Nonlinear Swing-up Controllers for the Pendulum, Acrobot, and Cart Pole

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***Abstract*—Nonlinear swing-up controllers are studied for a number of standard mechanical systems. The actuated pendulum example shows how feedback linearization can be used to replace the dynamics of a fully actuated system, making swing-up control trivial. An swing-up controller for an underactuated pendulum is also given, with an original proof of asymptotic stability. Next, partial feedback linearization and energy based nonlinear swing-up techniques from the literature are reviewed. These techniques are applied to both the Acrobot and Cart Pole systems. LQR controllers are used to drive the underactuated systems to their target equilibrium points. Simulations are provided for the underactuated systems.**

I. Introduction

A formal definition of an underactuated system is given in [2]. Informally, an underactuated system is a system with more degrees of freedom than control inputs, or a system which has saturated control inputs. Nearly every robot that can walk, fly, or swim is underactuated, and its control is inherently more difficult. This project simulates simple underactuated mechanical systems using control techniques from [1].

II. Pendulum

The equation for a rod pendulum is given by

$$I\ddot{q} = -mgL_c \cos q + \tau$$

where q is the angle of the pendulum taken from the horizontal axis counterclockwise, m is the mass of the rod, I is the inertia of the rod taken around the pivot, $L_c = L/2$ where L is the length of the rod, and g is the acceleration due to gravity. $q = \pi/2$ corresponds to the vertical position of the pendulum. The input τ is a torque applied to the pivot.

A. Fully Actuated Controller

Suppose that τ does not saturate. Then, the system is fully actuated, and using the controller

$$\tau = Iu + mgL_c \cos q$$

the closed loop system can be written as

$$\ddot{q} = u.$$

Therefore, we can "override" the dynamics of our system and replace them with our own. For example, gravity can be inverted with

$$Iu = mgL_c \cos q$$

or the pendulum can be put into the upright position using the linear PD controller

$$u = -21(q - \frac{\pi}{2}) - 10\dot{q}.$$

When the system is fully actuated, we can completely replace the system dynamics. The study of underactuated systems is much more interesting.

B. Underactuated Controller

Suppose that τ has saturation, but we still want to balance the pendulum at the unstable equilibrium point. In order to get the pendulum to the vertical position, we need to take advantage of the underlying dynamics of the pendulum. We use energy-based control to achieve this.

The basic idea is that we will pump energy into the pendulum by applying torque to the pendulum pivot in its direction of travel until the energy is equal to the potential energy of the upright position. The pendulum's (homoclinic) orbit will visit the unstable (saddle) equilibrium point, at which point an LQR controller can be used to stabilize the pendulum in the vertical position.

Let $x_1 = q$ and $x_2 = \dot{q}$. The nonlinear swing-up controller [2] is given by

$$\tau = \mathrm{sat}[(E_c - E)x_2]$$

where E_c is the potential energy of the pendulum in the vertical position, and E is the total energy of the pendulum as a function of its current state. Once the pendulum has reached its homoclinic orbit and x_1 is within 5 degrees of vertical, an LQR controller is used to balance the pendulum in the vertical position. A simulation is given in Fig. 1.

We now show that this controller makes the vertical equilibrium point asymptotically stable, by driving $E_c - E$ to zero before a linear controller is used to balance the pendulum. Take all coefficients to be unity. Then, the closed loop system is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\cos x_1 + \mathrm{sat}[(E_c - E)x_2].\end{aligned}$$

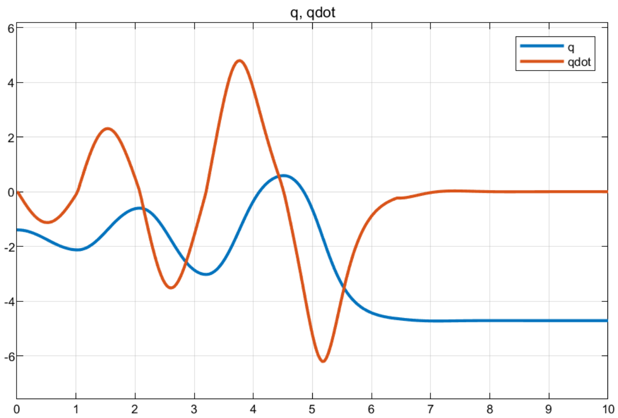
The energy of the system is given by

$$E = \frac{1}{2}x_2^2 + (1 + \sin x_1)$$

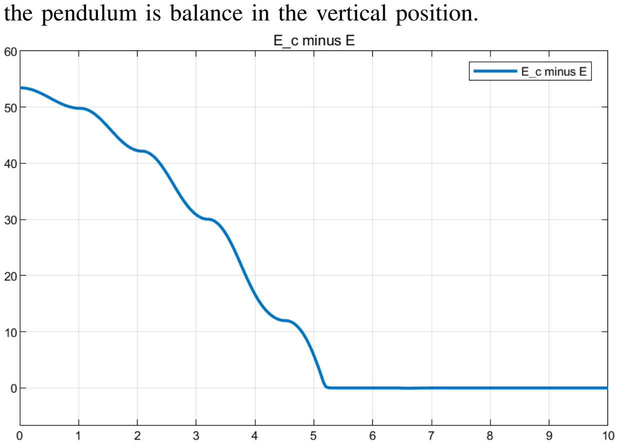
and $E_c = 2$. We see that

$$\frac{d}{dt}(E - E_c) = \dot{E} = -x_2\,\mathrm{sat}[x_2(E - E_c)].$$

Clearly, $\dot{E} = 0$ if $x_2 = 0$ or $E = E_c$. If $x_2 = 0$ then $\cos x_1 = 0$ and $x_1 = \pm\frac{\pi}{2}$, which are the two equilibrium points of the system. By LaSalle's invariance principle, we see that no trajectory can have $x_2 = 0$ for all time except the equilibrium



(a) Pendulum Angles. We see that $q = x_1$ is driven to $\frac{\pi}{2}$ (plus some integer multiple of 2π) and $\dot{q} = x_2$ is driven to 0. Thus, the pendulum is balance in the vertical position.



(b) Difference in energy $E_c - E$. We see that the pendulum enters its homoclinic orbit just after $t = 5$ seconds. This occurs before the pendulum is balanced.

Fig. 1: Pendulum with Swing-up and LQR Controllers

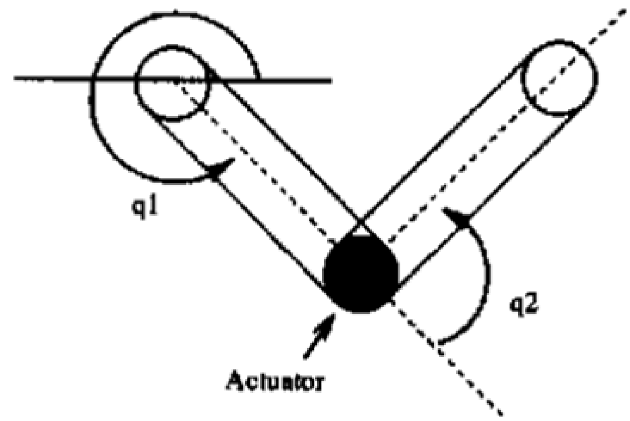


Fig. 2: The Acrobot [1]

points. If $E = E_c$, the pendulum is already in its homoclinic orbit. Consider $x_2 \neq 0$. If $E > E_c$, then $\dot{E} < 0$. If $E < E_c$, then $\dot{E} > 0$. Therefore, we conclude that all trajectories that do not start at the (downwards) equilibrium point will asymptotically converge to the manifold $E = E_c$, given by

$$\sin x_1 + \frac{1}{2}x_2^2 = 1.$$

By switching to a linear controller when the trajectory enters a ball near the vertical equilibrium point, we also conclude that the vertical equilibrium point is asymptotically stable. It is worth noting that the controller used makes the downward equilibrium point unstable, and the vertical equilibrium point unstable. In effect, we have reversed the stability of the equilibrium points with our controller.

III. Acrobot

The Acrobot is a classic underactuated mechanical system which resembles an acrobat on a horizontal bar, pictured in Fig. 2. The Acrobot is modeled as a double pendulum with a motor located at the ”hip”, which joins the two pendulum arms. The control design problem is to balance the Acrobot in the vertical position. To achieve this, [1] used partial feedback linearization and energy-based control to swing-up the pendulum, before switching to an LQR controller to balance the Acrobot in the vertical position. This section summarizes the work, and provides a simulation.

First, partial feedback linearization is used to kill the dynamics of the Acrobot’s lower leg (i.e. $\ddot{q}_2 = 0$). Then a PD controller is used to straighten the lower leg, causing the Acrobot to behave like a pendulum. An energy-based controller is used put the pendulum in its homoclinic orbit, before an LQR controller is used to balance the Acrobot.

A. Partial Feedback Linearization

The equation of motion for the Acrobot is given in [1] with

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + h_1 + \phi_1 &= 0 \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + h_2 + \phi_2 &= \tau \end{aligned}$$

where the coefficients are functions of q and $\dot{q}$. Their particular values are not important to illustrate the partial feedback linearization technique. We see that q_1 is unactuated, and q_2 is actuated. We solve the top equation for $\ddot{q}_1$ and substitute into the bottom equation to obtain

$$\begin{aligned} \ddot{q}_1 &= -m_{11}^{-1}(m_{12}\ddot{q}_2 + h_1 + \phi_1) \\ \tau &= \bar{m}_{22}\ddot{q}_2 + \bar{h}_2 + \bar{\phi}_2 \end{aligned}$$

where

$$\begin{aligned} \bar{m}_{22} &= m_{22} - m_{21}m_{11}^{-1}m_{12} \\ \bar{h}_2 &= h_2 - m_{21}m_{11}^{-1}h_1 \\ \bar{\phi}_2 &= \phi_2 - m_{21}m_{11}^{-1}\phi_1. \end{aligned}$$

The choice of $\tau = \bar{m}_{22}u + \bar{h}_2 + \bar{\phi}_2$ allows us to rewrite the system as

$$\begin{aligned} \ddot{q}_1 &= -m_{11}^{-1}(m_{12}u + h_1 + \phi_1) \\ \ddot{q}_2 &= u. \end{aligned}$$

We have achieved partial feedback linearization: the actuated pendulum arm has linear dynamics. Now, choose $u =$

$\bar{u} - k_1\dot{q}_2 - k_2 q_2$ with k_1 and k_2 chosen to make the linear subsystem $\ddot{q}_2 = u$ stable with $\bar{u} = 0$. (For simulations, $k_1 = 10$ and $k_2 = 21$ to given the transfer functions poles at -3 and -7.) Now, the Acrobot will straighten its legs and swing like a pendulum.

B. Swing-up Controller

The nonlinear swing-up controller given in [1]

$$\bar{u} = \mathrm{sat}((E_c - E)\dot{q}_1)$$

is of a similar nature to the controller for the pendulum, where E is the total energy of the Acrobot and E_c is the energy of the Acrobot in the vertical configuration. Using this controller, the Acrobot swings up and enters its homoclinic orbit. To stabilize the Acrobot, the swing-up controller was switched to an LQR controller when $|q_1 - \frac{\pi}{2}| < 5$ degrees. A simulation of the Acrobot swinging up and balancing is given in Fig. 3.

C. State Manipulator Equation Linearization

The LQR controller required the system to be linearized about the vertical equilibrium point. A faster way to linearize the Acrobot is to use the method described in Chapter 3 of [2]. We summarize this method below. Most mechanical systems can be written using state manipulator equations of the form

$$m(q)\ddot{q} + h(q,\dot{q}) + \phi(q) = b\tau$$

where m is a mass-inertia matrix, h is a matrix of Coriolis forces, ϕ is a gravity vector, b is a constant matrix, and τ is an input vector. The vector q represents each degree of freedom in the system. The system can be written in state-space form with

$$\dot{x} = \begin{bmatrix}\dot{q}\\ \ddot{q}\end{bmatrix} = \begin{bmatrix}\dot{q}\\ m^{-1}(b\tau - h - \phi)\end{bmatrix} = f(x,\tau)$$

By linearizing about an equilibrium point with $\dot{q}^* = 0$, we obtain

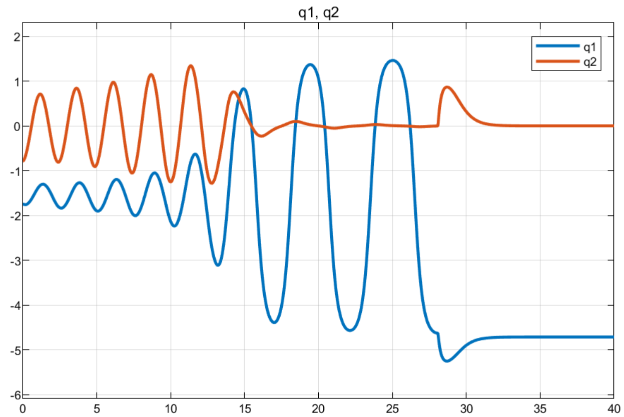
$$A = \begin{bmatrix}0 & I\\ -m^{-1}\frac{\partial\phi}{\partial q} & 0\end{bmatrix}$$

$$B = \begin{bmatrix}0\\ m^{-1}b.\end{bmatrix}$$

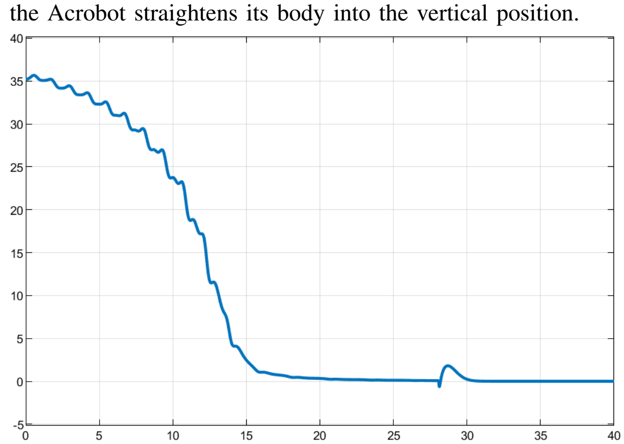
The h terms drop out because the Coriolis forces are zero at the equilibrium points (by construction). Similarly, the $\frac{\partial M^{-1}}{\partial q}$ term drops out because $b\tau - h - \phi = 0$ at the equilibrium points (since $\ddot{q} = \dot{q} = 0$). Thus, the system can be linearized by only computing $\frac{\partial\phi}{\partial q}$.

IV. Cart Pole

The cart pole (also called the inverted pendulum) is another classic example of an underactuated system, pictured in Fig. 4. The goal is to swing up and balance the pendulum above the cart. A nonlinear swing-up controller was provided in [1], which used partial feedback linearization and energy based methods similar to the Acrobot. In [1], the cart pole system modeled the pendulum as a point mass on a massless rod. Further, all parameter values were set to unity.



(a) Acrobot angles. The Acrobot kips its lower leg until $t = 20$ seconds. Then, the Acrobot straightens its legs and waits until q_1 is within 5 degrees of vertical. The LQR controller turns on at $t = 28$ seconds, which causes the Acrobot to move its collective center of mass above the central pivot point. Then, the Acrobot straightens its body into the vertical position.



(b) Difference in energy $E_c - E$. The kipping leg causes the energy difference to rapidly drop. At $t = 20$ seconds, the energy difference is nearly zero, and the Acrobot behaves like a pendulum in its homoclinic orbit. The LQR controller turns on at $t = 28$ seconds.

Fig. 3: Acrobot with Swing-up and LQR Controllers

In this section, the cart pole swing-up and balance is demonstrated for a more accurate model. Let M be the mass of the cart, m be the mass of the thin pendulum rod, L be the length of the rod. The equations of motion were derived using the Euler-Lagrange method, which gave

$$\frac{1}{3}mL^2\ddot{q}_1 + \frac{1}{2}mL\cos q_1\ddot{q}_2 - \frac{1}{2}mgL\sin q_1 = 0$$
$$\frac{1}{2}mL\ddot{q}_1\cos q_1 + (M+m)\ddot{q}_2 - \frac{1}{2}mL\dot{q}_1^2\sin q_1 = F$$

where $q_1 = \pi - \theta$ is the (clockwise) angle of the pendulum from the vertical, and $q_2 = x$ is the position of the cart. Now, partial feedback linearization is applied. Solve the first equation for $\ddot{q}_1$.

$$\ddot{q}_1 = \frac{3}{2L}(g\sin q_1 - \ddot{q}_2\cos q_1)$$

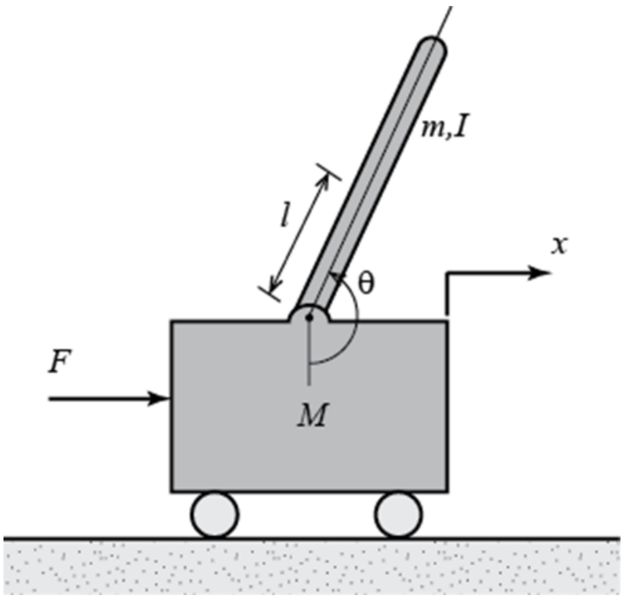


Fig. 4: Cart Pole [3]

Then, substitute into the second equation and simplify.

$$\ddot{q}_2 = \frac{F + \frac{1}{2}mL\dot{q}_1^2 \sin q_1 - \frac{3}{4}mg\cos q_1 \sin q_1}{M + m(1 - \frac{3}{4}\cos^2 q_1)}$$

F can be chosen so that $\ddot{q}_2 = u$. Partial feedback linearization kills the dynamics influenced on the cart from the pendulum, allowing the pendulum to swing freely with the cart stationary when $u = 0$.

The nonlinear swing-up controller for the cart pole in [1] is given by

$$u = \mathrm{sat}((E - E_c)\dot{q}_1 \cos q_1 - k_1\dot{q}_2 - k_2 q_2)$$

where $E - E_c = \frac{1}{2}(\frac{1}{3}mL^2)\dot{q}_1^2 + \frac{1}{2}mgL(1 + \cos q_1)$ is energy of the pendulum, and $E_c = mgL$ is the potential energy of the pendulum in the vertical position. The coefficients $k_1 = 10$ and $k_2 = 21$ were chosen to make keep the cart close the origin during the swing-up phase. An LQR controller was used to stabilize the pendulum when $|q_1| < 5$ degrees. From the equations of motion, we see that $b = [0, 1]^T$,

$$\phi = \begin{bmatrix} -\frac{1}{2}mgL\sin q_1 \\ 0 \end{bmatrix},$$

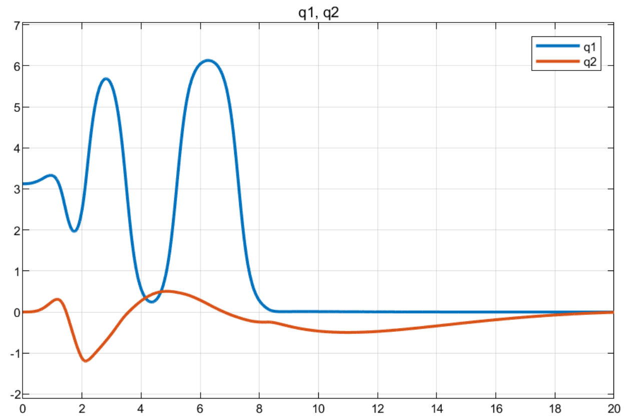
and

$$M^{-1} = \begin{bmatrix} \frac{1}{3}mL^2 & \frac{1}{2}mL\cos q_1 \\ \frac{1}{2}mL\cos q_1 & M + m \end{bmatrix}^{-1}.$$

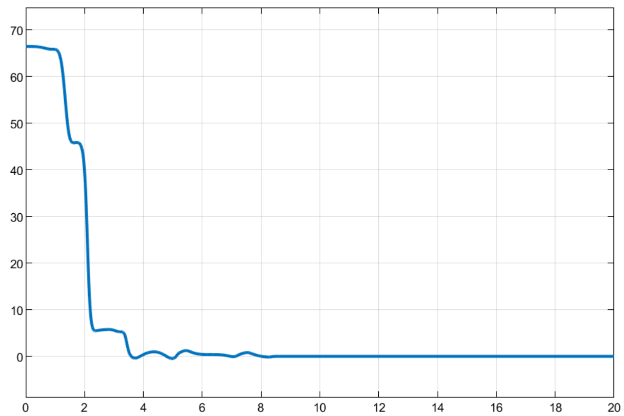
Linearization was performed, and Q was chosen to penalize q_1 and $\dot{q}_1$ more than q_2 and $\dot{q}_2$. A simulation of the nonlinear swing-up controller and LQR controller is given in Fig. 5.

V. Conclusion

In this work, underactuated nonlinear control techniques in [1] such as partial feedback linearization and energy based swing-up control were explored for the pendulum, Acrobot, and cart pole system. These systems were simulated with various levels of control, from no input through swing-up and LQR control. Videos of all simulations are available through the author's GitHub page listed in the Appendix.



(a) Pendulum angle and cart position. The cart moves right, left, right to put the pendulum into its homoclinic orbit. Then, when $|q_1| < 5$ degrees at $t = 7.5$ seconds, the LQR controller turns on and balances the pendulum. The cart slowly makes its way back to the starting position, as the q_2 and $\dot{q}_2$ states are penalized lightly.



(b) Difference in energy $E_c - E$. Movement of the cart adds energy to the pendulum until it is in its homoclinic orbit. A slight bump in the energy is observed at $t = 7.5$ seconds when the LQR controller is turned on.

Fig. 5: Cart Pole Swing-up and LQR Controllers

Appendix

MATLAB and Simulink files used in this work are available at https://github.com/logdog/EE_505/classProject. Videos of all simulations are available within the PowerPoint presentation in the same location.

References

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