EE 505 HW 3

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Problem 3.1

This problem has multiple parts. For each function $f(x)$, check the following properties: a) Continuously differentiable (CD)? b) Locally Lipschitz (LL)? c) Continuous (C)? d) Globally Lipschitz (GL)?

3.1.1

$$f(x) = x^2 + |x|$$

a) Not CD at $x = 0$.
b) LL on $W : \{x, y \in \mathbb{R}, |x| < a_1, |y| < a_2\}$. Proof: The term x^2 is CD, hence LL. The term $|x|$ is GL by definition (i.e. $|f(x) - f(y)| = |x - y|$), so $f(x)$ is LL.
c) LL, hence C.
d) Not GL since the x^2 term is not GL ($\frac{d}{dx}x^2 = 2x$ is not bounded).

3.1.2

$$f(x) = x + \operatorname{sgn}(x)$$

a) Not C at $x = 0$, hence not CD.
b) Not C at $x = 0$, hence not LL.
c) Not C at $x = 0$.
d) Not C at $x = 0$, hence not GL.

3.1.3

$$f(x) = \operatorname{sgn}(x)\sin(x)$$

a) Not CD at $x = 0$.
b) GL, hence LL.
c) GL, hence C.
d) GL. Proof: by symmetry, we only need to consider two cases for the Lipschitz condition: when $x \geq y \geq 0$ and when $x \geq 0 \geq y$. In the first case $x \geq y \geq 0$,

$$|f(x) - f(y)| = |\sin(x) - \sin(y)| = 2\left|\sin\frac{x-y}{2}\cos\frac{x+y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right| \leq |x - y|$$

Here, we used the facts that $\cos(x) \leq 1$ and $|\sin(x)| \leq |x|$. In the second case $x \geq 0 \geq y$,

$$|f(x) - f(y)| = |\sin(x) + \sin(y)| \leq |x + y| \leq |x - y|$$

We used the fact that when $x \geq 0 \geq y$, we have $|x + y| \leq |x - y|$.

3.1.4

$$f(x) = -x + a \sin x$$

a) CD.
b) CD, hence LL.
c) CD, hence C.
d) CD and $|\frac{df}{dx}| = |-1 + a \cos x| \leq |1 + a|$ is bounded, hence GL.

3.1.5

$$f(x) = -x + 2|x|$$

a) Not CD at $x = 0$.
b) GL, hence LL.
c) GL, hence C.
d) GL since the $-x$ term is GL (i.e. CD and $|\frac{d}{dx}(-x)| = 1$) and the $2|x|$ term is GL (i.e. $|f(x) - f(y)| = 2|x - y|$).

3.1.6

$$f(x) = \tan x$$

a) CD over the domain $D : -\pi/2 < x < \pi/2$.
b) CD over D, hence LL over D.
c) CD over D, hence C over D.
d) Not GL over D, since $\frac{df}{dx} = \sec^2 x$ is not bounded near $x = 0$.

3.1.7

$$f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) - \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$$

The Jacobian is given by

$$\frac{df}{dx} = \begin{bmatrix} a + b \operatorname{sech}^2(bx_1) & -b \operatorname{sech}^2(bx^2) \\ b \operatorname{sech}^2(bx_1) & a + b \operatorname{sech}^2(bx_2) \end{bmatrix}$$

a) CD.
b) CD, hence LL.
c) CD, hence C.
d) GL, since CD and the Jacobian is globally bounded.

3.1.8

$$f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a + b)x_1 + bx_1^2 - x_1 x_2 \end{bmatrix}$$

a) Not CD because f_1 is not CD.
b) Each term is LL, so $f(x)$ is LL.
c) LL, hence C.
d) Not GL since f_2 is CD but $\frac{df_2}{dx}$ is not globally bounded.

Problem 3.2

Consider the ball D_r centered at the origin with radius r. For each of the state space models $\dot{x} = f(t, x)$, determine whether the system is a) LL on D_r for sufficiently small (ss) r? b) LL on D_r for any $r > 0$? c) GL in x?

3.2.1

The pendulum with friction and constant torque is given by

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{m\ell^2}T \end{bmatrix}$$

and the Jacobian is easily computed as

$$\frac{df}{dx} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

a-b) GL, hence LL.
c) GL, since f is CD and the Jacobian is bounded.

3.2.2

The tunnel diode circuit is given by

$$f(x) = \begin{bmatrix} \frac{1}{C}(-h(x_1) + x_2) \\ \frac{1}{L}(-x_1 - Rx_2 + u) \end{bmatrix}$$

where $h(x)$ is a 5th order polynomial and u is a constant. The Jacobian is given by

$$\frac{df}{dx} = \begin{bmatrix} -\frac{1}{C}h'(x_1) & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$$

a-b) LL for any $r > 0$ since f is CD.
c) Not GL since f is CD but Jacobian is not bounded because $h'(x)$ is a 4th order polynomial.

3.2.3

The mass-spring oscillator is given by

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

a-b) The system is not LL at the origin because $\eta(x_1, x_2)$ is discontinuous along the line $x_2 = 0$.
c) Not LL, hence not GL.

3.2.4

The Van-der-Pol equation is given by

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + \varepsilon(1 - x_1^2)x_2 \end{bmatrix}$$

and the Jacobian is

$$\frac{df}{dx} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon(1 - x_1^2) \end{bmatrix}$$

a-b) LL since f is CD.
c) Not GL because f is CD but the Jacobian is not bounded.

3.2.5

The state space model can be written as

$$f(t,x) = \begin{bmatrix} a_m x_0 + k_p x_1 r(t) + k_p x_2 (x_0 + y_m(t)) \\ -\gamma x_0 r(t) \\ -\gamma x_0 (x_0 + y_m(t)) \end{bmatrix}$$

The Jacobian is given by

$$\frac{df}{dx} = \begin{bmatrix} a_m + k_p x_2 & k_p r(t) & k_p (x_0 + y_m(t)) \\ -\gamma r(t) & 0 & 0 \\ -\gamma (2x_0 + y_m(t)) & & \end{bmatrix}$$

a-b) Since f is CD and bounded given bounded $r(t)$ and $y_m(t)$, we know f is LL.
c) Since f is CD but the Jacobian is not bounded in x, we know f is not GL.

3.2.6

The state space equation is given by

$$f(x) = Ax + B\psi(Cx)$$

where $\psi(\cdot)$ is the standard dead-zone non-linearity.
a-b) GL, hence LL.
c) The system is GL because the both terms in the non-linearity $\psi(x) = x - \mathrm{sat}(x)$ are GL.

Problem 3.5

3.5 Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two different p-norms on R^n. Show that $f : R^n \to R^m$ is Lipschitz in $\|\cdot\|_\alpha$ if and only if it is Lipschitz in $\|\cdot\|_\beta$.

By the equivalence of p-norms in finite dimensional vector spaces, we know that there exists some constants c_1 and c_2 such that

$$c_1 ||x||_\alpha \le ||x||_\beta \le c_2 ||x||_\alpha$$

If f is Lipschitz in $||\cdot||_\alpha$, then

$$||f(x) - f(y)||_\alpha \le L_\alpha ||x - y||_\alpha$$

Thus,

$$c_1 ||f(x) - f(y)||_\beta \le ||f(x) - f(y)||_\alpha \le L_\alpha ||x - y||_\alpha \le c_2 L_\alpha ||x - y||_\beta$$

so

$$||f(x) - f(y)||_\beta \le \frac{c_2}{c_1} L_\alpha ||x - y||_\beta$$

Thus, f is Lipschitz in $||\cdot||_\beta$. Of course, α and β were chosen arbitrarily, so the analysis holds the same in the opposite direction.

Problem 3.6

3.6 Let $f(t, x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t, x)\| \le k_1 + k_2\|x\|, \quad \forall\ (t, x) \in [t_0, \infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \le \|x_0\| \exp[k_2(t - t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t - t_0)] - 1\}$$

for all $t \ge t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

a) This problem is simply an application of the Bellman inequality followed by some tedious integration. First, we notice that $f(t, x)$ is Lipschitz since

$$||f(t, x) - f(t, y)|| \le ||\, k_2||x|| - k_2||y||\, || \le |k_2|||x - y||$$

Using the integral form of the differential equation, we have

$$||x(t)|| \le ||x_0 + \int_{t_0}^{t} f(s, x)\, ds||$$

Substituting the given inequality and expanding the RHS, we have

$$||x(t)|| \le ||x_0 + k_1(t - t_0) + \int_{t_0}^{t} k_2||x_0||\, ds||$$

We apply the Bellman inequality:

$$||x(t)|| \le ||x_0 + k_1(t - t_0) + \int_{t_0}^{t} (||x_0|| + k_1(s - t_0))k_2 e^{k_2(t-s)}\, ds||$$

We split the integral into two parts. The first integral is trivial and the second is performed by parts. After lots of algebra, we arrive at the inequality we wished to show.

b) The upper bound on $||x(t)||$ is finite for any finite k_1, k_2, t therefore the solution cannot escape in finite time.

Problem 3.8

3.8 Show that the state equation

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2x_2}{1 + x_2^2}, \quad x_1(0) = a \\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1 + x_1^2}, \quad x_2(0) = b \end{aligned}$$

has a unique solution defined for all $t \ge 0$.

We observe that $f(x)$ is continuously differentiable for all $x \in \mathbb{R}^2$, therefore it is locally Lipschitz. Further, we see that

$$||f(x)|| \leq k_1 + k_2||x||$$

so by the result shown in Problem 3.6, it has a unique solution for $t \geq 0$.

Problem 3.9

3.9 Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

Any trajectory that starts in the region W enclosed in a limit cycle must remain it in for all time. Since $f(x)$ is LL over this region, and W is a closed compact set, we can apply Theorem 3.3. This proves that the trajectories that start in the region enclosed by the limit cycle will not have a finite escape time.

Problem 3.10

The original system is given by

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\
\dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u]
\end{aligned}$$

where $R = 1.5$, $u = 1.2$, and the nominal values of C and L are $C_0 = 2$, $L_0 = 5$. Let $\lambda = [C, L]^T$. The sensitivity equation is

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}}$$

where the Jacobians are defined as

$$\left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} = \begin{bmatrix} -\frac{1}{C}h'(x_1) & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}_{\text{nominal}} = \begin{bmatrix} -\frac{1}{2}h'(x_1) & \frac{1}{2} \\ -\frac{1}{5} & -\frac{3}{10} \end{bmatrix}$$

$$\begin{aligned}
\left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}} &= \begin{bmatrix} -\frac{1}{C^2}[-h(x_1) + x_2] & 0 \\ 0 & -\frac{1}{L^2}[-x_1 - Rx_2 + u] \end{bmatrix}_{\text{nominal}} \\
&= \begin{bmatrix} -\frac{1}{4}[-h(x_1) + x_2] & 0 \\ 0 & -\frac{1}{25}[-x_1 - 1.5x_2 + 1.2] \end{bmatrix}
\end{aligned}$$

We can write the augmented system in state space form with

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{2}[-h(x_1) + x_2] \\
\dot{x}_2 &= \frac{1}{5}[-x_1 - 1.5x_2 + 1.2] \\
\dot{x}_3 &= \frac{1}{2}[x_4 - h'(x_1)x_3] + \frac{1}{4}[h(x_1) - x_2] \\
\dot{x}_4 &= -\frac{1}{5}x_3 - \frac{3}{10}x_4 \\
\dot{x}_5 &= \frac{1}{2}[x_6 - h'(x_1)x_5] \\
\dot{x}_6 &= -\frac{1}{5}x_5 - \frac{3}{10}x_6 + \frac{1}{25}[x_1 + 1.5x_2 - 1.2]
\end{aligned}
$$

I simulated the augmented system with $x_1(0) = x_2(0) = 1$ and all other initial conditions set to 0. We see that x_6 varies the most, which corresponds to the effect on perturbations to L on variable x_2. It is not surprising that the transient effects disappear because the system approaches a stable equilibrium point.

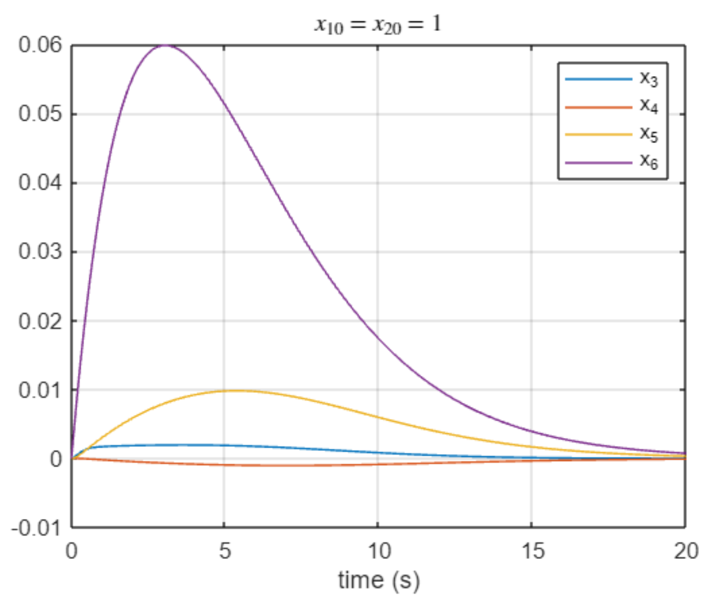

Figure 1: Problem 3.10

Problem 3.11

The original system is given by

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2
\end{aligned}
$$

The Jacobians are

$$
\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon(1 - x_1^2) \end{bmatrix} \quad \frac{\partial f}{\partial \varepsilon} = \begin{bmatrix} 0 \\ (1 - x_1^2)x_2 \end{bmatrix}
$$

The augmented system is given by

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon_0(1 - x_1^2)x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= (-1 - 2\varepsilon_0 x_1 x_2)x_3 + \varepsilon_0(1 - x_1^2)x_4 + (1 - x_1^2)x_2
\end{aligned}
$$

I simulated the augmented system with $x_1(0) = x_2(0) = 1$, $x_3(0) = x_4(0) = 0$ and $\varepsilon_0 = 0.2$ for $t = [0, 90]$ seconds, as shown below. Note that both x_1 and x_2 become increasingly sensitive to perturbations in ε as t increases.

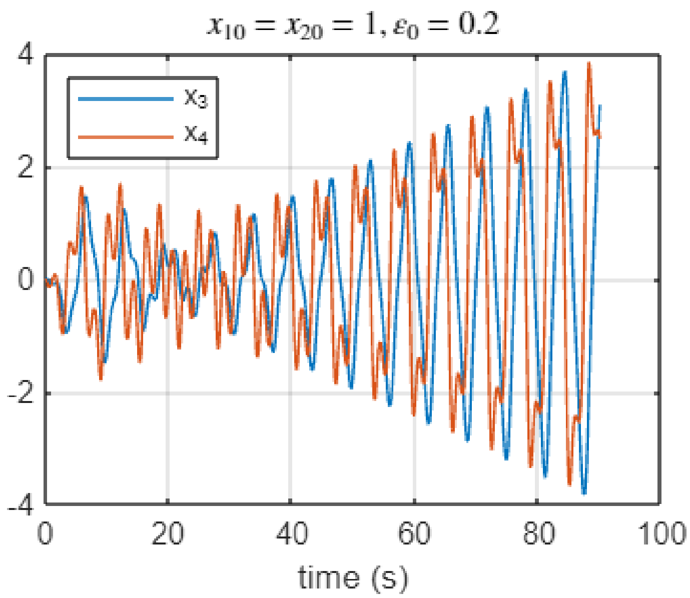

Figure 2: Problem 3.11

Problem 3.13

The original system is given by

$$
\begin{aligned}
\dot{x}_1 &= \arctan(ax_1) - x_1 x_2 \\
\dot{x}_2 &= bx_1^2 - cx_2
\end{aligned}
$$

The Jacobians are

$$
\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{a}{1+(ax_1)^2} - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix} \quad \frac{\partial f}{\partial \varepsilon} = \begin{bmatrix} \frac{x_1}{1+(ax_1)^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}
$$

The augmented system is given by

$$\begin{aligned}
\dot{x}_1 &= \arctan(x_1) - x_1 x_2 \\
\dot{x}_2 &= -x_2 \\
\dot{x}_3 &= \sigma(x_1 + x_3) - x_1 x_4 \\
\dot{x}_4 &= -x_4 \\
\dot{x}_5 &= \sigma x_5 - x_1 x_6 \\
\dot{x}_6 &= -x_6 + x_1^2 \\
\dot{x}_7 &= \sigma x_7 - x_1 x_8 \\
\dot{x}_8 &= -x_8 - x_2
\end{aligned}$$

where

$$\sigma = \frac{1}{1 + x_1^2}$$

I simulated the augmented system with $x_1(0) = 0.001, x_2(0) = -1$, and all other state variables initially set to zero. This system is unstable in nature.

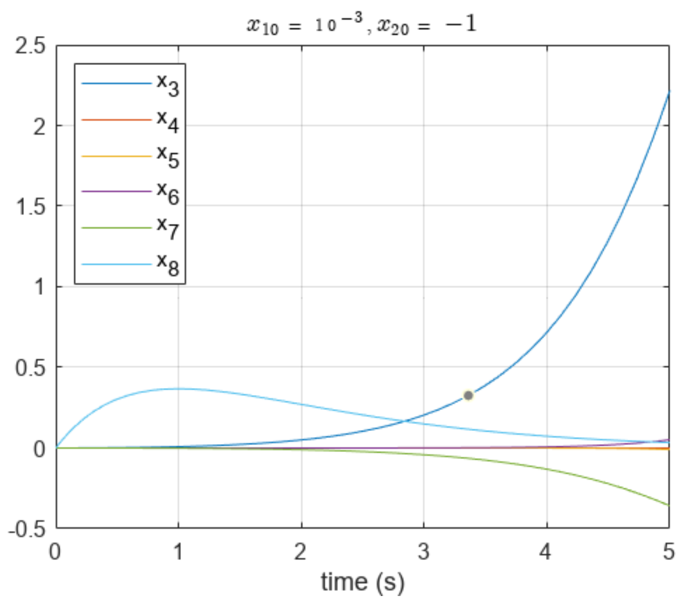

Figure 3: Problem 3.13

Problem 3.20

3.20 Show that if $f : R^n \rightarrow R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

Uniformly continuous means that given $||x - y|| \leq \delta$, we have

$$||f(x) - f(y)|| \leq \varepsilon$$

Since f is Lipschitz on W, we have

$$||f(x) - f(y)|| \leq L||x - y|| \leq L\delta$$

By choosing $\delta \geq \varepsilon/L$, we have

$$||f(x) - f(y)|| \leq \varepsilon \leq L\delta$$

therefore $f(x)$ is uniformly continuous on W. Note that this is indeed uniform continuity and not just ordinary continuity because δ is not a function of x, y.

Problem 3.21

3.21 For any $x \in R^n - \{0\}$ and any $p \in [1, \infty)$, define $y \in R^n$ by

$$y_i = \frac{x_i^{p-1}}{\|x\|_p^{p-1}} \operatorname{sign}(x_i^p)$$

Show that $y^T x = \|x\|_p$ and $\|y\|_q = 1$, where $q \in (1, \infty]$ is determined from $1/p + 1/q = 1$. For $p = \infty$, find a vector y such that $y^T x = \|x\|_\infty$ and $\|y\|_1 = 1$.

a) Show that $y^T x = ||x||_p$.

$$\begin{aligned} y^T x &= y_1 x_1 + \cdots + y_n x_n \\ &= \frac{1}{||x||_p^{p-1}} \left(x_1^{p-1} \operatorname{sgn}(x_1^p) x_1 + \cdots + x_n^{p-1} \operatorname{sgn}(x_n^p) x_n \right) \\ &= \frac{1}{||x||_p^{p-1}} \left(|x_1^p| + \cdots + |x_n^p| \right) = \frac{||x||_p^p}{||x||_p^{p-1}} = ||x||_p \end{aligned}$$

b) Show that $||y||_q = 1$ with $1/p + 1/q = 1$:

$$||y||_q^q = \frac{|x_1|^{(p-1)q} + \cdots + |x_n|^{(p-1)q}}{||x||_p^{(p-1)q}} = \frac{|x_1|^p + \cdots + |x_n|^p}{||x||_p^p} = \frac{||x||_p^p}{||x||_p^p} = 1$$

and it immediately follows that

$$||y||_q = 1$$

c) When $p = \infty$ and $q = 1$ we know that

$$||x||_\infty = \max(x_1, \ldots, x_n)$$

and

$$||y||_1 = |y_1| + \cdots + |y_n|$$

To ensure that $y^T x = ||x||_\infty$ and $||y||_1 = 1$, we can just choose

$$y_i = \begin{cases} 1 & i = \min \arg\max |x_i| \\ 0 & \text{otherwise} \end{cases}$$

In English, y will be a vector of zeros with exactly one entry equal to 1. This entry corresponds to the first instance where $x_i = \max |x_i|$.