4.2.3 Exact Solution/ Exponential Scheme

The 1-D convection-diffusion Equation 4.2.26 can be integrated in a closed form between two adjacent nodes $[0, L]$. Consider

$$x = 0, \phi = \phi_0 \tag{4.2.43a}$$
$$x = L, \phi = \phi_L \tag{4.2.43b}$$

Then solution to Eq. 4.2.26 is

$$\phi = C_1 + C_2 e^{Px/L} \tag{4.2.44}$$

using boundary conditions 4.2.43a and 4.2.43b

$$\phi_0 = C_1 + C_2 \tag{4.2.45a}$$
$$\phi_L = C_1 + C_2 e^P \tag{4.2.45b}$$

Solving for C_1 and C_2

$$C_2 = \frac{\phi_L - \phi_0}{e^P - 1} \tag{4.2.46a}$$
$$C_1 = \phi_L - \frac{\phi_L - \phi_0}{e^P - 1} e^P \tag{4.2.46b}$$

$$\therefore \phi = \phi_L - \frac{\phi_L - \phi_0}{e^P - 1} e^P + \frac{\phi_L - \phi_0}{e^P - 1} e^{Px/L} \tag{4.2.47}$$

$$\therefore \phi = \frac{\phi_L e^P - \phi_L - e^P \phi_L + e^P \phi_0 + e^{Px/L} \phi_L - e^{Px/L} \phi_0}{e^P - 1} \tag{4.2.48}$$

Subtract ϕ_0 from both sides

$$\phi - \phi_0 = \frac{\phi_L (e^{Px/L} - 1) + \phi_0 (e^P - e^{Px/L}) - \phi_0 (e^P - 1)}{e^P - 1} \tag{4.2.49}$$

After rearranging,

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{e^{Px/L} - 1}{e^P - 1} \tag{4.2.50}$$

When $P = 0$, convection is absent and it is pure diffusion. Using L' Hospitals' rule as $P \to 0$ Eq. 4.2.50 gives rise to a linear function.

Again, using L' Hospitals' rule it is evident that $\phi \to \phi_L$ as $P \to -\infty$ and $\phi \to \phi_0$ as $P \to \infty$. When $|P|$ is large, $d\phi/dx$ is measuring zero at $x = L/2$. Thus, the diffusion is almost absent (Patankar 1980). However, upwind scheme always calculates diffusion term from a linear profile $\phi \sim x$ and thus over-estimates diffusion at large values of $|P|$ (Patankar 1980). Hence, a discretization scheme based on the exact solution would not have these pitfalls. A discretization scheme based on the exact solution is called the *exponential scheme.*

Integrating the 1-D steady convection-diffusion equation over the CV shown in Figure 4.1.1 we get

$$(\rho u \phi)_e - (\rho u \phi)_w = \left(\Gamma \frac{d\phi}{dx}\right)_e - \left(\Gamma \frac{d\phi}{dx}\right)_w \tag{4.2.51}$$

$$F_e \phi_e - F_w \phi_w = \left(\Gamma \frac{d\phi}{dx}\right)_e - \left(\Gamma \frac{d\phi}{dx}\right)_w \tag{4.2.52}$$

Making use of the exact solution we can write

$$\phi_e = \phi_P + (\phi_E - \phi_P)\frac{e^{P_e/2} - 1}{e^{P_e} - 1} \tag{4.2.53a}$$

$$\phi_w = \phi_W + (\phi_P - \phi_W)\frac{e^{P_w/2} - 1}{e^{P_w} - 1} \tag{4.2.53b}$$

$$\left(\frac{d\phi}{dx}\right)_e = \frac{e^{P_e/2}(\phi_E - \phi_P)}{e^{P_e} - 1} \tag{4.2.53c}$$

$$\left(\frac{d\phi}{dx}\right)_w = \frac{e^{P_w/2}(\phi_P - \phi_W)}{e^{P_w} - 1} \tag{4.2.53d}$$

Substituting Eqs. 4.2.53a - 4.2.53d into Eq. 4.2.52 and simplifying,

$$\phi_P\left(F_e + \frac{F_e}{e^{P_e} - 1} + \frac{F_w}{e^{P_w} - 1}\right) = \phi_W\left(F_w + \frac{F_w}{e^{P_w} - 1}\right) + \phi_E\left(\frac{F_e}{e^{P_e} - 1}\right) \tag{4.2.54}$$

Therefore, we can write the above equation in the following familiar form,

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \tag{4.2.55a}$$

$$a_W = F_w + \frac{F_w}{e^{P_w} - 1} \tag{4.2.55b}$$

$$a_E = \frac{F_e}{e^{P_e} - 1} \tag{4.2.55c}$$

$$a_P = F_e + \frac{F_e}{e^{P_e} - 1} + \frac{F_w}{e^{P_w} - 1} \tag{4.2.55d}$$

Substituting Eqs. 4.2.55b and 4.2.55c into Eqs. 4.2.55d we get,

$$a_P = a_W + a_E + (F_e - F_w) \tag{4.2.56}$$

But $F_e - F_w = 0$, because of continuity. Therefore, $a_P = a_W + a_E$.

The next step is to determine $A(|P|)$ for the exponential scheme. Consider Eq4.2.55a - 4.2.55d

$$a_W = F_w + \frac{F_w}{e^{P_w} - 1} \tag{4.2.57a}$$

$$a_W = \frac{F_w}{e^{P_w} - 1} + \left[\!\left[F_w, 0\right]\!\right] - \left[\!\left[-F_w, 0\right]\!\right] \tag{4.2.57b}$$

$$a_W = D_w \left\{ \frac{P_w}{e^{P_w} - 1} - \left[\!\left[-P_w, 0\right]\!\right] \right\} + \left[\!\left[F_w, 0\right]\!\right] \tag{4.2.57c}$$

Comparing Eq. 4.3.85c with Eq. 4.2.25

$$A(P_w) = \frac{P_w}{e^{P_w} - 1} - \left[\!\left[-P_w, 0\right]\!\right] \tag{4.2.58}$$

For $P_w > 0$

$$A(P_w) = \frac{P_w}{e^{P_w} - 1} \tag{4.2.59}$$

For $P_w < 0$

$$A(P_w) = \frac{P_w}{e^{P_w} - 1} + P_w = \frac{P_w e^{P_w}}{e^{P_w} - 1} \tag{4.2.60}$$

For $P_w > 0$

$$A(|P_w|) = \frac{|P_w|}{e^{|P_w|} - 1} \tag{4.2.61}$$

For $P_w < 0$

$$A(|P_w|) = \frac{-|P_w| e^{-|P_w|}}{e^{-|P_w|} - 1} = \frac{|P_w|}{e^{|P_w|} - 1} \tag{4.2.62}$$

Thus it is verified for both positive and negative values of P_w that For $P_w > 0$

$$A(|P|) = \frac{|P|}{e^{|P|} - 1} \tag{4.2.63}$$

4.2.4 Hybrid Scheme

The hybrid scheme was originally developed by (Spalding 1972). This method can best be understood by examining a_E/D_e for the exponential scheme. Consider a plot of $A(|P|) = a_E/D_e$ as a function of $|P|$ as shown in Figure 4.2.3 (Patankar 1980). One can make the following observations: $a_E/D_e \to 0$ as $P_e \to \infty$, $a_E/D_e \to -P_e$ as $P_e \to -\infty$, and $a_E/D_e = 1 - P_e/2$ is tangent to the exponential scheme at $P_e = 0$. As $P_e \to \infty$,

$\frac{a_E}{D_e} = A(|P|) + [\![-P_e, 0]\!]$

$A(|P|) = \frac{a_E}{D_e} + [\![-P_e, 0]\!]$