

## 24. CAMPOS CONSERVATIVOS

1 Teorema fundamental del cálculo en integrales de línea

$$\text{TFc: } F(x) = \int_a^x f(t) dt \longrightarrow F'(x) = f(x)$$

De modo que:

$$\int_a^b f(t) dt = F(b) - F(a)$$

Segundo TFC o  
regla de Barrow

Nota:

$$\int_a^b \frac{ds}{dt} dt = f(b) - f(a)$$

Extensión a integrales curvilineas

Sea  $f: \mathbb{R}^3 \rightarrow \mathbb{R} \in C^1$ , y sea  $C: r(t): I = [a, b] \rightarrow \mathbb{R}^3$  una curva regular, entonces:

$$\int_C \nabla f \cdot d\bar{s} = f(r(b)) - f(r(a))$$

Demostración:

$$\bar{r}(t) = (x(t), y(t), z(t)) \rightarrow \bar{r}'(t) = (x'(t), y'(t), z'(t))$$

$$\begin{aligned} \int_C \nabla f \cdot d\bar{s} &= \int_a^b \left( \nabla f \Big|_{\bar{r}(t)} \right) \cdot \bar{r}'(t) dt = \int_a^b \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \Big|_{\bar{r}(t)} \right) \cdot \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) dt = \\ &= \int_a^b \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt = \int_a^b \frac{df(\bar{r}(t))}{dt} dt = f(\bar{r}(b)) - f(\bar{r}(a)) \end{aligned}$$

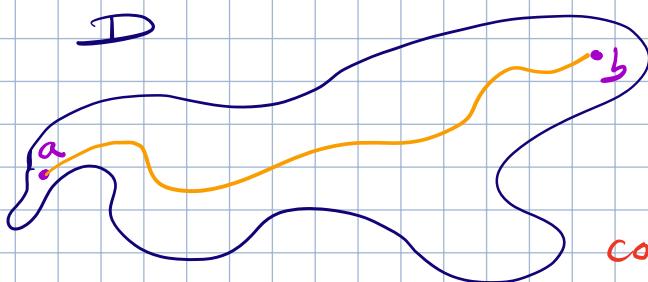
Regla de la cadena

$$* f(\bar{r}(t)) \rightarrow \frac{d}{dt} f(\bar{r}(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

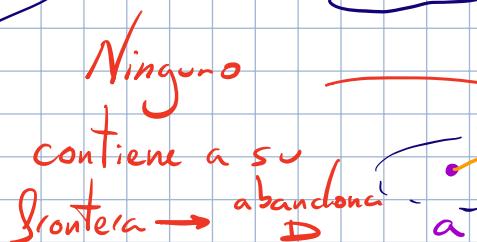
## 2 Dominios simplemente conexos

Definición: un conjunto  $D \subset \mathbb{R}^p$  es conexo por arcos si dados dos pts cualesquiera  $a, b \in D$ , es posible encontrar una curva continua que lleve de un pto a otro sin abandonar  $D$ .

Conexo por arcos



No conexo por arcos



Un dominio  $D$  es simplemente conexo si es conexo por arcos y cualquier curva  $C \subset D$ , cerrada y continua, puede reducirse de forma continua sin salirse de  $D$  hasta convertirse en un pto que pertenece a  $D$ .

EJEMPLOS:

a)  $\mathbb{R}^2$  y  $\mathbb{R}^3$  son simplemente conexos.

b)  $\mathbb{R}^3 - \{(0,0,0)\}$  es simplemente conexo

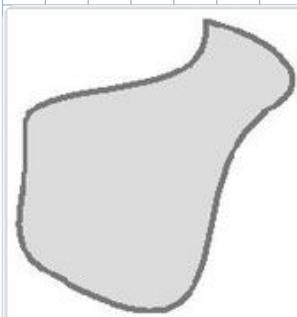
c)  $\mathbb{R}^3 - \{\text{un conjunto finito de puntos}\}$  es simplemente conexo

d)  $\mathbb{R}^2 - \{(0,0)\}$  no es simplemente conexo.

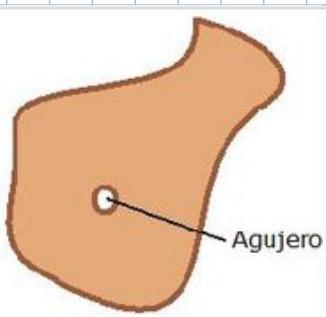
e)  $\mathbb{R}^2 - \{\text{un conjunto finito de puntos}\}$  no es simplemente conexo

g) En  $\mathbb{R}^2$ , un círculo es simplemente conexo, pero una corona circular no lo es.

g) En  $\mathbb{R}^3$ , una esfera es simplemente conexo y una corona esférica ~~también~~ lo es. Sin embargo una esfera con un taladro que la atraviese completamente no lo es.

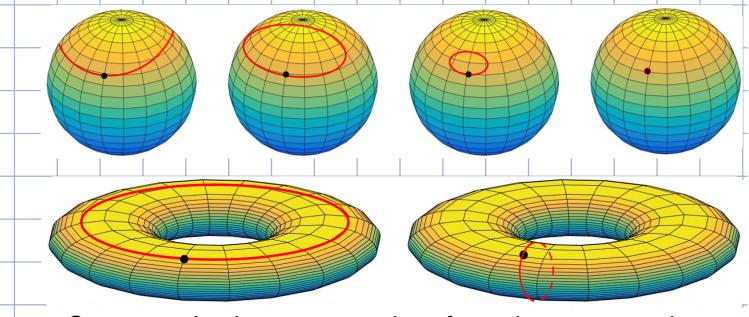


Simplemente conexo



No simplemente conexo

Hacer un agujero, o quitar un punto interior a un conjunto simplemente conexo en dos dimensiones hace que deje de ser simplemente conexo



Como puede observarse, en la esfera, al no tener agujeros siempre es posible "tirar" del lazo y deformarlo hasta reducirlo a un único punto. En el toro, pasa justo lo contrario, al tener un agujero, no podemos reducir cualquier curva a un punto. Para que la esfera (tres dimensiones) deje de ser simplemente conexo, no vale con hacerle un agujero (o quitarle un punto) hay que atravesarla por completo.

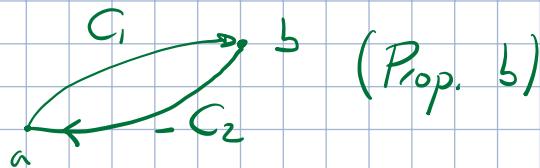
### 3 Campos conservativos

Sea  $\bar{F}: \mathbb{R}^p \rightarrow \mathbb{R}^p$  un campo vectorial de clase  $C^1$  definido en  $D \subset \mathbb{R}^p$  ( $p=2$  o  $p=3$ ), siendo  $D$  un conjunto simplemente conexo.

Se dice que  $\bar{F}$  es un campo vectorial conservativo si cumple cualquiera de las siguientes condiciones equivalentes entre sí:

a) La circulación de  $\bar{F}$  a lo largo de una curva cerrada y simple a trozos es nula. Es decir,

$$\oint_C \bar{F} \cdot d\bar{s} = 0$$



(Prop. 5)

**b** La circulación es independiente del camino. Es decir, para cualesquiera  $C_1$  y  $C_2$  curvas simples de clase  $C^k$  ( $k \geq 1$ ) a trozos, con los mismos pts extremos y la misma orientación, se verifica que: (Prop. c)

$$\int_{C_1} \bar{F} \cdot d\bar{s} = \int_{C_2} \bar{F} \cdot d\bar{s} = \int_C \nabla U \cdot d\bar{s} = U(\bar{r}(b)) - U(\bar{r}(a))$$

**c**  $\bar{F}$  es un campo gradiente, es decir existe alguna función escalar  $U: \mathbb{R}^P \rightarrow \mathbb{R}$  de clase  $C^2$  tal que:

$$\bar{F} = \nabla U$$

**d** El campo vectorial  $\bar{F}$  es irrotacional, es decir

$$\nabla \times \bar{F} = 0 \quad \text{Prop. c. } (\nabla \times (\nabla U)) = 0 \rightarrow \text{Por ser VEC}^2$$

Caso de que el dominio de definición del campo no sea un conjunto simplemente conexo.

Si  $D$  no es un conjunto simplemente conexo y  $\bar{F}$  es irrotacional ( $\nabla \times \bar{F} = 0$  en  $D$ ) puede ocurrir que  $\bar{F}$  no derive de un potencial (condición c) y tampoco se cumplen las condiciones a y b.

Sin embargo todas las propiedades seguirían siendo aplicables en subconjuntos de  $D$ , siempre que estos sean simplemente conexos.

En estos casos,  $\nabla \times \bar{F} = 0$  sigue siendo cond necesaria, ya q-e si no, es imposible q-e  $\nabla f = \bar{F}$ .  $\nabla \times (\nabla f)$  siempre es igual a 0.

### ESEMPIO

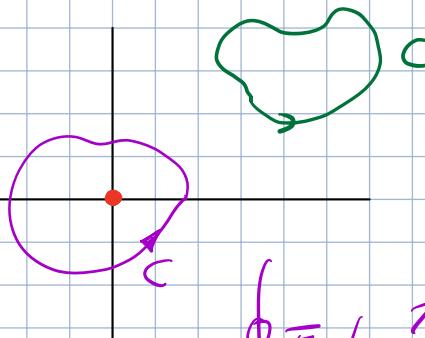
Sea  $\bar{F} = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$  definida en  $D = \mathbb{R}^2 - \{0,0\}$   
(dominio no simplemente conexo)

$$\nabla \times \bar{F} = (0, 0, F_{2x}' - F_{1y}') = 0$$

$$F_{2x}' = \frac{x^2+y^2 - 2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$F_{1y}' = \frac{-x^2-y^2 + 2y^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

Entonces, ¿se puede asegurar qe  $\oint_C \bar{F} \cdot d\bar{s} = 0$ ? No necesariamente



$\oint_C \bar{F} \cdot d\bar{s} = 0 \rightarrow$  Si no rodeo el punto conflictivo, todo va bien

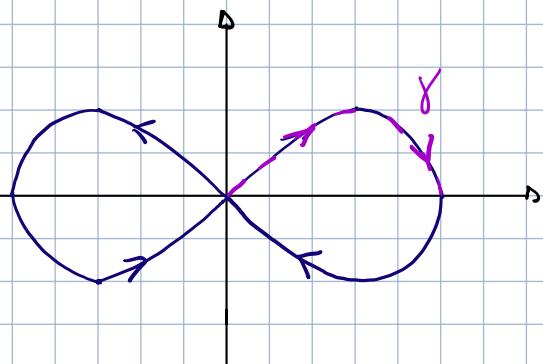
$\oint_C \bar{F} \cdot d\bar{s} \neq 0 \rightarrow$  En este caso, sin embargo, no puedo asegurar nada.

Probamos a calcular una:  $C \equiv \bar{r}(t) = (\cos \theta, \sin \theta)$

$$\begin{aligned} \bar{F}(\bar{r}(t)) &= (-\sin \theta, \cos \theta) \\ d\bar{s} &= (-\sin \theta, \cos \theta) d\theta \end{aligned} \quad \left| \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = 2\pi \neq 0 \right.$$

## EXERCICIO:

Calcular  $\int_{\gamma} \bar{F} \cdot d\bar{s}$  siendo  $\gamma$  la porción de la lemniscata de Bernoulli que va de  $(0,0)$  a  $(1,0)$  y  $\bar{F} = (2x, -2y)$ .



$$x = \sin t / (1 + \cos^2 t)$$

$$y = \sin t \cos t / (1 + \cos^2 t)$$

$$0 \leq t \leq \pi/2$$

Primero vamos a intentar resolver este problema integrando sobre la curva que nos dicen:

$$\int_{\gamma} \bar{F} \cdot d\bar{s} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) dt$$

$$1.- [a, b] = [0, 2\pi]$$

$$2.- \bar{F}(\bar{r}(t)) = (2x(t), -2y(t)) = \frac{\sin t}{1 + \cos^2 t} (1, \cos t)$$

$$3.- d\bar{s} = \bar{r}'(t) dt$$

$$x'(t) = \frac{\cos t (1 + \cos^2 t) + \sin t \cdot 2 \cos t \sin t}{(1 + \cos^2 t)^2}$$

$$y'(t) = \frac{(\cos^2 t - \sin^2 t)(1 + \cos^2 t) + \sin t \cos t \cdot 2 \cos t \sin t}{(1 + \cos^2 t)^2}$$

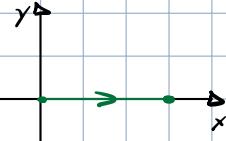
[...] ¿No hay una opción más sencilla?

Vamos a comprobar si el campo es conservativo. El dominio de  $\vec{F}$  es  $\mathbb{R}^2$  (dominio simplemente conexo) y es de clase  $C^\infty$ . De forma que si es irrotacional, será conservativo.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2x & -2y & 0 \end{vmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \vec{0} \rightarrow \vec{F} \text{ conservativo de modo que: } \int_C \vec{F} \cdot d\vec{s} \text{ indep. del camino.}$$

Puedo usar una curva más sencilla siempre que coincida el punto de partida y de final con  $\gamma$ .

① Recta que une  $(0,0)$  y  $(1,0)$



$$\vec{r}(t) = (0,0) + t((1,0) - (0,0)) = (t,0), \quad t \in [0,1]$$

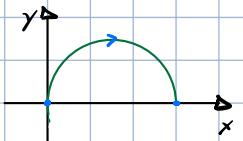
1.-  $[a,b] = [0,1]$

2.-  $\vec{F}(\vec{r}(t)) = (2x(t), -2y(t)) = (2t,0)$

3.-  $d\vec{s} = \vec{r}'(t) dt = (1,0) dt$

$$\int_0^1 (2t,0) \cdot (1,0) dt = \int_0^1 2t dt = 1$$

② Circunferencia de radio  $\frac{1}{2}$  y centro en  $\frac{1}{2}$



$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4} \quad \left| \begin{array}{l} x - \frac{1}{2} = \frac{1}{2} \cos \theta \\ y = \frac{1}{2} \sin \theta \end{array} \right.$$

Cuidado, que esta parametrización tiene sentido contrario.  
Tendré que cambiar al final el signo.

$$1.- [a, b] = [0, \pi]$$

$$2.- \bar{F}(\bar{r}(t)) = (2(\frac{1}{2}\cos\theta + \frac{1}{2}), -2\frac{1}{2}\sin\theta)$$

$$3.- d\bar{s} = \bar{r}'(t) dt = (-\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta) d\theta$$

$$\begin{aligned} & \int_0^\pi (\cos\theta + 1, -\sin\theta) \cdot \frac{1}{2} (-\sin\theta, \cos\theta) d\theta = \\ &= \int_0^\pi (-\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta) d\theta = \int_0^\pi (-\sin 2\theta - \sin\theta) d\theta = \\ &= \left. \frac{1}{2} \cos 2\theta + \cos\theta \right|_0^\pi = \frac{1-1}{2} + \frac{-1-(+1)}{2} = -1 \end{aligned}$$

Otra posible opción es sacar el potencial del que deriva el campo y así:

$$\nabla f = \bar{F} \implies \int_S \bar{F} \cdot d\bar{s} = \int_S \nabla f \cdot d\bar{s} = f(\bar{r}(b)) - f(\bar{r}(a))$$

$$\frac{\partial f}{\partial x} = 2x \implies f(x, y) = x^2 + c(y) \quad \frac{\partial f}{\partial y} \rightarrow f_y(x, y) = c'(y)$$

$$\frac{\partial f}{\partial y} = -2y \quad \frac{dc}{dy}(y) = -2y \rightarrow c(y) = -y^2 + K$$

$$f(x, y) = x^2 - y^2 + K$$

$$\hookrightarrow \int_S \nabla f \cdot d\bar{s} = f(\bar{r}(b)) - f(\bar{r}(a)) = x^2 - y^2 + K \Big|_{(0,0)}^{(1,0)} = 1 + K - K = 1$$

VII.21 a) Analizar si son independientes del camino las siguientes integrales

$$I_1 = \int_C (\operatorname{sen} z + z \operatorname{sen} x) dx + \cos z dy + (x \cos z - \cos x - y \operatorname{sen} z) dz,$$

$$I_2 = \int xy dx + y^2 dy + z dz,$$

calculando, en caso afirmativo, la función potencial.

b) Demostrar que  $I_3 = \int_C F(x, y, z)(dx + dy + dz)$ , con  $F: \mathbb{R}^3 \rightarrow \mathbb{R}$  de clase  $\mathcal{C}^2$ , es independiente del camino si y sólo si  $F(x, y, z) = \phi(x + y + z)$  con  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  de clase  $\mathcal{C}^2$

**Resultados:**  $I_1$  no depende del camino.  $I_2$  depende del camino.

a)  $\bar{F}_1(x, y, z) = (\operatorname{sen} z + z \operatorname{sen} x, \cos z, x \cos z - \cos x - y \operatorname{sen} z)$

Si  $\nabla \times \bar{F}_1 = 0$  en  $\mathbb{R}^3$ , entonces el campo es conservativo

$$\begin{array}{ccc} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \operatorname{sen} z + z \operatorname{sen} x & \cos z & x \cos z - \cos x - y \operatorname{sen} z \end{array} = \begin{bmatrix} -\operatorname{sen} z + z \operatorname{sen} z \\ \cos z + \operatorname{sen} x - \cos z - \operatorname{sen} x \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

El campo es conservativo, luego procedemos a buscar la función potencial de la cual deriva.

$$\bar{F}_1 = \nabla f(x, y, z) \rightarrow \frac{\partial f}{\partial x} = \operatorname{sen} z + z \operatorname{sen} x$$

$$\frac{\partial f}{\partial y} = \cos z$$

$$\frac{\partial f}{\partial z} = x \cos z - \cos x - y \operatorname{sen} z$$

(Integrando en  $\frac{\partial f}{\partial y}$ )

$$f = y \cos z + g(x, z) \rightarrow \frac{\partial f}{\partial x} = g'_x(x, z) = \operatorname{sen} z + z \operatorname{sen} x$$

Integrando  $\left( \frac{\partial g}{\partial x} \right)$

$$g(x, z) = x \sin z - z \cos x + h(z)$$

Luego

$$f = y \cos z + x \sin z - z \cos x + h(z) \rightarrow \frac{\partial f}{\partial z} = -y \sin z + x \cos z - \cos x + h'(z)$$

Luego  $h'(z) = 0$

$$f = y \cos z + x \sin z - z \cos x + K \rightarrow \text{función potencial}$$

$$I_1 = f(c_{end}) - f(c_{start})$$

$F_2$  no derive de un potencial puesto que  $\nabla \times F \neq \vec{0}$

# Finding a potential function for conservative vector fields

Este ejemplo está tomado de:

[https://mathinsight.org/conservative\\_vector\\_field\\_find\\_potential](https://mathinsight.org/conservative_vector_field_find_potential)

The process of finding a potential function of a **conservative vector field** is a multi-step procedure that involves both integration and differentiation, while paying close attention to the variables you are integrating or differentiating with respect to. For this reason, given a vector field

$\mathbf{F}$ , we recommend that you first **determine that** that

$\mathbf{F}$  is indeed conservative before beginning this procedure. That way you know a potential function exists so the procedure should work out in the end.

In this page, we focus on finding a potential function of a two-dimensional conservative vector field. We address three-dimensional fields in [another page](#).

We introduce the procedure for finding a potential function via an example. Let's use the vector field

$$\mathbf{F}(x, y) = (y \cos x + y^2, \sin x + 2xy - 2y).$$

The first step is to check if  $\mathbf{F}$  is conservative. Since

$$\begin{aligned}\frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(\sin x + 2xy - 2y) = \cos x + 2y \\ \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(y \cos x + y^2) = \cos x + 2y,\end{aligned}$$

we conclude that the scalar curl of  $\mathbf{F}$  is zero, as

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

Next, we observe that  $\mathbf{F}$  is defined on all of  $\mathbf{R}^2$ , so there are no **tricks to worry about**. The vector field  $\mathbf{F}$  is **indeed conservative**.

Since  $\mathbf{F}$  is conservative, we know there exists some potential function  $f$  so that  $\nabla f = \mathbf{F}$ . As a first step toward finding  $f$ , we observe that the condition  $\nabla f = \mathbf{F}$  means that

$$\begin{aligned}\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) &= (F_1, F_2) \\ &= (y \cos x + y^2, \sin x + 2xy - 2y).\end{aligned}$$

This vector equation is two scalar equations, one for each component. We need to find a function  $f(x, y)$  that satisfies the two conditions

$$\frac{\partial f}{\partial x}(x, y) = y \cos x + y^2 \quad (1)$$

and

$$\frac{\partial f}{\partial y}(x, y) = \sin x + 2xy - 2y. \quad (2)$$

Let's take these conditions one by one and see if we can find an  $f(x, y)$  that satisfies both of them. (We know this is possible since  $\mathbf{F}$  is conservative. If  $\mathbf{F}$  were path-dependent, the procedure that follows would hit a snag somewhere.)

Let's start with condition (1). We can take the equation

$$\frac{\partial f}{\partial x}(x, y) = y \cos x + y^2,$$

and treat  $y$  as though it were a number. In other words, we pretend that the equation is

$$\frac{df}{dx}(x) = a \cos x + a^2$$

for some number  $a$ . We can integrate the equation with respect to  $x$  and obtain that

$$f(x, y) = a \sin x + a^2 x + C.$$

But, then we have to remember that  $a$  really was the variable  $y$  so that

$$f(x, y) = y \sin x + y^2 x + C.$$

But actually, that's not right yet either. Since we were viewing  $y$  as a constant, the integration "constant"  $C$  could be a function of  $y$  and it wouldn't make a difference. The **partial derivative** of any function of  $y$  with respect to  $x$  is zero. We can replace  $C$  with any function of  $y$ , say  $g(y)$ , and condition (1) will be satisfied. A new expression for the potential function is

$$f(x, y) = y \sin x + y^2 x + g(y). \quad (3)$$

If you are still skeptical, try taking the partial derivative with respect to  $x$  of  $f(x, y)$  defined by equation (3). Since  $g(y)$  does not depend on  $x$ , we can conclude that  $\frac{\partial}{\partial x}g(y) = 0$ . Indeed, condition (1) is satisfied for the  $f(x, y)$  of equation (3).

Now, we need to satisfy condition (2). We can take the  $f(x, y)$  of equation (3) (so we know that condition (1) will be satisfied) and take its partial derivative with respect to  $y$ , obtaining

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}(y \sin x + y^2 x + g(y)) \\ &= \sin x + 2yx + \frac{dg}{dy}(y).\end{aligned}$$

Comparing this to condition (2), we are in luck. We can easily make this  $f(x, y)$  satisfy condition (2) as long as

$$\frac{dg}{dy}(y) = -2y.$$

If the vector field  $\mathbf{F}$  had been path-dependent, we would have found it impossible to satisfy both condition (1) and condition (2). We would have run into trouble at this point, as we would have found that  $\frac{dg}{dy}$  would have to be a function of  $x$  as well as  $y$ . Since  $\frac{dg}{dy}$  is a function of  $y$  alone, our calculation verifies that  $\mathbf{F}$  is conservative.

If we let

$$g(y) = -y^2 + k$$

for some constant  $k$ , then

$$\frac{\partial f}{\partial y}(x, y) = \sin x + 2yx - 2y,$$

and we have satisfied both conditions.

Combining this definition of  $g(y)$  with equation (3), we conclude that the function

$$f(x, y) = y \sin x + y^2 x - y^2 + k$$

is a potential function for  $\mathbf{F}$ . You can verify that indeed

$$\nabla f = (y \cos x + y^2, \sin x + 2xy - 2y) = \mathbf{F}(x, y).$$

With this in hand, calculating the integral

$$\int_C \mathbf{F} \cdot ds$$

is simple, no matter what path  $C$  is. We can apply the **gradient theorem** to conclude that the integral is simply  $f(\mathbf{q}) - f(\mathbf{p})$ , where  $\mathbf{p}$  is the beginning point and  $\mathbf{q}$  is the ending point of  $C$ . (For this reason, if  $C$  is a closed curve, the integral is zero.)

We might like to give a problem such as find

$$\int_C \mathbf{F} \cdot ds$$

where  $C$  is the curve given by the following graph.



The answer is simply

$$\begin{aligned}\int_C \mathbf{F} \cdot ds &= f(\pi/2, -1) - f(-\pi, 2) \\ &= -\sin \pi/2 + \frac{\pi}{2} - 1 + k - (2 \sin(-\pi) - 4\pi - 4 + k) \\ &= -\sin \pi/2 + \frac{9\pi}{2} + 3 = \frac{9\pi}{2} + 2\end{aligned}$$

(The constant  $k$  is always guaranteed to cancel, so you could just set  $k = 0$ .)

If the curve  $C$  is complicated, one hopes that  $\mathbf{F}$  is conservative. It's **always** a good idea to check if  $\mathbf{F}$  is conservative before computing its line integral

$$\int_C \mathbf{F} \cdot ds.$$

You might save yourself a lot of work.