EE 505 HW 1

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Problem 1.8

We are given a model for a generator

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin\delta$$
$$\tau \dot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E_{FD}$$

a) Let $x_1 = \delta$, $x_2 = \dot{\delta}$, and $x_3 = E_q$. The state equation becomes

$$f(x) = \begin{bmatrix} x_2 \\ \frac{1}{M}(\eta_1 x_3 \sin x_1 - D x_2 + P) \\ \frac{1}{\tau}(\eta_3 \cos x_1 - \eta_2 x_3 + E_{FD}) \end{bmatrix}$$

b) The equilibrium points occur when $f(x) = 0$. Therefore, $x_2 = 0$, while x_1 and x_3 obey

$$\eta_1 x_3 \sin x_1 + P = 0$$
$$\eta_3 \cos x_1 - \eta_2 x_3 + E_{FD} = 0$$

Writing $x_3 = \frac{\eta_3}{\eta_2}\cos x_1 + \frac{1}{\eta_2}E_{FD}$ and substituting, we see that x_1 obeys

$$\frac{\eta_1}{\eta_2}(\eta_3 \cos x_1 + E_{FD}) \sin x_1 - P = 0$$

which is periodic with $T = 2\pi$. Using the given values of the constants, we can plot the equation above to find when x_1 is zero, as shown in Fig. 1. Using the `fzero` command, we see that

$$x_1 = 0.4067 + 2\pi n \text{ or } x_1 = 1.6398 + 2\pi n$$

Under these conditions,

$$x_3 = 1.0301 \text{ or } x_3 = 0.4085$$

Therefore, the equilibrium points are

$$x = (0.4067 + 2\pi n, 0, 1.0301) \text{ or}$$
$$x = (1.6398 + 2\pi n, 0, 0.4085)$$

c) When $\dot{E}_q = 0$, the system is described with

$$M\ddot{\delta} + D\dot{\delta} + \eta_1 E_q \sin\delta = P$$

The equation of motion for a damped pendulum is given as

$$\ddot{y} + \frac{k}{m}\dot{y} + \frac{g}{\ell}\sin y = \frac{1}{ml^2}T$$

It is clear that both equations can be written in the form

$$\ddot{x} + b\dot{x} + c\sin x = F$$

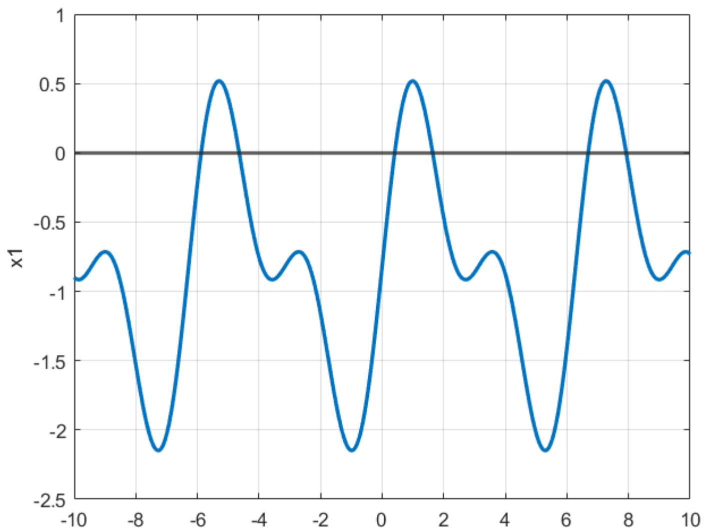

Figure 1: Zeros of x_1

Problem 1.9

a) Using the state variables ϕ_L and v_c, we find an expression for $f(t,x)$. From KCL, we see that

$$i_S(t) = i_R + i_C + i_L = \frac{1}{R}v_C + C\dot{v}_C + I_0 \sin k\phi_L$$
$$\Rightarrow \dot{v}_C = -\frac{1}{RC}v_C - \frac{I_0}{C}\sin k\phi_L + \frac{1}{C}i_S(t)$$

Since the inductor and capacitor are in parallel,

$$v_C = v_L = L\frac{di_L}{dt} = L\frac{d}{dt}(\sin k\phi_L) = kL\dot{\phi}_L \cos k\phi_L$$
$$\Rightarrow \dot{\phi}_L = \frac{v_C}{kL\cos k\phi_L}$$

Using $x_1 = \phi_L$ and $x_2 = v_C$, we have

$$f(t,x) = \left[\frac{x_2}{kL\cos kx_1}, \quad -\frac{I_0}{C}\sin kx_1 - \frac{1}{RC}x_2 + \frac{1}{C}i_S(t)\right]^T$$

b) It is easier to obtain an expression for $f(t,x)$ using the state variables i_L and v_C. The state equations become

$$\dot{v}_C = -\frac{1}{RC}v_C - \frac{1}{C}i_L + \frac{1}{C}i_S(t)$$

and

$$\frac{di_L}{dt} = \frac{1}{L}v_C$$

Problem 1.10

a) When $i_L = L\phi_L + \mu\phi_L^3$, the first state equation is simply

$$\dot{v}_C = -\frac{1}{RC}v_C - \frac{1}{C}(L\phi_L + \mu\phi_L^3) + \frac{1}{C}i_S(t)$$

The second state equation is

$$v_C = L\frac{di_L}{dt} = L\frac{d}{dt}(L\phi_L + \mu\phi_L^3) = L(L\dot{\phi}_L + 3\mu\phi_L^2\dot{\phi}_L) = \dot{\phi}_L(L^2 + 3\mu L\phi_L^2)$$
$$\Rightarrow \dot{\phi}_L = \frac{v_C}{L^2 + 3\mu L\phi_L^2}$$

b) When $i_S(t) = 0$, the equilibrium points are found by setting $\dot{v}_C = \dot{\phi}_L = 0$. From the second state equation, it is clear that equilibrium is reached only when $v_C = 0$. This implies

$$\phi_L(L + \mu\phi_L^2) = 0$$

Since μ and L are positive constants and ϕ_L is real, this expression holds only when $\phi_L = 0$. Therefore, the equilibrium point is located at the origin:

$$(\phi_L, v_C) = (0, 0)$$

Problem 1.11

a) Show that $\dot{z} = Az + B\sin e$ and $\dot{e} = -Cz$. From the block diagram, the following relationships hold:

$$e = \theta_i - \theta_0,\ u = \sin e,\ \theta_o = \int_{-\infty}^{t} y\,dt$$

Since $G(s)$ represents the stable system $\{A, B, C\}$ and z is the realization of $\{A, B, C\}$,

$$\dot{z} = Az + Bu$$
$$y = Cz$$

Substituting for u,

$$\dot{z} = Az + B\sin e$$

Taking a derivative of e,

$$\dot{e} = -\dot{\theta}_o = -y \Rightarrow \dot{e} = -Cz$$

b) The equilibrium points are found by setting $\dot{z} = \dot{e} = 0$. Further, A, B, C are constant coefficient matrices. From $\dot{e} = 0 \Rightarrow Cz = 0$, we see that $z \in \mathcal{N}(C)$. (z is in the null space of C). Next, we have

$$0 = Az + B\sin e$$

Rearranging and multiplying on the left by A^{-1}, we see that

$$z = -A^{-1}B\sin e$$

Now multiply on the left by C. We have

$$Cz = 0 = -CA^{-1}B\sin e$$

We know that $G(s) = C(sI - A)^{-1}B$, and $G(0) = -CA^{-1}B \neq 0$. Therefore, $\dot{z} = 0$ only when $\sin e = 0$. This occurs at $e = k\pi$, with $k \in \mathcal{Z}$. (k is any integer.)

c) Show that with $G(s) = 1/(\tau s+1)$ the closed-loop model coincides with the model of a pendulum. We can represent this transfer function in state space with $A = -\frac{1}{\tau}$, $B = \frac{1}{\tau}$, and $C = 1$. Proof:

$$G(s) = C(sI - A)^{-1}B = \frac{1/\tau}{s + 1/\tau} = \frac{1}{\tau s + 1}$$

Plugging into the state space model with $y = z$, we have

$$\dot{y} = -\frac{1}{\tau}y + \frac{1}{\tau}\sin e$$

From $e = \theta_i - \int y\,dt$, we have $\dot{e} = -y$ and $\ddot{e} = -\dot{y}$. Substituting for e and simplifying, we have

$$\ddot{e} + \frac{1}{\tau}\dot{e} + \frac{1}{\tau}\sin e = 0$$

The equation of motion for an unforced damped pendulum is given as

$$\ddot{y} + \frac{k}{m}\dot{y} + \frac{g}{\ell}\sin y = 0$$

It is clear that both equations can be written in the form

$$\ddot{x} + b\dot{x} + c\sin x = 0$$

Problem 1.12

The spring is described by the equation

$$m\ddot{y} = F_s + F_d = c_0 y + c_1\dot{y} + c_2\dot{y}|\dot{y}|$$

Letting $x_1 = y$ and $x_2 = \dot{y}$, the state equation is

$$f(x) = \begin{bmatrix} x_2 \\ c_0x_1 + c_1x_2 + c_2x_2|x_2| \end{bmatrix}$$

This can be written without the absolute values as

$$f(x) = \begin{bmatrix} x_2 \\ c_0x_1 + c_1x_2 + \operatorname{sgn}(x_2)c_2x_2^2 \end{bmatrix}$$

Problem 1.13

a) The equation of motion for the mass is given by

$$\begin{aligned} m\ddot{y} &= F_{k1} + F_{k2} + F_{drag} + F_{friction} \\ &= -k_1y - k_2y - b\dot{y} + h(v_0 - \dot{y}) \end{aligned}$$

Note that the $h(\cdot)$ term is positive because when $\dot{y} < v_0$, the friction force is to the right (positive y direction). When $\dot{y} > v_0$, the friction force is to the left.
b) When $|\dot{y}| \ll v_0$, $h(v) = h(v_0) - \dot{y}h'(v_0)$. The equation of motion becomes

$$\begin{aligned} m\ddot{y} &= -k_1y - k_2y - b\dot{y} + h(v_0) - \dot{y}h'(v_0) \\ &= -(k_1 + k_2)y - (b + h'(v_0))\dot{y} + h(v_0) \end{aligned}$$

c) The negative friction will occur when $b + h'(v_0) < 0$. For small b, this condition is met when $h'(v_0) < 0$ and $\dot{y} > 0$. From section 1.2.3, we see this condition will occur when Coulomb and Linear Friction is present along with the Stribeck effect.

Problem 1.14

The y be the distance traveled along the inclined road to the left. The equation of motion for the vehicle is

$$\begin{aligned} M\ddot{y} &= F_\mu + F_v + F_d + F_g + F \\ &= -\mu M g \cos\theta - b\dot{y} - c\dot{y}^2 - Mg\sin\theta + F \end{aligned}$$

Let $x_1 = y$ and $x_2 = \dot{y}$. The state model with input F and disturbance θ is

$$\dot{x} = f(x, \theta, F) = \begin{bmatrix} x_2 \\ -bx_2 - cx_2^2 - \mu M g \cos\theta - Mg\sin\theta + F \end{bmatrix}$$

Problem 1.15

a) Carrying out the indicated differentiation, we have

$$\begin{aligned} H &= m\ddot{y} + mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\ V &= mg - mL(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \end{aligned}$$

Now we multiply H by $L\cos\theta$ and multiply V by $L\sin\theta$ and plug into the moment equation. We simplify using the familiar identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} I\ddot{\theta} &= mgL\sin\theta - mL^2(\ddot{\theta}\sin^2\theta + \dot{\theta}^2\sin\theta\cos\theta) - mL^2(\ddot{\theta}\cos^2\theta - \dot{\theta}^2\sin\theta\cos\theta) - mL\ddot{y}\cos\theta \\ &= mgL\sin\theta - mL^2\ddot{\theta} - mL\ddot{y}\cos\theta \end{aligned}$$

Substituting H into the equation of motion for the cart immediately gives

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y}$$

b) Now we can solve for $\ddot{\theta}$ and $\ddot{y}$. First, let us define Eqs. (1) and (2) as

$$I\ddot{\theta} = mgL\sin\theta - mL^2\ddot{\theta} - mL\ddot{y}\cos\theta \tag{1}$$

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y} \tag{2}$$

To solve for $\ddot{\theta}$, we use the elimination method with $(m+M)(Eq.1)+(-mL\cos\theta)(Eq.2)$ and obtain

$$\ddot{\theta}[(m+M)(I+mL^2) - m^2L^2\cos^2\theta] = (m+M)mgL\sin\theta - mL\cos\theta(F + mL\dot{\theta}^2\sin\theta - k\dot{y})$$

To solve for $\ddot{y}$, we use $(-mL\cos\theta)(Eq.1) + (I+mL^2)(Eq.2)$ and obtain

$$\ddot{y}[(m+M)(I+mL^2) - m^2L^2\cos^2\theta] = (-mL\cos\theta)mgL\sin\theta + (I+mL^2)(F + mL\dot{\theta}^2\sin\theta - k\dot{y})$$

Recognizing the common factors, we can write this as

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix} \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

where $\Delta(\theta)$ is the scalar quantity

$$\Delta(\theta) = (m+M)(I+mL^2) - m^2L^2\cos^2\theta$$

Because $0 \le \cos^2\theta \le 1$,

$$\Delta(\theta) \ge (I+mL^2)M + mI > 0$$

c) With $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$, the state equation is

$$\dot{x} = f(t,x) = \begin{bmatrix} x_2 \\ \frac{1}{\Delta(x_1)}\big((m+M)mgL\sin x_1 - mL\cos x_1(F(t) + mLx_2^2\sin x_1 - kx_4)\big) \\ x_4 \\ \frac{1}{\Delta(x_1)}\big((-mL\cos x_1)mgL\sin x_1 + (I+mL^2)(F(t) + mLx_2^2\sin x_1 - kx_4)\big) \end{bmatrix}$$

where

$$\Delta(x_1) = (m+M)(I+mL^2) - m^2L^2\cos^2 x_1$$

Problem 1.16

a) Carrying out the differentiation gives

$$\begin{aligned} F_x &= m\ddot{x}_c + mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\ F_y &= mL(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \end{aligned}$$

Plugging into the torque equation and simplifying using the same technique as Problem 1.15 gives

$$\begin{aligned} (I+mL^2)\ddot{\theta} &= u - mL\cos\theta\ddot{x}_c \\ \Rightarrow (I+mL^2)\ddot{\theta} + mL\cos\theta\ddot{x}_c &= u \end{aligned}$$

Plugging into the equation of motion for the platform gives

$$\begin{aligned} (m+M)\ddot{x}_c &= -mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) - kx_c \\ \Rightarrow (mL\cos\theta)\ddot{\theta} + (m+M)\ddot{x}_c &= mL\dot{\theta}^2\sin\theta - kx_c \end{aligned}$$

We can rewrite the ($\Rightarrow$) equations in the form

$$D(\theta)\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix}$$

where

$$D(\theta) = \begin{bmatrix} I+mL^2 & mL\cos\theta \\ mL\cos\theta & m+M \end{bmatrix}$$

b) Multiplying each side by $D^{-1}(\theta)$, we see that

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta D(\theta)}\begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}\begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix}$$

where

$$\Delta D(\theta) = (I+mL^2)(m+M) - m^2L^2\cos^2\theta \geq (I+mL^2)M + mI > 0$$

c) With $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x_c$, and $x_4 = \dot{x}_c$, the state equation is

$$\dot{x} = f(t,x) = \begin{bmatrix} x_2 \\ \frac{1}{\Delta D(x_1)}\big((m+M)u(t) - mL\cos x_1(mLx_2^2 - kx_3)\big) \\ x_4 \\ \frac{1}{\Delta D(x_1)}\big((-mL\cos x_1)u(t) + (I+mL^2)(mLx_2^2 - kx_3)\big) \end{bmatrix}$$

where

$$\Delta D(x_1) = (I+mL^2)(m+M) - m^2L^2\cos^2 x_1$$

d) The equilibrium points occur when $f(x) = 0$ and $u(t) = 0$. This forces $x_2 = x_4 = 0$. The state equation reduces to

$$\begin{aligned}0 &= kx_3 \cos x_1 \\ 0 &= kx_3\end{aligned}$$

so equilibrium points occur on the line $x = [x_1, 0, 0, 0]^T$, with $x_1 \in \mathbb{R}$.

Problem 1.17

a) Choose $x_1 = i_f$, $x_2 = i_a$, $x_3 = \omega$, $u_1 = v_f$, and $u_2 = v_a$. Then the state equation is

$$\dot{x} = f(t, x) = [f_1(t, x), f_2(t, x), f_3(x)]^T$$

where

$$\begin{aligned}f_1(t, x) &= -\frac{R_f}{L_f} x_1 + \frac{1}{L_f} u_1(t) \\ f_2(t, x) &= -\frac{c_1}{L_a} x_1 x_3 - \frac{R_a}{L_a} x_2 + u_2(t) \\ f_3(x) &= \frac{c_2}{J} x_1 x_2 - \frac{c_3}{J} x_3\end{aligned}$$

b) When v_a is held constant, the state equation is

$$\dot{x} = f(t, x) = [f_1(t, x), f_2(x), f_3(x)]^T$$

where

$$\begin{aligned}f_1(t, x) &= -\frac{R_f}{L_f} x_1 + \frac{1}{L_f} u_1(t) \\ f_2(x) &= -\frac{c_1}{L_a} x_1 x_3 - \frac{R_a}{L_a} x_2 + v_a \\ f_3(x) &= \frac{c_2}{J} x_1 x_2 - \frac{c_3}{J} x_3\end{aligned}$$

c) When v_f is held constant, the state equation is

$$\dot{x} = f(t, x) = [f_1(x), f_2(t, x), f_3(x)]^T$$

where

$$\begin{aligned}f_1(x) &= -\frac{R_f}{L_f} x_1 + \frac{1}{L_f} v_f \\ f_2(t, x) &= -\frac{c_1}{L_a} x_1 x_3 - \frac{R_a}{L_a} x_2 + u_2(t) \\ f_3(x) &= \frac{c_2}{J} x_1 x_2 - \frac{c_3}{J} x_3\end{aligned}$$

Notice that $\dot{x}_1 = f_1(x)$ does not depend on any other state variables or control inputs, so we can obtain a closed form expression for x_1 (WLOG let t_0=0).

$$\begin{aligned}
\dot{x}_1 &= -\frac{R_f}{L_f}x_1 + \frac{1}{L_f}v_f \\
sX_1(s) &= -\frac{R_f}{L_f}X_1(s) + \frac{v_f}{L_f}\frac{1}{s} \\
X_1(s) &= \frac{v_f}{L_f}\frac{1}{s(s+R_f/L_f)} \\
&= \frac{v_f}{R_f}\left(\frac{1}{s} - \frac{1}{s+R_f/L_f}\right) \\
\Rightarrow x_1(t) &= \frac{v_f}{R_f}\left(1 - e^{-(R_f/L_f)t}\right)
\end{aligned}$$

This can be substituted into the other state equations to obtain a 2nd order state equation

$$\dot{x} = f(t,x) = [f_1(t,x), f_2(x)]^T$$

where

$$\begin{aligned}
f_1(t,x) &= -\frac{c_1}{L_a}\frac{v_f}{R_f}\left(1 - e^{-(R_f/L_f)t}\right)x_3 - \frac{R_a}{L_a}x_2 + u_2(t) \\
f_2(x) &= \frac{c_2}{J}\frac{v_f}{R_f}\left(1 - e^{-(R_f/L_f)t}\right)x_2 - \frac{c_3}{J}x_3
\end{aligned}$$

d) Substituting $v = v_a = v_f + R_x i_f$ into the original state equation and letting $u(t) = v(t)$, the state equation is

$$\dot{x} = f(t,x) = [f_1(t,x), f_2(t,x), f_3(x)]^T$$

where

$$\begin{aligned}
f_1(t,x) &= -\frac{R_f + R_x}{L_f}x_1 + \frac{1}{L_f}u(t) \\
f_2(t,x) &= -\frac{c_1}{L_a}x_1x_3 - \frac{R_a}{L_a}x_2 + u(t) \\
f_3(x) &= \frac{c_2}{J}x_1x_2 - \frac{c_3}{J}x_3
\end{aligned}$$

Problem 1.18

a) The two equations that govern the behavior of the magnetic levitation system are

$$\begin{aligned}
m\ddot{y} &= -k\dot{y} + mg - \frac{L_0 i^2}{2a(1+y/a)^2} \\
v &= \frac{d}{dt}\left(L_1 i + \frac{L_0 i}{1+y/a}\right) - Ri
\end{aligned}$$

Expanding the second equation and solving for $\frac{di}{dt}$ gives

$$\frac{di}{dt} = \frac{1}{L(y)}\left(v + Ri + \frac{L_0 i\dot{y}}{(1+y/a)^2}\right)$$

where $L(y)$ is defined as

$$L(y) = L_1 + \frac{L_0}{1+y/a}$$

Now let $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i$, and $u = v$. The state equation is $\dot{x} = f(t, x) = [f_1(x), f_2(x), f_3(t, x)]^T$ where

$$
\begin{aligned}
f_1(x) &= x_2 \\
f_2(x) &= -\frac{k}{m}x_2 + g - \frac{L_0 x_3^2}{2ma(1 + x_1/a)^2} \\
f_3(t, x) &= \frac{1}{L(x)}\left(Rx_3 + \frac{L_0 x_3 x_2}{(1 + x_1/a)^2} + u(t)\right)
\end{aligned}
$$

and $L(x) = L(y)\big|_{y=x_1}$.

b) When the ball is stationary at $x_1 = r$, $f_1(x) = f_2(x) = f_3(t, x) = 0$. That implies $x_2 = 0$. Solving the remaining state equations for x_3 and $u(t)$ give

$$
\begin{aligned}
x_3 = I_{ss} &= \frac{1}{1 + r/a}\sqrt{\frac{2gma}{L_0}} \\
u(t) = V_{ss} &= -RI_{ss} = -\frac{R}{1 + r/a}\sqrt{\frac{2gma}{L_0}}
\end{aligned}
$$