MEEN 644 - Homework 2

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Problem statement

Consider one-dimensional heat conduction in a cylindrical copper rod of length 1.0 m long. The diameter of the rod is 0.05 m. The left end of the rod is held at 100 °C and the ambient temperature is 25 °C. Heat is transported from the surface of the rod and the right end of the rod through natural convection to the ambient. The natural convection heat transfer coefficient is 0.5 W/m^2 °C. Write a finite volume code to predict temperature distribution as a function of length. Use TDMA to solve a set of discretization equations. Make calculations using ITMAX: 6, 11, 21, 41, and 81 nodes. Plot your results.

Preliminaries

ODE definition

With one-dimensional heat conduction with convection and constant material properties, we have the ODE:

$$\begin{cases} \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + \frac{h}{kd}(T - T_\infty) = 0\,, \\ T(0) = T_0\,, \\ \frac{\mathrm{d}T}{\mathrm{d}x}\Big|_{x=L} = -\frac{h}{k}(T - T_\infty)\,, \end{cases} \tag{1}$$

where

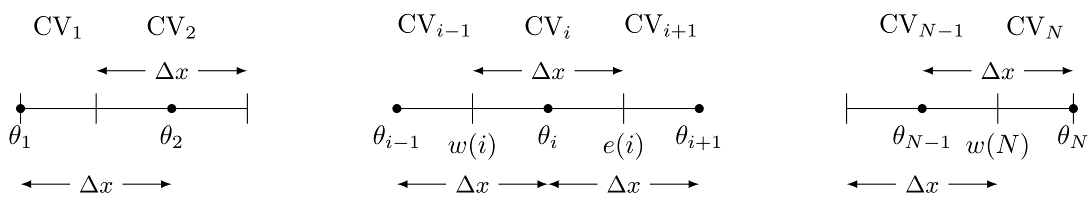
$$k \equiv 400 \text{ W/m °C}\,, \qquad h \equiv 0.5 \text{ W/m}^2 \text{ °C}\,, \qquad d \equiv 0.05 \text{ m}\,,$$
$$L \equiv 1.0 \text{ m}\,, \qquad T_0 \equiv 100 \text{ °C}\,, \qquad T_\infty \equiv 25 \text{ °C}\,.$$

We then make the substitutions $\theta(x) = T(x) - T_\infty$ and $m = h/kd$ to obtain the simplification

$$\begin{cases} \frac{\mathrm{d}^2 \theta}{\mathrm{d}x^2} + m\theta = 0\,, \\ \theta(0) = T_0 - T_\infty\,, \\ \frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{x=L} = -\frac{h}{k}\theta\,. \end{cases} \tag{2}$$

Grid generation

We discretize the region on $x = [0, L]$ by N (also defined as ITMAX) nodes and N control volumes, as follows in Figure 1 with $\Delta x = L/(N-1)$.



(a) Left CV (b) Interior CV ($1 < i < N$) (c) Right CV

Figure 1: The control volumes defined for discretization of the problem.

Equation discretization

Internal control volume equation

We start with the integration over an interior control volume, as

$$\int_{\mathrm{CV}_i} \left[-\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + m\theta \right] dx = 0\,, \quad 1 < i < N\,,$$

in which we know that the material properties are independent and we assume θ_i to be constant over the cell for the second term to obtain

$$-\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{e(i)} - \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{w(i)} \right) + m\Delta x \theta_i = 0\,, \quad 1 < i < N\,.$$

Use the two node formulation for the derivative terms and simplify as

$$-\left(\frac{\theta_{i+1} - \theta_i}{\Delta x} - \frac{\theta_i - \theta_{i-1}}{\Delta x} \right) + m\Delta x \theta_i = 0\,, \quad 1 < i < N\,,$$

$$-\frac{1}{\Delta x}\theta_{i-1} + \left(m\Delta x + \frac{2}{\Delta x} \right)\theta_i - \frac{1}{\Delta x}\theta_{i+1} = 0\,, \quad 1 < i < N\,.$$

Take note that at the $i = 2$ equation, θ_1 is known therefore we have

$$\boxed{\left(m\Delta x + \frac{2}{\Delta x} \right)\theta_2 - \frac{1}{\Delta x}\theta_3 = \frac{T_0 - T_\infty}{\Delta x}\,,} \tag{3}$$

$$\boxed{-\frac{1}{\Delta x}\theta_{i-1} + \left(m\Delta x + \frac{2}{\Delta x} \right)\theta_i - \frac{1}{\Delta x}\theta_{i+1} = 0\,, \quad 2 < i < N\,.} \tag{4}$$

Right control volume equation

We start with the integration over the right control volume, CV_N, as

$$\int_{\mathrm{CV}_N} \left[-\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + m\theta \right] dx = 0\,,$$

in which for the second term we will assume θ_N to be constant over CV_N to obtain

$$-\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{x=L} - \frac{\mathrm{d}\theta}{\mathrm{d}x}\bigg|_{w(N)} \right) + \frac{1}{2} m\Delta x \theta_N = 0\,.$$

Use the two node formulation for the derivative term at $w(N)$ and the right boundary condition for the derivative term at $x = L$ m to obtain

$$\frac{h}{k}\theta_N + \frac{\theta_N - \theta_{N-1}}{\Delta x} + \frac{1}{2}m\Delta x\theta_N = 0\,,$$

$$\boxed{-\frac{1}{\Delta x}\theta_{N-1} + \left(\frac{1}{2}m\Delta x + \frac{h}{k} + \frac{1}{\Delta x}\right)\theta_N = 0\,.} \tag{5}$$

Results

The plotted results as requested follow below in Figure 2.

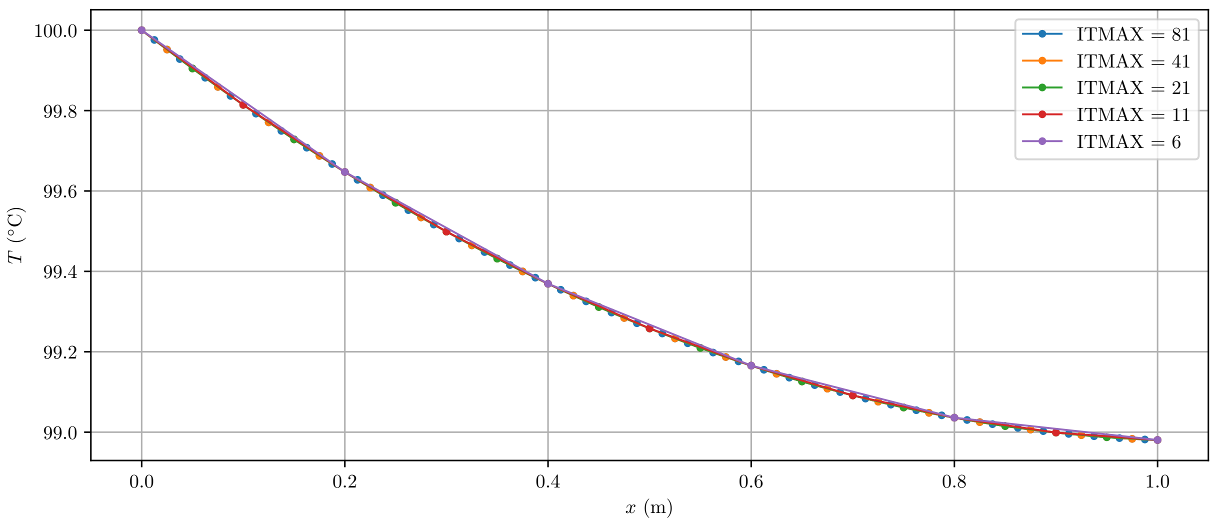


Figure 2: The plotted solution.