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Discontinuous behavior of liquids between parallel and tilted plates

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Discontinuous behavior of liquids between parallel and tilted plates in the absence of gravity is discussed. A principal finding, derived mathematically from the classical Young–Laplace–Gauss formulation for capillary free surfaces, is that in a large range of configurations liquid bridges between parallel plates are unstable with respect to small, even infinitesimal, tilting of one of the plates. Under a computationally based hypothesis of uniqueness of spherical bridges in a wedge, it is shown that the same discontinuous behavior prevails for all but very particular circumstances. The various liquid configurations, which form the basis for an experiment on board the Space Station *Mir*, are characterized and illustrated.
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I. INTRODUCTION

In reduced gravity, fluids can behave in striking ways that are different from what occurs in common experience in a terrestrial environment. Discussed here are some of the mathematical results underlying such behavior for equilibrium fluid configurations in a wedge and between parallel plates. The results are described in connection with the Angular Liquid Bridge (ALB) experimental investigation, designed jointly with Mark Weislogel of NASA Lewis Research Center, and scheduled for the glovebox experiment facility on Space Station *Mir* as part of the *Mir*-23/NASA-4 mission.

II. FORMULATION

Our mathematical work is based on the classical Young–Laplace–Gauss formulation for an equilibrium free surface of liquid partly filling a container or otherwise in contact with solid support surfaces. In this formulation, when gravity is absent or can be neglected, which is the situation we discuss here, the mechanical energy E of the system is given by

$$E=\sigma(S-S^{*}\cos\gamma). \tag{1}$$

The interfacial liquid–vapor surface tension parameter σ and the relative adhesion coefficient $\cos\gamma$ of the liquid with the container walls are assumed to depend only on the material properties, which are taken here to be homogeneous (the same value of $\cos\gamma$ on all parts of the container, as is the case for the experiment). S and S^{*} are, respectively, the areas of the liquid–vapor free surface and of the solid–liquid interface.

Equilibrium configurations are those providing stationary values of the energy functional E subject to the condition of fixed liquid volume.[1] The equilibrium liquid–vapor free surfaces so determined are surfaces of constant mean curvature meeting the bounding walls with contact angle γ. We consider here values of the contact angle $0<\gamma<\pi$. Of particular interest in our mathematical studies are situations in which small changes in contact angle or geometry can result in large changes, possibly discontinuous, of the equilibrium fluid configuration.

III. THE WEDGE – DISCONTINUOUS BEHAVIOR

The first part of the ALB investigation, which is a continuation of one initiated earlier on the Second United States Microgravity Laboratory (USML-2) Space Shuttle flight, concerns discontinuous behavior of a free surface in a wedge, such as in the configuration shown in Fig. 1.[2] Consider a cylindrical container, with cross section Ω, closed at one end and partly filled with liquid forming a free surface $\mathcal{S}$. Suppose the cross-section boundary has an isolated corner P of opening angle 2α, $0<2\alpha<\pi$, forming a local ''wedge domain'' at P. One seeks a free surface $\mathcal{S}$ that is (locally) represented by a single-valued function over some neighborhood of P in Ω, and which meets the walls that abut at P in a prescribed angle γ. Our results state that *for such a surface to exist the condition* $|\gamma-\pi/2|\leq\alpha$ *must hold*.[1–3]

Discontinuous change in behavior at $|\gamma-\pi/2|=\alpha$ can be illustrated directly for the container in Fig. 2, for which the boundary of the section Ω is completed by joining smoothly to a wedge a circular arc with center on the angle bisector. Over the entire range $|\gamma-\pi/2|\leq\alpha$ an explicit closed-form solution can be given for $\mathcal{S}$, which, for sufficient specified liquid volume, covers the base entirely. It is a portion of a hemispherical surface that meets the walls with angle γ. Thus the free surface height is bounded uniformly at P over this range. However, for $|\gamma-\pi/2|>\alpha$ such a surface cannot exist, and the liquid will necessarily move to the corner and uncover the base, rising arbitrarily high at the vertex if $\gamma<\pi/2$ (or falling arbitrarily low if $\gamma>\pi/2$), regardless of liquid volume. *The surface behavior changes discontinuously when* $|\gamma-\pi/2|-\alpha$ *crosses the value zero*.

The discontinuous change in behavior has been corroborated for certain cases in microgravity experiments[1,4,5] and is investigated further in one of the vessels making up the ALB space experiment. The vessel has a corner, as in the configuration of Fig. 2, and is constructed so that the vertex angle

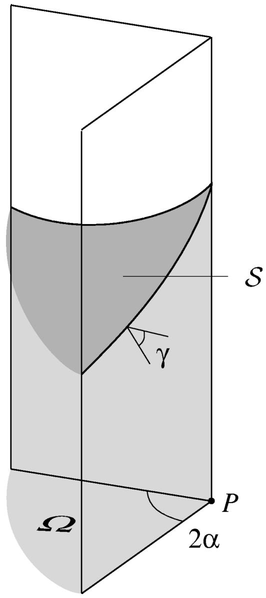

FIG. 1. Wedge configuration.

2α can be varied during the experiment. This allows the cases of advancing and receding liquid motion to be studied. Discontinuous behavior in the wedge enters into other parts of the experiment as well, as described below.

IV. LIQUID BRIDGES

The ALB investigation explores to a larger measure more general liquid configurations in a wedge than the one shown in Fig. 1. Impetus for this part of the experiment arises largely from recent doctoral dissertations[6,7] of two students associated with our study, J. McCuan[6] and L. Zhou,[7] from whose contrasting results striking inferences can be drawn. A detailed mathematical study of some of the results presented here is under preparation jointly with McCuan.

A. Liquid bridge in a wedge

In his work, McCuan found conditions under which an equilibrium tubular bridge in a wedge domain would be possible in zero gravity (Fig. 3; cf. Fig. 1), and he gave the shape such a bridge might take. As before, consider a wedge domain with opening angle 2α, $0<2\alpha<\pi$. The results McCuan proved contain the following (if the contact angles on the two sides of the wedge are different, similar results hold with γ on the left of the inequalities replaced by their average).

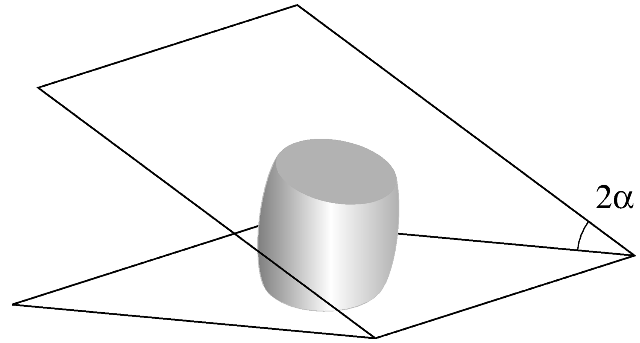

FIG. 3. Liquid bridge in a wedge.

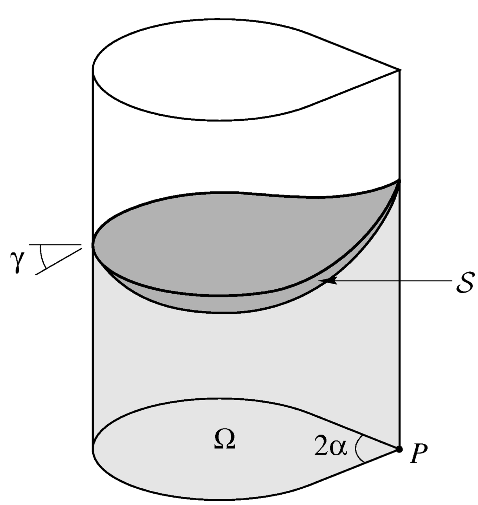

FIG. 2. Wedge container.

If $\gamma>\pi/2+\alpha$, a bridge in the shape of a portion of a sphere making contact angle γ with the walls exists.

If $\gamma\leqslant\pi/2+\alpha$, no physically realizable bridge is possible.

Note that these results complement, in respects, the ones given for the wedge in Sec. III. It has not yet been proved whether or not other shape bridges may be possible when $\gamma>\pi/2+\alpha$, or whether the spherical bridges are stable (provide a local minimum for the energy). However, our numerical results and those of Mittelmann,[8] obtained using the Surface Evolver software package,[9] indicate that the spherical bridges are stable, at least for the representative cases we considered, and that no other bridge shapes occur. The Surface Evolver software seeks local minima of a discretized energy functional, such as (1), subject to prescribed constraints, by employing gradient-descent type methods. Surfaces are approximated by a piecewise-linear triangulation, the form of which can be controlled to various degrees with commands available to the user. Under control of the user, the program adjusts the triangulated surfaces, step-by-step, in an attempt to decrease the energy. From the numerical and graphical output provided by the program, a user interprets whether a local minimum has been found.

Note that McCuan's results imply that a bridge is possible only for $\gamma>\pi/2$. A spherical liquid bridge, as calculated with the Surface Evolver, is shown in Fig. 6 for the case $\alpha=25°$, $\gamma=130°$.

B. Bridge between parallel plates

The above results for liquid bridges in a wedge compare in a remarkable way with those for bridges between parallel plates (Fig. 4). This latter problem was studied initially from a rigorous mathematical point of view by Athanassenas[10] and by Vogel,[11] and later using a more physical approach by Langbein.[12] [Note that in these papers, as is the case in Ref. 7 and here, the boundary conditions at the plates are prescribed contact angle, which arises from the variational condition for (1). For fixed end conditions, as considered in much of the materials science literature, the behavior of so-

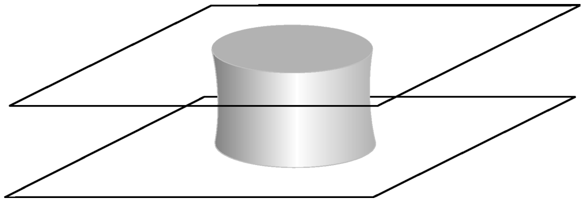

FIG. 4. Bridge between parallel plates.

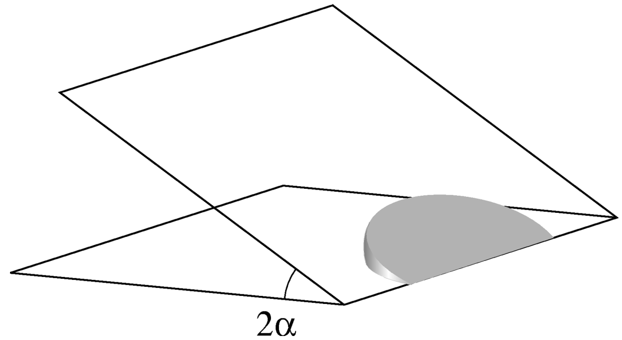

FIG. 5. Edge blob in a wedge.

lutions is different.] In her doctoral dissertation, Zhou obtained definitive mathematical results that imply the following .

For any value of the contact angle γ and for any liquid volume V greater than or equal to a critical value $V_0(\gamma)$, a unique stable liquid bridge exists between two parallel plates of given separation.

It is known that any equilibrium bridge must be rotationally symmetric,[11,13] and that its free surface is a Delaunay surface.[7,14,3] These are rotation surfaces of constant mean curvature, known as nodoids and unduloids, with the sphere and cylinder appearing as limiting cases. For $\gamma > \pi/2$ and for a specific liquid volume V_s depending on the plate spacing and γ, the free surface is a portion of the surface of a sphere. For other values of the volume the Delaunay surface for the same plate spacing and γ is different from a sphere.

C. Discontinuous behavior

From the results in Secs. IV A and IV B we obtain directly that a bridge between parallel plates can change its configuration and position markedly when one of the plates is tilted, even by a small amount, or, for a range of cases, it even may cease to exist as a bridge altogether.

A liquid bridge between parallel plates can behave discontinuously with respect to tilting of the plates.

The physical interpretation that has been given previously to stability studies limited to the parallel plate geometry, such as in Refs. 7 and 10–12, is thus largely illusory, as the studies do not take account of possible tilting of the plates. Except in very particular circumstances, regardless of the initial liquid bridge configuration, large fluid displacements can be expected to result from very small (even infinitesimal) tilting of one of the plates. This behavior is mathematically rigorous based on the exact equations when $\gamma \leqslant \pi/2$, in which case a wedge domain admits no liquid bridge (Sec. IV A). In the complementary case for which spherical bridges exist in a wedge, the behavior is a consequence of the numerically indicated uniqueness (for given volume) of such bridges.

As a specific example to illustrate the possibilities, consider the case $\gamma > \pi/2$ and a bridge between parallel plates of given spacing. Let the volume of the bridge have the value V_s defined above, so that the bridge is spherical for the given spacing and contact angle. In such a configuration, if one of the plates is tilted, the position of the bridge can change continuously so as to maintain a spherical shape and continue to meet the plates at the same contact angle. In fact, if the plate is tilted about an axis that is parallel to the plates and through the center of the sphere, then the bridge will not move at all, remaining fixed on the other plate. These are however exceptional cases that will not in general occur.

If the bridge has initial volume V different from V_s, then the bridge spanning the parallel plates will not be spherical. But assuming the numerically indicated hypothesis (see Sec. IV A) that spheres are the only possibility for the wedge, it follows that if the configuration continues to exist as a bridge spanning the plates, then with any tilt of either of the plates, no matter how small, the shape must change into a spherical one with the given contact angle. The change in shape and position from the parallel plate configuration is thus necessarily discontinuous. In particular configurations with $V < V_s$ it is easy to see that the liquid as a whole must displace toward the intersection line of the plates, and with $V > V_s$ the displacement will be away from the intersection line. The amount of displacement depends on how much V differs from V_s. In a physical experiment for which the plates do not extend to the intersection line, the fluid must be expected to displace to the edge of the plates in the direction of that line.

If $\gamma \leqslant \pi/2$ then the discussion becomes completely rigorous. An initial bridge always behaves discontinuously on the tilting of a plate, regardless of its volume, as it cannot persist as a bridge. In this case it has to be expected that the liquid will move to the edge of the plates. When the tilted plates touch forming a wedge, two further cases then have to be distinguished, as described in the following section.

V. OTHER CONFIGURATIONS

When the conditions for a bridge in a wedge are not satisfied, the liquid may assume a position as a blob in the shape of a portion of a sphere in contact with the edge; see Fig. 5. The condition for such a configuration to be possible is that $|\gamma - \pi/2| \leqslant \alpha$. (Recall we consider here only the case $0 < 2\alpha < \pi$.) The edge blobs have been noted in Refs. 15 and 16 and for some examples studied numerically. Our numerical computations indicate that, as for the angular bridges, the spherical edge blobs are stable; in the joint work with McCuan mentioned above it is proved that no other edge blob shapes can occur.

In our earlier work, discussed in Sec. III, we have shown that if $\alpha + \gamma < \pi/2$, then fluid cannot remain as a blob in the edge but must spread arbitrarily far along the edge. See also Ref. 16 and the references there for a discussion of stability of liquid columns in a wedge.

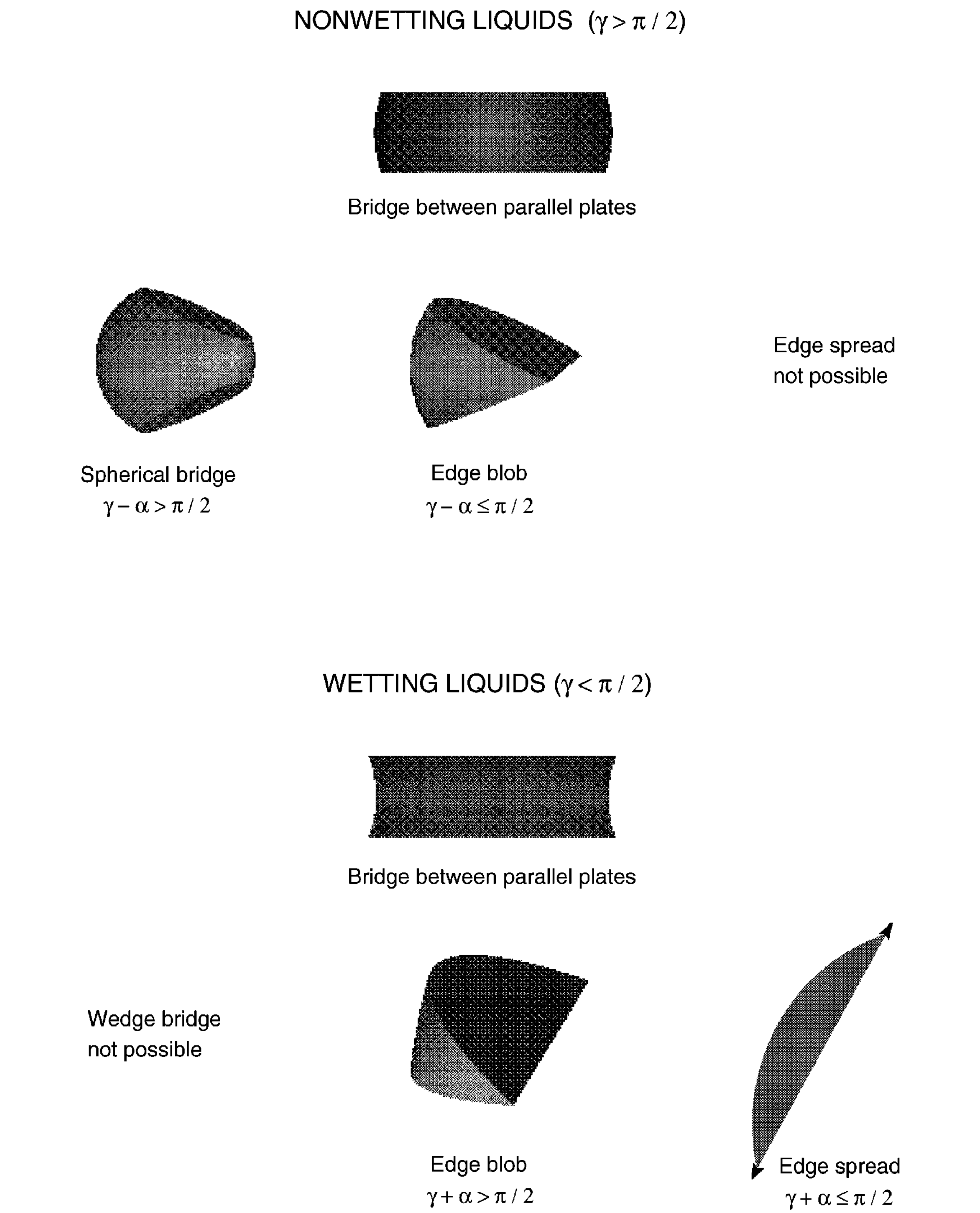

FIG. 6. Predicted liquid configurations. Upper two rows: nonwetting liquids; lower two rows: wetting liquids.

VI. PROJECTED EXPERIMENT BEHAVIOR

The liquid behavior one expects, based on the Young–Laplace–Gauss formulation, for a physical experiment in space, is summarized in Fig. 6. This figure illustrates the information discussed above, based in part on mathematically rigorous results and, where these are not available, on computational evidence for particular cases. The numerical solutions depicted in Fig. 6 were obtained using the Surface Evolver software package. The computations were carried out with initial approximations and transitions between configurations similar to those in which the experiment is designed to proceed, thereby enhancing appropriateness of the numerically based predictions on uniqueness and stability.

The upper two rows of Fig. 6 depict the nonwetting case $\gamma > \pi/2$: A liquid bridge between parallel plates is convex (part of a sphere for a specific fluid volume). Spherical tubular bridges and edge blobs exist for tilted plates, for the range of values indicated. Edge spread is not possible. For fixed $\gamma > \pi/2$, a transition from tubular bridges to edge blobs occurs as α increases through the value $\gamma - \pi/2$.

For the wetting case $\gamma < \pi/2$, a liquid bridge between parallel plates is concave. A tubular bridge between tilted plates is not possible, but the (spherical) edge blob and edge spread are. For fixed $\gamma < \pi/2$, the transition from edge blob to unbounded edge spread occurs as α decreases through the value $\pi/2 - \gamma$. Computed edge blobs are shown (from different viewing perspectives) for the case $\alpha = 25°$, $\gamma = 100°$ in the second row and for $\alpha = 20°$, $\gamma = 75°$ in the bottom row.

The ALB experiment explores the transition between the configurations for a nonwetting and for a wetting liquid. As discussed above, when initially parallel plates are tilted, the liquid is predicted to behave discontinuously in general, the exception being the special case of a spherical initial bridge. The other transitions, horizontally across the second and fourth rows of Fig. 6 as α changes value, are gradual, as can be demonstrated from the explicit spherical solutions.

VII. CONCLUDING REMARKS

We have described fluid behavior predicted mathematically and computationally for the Angular Liquid Bridge investigation on the *Mir*-23/NASA-4 Mission. The predictions, which include discontinuous behavior, are based on the idealized classical Young–Laplace–Gauss formulation. In the experiments there is an opportunity to compare the predictions with physical behavior and to observe the effects of hysteresis and other phenomena not included in the classical formulation.

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