

INTEGRACIÓN EN \mathbb{R}^P IV

19. MÉTODOS DE INTEGRACIÓN EN CONJUNTOS ACOTADOS B

1. Integración por cambio de variable. Recordatorio en \mathbb{R} .

Sea la función $g: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ y la integral $\int_a^b g(x) dx$.

Definimos un cambio de variable invertible $\begin{cases} c' \\ \varphi'(t) \neq 0 \text{ en } [c, d] \end{cases}$ de modo que $\varphi: [c, d] \rightarrow I = [a, b]$

$$t \rightarrow x = \varphi(t)$$

De esta forma, la integral se transforma del siguiente modo:

$$\int_a^b g(x) dx \xrightarrow[t=\varphi^{-1}(x)]{} \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} g(\varphi(t)) \cdot \varphi'(t) dt$$

① Cambio en la región de integración:

$$[x=a, x=b] \rightarrow [t=\varphi^{-1}(x=a)=c, t=\varphi^{-1}(x=b)=d]$$

② Cambio en la función:

$$g(x) \rightarrow g(\varphi(t))$$

③ Cambio en el diferencial de longitud:

$$x = \varphi(t) \rightarrow dx = \varphi'(t) dt$$

ESEMPIO $t = x^2 \xrightarrow{(x>0)} x = \sqrt{t}$

$$\int_0^{\pi^2} 2x \cos x^2 dx = \int_0^{\pi^4} \cos t dt =$$

$$= \sin t \Big|_0^{\pi^4} = \sin(\pi^4)$$

① $I = [0, \pi^2] \rightarrow J = [0, \pi^4]$

② $2x \cos x^2 \rightarrow 2x \cos t$

③ $dt = 2x dx$

2 Integración por cambio de variable. Caso \mathbb{R}^P

El objetivo es integrar una función $f: C \subset \mathbb{R}^P \rightarrow \mathbb{R}$. En este desarrollo asumiremos que dicha función es continua en C y que C es un conjunto simple.

Definimos la función cambio de variable $\varphi: C^* \subset \mathbb{R}^P \rightarrow \mathbb{R}^P$, que pedimos que sea:

* Inyectiva en C^*

* C^1 en C^*

* $|J\varphi| \neq 0$ en C^* (salvo, a lo sumo en un conjunto de contenido 0)

Analizaremos el efecto de esta última condición con el cambio de variable en polares.

Si se cumple lo anterior, tenemos que:

$$\int_C f(\bar{x}) d\bar{x} = \int_{C^*} f(\varphi(\bar{u})) \left| \det J\varphi(\bar{u}) \right| d\bar{u}$$

① Cambio en la región de integración: $C \rightarrow C^*$

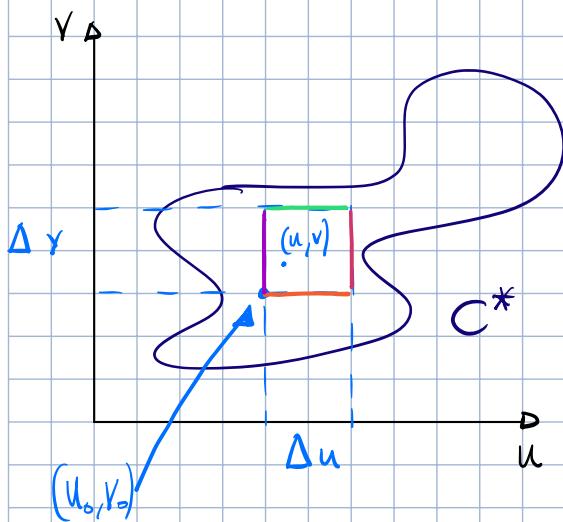
② Cambio en la función: $f(\bar{x}) \rightarrow f(\varphi(\bar{u}))$

③ Cambio en el diferencial: $d\bar{x} \rightarrow |\det J\varphi(\bar{u})| d\bar{u}$

Explicación de origen de la fórmula en \mathbb{R}^2 :

$$\varphi: C^* \subset \mathbb{R}^2 \rightarrow C \subset \mathbb{R}^2$$

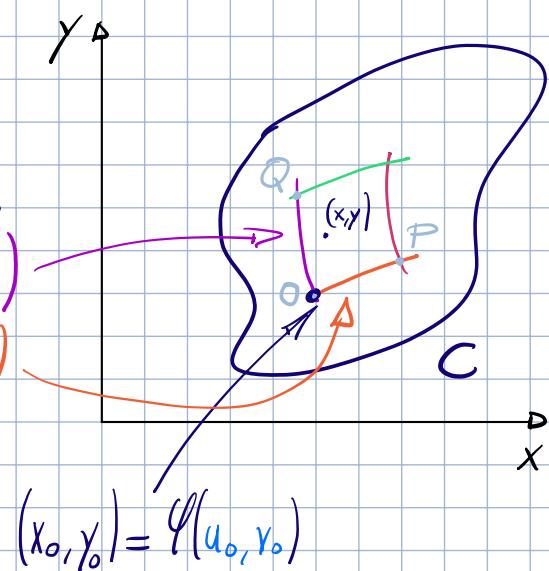
$$(u, v) \longrightarrow (x, y)$$



$$(x, y) = \varphi(u, v)$$

$$(x, y) = \varphi(u_0, v)$$

$$(x, y) = \varphi(u, v_0)$$



$$(x_0, y_0) = \varphi(u_0, v_0)$$

$$dA^* = \Delta u \Delta v$$

$$dA \approx \|\overline{OP} \times \overline{OQ}\|$$

$$\begin{aligned} \overline{OP} &= (x_p, y_p) - (x_0, y_0) = \varphi(u_0 + \Delta u, v_0) - \varphi(u_0, v_0) = \\ &= \frac{\varphi(u_0 + \Delta u, v_0) - \varphi(u_0, v_0)}{\Delta u} \cdot \Delta u \approx \varphi'_u \Delta u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) \Delta u \end{aligned}$$

$$\begin{aligned} \overline{OQ} &= (x_q, y_q) - (x_0, y_0) = \varphi(u_0, v_0 + \Delta v) - \varphi(u_0, v_0) = \\ &= \frac{\varphi(u_0, v_0 + \Delta v) - \varphi(u_0, v_0)}{\Delta v} \cdot \Delta v \approx \varphi'_v \Delta v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right) \Delta v \end{aligned}$$

$$dA \approx \|\overline{OP} \times \overline{OQ}\| = \begin{vmatrix} i & j & k \\ x_u' \Delta u & y_u' \Delta u & 0 \\ x_v' \Delta v & y_v' \Delta v & 0 \end{vmatrix} = \begin{vmatrix} x_u' \Delta u & y_u' \Delta u \\ x_v' \Delta v & y_v' \Delta v \end{vmatrix} = \begin{vmatrix} x_u' & y_u' \\ x_v' & y_v' \end{vmatrix} \Delta u \Delta v$$

De modo que, $dA = dx dy \xrightarrow{\text{se transforma en}} |\det J\varphi| du dv = |\det J\varphi| dA^*$

En resumen:

$$\textcircled{1} C \rightarrow C^* = \varphi^{-1}(C)$$

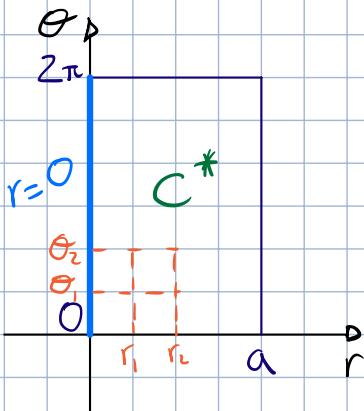
$$\textcircled{2} f(x,y) \rightarrow f(x(u,v), y(u,v)) = f^*(u,v)$$

$$\textcircled{3} dA = dx dy \rightarrow |\det J\varphi| du dv$$

$$\int_C f(x,y) dx dy = \int_{C^*} f^*(u,v) |\det J\varphi| du dv$$

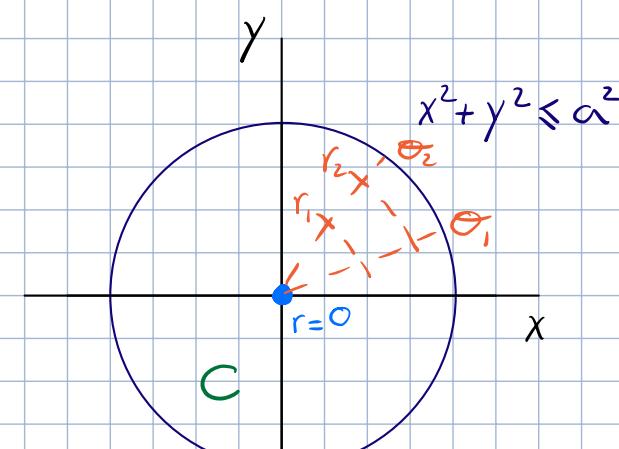
3 Cambios de variable usuales

\textcircled{1} Cambio a polares:



$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$\varphi: C^* = \mathbb{R}_+ \times [0, 2\pi] \rightarrow \mathbb{R}^2$$

$$(r, \theta) \rightarrow (x, y) = (r \cos \theta, r \sin \theta)$$

Veamos cómo se transforma en este caso un dA .

$$dA = dx dy = |\det J\varphi| dr d\theta \rightarrow r dr d\theta$$

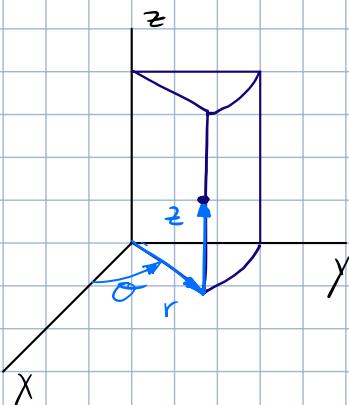
$$|\det J\varphi| = \begin{vmatrix} x'_r & y'_r \\ x'_\theta & y'_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

NOTA: $r=0$, $|\det J\varphi|=0$. Sin embargo, el segmento $r=0$ tiene contenido nulo, por lo que se puede aplicar la fórmula de cambio de variable sin problema.

②

Cambio a cilíndricas:

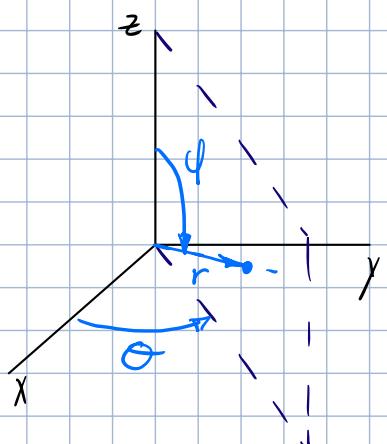
Es exactamente igual que el de polares, incluyendo el eje z .



$$\left. \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \right\} dV = r dr d\theta dz$$

③

Cambio a coordenadas esféricas.



$$\begin{aligned} x &= r\cos\theta\sin\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\varphi \end{aligned}$$

* NOTA: $\det(SY) = \det(SY^T)$

$$\det SY = \begin{vmatrix} x'_r & y'_r & z'_r \\ x'_\theta & y'_\theta & z'_\theta \\ x'_\varphi & y'_\varphi & z'_\varphi \end{vmatrix} = \begin{vmatrix} \cos\theta \sin\varphi & \sin\theta \sin\varphi & \cos\varphi \\ -r\sin\theta \sin\varphi & r\cos\theta \sin\varphi & 0 \\ r\cos\theta \cos\varphi & r\sin\theta \cos\varphi & -r\sin\varphi \end{vmatrix} =$$

Calculo el determinante por adjuntos

$$= \cos\varphi \begin{vmatrix} -r\sin\theta \sin\varphi & r\cos\theta \sin\varphi \\ r\cos\theta \cos\varphi & r\sin\theta \cos\varphi \end{vmatrix} +$$

$$-r\sin\varphi \begin{vmatrix} \cos\theta \sin\varphi & \sin\theta \sin\varphi \\ -r\sin\theta \sin\varphi & r\cos\theta \sin\varphi \end{vmatrix} = \cos\varphi (-r^2 \sin^2\theta \sin\varphi \cos\varphi)$$

$$-r^2 \cos^2\theta \sin\varphi \cos\varphi) - r\sin\varphi \underbrace{\left(r\cos^2\theta \sin^2\varphi + r\sin^2\theta \sin^2\varphi\right)}_1 =$$

$$-r^2 \sin\varphi \cos^2\varphi \underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_1 - r^2 \sin^3\varphi \underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_1 =$$

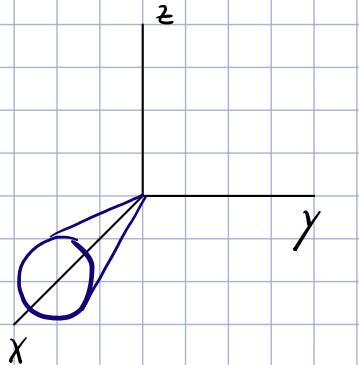
$$= -r^2 \sin\varphi (\cos^2\varphi + \sin^2\varphi) = -r^2 \sin\varphi$$

De modo que

$$dV = dx dy dz = r^2 \sin\varphi dr d\theta d\varphi$$

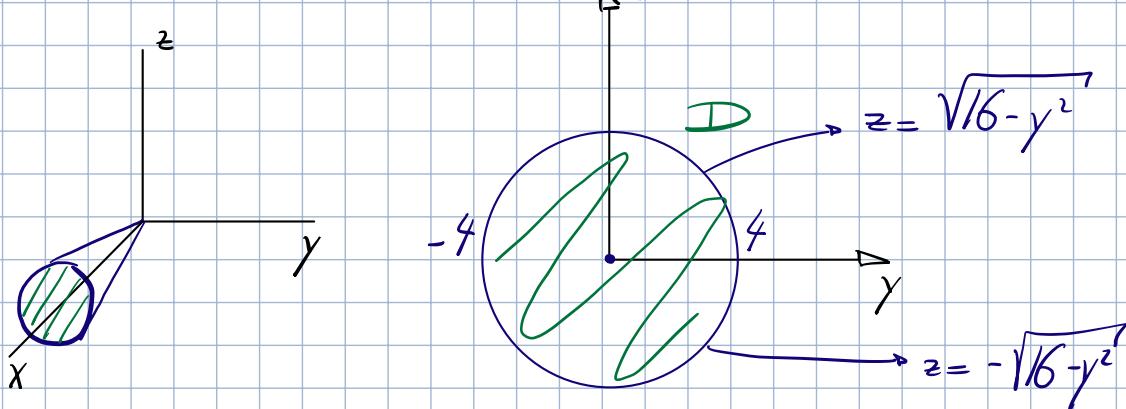
4 EJERCICIOS

① Calcular el volumen del sólido $E = \begin{cases} x \geq \sqrt{y^2 + z^2} \\ 0 \leq x \leq 4 \end{cases}$



Este problema ya lo hemos planteado anteriormente.
Haremos uso aquí del cambio de variable para calcular el resultado

Si planteamos este problema como uno de tipo ③, tenemos:

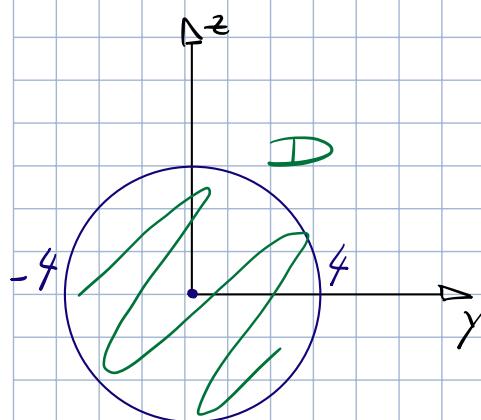


$$S = (x, y, z) \in \mathbb{R}^3 / \begin{cases} (y, z) \in D \\ \text{cono} \leq x \leq 4 \end{cases} \rightarrow \begin{cases} -\sqrt{16-y^2} \leq z \leq \sqrt{16-y^2} \\ -4 \leq y \leq 4 \\ \sqrt{y^2+z^2} \leq x \leq 4 \end{cases} \quad | D$$

De modo que

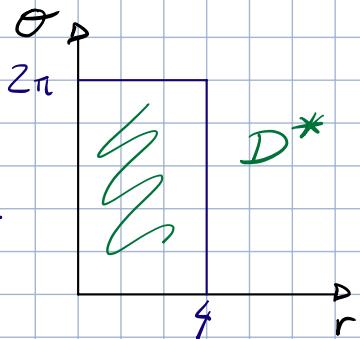
$$\iiint_E 1 dV = \iint_D \left(\int_{\sqrt{y^2+z^2}}^4 dx \right) dA = \iint_D (4 - \sqrt{y^2+z^2}) dA$$

Si usamos polares:



$$z = r \sin \theta$$

$$y = r \cos \theta$$



De modo que:

$$D^* = [0, 4] \times [0, 2\pi]$$

$$g = 4 - \sqrt{y^2 + z^2} = 4 - r$$

$$dA = r dr d\theta$$

$$\iint_D (4 - \sqrt{y^2 + z^2}) dA = \int_0^4 \int_0^{2\pi} (4 - r) r d\theta dr = 2\pi \int_0^4 (4r - r^2) dr =$$

$$2\pi \left(2r^2 - \frac{1}{3}r^3 \right) \Big|_0^4 = 2\pi \left(32 - \frac{1}{3}64 \right) = 64\pi \left(1 - \frac{2}{3} \right) = \frac{64}{3}\pi$$

Otra opción (con idéntico resultado) habría sido usar coordenadas cilíndricas.

$$x = x$$

$$y = r \cos \theta \xrightarrow{E}$$

$$z = r \sin \theta$$

$$r \leq x \leq 4$$

$$0 \leq r \leq 4$$

$$0 \leq \theta \leq 2\pi$$

$$\iiint_E dV = \int_0^4 \int_0^{2\pi} \int_r^4 1 \cdot r dx d\theta dr =$$

$$= \int_0^4 \int_0^{2\pi} r(4-r) d\theta dr = \dots$$

(2)

Calcular el volumen del sólido encerrado por:

$$(x^2 + y^2 = z^2 ; \quad x^2 + y^2 + z^2 = 9; \quad x^2 + y^2 + z^2 = 4)$$

cono

esfera 1

esfera 2

Este problema se puede expresar de forma sencilla usando coordenadas esféricas.

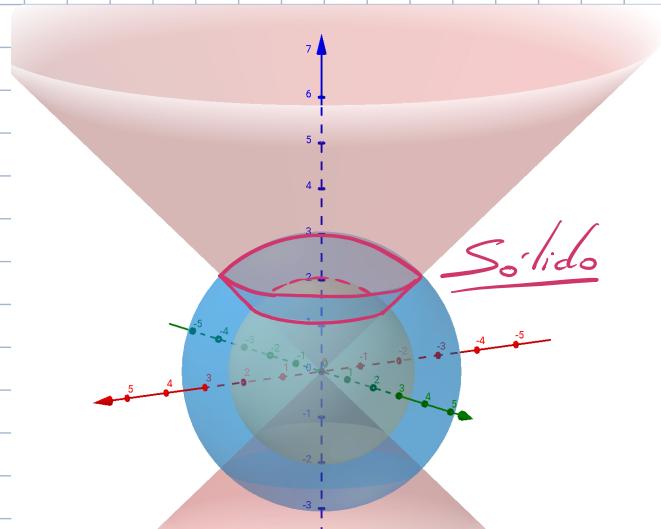
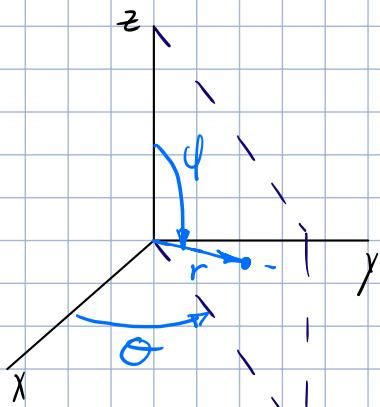


Imagen generada con Geogebra



$$x = r \cos \theta \sin \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \varphi$$

Sólido

$$0 \leq \theta \leq 2\pi$$

$$(\text{esfera 1}) \quad 2 \leq r \leq 3 \quad (\text{esfera 2})$$

$$0 \leq \varphi \leq \pi/4 \quad (\text{cono})$$

*

$$x^2 + y^2 = z^2$$

$$r^2 \cos^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta \sin^2 \varphi = r^2 \sin^2 \varphi = r^2 \cos^2 \varphi \rightarrow \sin^2 \varphi = \cos^2 \varphi \rightarrow$$

$$\rightarrow \varphi = \pi/4 *$$

De modo que la integral queda:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_2^3 r^2 \sin \varphi dr d\varphi d\theta = \int_0^{2\pi} d\theta \int_2^3 r^2 dr \int_0^{\pi/4} \sin \varphi d\varphi =$$

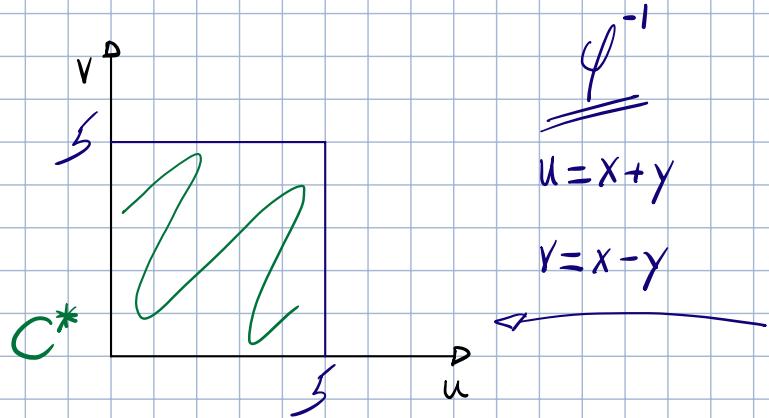
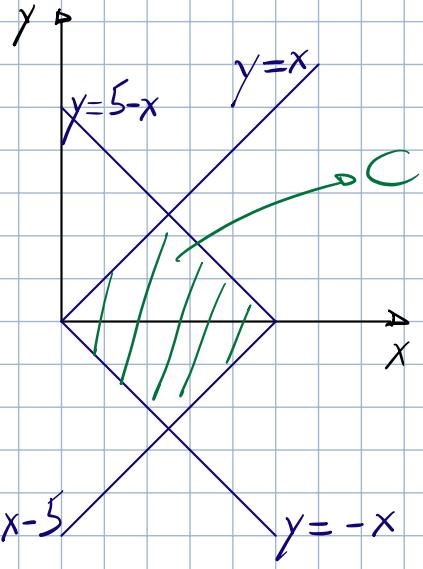
$$= 2\pi \left. \frac{r^3}{3} \right|_2^3 \left. (-\cos \varphi) \right|_0^{\pi/4} = \frac{2\pi}{3} (27 - 8) \left(1 - \frac{\sqrt{2}}{2} \right) = \frac{19}{3}\pi (2 - \sqrt{2}).$$

*

$$\int_c^d \int_a^b f(x) \cdot g(y) dx dy = \int_a^b f(x) dx \int_c^d g(y) dy$$

③ Calcular $\iint_C \frac{x+y}{x-y+1} dx dy$, con C

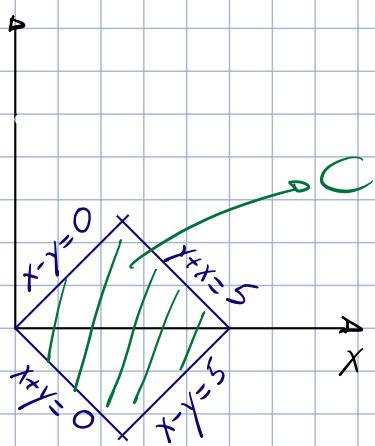
¿Puedo simplificar el problema con algún cambio de variable?



$$\varphi^{-1}$$

$$u = x + y$$

$$v = x - y$$



$$C^* = [0, 5] \times [0, 5]$$

La función se transforma en:

$$f(x,y) = \frac{x+y}{x-y+1} = \frac{u}{v+1} = f^*(u,v)$$

Y el diferencial de área es $dA = |\det J\varphi| du dv = \frac{1}{2} du dv$

$$\begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ u'_x & v'_x \\ u'_y & v'_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{2}$$

Teorema
de la función
inversa

(Jacobiano es constante por ser función lineal)

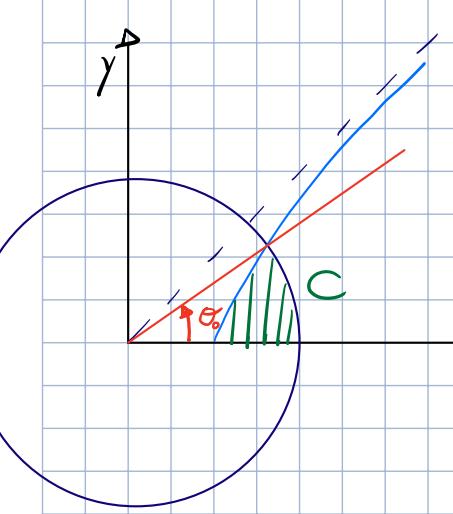
Luego la integral es:

$$\int_0^5 \int_0^5 \frac{u}{v+1} \frac{1}{2} du dv = \frac{1}{2} \int_0^5 u du \int_0^5 \frac{1}{v+1} dv = \frac{1}{4} 25 \ln 6$$

④ (V.8) Calcular $I = \iint_C g$ con $C \subset \mathbb{R}^2$ y $g: C \rightarrow \mathbb{R}^2$:

$C \Rightarrow$ $x \geq 0, y \geq 0, x^2 + y^2 = 2, x^2 - y^2 = 1$ | Conjunto limitado
por estas cuatro curvas

$$g(x,y) = \frac{xy}{(x^2+y^2)\sqrt{x^2-y^2}}$$



En este caso, un cambio de variable a polares podría ser útil.

Expresamos C en polares:

$$0 \leq \theta \leq \theta_0$$

hipérbola $\leq r \leq \sqrt{2}$

Tenemos que calcular el valor de θ_0 y expresar la hipérbola en polares:

$$x^2 - y^2 = 1 \rightarrow r^2 (\cos^2 \theta - \sin^2 \theta) = 1 \rightarrow \text{hipérbola}$$

$$r = \sqrt{2} \rightarrow \text{circunferencia}$$

$\theta_0 \rightarrow$ Corte entre hipérbola y circunferencia.

θ para el que $r = \sqrt{2}$ en ec. de hipérbola.

$$(\cos^2 \theta_0 - \sin^2 \theta_0) = 1/2 \rightarrow \cos 2\theta_0 = \frac{1}{2} \rightarrow$$

$$\rightarrow 2\theta_0 = \frac{\pi}{3} \rightarrow \theta_0 = \frac{\pi}{6}$$

Por otro lado, podemos expresar el valor r de la hipérbola

$$r_{\text{hip}} = \frac{1}{\sqrt{\cos^2 \theta - \sin^2 \theta}}, \text{ de modo que } C^* \text{ es}$$

$$C^* = \begin{cases} 1 & 0 \leq \theta \leq \frac{\pi}{6} \\ \frac{1}{\sqrt{\cos^2 \theta - \sin^2 \theta}} & \leq r \leq \sqrt{2} \end{cases}$$

En lo que se refiere a la función, tenemos que:

$$f(x,y) = \frac{xy}{(x^2+y^2)\sqrt{x^2-y^2}} = \frac{r^2 \cos \theta \sin \theta}{r^2 r \sqrt{\cos^2 \theta - \sin^2 \theta}} = \frac{\cos \theta \sin \theta}{r \sqrt{\cos^2 \theta - \sin^2 \theta}}$$

Y el diferencial de área es:

$$dA = r dr d\theta$$

De modo que la integral queda:

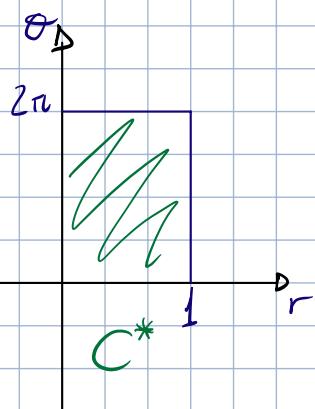
$$\begin{aligned} I &= \int_0^{\pi/6} \int_{\sqrt{2}}^{1/2} \frac{\cos \theta \sin \theta}{r \sqrt{\cos^2 \theta - \sin^2 \theta}} r dr d\theta = \int_0^{\pi/6} \frac{1/2 \sin 2\theta}{\sqrt{\cos 2\theta}} \left(\sqrt{2} - \frac{1}{\sqrt{\cos 2\theta}} \right) d\theta = \\ &= \frac{\sqrt{2}}{2} \int_0^{\pi/6} \frac{\sin 2\theta}{\sqrt{\cos 2\theta}} d\theta - \frac{1}{2} \int_0^{\pi/6} \frac{\sin 2\theta}{\cos 2\theta} d\theta = \\ &= -\frac{\sqrt{2}}{2} \left[\sqrt{\cos 2\theta} \right]_0^{\pi/6} + \frac{1}{4} \left[\ln |\cos 2\theta| \right]_0^{\pi/6} = -\frac{\sqrt{2}}{2} \left(\sqrt{\frac{1}{2}} - 1 \right) \\ &\quad + \frac{1}{4} \ln \left(\frac{1}{2} \right) = -\frac{1}{2} + \frac{\sqrt{2}}{2} - \frac{1}{4} \ln 2 \end{aligned}$$

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

⑤ (V9) Calcular la integral doble:

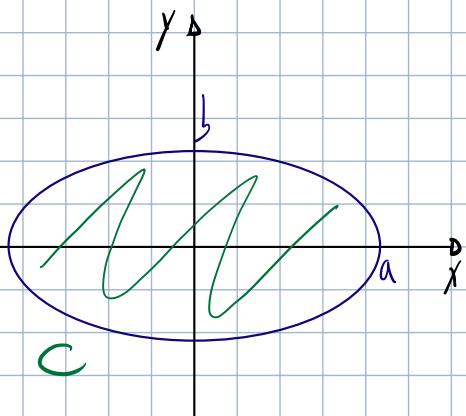
$$I = \iint_C \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} dx dy \quad \text{con } C = \left\{ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \right\}$$

Para este problema, puedo crear un cambio de variable inspirado en polares.



$$\begin{aligned} \frac{x}{a} &= r \cos \theta \\ \frac{y}{b} &= r \sin \theta \end{aligned}$$

Diagram illustrating the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) . A blue arrow points from the origin to a point on the curve, with a blue bracket indicating the radius r and a blue arrow indicating the angle θ .



Vemos que $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 \leq 1$

De modo que la función $f(x, y) \rightarrow f^*(r, \theta)$

$$f(x, y) = \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} = \sqrt{1 - r^2} = f^*(r, \theta)$$

Y el diferencial de área $dA = |J| d\theta dr = abr dr d\theta$

$$|J| = \begin{vmatrix} x'_r & y'_r \\ x'_{\theta} & y'_{\theta} \end{vmatrix} = \begin{vmatrix} a \cos \theta & b \sin \theta \\ -a r \sin \theta & b r \cos \theta \end{vmatrix} = ab \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = abr$$

De modo que la integral queda:

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} \cdot abr dr d\theta = 2\pi ab \left(-\frac{1}{2}\right) \frac{2}{3} \int_0^1 \frac{3}{2} (-2r) \sqrt{1 - r^2} dr = \\ &= -\frac{2}{3} \pi ab \left[\left(1 - r^2\right)^{3/2} \right] \Big|_0^1 = \frac{2\pi}{3} ab \end{aligned}$$

⑥ (V10) Sea D la región definida mediante

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 16y^2 + z^2 \leq 25, x^2 + 16y^2 \leq 16, z \geq 0\}$$

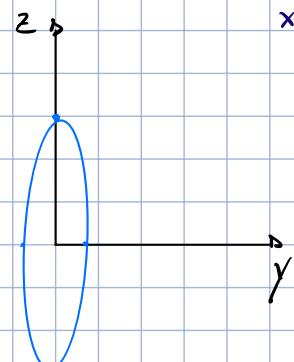
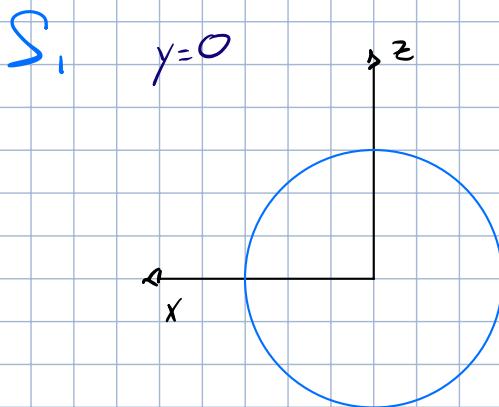
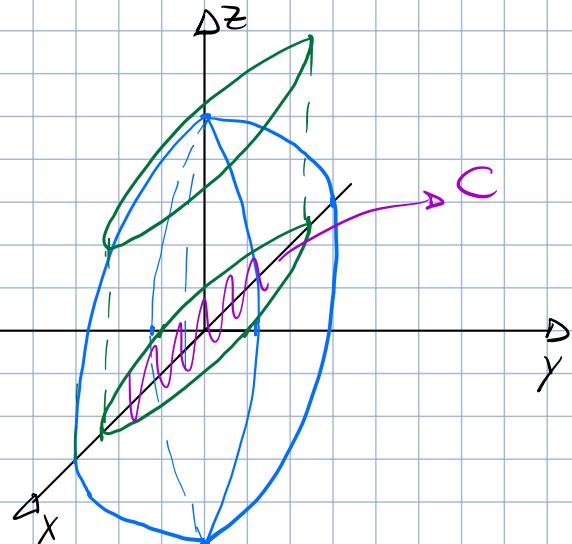
Σ_1 Σ_2

Sea $f: D \rightarrow \mathbb{R}$ la función definida mediante

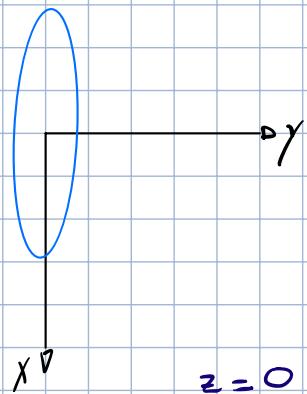
$$f(x, y, z) = \frac{x^2}{\sqrt{25 - x^2 - 16y^2}}$$

Calcular $I = \int_D f$.

La región de integración
son los pts de \mathbb{R}^3 que estén
dentro de ambas regiones a la
vez.

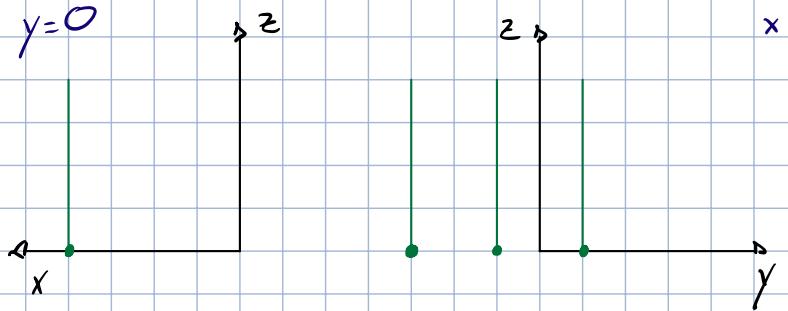


$$x^2 + y^2 \leq 16$$

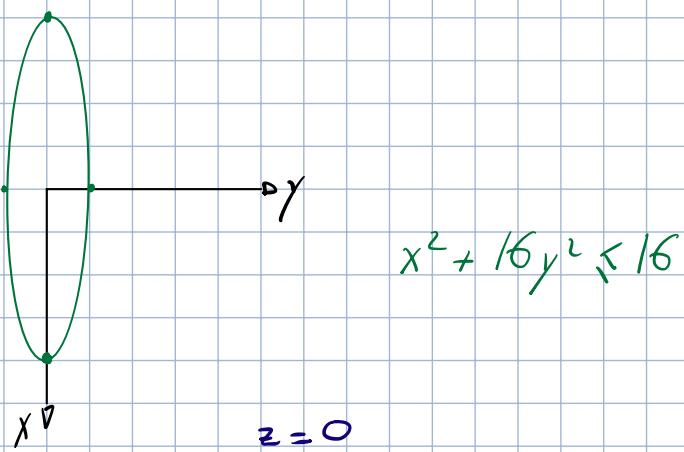


$$16y^2 + x^2 \leq 25$$

S_2 $y=0$ $\rightarrow z$ $z \rightarrow$ $x=0$



$$x^2 \leq 16$$



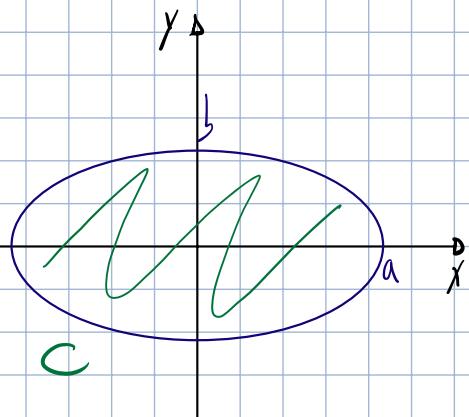
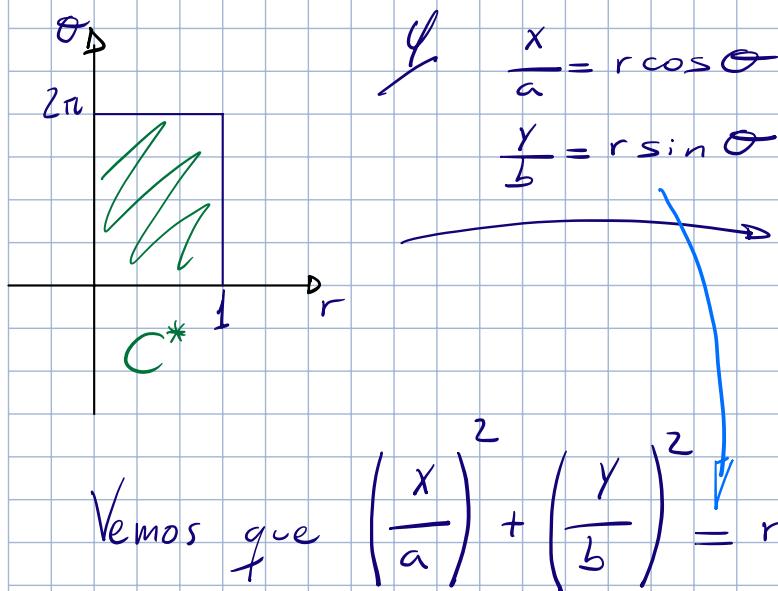
$$x^2 + 16y^2 \leq 16$$

$$\int_C \left(\iint_0^{\sqrt{25-x^2-16y^2}} f(x, y, z) dz \right) dA =$$

$$\int_C \left(\iint_0^{\sqrt{25-x^2-16y^2}} \frac{x^2}{\sqrt{25-x^2-16y^2}} dz \right) dA =$$

$$= \int_C \frac{x^2 \sqrt{25 - x^2 - 16y^2}}{\sqrt{25 - x^2 - 16y^2}} dA = \int_C x^2 dA$$

Para este problema, puedo crear un cambio de variable inspirado en polares.



Vemos que $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 \leq 1$

En este caso particular $x^2 + 16y^2 = 16 \rightarrow \left(\frac{x}{4}\right)^2 + y^2 = 1$, luego:

$$\begin{cases} \frac{x}{4} = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \rightarrow \text{con } r \in [0, 1], \theta \in [0, 2\pi]$$

$$\begin{cases} dA = 4r dr d\theta \\ f(x, y) = x^2 = 16r^2 \cos^2 \theta \end{cases} \rightarrow I = 16 \cdot 4 \int_0^1 r^3 dr \int_{0}^{2\pi} \cos^2 \theta d\theta = 16\pi$$

$$\int_a^b \cos \theta \cos \theta d\theta = \cos \theta \sin \theta \Big|_a^b + \int_a^b \sin^2 \theta d\theta =$$

$u = \cos \theta \quad du = -\sin \theta d\theta$

$v = \sin \theta \quad dv = \cos \theta d\theta$

$$= \cos \theta \sin \theta \Big|_a^b + \int_a^b 1 - \cos^2 \theta d\theta = \cos \theta \sin \theta \Big|_a^b + \theta \Big|_a^b - \int_a^b \cos^2 \theta d\theta$$

$$\int_a^b \cos^2 \theta d\theta + \int_a^b \cos^2 \theta d\theta = \cos \theta \sin \theta \Big|_a^b + \theta \Big|_a^b$$

$$\int_a^b \cos^2 \theta d\theta = \frac{1}{2} \left(\cos \theta \sin \theta \Big|_a^b + \theta \Big|_a^b \right)$$

⑦ Ejemplo adicional Para caso
 (Marsden-Tromba)

example 7

Let W be the ball of radius R and center $(0, 0, 0)$ in \mathbb{R}^3 . Find the volume of W .

solution

The volume of W is $\iiint_W dx dy dz$. This integral may be evaluated by reducing it to iterated integrals or by regarding W as a volume of revolution, but let us evaluate it here by using spherical coordinates. We get

$$\begin{aligned} \iiint_W dx dy dz &= \int_0^\pi \int_0^{2\pi} \int_0^R \rho^2 \sin \phi d\rho d\theta d\phi = \frac{R^3}{3} \int_0^\pi \int_0^{2\pi} \sin \phi d\theta d\phi \\ &= \frac{2\pi R^3}{3} \int_0^\pi \sin \phi d\phi = \frac{2\pi R^3}{3} \{-[\cos(\pi) - \cos(0)]\} = \frac{4\pi R^3}{3}, \end{aligned}$$

which is the standard formula for the volume of a solid sphere. ▲