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Stochastic Calculus for Finance II
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Question 1

Suppose $M(t)$, $0 \le t \le T$, is a martingale with respect to some filtration $\mathcal{F}(t)$, $0 \le t \le T$. Let $\Delta(t)$, $0 \le t \le T$, be a simple process adapted to $\mathcal{F}(t)$ (i.e. there is a partion $\Pi_n = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$ such that, for every j, $\Delta(t_j)$ is $\mathcal{F}(t_j)$-measurable and $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$). For $t \in [t_k, t_{k+1})$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)\big[M(t_{j+1}) - M(t_j)\big] + \Delta(t_k)\big[M(t) - M(t_k)\big].$$

We think of $M(t)$ as the price of an asset at time t and $\Delta(t_j)$ as the number of shares of the asset held by an investor between times t_j and t_{j+1}. Then $I(t)$ is the capital gains that accrue to the investor between times 0 and t. Show that $I(t)$, $0 \le t \le T$, is a martingale.

Answer

Let s and t be positive constants with $0 \le s \le t \le T$, and further let $[t_\ell, t_{\ell+1})$ be the subinterval that contains s. We then have that

$$\begin{aligned} &I(t) - I(s) \\ &\quad = \Delta(t_\ell)\big[M(t_{\ell+1}) - M(s)\big] + \sum_{j=\ell+1}^{k-1} \Delta(t_j)\big[M(t_{j+1}) - M(t_j)\big] + \Delta(t_k)\big[M(t) - M(t_k)\big]. \end{aligned}$$

As $s \ge t_\ell$, the allocation $\Delta(t_\ell)$ is known at time s, and because $M(t)$ is a martingale with respect to $\mathcal{F}(t)$, the first term's conditional expectation is zero: $\mathbb{E}\big[\Delta(t_\ell)(M(t_{\ell+1}) - M(t_\ell))|\mathcal{F}(s)\big] = \Delta(t_\ell)\big(\mathbb{E}[M(t_{\ell+1})|\mathcal{F}(s)] - M(s)\big) = \Delta(t_\ell)\big(M(s) - M(s)\big) = 0.$

Next, take j to be an integer with $\ell + 1 \le j \le k - 1$. Because $M(u) - M(v)$, $u \ge v$, is mean-zero conditional on time-v information, we have

$$\begin{aligned} \mathbb{E}\left[\Delta(t_j)\big(M(t_{j+1}) - M(t_j)\big) \,|\, \mathcal{F}(s)\right] &= \mathbb{E}\left[\mathbb{E}\big[\Delta(t_j)\big(M(t_{j+1}) - M(t_j)\big) \,|\, \mathcal{F}(t_j)\big] \,|\, \mathcal{F}(s)\right] \\ &= \mathbb{E}\left[\Delta(t_j)\mathbb{E}\big[M(t_{j+1}) - M(t_j) \,|\, \mathcal{F}(t_j)\big] \,|\, \mathcal{F}(s)\right] \\ &= \mathbb{E}\big[\Delta(t_j) \times 0 \,|\, \mathcal{F}(s)\big] = 0. \end{aligned}$$

Using the same chain of logic, it can be shown that the final term is zero conditional on time-s information. Thus,

$$\mathbb{E}\big[I(t) - I(s) \,|\, \mathcal{F}(s)\big] = 0 \qquad \Longleftrightarrow \qquad \mathbb{E}\big[I(t) \,|\, \mathcal{F}(s)\big] = \mathbb{E}\big[I(s) \,|\, \mathcal{F}(s)\big] = I(s),$$

showing that $I(s)$ is indeed a martingale.

Question 2

Let $W(t)$, $0 \le t \le T$, be a Brownian motion, and let $\mathcal{F}(t)$, $0 \le t \le T$, be an associated filtration. Let $\Delta(t)$, $0 \le t \le T$, be a *non-random* simple process (i.e. there is a partion $\Pi_n = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$ such that, for every j, $\Delta(t_j)$ is a nonrandom quantity and $\Delta(t) = \Delta(t_j)$ is constant in t on each subinterval $[t_j, t_{j+1})$). For $t \in [t_k, t_{k+1}]$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \Delta(t_k)\big[W(t) - W(t_k)\big].$$

(i) Show that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.
(ii) Show that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is a normally distributed random variable with mean zero and variance $\int_s^t \Delta^2(u)\,\mathrm{d}u$.
(iii) Use (i) and (ii) to show that $I(t)$, $0 \le t \le T$, is a martingale.
(iv) Show that $I^2(t) - \int_0^t \Delta^2(u)\,\mathrm{d}u$, $0 \le t \le T$, is a martingale.

Answer

(i) It is sufficient to show that $I(t_k) - I(t_\ell)$ is independent of $\mathcal{F}(t_\ell)$ for partition points t_k and t_ℓ, with $t_\ell < t_k$. Indeed, to see this is the case, let Π_n be a partition of $[0, T]$ with n points. If the given s resides in a subinterval $[t_\ell, t_{\ell+1})$ of Π_n, we can construct a partition $\Pi_{n+1} \equiv \Pi_n \cup \{s\}$ and set $\Delta(s) = \Delta(t_\ell)$. The stochastic integral over this partition evaluates to the same value as that over the Π_n partition. The same logic applies for $t \in [t_k, t_{k+1})$.

Take $t_\ell < t_k$ to be two parition points. Then

$$I(t_k) - I(t_\ell) = \sum_{j=\ell}^{k-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big].$$

Recall that $\Delta(t)$ is a *non-random* process, so the independence of $\Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big]$ and $\mathcal{F}(t_\ell)$ only hinges on the independence of $W(t_{j+1}) - W(t_j)$ and $\mathcal{F}(t_\ell)$. For $t \ge t_\ell$, the filtration satisfies $\mathcal{F}(t) \supseteq \mathcal{F}(t_\ell)$, and because $W(t)$ is a Brownian motion, $W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_j)$. But $t_j \ge t_\ell$, so $W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_\ell) \subseteq \mathcal{F}(t_j)$ for all j. Thus $I(t_k) - I(t_\ell)$ must also be independent of $\mathcal{F}(t_\ell)$.

(ii) Without loss of generality, we can take t and s to be partition points. Then $t = t_k$ and $s = t_\ell$ for some integers k and ℓ, and

$$I(t) - I(s) = \sum_{j=\ell}^{k-1} \Delta(t_j)\big[W(t_{j+1}) - W(t_j)\big].$$

That is, $I(t) - I(s)$ is a linear combination of independent normal random variables, with coefficients given by the non-random $\Delta(t_j)$, meaning $I(t) - I(s)$ is itself a normal random variable. The component of the sum with index j has mean and variance

$$\mathbb{E}\big[\Delta(t_j)(W(t_{j+1})-W(t_j))\big] = \Delta(t_j)\mathbb{E}\big[(W(t_{j+1})-W(t_j))\big] = 0 \qquad \text{and}$$
$$\operatorname{Var}\big(\Delta(t_j)(W(t_{j+1})-W(t_j))\big) = \Delta^2(t_j)\big(t_{j+1}-t_j\big) = \int_{t_j}^{t_{j+1}} \Delta^2(u)\,\mathrm{d}u,$$

respectively. Thus $I(t)-I(s)$ has mean zero and variance

$$\sum_{j=\ell}^{k-1}\int_{t_j}^{t_{j+1}} \Delta^2(u)\,\mathrm{d}u = \int_{t_\ell}^{t_k} \Delta^2(u)\,\mathrm{d}u = \int_s^t \Delta^2(u)\,\mathrm{d}u.$$

(iii) $I(t)-I(s)$ is independent of $\mathcal{F}(s)$, so conditioning on $\mathcal{F}(s)$ gives no extra information: $\mathbb{E}\big[I(t)-I(s)\,|\,\mathcal{F}(s)\big] = \mathbb{E}\big[I(t)-I(s)\big] = 0$. Thus $\mathbb{E}\big[I(t)\,|\,\mathcal{F}(s)\big] = \mathbb{E}\big[I(s)\,|\,\mathcal{F}(s)\big] = I(s)$.

(iv) Taking a cue from the text, define $D_j = W(t_{j+1})-W(t_j)$ for $j=0,\ldots,k-1$ and $D_k = W(t)-W(t_k)$. Then $I(t)=\sum_{j=0}^k \Delta(t_j)D_j$, and

$$I^2(t) = \sum_{j=0}^k \Delta^2(t_j)D_j^2 + 2\sum_{0\le i<j\le k}\Delta(t_i)\Delta(t_j)D_iD_j.$$

Fix an $s<t$ and let $[t_\ell, t_{\ell+1})$ be the subinterval that contains s.

The square terms can be written as

$$\sum_{j=0}^k \Delta^2(t_j)D_j^2 = \sum_{j=0}^{\ell-1}\Delta^2(t_j)D_j^2 + \Delta^2(t_\ell)\big[W(s)-W(t_\ell)\big]^2$$
$$+ \Delta^2(t_{\ell+1})\big[W(t_{\ell+1})-W(s)\big]^2 + \sum_{j=\ell+1}^k \Delta^2(t_j)D_j^2.$$

Conditional on $\mathcal{F}(s)$, the first two terms on the right-hand side above are known. Relying on the non-random nature of $\Delta(t)$, the latter two terms are

$$\mathbb{E}\left[\Delta^2(t_{\ell+1})\big[W(t_{\ell+1})-W(s)\big]^2 + \sum_{j=\ell+1}^k \Delta^2(t_j)D_j^2 \,\middle|\, \mathcal{F}(s)\right]$$
$$= \Delta^2(t_{\ell+1})\mathbb{E}\big[(W(t_{\ell+1})-W(s))^2\,|\,\mathcal{F}(s)\big] + \sum_{j=\ell+1}^k \Delta^2(t_j)\mathbb{E}\big[D_j^2\,|\,\mathcal{F}(s)\big]$$

Since future increments are independent of $\mathcal{F}(s)$, the expectation terms above are the variance of those increments, reducing the above expression to

$$\Delta^2(t_{\ell+1})\big(t_{\ell+1}-s\big) + \sum_{j=\ell+1}^{k-1}\Delta^2(t_j)\big(t_{j+1}-t_j\big) + \Delta^2(t_k)\big(t-t_k\big) = \int_s^t \Delta^2(u)\,\mathrm{d}u.$$

The cross terms with indices $i<j$ that satisfy $j>\ell$ are zero in conditional expectation, given the independent and mean-zero qualities of the increments:

$$\mathbb{E}\big[D_iD_j\,|\,\mathcal{F}(s)\big] = \mathbb{E}\big[D_i\,|\,\mathcal{F}(s)\big]\mathbb{E}\big[D_i\,|\,\mathcal{F}(s)\big] = 0, \qquad \begin{matrix} i & <j \\ j & >\ell \end{matrix}\ .$$

Note that it is not always the case that both of the expectations in the middle expression are zero; that only occurs when $j > i > \ell$, where $[t_\ell, t_{\ell+1})$ is the subinterval containing s. For cross terms with indices $\ell \geq j > i$, the D_i increments are known; in short we have

$$\mathbb{E}\left[2 \sum_{0\leq i<j\leq k} \Delta(t_i)\Delta(t_j)D_iD_j\right] = 2 \sum_{0\leq i<j\leq \ell} \Delta(t_i)\Delta(t_j)D_iD_j,$$

where we have abused notation a little bit in using D_ℓ to represent $W(t) - W(t_\ell)$. We continue to make this choice in what follows.

At this point, we have shown, for $s < t$,

$$\begin{aligned}\mathbb{E}\left[I^2(t) \,|\, \mathcal{F}(s)\right] = \sum_{j=0}^{\ell-1} \Delta^2(t_j)D_j^2 + \Delta^2(t_\ell)\big[W(s) - W(t_\ell)\big]^2 \\ + 2 \sum_{0\leq i<j\leq \ell} \Delta(t_i)\Delta(t_j)D_iD_j + \int_s^t \Delta^2(u)\,\mathrm{d}u.\end{aligned}$$

But! Notice the first three terms are precisely $I^2(s)$. Thus,

$$\begin{aligned}\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\,\mathrm{d}u \,\middle|\, \mathcal{F}(s)\right] &= I(s) + \int_s^t \Delta^2(u)\,\mathrm{d}u - \int_0^t \Delta^2(u)\,\mathrm{d}u \\ &= I(s) - \int_0^s \Delta^2(u)\,\mathrm{d}u,\end{aligned}$$

thereby confirming $I^2(t) - \int_0^t \Delta^2(u)\,\mathrm{d}u$ is a martingale.

There is an alternative, less mechanical way to show this, too:

$$\begin{aligned}&\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\,\mathrm{d}u - \left(I^2(s) - \int_0^s \Delta^2(u)\,\mathrm{d}u\right) \,\middle|\, \mathcal{F}(s)\right] \\ &= \mathbb{E}\left[I^2(t) - I^2(s) - \int_s^t \Delta^2(u)\,\mathrm{d}u \,\middle|\, \mathcal{F}(s)\right] \\ &= \mathbb{E}\left[\big(I(t) - I(s)\big)^2 + 2I(t)I(s) - 2I^2(s) \,|\, \mathcal{F}(s)\right] - \int_s^t \Delta^2(u)\,\mathrm{d}u\end{aligned}$$

Because $I(t) - I(s)$ is a mean-zero variable with variance $\int_s^t \Delta^2(u)\,\mathrm{d}u$, independent of $\mathcal{F}(s)$,

$$\mathbb{E}\left[\big(I(t) - I(s)\big)^2 \,|\, \mathcal{F}(s)\right] = \mathbb{E}\left[\big((I(t) - I(s)) - 0\big)^2\right] = \int_s^t \Delta^2(u)\,\mathrm{d}u.$$

As shown in part (iii), I is a martingale – therefore,

$$\mathbb{E}\big[I(t)I(s) \,|\, \mathcal{F}(s)\big] = I(s)\mathbb{E}\big[I(t) \,|\, \mathcal{F}(s)\big] = I^2(s).$$

Thus,

$$\begin{aligned}&\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\,\mathrm{d}u - \left(I^2(s) - \int_0^s \Delta^2(u)\,\mathrm{d}u\right) \,\middle|\, \mathcal{F}(s)\right] \\ &\quad = \int_s^t \Delta^2(u)\,\mathrm{d}u + 2I^2(s) - 2I^2(s) - \int_s^t \Delta^2(u)\,\mathrm{d}u = 0.\end{aligned}$$

Question 3

We now consider a case in which $\Delta(t)$ in Exercise 4.2 is simple but random. In particular, let $t_0 = 0$, $t_1 = s$, and $t_2 = t$ and let $\Delta(0)$ be nonrandom and $\Delta(s) = W(s)$. Which of the following assertions is true? Justify your answers.

(i) $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.
(ii) $I(t) - I(s)$ is normally distributed.
(iii) $\mathbb{E}\big[I(t) \,|\, \mathcal{F}(s)\big] = I(s)$.
(iv) $\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\,\mathrm{d}u \,|\, \mathcal{F}(s)\right] = I^2(s) - \int_0^s \Delta^2(u)\,\mathrm{d}u$.

Answer

The Itô integrals at times s and t are

$$\begin{aligned} I(t) &= \Delta(0)\big[W(s) - W(0)\big] + \Delta(s)\big[W(t) - W(s)\big] \\ &= \Delta(0)W(s) + W(s)\big[W(t) - W(s)\big] \\ \text{and} \qquad I(s) &= \Delta(0)\big[W(s) - W(0)\big] = \Delta(0)W(s). \end{aligned}$$

Thus $I(t) - I(s) = W(s)\big[W(t) - W(s)\big]$.

(i) **False.** $W(s) \in \mathcal{F}(s)$, so $I(t) - I(s)$ is not independent of $\mathcal{F}(s)$.

(ii) **False.** A normal random variable X with mean zero satisfies $\mathbb{E}[X^4] = 3\mathbb{E}[X^2]$. We will show that $I(t) - I(s)$ does not exhibit this property.

Throughout we use the fact that $W(s)$ is independent of the future increment $W(t) - W(s)$. The fourth moment of $I(t) - I(s)$ is

$$\begin{aligned} \mathbb{E}\left[\big(I(t) - I(s)\big)^4\right] &= \mathbb{E}\left[W(s)^4\big(W(t) - W(s)\big)^4\right] \\ &= \mathbb{E}\left[W(s)^4\right]\mathbb{E}\left[\big(W(t) - W(s)\big)^4\right] = (3s)\big[3(t-s)\big] = 9s(t-s). \end{aligned}$$

Whereas the second moment of $I(t) - I(s)$ is

$$\mathbb{E}\left[\big(I(t) - I(s)\big)^2\right] = \mathbb{E}\left[W(s)^2\right]\mathbb{E}\left[\big(W(t) - W(s)\big)^2\right] = s(t-s).$$

As $9s(t-s) \neq 3s(t-s)$, $I(t) - I(s)$ is not normally distributed.

(iii) **True.** Because the increment $W(t) - W(s)$ is independent of $\mathbb{E}\big[W(t) - W(s) \,|\, \mathcal{F}(s)\big] = \mathbb{E}\big[W(t) - W(s)\big] = 0$. Therefore,

$$\begin{aligned} \mathbb{E}\big[I(t) - I(s) \,|\, \mathcal{F}(s)\big] &= \mathbb{E}\left[W(s)\big(W(t) - W(s)\big) \,|\, \mathcal{F}(s)\right] \\ &= W(s)\mathbb{E}\big[W(t) - W(s) \,|\, \mathcal{F}(s)\big] = 0. \end{aligned}$$

(iv) **True.** Consider how

$$\begin{aligned} &\mathbb{E}\left[\left(I(t)^2 - \int_0^t \Delta(u)^2\,\mathrm{d}u\right) - \left(I(s)^2 - \int_0^s \Delta(u)^2\,\mathrm{d}u\right) \,\middle|\, \mathcal{F}(s)\right] \\ &\mathbb{E}\left[\big(I(t) - I(s)\big)^2 + 2I(t)I(s) - 2I(s)^2 - \int_s^t \Delta(u)^2\,\mathrm{d}u \,\middle|\, \mathcal{F}(s).\right] \end{aligned}$$

In expectation the first term is

$$\begin{aligned}\mathbb{E}\left[\big(I(t)-I(s)\big)^2 \,|\, \mathcal{F}(s)\right] &= \mathbb{E}\left[W(s)^2\big(W(t)-W(s)\big)^2 \,|\, \mathcal{F}(s)\right] \\ &= W(s)^2\mathbb{E}\left[\big(W(t)-W(s)\big)^2\right] = W(s)^2(t-s).\end{aligned}$$

The middle two terms are zero: $\mathbb{E}\big[2I(t)I(s)-2I(s)^2 \,|\, \mathcal{F}(s)\big] = 2\mathbb{E}[I(t) \,|\, \mathcal{F}(s)]I(s) - 2I(s)^2 = 0$. Finally, the integral term is known conditional on $\mathcal{F}(s)$:

$$\mathbb{E}\left[\int_s^t \Delta(u)^2 \,\mathrm{d}u \,\middle|\, \mathcal{F}(s)\right] = \mathbb{E}\left[\Delta(s)^2\big(t-s\big) \,|\, \mathcal{F}(s)\right] = W(s)^2\big(t-s\big)$$

Thus,

$$\mathbb{E}\left[I(t)^2 - \int_0^t \Delta(u)^2 \,\mathrm{d}u \,\middle|\, \mathcal{F}(s)\right] = I(s)^2 - \int_0^s \Delta(u)^2 \,\mathrm{d}u.$$

Question 4 - Stratonovich Integral

Let $W(t), t \geq 0$ be a Brownian motion. Let T be a fixed positive number and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0,T]$ (i.e. $0 = t_0 < t_1 < \cdots < t_n = T$). For each j, define $t_j^* = {}^{(t_j+t_{j+1})}/_2$ to be the midpoint of the interval $[t_j, t_{j+1}]$.

(i) Define the *half-sample quadratic variation* corresponding to Π to be

$$Q_{\Pi/2} = \sum_{j=0}^{n-1} \big(W(t_j^*) - W(t_j)\big)^2.$$

Show that $Q_{\Pi/2}$ has limit ${}^T/_2$ as $\|\Pi\| \to 0$.

(ii) Define the Stratonovich integral of $W(t)$ with respect to $W(t)$ to be

$$\int_0^T W(t) \circ dW(t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} W(t_j^*)\big(W(t_{j+1}) - W(t_j)\big).$$

In contrast to the Itô integral $\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - {}^T/_2$, which evaluates the integrand at the left endpoint of each subinterval $[t_j, t_{j+1}]$, here we evaluate the integrand at the midpoint t_j^*. Show that

$$\int_0^T W(t) \circ dW(t) = \tfrac{1}{2}W^2(T).$$

Answer

(i) For brevity, define $D_j = W(t_j^*) - W(t_j)$ for $j = 0, \dots, n-1$. The expected value of the half-sample quadratic variation is

$$\mathbb{E}\left[Q_{\Pi/2}\right] = \sum_{j=0}^{n-1} \mathbb{E}\left[D_j^2\right] = \sum_{j=0}^{n-1} \left(\tfrac{t_{j+1}+t_j}{2} - t_j\right) = \tfrac{1}{2}\sum_{j=0}^{n-1}(t_{j+1}-t_j) = \tfrac{T}{2},$$

where the second equality follows from the fact that $D_j = W(t_j^*) - W(t_j)$ is a Brownian increment.

Next consider the variance of the half-sample quadratic variance:

$$\mathrm{Var}\left(Q_{\Pi/2}\right) = \mathbb{E}\left[\left(\sum_{j=0}^{n-1} D_j^2 - \tfrac{T}{2}\right)^2\right] = \mathbb{E}\left[\left(\sum_{j=0}^{n-1} D_j^2 - \sum_{j=0}^{n-1} \tfrac{t_{j+1}-t_j}{2}\right)^2\right].$$

We can combine the two summations to derive

$$\begin{aligned}\mathrm{Var}\left(Q_{\Pi/2}\right) &= \mathbb{E}\left[\left(\sum_{j=0}^{n-1}\left[D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right]\right)^2\right] \\ &= \sum_{j=0}^{n-1} \mathbb{E}\left[\left(D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right)^2\right] + 2\sum_{0\le i<j\le n-1} \mathbb{E}\left[\left(D_i^2 - \tfrac{t_{i+1}-t_i}{2}\right)\left(D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right)\right]\end{aligned}$$

For every $j = 0, \dots, n-1$, we have that

$$\mathbb{E}\left[D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right] = \mathbb{E}\left[\left(W(t_j^*) - W(t_j)\right)^2\right] - \tfrac{t_{j+1}-t_j}{2} = \left(\tfrac{t_{j+1}+t_j}{2} - t_j\right) - \tfrac{t_{j+1}-t_j}{2} = 0.$$

This is especially useful when considering the cross-terms in the expression for $\mathrm{Var}(Q_{\Pi/2})$, because the D_i^2 and D_j^2 terms are non-overlapping. Thus they're functions of independent Brownian increments, and are therefore independent themselves. For the $i<j$ term of the cross-term summation, then,

$$\mathbb{E}\left[\left(D_i^2 - \tfrac{t_{i+1}-t_i}{2}\right)\left(D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right)\right] = \mathbb{E}\left[D_i^2 - \tfrac{t_{i+1}-t_i}{2}\right]\mathbb{E}\left[D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right] = 0.$$

Therefore $\mathrm{Var}(Q_{\Pi/2})$ reduces to

$$\mathrm{Var}\left(Q_{\Pi/2}\right) = \sum_{j=0}^{n-1} \mathbb{E}\left[\left(D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right)^2\right].$$

Consider the j-th term in that summation;

$$\begin{aligned}\mathbb{E}\left[\left(D_j^2 - \tfrac{t_{j+1}-t_j}{2}\right)^2\right] &= \mathbb{E}\left[D_j^4\right] - 2\tfrac{t_{j+1}-t_j}{2}\mathbb{E}\left[D_j^2\right] + \left(\tfrac{t_{j+1}-t_j}{2}\right)^2 \\ &= 3\left(\mathbb{E}\left[D_j^2\right]\right)^2 - \left(\tfrac{t_{j+1}-t_j}{2}\right)^2 = 3\left(\tfrac{t_{j+1}-t_j}{2}\right)^2 - \left(\tfrac{t_{j+1}-t_j}{2}\right)^2 = 2\left(\tfrac{t_{j+1}-t_j}{2}\right)^2.\end{aligned}$$

Therefore we can bound the variance from above by

$$\begin{aligned}\mathrm{Var}\left(Q_{\Pi/2}\right) &= \sum_{j=0}^{n-1} 2\left(\tfrac{t_{j+1}-t_j}{2}\right)^2 = \tfrac{1}{2}\sum_{j=0}^{n-1}\left(t_{j+1}-t_j\right)^2 \\ &\le \tfrac{1}{2}\max_i |t_{i+1}-t_i| \sum_{j=0}^{n-1}(t_{j+1}-t_j) = \tfrac{1}{2}T\max_i |t_{i+1}-t_i|.\end{aligned}$$

Taking $\|\Pi\| \to 0$, the maximum gap between successive partition points converges to zero. Thus $\mathrm{Var}(Q_{\Pi/2}) \to 0$.

Thus $Q_{\Pi/2}$ converges almost surely to $T/2$.

(ii)

Question 5 - Solving the Generalized Geometric Brownian Motion Equation

Let $S(t)$ be a positive stochastic process that satisfies the generalized geometric Brownian motion differential eqquation (see Example 4.4.8)

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t),$$

where $\alpha(t)$ and $\sigma(t)$ are processes adapted to the filtration $\mathcal{F}(t), t \geq 0$, associated with the Brownian motion $W(t), t \geq 0$. In this exercise, we show that $S(t)$ must be given by formula (4.4.26), reproduced here:

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \tfrac{1}{2}\sigma(s)^2\right) ds\right\}$$

(i.e. that formula provides the only solution to the stochastic differential equation (4.10.2)). In the process, we provide a method for solving this equation.

(i) Using the above formula, and the Itô-Doeblin formula, compute $d\log S(t)$. Simplify so that you have a formula for $d\log S(t)$ that does not involve $S(t)$.

(ii) Integrate the formula you obtained in (i), and then exponentiate the answer to obtain (4.4.26).

Answer

(i) With $dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$ and $f(x) = \log x$, the Itô-Doeblin formula $df(S(t)) = f'(S(t))dS(t) + \frac{1}{2}f''(S(t))dS(t)dS(t)$ implies

$$\begin{aligned} d\log S(t) &= \tfrac{1}{S(t)}\big(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\big) + \tfrac{1}{2}\left(-\tfrac{1}{S(t)^2}\right)\sigma(t)^2S(t)^2dt \\ &= \left(\alpha(t) - \tfrac{1}{2}\sigma(t)^2\right)dt + \sigma(t)dW(t). \end{aligned}$$

(ii) We integrate the expression above from 0 to $t > 0$:

$$\log S(t) = \int_0^t d\log S(s)\,ds = \widetilde{S}(0) + \int_0^t \sigma(s)\,dW(s) + \int_0^t \alpha(s) - \tfrac{1}{2}\sigma(s)^2\,ds,$$

where $\widetilde{S}(0)$ is some nonrandom constant. Exponentiating, we have

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) + \int_0^t \alpha(s) - \tfrac{1}{2}\sigma(s)^2\,ds,\right\},$$

where $S(0) = \exp(\widetilde{S}(0)) > 0$.

Question 6

Let $S(t) = S(0)\exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}$ be a geometric Brownian motion. Let p be a positive constant. Compute $d(S^p(t))$, the differential of $S(t)$ raised to the point p.

Answer

Given a $p > 0, p \neq 1$, define $f(x) = x^p$. The Itô-Doeblin formula guarantees that

$$d\big(S(t)^p\big) = df(S(t)) = pS(t)^{p-1}dS(t) + \tfrac{1}{2}p(p-1)S(t)^{p-2}dS(t)dS(t).$$

Then, using the identities

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \quad \text{and} \quad dS(t)dS(t) = \sigma^2 S(t)^2 dt,$$

we have

$$\begin{aligned} d\big(S(t)^p\big) &= p\alpha S(t)^p dt + p\sigma S(t)^p dW(t) + \tfrac{1}{2}p(p-1)\sigma^2 S(t)^p dt \\ &= pS(t)^p \left[\sigma dW(t) + \left(\alpha + \tfrac{p-1}{2}\sigma^2\right) dt\right]. \end{aligned}$$

Question 7

(i) Compute $dW^4(t)$ and then write $W^4(T)$ as the sum of an ordinary (Lebesgue) integral with respect to time and an Itô integral.

(ii) Take expectations on both sides of the formula you obtained in (i), use the fact that $\mathbb{E}[W^2(t)] = t$, and derive the formula $\mathbb{E}[W^4(T)] = 3T^2$.

(iii) Use the method of (i) and (ii) to derive a formula for $\mathbb{E}[W^6(T)]$.

Answer

(i) Applying the Itô-Doeblin to $f(W(t)) = W^4(t)$,

$$dW^4(t) = 4W^3(t)dW(t) + \tfrac{1}{2}12W^2(t)dW(t)dW(t) = 4W^3(t)dW(t) + 6W^2(t)dt.$$

Integrating from 0 to T and using the fact that $W(0) = 0$, we have

$$W^4(T) = 4\int_0^T W^3(t)\,dW(t) + 6\int_0^T W^2(t)\,dt,$$

where the first term is the Itô integral and the second is the Lebesgue.

(ii) Recall that odd-numbered centered moments of normal random variables (except for the first) are always zero. Thus

$$\begin{aligned} \mathbb{E}\big[W^4(t)\big] &= 4\int_0^T \mathbb{E}\big[W^3(t)\big]\,dW(t) + 6\int_0^T \mathbb{E}\big[W^2(t)\big]dt, \\ &= 0 + 6\int_0^T t\,dt = 3t^2\Big|_{t=0}^{T} = 3T^2. \end{aligned}$$

(iii) From Itô-Doeblin,

$$\begin{aligned} &dW^6(t) = 6W^5(t)dW(t) + 15W^4(t)dt \\ \implies \quad &\mathbb{E}\big[W^6(T)\big] = 15\int_0^T \mathbb{E}\big[W^4(t)\big]\,dt = 15\int_0^T 3t^2\,dt = 15T^3. \end{aligned}$$

Question 8

The Vasicek interest rate stochastic differential equation (4.4.32) is

$$dR(t) = \big(\alpha - \beta R(t)\big)dt + \sigma dW(t),$$

where α, β, and σ are positive constants. The solution to this equation is given in Example 4.4.10. This exercise shows how to derive this solution.

(i) Use (4.4.32) and the Itô-Doeblin formula to compute $d[e^{\beta t}R(t)]$. Simplify it so that you have a formula for $d[e^{\beta t}R(t)]$ that does not involve $R(t)$.

(ii) Integrate the equation you obtained in (i) and solve for $R(t)$ to obtain (4.4.33).

Answer

(i) From the product rule, $d[e^{\beta t}R(t)] = \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t)$. From (4.4.32), then, $d\big[e^{\beta t}R(t)\big] = \sigma e^{\beta t}dW(t) + \alpha e^{\beta t}dt = e^{\beta t}\big[\sigma dW(t) + \alpha dt\big]$.

(ii) Integrating from 0 to $t > 0$,

$$e^{\beta t}R(t) = \int_0^t d\big[e^{\beta t}R(t)\big]\,dt = R(0) + \sigma\int_0^t e^{\beta s}\,dW(s) + \alpha\int_0^t e^{\beta s}\,ds.$$

With $\int_0^t e^{\beta s}ds = \beta^{-1}(e^{\beta t} - 1)$, we can divide both sides by $e^{\beta t}$ to derive

$$R(t) = e^{-\beta t}R(0) + \tfrac{\alpha}{\beta}\big(1 - e^{-\beta t}\big) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW(s),$$

which is precisely (4.4.33).

Question 9

For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x, is

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big),$$

where

$$d_+(\tau,x) = \tfrac{1}{\sigma\sqrt{\tau}}\left[\log\tfrac{x}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)\tau\right],$$

$$d_-(\tau,x) = d_+(\tau,x) - \sigma\sqrt{\tau},$$

and $N(y)$ is the cumulative standard normal distribution

$$N(y) = \tfrac{1}{2\pi}\int_{-\infty}^{y} e^{-z^2/2}\,dz = \tfrac{1}{2\pi}\int_{-y}^{\infty} e^{-z^2/2}\,dz.$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t,x) + rxc_x(t,x) + \tfrac{1}{2}\sigma^2x^2c_{xx}(t,x) = rc(t,x),$$

the *terminal condition*

$$\lim_{t\uparrow T} c(t,x) = (x-K)^+, \qquad x>0, x\neq K,$$

and the *boundary conditions*

$$\lim_{t\downarrow 0} c(t,x) = 0, \qquad \lim_{x\to\infty} c(t,x) = x - e^{-r(T-t)}K, \quad 0\leq t<T.$$

The terminal condition and the first boundary condition are usually written more simply but less precisely as

$$c(T,x) = (x-K)^+, \quad x\geq 0, \quad \text{and} \quad c(t,0)=0, \quad 0\leq t\leq T.$$

For this exercise, we abbreviate $c(t,x)$ as simply c and $d_\pm(T-t,x)$ as simply d_π.

(i) Verify first the equation

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+).$$

(ii) Show that $c_x = N(d_+)$. This is the *delta* of the option.

(iii) Show that

$$c_t = -rKe^{-r(T-t)}N(d_-) - \tfrac{\sigma x}{2\sqrt{T-t}}N'(d_+).$$

This is the *theta* of the option.

(iv) Use the formulas above to show that c satisfies the Black-Scholes-Merton differential equation.

(v) Show that for $x>K$, $\lim_{t\uparrow T} d_\pm = \infty$, but for $0<x<K$, $\lim_{t\uparrow T} d_\pm = -\infty$. Use these equalities to derive the terminal condition.

(vi) Show that for $0\leq t<T$, $\lim_{x\downarrow 0} d_\pm = -\infty$. Use this fact to verify the first part of the boundary condition as $x\downarrow 0$.

(vii) Show that for $0\leq t<T$, $\lim_{x\uparrow\infty} d_\pm = \infty$. Use this fact to verify the second part of the boundary condition as $x\uparrow\infty$. In this verification, you will need to show that

$$\lim_{x\to\infty}\frac{N(d_+)-1}{x^{-1}} = 0.$$

This is an indeterminate form $0/0$, and L'Hôpital's rule implies that this limit is

$$\lim_{x\to\infty}\frac{\frac{d}{dx}\left[N(d_+)-1\right]}{\frac{d}{dx}x^{-1}} = 0.$$

Work out this expression and use the fact that

$$x = K\exp\left\{\sigma\sqrt{T-t}d_+ - (T-t)\left(r+\tfrac{1}{2}\sigma^2\right)\right\}$$

to write this expression solely in terms of d_+ (i.e., without the appearance of any x except the x in the argument of $d_+(T-t,x)$.) Then argue that the limit is zero as $d_+\to\infty$.

Answer

(i) Throughout we use τ to denote $T-t$. The function $d_-(\tau, x)$ is explicitly written as

$$d_+(\tau, x) = \tfrac{1}{\sigma\sqrt{\tau}} \left[\log \tfrac{x}{K} + \left(r - \tfrac{1}{2}\sigma^2\right)\tau\right].$$

As a preliminary exercise, consider the quantity

$$\left(r - \tfrac{1}{2}\sigma^2\right)^2 + 2r\sigma^2 = \left(r + \tfrac{1}{2}\sigma^2\right)^2.$$

Using this identity, we can write the argument of the exponential operator in $e^{-r\tau}N'(d_-) = (2\pi)^{-1}\exp\left\{\tfrac{-1}{2}d_-^2 - r\tau\right\}$ as

$$\begin{aligned} -\tfrac{1}{2}\left(d_-^2 + 2r\tau\right) &= -\tfrac{1}{2\sigma^2\tau}\left[\left(\exp\tfrac{x}{K}\right)^2 + 2\left(r - \tfrac{\sigma^2}{2}\right)\tau\exp\tfrac{x}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)^2\tau^2\right] \\ &= -\tfrac{1}{2\sigma^2\tau}\left[\left(\log\tfrac{x}{K} + \left(r + \tfrac{1}{2}\sigma^2\right)\tau\right)^2 - 2\sigma^2\tau\log\tfrac{x}{K}\right] \\ &= -\tfrac{1}{2}d_+^2 + \log\tfrac{x}{K}. \end{aligned}$$

The first equality makes use of our preliminary result, and the second follows from making the judicious addition of $0 = 2\sigma^2\tau\log(x/K) - 2\sigma^2\tau\log(x/K)$ and rearranging the cross-term.

Thus,

$$e^{-r\tau}N'(d_-) = \tfrac{1}{2\pi}\exp\left\{-\tfrac{1}{2}d_+^2 + \log\tfrac{x}{K}\right\} = N'(d_+)\tfrac{x}{K},$$

from which the desired equality follows swiftly.

(ii) The only difference between d_+ and d_- is the term with the τ coefficient; hence

$$\tfrac{\partial}{\partial x}\left[d_+\right] = \tfrac{\partial}{\partial x}\left[d_-\right].$$

Turning to the partial derivative of the call price, c_x, we have

$$\begin{aligned} c_x &= N(d_+) + xN'(d_+)\tfrac{\partial}{\partial x}\left[d_+\right] - Ke^{-r\tau}N'(d_-)\tfrac{\partial}{\partial x}\left[d_-\right] \\ &= N(d_+) + \left[xN'(d_+) - Ke^{-r\tau}N'(d_-)\right]\tfrac{\partial}{\partial x}\left[d_+\right]. \end{aligned}$$

Using our result in part (i), the second term on the right-hand side is zero, so we are forced to conclude $c_x = N(d_+)$.

(iii)