MEEN 644 - Homework 2

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Problem statement

Consider one-dimensional heat conduction in a cylindrical copper rod of length 1.0 m long. The diameter of the rod is 0.05 m. The left end of the rod is held at 100 °C and the ambient temperature is 25 °C. Heat is transported from the surface of the rod and the right end of the rod through natural convection to the ambient. The natural convection heat transfer coefficient is 0.5 W/m^2 °C. Write a finite volume code to predict temperature distribution as a function of length. Use TDMA to solve a set of discretization equations. Make calculations using ITMAX: 6, 11, 21, 41, and 81 nodes. Plot your results.

Preliminaries

ODE definition

With one-dimensional heat conduction with convection and constant material properties, we have the ODE:

$$\begin{cases} kA\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + hp(T - T_\infty) = 0\,, \\ T(0) = T_0\,, \\ \frac{\mathrm{d}T}{\mathrm{d}x}\Big|_{x=1\text{ m}} = -\frac{k}{h}(T - T_\infty)\,, \end{cases} \tag{1}$$

where

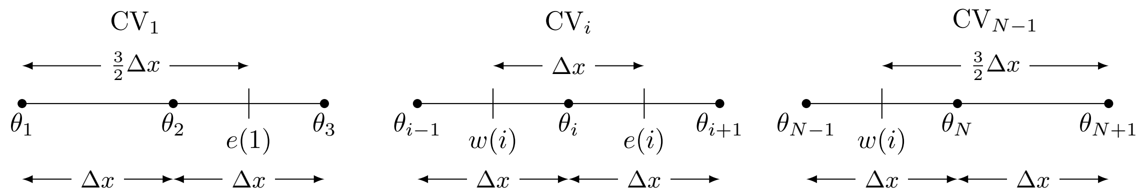
$$k \equiv 400 \text{ W/m °C}\,, \qquad h \equiv 0.5 \text{ W/m}^2 \text{ °C}\,, \qquad A \equiv 0.25^2\pi \text{ m}\,,$$
$$p \equiv 0.5\pi \text{ m}\,, \qquad T_0 \equiv 100 \text{ °C}\,, \qquad T_\infty \equiv 25 \text{ °C}\,.$$

We then make the substitution $\theta(x) = T(x) - T_\infty$ to obtain the simplification

$$\begin{cases} kA\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + hp\theta = 0\,, \\ \theta(0) = T_0 - T_\infty\,, \\ \frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{x=1\text{ m}} = -\frac{k}{h}\theta\,. \end{cases} \tag{2}$$

Discretization

We discretize the region on $x = [0, L]$ by N (also defined as ITMAX) nodes and $N - 1$ control volumes, as follows in Figure 1.



(a) Left CV (b) Interior CV ($1 < i < N-1$) (c) Right CV

Figure 1: The control volumes defined for discretization of the problem.

Internal control volume equation

We start with the integration over an interior control volume, as

$$\int_{\mathrm{CV}_i} \left[-\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \frac{hp}{kA}\theta \right] dx = 0\,, \quad 1 < i < N-1\,,$$

in which we know that the material properties are independent, to obtain

$$-\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{e(i)} - \frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{w(i)} \right) + \frac{hp\Delta x}{kA}\theta_i = 0\,, \quad 1 < i < N-1\,.$$

Use the two node formulation for the derivative terms and simplify as

$$-\left(\frac{\theta_{i+1} - \theta_i}{\Delta x} - \frac{\theta_i - \theta_{i-1}}{\Delta x} \right) + \frac{hp\Delta x}{kA}\theta_i = 0\,, \quad 1 < i < N-1\,,$$

$$-\frac{1}{\Delta x}\theta_{i-1} + \left(\frac{hp\Delta x}{kA} + \frac{2}{\Delta x} \right)\theta_i - \frac{1}{\Delta x}\theta_{i+1} = 0\,, \quad 1 < i < N-1\,. \tag{3}$$

Left boundary control volume equation

Utilize Equation 3 for $i = 1$ with a known value of $\theta_1 = T_0 - T_\infty$ to obtain

$$\left(\frac{hp\Delta x}{kA} + \frac{2}{\Delta x} \right)\theta_1 - \frac{1}{\Delta x}\theta_2 = \frac{1}{\Delta x}(T_0 - T_\infty)\,. \tag{4}$$

Right boundary control volume equation

At the right boundary we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{x=1\ \mathrm{m}} = -\frac{k}{h}\theta\,.$$

Use a backward difference for the derivative term to obtain

$$\frac{\theta_{N+1} - \theta_N}{\frac{1}{2}\Delta x} = -\frac{h}{k}\theta_{N+1}\,,$$

$$-\theta_N + \left(1 + \frac{h\Delta x}{2k} \right)\theta_{N+1} = 0\,. \tag{5}$$

Simplified control volume equations

First, define

$$a_w = a_e = \frac{1}{\Delta x} \quad \text{and} \quad a_p = \frac{hp\Delta x}{kA} + \frac{2}{\Delta x}\,,$$

in which we are then solving the system

$$\begin{cases} a_p\theta_1 - a_e\theta_2 = a_w(T_0 - T_\infty)\,, \\ a_w\theta_{i-1} - a_p\theta_i + a_e\theta_{i+1} = 0\,, \quad 1 < i < N-1\,, \\ -\theta_N + \left(1 + \frac{h\Delta x}{2k}\right)\theta_{N+1} = 0\,. \end{cases} \tag{6}$$

Results

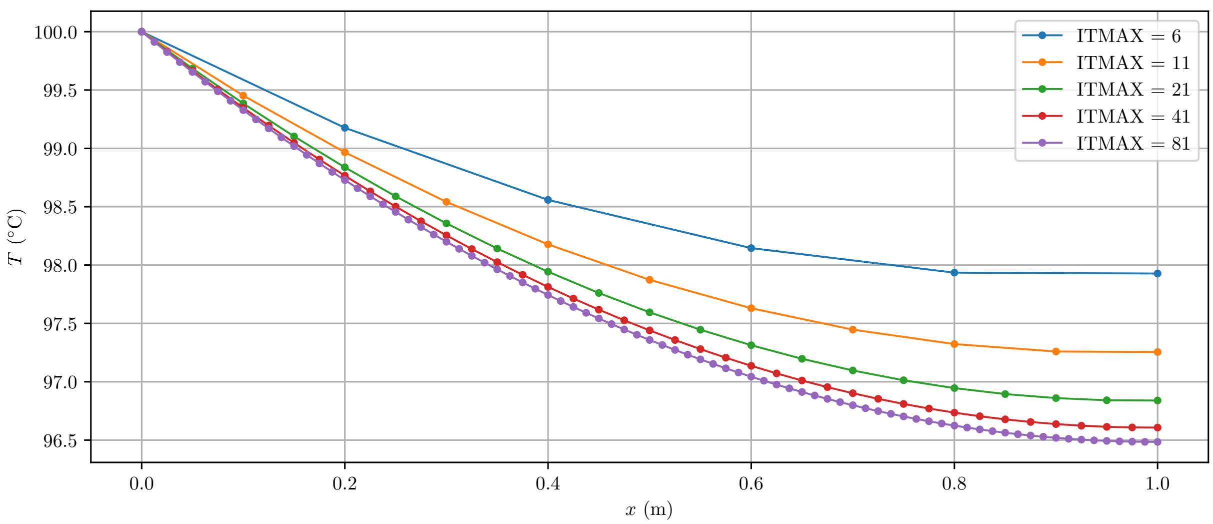


Figure 2: The plotted solution.