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Stochastic Calculus for Finance II
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Question 1

According to Definition 3.3.3(iii), for $0 \leq t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the σ-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \leq t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

Answer

From property (iii) of the Definition 3.3.3, we can deduce that $W(u_2) - W(u_1)$ is independent of the σ-algebra $\mathcal{F}(u_1)$. Stated in terms of events in the sample space, this independence means that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \quad \text{for all} \quad A \in \mathcal{F}(u_1), B \in \sigma(W(u_2) - W(u_1)).$$

Moreover, because information in a filtration accumulates, $\mathcal{F}(t) \subseteq \mathcal{F}(u_1)$, the property above must hold for all $A \in \mathcal{F}(t), B \in \sigma(W(u_2) - W(u_1))$. Thus we conclude $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

Question 2

Let $W(t), t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$ be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale.

Answer

The process $V(t) = W^2(t) - t$ is a martingale if, for $0 \leq s \leq t$, $\mathbb{E}[V(t) \,|\, \mathcal{F}(s)] = V(s)$. Take $0 \leq s \leq t$ as given, and note that $W^2(t) = \big[(W(t) - W(s)) + W(s)\big]^2$, which we can further rearrange to resemble

$$\begin{aligned} W^2(t) &= \big(W(t) - W(s)\big)^2 + 2\big(W(t) - W(s)\big)W(s) + W^2(s) \\ &= \big(W(t) - W(s)\big)^2 + 2W(t)W(s) - W^2(s) \end{aligned}$$

Now consider

$$\begin{aligned} \mathbb{E}\big[W^2(t) - t \,|\, \mathcal{F}(s)\big] &= \mathbb{E}\left[\big(W(t) - W(s)\big)^2 + 2W(t)W(s) - W^2(s) - t \,|\, \mathcal{F}(s)\right] \\ &= \mathbb{E}\left[\big(W(t) - W(s)\big)^2 \,|\, \mathcal{F}(s)\right] + 2W(s)\mathbb{E}\big[W(t) \,|\, \mathcal{F}(s)\big] - t - W^2(s). \end{aligned}$$

Recall that, $W(u)$ is $\mathcal{F}(u)$-measureable, so that $W(s)$ can be treated as a constant when taking expectations conditional on $\mathcal{F}(s)$. Also, the t term is not a random variable, and can be extracted from the expectations operator without difficulty. The first term above, $\mathbb{E}\big[(W(t)-W(s))^2 \,|\, \mathcal{F}(s)\big]$, is the variance of the Brownian incrememnt from time-s to time-t. Since W is a martingale, $\mathbb{E}\big[W(t) \,|\, \mathcal{F}(s)\big] = W(s)$. Thus we have

$$\mathbb{E}\big[W^2(t) - t \,|\, \mathcal{F}(s)\big] = (t-s) + 2W^2(s) - t - W^2(s) = W^2(s) - s,$$

as desired.

Question 3 - Normal Kurtosis

The *kurtosis* of a random variable is defined to be the ratio of its fourth central moment to the square of its variance. For a normal random variable, the kurtosis is 3. This fact was used to obtain (3.4.7). This exercise verifies that fact.

Let X be a normal random variable with mean μ, so that $X-\mu$ has mean zero. Let the variance of X, which is also the variance of $X-\mu$, be σ^2. In (3.2.13), we computed the moment-generating function of $X-\mu$ to be $\varphi(u) = \mathbb{E}[e^{u(X-\mu)}] = e^{\frac{1}{2}\sigma^2u^2}$, where u is a real variable. Differentiating this function with respect to u, we obtain

$$\varphi'(u) = \mathbb{E}\left[(X-\mu)e^{u(X-\mu)}\right] = \sigma^2 u e^{\frac{1}{2}\sigma^2u^2},$$

and, in particular, $\varphi'(0) = \mathbb{E}[X-\mu] = 0$. Differentiating once again, we obtain

$$\varphi''(u) = \mathbb{E}\left[(X-\mu)^2e^{u(X-\mu)}\right] = (\sigma^2+\sigma^4u^2)e^{\frac{1}{2}\sigma^2u^2},$$

and, in particular, $\varphi''(0) = \mathbb{E}[(X-\mu)^2] = \sigma^2$. Differentiate two more ties and obtain the normal kurtosis formula $\mathbb{E}[(X-\mu)^4] = 3\sigma^4$.

Answer

The third derivative of the moment-generating function is

$$\varphi^{(3)}(u) = (2\sigma^4u)e^{\frac{1}{2}\sigma^2u^2} + (\sigma^2+\sigma^4u^2)\left(\tfrac{1}{2}\sigma^2 2u\right)e^{\frac{1}{2}\sigma^2u^2} = \big(3\sigma^4u + \sigma^6u^3\big)e^{\frac{1}{2}\sigma^2u^2},$$

and its fourth derivative is

$$\begin{aligned}\varphi^{(4)}(u) &= (3\sigma^4 + 3\sigma^6u^2)e^{\frac{1}{2}\sigma^2u^2} + \big(3\sigma^4u + \sigma^6u^3\big)\left(\tfrac{1}{2}\sigma^2 2u\right)e^{\frac{1}{2}\sigma^2u^2} \\ &= \big(3\sigma^4 + 6\sigma^6u^2 + \sigma^8u^4\big)e^{\frac{1}{2}\sigma^2u^2}.\end{aligned}$$

Evaluating this at $u=0$, we have $\varphi^{(4)}(0) = \mathbb{E}\big[(X-\mu)^4\big] = 3\sigma^4$. Dividing this fourth central moment by its squared variance, σ^4, we can confirm the kurtosis of a normal random variable is three.

Question 4 - Other variations of Brownian Motion

Theorem 3.4.3 asserts that if T is a positive number and we choose a partition Π with points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, then as the number n of partition points approaches infinity and the length of the longest subinterval $\|\Pi\|$ approaches zero, the sample quadratic variation

$$\sum_{j=0}^{n-1} \left[W(t_{j+1}) - W(t_j)\right]^2$$

approaches T for almost every path of the Brownian motion W. In Remark 3.4.5, we further showed that $\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)](t_{j+1} - t_j)$ and $\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2$ have limit zero. We summarize these facts by the multiplication rules

$$dW(t)dW(t) = dt, \qquad dW(t)dt = 0, \qquad dtdt = 0.$$

(i) Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|$$

approaches ∞ for almost every path of the Brownian motion W.

(ii) Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|^3$$

approaches zero for almost every path of the Brownian motion W.

Answer

(i) For an n-partition Π with points $0 = t_0 < t_1 < \cdots < t_n = T$, define

$$Q_\Pi^1 = \sum_{j=0}^{n-1} |W(t_{n+1}) - W(t_n)|.$$

Towards a contradiction, suppose $Q_\Pi^1 \xrightarrow{\text{a.s.}} Q$ for some real number Q as $\|\Pi\| \to 0$. Note

$$\sum_{j=0}^{n-1} [W(t_{n+1}) - W(t_n)]^2 \le \left(\max_{0\le k\le n-1} |W(t_{n+1}) - W(t_n)|\right) Q_\Pi^1.$$

As W is continuous with respect to time, $|W(t_{n+1}) - W(t_n)| \to 0$ as $\|\Pi\| \to 0$. The left-hand side of the inequality above converges to T, whereas the right-hand side converges to $(0)(Q)$ by supposition. This is a contradiction; hence Q_Π^1 diverges to infinity.

(ii) Define

$$Q_{\Pi}^3 = \sum_{j=0}^{n-1} |W(t_{n+1}) - W(t_n)|^3$$

and note that

$$Q_{\Pi}^3 \leq \left(\max_{0 \leq k \leq n-1} |W(t_{n+1}) - W(t_n)| \right) \sum_{j=0}^{n-1} |W(t_{n+1}) - W(t_n)|^2.$$

The first term converges to zero; the second to T. Hence Q_{Π}^3 converges to zero almost surely.

Question 5 - Black-Scholes-Merton Formula

Let the interest rate r and the volatility $\sigma > 0$ be constant. Let

$$S(t) = S(0) \exp\left[\left(r - \tfrac{1}{2}\sigma^2 \right) t + \sigma W(t) \right]$$

be a geometric Brownian motion with mean rate of return r, where the initial stock price $S(0)$ is positifve. Let K be a positive constant. Show that, for $T > 0$,

$$\mathbb{E}\left[e^{-rT} \big(S(T) - K\big)^+ \right] = S(0) N\big(d_+(T, S(0))\big) - K e^{-rT} N\big(d_-(T, S(0))\big),$$

where

$$(x)^+ = \max\{x, 0\}, \qquad d_\pm\big(T, S(0)\big) = \tfrac{1}{\sigma\sqrt{T}} \left[\ln \tfrac{S(0)}{K} + \left(r \pm \tfrac{\sigma^2}{2} \right) T \right],$$

and N is the cumulative standard normal distribution function

$$N(y) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}z^2}\,\mathrm{d}z = \tfrac{1}{\sqrt{2\pi}} \int_{-y}^{\infty} e^{-\frac{1}{2}z^2}\,\mathrm{d}z.$$

Answer

We take expectations with respect to $W(T)$, that is, with respect to the Brownian motion of length T, which has density $p(w) = (2\pi T)^{-1/2} e^{-(2T)^{-1}w^2}$. Hence,

$$\mathbb{E}\left[e^{-rT} \big(S(T) - K\big)^+ \right] = \tfrac{e^{-rT}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \left\{ S(0) \exp\left[\left(r - \tfrac{1}{2}\sigma^2 \right) T + \sigma w \right] - K \right\}^+ e^{-\frac{w^2}{2T}}\,\mathrm{d}w.$$

The integrand is zero for all $w < w^*$, where w^* satisfies

$$S(0) \exp\left[\left(r - \tfrac{1}{2}\sigma^2 \right) T + \sigma w^* \right] = K \quad \Longleftrightarrow \quad w^* = \tfrac{1}{\sigma} \left[\ln \tfrac{K}{S(0)} - \left(r - \tfrac{1}{2}\sigma^2 \right) T \right],$$

meaning we can further rewrite the discounted expected value of the call option as

$$\begin{aligned}\mathbb{E}\left[e^{-rT}\big(S(T)-K\big)^+\right] &= \tfrac{e^{-rT}}{\sqrt{2\pi T}}\int_{w^*}^{\infty}\left\{S(0)\exp\left[\left(r-\tfrac{1}{2}\sigma^2\right)T+\sigma w\right]-K\right\}e^{-\frac{w^2}{2T}}\,\mathrm{d}w \\ &= \tfrac{e^{-rT}}{\sqrt{2\pi T}}S(0)\int_{w^*}^{\infty}\exp\left[\left(r-\tfrac{1}{2}\sigma^2\right)T+\sigma w-\tfrac{w^2}{2T}\right]\mathrm{d}w \\ &\qquad -\tfrac{e^{-rT}}{\sqrt{2\pi T}}K\int_{w^*}^{\infty}e^{-\frac{w^2}{2T}}\,\mathrm{d}w\end{aligned}$$

Making the change of variables $z=T^{-1/2}w$ to the second integral, we see it reduces to

$$Ke^{-rT}\tfrac{1}{\sqrt{2\pi T}}\int_{w^*}^{\infty}e^{-\frac{w^2}{2T}}\,\mathrm{d}w = Ke^{-rT}\tfrac{1}{\sqrt{2\pi T}}\int_{T^{-1/2}w^*}^{\infty}e^{-\frac{z^2}{2}}\sqrt{T}\,\mathrm{d}z = Ke^{-rT}N\left(-\tfrac{w^*}{\sqrt{T}}\right).$$

The exponent in the first integral above can be rewritten via completing the square as

$$-\tfrac{w^2}{2T}+\sigma w+\left(r-\tfrac{1}{2}\sigma^2\right)T = -\tfrac{1}{2T}(w-\sigma T)^2+rT.$$

Hence,

$$\begin{aligned}&S(0)e^{-rT}\tfrac{1}{\sqrt{2\pi T}}\int_{w^*}^{\infty}\exp\left[\left(r-\tfrac{1}{2}\sigma^2\right)T+\sigma w-\tfrac{w^2}{2T}\right]\mathrm{d}w = \\ &\quad S(0)e^{-rT}\tfrac{1}{\sqrt{2\pi T}}\int_{w^*}^{\infty}e^{-\frac{1}{2T}(w-\sigma T)^2}e^{rT}\,\mathrm{d}w = S(0)\tfrac{1}{\sqrt{2\pi T}}\int_{w^*}^{\infty}e^{-\frac{1}{2T}(w-\sigma T)^2}\,\mathrm{d}w.\end{aligned}$$

Adopting a similar transformation as above, $z=T^{-1/2}(w-\sigma T)$, this integral simplifies to

$$S(0)\tfrac{1}{\sqrt{2\pi T}}\int_{w^*}^{\infty}e^{-\frac{1}{2T}(w-\sigma T)^2}\,\mathrm{d}w = S(0)\tfrac{1}{\sqrt{2\pi}}\int_{\frac{w^*-\sigma T}{\sqrt{T}}}^{\infty}e^{-\frac{z^2}{2}}\sqrt{T}\,\mathrm{d}z = S(0)N\left(-\tfrac{w^*-\sigma T}{\sqrt{T}}\right).$$

Finally,

$$\begin{aligned}-\tfrac{w^*}{\sqrt{T}} &= \tfrac{1}{\sigma\sqrt{T}}\left[\ln\tfrac{S(0)}{K}+\left(r-\tfrac{1}{2}\sigma^2\right)T\right] = d_-\big(T,S(0)\big) \\ -\tfrac{w^*-\sigma T}{\sqrt{T}} &= \tfrac{1}{\sigma\sqrt{T}}\left[\ln\tfrac{S(0)}{K}+\left(r-\tfrac{1}{2}\sigma^2\right)T\right]+\sigma\sqrt{T} = d_+\big(T,S(0)\big).\end{aligned}$$

Thus we conclude

$$\mathbb{E}\left[e^{-rT}\big(S(T)-K\big)^+\right] = S(0)N\big(d_+(T,S(0))\big)-Ke^{-rT}N\big(d_-(T,S(0))\big).$$

Question 6

Let $W(t)$ be a Brownian motion and let $\mathcal{F}(t), t\geq 0$, be an associated filtration.
(i) For $\mu\in\mathbb{R}$, consider the *Brownian motion with drift* μ:

$$X(t)=\mu t+W(t).$$

Show that for any Borel-measurable function $f(y)$, and for any $0 \le s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{\big(y - x - \mu(t-s)\big)^2}{2(t-s)}\right\} \mathrm{d}y$$

satisfies $\mathbb{E}[f(X(t)) \,|\, \mathcal{F}(s)] = g(X(s))$ and hence X has the Markov property.

We may rewrite $g(x)$ as $g(x) = \infty_{-\infty}^{\infty} f(y) p(\tau, x, y)\,\mathrm{d}y$, where $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{-\frac{(y - x - \mu\tau)^2}{2\tau}\right\}$$

is the *transition density* for Brownian motion with drift μ.

(ii) For $\nu \in \mathbb{R}$ and $\sigma > 0$, consider the *geometric Brownian motion*

$$S(t) = S(0)e^{\sigma W(t) + \nu t}.$$

Let $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sigma y\sqrt{2\pi\tau}} \exp\left\{\frac{(\log(y/x) - \nu\tau)^2}{2\sigma^2\tau}\right\}.$$

Show that for any Borel-measurable function $f(y)$ and for any $0 \le s < t$ the function $g(x) = \int_0^\infty h(y) p(\tau, x, y)\,\mathrm{d}y$ satisfies $\mathbb{E}[f(S(t)) \,|\, \mathcal{F}(s)] = g(S(s))$ and hence S has the Markov property and $p(\tau, x, y)$ is its transition density.

Answer

Throughout, let f be a Borel-measurable function and let $t, s \in \mathbb{R}$ abide by $0 \le s < t$.

(i) To begin, note that

$$\mathbb{E}\big[f(X(t)) \,\big|\, \mathcal{F}(s)\big] = \mathbb{E}\Big[f\big(X(t) - X(s) + X(s)\big) \,\Big|\, \mathcal{F}(s)\Big].$$

Because $X(t) - X(s)$ is independent of $\mathcal{F}(s)$ and $X(s)$ is $\mathcal{F}(s)$-measurable, we can apply the Independence Lemma. To that end, define the function g as

$$g(x) = \mathbb{E}\big[f(X(t) - X(s) + x)\big] = \mathbb{E}\Big[f\big((W(t) - W(s)) + \mu(t-s) + x\big)\Big].$$

Recognizing that $W(t) - W(s)$ is normally distributed with mean zero and variance $\tau \equiv t - s$, we can reexpress g as

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(z + \mu\tau + x) \exp\left\{-\tfrac{z^2}{2\tau}\right\} \mathrm{d}z. \\ &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{(y - x - \mu\tau)^2}{2\tau}\right\} \mathrm{d}y = \int_{-\infty}^{\infty} f(y) p(\tau, x, y)\,\mathrm{d}y, \end{aligned}$$

where the second equality makes use of the substitution $y = z + \mu\tau + x$. Taking $x = X(s)$, we can confirm X has the Markov property.

(ii) We first express $S(t)$ in terms of the Brownian motion with drift parameter $\mu = \nu/\sigma$, i.e. $X(t) = (\nu/\sigma)t + W(t)$:

$$S(t) = S(0)\exp\left[\sigma W(t) + \nu t\right] = S(0)\exp\left[\sigma(W(t) + \mu t)\right] = S(0)e^{\sigma X(t)}$$

Applying the result from part (i) to $S(t) = S(0)e^{\sigma X(t)}$, immediately we have

$$\mathbb{E}\left[f(S(t))\,|\,\mathcal{F}(s)\right] = \int_{-\infty}^{\infty} f((S(0)e^{\sigma z})\,\tfrac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\tfrac{(z-X(s)-\mu(t-s))^2}{2(t-s)}\right\}\,\mathrm{d}z.$$

Define $\tau \equiv t-s$ and make the substitution $y = S(0)e^{\sigma z}$, so that $\mathrm{d}y = S(0)e^{\sigma z}\sigma\mathrm{d}z = y\sigma\mathrm{d}z$. The conditional expectation is then written

$$\int_0^{\infty} f(y)\tfrac{1}{\sqrt{2\pi\tau}}\exp\left\{-\frac{(\sigma^{-1}(\ln y - \ln S(0)) - X(s) - \mu\tau)^2}{2\tau}\right\}\tfrac{1}{y\sigma}\,\mathrm{d}y.$$

The numerator in the exponential term can be more compactly expressed as

$$\sigma^{-1}\left(\ln y - \ln S(0)\right) - \sigma^{-1}\left(\ln S(s) - \ln S(0)\right) - \mu\tau = \sigma^{-1}\big[\ln y - \ln S(s) - \nu\tau\big].$$

Thus,

$$\begin{aligned}\mathbb{E}\left[f(S(t))\,|\,\mathcal{F}(s)\right] &= \tfrac{1}{\sigma y\sqrt{2\pi\tau}}\int_0^{\infty} f(y)\exp\left\{-\tfrac{[\sigma^{-1}(\ln y - \ln S(s) - \nu\tau]^2}{2\tau}\right\}\mathrm{d}y \\ &= \tfrac{1}{\sigma y\sqrt{2\pi\tau}}\int_0^{\infty} f(y)\exp\left\{-\tfrac{(\ln(y/S(s)) - \nu\tau)^2}{2\tau\sigma^2}\right\}\mathrm{d}y.\end{aligned}$$

Taking $x = S(s)$, we see that S indeed satisfies the Markov property, with $p(\tau, x, y)$ as its transition density.

Question 7

Theorem 3.6.2 provides the Laplace transform of the density of the first passage time for Brownian motion. This problem derives the analogous formula for Brownian motions with drift. Let W be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \le t < \infty$, define

$$X(t) = \mu t + W(t), \qquad \text{and} \qquad \tau_m = \min\{t \ge 0\,|\,X(t) = m\}.$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level m. Let σ be a positive number and set

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)t\right\}.$$

(i) Show that $Z(t), t \ge 0$, is a martingale.

(ii) Use (i) to conclude that

$$\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1, \qquad t \ge 0.$$

(iii) Now suppose $\mu \ge 0$. Show that, for $\sigma > 0$,

$$\mathbb{E}\left\{\exp\left[\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} = 1.$$

Use this fact to show $\mathbb{P}(\tau_m < \infty) = 1$ and to obtain the Laplace transform

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = \exp\left(m\mu - m\sqrt{2\alpha + \mu^2}\right) \qquad \text{for all} \qquad \alpha > 0.$$

(iv) Show that if $\mu > 0$, then $\mathbb{E}[\tau_m] < \infty$. Obtain a formula for $\mathbb{E}[\tau_m]$.

(v) Now suppose $\mu < 0$. Show that, for $\sigma > -2\mu$,

$$\mathbb{E}\left\{\exp\left[\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} = 1.$$

Use this fact to show that $\mathbb{P}(\tau_m < \infty) = e^{-2m|\mu|}$, which is strictly less than one, and to obtain the Laplace transform

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = \exp\left(m\mu - m\sqrt{2\alpha + \mu^2}\right) \qquad \text{for all} \qquad \alpha > 0.$$

Answer

Recall that $x \wedge y = \min\{x, y\}$.

(i) We begin this proof in the usual way: write $Z(t)$ as $\big(Z(t)Z(s)^{-1}\big)Z(s)$, factor the second term out of the conditional expectation, and rely on the independence of $W(t) - W(s)$ (and thus functions of that difference, although we are jumping the gun a bit by not being rigorous on this point) and $\mathcal{F}(s)$:

$$\mathbb{E}\big[Z(t)\,|\,\mathcal{F}(s)\big] = Z(s)\mathbb{E}\big[Z(t)Z(s)^{-1}\,|\,\mathcal{F}(s)\big] = Z(s)\mathbb{E}\big[Z(t)Z(s)^{-1}\big]$$

Then, because

$$\begin{aligned} Z(t)Z(s)^{-1} &= \exp\left\{\sigma\big(X(t) - X(s)\big) - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)(t-s)\right\} \\ &= \exp\left\{\sigma\big(W(t) - W(s)\big) - \tfrac{1}{2}\sigma^2(t-s)\right\}, \end{aligned}$$

we have

$$\mathbb{E}\big[Z(t)\,\big|\,\mathcal{F}(s)\big] = Z(s)e^{-\frac{1}{2}\sigma^2(t-s)}\mathbb{E}\left[e^{\sigma(W(t)-W(s))}\right].$$

The final expectations term above is the moment-generating function of the $N(0, t-s)$ random variable $W(t) - W(s)$ evaluated at σ, so that

$$\mathbb{E}\big[Z(t)\,\big|\,\mathcal{F}(s)\big] = Z(s)e^{-\frac{1}{2}\sigma^2(t-s)}e^{\frac{1}{2}\sigma^2(t-s)} = Z(s),$$

as desired.

(ii) The optional stopping theorem states that $\mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}[Z(0)] = 1$. Thus

$$\mathbb{E}\big[Z(t \wedge \tau_m)\big] = \mathbb{E}\left[\exp\left\{\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\}\right] = 1, \qquad t \geq 0.$$

Another route to showing this that does not rely on a direct appeal to the optional stopping theorem goes as follows. Since $m > 0$, τ_m is strictly greater than zero, although it can be arbitrarily close for sufficiently small m (this follows from the fact that W, and thus X and Z, are continuous with respect to time). If $t = 0$, then $t \wedge \tau_m = 0$ and $\mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}[Z(0)] = 1$. If instead $t > 0$, then $Z(t \wedge \tau_m)$ is independent of the sub-σ-algebra $\mathcal{F}(0)$ so that $\mathbb{E}[Z(t \wedge \tau_m)| \,|\, \mathcal{F}(0)] = \mathbb{E}[Z(t \wedge \tau_m)]$. As Z is a martingale (see (i)),

$$\mathbb{E}\big[Z(t \wedge \tau_m)\big] = \mathbb{E}\big[Z(t \wedge \tau_m) \,|\, \mathcal{F}(0)\big] = Z(0) = 1.$$

(iii) With $\mu \geq 0$ and $\sigma > 0$, the term $-(\sigma\mu + \sigma^2/2)t$ is guaranteed to be negative. Thus $Z(t) \leq e^{\sigma X(t)}$ for all t, and moreover for all $t \in [0, \tau_m]$ we have that $Z(t) \leq e^{\sigma m}$, since $X(t) \leq m$ for all such t.

If the stopping time τ_m is finite, then

$$\lim_{t\to\infty} Z(t \wedge \tau_m) = Z(\tau_m) = \exp\left\{\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right\}$$

. If instead τ_m is infinite, then $0 \leq Z(t) \leq e^{\sigma X(t)} \leq e^{\sigma m}$ as $t \to \infty$, as noted above. Thus in the limit, $Z(t)$ is bounded, as shown below,

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} e^{\sigma X(t)} \lim_{t\to\infty} \exp\left\{-\left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)t\right\} \leq e^{\sigma m} \times 0 = 0$$

We can harmonize these two cases by adding a term that indicates whether or not X reaches level m:

$$\lim_{t\to\infty} Z(t \wedge \tau_m) = \exp\left\{\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right\} \mathbf{1}[\tau_m < \infty]$$

Hence by the bounded convergence theorem, we can interchange the limit and expectation operator to conclude

$$\begin{aligned}\mathbb{E}\left\{\exp\left[\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} &= \mathbb{E}\left\{\lim_{t\to\infty} Z(t \wedge \tau_m)\right\} \\ &= \lim_{t\to\infty} \mathbb{E}\left[Z(t \wedge \tau_m)\right] = \lim_{t\to\infty} 1 = 1.\end{aligned}$$

The second-to-last equality uses the result from (ii).

As $\sigma \downarrow 0$, $\exp\left[\sigma m - (\sigma\mu + \sigma^2/2)\tau_m\right]$ converges to one. Thus

$$\begin{aligned}1 = \lim_{\sigma\downarrow 0} \mathbb{E}\left\{\exp\left[\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} &= \mathbb{E}\left\{\mathbf{1}[\tau_m < \infty]\right\} \\ &= \mathbb{P}(\tau_m < \infty).\end{aligned}$$

Since τ_m is finite almost surely, we can drop the indicator function in what follows. Factoring out the constant $e^{-\sigma m}$ from the expectation yields

$$\mathbb{E}\left\{\exp\left[-\left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\right\} = e^{-\sigma m},$$

prompting us to define $\alpha = \sigma\mu + \sigma^2/2 > 0$, so that $\sigma = -\mu + \sqrt{2\alpha + \mu^2}$. Thus, $\mathbb{E}[e^{-\alpha\tau_m}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}}$ for all α, as desired.

(iv) Take the derivative of the equality in part (iii) with respect to α and use the dominated convergence theorem to exchange the derivative and expectations operators:

$$\tfrac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\mathbb{E}\left[e^{-\alpha\tau_m}\right]\right\} = \mathbb{E}\left[\tfrac{\mathrm{d}}{\mathrm{d}\alpha}\left\{e^{-\alpha\tau_m}\right\}\right] = -\mathbb{E}\left[\tau_m e^{-\alpha\tau_m}\right] = -\frac{m}{\sqrt{2\alpha + \mu^2}} e^{m\mu - m\sqrt{2\alpha+\mu^2}}.$$

Next we take the limit $\alpha \downarrow 0$:

$$\lim_{\alpha\downarrow 0} \mathbb{E}\left[\tau_m e^{-\alpha\tau_m}\right] = \mathbb{E}\left[\lim_{\alpha\downarrow 0} \tau_m e^{-\alpha\tau_m}\right] = \mathbb{E}[\tau_m] = \tfrac{m}{\mu}.$$

We assumed $\mu > 0$, guaranteeing this expectation is finite.

(v) The assumptions that $\mu < 0$ and $\sigma > -2\mu$ imply $\sigma\mu + \sigma^2/2 > 0$. Thus we can follow many of the same steps as in part (ii) to answer this question.

It is still the case that for $t \in [0, \tau_m]$, $e^{\sigma X(t)}$ is bounded above by $e^{\sigma m}$, so that $\lim_{t\to\infty} e^{\sigma X(t)}$ is bounded above whether or not τ_m is infinite. Furthermore, because $\sigma\mu + \sigma^2/2 > 0$, when τ_m is infinite,

$$\lim_{t\to\infty} \exp\left\{-\left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\} = \lim_{t\to\infty} \exp\left\{-\left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)t\right\} = 0.$$

Hence we can use all the same monotone convergence arguments as in part (ii) to conclude

$$\mathbb{E}\left\{\exp\left[\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} = 1.$$

Where we depart from part (iii)'s path is here: because $\sigma > -2\mu$, we cannot take the limit $\sigma \downarrow 0$ as before. Instead, consider taking the limit of the above expression as $\sigma \downarrow -2\mu$:

$$\begin{aligned}
1 &= \lim_{\sigma\downarrow -2\mu} \mathbb{E}\left\{\exp\left[\sigma m - \left(\sigma\mu + \tfrac{1}{2}\sigma^2\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} \\
&= \mathbb{E}\left\{\exp\left[-2m\mu - \left(-2\mu^2 + \tfrac{1}{2}(4\mu^2)\right)\tau_m\right]\mathbf{1}[\tau_m < \infty]\right\} \\
&= \mathbb{E}\left\{e^{-2m\mu}\mathbf{1}[\tau_m < \infty]\right\}.
\end{aligned}$$

Moving the constant $e^{-2m\mu}$ term to the other side of the equation, we have that $\mathbb{P}(\tau_m < \infty) = e^{2m\mu} = e^{-2m|\mu|}$.

As before we define $\alpha = \sigma\mu + \sigma^2/2$ to derive the Laplace expression.

Question 8

This problem presents the convergence of the distribution of stock prices in a sequence of binomial models to the distribution of geometric Brownian motion. In contrast to the analysis of Subsection 3.2.7, here we allow the interest rate to be different from zero.

Let $\sigma > 0$ and $r \geq 0$ be given. For each positive integer n, we consider a binomial model taking n steps per unit time. In this model, the interest rate per period is r/n, the up factor is $u_n = e^{\sigma/\sqrt{n}}$, and the down factor is $d_n = e^{-\sigma/\sqrt{n}}$. The risk-neutral probabilities are then

$$\tilde{p}_n = \frac{r/n + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \quad \text{and} \quad \tilde{q}_n = \frac{e^{\sigma/\sqrt{n}} - r/n - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}$$

Let t be an arbitrary positive rational number, and for each positive integer n for which nt is an integer, define

$$M_{nt,n} = \sum_{k=1}^{nt} X_{k,n},$$

where $X_{1,n}, \ldots, X_{n,n}$ are independent, identically distributed random variables with

$$\widetilde{\mathbb{P}}(X_{k,n} = 1) = \tilde{p}_n, \qquad \widetilde{\mathbb{P}}(X_{k,n} = -1) = \tilde{q}_n, \qquad k = 1, \ldots, n.$$

The stock price at time t in this binomial model, which is the result of nt steps from the initial time, is given by

$$
\begin{aligned}
S_n(t) &= S(0) u_n^{\frac{1}{2}(nt+M_{nt,n})} d_n^{\frac{1}{2}(nt-M_{nt,n})} \\
&= S(0) \exp\left\{ \tfrac{\sigma}{2\sqrt{n}}(nt+M_{nt,n}) \right\} \exp\left\{ -\tfrac{\sigma}{2\sqrt{n}}(nt-M_{nt,n}) \right\} \\
&= S(0) \exp\left\{ \tfrac{\sigma}{\sqrt{n}} M_{nt,n} \right\}.
\end{aligned}
$$

This problem shows that as $n \to \infty$, the distribution of the sequence of random variables $({}^{\sigma}/\sqrt{n}) M_{nt,n}$ appearing in the exponent above converges to the normal distribution with mean $(r - {}^{\sigma^2}/2)t$ and variance $\sigma^2 t$. Therefore, the limiting distribution of $S_n(t)$ is the same as the distribution of the geometric Brownian motion $S(0)\exp\{\sigma W(t) + (r - {}^{\sigma^2}/2)t\}$ at time t.

(i) Show that the moment-generating function $\varphi_n(u)$ of $n^{-1/2} M_{nt,n}$ is given by

$$
\varphi_n(u) = \left[e^{u/\sqrt{n}} \left(\frac{{}^{r}/n + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) + e^{-u/\sqrt{n}} \left(\frac{{}^{r}/n + 1 - e^{\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) \right]^{nt}.
$$

(ii) We want to compute

$$
\lim_{n\to\infty} \varphi_n(u) = \lim_{x \downarrow 0} \varphi_{1/x^2}(u),
$$

where we have made the change of variable $x = n^{-1/2}$. To do this, we will compute $\log \varphi_{1/x^2}(u)$ and then take the limit as $x \downarrow 0$. Show that

$$
\log \varphi_{1/x^2}(u) = \tfrac{t}{x^2} \log \left[\frac{(rx^2+1)\sinh(ux) + \sin[(\sigma-u)x]}{\sinh(\sigma x)} . \right]
$$

The definitions are $\sinh z = (1/2)(e^z - e^{-z})$ and $\cosh z = (1/2)(e^z + e^{-z})$. Use the formula

$$
\sinh(A-B) = \sinh A \cosh B - \cosh A \sinh B
$$

to rewrite this as

$$
\log \varphi_{1/x^2}(u) = \tfrac{t}{x^2} \log \left[\cosh(ux) + \frac{(rx^2 + 1 - \cosh(\sigma x))\sinh(ux)}{\sinh(\sigma x)} \right]
$$

.

(iii) Use the Taylor series expansions

$$
\cosh z = 1 + \tfrac{1}{2} z^2 + O(z^4), \qquad \text{and} \qquad \sinh z = z + O(z^3),
$$

to show that

$$
\cosh(ux) + \frac{(rx^2 + 1 - \cosh(\sigma x))\sinh(ux)}{\sinh(\sigma x)} = 1 + \tfrac{1}{2}u^2x^2 + \tfrac{rux^2}{\sigma} - \tfrac{1}{2}ux^2\sigma + O(x^4).
$$

The notation $O(x^j)$ is used to represent terms of the order x^j.

(iv) Use the Taylor series expansion $\log(1+x) = x + O(x^2)$ to compute $\lim_{x\downarrow 0} \log \varphi_{1/x^2}(u)$. Now explain how you know that the limiting distribution for $({}^{\sigma}/\sqrt{n}) M_{nt,n}$ is normal with mean $(r - {}^{\sigma^2}/2)t$ and variance $\sigma^2 t$.

Answer

(i) The moment-generating function for any of the $X_{k,n}$ random variables is

$$\varphi_{X,n}(u) = \mathbb{E}\big[e^{uX_{k,n}}\big] = e^u\tilde{P}(X_{k,n}=1) + e^{-u}\tilde{P}(X_{k,n}=-1) = e^u\tilde{p}_n + e^{-u}\tilde{q}_n.$$

Recall that if $\varphi_Y(u)$ is the moment-generating function for a random variable Y, then the moment-generating function for the random variable $aY+b$ with constants a and b is $\varphi_{aY+b}(u) = e^{bu}\varphi_Y(au)$. Thus, the moment-generating function of $n^{-1/2}X_{k,n}$ is

$$\varphi_{n^{-1/2}X,n}(u) = e^{u/\sqrt{n}}\tilde{p}_n + e^{-u/\sqrt{n}}\tilde{p}_n.$$

Recall also the moment-generating function of a sum of independent random variables is the product of the the moment-generating functions of the individual random variables. Thus, the moment-generating function of $n^{-1/2}M_{nt,n} = \sum_k n^{-1/2}X_{k,n}$ is indeed

$$\varphi_n(u) = \prod_{k=1}^{nt}\varphi_{n^{-1/2}X,n}(u) = \varphi_{n^{-1/2}X,n}(u)^{nt} = \left(e^{u/\sqrt{n}}\tilde{p}_n + e^{-u/\sqrt{n}}\tilde{q}_n\right)^{nt},$$

so we are done.

(ii) With the change of variable $x = n^{-1/2}$, the risk-neutral probabilities are

$$\tilde{p}_{1/x^2} = \frac{rx^2+1-e^{-\sigma x}}{e^{\sigma x}-e^{-\sigma x}} \quad \text{and} \quad \tilde{q}_{1/x^2} = \frac{e^{\sigma x}-rx^2-1}{e^{\sigma x}-e^{-\sigma x}}.$$

Note the denominator of each of the probabilities is $2\sinh(\sigma x)$. The up and down factors are e^{ux} and e^{-ux} respectively. The numerator of the expression $e^{ux}\tilde{p}_{1/x^2} + e^{-ux}\tilde{q}_{1/x^2}$ is

$$\begin{aligned} &e^{ux}\left(rx^2+1-e^{-\sigma x}\right) + e^{-ux}\left(e^{\sigma x}-rx^2-1\right) \\ &\quad = \left(e^{ux}-e^{-ux}\right)(rx^2+1) + e^{(\sigma-u)x} - e^{-(\sigma-u)x} \end{aligned}$$

which is equal to $2\sinh(ux)(rx^2+1) + 2\sinh\big((\sigma-u)x\big)$. Thus,

$$e^{ux}p_{1/x^2} + e^{-ux}q_{1/x^2} = \frac{\sinh(ux)(rx^2+1)+\sinh\big((\sigma-u)x\big)}{\sinh(\sigma x)}.$$

Taking logs of $\varphi_n(u) = \varphi_{1/x^2}(u)$ yields the first desired equality.

Applying the identity $\sinh(A-B) = \sinh A\cosh B - \cosh A\sinh B$ to the $\sinh\big((\sigma-u)x\big) = \sinh(\sigma x - ux)$ term in the numerator, we have

$$\begin{aligned} &\sinh(ux)(rx^2+1) + \sinh\big((\sigma-u)x\big) \\ &\quad = \sinh(\sigma x)\cosh(ux) + \big[rx^2+1-\cosh(\sigma x)\big]\sinh(ux) \end{aligned}$$

Thus,

$$\log\varphi_{1/x^2}(u) = \tfrac{t}{x^2}\log\left[\cosh(ux) + \frac{(rx^2+1-\cosh(\sigma x))\sinh(ux)}{\sinh(\sigma x)}\right].$$

(iii) Using the Taylor series expansion $\cosh z = 1 + {}^{z^2}\!/\!{}_2 + O(z^4)$, we can write

$$rx^2 + 1 - \cosh(\sigma x) \qquad \text{as} \qquad rx^2 + \tfrac{1}{2}\sigma^2 x^2 + O(x^4).$$

Multiply this by $\sinh(z) = z + O(z^4)$ with $z = ux$ to derive

$$\big(rx^2 + 1 - \cosh(\sigma x)\big)\sinh(ux) = rux^3 + \tfrac{1}{2}\sigma^2 ux^3 + O(x^5),$$

then divide by $\sinh(\sigma x)$ to arrive at

$$\frac{\big(rx^2 + 1 - \cosh(\sigma x)\big)\sinh(ux)}{\sinh(\sigma x)} = \tfrac{ru}{\sigma}x^2 + \tfrac{1}{2}\sigma ux^2 + O(x^4).$$

Finally we add $\cosh(ux)$ to create the desired expression

$$\cosh(ux) + \frac{\big(rx^2 + 1 - \cosh(\sigma x)\big)\sinh(ux)}{\sinh(\sigma x)} = 1 + \tfrac{1}{2}u^2x^2 + \tfrac{ru}{\sigma}x^2 - \tfrac{1}{2}\sigma ux^2 + O(x^4).$$

(iv) We follow the instructions to write

$$\begin{aligned}
\log \varphi_{1/x^2}(u) &= \tfrac{t}{x^2}\log\left[1 + \tfrac{1}{2}u^2x^2 + \tfrac{ru}{\sigma}x^2 - \tfrac{1}{2}\sigma ux^2\right] \\
&= \tfrac{t}{x^2}\left(\tfrac{1}{2}u^2x^2 + \tfrac{ru}{\sigma}x^2 - \tfrac{1}{2}\sigma ux^2 + O(x^{2^2})\right) \\
&= \tfrac{t}{2}u^2 + \left(\tfrac{rt}{\sigma} - \tfrac{\sigma t}{2}\right)u + O(x^2)
\end{aligned}$$

Taking the limit as $x \downarrow 0$, we have

$$\log \varphi_{1/x^2}(u) = \tfrac{t}{2}u^2 + t\left(\tfrac{r}{\sigma} - \tfrac{\sigma}{2}\right)u \iff \varphi_{1/x^2}(u) = \exp\left[\tfrac{t}{2}u^2 + t\left(\tfrac{r}{\sigma} - \tfrac{\sigma}{2}\right)u\right]$$

This last expression is precisely the moment-generating function for a Gaussian random variable with mean $t({}^{r}\!/\!{}_{\sigma} - {}^{\sigma}\!/\!{}_{2})$ and variance t; this is the distribution of $n^{-1/2}M_{nt,n}$. Multiplying this random variable by σ, we see that $\sigma n^{-1/2}M_{nt,n}$ is also a Gaussian variable, with mean $t(r - {}^{\sigma^2}\!/\!{}_{2})$ and variance $\sigma^2 t$.

Question 9 - Laplace Transform of the First Passage Density

Let $m > 0$ be given, and define

$$f(t,m) = \tfrac{m}{t\sqrt{2\pi t}}\exp\left\{-\tfrac{m^2}{2t}\right\}.$$

According to (3.7.3) in Theorem 3.7.1, $f(t,m)$ is the density in the variable t of the first passage time $\tau_m = \min\{t \geq 0 \,|\, W(t) = m\}$, where W is a Brownian motion without drift. Let

$$g(\alpha, m) = \int_0^\infty e^{-\alpha t} f(t,m)\,\mathrm{d}t, \qquad \alpha > 0,$$

be the Laplace transform of the density $f(t,m)$. This problem verifies that $g(\alpha,m) = e^{-m\sqrt{2\alpha}}$, which is the formula drived in Theorem 3.6.2.

(i) For $k \geq 1$, define

$$a_k(m) = \tfrac{1}{\sqrt{2\pi}} \int_0^\infty t^{-k/2} \exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}, \mathrm{d}t,$$

so that $g(\alpha, m) = m a_3(m)$. Show that

$$g_m(\alpha, m) = a_3(m) - m^2 a_5(m),$$
$$g_{mm}(\alpha, m) = -3m a_5(m) + m^3 a_7(m)^{\cdot}$$

(ii) Use integration by parts to show that

$$a_5(m) = -\tfrac{2\alpha}{3} a_3(m) + \tfrac{m^2}{3} a_7(m).$$

(iii) Use (i) and (ii) to show that g satisfies the second-order ordinary differential equation

$$g_{mm}(\alpha, m) = 2\alpha g(\alpha, m).$$

(iv) The general solution to a second-order ordinary differential equation of the form

$$ay''(m) + by'(m) + cy(m) = 0$$

is

$$y(m) = A_1 e^{\lambda_1 m} + A_2 e^{\lambda_2 m},$$

where λ_1 and λ_2 are roots of the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0.$$

here we are assuming that these roots are distinct. Find the general solution of the equation in (iii) when $\alpha > 0$. This solution has two undetermined parameters A_1 and A_2, and these may depend on α.

(v) Derive the bound

$$g(\alpha, m) \leq \tfrac{m}{\sqrt{2\pi}} \int_0^m \sqrt{\tfrac{m}{t}} t^{-3/2} \exp\left\{-\tfrac{m^2}{2t}\right\} \mathrm{d}t + \tfrac{1}{\sqrt{2\pi m}} \int_m^\infty e^{-\alpha t} \, \mathrm{d}t$$

and use it to show that, for every $\alpha > 0$,

$$\lim_{m \uparrow \infty} g(\alpha, m) = 0.$$

Use this fact to determine one of the parameters in the general solution to the equation in (iii).

(vi) Using first the change of variable $s = {}^{t}\!/_{m^2}$ and then the change of variable $y = {}^{1}\!/_{\sqrt{s}}$, show that

$$\lim_{m \downarrow 0} g(\alpha, m) = 1.$$

Use this fact to determine the other parameter in the general solution to the equation in (iii).

Answer

(i) Differentiating under the integral sign, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}m}\left[a_3(m)\right] = \tfrac{1}{\sqrt{2\pi}}\int_0^\infty t^{-3/2}\left(\tfrac{-2m}{2t}\right)\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}, \mathrm{d}t, = -ma_5(m).$$

Doing the same to $a_5(m)$, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}m}\left[a_5(m)\right] = \tfrac{1}{\sqrt{2\pi}}\int_0^\infty t^{-5/2}\left(\tfrac{-2m}{2t}\right)\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}, \mathrm{d}t, = -ma_7(m).$$

The product rule yields our desired answers straightaway:

$$g_m(\alpha, m) = \tfrac{\mathrm{d}}{\mathrm{d}m}\left[ma_3(m)\right] = a_3(m) + m\tfrac{\mathrm{d}}{\mathrm{d}m}\left[a_3(m)\right] = a_3(m) - m^2 a_3(m)$$

and

$$g_{mm}(\alpha, m) = -ma_5(m) - \big(2ma_5(m) - m^3 a_7(m)\big) = -3ma_5(m) + m^3 a_7(m).$$

(ii) We use integration by parts to reexpress $a_k(m)$ for $k \geq 3$, where the restriction is in place to guarantee $k - 2 \geq 1$. Here,

$$\sqrt{2\pi}a_k(m) = \int_0^\infty \underbrace{\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}}_{\equiv u}\underbrace{t^{-k/2}\,\mathrm{d}t}_{\equiv \mathrm{d}v}.$$

Using integration by parts, we have

$$\begin{aligned}\sqrt{2\pi}a_k(m) &= uv\big|_{t=0}^{t=\infty} - \int_0^\infty v\,\mathrm{d}u \\ &= \tfrac{2}{-k+2}t^{\frac{-k+2}{2}}\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}\Bigg|_{t=0}^{t=\infty} \\ &\quad - \int_0^\infty \tfrac{2}{-k+2}t^{\frac{-k+2}{2}}\left(-\alpha + \tfrac{m^2}{2t^2}\right)\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}\,\mathrm{d}t\end{aligned}$$

The first term evaluates to $0 \times 0 - \infty \times 0 = 0$, and the integral term on the right-hand side can be split into two:

$$\tfrac{2\alpha}{-k+2}\int_0^\infty t^{\frac{-k+2}{2}}\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}\,\mathrm{d}t - \tfrac{m^2}{-k+2}\int_0^\infty t^{\frac{-k+2}{2}-2}\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\}\,\mathrm{d}t.$$

Recognizing the first integral as $\sqrt{2\pi}a_{k-2}(m)$ and the second as $\sqrt{2\pi}a_{k+2}(m)$, we have arrived at the identity

$$a_k(m) = \tfrac{2\alpha}{-k+2}a_{k-2}(m) - \tfrac{m^2}{-k+2}a_{k+2}(m).$$

Setting $k = 5$, we are done.

(iii) Substitute $a_5(m) = -\frac{2\alpha}{3}a_3(m) + \frac{m^2}{3}a_7(m)$ into $g_{mm}(\alpha, m) = -3ma_5(m) + m^3 a_7(m)$ to derive

$$\begin{aligned}g_{mm}(\alpha, m) &= -3m\left[-\tfrac{2\alpha}{3}a_3(m) + \tfrac{m^2}{3}a_7(m)\right] + m^3 a_7(m) \\ &= 2\alpha m a_3(m) - m^3 a_7(m) + m^3 a_7(m) = 2\alpha g(\alpha, m).\end{aligned}$$

The final equality follows from the identity $g(\alpha, m) = ma_3(m)$.

(iv) Comparing the identity $g_{mm}(\alpha, m) = 2\alpha g(\alpha, m)$ from (iii) to the second-order differential equation in (iv), we see $a = 1$, $b = 0$, and $c = -2\alpha$. The roots of the characteristic equation are

$$\lambda = \pm\tfrac{1}{2}\sqrt{-4(1)(-2\alpha)} = \pm\sqrt{2\alpha}.$$

Thus, the general solution is given by

$$y(m) = g(\alpha, m) = A_1 e^{-\sqrt{2\alpha}m} + A_2 e^{\sqrt{2\alpha}m},$$

where A_1 and A_2 are yet to be determined.

(v) Since m, α, and t are all non-negative, we have the following inequalities:

$$\exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\} \leq \exp\left\{-\tfrac{m^2}{2t}\right\} \qquad \text{and} \qquad \exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\} \leq e^{-\alpha t}.$$

Furthermore, if $t \in (0, m]$,

$$mt^{-3/2} = m\sqrt{\tfrac{t}{t}}t^{-3/2} \leq m\sqrt{\tfrac{m}{t}}t^{-3/2},$$

and if instead $t \in (m, \infty)$, $mt^{-3/2} \leq mm^{-3/2} = m^{-1/2}$.

We have all the ingredients we need to derive the bound. First, partition the integral in g's definition as

$$g(\alpha, m) = \int_0^m \tfrac{m}{t\sqrt{2\pi t}} \exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\} \mathrm{d}t + \int_m^\infty \tfrac{m}{t\sqrt{2\pi t}} \exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\} \mathrm{d}t$$

From our inequalities above, the first integral is bounded above by

$$\begin{aligned}\int_0^m \tfrac{m}{t\sqrt{2\pi t}} \exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\} \mathrm{d}t &\leq \int_0^m m\sqrt{\tfrac{m}{t}}t^{-3/2} \exp\left\{-\tfrac{m^2}{2t}\right\} \mathrm{d}t \\ &= \tfrac{m}{\sqrt{2\pi}} \int_0^\infty \sqrt{\tfrac{m}{t}}t^{-3/2} \exp\left\{-\tfrac{m^2}{2t}\right\} \mathrm{d}t,\end{aligned}$$

which is precisely the first integral in the bound. Similarly, the second integral in g's partitioned definition bounded from above by

$$\int_m^\infty \tfrac{m}{t\sqrt{2\pi t}} \exp\left\{-\alpha t - \tfrac{m^2}{2t}\right\} \mathrm{d}t \leq \tfrac{1}{\sqrt{2\pi m}} \int_m^\infty e^{-\alpha t}\, \mathrm{d}t.$$

Thus, we have established the given bound.