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Stochastic Calculus for Finance II
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Question 1

Use the properties of Definition 1.1.2 for a probability measure $\mathbb{P}$, show the following

(i) If $A \in \mathcal{F}, B \in \mathcal{F}$, and $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(ii) If $A \in \mathcal{F}$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ with $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$ and $A \subset A_n$ for every n, then $\mathbb{P}(A) = 0$.

Answer

(i) Write B as $B = A \cup (B \setminus A)$, and note that A and $B \setminus A$ are disjoint. From part (ii) of Definition 1.1.2, then, we have $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$. Because probabilities are weakly greater than zero, $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$, as desired.

(ii) From part (i), $\mathbb{P}(A) \leq \mathbb{P}(A_n)$ for all $n \in \mathbb{N}$. Again making use of the fact $\mathbb{P}(A) \geq 0$, we have

$$0 = \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \mathbb{P}(A) \leq \lim_{n\to\infty} \mathbb{P}(A_n) = 0.$$

Thus, $\lim_{n\to\infty} \mathbb{P}(A) = P(A) = 0$.

Question 2

The infinite coin-toss space Ω_∞ of Example 1.1.4 is *uncountably infinite*. In other words, we cannot list all its elements in a sequence. To see that this is impossible, suppose there such a sequential list of all elements of Ω_∞:

$$\begin{aligned}
\omega^{(1)} &= \omega_1^{(1)}\omega_2^{(1)}\omega_3^{(1)}\omega_4^{(1)} \cdots, \\
\omega^{(2)} &= \omega_1^{(2)}\omega_2^{(2)}\omega_3^{(2)}\omega_4^{(2)} \cdots, \\
\omega^{(3)} &= \omega_1^{(3)}\omega_2^{(3)}\omega_3^{(3)}\omega_4^{(3)} \cdots, \\
&\vdots
\end{aligned}$$

An element that does not appear in this list is the sequence whose first component is H if $\omega_1^{(1)}$ is T and is T if $\omega_1^{(1)}$ is H, whose second component is H if $\omega_2^{(2)}$ is T and is T if $\omega_2^{(2)}$ is H, whose third component is H if $\omega_3^{(3)}$ is T and is T if $\omega_3^{(3)}$ is H, etc. Thus, the list does not include every element of Ω_∞.

Now consider the set of sequences of coin tosses in which the outcome on each even-numbered toss matches the outcome of the toss preceding it, i.e.,

$$A = \big\{\omega = \omega_1\omega_2\omega_3\omega_4\omega_5 \cdots \,|\, \omega_1 = \omega_2,\, \omega_3 = \omega_4 \cdots\big\}.$$

(i) Show that A is uncountably infinite.

(ii) Show that, when $0 < p < 1$, we have $\mathbb{P}(A) = 0$.

Uncountably infinite sets can have any probability between zero and one, including zero and one. The uncountability of a set does not determine its probability.

Answer

(i) We use the same type of diagonalization argument as we did for Ω_∞.

Define the "negation" operator for coin flips as $\neg$ as $\neg H = T$ and $\neg T = H$ for convenience. Suppose $L = \{\omega^{(n)}\}$ is a sequential list of all the elements of A, in the same manner as in the question statement. Consider the element $\widetilde{\omega} \in \Omega_\infty$ with the n-th coinflip given by $\widetilde{\omega}_n = \neg\omega_{2n-1}^{(n)}$. That is, the n-th coinflip in $\widetilde{\omega}$ is the negation of the $(2n-1)$-th paired coinflip of the n-th element of L. Clearly, $\widetilde{\omega}$ is not in L, and thus A is uncountably infinite.

(ii) Recall p is the probability of a heads. For $n \in \mathbb{N}$, define $A_n = \big\{\omega = \omega_1\omega_2\cdots\omega_{2n-1}\omega_{2n} \,|\, \omega_1 = \omega_2 \cdots \omega_{2n-1} = \omega_{2n} \cdots\big\}$ to be the set of n paired coinflips. The sequence of sets $\{A_n\}$ converges to A as $n \to \infty$ Thus

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \mathbb{P}(\omega_1 = \omega_2)\times\cdots\times\mathbb{P}(\omega_{2n-1} = \omega_{2n}) = \lim_{n\to\infty} \big[p^2+(1-p)^2\big]^n.$$

For $0 < p < 1$, the term $p^2 + (1-p)^2$ is strictly less than one. Thus $\lim_{n\to\infty} \big[p^2 + (1-p)^2\big]^n$ converges to zero, and $\mathbb{P}(A) = 0$.

Question 3

Consider the set function $\mathbb{P}$ defined for every subset of $[0, 1]$ by the formula that $\mathbb{P}(A) = 0$ if A is finite and $\mathbb{P}(A) = \infty$ if A is an infinite set. Show that $\mathbb{P}$ satisfies (1.1.3) - (1.1.5), but $\mathbb{P}$ does not have the countable additive property (1.1.2). We see then that the finite additivity property (1.1.5) does not imply the countable additivity property (1.1.2).

Answer

The four properties are:

- (1.1.2): Whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

- (1.1.3): $\mathbb{P}(\varnothing) = 0$.
- (1.1.4): If A and B are disjoint sets in $\mathcal{F}$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.
- (1.1.5): If $A_1, A_2, \ldots, A_N$ are finitely many disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mathbb{P}(A_n).$$

The empty set is finite, hence $\mathbb{P}(\varnothing) = 0$ is immediate.

Let $A, B \subseteq [0,1]$. If A and B are finite, so too is $A \cup B$, so that $\mathbb{P}(A \cup B) = 0 = 0 + 0 = \mathbb{P}(A) + \mathbb{P}(B)$. If instead A is finite and B is infinite, so is $A \cup B$, meaning $\mathbb{P}(A \cup B) = \infty = 0 + \infty = \mathbb{P}(A) + \mathbb{P}(B)$. The case where A is infinite and B is finite is similar. If both A and B are infinite, then so too is $A \cup B$. Thus $\mathbb{P}(A \cup B) = \infty = \infty + \infty = \mathbb{P}(A) + \mathbb{P}(B)$.

Let $A_1, \dots A_N$ be a collection of disjoint sets in $[0,1]$. If at least one of the A_n is infinite, then $\cup_n A_n$ is as well. Thus $\mathbb{P}\left(\cup_{n=1}^N A_n\right) = \infty = \sum_{n=1}^N A_n$.If instead all the A_n are finite, then the finite union $\cup_n A_n$ is too. Therefore we can iteratively apply (1.1.4) to $\cup_n A_n$ to deduce that the probability of the union and the sum of the individual probabilities are both zero.

To show that this set function does not satisfy the countable additivity property, let $A_n = {}^1\!/\!_n$ for $n \in \mathbb{N}$. Then $\mathbb{P}(A_n) = 0$ for all n, but $A = \cup_n A_n = \{{}^1\!/\!_n \,|\, n \in \mathbb{N}\}$ is an infinite set, and thus we have $\mathbb{P}(A) = \infty$.

Question 4

(i) Construct a standard normal random variable Z on the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P})$ of Example 1.1.4 under the assumption that the probability for heads is $p = {}^1\!/\!_2$.

(ii) Define a sequence of random variables $\{Z_n\}_{n=1}^\infty$ on Ω_∞ such that

$$\lim_{n\to\infty} Z_n(\omega) = Z(\omega) \quad \text{for every} \quad \omega \in \Omega_\infty$$

and, for each n, Z_n depends only on the first n coin tosses. (This gives us a procedure for approximating a standard normal random variable by random variables generated by a finite number of coin tosses, a useful algorithm for Monte Carlo simulation.)