MEEN 644 - Homework 3

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Problem statement

Consider a thin copper square plate of dimensions 0.5 m × 0.5 m. The temperature of the west and south edges are maintained at 50 °C and the north edge is maintained at 100 °C. The east edge is insulated. Using finite volume method, write a program to predict the steady-state temperature solution.

(a) **(35 points)** Set the over relaxation factor α from 1.00 to 1.40 in steps of 0.05 to identify α_{opt}. Plot the number of iterations required for convergence for each α.

(b) **(15 points)** Solve the same problem using $21^2, 25^2, 31^2$, and 41^2 CVs, respectively. Plot the temperature at the center of the plate (0.25 m, 0.25 m) vs CVs.

(c) **(10 points)** Plot the steady state temperature contour in the 2D domain with the 41^2 CV solution.

Preliminaries

Two-dimensional heat conduction

With two-dimensional heat conduction with constant material properties, insulation on the right and prescribed temperatures on all other sides, we have the PDE

$$\begin{cases} k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} = 0\,, \\ T(x,0) = T_B\,, \\ T(0,y) = T_L\,, \\ T(0,L) = T_T\,, \\ -k\frac{\partial T}{\partial x}\Big|_{x=L} = 0\,, \end{cases} \tag{1}$$

where

$$T_B \equiv 50\ ^\circ\text{C}\,, \qquad T_L \equiv 50\ ^\circ\text{C}\,, \qquad T_T \equiv 100\ ^\circ\text{C}\,.$$
$$k \equiv 386\ \text{W/m}\ ^\circ\text{C}\,, \qquad L \equiv 0.5\ \text{m}\,.$$

We discretize the region on $x \times y = [0,L]^2$ by N^2 internal nodes with $\Delta x = x/N, \Delta y = y/N$.

Control volume equations

Integrate over an internal control volume (i, j) and use the two node formulation for the derivative to obtain

$$k\Delta y\left[\frac{T_{E_{ij}}-T_{P_{ij}}}{\Delta x}-\frac{T_{P_{ij}}-T_{W_{ij}}}{\Delta x}\right]+k\Delta x\left[\frac{T_{N_{ij}}-T_{P_{ij}}}{\Delta y}-\frac{T_{P_{ij}}-T_{S_{ij}}}{\Delta y}\right]=0\,,\quad (i,j)\in[2,3,\ldots,N]^2\,.$$

Collect like terms and modify the index to obtain

$$T_{i,j}a_p - T_{i,j+1}a_n - T_{i+1,j}a_e - T_{i,j-1}a_s - T_{i-1,j}a_w = 0\,,\quad (i,j)\in[2,3,\ldots,N]^2\,, \tag{2}$$

where

$$a_n \equiv \frac{k\Delta y}{\Delta x}\,,\quad a_e \equiv \frac{k\Delta x}{\Delta y}\,,\quad a_s \equiv \frac{k\Delta y}{\Delta x}\,,\quad a_w \equiv \frac{k\Delta x}{\Delta y}\,,\quad a_p \equiv a_n + a_e + a_s + a_w\,.$$

The remaining equations are solved similarly but with slight differences depending on which boundary the CV is on.

Solving method

The problem is to be solved by the line-by-line method. In specific, the sweeping arrangement is: **south to north, west to east, north to south, east to west**. In this method, the contribution from one direction in a given control volume is lagged and moved to the right hand side in order to solve a tri-diagonal system. Convergence is declared when

$$R = \sum_{\mathrm{CV}}\left|a_pT_p - \sum_{\mathrm{nb}} a_{\mathrm{nb}}T_{\mathrm{nb}}b_p\right| \le 10^{-5}\,. \tag{3}$$

Upon solving an individual system $Ax^{\ell+1} = b$ with relaxation (where ℓ is the iteration index), the system is relaxed with the coefficient α by modifying it after construction by

$$\begin{cases} a_{ii} = a_{ii}/\alpha\,, \\ b_i = b_i + (\alpha^{-1}-1)a_{ii}x_i^{\ell}\,, \end{cases} \qquad i = 1,\ldots,N\,,$$

and it is then solved using the standard TDMA algorithm.

Results

Part a

With the given range of α, it was determined for this specific problem with 15^2 CVs that $\alpha_{\mathrm{opt}} \approx 1.3$. The requested figure showing the iteration need for each relaxation parameter follows in Figure 1.

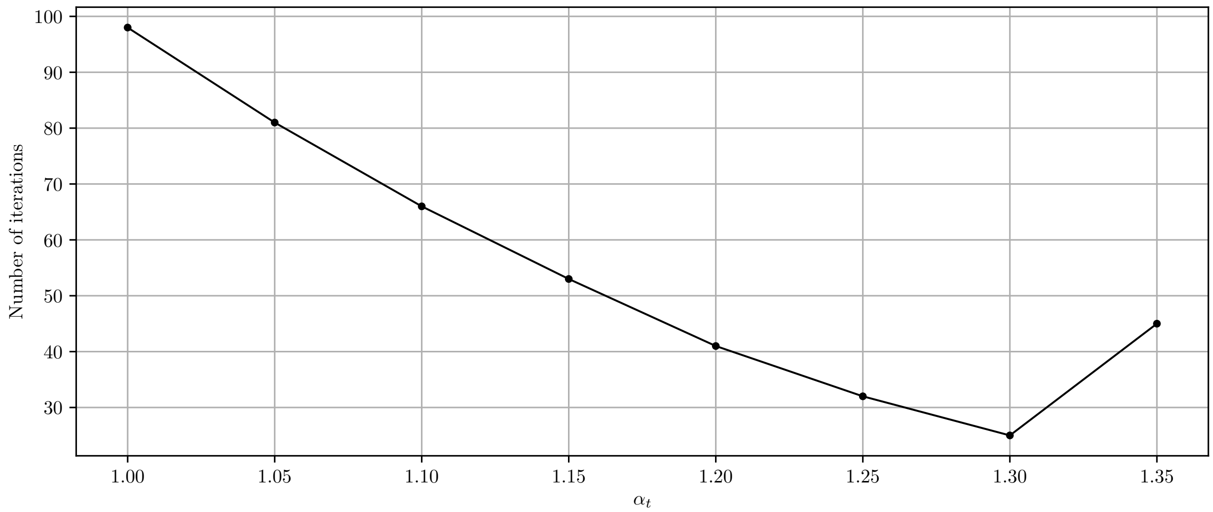

Figure 1: Plot of the required iterations for each over relaxation factor.

Part b

With a mesh refinement of $21^2, 25^2, 31^2$, and 41^2 CVs, the center temperature for each refinement is plotted below in Figure 2.

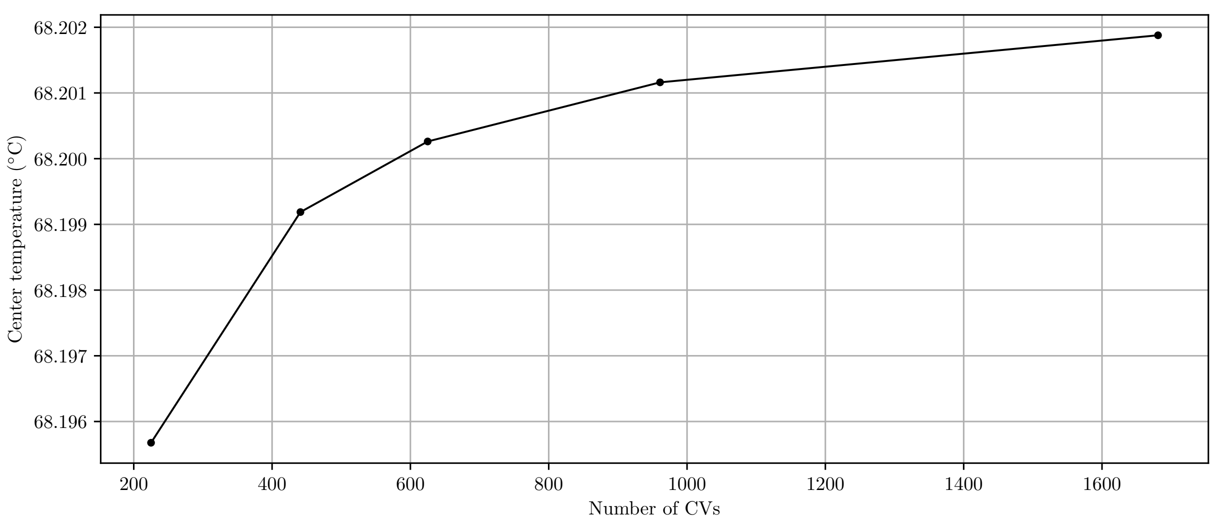

Figure 2: Plot of the center temperature with mesh refinement.

Part c

With the final mesh refinement of 41^2 CVs, a colored contour plot of the temperature solution follows in Figure 3.

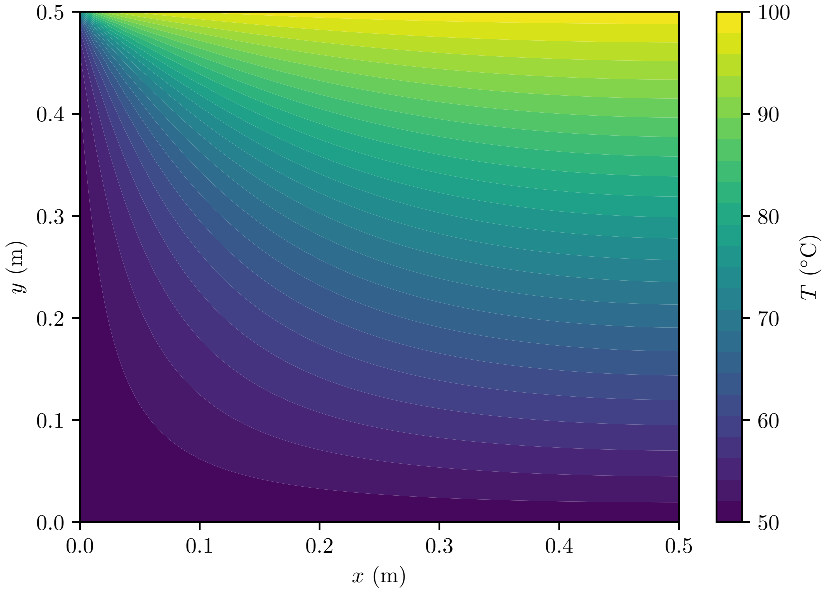

Figure 3: Plot of the solution with 41^2 CVs.