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New Analytical Results of the Energy Based Swinging up Control of the Acrobot

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Abstract— **This paper addresses the energy based swinging up control problem of the Acrobot, which is a typical example of underactuated mechanical systems. This paper provides a complete analysis of the convergence of the energy and the motion of the Acrobot, and illustrates clearly several unique characteristics of the closed-loop system of the Acrobot under the energy based control. Specifically, this paper shows clearly how to choose the control parameters such that starting from *any initial state*, the Acrobot will eventually either be swung up to an arbitrarily small neighborhood of the upright equilibrium point, or remain at the downward equilibrium point which is shown to be unstable for the closed-loop system. This proves theoretically that the energy based control is effective for swinging up the Acrobot.**

I. Introduction

The Acrobot, which is a two link planar robot with single actuator at the joint of two links, has been studied as a typical example of underactuated mechanical systems, see e.g., [1], [2], [5], [8], [10]. Many research results have been reported on the swing up control problem for the Acrobot, i.e., to swing the Acrobot from the downward equilibrium point to the upright equilibrium point and balance it about the vertical [10]. For solving such problem, the approach of combining the partial linearization control for the swinging up phase and the LQR controller for the balancing phase was proposed in [10]. An energy based swing up algorithm was also proposed for the swinging up phase in [10]; however, strict analysis of the energy or the motion of the Acrobot was not provided there. A difficulty as indicated in [10] is to tune the control parameters to accomplish the capture and balancing phase successfully.

To overcome such difficulty motivated us to study whether there exists a controller under which the Acrobot can be swung up to an arbitrarily small neighborhood of the upright equilibrium point; this guarantees that the capture and balancing phase can be easily accomplished.

On the other hand, [3] and [7] reported interesting energy based control solutions to the swing up control problem of the Pendubot, which is a two link planar robot with single actuator at the shoulder (link 1) [9]. The solution provided in [3] requires that the initial state of the Pendubot should satisfied some conditions, while the solution given in [7] is free of conditions on the initial state. These results stimulated our endeavor starting in [12] to extend the energy based control results of the Pendubot to the Acrobot. Note that the applications of the energy based control approach to the cart-pole system, the Furuta pendulum, the reaction wheel pendulum, the ball and beam system etc. can be found in [4]; however, the application to *the Acrobot* remained open. As shown in [12] and in this paper, the application is by no means trivial. Indeed, the difficulty is not to find the energy based control law, but to analyze the convergence of the energy and the motion of the Acrobot due to its unique structure of actuation. Under the assumption on the mechanical parameters of the Acrobot, [12] proposed the condition on the control parameters such that the Acrobot can be swung up to an arbitrarily small neighborhood of the upright equilibrium point. Note that there is no corresponding assumption needed in the case of the Pendubot. It is remained open whether the assumption on the mechanical parameters is necessary. Moreover, the condition on the control parameters in [12] is somewhat complicated.

This paper provides a complete analysis of the convergence of the energy and the motion of the Acrobot under the energy based control. This paper shows clearly how to choose the control parameters such that starting from *any initial state*, the Acrobot will eventually either be swung up to an arbitrarily small neighborhood of the upright equilibrium point, or remain at the downward equilibrium point which is shown to be unstable for the closed-loop system. This proves that the energy based control is effective for swinging up the Acrobot. Note that in this paper the assumption on the mechanical parameters in [12] is not needed and the condition on the control parameters is weaker and much simpler than the corresponding one provided in [12].

II. Preliminaries

With notations shown in Fig. 1, the motion equations of the two-link planar robot [10] are:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \tag{1}$$

where $q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$, $\tau = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}^T$, and

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

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$$= \left[\begin{array}{cc} c_1 + c_2 + 2c_3 \cos q_2 & c_2 + c_3 \cos q_2 \\ c_2 + c_3 \cos q_2 & c_2 \end{array} \right], \quad (2)$$

$$C(q, \dot{q})\dot{q} = \left[\begin{array}{c} h_1 \\ h_2 \end{array} \right] = c_3 \left[\begin{array}{c} -2\dot{q}_1 \dot{q}_2 - \dot{q}_2^2 \\ \dot{q}_1^2 \end{array} \right] \sin q_2, \quad (3)$$

$$G(q) = \left[\begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right] = \left[\begin{array}{c} c_4 g \cos q_1 + c_5 g \cos(q_1 + q_2) \\ c_5 g \cos(q_1 + q_2) \end{array} \right], \quad (4)$$

where c_i $(i = 1, \cdots, 5)$ determined by the mechanical parameters of the Acrobot are

$$\begin{cases} c_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1, \quad c_2 = m_2 l_{c2}^2 + I_2, \\ c_3 = m_2 l_1 l_{c2}, \quad c_4 = m_1 l_{c1} + m_2 l_1, \quad c_5 = m_2 l_{c2} \end{cases}, \quad (5)$$

and g is the acceleration of the gravity.

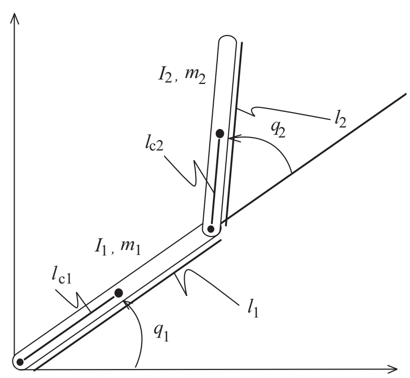

Fig. 1. The Acrobot.

The robot is called the *Acrobot* if $\tau_1 \equiv 0$ in [5], and is called the *Pendubot* if $\tau_2 \equiv 0$ in [9].

III. THE ENERGY BASED CONTROL LAW FOR THE ACROBOT

This paper studies whether there exists the controller under which the Acrobot can be swung up to an arbitrarily small neighborhood of the following upright equilibrium point:

$$q_1 = \pi/2 \ (\mathrm{mod}\, 2\pi), \ q_2 = 0 \ (\mathrm{mod}\, 2\pi), \ \dot{q}_1 = 0, \ \dot{q}_2 = 0. \quad (6)$$

To this end, first by setting the potential energy of the Acrobot at the upright equilibrium point to be 0, the total energy of the Acrobot is given by

$$E = \frac{1}{2}\dot{q}^T D(q)\dot{q} + c_4 g(\sin(q_1) - 1) + c_5 g(\sin(q_1 + q_2) - 1). \quad (7)$$

Next, by observing $\dot{E} = \dot{q}^T \tau = \dot{q}_2 \tau_2$, we define the following Lyapunov function candidate:

$$V = \frac{1}{2}k_E E^2 + \frac{1}{2}k_D \dot{q}_2^2 + \frac{1}{2}k_P q_2^2, \quad (8)$$

where k_E, k_D, and k_P are positive constants. If we can find τ_2 such that $\lim_{t\to\infty} E = 0$ and $\lim_{t\to\infty} q_2 = 0$ hold, then it is easy to see the Acrobot can be swung up to an arbitrarily small neighborhood of the upright equilibrium point.

Taking the time derivative of V along (1) yields $\dot{V} = \dot{q}_2(k_E E \tau_2 + k_D \ddot{q}_2 + k_P q_2)$. If we can choose τ_2 such that

$$k_E E \tau_2 + k_D \ddot{q}_2 + k_P q_2 = -k_V \dot{q}_2, \quad (9)$$

where k_V is a positive constant, then

$$\dot{V} = -k_V \dot{q}_2^2 \le 0. \quad (10)$$

In what follows, we study when (9) is solvable with respect to τ_2. To this end, obtaining $\ddot{q}_2$ from (1) and putting it into (9), we have

$$\left(k_E E + k_D \frac{d_{11}}{\Delta}\right)\tau_2 = -k_V \dot{q}_2 - k_P q_2$$
$$-k_D \frac{d_{21}(h_1 + \phi_1) - d_{11}(h_2 + \phi_2)}{\Delta}, \quad (11)$$

where $\Delta = d_{11}d_{22} - d_{12}^2 > 0$ owing to $D(q) > 0$. To derive τ_2 from (11), we study when

$$k_E E + k_D \frac{d_{11}}{\Delta} \neq 0 \quad (12)$$

holds for all $t \ge 0$. To begin with, from (7), we obtain

$$E \ge -2(c_4 + c_5)g. \quad (13)$$

Next, we have

$$\frac{d_{11}}{\Delta} \ge \min_{q_2 \in [0,\, 2\pi]} \frac{c_1 + c_2 + 2c_3 \cos q_2}{c_1 c_2 - c_3^2 \cos^2 q_2} := \rho. \quad (14)$$

Note that the equalities in (13) and (14) are not satisfied simultaneously. Therefore, we obtain $k_E E + k_D d_{11}/\Delta > -2k_E(c_4 + c_5)g + k_D \rho$. This gives that if

$$\frac{k_D}{k_E} \ge 2(c_4 + c_5)g/\rho \quad (15)$$

holds, then $k_E E + k_D d_{11}/\Delta > 0$ holds, and τ_2 from (11) can be obtained as follows:

$$\tau_2 = \frac{-(k_V \dot{q}_2 + k_P q_2)\Delta}{k_E E \Delta + k_D d_{11}} - \frac{k_D(d_{21}(h_1 + \phi_1) + d_{11}(h_2 + \phi_2))}{k_E E \Delta + k_D d_{11}}. \quad (16)$$

We are ready to present the following result.

LEMMA 1: Consider the Acrobot system (1). Suppose that $k_E > 0$, $k_D > 0$ satisfy (15), where ρ is defined in (14). Then the control law (16) with $k_P > 0$, $k_V > 0$ has no singular point, and the solution of the closed-loop system converges to the invariant set M described by

$$\begin{cases} \dot{q}_1^2 = \dfrac{2g\left(c_4(1 - \sin q_1) + c_5(1 - \sin(q_1 + q_2^*))\right) + 2E^*}{c_1 + c_2 + 2c_3 \cos q_2^*} \\ q_2 \equiv q_2^* \end{cases} \quad (17)$$

where q_2^* and E^* are constants and are convergent values of q_2 and E, respectively.

Proof: From (10), let M be the largest invariant set in the set of all points $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ satisfying $\dot{V} = 0$. Then using LaSalle's theorem [6], we know that every solution

$(q_1, \dot{q}_1, q_2, \dot{q}_2)$ of the closed-loop system consisted of (1) and (16) approaches M as $t \to \infty$.

From $\dot{V}(t) = 0$, we obtain $\dot{q}_2(t) = 0$. This follows that V and q_2 will converge to some constants denoted as V^* and q_2^*, respectively. Then using (8), we know that E will converge to the constant denoted as E^*. Finally, (17) can be obtained directly from (7) with $q_2 \equiv q_2^*$. ∎

To investigate whether the Acrobot can be swung up under the energy control law (16), we will analyze the invariant set M by considering the cases $E^* \neq 0$ and $E^* = 0$, separately.

IV. THE CONVERGENT VALUES OF THE ENERGY AND THE MOTION OF THE ACROBOT

A. On nonzero convergence of the energy of the Acrobot

We will analyze the case of $E^* \neq 0$. We give the following proposition.

Proposition 1: Consider the Acrobot system (1). Under the control law (16) with $k_E > 0$, $k_D > 0$ satisfying (15) and $k_P > 0$, $k_V > 0$, when $E^* \neq 0$ holds, then the Acrobot stay at the following equilibrium point:

$$q_1 \equiv q_1^*, \quad q_2 \equiv q_2^*, \tag{18}$$

where q_1^* and q_2^* are constants satisfying

$$c_4 g \cos q_1^* + c_5 g \cos(q_1^* + q_2^*) = 0, \tag{19}$$

$$(q_1^*, q_2^*) \neq (\pi/2, 0) \quad (mod\, 2\pi). \tag{20}$$

In this case, E^* and τ_2^* (the convergent value of τ_2) satisfy

$$\begin{cases} E^* = c_4 g(\sin q_1^* - 1) + c_5 g(\sin(q_1^* + q_2^*) - 1) \\ \tau_2^* = -c_4 g \cos q_1^* \end{cases}. \tag{21}$$

Proof: Since $E^* \neq 0$, it follows from (9) that τ_2 converges to constant τ_2^* satisfying

$$k_E E^* \tau_2^* + k_P q_2^* = 0. \tag{22}$$

Let us analyze the motion of link 1. To this end, using $q_2 = q_2^*$ and (1), we have

$$d_{11}\ddot{q}_1 + c_4 g \cos q_1 + c_5 g \cos(q_1 + q_2^*) = 0, \tag{23}$$
$$d_{21}\ddot{q}_1 + c_3 \dot{q}_1^2 \sin q_2^* + c_5 g \cos(q_1 + q_2^*) = \tau_2^*. \tag{24}$$

It is worth mentioning that $d_{11}(q_2^*)$ and $d_{21}(q_2^*)$ are constant. Obtaining $\ddot{q}_1$ follows from (23) as

$$\ddot{q}_1 = -[c_4 g \cos q_1 + c_5 g \cos(q_1 + q_2^*)]/d_{11}, \tag{25}$$

Taking time derivative of (24) after having put $\ddot{q}_1$ given in (25), we obtain

$$\dot{q}_1 \Bigg(\frac{d_{21}}{d_{11}} [c_4 g \sin q_1 + c_5 g \sin(q_1 + q_2^*)] + 2c_3 \ddot{q}_1 \sin q_2^* - c_5 g \sin(q_1 + q_2^*) \Bigg) = 0. \tag{26}$$

Putting (25) into (26) and performing a straightforward calculation yields $\dot{q}_1 (A(q_2^*) \sin q_1 + B(q_2^*) \cos q_1) = 0$, that is,

$$\dot{q}_1 \sqrt{A^2(q_2^*) + B^2(q_2^*)}\, \sin(q_1 + \phi(q_2^*)) = 0, \tag{27}$$

where

$$\begin{aligned} A(q_2^*) &= 3c_3c_5 \cos^2 q_2^* - (c_3c_4 - c_1c_5) \cos q_2^* \\ &\quad -(c_2c_4 + 2c_3c_5), & (28) \\ B(q_2^*) &= (3c_3c_5 \cos q_2^* + 2c_3c_4 + c_1c_5) \sin q_2^*, & (29) \end{aligned}$$

and $\phi(q_2^*)$ is a constant determined by $A(q_2^*)$ and $B(q_2^*)$. Now we will show $\dot{q}_1 \equiv 0$. On the contrary, we assume $\dot{q}_1 \neq 0$, then

$$A(q_2^*) = 0 \quad \text{and} \quad B(q_2^*) = 0 \tag{30}$$

holds due to (27). From Lemma A1 in Appendix A, there exists q_2^* satisfying (30) if and only if

$$(c_1 - c_3)c_5 = (c_3 - c_2)c_4, \tag{31}$$

holds; in this case,

$$q_2^* = \pi \pmod{2\pi} \tag{32}$$

holds. Therefore, for the Acrobot whose parameters do not satisfy (31), we know (30) does not hold; this gives a contradiction. On the other hand, for the Acrobot whose parameters do satisfy (31), it follows that (31) and (32) together with (24) and (25) give

$$\tau_2^* = \frac{(c_3 - c_2)c_4 - (c_1 - c_3)c_5}{c_1 + c_2 - 2c_3} g \cos q_1 = 0. \tag{33}$$

which yields $q_2^* = 0$ from (22). This contradicts (32).

Therefore, $\dot{q}_1 \equiv 0$ holds, this yields that q_1 converges to a constant denoted as q_1^*. This means that the Acrobot will stay at an equilibrium point. (19) holds due to (23), and (20) holds due to $E^* \neq 0$. Finally, τ_2^* in (21) can be derived via (19) and (24). ∎

Remark 1: The exposed properties of the mechanical parameters in Appendix A are vital to accomplishing the proof of Proposition 1. The analysis of the motion of the Acrobot for $E^* \neq 0$ is much more difficult than the related analysis for the Pendubot given in [3] or [7]. This is due to the different actuation of the two robots.

B. The effectiveness of the energy based control of the Acrobot

We present a condition on k_E and k_P in (34) such that E^* takes a value of either 0 or its minimum given in (13). Now we give the following results.

Proposition 2: Consider the Acrobot system (1). Under the control law (16) with positive parameters k_E, k_D, k_P satisfying (15) and

$$\frac{k_P}{k_E} \geq 2c_4c_5g^2, \tag{34}$$

and $k_V > 0$, then **either** *of the following statements holds:*

i). The solution of the closed-loop system converges to the invariant set M_0 described by

$$\dot{q}_1^2 = \frac{2(c_4+c_5)g}{c_1+c_2+2c_3}(1-\sin q_1), \quad q_2 \equiv 0, \tag{35}$$

and

$$\tau_2 \to -\frac{(c_2c_4+c_3c_4-c_1c_5-c_3c_5)g\cos q_1}{c_1+c_2+2c_3}, \quad \text{as } t \to \infty. \tag{36}$$

ii). The solution of the closed-loop system converges to the downward equilibrium point $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (-\pi/2, 0, 0, 0)$, which is an unstable equilibrium point of the closed-loop system.

Proof: Under the control law (16), from Lemma 1, E is convergent to E^*.

i). For the case of $E^* = 0$, we obtain $q_2^* = 0$ from (9). We can obtain (35) and (36) by putting $E = E^* = 0$ and $q_2 = q_2^* = 0$ into (17) and (16), respectively. This shows the statement i).

ii). Consider the case $E^* \neq 0$. It follows from Lemma 1 and Proposition 1 that q_1, q_2 are convergent to some constants q_1^* and q_2^*, respectively. In what follows * is dropped for simplicity.

We shall investigate the solutions of (19) and (22). Define

$$\beta = \frac{c_4}{c_5}, \quad \gamma = \frac{k_P}{c_4c_5g^2k_E}. \tag{37}$$

We rewrite (19) and (22) as

$$\beta\cos q_1 + \cos(q_1+q_2) = 0, \tag{38}$$

$$\gamma q_2 + (\beta+1-G_y)\cos q_1 = 0, \tag{39}$$

respectively, where (21) has been used in obtaining (39), and

$$G_y = \beta\sin q_1 + \sin(q_1+q_2). \tag{40}$$

Note that (34) is equivalent to $\gamma \geq 2$. Under such condition, we will show that (38) and (39) have only the solution

$$q_1 = -\pi/2 \ (\mathrm{mod}\ 2\pi), \quad q_2 = 0. \tag{41}$$

To start with, we try to delete q_1 from (38) and (39). From (38), we have $(\beta+\cos q_2)\cos q_1 = \sin q_2 \sin q_1$, which gives

$$G_y\cos q_1 = \sin q_2. \tag{42}$$

This yields that (39) is equivalent to

$$\gamma q_2 + (\beta+1)\cos q_1 - \sin q_2 = 0. \tag{43}$$

Adding the square of the left hand of (38) to the square of the right hand of (40) yields

$$G_y^2 = \delta^2, \tag{44}$$

where

$$\delta(q_2) = \sqrt{1+\beta^2+2\beta\cos q_2} \geq 0. \tag{45}$$

When $\delta(q_2) \neq 0$, we can obtain $\cos q_1$ as a function of q_2 by using (42) and (44). Note that $\delta(q_2) = 0$ holds only at

$$\beta = 1 \quad \text{and} \quad \cos q_2 = -1. \tag{46}$$

If (46) holds, we can show that (43) has no solution. Indeed, (43) is now reduced to $\gamma q_2 + 2\cos q_1 = 0$. This is impossible when $\gamma \geq 2$ and $\cos q_2 = -1$ hold.

Therefore, we will only need to consider the case when (46) does not hold; this guarantees $\delta > 0$. From (44), we need to treat the cases $G_y = -\delta$ and $G_y = \delta$ separately.

Case i. $G_y = -\delta$

Since the Y-axis coordinate of the center of mass (CM) of the Acrobot is $c_5G_y/(m_1+m_2)$, then $G_y < 0$ implies that the CM of the Acrobot is below the horizontal. From (42), we have

$$\cos q_1 = -\sin(q_2)/\delta(q_2). \tag{47}$$

This yields with (43) that

$$\gamma q_2 - (\delta(q_2)+\beta+1)\sin(q_2)/\delta(q_2) = 0,$$

which can be rewritten as

$$q_2(\gamma - \xi(q_2)) = 0, \tag{48}$$

where

$$\xi(q_2) = \frac{(\delta(q_2)+\beta+1)\sin q_2}{\delta(q_2)q_2}. \tag{49}$$

Note that $\xi(q_2)$ is well defined for all q_2. Moreover, we can show in Appendix B that

$$\sup_{q_2\neq 0} \xi(q_2) = 2; \quad \xi(q_2) < 2, \text{ for } q_2 \neq 0, \tag{50}$$

which is free of β. Therefore, when $\gamma \geq 2$ holds, (48) has only solution $q_2 = 0$. Moreover, from (47) and $G_y < 0$, we obtain $\cos q_1 = 0$ and $\sin q_1 < 0$. Therefore, $q_1 = -\pi/2$ $(\mathrm{mod}\ 2\pi)$.

Case ii. $G_y = \delta$

In this case, the CM of the Acrobot is above the horizontal. From (42), we obtain

$$\cos q_1 = \sin(q_2)/\delta(q_2). \tag{51}$$

Putting it into (43) gives

$$\gamma q_2 - (\delta(q_2)-\beta-1)\sin(q_2)/\delta(q_2) = 0,$$

which can be rewritten as

$$q_2(\gamma - \eta(q_2)) = 0, \tag{52}$$

where

$$\eta(q_2) = \frac{(\delta(q_2)-\beta-1)\sin q_2}{\delta(q_2)q_2}. \tag{53}$$

Note that $\eta(q_2)$ is well defined for all q_2. Since we can show in Appendix C that

$$\sup_{q_2} \eta(q_2) < \frac{4\beta}{(\beta+1)\pi} < \frac{4}{\pi} < 2 \tag{54}$$

holds, therefore, (52) has only solution $q_2 = 0$ when $\gamma \geq 2$. From (51), we have $\cos q_1 = 0$. Since $\sin q_1 > 0$ holds owing to $G_y > 0$, we obtain $q_1 = \pi/2$ $(\mathrm{mod}\ 2\pi)$. It yields from (21) that $E = 0$ which contradicts the assumption $E \neq 0$. Therefore, $G_y = \delta$ will not occur at all.

Finally, to show the downward equilibrium point (DEP) is unstable, consider the state $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (-\pi/2 + \epsilon_1, 0, \epsilon_2, 0)$ near the DEP, where $\epsilon_1^2 + \epsilon_2^2 > 0$. Let E_ϵ and $E_{\rm down}$ be the energies of the Acrobot at the above state and at the DEP, respectively. From (7), we have

$$\begin{aligned} E_\epsilon &= (c_1 + c_2 + 2c_3)\epsilon_2^2/2 - (c_4 + c_5)g(\cos\epsilon_1 + 1) \\ &> -2(c_4 + c_5)g = E_{\rm down} \end{aligned}$$

Since there exists $\epsilon > 0$ such that $E_\epsilon < 0$ holds for any (ϵ_1, ϵ_2) satisfying $0 < \epsilon_1^2 + \epsilon_2^2 < \epsilon^2$, therefore, $V(-\pi/2 + \epsilon_1, 0, \epsilon_2, 0) < V(-\pi/2, 0, 0, 0)$ holds no matter how small the ϵ is. Note V is non-increasing under the control law (16), then starting from $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (-\pi/2 + \epsilon_1, 0, \epsilon_2, 0)$, the motion of the Acrobot will not converge to the DEP, but converge to the set M_0 described in (35). This shows the DEP is unstable. ■

According to Proposition 2, since the Acrobot can not be maintained at the DEP in practice, and from the homoclinic orbit (35), then the Acrobot starting from *any initial state*, will eventually be swung up to an arbitrarily small neighborhood of the upright equilibrium point, where the controller can be switched to any locally stabilizing one. Therefore, the objective of the swing up control for the Acrobot can be achieved under the conditions shown in Proposition 2.

Finally, we present the following remarks.

Remark 2: The key point in the proof of Proposition 2 is to show that if $\gamma \geq 2$ holds, then (38) (describing the equilibrium points of the Acrobot) and (39) have only the solution given by (41), where the elucidation of **Cases i and ii** *is interesting.*

Remark 3: Note that the assumption on the mechanical parameters needed in [12] is that none of (A4), (A5) and (31) holds. Such assumption is not needed here. Also, the condition $\gamma \geq 2$ in this paper is weaker and much simpler than the corresponding one provided in [12], where $\gamma > \max\{\sup_{q_2\in(0,\pi)} \xi(q_2), \sup_{q_2\in(\pi,2\pi)} \eta(q_2)\}$ was expressed, and the numerical computation was suggested.

V. Simulation Results

The validity of the developed theoretical results is verified via numerical simulation investigation to three types of the Acrobot described in [1], [5], [10]. Here we only introduce our numerical simulation results for the Acrobot in [10], where $m_1 = 1$[kg], $m_2 = 1$[kg], $l_1 = 1$[m], $l_2 = 2$[m], $l_{c1} = 0.5$[m], $l_{c2} = 1$[m], $I_1 = 0.083$[kg·m^2], $I_2 = 0.33$[kg·m^2] and $g = 9.8$[m/s^2].

Numerical computation of ρ^* in (14) yields $\rho^* = 0.8578$. For an initial condition $q_1(0) = -8\pi/9$, $q_2(0) = 0$, $\dot{q}_1(0) = 0$, $\dot{q}_2(0) = 0$, we choose $k_E = 0.5$, $k_V = 45$, and according to (15) and (34), we choose $k_D = 2(c_4 + c_5)k_E/\rho^* = 14.2804$, $k_P = 2c_4c_5g^2k_E = 144.0600$.

The simulation results under controller (16) with the above control parameters are depicted in Figs. 2 and 3. From Fig. 2, we know that q_2 converges to 0, and link

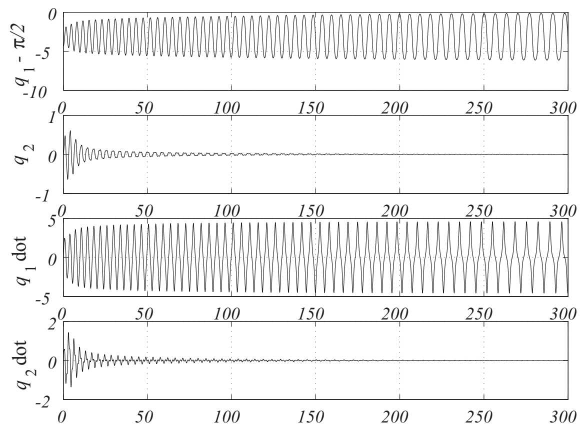

Fig. 2. Time responses of states of the Acrobot.

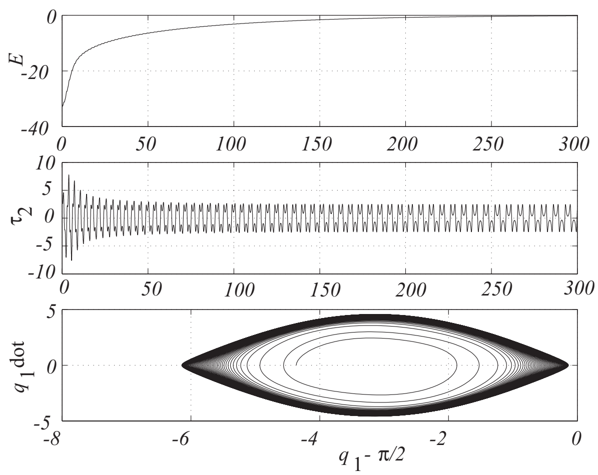

Fig. 3. Time responses of E, τ_2, and phase plot related to q_1.

1 remains swinging while approaching closer and closer to the vertical. From Fig. 3, we can observe that E converges to zero, and $(q_1, \dot{q}_1)$ converges to homoclinic orbit (35).

VI. Conclusions

This paper has provided the complete analysis of the convergence of the energy and the motion of the Acrobot, and has illustrated clearly several unique characteristics of the closed-loop system for the Acrobot under the energy based control. This paper has showed clearly and elegantly how to choose the control parameters to swing up the Acrobot. This paper has proved that the energy based control is effective for swinging up the Acrobot. The simulation results have also provided to demonstrate the validity of the theoretical results. Finally, it is expected that the results attained here will fuel the further studies of the energy based control for more complicated underactuated mechanical systems.

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Appendix A. On solvability of (30) with respect to the mechanical parameters

LEMMA A1: There exists q_2^* satisfying (30) if and only if (31) holds. In this case, $q_2^* = \pi \ (mod\, 2\pi)$.

Proof: From $B(q_2^*) = 0$, the possible solutions of (30) are q_2^* satisfying one of

$$\cos q_2^* = -\alpha_0, \tag{A1}$$

$$\cos q_2^* = 1, \tag{A2}$$

$$\cos q_2^* = -1, \tag{A3}$$

where $\alpha_0 = (2c_3c_4 + c_1c_5)/(3c_3c_5)$. We will show: if (A1) or (A2) holds, then $A(q_2^*) \neq 0$ holds for any c_i in (5); if (A3) holds, then $A(q_2^*) = 0$ holds if and only if (31) holds.

First, note that q_2^* satisfying (A1) is a solution of (30) if and only if $\alpha_0 \leq 1$ and $A\big|_{\cos q_2^* = -\alpha_0} = 0$. Via a direct calculation, this is equivalent to

$$\begin{cases} c_1 + 2l_1c_4 \leq 3l_1c_5 \\ c_2c_4 + 2l_1c_5^2 = (c_1 + 2l_1c_4)c_4 \end{cases}. \tag{A4}$$

However, (A4) does not hold. The detail of proving this fact is omitted here.

Second, note that q_2^* satisfying (A2) is a solution of (30) if and only if

$$A\big|_{\cos q_2^* = 1} = c_3c_5 + c_1c_5 - c_2c_4 - c_3c_4 = 0 \tag{A5}$$

However, $c_3c_5 + c_1c_5 - c_2c_4 - c_3c_4 < 0$ follows from the following inequalities:

$$c_3c_5 < c_2c_4, \quad c_1c_5 \leq c_3c_4, \tag{A6}$$

whose proof is omitted here.

Finally, observe that q_2^* satisfying (A3) is a solution of (30) if and only if $A\big|_{\cos q_2^* = -1} = 0$, which is equivalent to (31). This completes the proof of Lemma A1. ■

Here we give an example showing that there do exist c_i ($i = 1, \cdots, 5$) satisfying (31). Consider the Acrobot with $l_{c1} = l_1/2$, $I_1 = m_1 l_1^2/12$, $l_{c2} = l_2/2$, $I_2 = m_2 l_2^2/12$. From $(c_1 - c_3)c_5 - (c_3 - c_2)c_4 = -m_1 m_2 l_1 l_2 (l_1 - (2 + m_2/m_1)l_2)/12$, we know that (31) holds if and only if $l_1 = (2 + m_2/m_1)l_2$ holds.

Appendix B. Proof of (50)

Note that $\xi(q_2)$ is an even function; and $\xi(q_2) \leq 0$ for $q_2 \in [(2n-1)\pi, 2n\pi]$ due to $\sin q_2 \leq 0$, where n is a positive integer. Therefore, we obtain

$$\sup_{q_2 \neq 0} \xi(q_2) = \sup_n \left\{ \sup_{q_2 \in (2(n-1)\pi, (2n-1)\pi)} \xi(q_2) \right\} > 0.$$

Since $\sup_{q_2 \in (2(n-1)\pi, (2n-1)\pi)} \xi(q_2)$ is a strictly decreasing function of n, then $\sup_{q_2 \neq 0} \xi(q_2) = \sup_{q_2 \in (0,\pi)} \xi(q_2)$. Since $\lim_{q_2 \to 0} \xi(q_2) = 2$, it suffices to show $\xi(q_2) < 2$ for $q_2 \in (0, \pi)$.

Next, we establish the following inequality, whose proof is omitted.

$$4q_2(q_2 - \sin q_2) > (1 - \cos q_2)^2, \quad q_2 > 0. \tag{B1}$$

Multiplying $2\beta(1 + \cos q_2)$ to the both sides of (B1) gives

$$4(2\beta + 2\beta \cos q_2) q_2 (q_2 - \sin q_2) > 2\beta(1 - \cos q_2)\sin^2 q_2,$$

which yields together with the fact $1 + \beta^2 \geq 2\beta$ that

$$4\delta^2 q_2(q_2 - \sin q_2) > 2\beta(1 - \cos q_2)\sin^2 q_2.$$

By adding $\delta^2 \sin^2 q_2$ to the above inequality to complete the square, we obtain $\delta^2(2q_2 - \sin q_2)^2 > (\beta+1)^2 \sin^2 q_2$. Since $q_2 > \sin q_2 > 0$ holds for $q_2 \in (0, \pi)$, then $\delta(2q_2 - \sin q_2) > (\beta+1)\sin q_2$. This gives $2\delta q_2 - (\delta + \beta + 1)\sin q_2 > 0$ which shows $\xi(q_2) < 2$. This completes the proof of (50). ■

Appendix C. Proof of (54)

Note that $\eta(q_2)$ is an even function. Since $\delta \leq \beta + 1$, we obtain $\eta(q_2) \leq 0$ for $q_2 \in [2(n-1)\pi, (2n-1)\pi]$ due to $\sin q_2 \geq 0$, where n is a positive integer. Therefore,

$$\begin{aligned} \sup_{q_2} \eta(q_2) &= \sup_n \left\{ \sup_{q_2 \in ((2n-1)\pi, 2n\pi)} \eta(q_2) \right\} \\ &= \sup_{q_2 \in (\pi, 2\pi)} \eta(q_2). \end{aligned}$$

Using $\eta(q_2) = 2\beta(\cos q_2 - 1)\sin(q_2)/((\beta + 1 + \delta)\delta q_2)$ and (51), we obtain

$$\begin{aligned} &\sup_{q_2 \in (\pi, 2\pi)} \eta(q_2) \\ \leq &\sup_{q_2 \in (\pi, 2\pi)} \frac{2\beta(1 - \cos q_2)}{(\beta + 1 + \delta)} \cdot |\cos q_1| \cdot \sup_{q_2 \in (\pi, 2\pi)} \frac{1}{q_2}. \end{aligned}$$

This completes the proof of (54). ■