Pen-and-paper exercises - 2D position and orientation

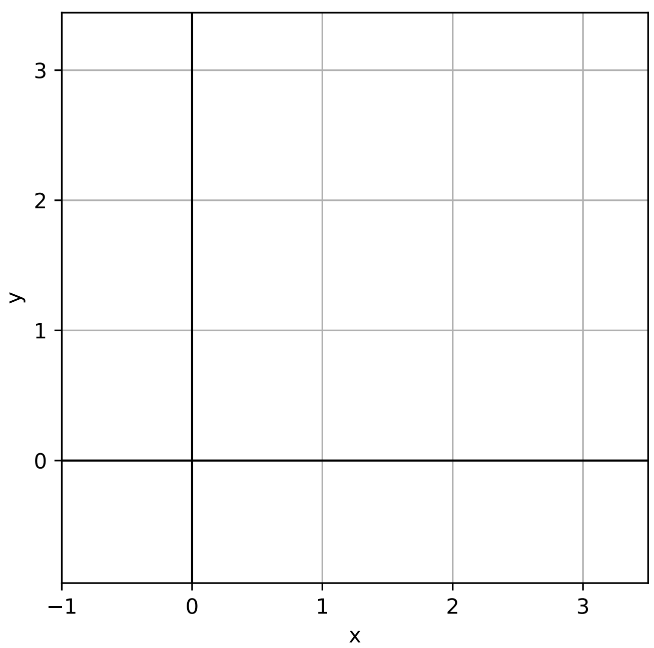
Exercise 1

We have two coordinate frames {A} and {B} defined by the following transformation matrices:

$$\boldsymbol{T}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

$$\boldsymbol{T}_B = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 2 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

Draw both coordinate frames in the plot below.



ANSWER:

Method 1: We know that a transformation matrix is defined by

$$\boldsymbol{T} = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$$

For $\boldsymbol{T}_A$:

$$\boldsymbol{T}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the identity matrix, which belongs to the default coordinate frame with the original at $(x, y) = (0, 0)$ and and orientation of 0 rad (x-axis to the right, y-axis upwards). We can also calculate that from the transformations matrix, because:

$$\cos\theta = 1$$
$$\sin\theta = 0$$

which means that $\theta = 0$ rad. And the translation can be directly taken from the matrix: $(x, y) = (0, 0)$.

For $\boldsymbol{T}_B$:

$$\boldsymbol{T}_B = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 2 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, we can derive that

$$\cos\theta = \tfrac{1}{2}\sqrt{2}$$
$$\sin\theta = \tfrac{1}{2}\sqrt{2}$$

Which means that $\theta = \frac{1}{4}\pi$ rad or 45^o. and the translation is $(x, y) = (2, 1)$.

Method 2: Instead of calculating the rotation from the transformation matrix, we can also directly get the vectors of the coordinate frame from the transformation matrix, because:

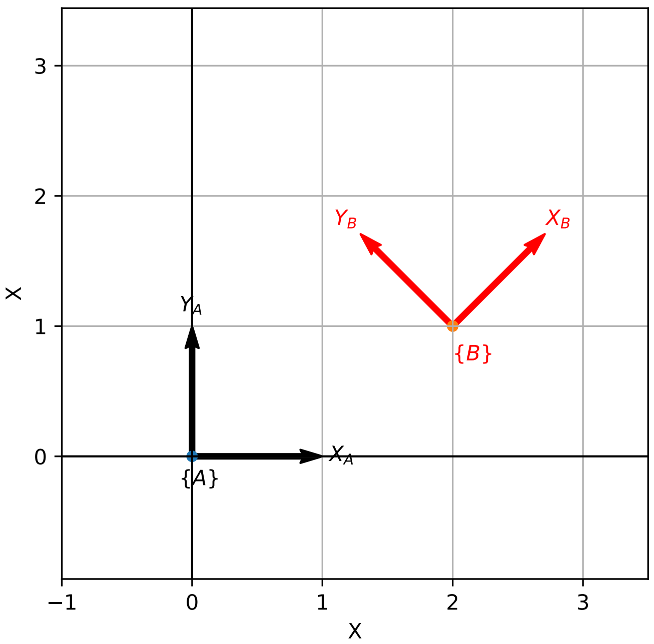
$$\boldsymbol{T} = \begin{pmatrix} \hat{\boldsymbol{x}}^x & \hat{\boldsymbol{y}}^x & x \\ \hat{\boldsymbol{x}}^y & \hat{\boldsymbol{y}}^y & y \\ 0 & 0 & 1 \end{pmatrix}$$

where the x-axis of the coordinate frame is defined by the vector: $\begin{pmatrix} \hat{\boldsymbol{x}}^x \\ \hat{\boldsymbol{x}}^y \end{pmatrix}$ and the y-axis of the coordinate frame is defined by the vector: $\begin{pmatrix} \hat{\boldsymbol{y}}^x \\ \hat{\boldsymbol{y}}^y \end{pmatrix}$.

Applying this to frame {A} results in $\begin{pmatrix} \hat{\boldsymbol{x}}_A^x \\ \hat{\boldsymbol{x}}_A^y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \hat{\boldsymbol{y}}_A^x \\ \hat{\boldsymbol{y}}_A^y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Applying this to frame {B} results in $\begin{pmatrix} \hat{\boldsymbol{x}}_B^x \\ \hat{\boldsymbol{x}}_B^y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} \hat{\boldsymbol{y}}_B^x \\ \hat{\boldsymbol{y}}_B^y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$

Drawing: Both methods result in these two coordinate frames:



Exercise 2

We start from the default coordinate frame:

$$T_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

We will then apply two consecutive transformations, ${}^{A}\boldsymbol{T}_B$ and ${}^{B}\boldsymbol{T}_C$:

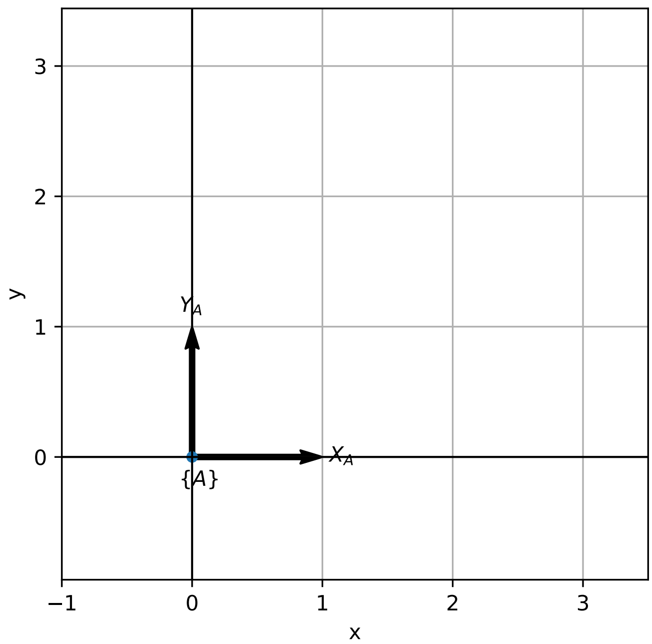
$${}^{A}\boldsymbol{T}_B = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

$${}^{B}\boldsymbol{T}_C = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

Draw the coordinate frames {B} and {C} that are defined by:

$$\boldsymbol{T}_B = \boldsymbol{T}_A \cdot {}^{A}\boldsymbol{T}_B \tag{2.4}$$

$$\boldsymbol{T}_C = \boldsymbol{T}_B \cdot {}^{B}\boldsymbol{T}_C \tag{2.5}$$



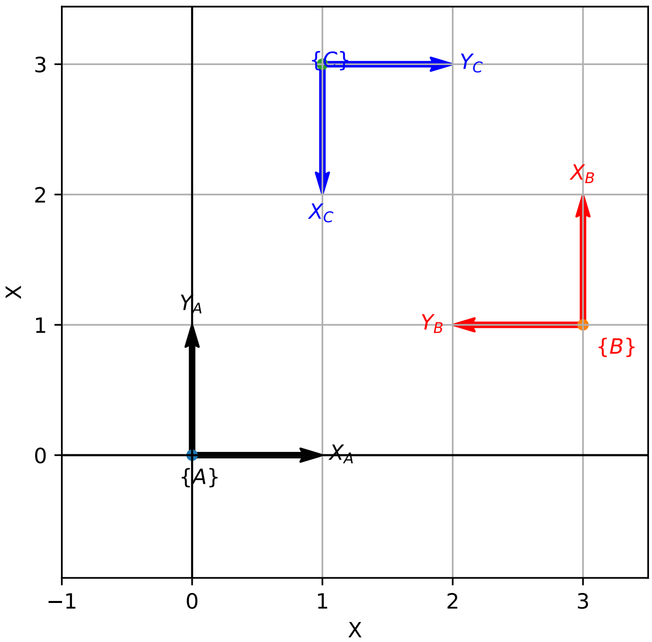
ANSWER:

$$T_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

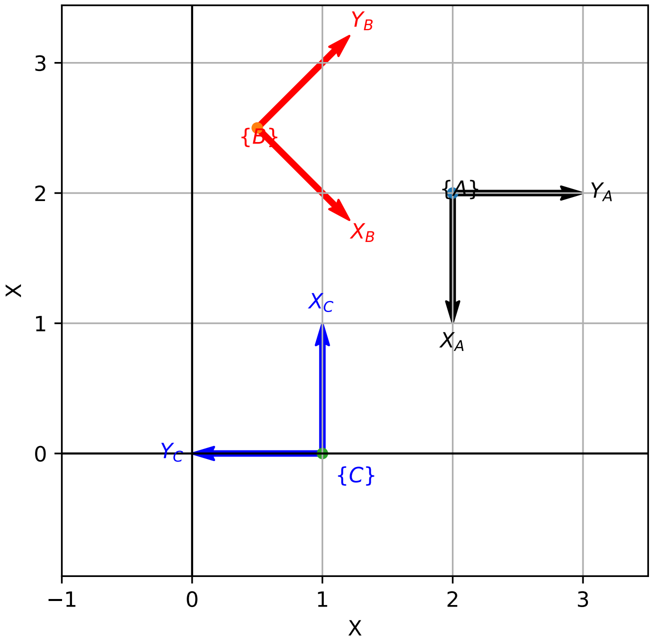
$$T_C = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Drawing these coordinate frames results in:



Exercise 3

Given the coordinate frames {A}, {B} and {C}.



Write down the transformation matrix for the three coordinate frames.

ANSWER:

Given that

$$T = \begin{pmatrix} \hat{\boldsymbol{x}}^x & \hat{\boldsymbol{y}}^x & x \\ \hat{\boldsymbol{x}}^y & \hat{\boldsymbol{y}}^y & y \\ 0 & 0 & 1 \end{pmatrix},$$

we can find the transformation matrices by looking at the origin of the coordinate frame and the vectors for the x and y axes:

$$T_A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_B = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 2\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_C = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$