RUNGE-KUTTA METHODS

Runge-Kutta Methods are **stable and explicit methods** of solution for ordinary differential equations. The objective is to obtain the solution for

$$\frac{dy}{dx} = f(x,y) \qquad \mathbf{1}$$

Initial condition (I.C.), $x_i = x_o$, $y_i = y_o$. If h is the step size of integration, then

$$y_{i+1} = y_i + \varphi(x_i, y_i, h)h \qquad \mathbf{2}$$

where φ (x_i, y_i, h) is called an increment function and can be viewed as an average slope over the interval of integration. The increment function can be written as:

$$\varphi = a_1k_1 + a_2k_2 + \dots\dots\dots + a_nk_n \qquad \mathbf{3}$$

where a's are constants and k's are:

$$k_1 = f\,(x_i,y_i)$$

$$k_2 = f\,(x_i + P_1h, y_i + q_{11}k_1h)$$

$$k_3 = f\,(x_i + P_2h, y_i + q_{21}k_1h + q_{22}k_2h) \qquad \mathbf{4}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$k_n = f(x_i + P_{n\text{-}1}h, y_i + q_{n\text{-}1,1}k_1h + q_{n\text{-}1,2}k_2h + \dots + q_{n\text{-}1,n\text{-}1}k_{n\text{-}1}h)$$

Note that k_1 appears in the equation for k_2, which appears in the equation for k_3 and so forth. This recurrence makes RK methods efficient for computer calculations. Depending on the number of terms (n) in the increment function, various types of Runge-Kutta Methods can be devised. For n = 1, results in a first order RK method which is in fact, Euler's explicit method. The value of n governs the order of the method. Once the value of n is chosen, values for a's, p's and q's are evaluated by setting equations for k_1, k_2, k_3 equal to terms in a Taylor series expansion. As an example, let us consider derivation of the second-order Runge-Kutta method, i.e.:

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2)h \qquad \mathbf{5}$$

where,

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + P_1 h, y_i + q_{11} k_1 h) \quad \mathbf{6}$$

In order to find y_{i+1} the values for a_1, a_2, P_1, q_{11} needs to be evaluated. Invoke second-order Taylor series for y_{i+1} in terms of y_i and $f(x_i, y_i)$

$$y_{i+1} = y_i + f(x_i, y_i)h + f'(x_i, y_i)\frac{h^2}{2} \quad \mathbf{7}$$

$f'(x_i, y_i)$8is determined by applying the chain rule.

$$f'(x_i, y_i) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad (8)$$

substituting for $f'(x_i, y_i)$10into Equation 7

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)\frac{h^2}{2} \quad (9)$$

a_1, a_2, P_1, and P_2 are evaluated by the following procedure. We first use a Taylor series to expand the equation for k_2. The Taylor-series for a two-variable function is defined as

$$g(x + r, y + s) = g(x, y) + r\frac{\partial g}{\partial x} + s\frac{\partial g}{\partial y} + \ldots\ldots. \quad (10)$$

applying the above to expand $f(x_i + P_1 h, y_i + q_{11} k_1 h)$ 13

$$f(x_i + P_1 h, y_i + q_{11} k_1 h) = f(x_i, y_i) + P_1 h\frac{\partial f}{\partial x} + q_{11} k_1 h\frac{\partial f}{\partial y} + 0(h^2) \quad (11)$$

Substituting for k_1 and k_2 into Equation 5 we get:

$$y_{i+1} = y_i + a_1 hf(x_i,y_i) + a_2 hf(x_i,y_i) + a_2 P_1 h^2 \frac{\partial f}{\partial y} + \\ a_2 q_{11} h^2 f(x_i,y_i) \frac{\partial f}{\partial y} + 0(h^3) \tag{12}$$

Rearranging Equation 12:

$$y_{i+1} = y_i + \Big(a_1 f(x_i,y_i) + a_2 f(x_i,y_i) \Big) h + \\ \left(a_2 P_1 \frac{\partial f}{\partial x} + a_2 q_{11} f(x_i,y_i) \frac{\partial f}{\partial y} \right) h^2 + 0(h^3) \tag{13}$$

Comparing for like terms in Equations 9 and 13 we get:

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_2 P_1 &= 1/2 \\ a_2 q_{11} &= 1/2 \end{aligned} \tag{14}$$

Here we have a set of three simultaneous equations and four unknowns. Hence, there is no unique solution. However, by assuming the value of one of the constants, we can determine the other three. Consequently, there is a family of second-order methods rather than a single version. Let us specify a value for a_2 then

$$\begin{aligned} a_1 &= 1 - a_2 \\ P_1 = q_{11} &= 1/2a_2 \end{aligned} \tag{15}$$

An infinite number of values can be assigned to a_2 and this will render an infinite number of second-order RK methods. Every version would yield exactly the same results if the solution to the ODE is less than or equal to the order 2. The most popular set of constants are:

$$a_2 = 1/2$$

$$a_1 = 1/2 \qquad (16)$$

$$P_1 = q_{11} = 1$$

$$.e.\ y_{i+1} = y_i + (k_1 + k_2)\frac{h}{2}$$

where:

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + h, y_i + hk_1)$

FOURTH-ORDER RK METHOD TO SOLVE AN ODE

$$y_{i+1} = y_i + \left(k_1 + 2k_2 + 2k_3 + k_4\right)\frac{h}{6} \qquad (17)$$

where:

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + h/2, y_i + hk_1/2)$

$k_3 = f(x_i + h/2, y_i + hk_2/2)$

$k_4 = f(x_i + h, y_i + hk_3)$

FOURTH-ORDER RK METHOD TO SOLVE A SYSTEM OF ODES

Consider a system of n coupled ordinary differential equations of the form,

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots .y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots .y_n) \qquad (19)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots .y_n)$$

The solution for such a system of equations requires n initial conditions,

$$x = 0,\ y_1 = C_1,\ y_2 = C_2, \ldots, y_n = C_n. \tag{20}$$

If n = 1, the RK fourth-order method is given by Equations 17 and 18. Extending the RK fourth order formulation to n = 2, with step size of integration as h:

$$\begin{aligned} y_1(h) &= y_1(0) + \left[k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\ \ h/6\right] \\ y_2(h) &= y_2(0) + \left[k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\ \ h/6\right] \end{aligned} \tag{21}$$

The first step is to calculate $k_{I,J}$'s, where $k_{I,J}$ is the ith value of k for the Jth dependent variable. Before calculating k values, the values of y_1 and y_2 needed to determine K's should be computed. The implementation of the RK fourth order method is illustrated by the following example.

AN EXAMPLE

Solve:

$$\frac{dy_1}{dx} = -0.5y_1$$

$$\frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1$$

I.C. x = 0, y_1 = 4, y_2 = 6, use h = 0.5. i.e., $y_1(0) = 4$ and $y_2(0) = 6$.

$$k_{1,1} = f_1[x(0), y_1(0), y_2(0)] = -2$$

$$k_{1,2} = f_2[x(0), y_1(0), y_2(0)] = 1.8$$

Next, calculate the values of y_1 and y_2 to calculate $k_{2,1}$ and $k_{2,2}$

$$y_1(4 + k_{1,1}h/2) = 3.5$$

$$y_2(6 + k_{1,2}h/2) = 6.45$$

then,

Error!

The next step is to calculate $k_{3,1}$ and $k_{3,2}$. For this purpose corresponding y_1 and y_2 should be calculated.

$$i.e.\ \ y_1(4 + k_{2,1}h/2) = 3.5625$$

$$y_2(6 + k_{2,2}h/2) = 6.42875$$

Error!

finally, $k_{4,1}$ and $k_{4,2}$ are calculated.

$$y_1(4 + hk_{3,1}) = 3.109375$$

$$y_2(6 + hk_{3,2}) = 6.8575625$$

$$then,\ \ k_{4,1} = f_1[x(0.5), y_1(3.109375), y_2(6.8575625)]$$

$$k_{4,1} = -1.5546875$$

$$= f_2[x(0.5), y_1(3.109375), y_2(6.8575625)]$$

$$k_{4,2} = 1.6317975$$

$$y_1(0.5) = y_1(0) + h[k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}]/6$$

$$y_2(0.5) = y_2(0) + h[k_{1,2} + 2k_{2,2} + 2k_{3,2} + 4_{4,2}]/6$$

$$y_1(0.5) = 3.1152344$$

$$y_2(0.5) = 6.857670625$$

The above set of steps enabled us to determine the values of y_1 and y_2 at $x = 0.5$. In order to evaluate y_1 and y_2 at x = 1.0, 1.5 and 2.0, the very same procedure should be repeated. Note that for the second step:

$$y_1(1.0) = y_1(0.5) + \ldots.$$

$$y_2(1.0) = y_2(0.5) + \ldots.$$

Summary of Calculations

x	y_1	y_2
0	4	6
0.5	3.1152344	6.85767
1.0	2.4261713	...
1.5	1.8895231	...
2.0	1.4715768	...

Reference

Chapra, S.C. and Canale, R.P., "Numerical Methods for Engineers," McGraw-Hill, 1985.

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