24. CAMPOS CONSERVATIVOS

1 Teorema fundamental del cálculo en integrales de línea

$\mathbb{R}$

TFC: $F(x)=\int_a^x f(t)\,dt \longrightarrow F'(x)=f(x)$

De modo que:

$$\int_a^b f(t)\,dt = F(b)-F(a)$$

Segundo TFC o regla de Barrow

Nota: $\int_a^b \frac{df}{dt}\,dt = f(b)-f(a)$

Extensión a integrales curvilíneas

Sea $f:\mathbb{R}^3\to\mathbb{R}\in C^1$, y sea $C: \bar{r}(t): I=[a,b]\to\mathbb{R}^3$ una curva regular, entonces:

$$\int_C \nabla f\cdot d\bar{s} = f(\bar{r}(b)) - f(\bar{r}(a))$$

Demostración:

$$\bar{r}(t)=(x(t),\,y(t),\,z(t)) \longrightarrow \bar{r}'(t)=(x'(t),\,y'(t),\,z'(t))$$

$$\int_C \nabla f\cdot d\bar{s} = \int_a^b \nabla f\Big|_{\bar{r}(t)}\cdot \bar{r}'(t)\,dt = \int_a^b \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\Big|_{\bar{r}(t)}\cdot\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)dt =$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt \underset{*}{=} \int_a^b \frac{df(\bar{r}(t))}{dt}\,dt = f(\bar{r}(b))-f(\bar{r}(a))$$

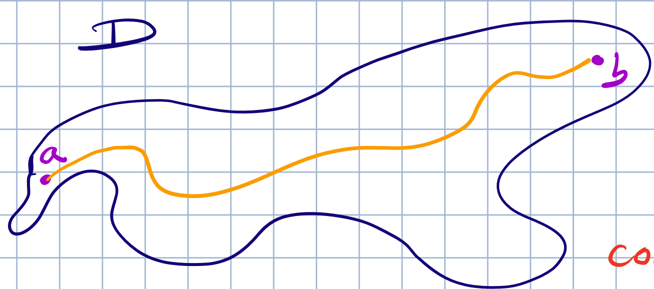
Regla de la cadena

$$* \quad f(\bar{r}(t)) \longrightarrow \frac{d}{dt}f(\bar{r}(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

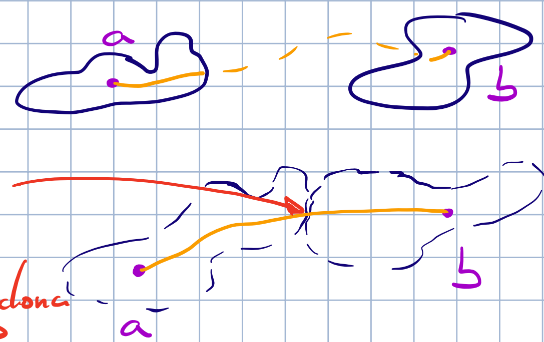
2 Dominios simplemente conexos

Definición: un conjunto $D \subset \mathbb{R}^p$ es conexo por arcos si dados dos pts cualesquiera $a, b \in D$, es posible encontrar una curva continua que lleve de un pto a otro sin abandonar D.

Conexo por arcos



No conexo por arcos



Un dominio D es simplemente conexo si es conexo por arcos y cualquier curva $C \subset D$, cerrada y continua, puede reducirse de forma continua sin salirse de D hasta convertirse en un pto que pertenece a D.

EJEMPLOS:

a $\mathbb{R}^2$ y $\mathbb{R}^3$ son simplemente conexos.

b $\mathbb{R}^3 - \{(0,0,0)\}$ es simplemente conexo

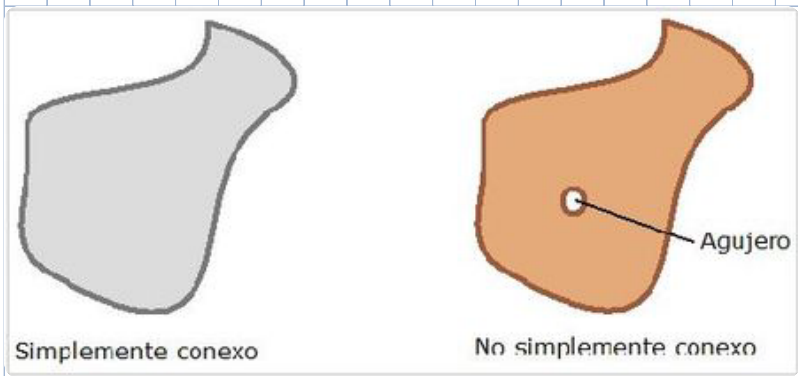
c $\mathbb{R}^3 - \{$Un conjunto finito de puntos$\}$ es simplemente conexo

d $\mathbb{R}^2 - \{(0,0)\}$ no es simplemente conexo.

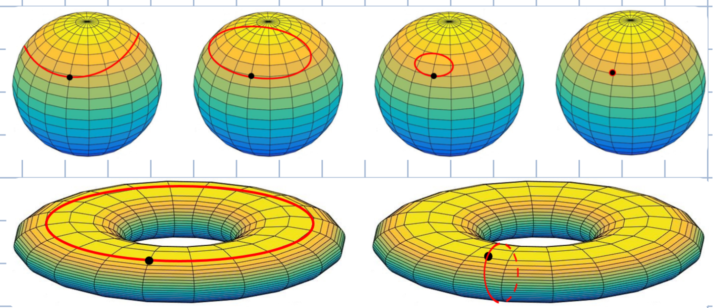
e $\mathbb{R}^2 - \{$Un conjunto finito de puntos$\}$ no es simplemente conexo

8 En $\mathbb{R}^2$, un círculo es simplemente conexo, pero una corona circular no lo es

9 En $\mathbb{R}^3$, una esfera es simplemente conexo y una corona esférica tb lo es. Sin embargo una esfera con un taladro que la atraviese completamente no lo es.



Hacer un agujero, o quitar un punto interior a un conjunto simplemente conexo en dos dimensiones hace que deje de ser simplemente conexo


Como puede observarse, en la esfera, al no tener agujeros siempre es posible "tirar" del lazo y deformarlo hasta reducirlo a un único punto. En el toro, pasa justo lo contrario, al tener un agujero, no podemos reducir cualquier curva a un punto. Para que la esfera (tres dimensiones) deje de ser simplemente conexo, no vale con hacerle un agujero (o quitarle un punto) hay que atravesarla por completo.

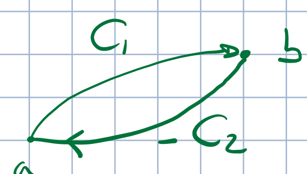
3 Campos conservativos

Sea $\bar{F}: \mathbb{R}^p \to \mathbb{R}^p$ un campo vectorial de clase C^1 definido en $D \subset \mathbb{R}^p$ ($p=2$ o $p=3$), siendo D un conjunto simplemente conexo.

Se dice que $\bar{F}$ es un campo vectorial conservativo si cumple cualquiera de las siguientes condiciones equivalentes entre sí:

a La circulación de F a lo largo de una curva C cerrada y simple a trozos es nula. Es decir,

$$\oint_C \bar{F} \cdot d\bar{s} = 0$$



(Prop. b)

b La circulación es independiente del camino. Es decir, para cualesquiera C_1 y C_2 curvas simples de clase C^k $(k \geq 1)$ a trozos, con los mismos pts extremos y la misma orientación, se verifica que:

$$\int_{C_1} \bar{F} \cdot d\bar{s} = \int_{C_2} \bar{F} \cdot d\bar{s} \overset{(Prop.\ c)}{=} \int_{C} \nabla U \cdot d\bar{s} = U(\bar{r}(b)) - U(\bar{r}(a))$$

c F es un campo gradiente, es decir existe alguna función escalar $U: \mathbb{R}^p \to \mathbb{R}$ de clase C^2 tal que:

$$\bar{F} = \nabla U$$

d El campo vectorial $\bar{F}$ es irrotacional, es decir

$$\nabla \times \bar{F} = 0 \qquad \overset{Prop.\ c.}{(\nabla \times (\nabla U)) = 0} \to \text{Por ser } U \in C^2$$

Caso de que el dominio de definición del campo <u>no</u> sea un conjunto simplemente conexo.

Si D <u>no</u> es un conjunto simplemente conexo y $\bar{F}$ es irrotacional ($\nabla \times \bar{F} = 0$ en D) puede ocurrir que $\bar{F}$ no derive de un potencial (condición c) y tampoco se cumplan las condiciones a y b.

Sin embargo todas las propiedades seguirían siendo aplicables en subconjuntos de D, siempre que estos sean simplemente conexos.

En estos casos, $\nabla \times \bar{F} = 0$ sigue siendo cond necesaria, ya que si no, es imposible que $\nabla f = \bar{F}$. $\nabla \times (\nabla f)$ siempre es igual a 0.

EJEMPLO

Sea $\bar{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ definida en $D = \mathbb{R}^2 - \{0,0\}$

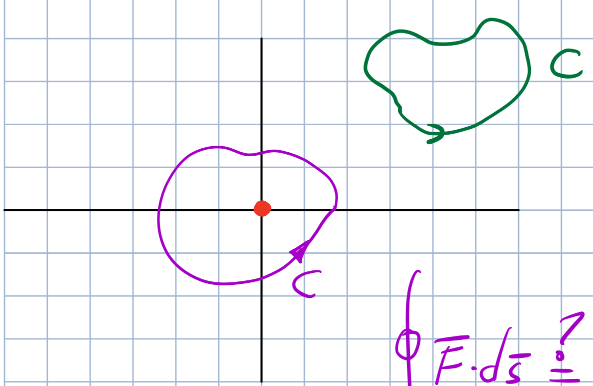
(dominio no simplemente conexo)

$$\nabla \times \bar{F} = (0, 0, F'_{2x} - F'_{1y}) = \bar{0}$$

$$F'_{2x} = \frac{x^2+y^2-2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$F'_{1y} = \frac{-x^2-y^2+2y^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

Entonces, ¿se puede asegurar que $\oint_C \bar{F} \cdot d\bar{s} = 0$? No necesariamente



$\oint_C \bar{F} \cdot d\bar{s} = 0 \longrightarrow$ Si no rodeo el punto conflictivo, todo va bien

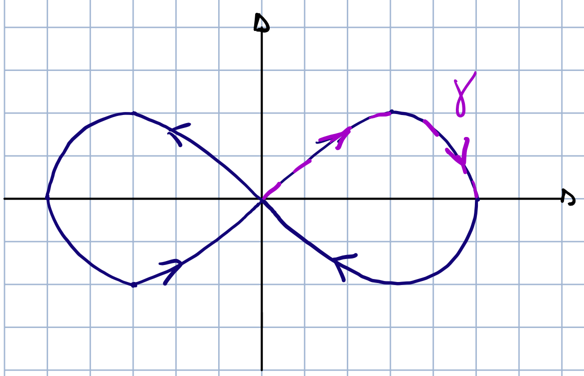
$\oint_C \bar{F} \cdot d\bar{s} \overset{?}{=} 0 \longrightarrow$ En este caso, sin embargo, no puedo asegurar nada.

Probamos a calcular una: $C \equiv \bar{r}(t) = (\cos\theta, \sin\theta)$

$$\left. \begin{array}{l} \bar{F}(\bar{r}(t)) = (-\sin\theta, \cos\theta) \\ d\bar{s} = (-\sin\theta, \cos\theta)\, d\theta \end{array} \right| \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = 2\pi \neq 0$$

EJERCICIO:

Calcular $\int_\gamma \bar{F}\cdot d\bar{s}$ siendo γ la porción de la lemniscata de Bernoulli que va de $(0,0)$ a $(1,0)$ y $\bar{F}=(2x,-2y)$.



$$x = \sin t / (1+\cos^2 t)$$

$$y = \sin t \cos t / (1+\cos^2 t)$$

$$0 \leq t \leq \pi/2$$

Primero vamos a intentar resolver este problema integrando sobre la curva que nos dican:

$$\int_\gamma \bar{F}\cdot d\bar{s} = \int_a^b \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt$$

1.- $[a,b]=[0,2\pi]$

2.- $\bar{F}(\bar{r}(t)) = (2x(t), -2y(t)) = \dfrac{\sin t}{1+\cos^2 t}(1, \cos t)$

3.- $d\bar{s} = \bar{r}'(t)\,dt$

$$x'(t) = \frac{\cos t\,(1+\cos^2 t) + \sin t\cdot 2\cos t \sin t}{(1+\cos^2 t)^2}$$

$$y'(t) = \frac{(\cos^2 t - \sin^2 t)(1+\cos^2 t) + \sin t\cos t\cdot 2\cos t\sin t}{(1+\cos^2 t)^2}$$

[...] ¿No hay una opción más sencilla?

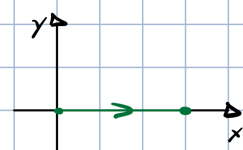
Vamos a comprobar si el campo es conservativo. El dominio de $\bar{F}$ es $\mathbb{R}^2$ (dominio simplemente conexo) y es de clase C^∞. De forma que si es irrotacional, será conservativo.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial x & \partial y & \partial z \\ 2x & -2y & 0 \end{vmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \bar{0} \longrightarrow \bar{F} \text{ conservativo de modo que:}$$

$\int_C \bar{F} \cdot d\bar{s}$ indep. del camino.

Puedo usar una curva más sencilla siempre que coincida el punto de partida y de final con γ.

① Recta que une $(0,0)$ y $(1,0)$



$$\bar{r}(t) = (0,0) + t((1,0) - (0,0)) = (t,0) \quad , \quad t \in [0,1]$$

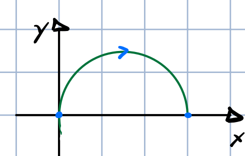
1.- $[a,b] = [0,1]$

2.- $\bar{F}(\bar{r}(t)) = (2x(t), -2y(t)) = (2t, 0)$

3.- $d\bar{s} = \bar{r}'(t)\,dt = (1,0)\,dt$

$$\int_0^1 (2t,0)\cdot(1,0)\,dt = \int_0^1 2t\,dt = 1$$

② Circunferencia de radio $1/2$ y centro en $1/2$



$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4} \quad \left| \begin{array}{l} x - \tfrac{1}{2} = \tfrac{1}{2}\cos\theta \\ y = \tfrac{1}{2}\sin\theta \end{array} \right| \quad \theta \in [0, \pi]$$

Cuidado, que esta parametrización tiene sentido contrario. Tendré que cambiar al final el signo.

1.- $[a,b] = [0, \pi]$

2.- $\bar{F}(\bar{r}(t)) = \left(2\left(\tfrac{1}{2}\cos\theta + \tfrac{1}{2}\right), -2\,\tfrac{1}{2}\sin\theta\right)$

3.- $d\bar{s} = \bar{r}'(t)\,dt = \left(-\tfrac{1}{2}\sin\theta, \tfrac{1}{2}\cos\theta\right) d\theta$

$$\int_0^{\pi} (\cos\theta + 1, -\sin\theta) \cdot \frac{1}{2}(-\sin\theta, \cos\theta)\, d\theta =$$

$$= \int_0^{\pi} (-\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta)\, d\theta = \int_0^{\pi} (-\sin 2\theta - \sin\theta)\, d\theta =$$

$$= \frac{1}{2}\cos 2\theta + \cos\theta \Big|_0^{\pi} = \frac{1-1}{2} + \frac{-1-(+1)}{2} = -1$$

Otra posible opción es sacar el potencial del que deriva el campo ya que:

$$\nabla f = \bar{F} \longrightarrow \int_{\gamma} \bar{F} \cdot d\bar{s} = \int_{\gamma} \nabla f \cdot d\bar{s} = f(\bar{r}(b)) - f(\bar{r}(a))$$

$$\frac{\partial f}{\partial x} = 2x \longrightarrow f(x,y) = x^2 + c(y) \xrightarrow{\frac{\partial}{\partial y}} f'_y(x,y) = c'(y)$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{dc}{dy}(y) = -2y \longrightarrow c(y) = -y^2 + K$$

$$f(x,y) = x^2 - y^2 + K$$

$$\int_{\gamma} \nabla f \cdot d\bar{s} = f(\bar{r}(b)) - f(\bar{r}(a)) = x^2 - y^2 + K \Big|_{(0,0)}^{(1,0)} = 1 + K - K = 1$$

VII.21 a) Analizar si son independientes del camino las siguientes integrales

$$I_1 = \int_C (\operatorname{sen} z + z \operatorname{sen} x)\,\mathrm{d}x + \cos z\,\mathrm{d}y + (x\cos z - \cos x - y\operatorname{sen} z)\,\mathrm{d}z,$$

$$I_2 = \int xy\,\mathrm{d}x + y^2\,\mathrm{d}y + z\,\mathrm{d}z,$$

calculando, en caso afirmativo, la función potencial.

b) Demostrar que $I_3 = \int_C F(x,y,z)(\,\mathrm{d}x + \mathrm{d}y + \mathrm{d}z)$, con $F:\mathbb{R}^3 \to \mathbb{R}$ de clase $\mathcal{C}^2$, es independiente del camino si y sólo si $F(x,y,z) = \phi(x+y+z)$ con $\phi:\mathbb{R}\to\mathbb{R}$ de clase $\mathcal{C}^2$

Resultados: I_1 no depende del camino. I_2 depende del camino.

ⓐ $\bar{F}_1(x,y,z) = (\sin z + z\sin x,\ \cos z,\ x\cos z - \cos x - y\sin z)$

Si $\nabla \times \bar{F}_1 = 0$ en $\mathbb{R}^3$, entonces el campo es conservativo

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin z + z\sin x & \cos z & x\cos z - \cos x - y\sin z \end{vmatrix} = \begin{bmatrix} -\sin z + \sin z \\ \cos z + \sin x - \cos z - \sin x \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

El campo es conservativo, luego procedemos a buscar la función potencial de la cual deriva.

$$\bar{F}_1 = \nabla f(x,y,z) \longrightarrow \frac{\partial f}{\partial x} = \sin z + z\sin x$$

$$\frac{\partial f}{\partial y} = \cos z$$

$$\frac{\partial f}{\partial z} = x\cos z - \cos x - y\sin z$$

(Integrando en $\frac{\partial f}{\partial y}$)

$$f = y\cos z + g(x,z) \longrightarrow \frac{\partial f}{\partial x} = g'_x(x,z) = \sin z + z\sin x$$

Integrando $\left(\frac{\partial f}{\partial x}\right)$

$$g(x,z) = x\sin z - z\cos x + h(z)$$

Luego

$$f = y\cos z + x\sin z - z\cos x + h(z) \rightarrow \frac{\partial f}{\partial z} = -y\sin z + x\cos z - \cos x + h'(z)$$

Luego $h'(z) = 0$

$$f = y\cos z + x\sin z - z\cos x + K \longrightarrow \text{función potencial}$$

$$I_1 = f(C_{end}) - f(C_{start})$$

F_2 no deriva de un potencial puesto que $\nabla \times \bar{F} \neq \bar{0}$

Finding a potential function for conservative vector fields

Este ejemplo está tomado de:
https://mathinsight.org/conservative_vector_field_find_potential

The process of finding a potential function of a conservative vector field is a multi-step procedure that involves both integration and differentiation, while paying close attention to the variables you are integrating or differentiating with respect to. For this reason, given a vector field
$\mathbf{F}$, we recommend that you first determine that that
$\mathbf{F}$ is indeed conservative before beginning this procedure. That way you know a potential function exists so the procedure should work out in the end.

In this page, we focus on finding a potential function of a two-dimensional conservative vector field. We address three-dimensional fields in another page.

We introduce the procedure for finding a potential function via an example. Let's use the vector field

$$\mathbf{F}(x, y) = (y \cos x + y^2, \sin x + 2xy - 2y).$$

The first step is to check if $\mathbf{F}$ is conservative. Since

$$\begin{aligned} \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(\sin x + 2xy - 2y) = \cos x + 2y \\ \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(y \cos x + y^2) = \cos x + 2y, \end{aligned}$$

we conclude that the scalar curl of $\mathbf{F}$ is zero, as

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

Next, we observe that $\mathbf{F}$ is defined on all of $\mathbf{R}^2$, so there are no tricks to worry about. The vector field $\mathbf{F}$ is indeed conservative.

Since $\mathbf{F}$ is conservative, we know there exists some potential function f so that $\nabla f = \mathbf{F}$. As a first step toward finding f, we observe that the condition $\nabla f = \mathbf{F}$ means that

$$\begin{aligned} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) &= (F_1, F_2) \\ &= (y \cos x + y^2, \sin x + 2xy - 2y). \end{aligned}$$

This vector equation is two scalar equations, one for each component. We need to find a function $f(x, y)$ that satisfies the two conditions

$$\frac{\partial f}{\partial x}(x, y) = y \cos x + y^2 \tag{1}$$

and

$$\frac{\partial f}{\partial y}(x, y) = \sin x + 2xy - 2y. \tag{2}$$

Let's take these conditions one by one and see if we can find an $f(x, y)$ that satisfies both of them. (We know this is possible since $\mathbf{F}$ is conservative. If $\mathbf{F}$ were path-dependent, the procedure that follows would hit a snag somewhere.)

Let's start with condition (1). We can take the equation

$$\frac{\partial f}{\partial x}(x, y) = y \cos x + y^2,$$

and treat y as though it were a number. In other words, we pretend that the equation is

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = a \cos x + a^2$$

for some number a. We can integrate the equation with respect to x and obtain that

$$f(x) = a \sin x + a^2 x + C.$$

But, then we have to remember that a really was the variable y so that

$$f(x, y) = y \sin x + y^2 x + C.$$

But actually, that's not right yet either. Since we were viewing y as a constant, the integration "constant" C could be a function of y and it wouldn't make a difference. The partial derivative of any function of y with respect to x is zero. We can replace C with any function of y, say $g(y)$, and condition (1) will be satisfied. A new expression for the potential function is

$$f(x, y) = y \sin x + y^2 x + g(y). \tag{3}$$

If you are still skeptical, try taking the partial derivative with respect to x of $f(x, y)$ defined by equation (3). Since $g(y)$ does not depend on x, we can conclude that $\frac{\partial}{\partial x} g(y) = 0$. Indeed, condition (1) is satisfied for the $f(x, y)$ of equation (3).

Now, we need to satisfy condition (2). We can take the $f(x, y)$ of equation (3) (so we know that condition (1) will be satisfied) and take its partial derivative with respect to y, obtaining

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}\left(y \sin x + y^2 x + g(y)\right) \\ &= \sin x + 2yx + \frac{\mathrm{d}g}{\mathrm{d}y}(y). \end{aligned}$$

Comparing this to condition (2), we are in luck. We can easily make this $f(x, y)$ satisfy condition (2) as long as

$$\frac{\mathrm{d}g}{\mathrm{d}y}(y) = -2y.$$

If the vector field $\mathbf{F}$ had been path-dependent, we would have found it impossible to satisfy both condition (1) and condition (2). We would have run into trouble at this point, as we would have found that $\frac{dg}{dy}$ would have to be a function of x as well as y. Since $\frac{dg}{dy}$ is a function of y alone, our calculation verifies that $\mathbf{F}$ is conservative.

If we let

$$g(y) = -y^2 + k$$

for some constant k, then

$$\frac{\partial f}{\partial y}(x, y) = \sin x + 2yx - 2y,$$

and we have satisfied both conditions.

Combining this definition of $g(y)$ with equation (3), we conclude that the function

$$f(x, y) = y \sin x + y^2 x - y^2 + k$$

is a potential function for $\mathbf{F}$. You can verify that indeed

$$\nabla f = (y \cos x + y^2, \sin x + 2xy - 2y) = \mathbf{F}(x, y).$$

With this in hand, calculating the integral

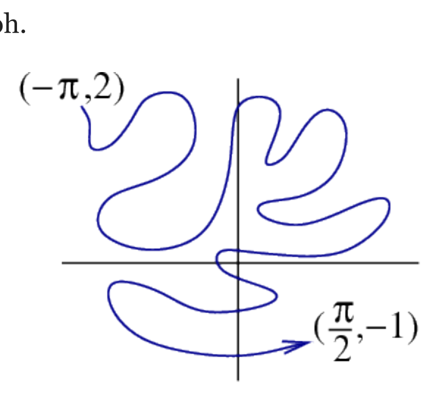
$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

is simple, no matter what path C is. We can apply the gradient theorem to conclude that the integral is simply $f(\mathbf{q}) - f(\mathbf{p})$, where $\mathbf{p}$ is the beginning point and $\mathbf{q}$ is the ending point of C. (For this reason, if C is a closed curve, the integral is zero.)

We might like to give a problem such as find

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

where C is the curve given by the following graph.



The answer is simply

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= f(\pi/2, -1) - f(-\pi, 2) \\ &= -\sin \pi/2 + \frac{\pi}{2} - 1 + k - (2\sin(-\pi) - 4\pi - 4 + k) \\ &= -\sin \pi/2 + \frac{9\pi}{2} + 3 = \frac{9\pi}{2} + 2 \end{aligned}$$

(The constant k is always guaranteed to cancel, so you could just set $k = 0$.)

If the curve C is complicated, one hopes that $\mathbf{F}$ is conservative. It's **always** a good idea to check if $\mathbf{F}$ is conservative before computing its line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

You might save yourself a lot of work.