MEEN 644 - Homework 3

Logan Harbour

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Problem statement

Complete

Preliminaries

Two-dimensional heat conduction

With two-dimensional heat conduction with constant material properties, insulation on the right and prescribed temperatures on all other sides, we have the PDE

$$\begin{cases} k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} = 0\,, \\ T(x,0) = T_B\,, \\ T(0,y) = T_L\,, \\ T(0,L) = T_T\,, \\ -k\frac{\partial T}{\partial x}\Big|_{x=L} = 0\,, \end{cases} \tag{1}$$

where

$$T_B \equiv 50\ ^\circ\mathrm{C}\,, \qquad T_L \equiv 50\ ^\circ\mathrm{C}\,, \qquad T_T \equiv 100\ ^\circ\mathrm{C}\,.$$
$$k \equiv 386\ \mathrm{W/m}\ ^\circ\mathrm{C}\,, \qquad L \equiv 0.5\ \mathrm{m}\,.$$

We discretize the region on $x \times y = [0, L]^2$ by N^2 internal nodes with $\Delta x = x/N, \Delta y = y/N$.

Equation discretization

Internal control volume equations

Integrate over an internal control volume (i, j) and use the two node formulation for the derivative to obtain

$$k\Delta y\left[\frac{T_{E_{ij}} - T_{P_{ij}}}{\Delta x} - \frac{T_{P_{ij}} - T_{W_{ij}}}{\Delta x}\right] + k\Delta x\left[\frac{T_{N_{ij}} - T_{P_{ij}}}{\Delta y} - \frac{T_{P_{ij}} - T_{S_{ij}}}{\Delta y}\right] = 0\,, \quad (i,j) \in [2,3,\ldots,N]^2\,.$$

Collect like terms and modify the index to obtain

$$T_{i,j}a_p - T_{i,j+1}a_n - T_{i+1,j}a_e - T_{i,j-1}a_s - T_{i-1,j}a_w = 0\,, \quad (i,j) \in [2,3,\ldots,N]^2\,, \tag{2}$$

where

$$a_n \equiv \frac{k\Delta y}{\Delta x}\,, \quad a_e \equiv \frac{k\Delta x}{\Delta y}\,, \quad a_s \equiv \frac{k\Delta y}{\Delta x}\,, \quad a_w \equiv \frac{k\Delta x}{\Delta y}\,, \quad a_p \equiv a_n + a_e + a_s + a_w\,.$$

The remaining equations are solved similarly.

Bottom internal control volume equations

$$T_{i,2}(a_n + a_e + 2a_s + a_w) - T_{i,3}a_n - T_{i+1,2}a_e - T_{i-1,2}a_w = 2T_B a_s\,, \quad i \in 3, 4, \ldots, N\,. \tag{3}$$

Bottom left control volume equation

$$T_{2,2}(a_n + a_e + 2a_s + 2a_w) - T_{2,3}a_n - T_{3,2}a_e = 2T_B a_s + 2T_L a_w\,. \tag{4}$$

Left internal control volume equations

$$T_{2,j}(a_n + a_e + a_s + 2a_w) - T_{2,j+1}a_n - T_{3,j}a_e - T_{2,j-1}a_s = 2T_L a_w\,, \quad j \in 3, 4, \ldots, N\,. \tag{5}$$

Top left control volume equation

$$T_{2,N+1}(2a_n + a_e + a_s + 2a_w) - T_{2,N}a_s - T_{3,N+1}a_e = 2T_T a_n + 2T_L a_w\,. \tag{6}$$

Top internal control volume equation

$$T_{i,N+1}(2a_n + a_e + a_s + a_w) - T_{i+1,N+1}a_e - T_{i,N}a_s - T_{i-1,N+1}a_w = 2T_T a_n\,, \quad i \in 3, 4, \ldots, N\,. \tag{7}$$

Top right control volume equation

$$T_{N+1,N+1}(2a_n + a_s + a_w) - T_{N+1,N}a_s - T_{N,N+1}a_w = 2T_T a_n\,. \tag{8}$$

Right internal control volume equations

$$T_{N+1,j}(a_n + a_s + a_w) - T_{N+1,j+1}a_n - T_{N+1,j-1}a_s - T_{N,j}a_w = 0\,, \quad j \in 3, 4, \ldots, N\,. \tag{9}$$

Bottom right control volume equation

$$T_{N+1,2}(a_n + 2a_s + a_w) - T_{N+1,3}a_n - T_{N,2}a_w = 2T_B a_s\,. \tag{10}$$

Results

Part a

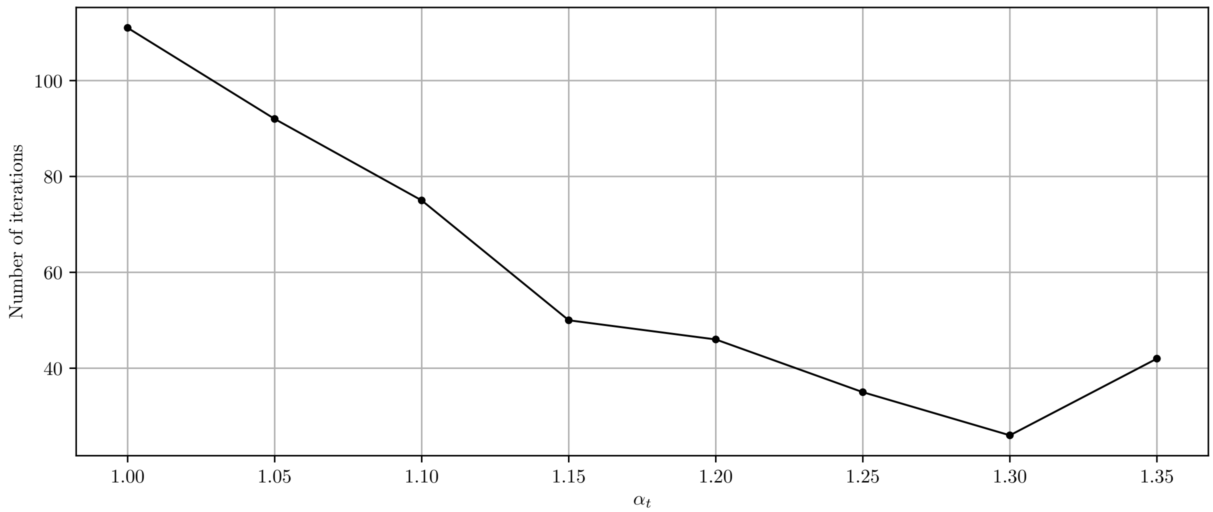

Figure 1: Plot of the required iterations for each over relaxation factor.

Part b

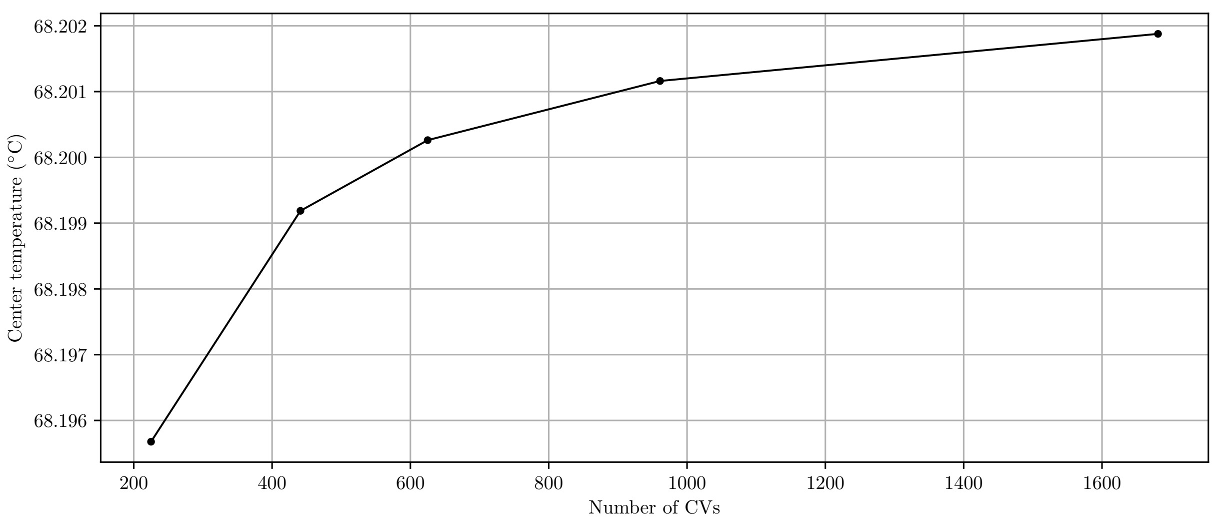

Figure 2: Plot of the center temperature with mesh refinement.

Part c

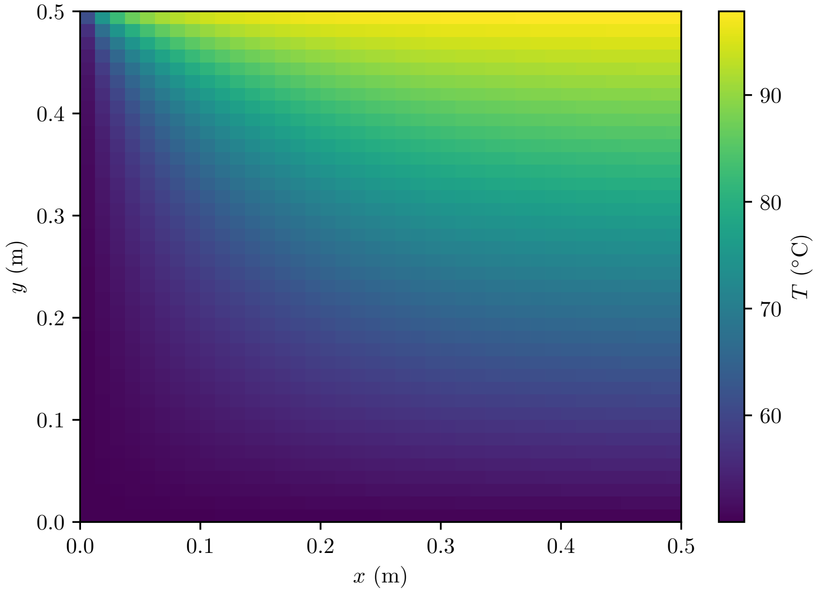

Figure 3: Plot of the solution with 41×41 CVs.