MEEN 644 - Homework 2

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Problem statement

Consider one-dimensional heat conduction in a cylindrical copper rod of length 1.0 m long. The diameter of the rod is 0.05 m. The left end of the rod is held at 100 °C and the ambient temperature is 25 °C. Heat is transported from the surface of the rod and the right end of the rod through natural convection to the ambient. The natural convection heat transfer coefficient is 0.5 W/m^2 °C. Write a finite volume code to predict temperature distribution as a function of length. Use TDMA to solve a set of discretization equations. Make calculations using ITMAX: 6, 11, 21, 41, and 81 nodes. Plot your results.

Preliminaries

One-dimensional heat conduction

With one-dimensional heat conduction with convection and constant material properties, we have the ODE

$$\begin{cases} \frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \frac{h}{kd}(T - T_\infty) = 0\,, \\ T(0) = T_0\,, \\ \frac{\mathrm{d}T}{\mathrm{d}x}\Big|_{x=L} = -\frac{h}{k}(T - T_\infty)\,, \end{cases} \tag{1}$$

where

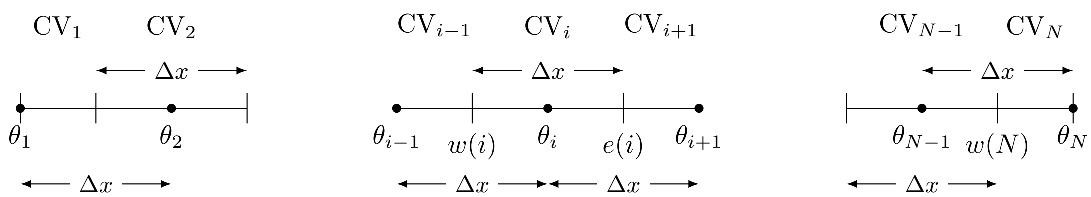
$$k \equiv 400 \text{ W/m °C}\,, \qquad h \equiv 0.5 \text{ W/m}^2 \text{ °C}\,, \qquad d \equiv 0.05 \text{ m}\,,$$
$$L \equiv 1.0 \text{ m}\,, \qquad T_0 \equiv 100 \text{ °C}\,, \qquad T_\infty \equiv 25 \text{ °C}\,.$$

We then make the substitutions $\theta(x) = T(x) - T_\infty$ and $m = 4h/kd$ to obtain the simplification

$$\begin{cases} \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + m\theta = 0\,, \\ \theta(0) = T_0 - T_\infty\,, \\ \frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{x=L} = -\frac{h}{k}\theta\,. \end{cases} \tag{2}$$

Grid generation

We discretize the region on $x = [0, L]$ by N (also defined as ITMAX) nodes and N control volumes, as follows in Figure 1 with $\Delta x = L/(N-1)$.



(a) Left CV (b) Interior CV ($1 < i < N$) (c) Right CV

Figure 1: The control volumes defined for discretization of the problem.

Equation discretization

Internal control volume equation

We start with the integration over an interior control volume, as

$$\int_{\mathrm{CV}_i} \left[-\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + m\theta \right] dx = 0\,, \quad 1 < i < N\,,$$

in which we know that the material properties are independent and we assume θ_i to be constant over the cell for the second term to obtain

$$-\left(\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{e(i)} - \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{w(i)} \right) + m\Delta x\theta_i = 0\,, \quad 1 < i < N\,.$$

Use the two node formulation for the derivative terms and simplify as

$$-\left(\frac{\theta_{i+1} - \theta_i}{\Delta x} - \frac{\theta_i - \theta_{i-1}}{\Delta x} \right) + m\Delta x\theta_i = 0\,, \quad 1 < i < N\,,$$

$$-\frac{1}{\Delta x}\theta_{i-1} + \left(m\Delta x + \frac{2}{\Delta x} \right)\theta_i - \frac{1}{\Delta x}\theta_{i+1} = 0\,, \quad 1 < i < N\,.$$

Take note that at the $i = 2$ equation, θ_1 is known therefore we have

$$\boxed{\left(m\Delta x + \frac{2}{\Delta x} \right)\theta_2 - \frac{1}{\Delta x}\theta_3 = \frac{T_0 - T_\infty}{\Delta x}\,,} \tag{3}$$

$$\boxed{-\frac{1}{\Delta x}\theta_{i-1} + \left(m\Delta x + \frac{2}{\Delta x} \right)\theta_i - \frac{1}{\Delta x}\theta_{i+1} = 0\,, \quad 2 < i < N\,.} \tag{4}$$

Right control volume equation

We start with the integration over the right control volume, CV_N, as

$$\int_{\mathrm{CV}_N} \left[-\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + m\theta \right] dx = 0\,,$$

in which for the second term we will assume θ_N to be constant over CV_N to obtain

$$-\left(\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=L} - \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{w(N)} \right) + \frac{1}{2}m\Delta x\theta_N = 0\,.$$

Use the two node formulation for the derivative term at $w(N)$ and the right boundary condition for the derivative term at $x = L$ m to obtain

$$\frac{h}{k}\theta_N + \frac{\theta_N - \theta_{N-1}}{\Delta x} + \frac{1}{2}m\Delta x\theta_N = 0\,,$$
$$\boxed{-\frac{1}{\Delta x}\theta_{N-1} + \left(\frac{1}{2}m\Delta x + \frac{h}{k} + \frac{1}{\Delta x}\right)\theta_N = 0\,.} \tag{5}$$

TDMA

The system we are solving is of the form

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}.$$

We will solve this system using the tridiagonal matrix algorithm (TDMA) given the fact that our system is positive definite. With our matrix in the form above, the algorithm follows in Algorithm 1.

for $i = 2, 3, \ldots, n$ **do**
  $w = a_i / b_{i-1}$;
  $b_i = b_i - wc_{i-1}$;
  $d_i = d_i - wd_{i-1}$;
end
$x_n = d_n / b_n$;
for $i = n-1, n-2, \ldots, 1$ **do**
  $x_i = (d_i - c_i x_{i+1}) / b_i$;
end

Algorithm 1: The tridiagonal matrix algorithm (TDMA).

Results

The plotted results as requested follow below in Figure 2.

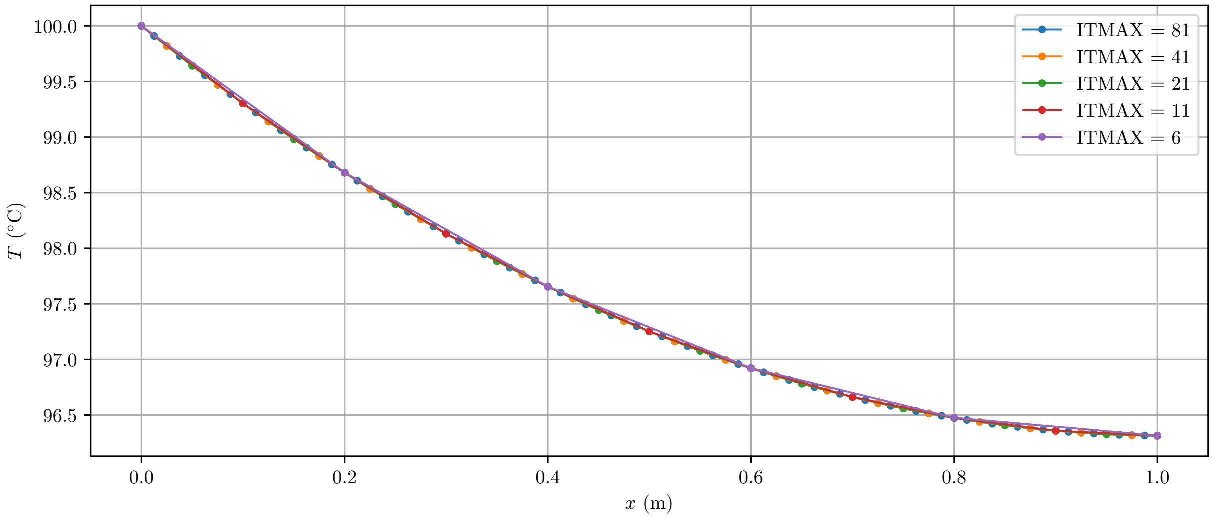


Figure 2: The plotted solution.