

MEEN 644 - Homework 3

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Problem statement

Complete

Preliminaries

Two-dimensional heat conduction

With two-dimensional heat conduction with constant material properties, insulation on the right and prescribed temperatures on all other sides, we have the PDE

$$\begin{cases} k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} = 0, \\ T(x, 0) = T_B, \\ T(0, y) = T_L, \\ T(0, L) = T_T, \\ -k \frac{\partial T}{\partial x} \Big|_{x=L} = 0, \end{cases} \quad (1)$$

where

$$\begin{aligned} T_B &\equiv 50 \text{ }^\circ\text{C}, & T_L &\equiv 50 \text{ }^\circ\text{C}, & T_T &\equiv 100 \text{ }^\circ\text{C}. \\ k &\equiv 386 \text{ W/m }^\circ\text{C}, & L &\equiv 0.5 \text{ m}. \end{aligned}$$

We discretize the region on $x \times y = [0, L]^2$ by N^2 internal nodes with $\Delta x = x/N, \Delta y = y/N$.

Equation discretization

Internal control volume equations

Integrate over an internal control volume (i, j) and use the two node formulation for the derivative to obtain

$$k\Delta y \left[\frac{T_{E_{ij}} - T_{P_{ij}}}{\Delta x} - \frac{T_{P_{ij}} - T_{W_{ij}}}{\Delta x} \right] + k\Delta x \left[\frac{T_{N_{ij}} - T_{P_{ij}}}{\Delta y} - \frac{T_{P_{ij}} - T_{S_{ij}}}{\Delta y} \right] = 0, \quad (i, j) \in [2, 3, \dots, N]^2.$$

Collect like terms and modify the index to obtain

$$T_{i,j}a_p - T_{i,j+1}a_n - T_{i+1,j}a_e - T_{i,j-1}a_s - T_{i-1,j}a_w = 0, \quad (i, j) \in [2, 3, \dots, N]^2, \quad (2)$$

where

$$a_n \equiv \frac{k\Delta y}{\Delta x}, \quad a_e \equiv \frac{k\Delta x}{\Delta y}, \quad a_s \equiv \frac{k\Delta y}{\Delta x}, \quad a_w \equiv \frac{k\Delta x}{\Delta y}, \quad a_p \equiv a_n + a_e + a_s + a_w.$$

The remaining equations are solved similarly.

Bottom internal control volume equations

$$T_{i,2}(a_n + a_e + 2a_s + a_w) - T_{i,3}a_n - T_{i+1,2}a_e - T_{i-1,2}a_w = 2T_B a_s, \quad i \in 3, 4, \dots, N. \quad (3)$$

Bottom left control volume equation

$$T_{2,2}(a_n + a_e + 2a_s + 2a_w) - T_{2,3}a_n - T_{3,2}a_e = 2T_B a_s + 2T_L a_w. \quad (4)$$

Left internal control volume equations

$$T_{2,j}(a_n + a_e + a_s + 2a_w) - T_{2,j+1}a_n - T_{3,j}a_e - T_{2,j-1}a_s = 2T_L a_w, \quad j \in 3, 4, \dots, N. \quad (5)$$

Top left control volume equation

$$T_{2,N+1}(2a_n + a_e + a_s + 2a_w) - T_{2,N}a_s - T_{3,N+1}a_e = 2T_T a_n + 2T_L a_w. \quad (6)$$

Top internal control volume equation

$$T_{i,N+1}(2a_n + a_e + a_s + a_w) - T_{i+1,N+1}a_e - T_{i,N}a_s - T_{i-1,N+1}a_w = 2T_T a_n, \quad i \in 3, 4, \dots, N. \quad (7)$$

Top right control volume equation

$$T_{N+1,N+1}(2a_n + a_s + a_w) - T_{N+1,N}a_s - T_{N,N+1}a_w = 2T_T a_n. \quad (8)$$

Right internal control volume equations

$$T_{N+1,j}(a_n + a_s + a_w) - T_{N+1,j+1}a_n - T_{N+1,j-1}a_s - T_{N,j}a_w = 0, \quad j \in 3, 4, \dots, N. \quad (9)$$

Bottom right control volume equation

$$T_{N+1,2}(a_n + 2a_s + a_w) - T_{N+1,3}a_n - T_{N,2}a_w = 2T_B a_s. \quad (10)$$