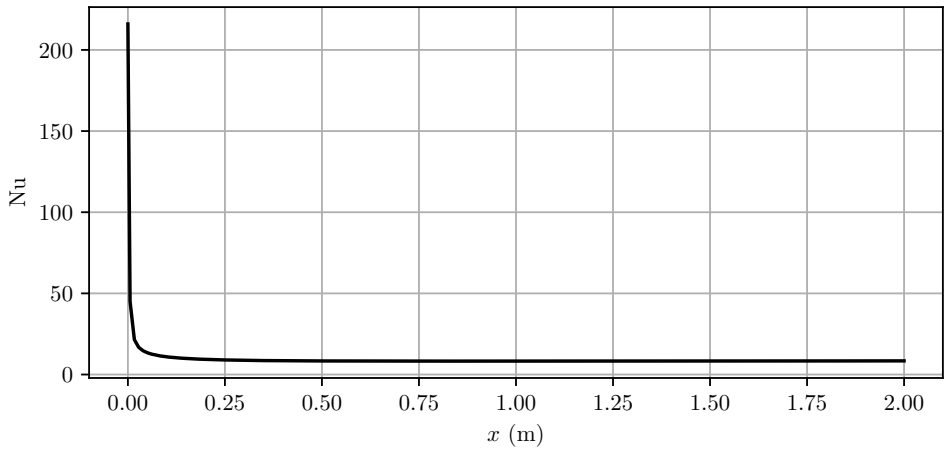


Figure 5: The Nusselt number plotted as a function of stream-wise distance for the 180 $\times$ 54 grid.

Code listing

For the implementation, we have the following files:

- `Makefile` – Allows for compiling the c++ project with `make`.
- `hwk4.cpp` – Contains the `main()` function that is required by C that runs the cases requested in this problem set.
- `Problem.h` – Contains the header for the `Problem` class which is the main driver for a `Flow2D::Problem`.
- `Variable.h` – Contains the `Flow2D::Variable` class, which is a storage container for a single variable (i.e., u).
- `Problem.cpp` – Contains the `run()` functions that executes a `Problem`.
- `Problem_coefficients.cpp` – Contains the functions for solving coefficients in a `Problem`.
- `Problem_corrections.cpp` – Contains the functions for correcting solutions in a `Problem`.
- `Problem_residuals.cpp` – Contains the functions for computing residuals in a `Problem`.
- `Problem_solvers.cpp` – Contains the functions for sweeping and solving in a `Problem`.
- `Matrix.h` – Contains the `Matrix` class which provides storage for a matrix with various standard matrix operations.
- `TriDiagonal.h` – Contains the `TriDiagonal` class which provides storage for a tri-diagonal matrix including the TDMA solver found in the member function `solveTDMA()`.
- `Vector.h` – Contains the `Vector` class for one-dimensional vector storage.
- `postprocess.py` - Produces the plots and tables in this report.

Makefile

```
src = $(wildcard *.cpp)
obj = $(src:.cpp=.o)
CXXFLAGS = -std=c++14
CCFLAGS = $(CXXFLAGS)

hwk-opt: $(obj)
	clang++ -o $@ $^

.PHONY: clean
clean:
	rm -f $(obj) hwk-opt
```

hwk5.cpp

```
#include "Problem.h"

using namespace Flow2D;

int
main()
{
  // Problem wide constants
  double Lx = 2;
  double Ly = 0.02;
  double cp = 4183;
  double k = 0.609;
  double rho = 998.3;
  double mu = 0.001002;
  double Re = 100;
  double u_bc_val = Re * mu / (2 * rho * Ly);
  double T_bc_val = 25;
  double q_bc_val = 500;

  // Standard inputs
  InputArguments input;
  input.Lx = Lx;
  input.Ly = Ly;
  input.cp = cp;
  input.k = k;
  input.mu = mu;
  input.rho = rho;
  input.u_ref = u_bc_val;
  input.L_ref = Lx;
  input.T_left = T_bc_val;
  input.u_bc = BoundaryCondition(0, 0, 0, u_bc_val);
  input.v_bc = BoundaryCondition(0, 0, 0, 0);
  input.q_top_bot = q_bc_val;

  // Problem 3: check symmetry
  std::cout << "Problem 3: check symmetry" << std::endl;
  Problem problem3(10, 5, input);
  problem3.run();
  problem3.save(Variables::u, "results/coarse_u.csv");
  problem3.save(Variables::p, "results/coarse_p.csv");
  problem3.save(Variables::T, "results/coarse_T.csv");

  // Problem 4: 180x54 grid
  std::cout << "Problem 4: 180 x 54 grid" << std::endl;
  Problem problem4(180, 54, input);
  problem4.run();
  problem4.save(Variables::u, "results/fine_u.csv");
  problem4.save(Variables::v, "results/fine_v.csv");
  problem4.save(Variables::T, "results/fine_T.csv");
}
```

Problem.h

```cpp
#ifndef PROBLEM_H
#define PROBLEM_H

#include <cmath>
#include <ctime>
#include <iomanip>
#include <iostream>
#include <map>

#include "Variable.h"

namespace Flow2D
{

using namespace std;

struct InputArguments
{
  double Lx, Ly;
  BoundaryCondition u_bc, v_bc;
  double T_left, q_top_bot;
  double L_ref, u_ref;
  double cp, k, mu, rho;
  bool debug = false;
  double alpha_p = 0.7;
  double alpha_uv = 0.5;
  unsigned int max_main_its = 50000;
  unsigned int max_aux_its = 20000;
  double tol = 1.0e-6;
};

class Problem
{
public:
  Problem(const unsigned int Nx, const unsigned int Ny, const InputArguments & input);

  void run();

  // Public access to printing and saving variable results
  void print(const Variables var,
             const string prefix = "",
             const bool newline = false,
             const unsigned int pr = 5) const
  {
    variables.at(var).print(prefix, newline, pr);
  }
  void save(const Variables var, const string filename) const { variables.at(var).save(filename); }

private:
  // Problem_corrections.cpp
  void correctMain();
  void correctAux();
  void pCorrect();
  void pBCCorrect();
  void TBCCorrect();
  void uCorrect();
  void uBCCorrect();
  void vCorrect();

  // Problem_coefficients.cpp
  void fillCoefficients(const Variable & var);
  void pcCoefficients();
  void TCoefficients();
  void uCoefficients();
  void vCoefficients();
  void fillPowerLaw(Coefficients & a,
                    const Coefficients & D,
                    const Coefficients & F,
                    const double & b = 0);

  // Problem_residuals.cpp
  void computeMainResiduals();
  void computeAuxResiduals();
  double pResidual() const;
  double TResidual() const;
  double velocityResidual(const Variable & var) const;
```

```
  // Problem_solvers.cpp
  void solveMain();
  void solveAux();
  void solve(Variable & var);
  void sweepColumns(Variable & var, const bool west_east = true);
  void sweepRows(Variable & var, const bool south_north = true);
  void sweepColumn(const unsigned int i, Variable & var);
  void sweepRow(const unsigned int j, Variable & var);
  void solveVelocities();

  // Quicker v^5 for velocityCoefficients() (yes, it's actually much faster...)
  static const double pow5(const double & v) { return v * v * v * v * v; }

protected:
  // Number of pressure CVs
  const unsigned int Nx, Ny;

  // Geometry [m]
  const double Lx, Ly, dx, dy;
  // Material properties
  const double cp, k, mu, rho;
  // Residual references
  const double L_ref, u_ref;
  // Other boundary conditions
  const double q_top_bot;
  // Mass inflow
  const double m_in;

  // Enable debug mode (printing extra output)
  const bool debug;

  // Maximum iterations
  const unsigned int max_main_its, max_aux_its;
  // Iteration tolerance
  const double tol;
  // Pressure relaxation
  const double alpha_p;
  // Number of iterations completed
  unsigned int main_iterations = 0;
  unsigned int aux_iterations = 0;

  // Variables
  Variable u, v, pc, p, T;
  // Variable map
  map<const Variables, const Variable &> variables;

  // Whether or not we converged
  bool main_converged = false;
  bool converged = false;
  // Run start time
  clock_t start;
};

} // namespace Flow2D
#endif /* PROBLEM_H */
```

Variable.h

```
#ifndef VARIABLE_H
#define VARIABLE_H

#include "Matrix.h"
#include "TriDiagonal.h"
#include "Vector.h"

namespace Flow2D
{

using namespace std;

// Storage for boundary conditions
struct BoundaryCondition
{
  BoundaryCondition() {}
  BoundaryCondition(const double top, const double right, const double bottom, const double left)
    : top(top), right(right), bottom(bottom), left(left)
  {
  }
  double top = 0, right = 0, bottom = 0, left = 0;
};

// Storage for coefficients for a single CV
struct Coefficients
{
  double p = 0, n = 0, e = 0, s = 0, w = 0, b = 0;
  void print(const unsigned int pr = 5) const
  {
    cout << setprecision(pr) << scientific << "n = " << n << ", e = " << e << ", s = " << s
         << ", w = " << w << ", p = " << p << ", b = " << b << endl;
  }
};

// Enum for variable types
enum Variables
{
  u,
  v,
  pc,
  p,
  T
};

// Conversion from variable type to its string
static string
VariableString(Variables var)
{
  switch (var)
  {
    case Variables::u:
      return "u";
    case Variables::v:
      return "v";
    case Variables::pc:
      return "pc";
    case Variables::p:
      return "p";
    case Variables::T:
      return "T";
  }
}

// General storage structure for primary and auxilary variables
struct Variable
{
  // Constructor for a primary variable
  Variable(const Variables name,
           const unsigned int Nx,
           const unsigned int Ny,
           const double alpha,
           const BoundaryCondition bc = BoundaryCondition())
    : name(name),
      string(VariableString(name)),
      Nx(Nx),
      Ny(Ny),
      Mx(Nx - 1),
```

```
      My(Ny - 1),
      w(1 / alpha),
      bc(bc),
      a(Nx, Ny),
      phi(Nx, Ny),
      Ax(Nx - 2),
      Ay(Ny - 2),
      bx(Nx - 2),
      by(Ny - 2)
  {
    // Apply initial boundary conditions
    if (bc.left != 0)
      phi.setColumn(0, bc.left);
    if (bc.right != 0)
      phi.setColumn(Mx, bc.right);
    if (bc.bottom != 0)
      phi.setRow(0, bc.bottom);
    if (bc.top != 0)
      phi.setRow(My, bc.top);
  }

  // Constructor for an auxilary variable (no solver storage)
  Variable(const Variables name, const unsigned int Nx, const unsigned int Ny)
    : name(name), string(VariableString(name)), Nx(Nx), Ny(Ny), Mx(Nx - 1), My(Ny - 1), phi(Nx, Ny)
  {
  }

  // Solution matrix operations
  void operator=(const double v) { phi = v; }
  const double & operator()(const unsigned int i, const unsigned int j) const { return phi(i, j); }
  double & operator()(const unsigned int i, const unsigned int j) { return phi(i, j); }
  void print(const string prefix = "", const bool newline = false, const unsigned int pr = 5) const
  {
    phi.print(prefix, newline, pr);
  }
  void save(const string filename) const { phi.save(filename); }
  void reset() { phi = 0; }

  // Coefficient debug
  void printCoefficients(const string prefix = "",
                         const bool newline = false,
                         const unsigned int pr = 5) const
  {
    for (unsigned int i = 1; i < Nx - 1; ++i)
      for (unsigned int j = 1; j < Ny - 1; ++j)
      {
        cout << prefix << "(" << i << ", " << j << "): ";
        a(i, j).print(pr);
      }
    if (newline)
      cout << endl;
  }

  // Variable enum name
  const Variables name;
  // Variable string
  const string string;
  // Variable size
  const unsigned int Nx, Ny;
  // Maximum variable index that is being solved
  const unsigned int Mx, My;
  // Relaxation coefficient used in solving linear systems
  const double w = 0;
  // Boundary conditions
  const BoundaryCondition bc = BoundaryCondition();
  // Matrix coefficients
  Matrix<Coefficients> a;
  // Variable solution
  Matrix<double> phi;
  // Linear system LHS for both sweep directions
  TriDiagonal<double> Ax, Ay;
  // Linear system RHS for both sweep directions
  Vector<double> bx, by;
};

} // namespace Flow2D
#endif /* VARIABLE_H */
```

Problem.cpp

```cpp
#include "Problem.h"

namespace Flow2D
{

Problem::Problem(const unsigned int Nx, const unsigned int Ny, const InputArguments & input)
  : // Number of pressure CVs
    Nx(Nx),
    Ny(Ny),
    // Domain sizes
    Lx(input.Lx),
    Ly(input.Ly),
    dx(Lx / Nx),
    dy(Ly / Ny),
    // Residual references
    L_ref(input.L_ref),
    u_ref(input.u_ref),
    // Other boundary conditions
    q_top_bot(input.q_top_bot),
    // Mass inflow
    m_in(input.u_bc.left * input.rho * Ly),
    // Material properties
    cp(input.cp),
    k(input.k),
    mu(input.mu),
    rho(input.rho),
    // Enable debug
    debug(input.debug),
    // Solver properties
    max_main_its(input.max_main_its),
    max_aux_its(input.max_aux_its),
    tol(input.tol),
    alpha_p(input.alpha_p),
    // Initialize variables for u, v, pc (solved variables)
    u(Variables::u, Nx + 1, Ny + 2, input.alpha_uv, input.u_bc),
    v(Variables::v, Nx + 2, Ny + 1, input.alpha_uv, input.v_bc),
    pc(Variables::pc, Nx + 2, Ny + 2, 1),
    T(Variables::T, Nx + 2, Ny + 2, 1),
    // Initialize aux variables
    p(Variables::p, Nx + 2, Ny + 2)
{
  // Add into variable map for access outside of class
  variables.emplace(Variables::u, u);
  variables.emplace(Variables::v, v);
  variables.emplace(Variables::pc, pc);
  variables.emplace(Variables::p, p);
  variables.emplace(Variables::T, T);

  // T initial condition
  T = input.T_left;
}

void
Problem::run()
{
  // Store start time
  start = clock();

  // Solve main variables
  for (unsigned int l = 0; l < max_main_its; ++l)
  {
    solveMain();
    correctMain();
    computeMainResiduals();

    // Break out if we've converged
    if (main_converged)
      break;
  }

  // Ensure main variables converged
  if (!main_converged)
    cout << "Main variables did not converge after " << max_main_its << " iterations!" << endl;

  // Solve aux variables
  for (unsigned int l = 0; l < max_aux_its; ++l)
  {
```

```
    solveAux();
    correctAux();
    computeAuxResiduals();

    // Exit if everything is converged
    if (converged)
      return;
  }

  // Oops. Didn't converge
  cout << "Aux variables did not converge after " << max_aux_its << " iterations!" << endl;
}
} // namespace Flow2D
```

Problem_coefficients.cpp

```cpp
#include "Problem.h"

namespace Flow2D
{

void
Problem::fillCoefficients(const Variable & var)
{
  if (var.name == Variables::pc)
    pcCoefficients();
  else if (var.name == Variables::T)
    TCoefficients();
  else if (var.name == Variables::u)
    uCoefficients();
  else if (var.name == Variables::v)
    vCoefficients();

  if (debug)
  {
    cout << var.string << " coefficients: " << endl;
    var.printCoefficients(var.string, true);
  }
}

void
Problem::pcCoefficients()
{
  for (unsigned int i = 1; i < pc.Mx; ++i)
    for (unsigned int j = 1; j < pc.My; ++j)
    {
      Coefficients & a = pc.a(i, j);

      if (i != 1)
        a.w = rho * dy * dy / u.a(i - 1, j).p;
      if (i != pc.Mx - 1)
        a.e = rho * dy * dy / u.a(i, j).p;
      if (j != 1)
        a.s = rho * dx * dx / v.a(i, j - 1).p;
      if (j != pc.My - 1)
        a.n = rho * dx * dx / v.a(i, j).p;
      a.p = a.n + a.e + a.s + a.w;
      a.b = rho * (dy * (u(i - 1, j) - u(i, j)) + dx * (v(i, j - 1) - v(i, j)));
    }
}

void
Problem::TCoefficients()
{
  Coefficients D, F;

  for (unsigned int i = 1; i < T.Mx; ++i)
    for (unsigned int j = 1; j < T.My; ++j)
    {
      // Diffusion coefficient
      D.n = (j == T.My - 1 ? 2 * dx * k / dy : dx * k / dy);
      D.e = (i == T.Mx - 1 ? 2 * dy * k / dx : dy * k / dx);
      D.s = (j == 1 ? 2 * dx * k / dy : dx * k / dy);
      D.w = (i == 1 ? 2 * dy * k / dx : dy * k / dx);

      // Heat flows
      F.n = dx * cp * rho * v(i, j);
      F.e = dy * cp * rho * u(i, j);
      F.s = dx * cp * rho * v(i, j - 1);
      F.w = dy * cp * rho * u(i - 1, j);

      // Compute and store power law coefficients
      fillPowerLaw(T.a(i, j), D, F);
    }
}

void
Problem::uCoefficients()
{
  Coefficients D, F;
  double W, dy_pn, dy_ps, b;

  for (unsigned int i = 1; i < u.Mx; ++i)
```

```
    for (unsigned int j = 1; j < u.My; ++j)
    {
      // Width of the cell
      W = (i == 1 || i == u.Mx - 1 ? 3 * dx / 2 : dx);
      // North/south distances to pressure nodes
      dy_pn = (j == u.My - 1 ? dy / 2 : dy);
      dy_ps = (j == 1 ? dy / 2 : dy);

      // Diffusion coefficients
      D.n = mu * W / dy_pn;
      D.e = mu * dy / dx;
      D.s = mu * W / dy_ps;
      D.w = mu * dy / dx;

      // East and west flows
      F.e = (i == u.Mx - 1 ? rho * dy * u(u.Mx, j) : rho * dy * (u(i + 1, j) + u(i, j)) / 2);
      F.w = (i == 1 ? rho * dy * u(0, j) : rho * dy * (u(i - 1, j) + u(i, j)) / 2);
      // North and south flows
      if (i == 1) // Left boundary
      {
        F.n = rho * W * (v(0, j) + 3 * v(1, j) + 2 * v(2, j)) / 6;
        F.s = rho * W * (v(0, j - 1) + 3 * v(1, j - 1) + 2 * v(2, j - 1)) / 6;
      }
      else if (i == u.Mx - 1) // Right boundary
      {
        F.n = rho * W * (2 * v(i, j) + 3 * v(i + 1, j) + v(i + 2, j)) / 6;
        F.s = rho * W * (2 * v(i, j - 1) + 3 * v(i + 1, j - 1) + v(i + 2, j - 1)) / 6;
      }
      else // Interior (not left or right boundary)
      {
        F.n = rho * W * (v(i, j) + v(i + 1, j)) / 2;
        F.s = rho * W * (v(i, j - 1) + v(i + 1, j - 1)) / 2;
      }

      // Pressure RHS
      b = dy * (p(i, j) - p(i + 1, j));

      // Compute and store power law coefficients
      fillPowerLaw(u.a(i, j), D, F, b);
    }
}

void
Problem::vCoefficients()
{
  Coefficients D, F;
  double H, dx_pe, dx_pw, b;

  for (unsigned int i = 1; i < v.Mx; ++i)
    for (unsigned int j = 1; j < v.My; ++j)
    {
      // Height of the cell
      H = (j == 1 || j == v.My - 1 ? 3 * dy / 2 : dy);
      // East/west distances to pressure nodes
      dx_pe = (i == v.Mx - 1 ? dx / 2 : dx);
      dx_pw = (i == 1 ? dx / 2 : dx);

      // Diffusion coefficient
      D.n = mu * dx / dy;
      D.e = mu * H / dx_pe;
      D.s = mu * dx / dy;
      D.w = mu * H / dx_pw;

      // North and east flows
      F.n = (j == v.My - 1 ? rho * dx * v(i, v.My) : rho * dx * (v(i, j + 1) + v(i, j)) / 2);
      F.s = (j == 1 ? rho * dx * v(i, 0) : rho * dx * (v(i, j - 1) + v(i, j)) / 2);
      // East and west flows
      if (j == 1) // Bottom boundary
      {
        F.e = rho * H * (u(i, 0) + 3 * u(i, 1) + 2 * u(i, 2)) / 6;
        F.w = rho * H * (u(i - 1, 0) + 3 * u(i - 1, 1) + 2 * u(i - 1, 2)) / 6;
      }
      else if (j == v.My - 1) // Top boundary
      {
        F.e = rho * H * (2 * u(i, j) + 3 * u(i, j + 1) + u(i, j + 2)) / 6;
        F.w = rho * H * (2 * u(i - 1, j) + 3 * u(i - 1, j + 1) + u(i - 1, j + 2)) / 6;
      }
      else // Interior (not top or bottom boundary)
      {
        F.e = rho * H * (u(i, j) + u(i, j + 1)) / 2;
        F.w = rho * H * (u(i - 1, j) + u(i - 1, j + 1)) / 2;
```

```
      }

      // Pressure RHS
      b = dx * (p(i, j) - p(i, j + 1));

      // Compute and store power law coefficients
      fillPowerLaw(v.a(i, j), D, F, b);
    }
}

void
Problem::fillPowerLaw(Coefficients & a,
                      const Coefficients & D,
                      const Coefficients & F,
                      const double & b)
{
  a.n = D.n * fmax(0, pow5(1 - 0.1 * fabs(F.n / D.n))) + fmax(-F.n, 0);
  a.e = D.e * fmax(0, pow5(1 - 0.1 * fabs(F.e / D.e))) + fmax(-F.e, 0);
  a.s = D.s * fmax(0, pow5(1 - 0.1 * fabs(F.s / D.s))) + fmax(F.s, 0);
  a.w = D.w * fmax(0, pow5(1 - 0.1 * fabs(F.w / D.w))) + fmax(F.w, 0);
  a.p = a.n + a.e + a.s + a.w;
  a.b = b;
}

} // namespace Flow2D
```

Problem_corrections.cpp

```cpp
#include "Problem.h"

namespace Flow2D
{

void
Problem::correctMain()
{
  uCorrect();
  vCorrect();
  pCorrect();
  pBCCorrect();
  uBCCorrect();
}

void
Problem::correctAux()
{
  TBCCorrect();
}

void
Problem::pCorrect()
{
  for (unsigned int i = 1; i < pc.Mx; ++i)
    for (unsigned int j = 1; j < pc.My; ++j)
      p(i, j) += alpha_p * pc(i, j);

  // Set pressure correction back to zero
  pc.reset();

  if (debug)
    p.print("p corrected = ", true);
}

void
Problem::pBCCorrect()
{
  // Apply the edge values as velocity is set
  for (unsigned int i = 0; i <= pc.Mx; ++i)
  {
    p(i, 0) = p(i, 1);
    p(i, pc.My) = p(i, pc.My - 1);
  }
  for (unsigned int j = 0; j <= pc.My; ++j)
  {
    p(0, j) = p(1, j);
    p(pc.Mx, j) = p(pc.Mx - 1, j);
  }

  if (debug)
    p.print("p boundary condition corrected = ", true);
}
void
Problem::TBCCorrect()
{
  for (unsigned int j = 0; j <= T.My; ++j)
    T(T.Mx, j) = 2 * T(T.Mx - 1, j) - T(T.Mx - 2, j);
  for (unsigned int i = 0; i <= T.Mx; ++i)
  {
    T(i, 0) = T(i, 1) + q_top_bot * dy / (2 * k);
    T(i, T.My) = T(i, T.My - 1) + q_top_bot * dy / (2 * k);
  }
}

void
Problem::uCorrect()
{
  for (unsigned int i = 1; i < u.Mx; ++i)
    for (unsigned int j = 1; j < u.My; ++j)
      u(i, j) += dy * (pc(i, j) - pc(i + 1, j)) / u.a(i, j).p;

  if (debug)
    u.print("u corrected = ", true);
}

void
```

```
Problem::uBCCorrect()
{
  double m_out = 0;
  for (unsigned int j = 0; j <= u.My; ++j)
    m_out += rho * dy * u(u.Mx - 1, j);
  for (unsigned int j = 0; j <= u.My; ++j)
    u(u.Mx, j) = m_in * u(u.Mx - 1, j) / m_out;

  if (debug)
    u.print("u boundary condition corrected = ", true);
}

void
Problem::vCorrect()
{
  for (unsigned int i = 1; i < v.Mx; ++i)
    for (unsigned int j = 1; j < v.My; ++j)
      v(i, j) += dx * (pc(i, j) - pc(i, j + 1)) / v.a(i, j).p;

  if (debug)
    v.print("v corrected = ", true);
}

} // namespace Flow2D
```

Problem_residuals.cpp

```
#include "Problem.h"

namespace Flow2D
{

void
Problem::computeMainResiduals()
{
  const double Rp = pResidual();
  const double Ru = velocityResidual(u);
  const double Rv = velocityResidual(v);

  if (Ru < tol && Rv < tol && Rp < tol)
  {
    if (debug)
      cout << "Main variables converged in " << main_iterations << " iterations" << endl;
    main_converged = true;
  }
}

void
Problem::computeAuxResiduals()
{
  double RT = TResidual();

  // Still not converged
  if (RT > tol)
    return;

  // Converged, finish up
  converged = true;

  // Print the result
  const double Rp = pResidual();
  const double Ru = velocityResidual(u);
  const double Rv = velocityResidual(v);
  cout << "Converged in " << noshowpos << fixed << setprecision(3)
       << 1.0 * (clock() - start) / CLOCKS_PER_SEC << " sec in " << main_iterations << " main and "
       << aux_iterations << " aux iterations: ";
  cout << noshowpos << setprecision(1) << scientific;
  cout << "p = " << Rp;
  cout << ", v = " << Rv;
  cout << ", p = " << Rp;
  cout << ", T = " << RT << endl;
}

double
Problem::pResidual() const
{
  double numer = 0;
  for (unsigned int i = 1; i < pc.Mx; ++i)
    for (unsigned int j = 1; j < pc.My; ++j)
      numer += abs(dy * (u(i - 1, j) - u(i, j)) + dx * (v(i, j - 1) - v(i, j)));
  return numer / (u_ref * L_ref);
}

double
Problem::TResidual() const
{
  double numer, numer_temp, denom = 0;
  for (unsigned int i = 1; i < T.Mx; ++i)
    for (unsigned int j = 1; j < T.My; ++j)
    {
      const Coefficients & a = T.a(i, j);
      numer_temp = a.p * T(i, j);
      denom += abs(numer_temp);
      numer_temp -= a.n * T(i, j + 1) + a.e * T(i + 1, j);
      numer_temp -= a.s * T(i, j - 1) + a.w * T(i - 1, j) + a.b;
      numer += abs(numer_temp);
    }
  return numer / denom;
}

double
Problem::velocityResidual(const Variable & var) const
{
  double numer, numer_temp, denom = 0;
```

```
  for (unsigned int i = 1; i < var.Mx; ++i)
    for (unsigned int j = 1; j < var.My; ++j)
    {
      const Coefficients & a = var.a(i, j);
      numer_temp = a.p * var(i, j);
      denom += abs(numer_temp);
      numer_temp -= a.n * var(i, j + 1) + a.e * var(i + 1, j);
      numer_temp -= a.s * var(i, j - 1) + a.w * var(i - 1, j) + a.b;
      numer += abs(numer_temp);
    }
  return numer / denom;
}

} // namespace Flow2D
```

Problem_solvers.cpp

```
#include "Problem.h"

namespace Flow2D
{

void
Problem::solveMain()
{
  ++main_iterations;
  if (debug)
    cout << endl << "Main iteration " << main_iterations << endl << endl;

    solve(u);
    solve(v);
    solve(pc);
}

void
Problem::solveAux()
{
  ++aux_iterations;
  if (debug)
    cout << endl << "Aux iteration " << aux_iterations << endl << endl;

  solve(T);
}

void
Problem::solve(Variable & var)
{
  if (debug)
    cout << "Solving variable " << var.string << endl << endl;

  // Fill the coefficients
  fillCoefficients(var);

  // Solve west to east
  sweepColumns(var);
  // Solve south to north
  sweepRows(var);
  // Solve east to west
  sweepColumns(var, false);

  if (debug)
    var.print(var.string + " sweep solution = ", true);
}

void
Problem::sweepRows(Variable & var, const bool south_north)
{
  if (debug)
    cout << "Sweeping " << var.string << (south_north ? " south to north" : " north to south")
         << endl;

  // Sweep south to north
  if (south_north)
    for (int j = 1; j < var.My; ++j)
      sweepRow(j, var);
  // Sweep north to south
  else
    for (int j = var.My - 1; j > 0; --j)
      sweepRow(j, var);
}

void
Problem::sweepColumns(Variable & var, const bool west_east)
{
  if (debug)
    cout << "Sweeping " << var.string << (west_east ? " east to west" : " west to east") << endl;

  // Sweep west to east
  if (west_east)
    for (int i = 1; i < var.Mx; ++i)
      sweepColumn(i, var);
  // Sweep east to west
  else
    for (int i = var.Mx - 1; i > 0; --i)
```

```
      sweepColumn(i, var);
}

void
Problem::sweepColumn(const unsigned int i, Variable & var)
{
  if (debug)
    cout << "Solving " << var.string << " column " << i << endl;

  auto & A = var.Ay;
  auto & b = var.by;

  // Fill for each cell
  for (unsigned int j = 1; j < var.My; ++j)
  {
    const Coefficients & a = var.a(i, j);
    b[j - 1] = a.b + a.w * var(i - 1, j) + a.e * var(i + 1, j);
    if (var.w != 1)
      b[j - 1] += a.p * var(i, j) * (var.w - 1);
    if (j == 1)
    {
      A.setTopRow(a.p * var.w, -a.n);
      if (var.name != Variables::pc)
        b[j - 1] += a.s * var(i, j - 1);
    }
    else if (j == var.My - 1)
    {
      A.setBottomRow(-a.s, a.p * var.w);
      if (var.name != Variables::pc)
        b[j - 1] += a.n * var(i, j + 1);
    }
    else
      A.setMiddleRow(j - 1, -a.s, a.p * var.w, -a.n);
  }

  if (debug)
  {
    A.print("A =");
    b.print("b =");
  }

  // Solve
  A.solveTDMA(b);

  if (debug)
    b.print("sol =", true);

  // Store solution
  for (unsigned int j = 1; j < var.My; ++j)
    var(i, j) = b[j - 1];
}

void
Problem::sweepRow(const unsigned int j, Variable & var)
{
  if (debug)
    cout << "Solving " << var.string << " row " << j << endl;

  auto & A = var.Ax;
  auto & b = var.bx;

  // Fill for each cell
  for (unsigned int i = 1; i < var.Mx; ++i)
  {
    const Coefficients & a = var.a(i, j);
    b[i - 1] = a.b + a.s * var(i, j - 1) + a.n * var(i, j + 1);
    if (var.w != 1)
      b[i - 1] += a.p * var(i, j) * (var.w - 1);
    if (i == 1)
    {
      A.setTopRow(a.p * var.w, -a.e);
      if (var.name != Variables::pc)
        b[i - 1] += a.w * var(i - 1, j);
    }
    else if (i == var.Mx - 1)
    {
      A.setBottomRow(-a.w, a.p * var.w);
      if (var.name != Variables::pc)
        b[i - 1] += a.e * var(i + 1, j);
    }
    else
```

```
      A.setMiddleRow(i - 1, -a.w, a.p * var.w, -a.e);
    }

    if (debug)
    {
      A.print("A =");
      b.print("b =");
    }

    // Solve
    A.solveTDMA(b);

    if (debug)
      b.print("sol =", true);

    // Store solution
    for (unsigned int i = 1; i < var.Mx; ++i)
      var(i, j) = b[i - 1];
  }

} // namespace Flow2D
```

Matrix.h

```
#ifndef MATRIX_H
#define MATRIX_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include <vector>

using namespace std;

/**
 * Class that holds a N x M matrix with common matrix operations.
 */
template <typename T>
class Matrix
{
public:
  Matrix() {}
  Matrix(const unsigned int N, const unsigned int M) : N(N), M(M), A(N, vector<T>(M)) {}

  // Const operator for getting the (i, j) element
  const T & operator()(const unsigned int i, const unsigned int j) const
  {
    assert(i < N && j < M);
    return A[i][j];
  }
  // Operator for getting the (i, j) element
  T & operator()(const unsigned int i, const unsigned int j)
  {
    assert(i < N && j < M);
    return A[i][j];
  }
  // Operator for setting the entire matrix to a value
  void operator=(const T v)
  {
    for (unsigned int j = 0; j < M; ++j)
      setRow(j, v);
  }

  // Prints the matrix
  void print(const string prefix = "", const bool newline = false, const unsigned int pr = 5) const
  {
    if (prefix.length() != 0)
      cout << prefix << endl;
    for (unsigned int j = 0; j < M; ++j)
    {
      for (unsigned int i = 0; i < N; ++i)
        cout << showpos << scientific << setprecision(pr) << A[i][j] << " ";
      cout << endl;
    }
    if (newline)
      cout << endl;
  }
  // Saves the matrix in csv format
  void save(const string filename, const unsigned int pr = 12) const
  {
    ofstream f;
    f.open(filename);
    for (unsigned int j = 0; j < M; ++j)
    {
      for (unsigned int i = 0; i < N; ++i)
      {
        if (i > 0)
          f << ",";
        f << setprecision(pr) << A[i][j];
      }
      f << endl;
    }
    f.close();
  }

  // Set the j-th row to v
  void setRow(const unsigned int j, const T v)
  {
    assert(j < M);
    for (unsigned int i = 0; i < N; ++i)
      A[i][j] = v;
```

```
  }
  // Set the i-th column to v
  void setColumn(const unsigned int i, const T v)
  {
    assert(i < N);
    for (unsigned int j = 0; j < M; ++j)
      A[i][j] = v;
  }

private:
  // The size of this matrix
  const unsigned int N = 0, M = 0;

  // Matrix storage
  vector<vector<T>> A;
};

#endif /* MATRIX_H */
```

TriDiagonal.h

```cpp
#ifndef TRIDIAGONAL_H
#define TRIDIAGONAL_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include "Vector.h"

using namespace std;

/**
 * Class that holds a tri-diagonal matrix and is able to perform TDMA in place
 * with a given RHS.
 */
template <typename T>
class TriDiagonal
{
public:
  TriDiagonal() {}
  TriDiagonal(const unsigned int N, const T v = 0) : N(N), A(N, v), B(N, v), C(N - 1, v) {}

  // Setters for the top, middle, and bottom rows
  void setTopRow(const T b, const T c)
  {
    B[0] = b;
    C[0] = c;
  }
  void setMiddleRow(const unsigned int i, const T a, const T b, const T c)
  {
    assert(i < N - 1 && i != 0);
    A[i] = a;
    B[i] = b;
    C[i] = c;
  }
  void setBottomRow(const T a, const T b)
  {
    A[N - 1] = a;
    B[N - 1] = b;
  }

  // Prints the matrix
  void print(const string prefix = "", const bool newline = false, const unsigned int pr = 6) const
  {
    if (prefix.length() != 0)
      cout << prefix << endl;
    for (unsigned int i = 0; i < N; ++i)
      cout << showpos << scientific << setprecision(pr) << (i > 0 ? A[i] : 0) << " " << B[i] << " "
           << (i < N - 1 ? C[i] : 0) << endl;
    if (newline)
      cout << endl;
  }
  // Saves the matrix in csv format
  void save(const string filename, const unsigned int pr = 12) const
  {
    ofstream f;
    f.open(filename);
    for (unsigned int i = 0; i < N; ++i)
    {
      if (i > 0)
        f << setprecision(pr) << A[i] << ",";
      else
        f << "0"
          << ",";
      f << setprecision(pr) << B[i] << ",";
      if (i != N - 1)
        f << setprecision(pr) << C[i] << endl;
      else
        f << 0 << endl;
    }
    f.close();
  }

  // Solves the system Ax = d in place where d eventually stores the solution
  void solveTDMA(Vector<T> & d)
  {
    // Forward sweep
    T tmp = 0;
```

```
    for (unsigned int i = 1; i < N; ++i)
    {
      tmp = A[i] / B[i - 1];
      B[i] -= tmp * C[i - 1];
      d[i] -= tmp * d[i - 1];
    }

    // Backward sweep
    d[N - 1] /= B[N - 1];
    for (unsigned int i = N - 2; i != numeric_limits<unsigned int>::max(); --i)
    {
      d[i] -= C[i] * d[i + 1];
      d[i] /= B[i];
    }
  }

protected:
  // Matrix size (N x N)
  unsigned int N = 0;

  // Left/main/right diagonal storage
  vector<T> A, B, C;
};

#endif /* TRIDIAGONAL_H */
```

Vector.h

```
#ifndef VECTOR_H
#define VECTOR_H

#define NDEBUG
#include <cassert>
#include <fstream>
#include <vector>

using namespace std;

/**
 * Class that stores a 1D vector and enables printing and saving.
 */
template <typename T>
class Vector
{
public:
  Vector(const unsigned int N) : v(N), N(N) {}
  Vector() {}

  const T & operator()(const unsigned int i) const
  {
    assert(i < N);
    return v[i];
  }
  T & operator()(const unsigned int i)
  {
    assert(i < N);
    return v[i];
  }
  const T & operator[](const unsigned int i) const
  {
    assert(i < N);
    return v[i];
  }
  T & operator[](const unsigned int i)
  {
    assert(i < N);
    return v[i];
  }

  // Prints the vector
  void print(const string prefix = "", const bool newline = false, const unsigned int pr = 6) const
  {
    if (prefix.length() != 0)
      cout << prefix << endl;
    for (unsigned int i = 0; i < v.size(); ++i)
      cout << showpos << scientific << setprecision(pr) << v[i] << " ";
    cout << endl;
    if (newline)
      cout << endl;
  }

  // Saves the vector
  void save(const string filename, const unsigned int pr = 12) const
  {
    ofstream f;
    f.open(filename);
    for (unsigned int i = 0; i < v.size(); ++i)
      f << scientific << v[i] << endl;
    f.close();
  }

private:
  vector<T> v;
  const unsigned int N = 0;
};

#endif /* VECTOR_H */
```

postprocess.py

```
import numpy as np
import matplotlib.pyplot as plt

plt.rc('text', usetex=True)
plt.rc('font', family='serif')

Lx = 2
Ly = 0.02
cp = 4183
rho = 998.3
q = 500
k = 0.609

################################################################################
# Problem 3 prints

# Load coarse results
u = np.loadtxt('results/coarse_u.csv', delimiter=',').T
p = np.loadtxt('results/coarse_p.csv', delimiter=',').T
T = np.loadtxt('results/coarse_T.csv', delimiter=',').T
print(p.shape)

# Normalize P
p -= p[p.shape[0] - 1, :]

for var in [u, p, T]:
    for i in range(var.shape[0]):
        line = '{} & '.format(i + 1)
        for j in range(var.shape[1]):
            if var is T:
                line += '{:.5f}'.format(var[i,j])
            else:
                line += '{:.5e}'.format(var[i,j])
            if j == var.shape[1] - 1:
                line += " \\\\"
            else:
                line += " & "
        print(line)
    print()

################################################################################
# Load refined results

u = np.loadtxt('results/fine_u.csv', delimiter=',').T
v = np.loadtxt('results/fine_v.csv', delimiter=',').T
T = np.loadtxt('results/fine_T.csv', delimiter=',').T

dx = Lx / (T.shape[0] - 2)
dy = Ly / (T.shape[1] - 2)

################################################################################
# Plot problem 4a

x_center = np.linspace(0, Lx, num = u.shape[0])
u_center = np.copy(u)[:, int(u.shape[1] / 2)]
u_center /= u_center[0]
fig, ax = plt.subplots(1)
fig.set_figwidth(6)
fig.set_figheight(3)
ax.plot(x_center, u_center, 'k')
ax.set_xlabel(r'$x$ (m)')
ax.set_ylabel('Centerline $u / u_{{\\mathrm{{in}}}}$')
ax.grid()
fig.tight_layout()
fig.savefig('results/u_centerline.pdf', bbox_inches='tight')

################################################################################
# Plot problem 4b

for var, name in [(u, 'u'), (v, 'v'), (T, 'T')]:
    if name == 'T':
        y_8 = np.hstack((0, (np.linspace(dy / 2, Ly - dy / 2, num = var.shape[1] - 2)), Ly))
    else:
        y_8 = np.linspace(0, Ly, num = var.shape[1])
    var_8 = var[int(0.8 * var.shape[0] / Lx), :]

    fig, ax = plt.subplots(1)
```

```
    fig.set_figwidth(6)
    fig.set_figheight(3)
    if name == 'y':
        ax.semilogy(y_8, var_8, 'k')
    else:
        ax.plot(y_8, var_8, 'k')
    ax.set_xlabel(r'$y$ (m)')
    ax.set_ylabel('${}(0.8$ m$, y)$'.format(name))
    ax.grid()
    fig.tight_layout()
    fig.savefig('results/{}_0p8.pdf'.format(name), bbox_inches='tight')

################################################################################
# Plot problem 4b

# Sample u at temperature nodes
uT = np.zeros(T.shape)
for i in range(1, T.shape[0] - 1):
    for j in range(u.shape[1]):
        uT[0, j] = u[0, j]
        uT[u.shape[0], j] = u[u.shape[0] - 1, j]
        uT[i, j] = (u[i - 1, j] + u[i, j]) / 2

# Compute Nusselt numbers
Nu = np.zeros(T.shape[0])
u_left = u[0, 0]
for i in range(T.shape[0]):
    Tw = T[i, 0]
    Tb = 0
    for j in range(1, T.shape[1] - 1):
        Tb += uT[i, j] * T[i, j]
    Tb *= rho * cp * dy / (u_left * cp * Ly * rho)
    Nu[i] = 2 * Ly * q / (k * (Tw - Tb))

# Plot
x = np.hstack((0, (np.linspace(dx / 2, Lx - dx / 2, num = len(Nu) - 2)), Lx))
fig, ax = plt.subplots(1)
fig.set_figwidth(6)
fig.set_figheight(3)
ax.plot(x, Nu, 'k')
ax.set_xlabel(r'$x$ (m)')
ax.set_ylabel('Nu')
ax.grid()
fig.tight_layout()
fig.savefig('results/Nu.pdf', bbox_inches='tight')
```