Electrical Noise due to Thermal Excitations in Resistive Materials

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Thermal electrical noise and its relationship to Boltzmann's constant is investigated through resistance- and temperature-dependent measurements of noise power spectral density (PSD). Using cross-correlational analysis to isolate thermally induced noise, we calculate $k_b = (1.377 \pm 0.005) \times 10^{-23}$ J/K, consistent with the accepted value. A temperature-dependent investigation yielded $k_b = (1.145 \pm 0.08) \times 10^{-23}$ J/K, deviating due to sub-optimal temperature control. Methods, error analysis, and future improvements are discussed.

Electrical noise is a fundamental component of any circuit. Among the various types of electrical noise, the thermal noise investigated in this paper is particularly important as it is a fundamental result of the thermodynamics of resistive materials. This noise can be quantitatively analyzed through its power spectral density (PSD), which is proportional to the temperature and resistance of the material. Understanding this noise is essential to the design of sensitive electrical components and for its applications in microscopic thermometry.

The relationship between this electrical noise and Boltzmann's constant k_b and temperature forms the basis of this study. Boltzmann's constant is foundational in physics, bridging the macroscopic and microscopic worlds by relating temperature to energy at the particle level. Accurate determination of k_b is essential for various applications, including thermodynamics and quantum mechanics, as it provides insights into the behavior of systems at thermal equilibrium. In this study, we aim to precisely calculate Boltzmann's constant through the lens of thermally induced electrical noise and to quantify its relationship to resistance and temperature.

The thermal excitation of charge carriers in any circuit induces current. As shown by H. Nyquist [1], when these excited states are treated as oscillatory modes in a 1-dimensional resistor, from the equipartition theorem, each mode of oscillation contributes approximately k_bT average energy. This means that each frequency band δf contributes k_bT average energy. If $\Delta f = \sum \delta f$ then $P_{noise} = \sum \delta f k_b T = k_b T \sum \delta f = k_b T \Delta f$ this implies $P_{noise}/\Delta f = k_b T$. If we consider a scenario where this noisy resistor is connected transparently to a perfect cold resistor (of the same resistance R) such that all energy produced by one end of the noisy resistor is absorbed by the cold resistor, finding the PSD of the signal produced by the noisy resistor becomes the equivalent of finding the PSD absorbed by the cold resistor $P_{noise}/\Delta f = k_b T = P_{cold}/\Delta f$. By Ohm's law, the voltage from the ground end of the noisy resistor to the ground end of the cold resistor is $V_{noise} = I_{noise} 2R$. And since the power absorbed by the cold resistor is given by $P_{cold} = I_{noise}^2 R$, algebraic substitution yields $P_{cold} = V_{noise}^2/(4R)$. Dividing by Δf gives $P_{cold}/\Delta f = V_{noise}^2/(4R\Delta f) = k_b T$. Solving for the thermal noise PSD $V_{noise}^2/\Delta f$, which we will denote as S_V from now on, yields equation 1.

$$S_V = 4k_bTR \tag{1}$$

Other sources of noise also exist within the experimental setup, but they will not receive as thorough of a derivation as equation 1, as they are less central to the experimental goal of measuring Boltzmann's constant. The discretized current that leaks from the transistors present in the amplifier produces its own PSD, which we will denote as S_{IR} and the amplifier itself has a baseline level of noise that is roughly independent of frequency, which we will denote as S_A. These PSDs sum linearly to produce the final PSD $S_{out} = G^2(f)(S_{IR} + S_V + S_A)$ where $G(f)$ is the gain produced by the experimental setup. Since S_{IR} is produced by current noise across a resistor it can be expressed in terms of the current spectral density $S_{IR} = S_I R^2$, and excellent derivation of S_I has been done by Frank Rice [2] and it validates the above expression while also showing that we expect it to be roughly independent of frequency or 'white' noise, similar to thermal noise. Thus we expect the final PSD to be

$$S_{out} = G^2(f)(S_I R^2 + 4k_bTR + S_A) \tag{2}$$

The sound card within the experimental setup illustrated in Fig. 1 required calibration to produce a voltage signal that could then be analyzed. The calibration was done by measuring the amplitude of a sinusoidal voltage signal on an oscilloscope and then recording this same signal using the sound card. Using the fact that the recorded signal was not distorted, it was concluded that the sound card records the voltage signal multiplied by some linear scale factor $V_{sc} = CV$. This factor was found to vary with frequency and was characterized by first recording 11 signals at frequencies ranging from 100 Hz to 18 kHz. The amplitudes of these signals were then measured and compared with the calibration signal to find the calibration coefficient value at each frequency.

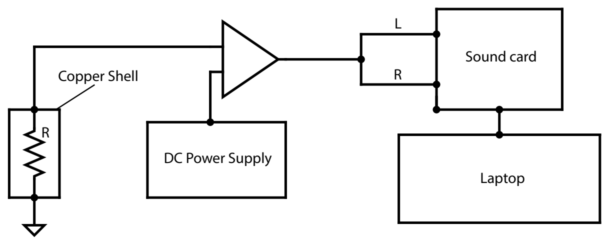

FIG. 1. The experimental setup used to collect noise signals from a resistor within an electronically insulating copper shell. The amplifier is connected to a battery bank producing a +3 V, -3 V DC differential. The amplifier is connected via left and right channels to a sound card which is then plugged into a laptop for data analysis.

These data points were then interpolated using a radial basis function and a Taylor series to approximate coefficients for different frequencies.

The amplifier's gain function $G(f)$ was also characterized by recording 20 data points of voltage gain over 20 evenly spaced sinusoidal signal frequencies in log space. These data points were then interpolated by a cubic spline to get a smooth approximation for voltage gain produced at other frequencies. The voltage gain data was obtained by measuring an initial sinusoidal voltage signal with amplitude V and frequency f, before sending the signal into a voltage divider circuit whose resistor values were calculated such that the resulting signal from the amplifier would have an amplitude of approximately 100 mV. The required resistances were estimated using the theoretical prediction of the amplifier's gain function $R_2 = 120k\Omega/(G_t(f)/2 - 1)$, this was done to ensure the amplifier was not overloaded and a consistent scale was used on the oscilloscope. The actual resistor values were measured using a multimeter before incorporation into the circuit. V_{in} was then the result of the voltage division such that $V_{in} = R_2V/(120k\Omega + R_2)$. The gain was then measured as $G_f = V_{out}/V_{in}$ where V_{out} was measured on an oscilloscope.

The noise signals from the resistor were then recorded twice at 13 different resistances ranging from 18 Ω - 1 MΩ. Signals were recorded using the sound card for 10 seconds at a 48 kHz sample rate. The noise signals were analyzed using the Welch method [3] implemented in Scipy's cross spectral density analysis function to generate cross spectral density (CSD) data between the left and right channels of the noise signal. The reason for finding the CSD is to eliminate uncorrelated noise between the two channels, which corresponds to isolating thermally induced noise in the resistor along with some residual noise due to interference between the two channels in the amplifier. This will be accounted for in the final fit by using a quadratic fit to account for any minimal current noise and amplifier noise contributions to the CSD. The resulting CSD distributions are shown in Fig. 2. Due to the finite sampling rate of the noise signal, we are presented with the fact that frequencies above the Nyquist frequency $f_{max} = f_s/2$ cannot be reliably captured, and frequencies above this value will fold back into lower frequencies, distorting the calculated CSD. This is avoided through the sound card containing a low pass filter that aggressively attenuates signals above $f \approx 20$ kHz. Since the noise signals were then in the frequency domain, the effect of the amplifier's frequency-dependent gain and the effects of the sound card could be readily reversed by dividing S_{out} by $G^2(f)$ denoting this new quantity as $S_f = S_{out}/G^2(f)$. The CSD data was then averaged in the frequency domain where the CSD distribution was approximately flat (500 Hz - 18 kHz). This was done because we wanted to avoid the region artificially attenuated by the sound card as well as the low-frequency region where external signals likely have a significant contribution to the CSD. These choices were justified because this domain still encompassed the vast majority of the data. A plot of $\langle S_f \rangle/(4T)$ versus resistance was then fitted using a quadratic model with 3 parameters. For resistance values above 11 kΩ, a strange distortion appeared in the CSD distributions, there was speculation that this may be caused by impedance mismatching at higher resistance values. Regardless, we are fairly confident this distortion is artificially induced and these data points were disregarded in the final fit. The thermal noise signals were also investigated as a function of temperature. A thermocouple was attached to the resistor within its copper shielding and a heat gun was

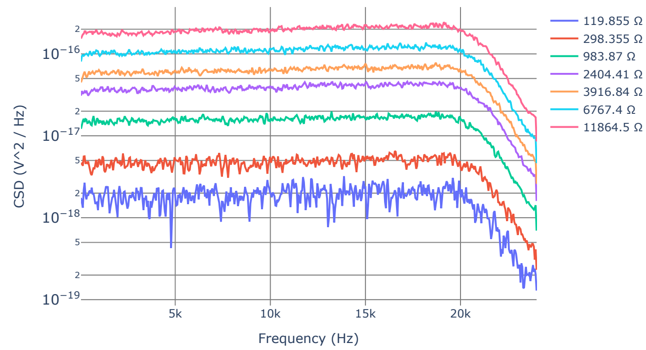

FIG. 2. The cross power spectral density (CSD) distributions of the left and right channels of a set of signals obtained from a variety of resistors. Each curve represents a different resistance as shown by the coloured legend. Notice the aggressive attenuation of frequencies larger than around 20 kHz. CSDs produced by 18 and 55 Ohm resistors were omitted from the plot for visual clarity.

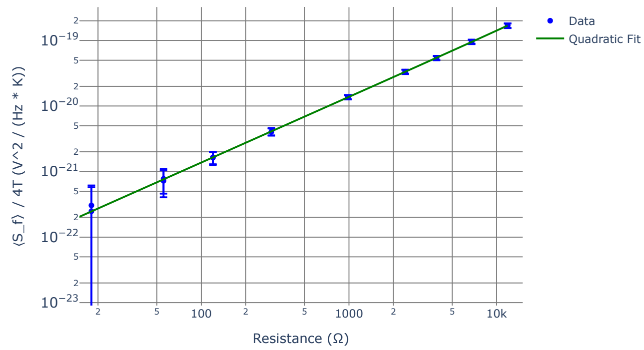

FIG. 3. The resulting plot of the collected data for $\langle S_{out}/G^2(f)\rangle/(4T) = \langle S_f\rangle/(4T)$ versus the resistance values for R in Fig. 1, which produces the analyzed noise signal. Theory predicts the quadratic coefficient to be k_b.

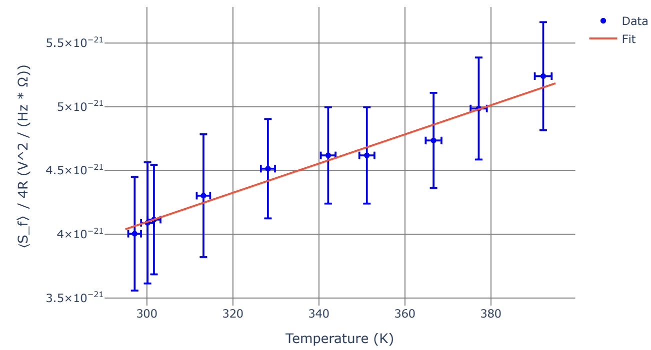

FIG. 4. The resulting plot of the collected data for $\langle S_{out}/G^2(f)\rangle/(4R) = \langle S_f\rangle/(4R)$ versus temperature of resistor R as shown in Fig. 1. Theory predicts the slope of the linear fit to be k_b.

used to change the temperature of the resistor. A 300 Ω resistor was used for signal generation. Signals were recorded for 10 different temperatures ranging from 24°C to 119°C. A similar process was used as the resistance-dependent investigation to obtain $\langle S_f\rangle/(4R)$ and thus k_b. The CSD was found using the left and right channels of each signal and the flat regions of the CSDs were averaged to produce $\langle S_f\rangle$ for each distribution. These values were then divided by $4R$ and a linear fit was performed on the resulting data points using 2 parameters to find k_b.

The resistance-dependent data as well the quadratic fit for that data is shown in Fig. 3. The quadratic fit produced an evaluation of $k_b = (1.377 \pm 0.005) \times 10^{-23}$ J/K. This is in good agreement with the contemporary literature value of k_b as it is within 1 standard deviation from the exact value $k_b = 1.380649 \times 10^{-23} J/K$ [4].

The temperature-dependent data as well as the linear fit for that data is shown in Fig. 4. The fit produced an evaluation of $k_b = (1.145 \pm 0.08) \times 10^{-23}$ J/K. This value is not in good agreement with $k_b = 1.380649 \times 10^{-23} J/K$ as it is approximately 3 standard deviations from the exact value. This disagreement is likely due to the sub-optimal temperature measurement procedure. During the experiment, the resistor's temperature fluctuated continuously, as it was nearly impossible to maintain a stable thermal equilibrium using the heat gun. These fluctuations introduced a time dependence in the temperature, leading to corresponding time-dependent distortions in the signals and their calculated CSDs.

To arrive at the final uncertainty calculated for k_b we will start with sound card calibration. The uncertainty in the conversion constant $C(f)$ was found via the uncertainty propagation $\sigma_{C(f)} = C(f)\frac{\sigma_V}{V}$. Where $\sigma_V = 3$ mV and $V = 40$ mV. The uncertainties in the voltage gain data G_f used in the fitting process were obtained via $\sigma_{G_f} = G_f\sqrt{(\sigma_{V_{out}}/V_{out})^2 + (\sigma_{V_{in}}/V_{in})^2}$. Where the voltage input into the voltage divider V_i and the observed voltage on the oscilloscope V_{out} had uncertainties estimated as $\sigma_V = 3$ mV + 1%. The uncertainties in the resistor values used were assumed to be dominated by random fluctuations in voltage, resulting in $\sigma_{V_{in}} = \sigma_V R_2/(120\text{k}\Omega + R_2)$. The uncertainty in the predictions of the fitted function $G(f)$ was estimated as the uncertainty in the fitted gain data such that $\sigma_{G(f)} = \sigma_{G_f}$. The uncertainty propagation then for S_f is then given by $\sigma_{S_f} = 2S_f\sqrt{(\sigma_{V_{sc}}/V_{sc})^2 + (\sigma_{C_f}/C_f)^2 + (\sigma_{G(f)}/G(f))^2}$ where σ_{sc} is the uncertainty in the voltage measurement of the sound card and V_{sc} is the sound card signal. σ_{sc} was assumed to be negligible resulting in $\sigma_{S_f} = 2S_f\sqrt{(\sigma_{C(f)}/C(f))^2 + (\sigma_{G(f)}/G(f))^2}$. The uncertainty in $\langle S_f\rangle$ can be found via $\sigma_{\langle S_f\rangle} = \sqrt{\sigma_{dist}^2 + \langle\sigma_{S_f}\rangle^2}$ where σ_{dist} is the standard deviation of the averaged CSD distribution and $\langle\sigma_{S_f}\rangle$ is the mean uncertainty in the CSD data points, given by $\langle\sigma_{S_f}\rangle = \sqrt{\sum_{n=1}^{N}\sigma_{S_{f_n}}^2}/N$. The uncertainty in temperature σ_T was estimated as $\sigma_T = 1$ K due to using the room's thermostat for temperature measurement. This resulted in in $\sigma_{\langle S_f\rangle/(4T)} = \frac{S_f}{4T}\sqrt{(\sigma_T/T)^2 + (\sigma_{\langle S_f\rangle}/\langle S_f\rangle)^2}$. The uncertainty in k_b was then calculated from the covariance matrix outputted by the fit of $\langle S_f\rangle/(4T)$ versus R. The uncertainty propagation for the temperature-dependence extension is very similar, differing only in the final step where instead of dividing by $4T$ we divide by $4R$. Thus $\sigma_{\langle S_f\rangle/(4R)} = \frac{S_f}{4R}\sqrt{(\sigma_R/R)^2 + (\sigma_{\langle S_f\rangle}/\langle S_f\rangle)^2}$. The temperature uncertainty was also deemed significant, so the uncertainty in the temperature along the x-axis was also included in the fit. The temperature uncertainty was estimated as $\sigma_T = 0.005T$ K to account for the variation in temperature over the course of a signal measurement. The final uncertainty in k_b was obtained from the covariance matrix outputted by the fitting function.

The primary measured value of k_b presented in this work is in alignment with the accepted literature value. However, the value obtained from the temperature dependence extension deviates considerably. Future resistance-dependent investigations of thermal noise could focus on characterizing the distortions in CSDs produced at high resistances and collecting more data points to improve the quality and credibility of results. Future temperature-dependent investigations should utilize a more rigorous temperature measurement procedure. This could be done by ensuring the resistor is in thermal equilibrium by using a water bath to maintain a consistent temperature over the course of signal measurement. This will enable us to better quantify thermal noise in conductors as a result of temperature and resistance, thus generating valuable insights that can be applied to the production of sensitive electronics and microscopic thermometry.

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