EE 505 HW 5

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4.1 Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable:

(1) stable node	**(2)** unstable node	**(3)** stable focus
(4) unstable focus	**(5)** center	**(6)** saddle

Justify your answer using phase portraits.

From the phase portraits, we draw a ball with radius delta around the equilibrium point and observe how the trajectories behave. From this analysis, the six types of equilibrium points obey the following behavior.

1. asymptotically stable
2. unstable
3. asymptotically stable
4. unstable
5. stable (not AS)
6. unstable

4.2 Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

Consider the Lyapunov candidate function $V = \frac{1}{2}x^2$. Then, $\dot{V} = x\dot{x} \leq ax^{p+1} + k|x|^{p+2}$. Near the origin with $a \neq 0$, the $k|x|^{p+2}$ term is smaller. Now consider the following cases, and let c be some positive constant.

Table 1: Comparison of Average Speed to Prior Work

Condition		Result	Stability
$a < 0$	p odd	$\dot{V} < 0$	AS
$a > 0$	p odd	$\dot{V} \leq c$	Unstable
$a \neq 0$	p even	$\dot{V} \leq c$	Unstable

4.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable:

$$
\begin{array}{llllll}
(1) & \dot{x}_1 & = & -x_1 + x_1 x_2, & \dot{x}_2 & = & -x_2 \\
(2) & \dot{x}_1 & = & -x_2 - x_1(1 - x_1^2 - x_2^2), & \dot{x}_2 & = & x_1 - x_2(1 - x_1^2 - x_2^2) \\
(3) & \dot{x}_1 & = & x_2(1 - x_1^2), & \dot{x}_2 & = & -(x_1 + x_2)(1 - x_1^2) \\
(4) & \dot{x}_1 & = & -x_1 - x_2, & \dot{x}_2 & = & 2x_1 - x_2^3
\end{array}
$$

Investigate whether the origin is globally asymptotically stable.

1) Choose the Lyapunov candidate function $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Then,

$$\dot{V} = -x_1^2 - x_2^2 + x_1^2 x_2$$

Now consider the ball with radius r, $\{x \in \mathbb{R}^2 \;\big|\; ||x||_2 \leq r\}$. We have $|x_1| \leq r$, so we can write

$$\dot{V} \leq -x_1^2 - x_2^2 + r|x_1||x_2| \leq -\begin{bmatrix} |x_1| & |x_2| \end{bmatrix} \begin{bmatrix} 1 & -r/2 \\ -r/2 & 1 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$$

We note that the matrix is row equivalent to

$$\begin{bmatrix} 1 & -r/2 \\ 0 & 1 - r^2/4 \end{bmatrix}$$

with eigenvalues on the diagonals. Therefore, this matrix is positive definite when $r < 2$, and we have local asymptotic stability at the origin. I don't know whether this equilibrium point is globally AS.

2) Choose the same Lyapunov candidate function as before. Then,

$$\dot{V} = -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) < 0$$

when $x_1^2 + x_2^2 < 1$. Therefore, we have local AS. The equilibrium point is clearly not globally AS, as $\dot{V} > 0$ for $x_1^2 + x_2^2 > 1$.

3) Choose the same Lyapunov candidate function as before. Then,

$$\dot{V} = -x_2^2(1 - x_1^2) < 0$$

when $x_1 < 1$. The equilibrium point is locally AS, but not globally AS because $x_1 = 1$ is an equilibrium point.

4) Choose a slightly modified Lyapunov candidate function $V = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2$. Then,

$$\dot{V} = -x_1^2 - 3/2x_2^2 < 0$$

so the equilibrium point is globally and locally AS.

4.4 ([151]) Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}$$

where ω_1 to ω_3 are the components of the angular velocity vector ω along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

a) The equilibrium point is stable, but not AS, as $V = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2) \Rightarrow \dot{V} = 0$.

b) The equilibrium point is globally AS. Use the same V as before. Then,

$$\dot{V} = -k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2 < 0$$

4.5 Let $g(x)$ be a map from R^n into R^n. Show that $g(x)$ is the gradient vector of a scalar function $V : R^n \to R$ if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, 2, \ldots, n$$

First, we prove this in the forward direction. Assume that $g(x) = \nabla V$, so $g_i(x) = \frac{\partial V}{\partial x_i}$. Now,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial V}{\partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j}$$

and

$$\frac{\partial g_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial V}{\partial x_j} = \frac{\partial^2 V}{\partial x_i \partial x_j}.$$

Next, we prove this in the reverse direction. Taking inspiration from example 4.5, we have

$$V = \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2, \ldots, 0)\, dy_2 + \cdots + \int_0^{x_n} g_1(x_1, x_2, \ldots, y_n)\, dy_n$$

Now take a derivative with respect to x_1, and use the symmetric Jacobian assumption:

$$\begin{aligned}
\frac{\partial V}{\partial x_1} &= g_1(x_1, 0, \ldots, 0) + \int_0^{x_2} \frac{\partial}{\partial x_1} g_2(x_1, y_2, \ldots, 0)\, dy_2 + \cdots + \int_0^{x_n} \frac{\partial}{\partial x_1} g_n(x_1, x_2, \ldots, y_n)\, dy_n \\
&= g_1(x_1, 0, \ldots, 0) + \int_0^{x_2} \frac{\partial}{\partial x_2} g_1(x_1, y_2, \ldots, 0)\, dy_2 + \cdots + \int_0^{x_n} \frac{\partial}{\partial x_n} g_1(x_1, x_2, \ldots, y_n)\, dy_n \\
&= g_1(x_1, 0, \ldots, 0) + g_1(x_1, x_2, \ldots, 0) - g_1(x_1, 0, \ldots, 0) + \ldots \\
&\qquad + g_1(x_1, x_2, \ldots, x_n) - g_1(x_1, x_2, \ldots, x_{n-1}, 0) \\
&= g_1(x_1, x_2, \ldots, x_n) = g_1(x)
\end{aligned}$$

So, we have $\frac{\partial V}{\partial x_1} = g_1(x)$. In a similar manner, we have

$$\frac{\partial V}{\partial x_i} = g_i(x)$$

for all $1 \leq i \leq n$. Therefore, $g(x) = \nabla V$ and we conclude our proof.

4.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

where h is continuously differentiable and $zh(z) > 0$ for all $z \neq 0$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

Taking inspiration from example 4.5 in the textbook, choose $g(x) = \nabla V$ as

$$g(x) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

Since $\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$, we have that $\beta = \gamma$. Now we investigate $\dot{V}$.

$$\begin{aligned} \dot{V} &= g_1(x)\dot{x}_1 + g_2(x)\dot{x_1} \\ &= (\alpha - \beta - \delta)x_1 x_2 + \beta x_2^2 - \delta x_2^2 - \beta x_1^2 - \beta x_1 h(x_1 + x_2) - \delta x_2 h(x_1 + x_2) \end{aligned}$$

To eliminate the cross terms, choose $\alpha = \beta + \delta$. To obtain the form $zh(z) > 0$, let $\beta = \delta$. Then,

$$\dot{V} = -\beta x_1^2 - \beta(x_1 + x_2)h(x_1 + h_2) < 0$$

on $x \in \mathbb{R}^2 \setminus 0$. Now, we integrate to obtain V.

$$\begin{aligned} V &= \int_0^{x_1} g_1(y_1, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2)\, dy_2 \\ &= \beta(x_1^2 + x_1 x_2 + \frac{1}{2}x_2^2) \\ &= \beta x^T \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} x = \beta x^T Q x \end{aligned}$$

and $Q > 0$ since its eigenvalues are both strictly positive. We just choose $\beta > 0$ and we obtain our Lyapunov function $V(x)$. Since $V(x)$ is radially unbounded, the origin is global AS.

4.7 Consider the system $\dot{x} = -Q\phi(x)$, where Q is a symmetric positive definite matrix and $\phi(x)$ is a continuously differentiable function for which the ith component ϕ_i depends only on x_i, that is, $\phi_i(x) = \phi_i(x_i)$. Assume that $\phi_i(0) = 0$ and $y\phi_i(y) > 0$ in some neighborhood of $y = 0$, for all $1 \le i \le n$.

(a) Using the variable gradient method, find a Lyapunov function that shows that the origin is asymptotically stable.

(b) Under what conditions will it be globally asymptotically stable?

(c) Apply to the case

$$n = 2,\ \phi_1(x_1) = x_1 - x_1^2,\ \phi_2(x_2) = x_2 + x_2^3,\ Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) Choose $g(x) = \nabla V = Q^{-1}x$. Then,

$$\dot{V} = -g^T(x)Qx = -x^T\phi(x) < 0.$$

Integrating to find V, we see that

$$V = \int_0^x Q^{-1}ydy = \frac{1}{2}x^TQ^{-1}x$$

Note that $Q^{-1} > 0$ because Q is positive definite. In summary, $\dot{V} < 0$ and $V > 0$ in some neighborhood of the origin. Thus, the origin is locally AS.

b) The origin is global AS when $y\phi_i(y) > 0$ for all $1 \le i \le n$ and $y \in \mathbb{R} \setminus 0$, as V is radially unbounded.

c) We see that $Q = Q^T > 0$, $\phi_i(x) = \phi_i(x_i)$, and $x_i\phi_i(x_i) > 0$ in the neighborhood of the origin. However, $x_1\phi_1(x_1) = x_1^2 - x_1^3$ is not negative definite for all possible x_1, therefore the origin is locally AS but not globally AS.

4.8 ([72]) Consider the second-order system

$$\dot{x}_1 = \frac{-6x_1}{u^2} + 2x_2, \qquad \dot{x}_2 = \frac{-2(x_1 + x_2)}{u^2}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

(b) Consider the hyperbola $x_2 = 2/(x_1 - \sqrt{2})$. Show, by investigating the vector field on the boundary of this hyperbola, that trajectories to the right of the branch in the first quadrant cannot cross that branch.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), show that $\dot{x}_2/\dot{x}_1 = -1/(1 + 2\sqrt{2}x_1 + 2x_1^2)$ on the hyperbola, and compare with the slope of the tangents to the hyperbola.

a) Obviously $V(x) > 0$. Now,

$$\dot{V} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \begin{bmatrix} \frac{2x_1}{u^2} & 2x_2 \end{bmatrix} \begin{bmatrix} -\frac{6x_1}{u^2} + 2x_2 \\ -\frac{2(x_1 + x_2)}{u^2} \end{bmatrix} = -\frac{12x_1^2}{u^4} - \frac{4x_2^2}{u^2} < 0$$

b) The slope of the vector field along the hyperbola is given by the hint:

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{-1}{2x_1^2 + 2\sqrt{2}x_1 + 1}$$

The tangent to the hyperbola is easily calculated as:

$$\frac{dx_2}{dx_1} = \frac{d}{dx_1}\left(\frac{2}{x_1 - \sqrt{2}}\right) = \frac{-1}{\frac{1}{2}x_1^2 - \sqrt{2}x_1 + 1}$$

For $x_1 > \sqrt{2}$, we see that

$$2x_1^2 + 2\sqrt{2}x_1 + 1 > \frac{1}{2}x_1^2 - \sqrt{2}x_1 + 1$$

so the slope of the vector field along the hyperbola is greater than the slope of the tangent line. However, we don't yet know whether the vector field points in the northeast or southwest direction. We see that on the hyperbola,

$$\dot{x}_1 = -\frac{6x_1}{(1 + x_1^2)^2} + \frac{4}{x_1 - \sqrt{2}} > 0$$

when $x_1 > \sqrt{2}$. Therefore, the vector field points in the northeast direction along the hyperbola, so trajectories to the right of the hyperbola cannot cross.

c) The origin is not globally AS, as trajectories to the right of the parabola can never reach the origin.

4.9 In checking radial unboundedness of a positive definite function $V(x)$, it may appear that it is sufficient to examine $V(x)$ as $\|x\| \to \infty$ along the principal axes. This is not true, as shown by the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

(a) Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

(b) Show that $V(x)$ is not radially unbounded.

a) First, let $x_1 = 0$. We see that

$$\lim_{x_2 \to \infty} \frac{x_2^2}{1 + x_2^2} + x_2^2 = \infty$$

clearly diverges. Then, let $x_2 = 0$. Again,

$$\lim_{x_1 \to \infty} \frac{x_1^2}{1 + x_1^2} + x_1^2 = \infty$$

b) Now we investigate what happens when we move away from the origin along the line $x_1 = x_2$. We see that,

$$\lim_{x_1 \to \infty} \frac{(2x_1)^2}{1 + (2x_1)^2} + (x_1 - x_1)^2 = 1$$

so $V(x)$ is not radially unbounded.

Note: $V(x)$ is unbounded, as we cannot put $V(x)$ inside of a ball. Radially bounded means that there exists a direction where we can walk in a straight line from the origin and hit the boundary. This is a very interesting case because for all directions except $x_1 = x_2$, we have $V(x)$ is radially bounded.

4.10 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\, d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x)Pf(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

a) We show this directly.

$$\begin{aligned}
x^T P f(x) + f^T(x) P x &= x^T P \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x\, d\sigma + \int_0^1 x^T \frac{\partial f}{\partial x}^T (\sigma x)\, d\sigma P x \\
&= x^T \int_0^1 P \frac{\partial f}{\partial x}(\sigma x)\, d\sigma x + x^T \int_0^1 \frac{\partial f}{\partial x}^T (\sigma x) P\, d\sigma x \\
&= x^T \int_0^1 \underbrace{\left(P \frac{\partial f}{\partial x}(\sigma x) + \frac{\partial f}{\partial x}^T (\sigma x) P\right)}_{\leq -I} d\sigma x \\
&\leq -x^T x
\end{aligned}$$

b) First, we show that $V(x) > 0$. By construction, $V(x) > 0$ whenever $f(x) \neq 0$. In order for $V(x) > 0$, there must not be any other equilibrium points. Proof by contradiction.

Proof. Suppose that there exists another equilibrium point not at the origin y. Then, by the previous argument,

$$y^T P f(y) + f^T(y) P y = 0 \leq -y^T y$$

but we know that $y^T y > 0$, which is a contradiction. Therefore, $V(x) > 0$ on $x \in \mathbb{R}^n \setminus 0$. □

Now we show that $V(x)$ is radially unbounded.

Proof. From the relationship

$$x^T P f(x) + f^T(x) P x \leq -x^T x = -||x||_2^2$$

we see that

$$\frac{x^T P f(x) + f^T(x) P x}{||x||_2^2} = \frac{2 x^T P f(x)}{||x||_2^2} \leq -1$$

Now consider

$$\frac{||x^T P f(x)||_2}{||x||_2^2} \leq \frac{||x||_2 ||P||_2 ||f(x)||_2}{||x||_2^2} \leq \frac{||P||_2 ||f(x)||_2}{||x||_2}$$

Since P is positive definite, $V(x)$ is radially unbounded if and only if $||f(x)||_2 \to \infty$ as $||x||_2 \to \infty$. That is, at least one component of $f(x)$ goes to infinity as $||x||_2 \to \infty$. We use a proof by contradiction to show that $V(x)$ is radially unbounded. Suppose that $||f(x)||_2$ is bounded by some positive constant. Then, the expression above will approach zero as $||x||_2 \to \infty$. However, we have

$$\lim_{||x||_2 \to \infty} \frac{2x^T P f(x)}{||x||_2^2} = 0 \not\leq -1$$

which is a contradiction. Therefore, $||f(x)||_2 \to \infty$ as $||x||_2 \to \infty$, so $V(x)$ is radially unbounded. $\square$

c) Since $V(x) > 0$ and $V(x)$ is radially unbounded, we just need to show that $\dot{V} < 0$ for all $x \neq 0$ for the origin to be global AS. Using a direct approach,

$$\begin{aligned}
\dot{V} &= \left(\frac{\partial}{\partial t} f(x)\right)^T P f(x) + f^T(x) P \frac{\partial}{\partial t}\Big(f(x)\Big) \\
&= \left(\frac{\partial f}{\partial x}(x) \frac{\partial x}{\partial t}\right)^T P f(x) + f^T(x) P \left(\frac{\partial f}{\partial x}(x) \frac{\partial x}{\partial t}\right) \\
&= f^T(x) \underbrace{\left(\frac{\partial f^T}{\partial x}(x) P + P \frac{\partial f}{\partial x}(x)\right)}_{\leq -I} f(x) \\
&\leq -f^T(x) f(x) < 0
\end{aligned}$$

So, the origin is global AS.