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Nonlinear Control of a Swinging Pendulum*

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An experimental investigation into the dynamics and control of a swinging pendulum on a cart leads to new theoretical insights into the control of nonlinear systems about a periodic orbit.

Key Words—Nonlinear control systems; periodic orbits, stability; transverse linearization; control system design; control system analysis; experimental control systems.

Abstract—We propose a nonlinear controller to regulate the swinging energy of the pendulum for a cart and pendulum system. Roughly speaking, the controller is designed to regulate an output (the swing energy) while providing internal stability (regulating the cart position). It is difficult to apply many of the standard nonlinear control design techniques, since the output zeroing manifold does not contain any equilibrium points and the relative degree of the system is not constant. In contrast to controllers that use a command generator and possibly a time-varying feedback, our control law is a simple autonomous nonlinear controller. We analyze the stability of the closed-loop system using an L_∞ small-gain approach on a transverse linearization of the system about the desired periodic orbit. One can easily extend this approach to analyze the robustness of the control system with respect to disturbances and parameter variations. Experimental results demonstrate the effectiveness of the nonlinear controller.

1. INTRODUCTION

In this paper, we propose a nonlinear controller to regulate the swinging energy of the pendulum for a cart and pendulum system. In contrast to the usual inverted balancing objective, the swinging energy cannot be regulated by a simple linear time-invariant controller.

In fact, many of the standard techniques of nonlinear control are also ineffective. For instance, since the relative degree (Isidori, 1989) of the system is not constant (when the output is chosen to be the swinging energy of the pendulum), the system is not input–output linearizable. Also, since certain system distributions are not involutive, the system is not feedback linearizable (Jakubczyk and Respondek, 1980; Hunt *et al.*, 1983). Furthermore, when the desired swing energy requires the pendulum to swing past the horizontal ($\pm\frac{1}{2}\pi$), the controllability distribution of the system does not even have constant rank.

In principle, a regulator for the swinging energy can be designed using the approach of Isidori and Byrnes (1990). The regulator equations are easily solved, since the natural exosystem is just an undamped pendulum and the required mapping is trivial. Since the linearization about the lower equilibrium point is controllable, local regulation about the desired trajectory can be accomplished using a constant feedback matrix (as suggested by Isidori and Byrnes, 1990) provided the desired energy level is sufficiently small. However, as the desired energy level is increased, (e.g. requiring a swing past the horizontal), the local (constant state feedback) solution to the regulator problem is no longer effective. Of course, provided the (time-varying) linearization about the desired trajectory is stabilizable (it is), a stabilizing feedback can be found, for example, by solving a (time-varying) linear quadratic optimal control problem.

In this paper, we propose an autonomous nonlinear state feedback control law to regulate the cart position as well as the swinging energy of the pendulum. The resulting closed-loop system possesses a stable periodic orbit; initial conditions near the orbit will converge to some periodic trajectory on the orbit—there is no notion of a phase error for the system. In contrast, the regulators discussed above (see e.g. Isidori and Byrnes, 1990) are nonautonomous and try to force the system to follow a specific periodic trajectory (determined by the exosystem) on the orbit. Thus, if the initial phase error is too large, the system may diverge even when the initial condition lies on the desired periodic orbit.

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This paper is organized as follows. The equations of motion for the system are given in Section 2. In Section 3, we propose a control law to regulate the swinging energy of the pendulum with no concern for stabilizing the cart motion. This nonlinear controller is shown to provide global stability (except for a set of measure zero) for almost all desired energy levels. In Section 4, we augment the control law of Section 3 with PD terms to stabilize the cart motion as well as the pendulum swing energy. A new method for establishing the exponential stability of a periodic orbit is developed and used to show that the new control system has the desired properties. In Section 5, we discuss numerical techniques for calculating the quantities we use to help select the controller parameters. Simulation results are presented in Section 6 to indicate the performance of the closed loop system under the proposed control law. In Section 7, experimental results are presented to demonstrate the effectiveness of this approach.

2. SYSTEM DYNAMICS

Consider the cart and pendulum system shown in Fig. 1. Under the usual assumptions (massless rod, point masses, etc.), the dynamics is given by

$$\begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -ml\sin\theta\,\dot{\theta}^2 \\ mgl\sin\theta \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad (1)$$

where f is the external force applied to the cart and g is the acceleration due to gravity. If we take as input the acceleration of the cart (rather than the external force), i.e., we take

$$f = (M + m\sin^2\theta)u - (ml\sin\theta\,\dot{\theta}^2 + mg\sin\theta\cos\theta),$$

the equations have the rather simple form

$$\ddot{\theta} = -\frac{g\sin\theta}{l} - \frac{\cos\theta}{l}u,$$
$$\ddot{x}_c = u.$$

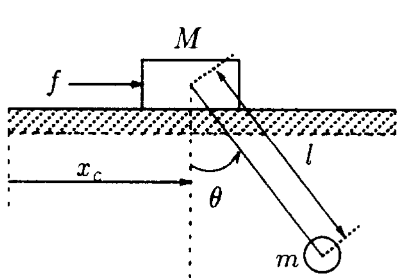

Fig. 1. The cart and pendulum system.

For simplicity of exposition, we take $g = 1$, $l = 1$, and $m = 1$.

3. REGULATION OF A SWINGING PENDULUM

In this section, we develop a control law to regulate the swinging energy of the pendulum without regard for the cart motion. The resulting control system is such that the swinging energy will converge to the desired energy from almost all initial conditions, Furthermore, the convergence to the desired energy level is asymptotically exponential with a rate that depends on the desired energy level.

Consider the control system

$$\dot{\theta} = \omega, \qquad \dot{\omega} = -\sin\theta - \cos\theta\,u, \quad (2)$$

defined on the cylindrical phase $(\theta, \omega) \in S^1 \times \mathbb{R}$. The undriven system $(u \equiv 0)$ has equilibrium points (with $\omega = 0$) at $\theta = 0$ and $\theta = \pi \equiv -\pi$ (we identify $+\pi$ and $-\pi$ to get the circle group S^1). We should like to design a feedback control u so that the swing energy

$$H(\theta, \omega) := \tfrac{1}{2}\omega^2 + (1 - \cos\theta)$$

is regulated to a desired swing energy $\hat{H}$.

Define

$$E(\theta, \omega) := H(\theta, \omega) - \hat{H}$$

and note that

$$\dot{E} = -\omega\cos\theta\,u.$$

Choosing the feedback control law

$$u = \alpha\omega\cos\theta\,E, \quad (3)$$

where $\alpha > 0$ is a design parameter, results in

$$\dot{E} = -\alpha\omega^2\cos^2\theta\,E$$

so that

$$E(t) = E(0)\exp\left[-\alpha\int_0^t \omega^2(s)\cos^2\theta(s)\,ds\right]. \quad (4)$$

In particular, if

$$\int_0^t \omega^2(s)\cos^2\theta(s)\,ds \to \infty \quad \text{as } t \to \infty$$

along the trajectories of the closed-loop system (2), (3) then

$$E(t) \to 0.$$

Note that (4) implies that $|E(t)|$ is nonincreasing.

We can use LaSalle's theorem (LaSalle and Lefschetz, 1961) to show that $E(t)$ does in fact converge to zero for almost all initial conditions. Indeed, setting

$$V(\theta, \omega) = E^2(\theta, \omega),$$

we see that

$$\dot{V}(\theta, \omega) = -2\alpha\omega^2 \cos^2\theta\, E^2(\theta, \omega) \le 0$$

so that

$$\{(\theta, \omega): V(v, \omega) \le V(\theta_0, \omega_0)\},$$

where $(\theta_0, \omega_0) = (\theta(0), \omega(0))$, is compact. Therefore, all trajectories of the system will converge to the largest invariant set where $\dot{V} = 0$, namely,

$$\{(\theta, \omega): E(\theta, \omega) = 0\} \cup \{(0, 0), (\pi, 0)\}.$$

In particular, since V is nonincreasing, it is easy to see that the only trajectories that do not converge to the periodic orbit $E(\theta, \omega) = 0$ are the trivial trajectory and trajectories that live on the stable manifold of the saddle equilibrium point $(\pi, 0)$. Therefore, the periodic orbit with energy $\hat{H}$ is locally asymptotically stable.

We can estimate the asymptotic rate of convergence using a Poincaré map. Suppose that $\hat{H} > 0$ and $\neq 2$. Then the closed-loop system (2), (3) has a periodic orbit (two distinct periodic orbits if $\hat{H} > 2$) that intersects the section $\Sigma = \{(\theta, \omega): \theta = 0\}$ at a point $\hat{\xi} = (0, \hat{\omega})$ with $\frac{1}{2}\hat{\omega}^2 = \hat{H}$. Thus, $\phi_{\hat{T}}(\hat{\xi}) = \hat{\xi}$, where ϕ_t denotes the flow of the closed-loop system (2), (3) and $\hat{T}$ is the period of the orbit through $\hat{\xi}$. Note that Σ is transverse to the flow ϕ_t except at the origin. Thus, for ξ in a neighborhood $U \subset \Sigma$ of $\hat{\xi}$, the Poincaré map $P: U \to \Sigma$ is well defined and is given by

$$P: \xi \mapsto \phi_{T(\xi)}(\xi),$$

where $T(\xi)$ is the first time that the orbit $\phi_t(\xi)$ returns to Σ Using (4) and the fact that $E = \frac{1}{2}(\omega^2 - \hat{\omega}^2)$ on Σ, we can express the ω portion of the Poincaré map $p_2(\omega) = \pi_2 P(0, \omega)$

$$p_2: \omega \mapsto \left\{(\omega^2 - \hat{\omega}^2) \times \exp\left[-\alpha \int_0^{T(0,\omega)} h(\phi_s(0, \omega))\, ds\right] + \hat{\omega}^2\right\}^{1/2}$$

where $h: (\theta, \omega) \mapsto \omega^2 \cos^2\theta$. Since

$$p_2'(\hat{\omega}) = e^{-\alpha\lambda\hat{T}} < 1,$$

where

$$\lambda = \frac{1}{\hat{T}} \int_0^{\hat{T}} h(\phi_s(0, \hat{\omega}))\, ds > 0,$$

we conclude that the asymptotic rate of convergence is $-\alpha\lambda$. That is, for each nonzero initial condition (not on the stable manifold of $(\pi, 0)$) and each $\delta > 0$, there exists $t_1 \ge 0$ such that

$$|E(t)| \le |E(t_1)|\, e^{-\alpha(\lambda - \delta)(t - t_1)}.$$

We have shown that the control law (3) is suitable for regulating the swing energy of the pendulum system (1) without regard for the cart motion. In order for such a system to be practical, we must also regulate the cart position. This is the topic of the next section.

4. REGULATION OF A CART AND SWINGING PENDULUM

In this section, we develop a nonlinear controller that regulates the position of cart as well as the swinging energy of pendulum. This is accomplished by adding low-bandwidth linear position and velocity terms to stabilize the cart position without destroying the regulation of the swinging energy.

Returning to the full cart and pendulum system, we rewrite the system dynamics (1) as

$$\begin{aligned}\dot{\theta} &= \omega,\\ \dot{\omega} &= -\sin\theta\cos\theta\, u,\\ \dot{x}_c &= v_c,\\ \dot{v}_c &= u.\end{aligned} \tag{5}$$

We seek to modify the control law (3) in such a way as to stabilize the cart while maintaining the swinging motion. It is clear that a proportional–derivative feedback law will stabilize the cart position. With this in mind, we consider modifying the nonlinear swinging control law by adding proportional and derivative terms to stabilize the cart position. The resulting control law is given by

$$u = \alpha\omega\cos\theta\, E - (2\zeta\omega_c v_c + \omega_c^2 x_c), \tag{6}$$

where ω_c is the cutoff frequency and ζ is the damping ratio. For simplicity, we shall fix $\alpha = 1$ and use a critically damped PD feedback ($\zeta = 1$). It is not clear that the addition of the PD terms to the control law will provide stabilization of the cart while maintaining regulation of the swinging energy. Note that the closed-loop system (5), (6) possesses a periodic orbit, namely,

$$\gamma = \{(\theta, \omega, x_c, v_c) \in S^1 \times \mathbb{R}^3: H(\theta, \omega) = \hat{H},\ x_c = v_c = 0\}. \tag{7}$$

We shall show that, for the case when $\alpha = 1$, ω_c can be chosen so that the periodic orbit is, in fact, (exponentially) stable for a wide range of $\hat{H}$. That is, both the swinging energy of the pendulum and the position of the cart are regulated.

One way to guarantee the exponential stability of γ is by using Floquet theory. In particular, we should linearize the full dynamics around a periodic trajectory on γ. Then, if all but one of the eigenvalues of the state transition matrix evaluated over a period have magnitude less than one, the periodic orbit is exponentially stable. This can, of course, be checked numerically for a particular selection of design parameters. However, using this method, we have little intuition about the set of suitable parameters.

For this problem, we can actually understand the system better by looking at the dynamics transverse to the periodic orbit γ. In particular, using x_c, v_c, and E as transverse coordinates, we shall find design parameters such that the linearization of the transverse dynamics along a trajectory on the periodic orbit is asymptotically stable. This, in turn, will imply that the periodic orbit itself is asymptotically stable.

First note that, since the differentials of the functions E, x_c, and v_c are linearly independent on γ, these functions form a well defined coordinate system for the transverse dynamics. Using the coordinates, the transverse dynamics of the system is given by

$$\begin{aligned}\dot{E} &= -\alpha\omega^2\cos^2\theta\, E + \omega\cos\theta\,(2\omega_c v_c + \omega_c^2 x_c),\\ \dot{x}_c &= v_c, \\ \dot{v}_c &= \alpha\omega\cos\theta\, E - (2\omega_c v_c + \omega_c^2 x_c).\end{aligned} \tag{8}$$

Note that we do not have a coordinate to indicate the relative phase of the system on the periodic orbit. That is, the control law is determined only by θ, ω, E, x, and $\dot{x}$. We plan to linearize about a trajectory on the periodic orbit, and the dynamics tangent to the periodic orbit (i.e., the phase) will be neutrally stable and hence does not play a role in the stability of the periodic orbit.

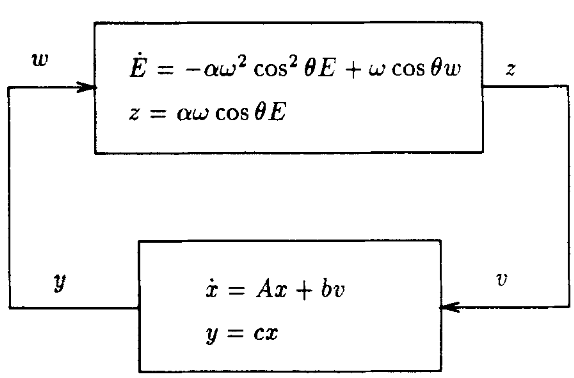

Fig. 2. Nonlinear feedback system: feedback equivalent of the closed-loop system.

As shown in Fig. 2, it is convenient to think of this dynamics as a feedback connection of two dynamic systems, namely,

$$\begin{aligned}\dot{E} &= -\alpha\omega^2\cos^2\theta\, E + \omega\cos\theta\,\omega,\\ z &= \alpha\omega\cos\theta\, E\end{aligned}$$

and (9)

$$\begin{aligned}\dot{x} &= Ax + bv,\\ y &= cx,\end{aligned}$$

with feedback connection

$$\begin{aligned}v &= z,\\ w &= y.\end{aligned} \tag{10}$$

In (9), $x = [x_c\ \ v_c]^T$ and

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_c^2 & -2\omega_c \end{bmatrix},\quad b = [0\ \ 1]^T,\quad c = [\omega_c^2\ \ 2\omega_c].$$

Now consider the linearization of (9) about a periodic trajectory $(\bar{\theta}(t), \bar{\omega}(t), 0, 0)$ on γ. This linearization of the transverse dynamics is given, in feedback form, by

$$\begin{aligned}\delta\dot{E} &= -\alpha\beta^2(t)\,\delta E + \beta(t)\,\delta w,\\ \delta z &= \alpha\beta(t)\,\delta E\end{aligned} \tag{11}$$

and

$$\begin{aligned}\delta\dot{x} &= A\,\delta x + b\,\delta v,\\ \delta y &= c\,\delta x,\end{aligned} \tag{12}$$

with feedback connection

$$\begin{aligned}\delta v &= \delta z,\\ \delta w &= \delta y\end{aligned} \tag{13}$$

and $\beta(t) := \bar{\omega}(t)\cos\bar{\theta}(t)$. Note the absence of variational terms tangent to γ in the linearization. This fortunate situation is, in fact, what happens in the general case for the transverse dynamics, as the following result shows.

Proposition 4.1. Let ϕ_t be the flow of a C^1 vector field $f: \mathbb{R}^n \to \mathbb{R}^n$ and let γ be a periodic orbit of ϕ_t. Let $h_1(x), \ldots, h_{n-1}(x)$ be $n-1$ independent C^2 functions that vanish on γ (i.e., for all $x \in \gamma$, $h_1(x) = \cdots = h_{n-1}(x) = 0$ and $dh_1(x), \ldots, dh_{n-1}(x)$ are linearly independent) and define $\xi_i = h_i(x)$ for $i = 1, \ldots, n-1$. Then the linearization of the ξ dynamics about a trajectory $\bar{x}(t)$ in γ is given by

$$\delta\dot{\xi} = A(t)\,\delta\xi \tag{14}$$

for some $A(t) \in \mathbb{R}^{(n-1)\times(n-1)}$. That is, the dynamics of $\delta\xi$ is independent of perturbations along the periodic orbit.

Proof. See the Appendix.

It is not immediately clear that asymptotic stability of (11)–(13) will imply exponential stability of γ. The following proposition shows that the asymptotic stability of the $(n-1)$-dimensional (linearized) transverse dynamics is sufficient to conclude the exponential stability of the periodic orbit. A similar result has been shown using a moving orthornormal coordinate system (Hale, 1980). We also remark that these ideas can be used to construct a family of converse Lyapunov functions proving the stability of an exponentially stable periodic orbit (Hauser and Chung, 1994).

Proposition 4.2. Let ϕ_t be the flow of a C^1 vector field f and let γ be a periodic orbit of the flow ϕ_t with period T. Let $(\xi, \eta) \in \mathbb{R}^{n-1} \times S^1$ be a set of coordinates valid on a neighborhood of γ and such that $\gamma = \{(\xi, \eta) : \xi = 0\}$. Then, if the linearization of the ξ dynamics about a periodic trajectory in γ given by

$$\dot{\delta\xi} = A(t)\,\delta\xi \tag{15}$$

is asymptotically (hence exponentially) stable, γ is an exponentially stable periodic orbit.

Proof. See the Appendix.

Many approaches can be used to decide whether or not the linear system (11)–(13) is exponentially stable. For example, using the theory of periodic linear systems (Burton, 1985), we may numerically determine stability for a given choice of parameters. However, such an approach provides little understanding to guide the choice of the design parameters α, ζ, and ω_c.

In order to gain a better understanding of the effect of the parameter choice, we shall use an approach based on the small-gain theorem (Desoer and Vidyasagar, 1975). In particular, we shall detail an approach to compute an upper bound on the L_∞ induced norm of the linear operator mapping $\delta w(\cdot)$ to $\delta y(\cdot)$ defined by (11), (12) and the connection $\delta v = \delta z$. Then, we shall show that, by choosing the cutoff frequency ω_c to be sufficiently small, we may be able to make the L_∞ induced norm of this operator less than one. The extra condition is a condition on the system (11) and will be stated precisely below. Since the individual subsystems are exponentially stable and bounded-input, bounded-output stable, this is sufficient to conclude that the linear feedback system is exponentially stable. Roughly speaking, after the parameter α is chosen to specify the rate of convergence of the swinging energy, a range of cutoff frequencies ω_c is determined to retain the stability of the periodic orbit.

As pointed out by an anonymous referee, one can also choose the parameters α, ζ, and ω_c to depend on a small parameter ϵ so that the averaging method can be applied. Indeed, this approach can be used to conclude that it is always possible to stabilize the desired periodic orbit for any valid energy level. However, this approach uses a different parameterized family of control laws where the gains for swinging energy regulation and cart regulation are directly coupled (e.g., ω_c and α become small simultaneously as ϵ is decreased). In contrast, our approach provides a method to decide, once α has been chosen (for swinging energy regulation), whether the system can be stabilized by a suitable choice of ω_c (Proposition 4.3 below). We note that, by the above mentioned averaging method, a valid α, ω_c pair is guaranteed to exist.

We begin by noting that the linear mapping of interest,

$$\mathbf{L}: \delta w(\cdot) \mapsto \delta y(\cdot),$$

is defined by

$$\delta y(t) = \int_{-\infty}^{t} h(t-s)\alpha\beta(s) \times \int_{-\infty}^{s} \phi_\alpha(s,\tau)\beta(\tau)\,\delta w(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

where $h(t)$ is the (causal) impulse response of (12) and

$$\phi_\alpha(t,s) = \exp\left[-\alpha \int_s^t \beta^2(\tau)\,\mathrm{d}\tau\right]$$

is the state transition function of (11). Suppose that $\delta w(\cdot)$ is bounded, say by b_w. Then

$$|\delta y(t)| \le \int_{-\infty}^{t} |h(t-s)|\,\alpha\,|\beta(s)| \times \int_{-\infty}^{s} \phi_\alpha(s,\tau)\,|\beta(\tau)|\,b_w\,\mathrm{d}\tau\,\mathrm{d}s, \tag{16}$$

where we have used the fact $\phi_\alpha(s,\tau) > 0$ $\forall s, \tau \in \mathbb{R}$. Defining

$$\bar{y}(t) = \int_{-\infty}^{t} |h(t-s)|\,\alpha\,|\beta(s)| \times \int_{-\infty}^{s} \phi_\alpha(s,\tau)\,|\beta(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s,$$

we see that

$$\|\mathbf{L}\|_{L_\infty,i} \le \sup_{t\in\mathbb{R}} \bar{y}(t) =: b_y. \tag{17}$$

Note that $\bar{y}(t)$ is periodic since

$$\bar{z}(t) := \alpha\,|\beta(t)| \int_{-\infty}^{t} \phi_\alpha(t,\tau)\,|\beta(\tau)|\,\mathrm{d}\tau$$

is periodic (Burton, 1985) and $\bar{z}(t)$ is filtered by the (infinite-dimensional) linear time-invariant (LTI) system

$$\bar{y}(t) = \int_{-\infty}^{t} |h(t-\tau)|\,\bar{z}(\tau)\,\mathrm{d}\tau$$
$$= \int_{-\infty}^{t} g(t-\tau)\bar{z}(\tau)\,\mathrm{d}\tau \qquad (18)$$

with $g(t) := |h(t)|$ to get $\bar{y}(\cdot)$. Thus, we can calculate b_y as

$$b_y = \max_{t\in[0,T]} \bar{y}(t).$$

We shall also make use of a bound on $\bar{z}(\cdot)$,

$$b_z := \max_{t\in[0,T]} \bar{z}(t),$$

and the average value of $\bar{z}(\cdot)$,

$$a_z := \frac{1}{T}\int_0^T \bar{z}(t)\,\mathrm{d}t.$$

Note that many of the above quantities actually depend on $\hat{H}$ and ω_c (and α if it is considered a design parameter—currently, we have set $\alpha = 1$). In the sequel, we shall indicate this dependence by adding the appropriate variables to the argument list of the functions. The functions of interest, therefore, become, in order of application, $T(\hat{H})$, $\beta(\hat{H},t)$, $\bar{z}(\hat{H},t)$, $b_z(\hat{H})$, $a_z(\hat{H})$, $h(\omega_c, t)$, $g(\omega_c, t)$, $\bar{y}(\hat{H}, \omega_c, t)$, and $b_y(\hat{H}, \omega_c)$. The dependence of the periodic orbit γ on $\hat{H}$ is made explicit using the notation $\gamma_{\hat{H}}$.

We now show that it may be possible to choose ω_c such that the closed-loop system is stable for a wide range of desired energy $\hat{H}$. We argue that, as long as the product of the constant component of $\bar{z}(\cdot)$ and the zero-frequency (DC) gain of the linear filter (18) is strictly less than one, b_y can be made less than one by choosing ω_c sufficiently small. This is true since the nonconstant component of $\bar{y}(\cdot)$ can be made arbitrarily small by choosing ω_c sufficiently small. In the critically damped case ($\zeta = 1$) under consideration, the function

$$h(\omega_c, t) = (2\omega_c e^{-\omega_c t} - \omega_c^2 t e^{-\omega_c t})\mathbf{1}(t)$$

passes through zero at $t = 2/\omega_c$, so that

$$g(\omega_c, t) = |h(\omega_c, t)|$$
$$= h(\omega_c, t) + f(\omega_c, t - 2/\omega_c)\mathbf{1}(t - 2/\omega_c),$$

where $f(\omega_c, t) = 2e^{-2}\omega_c^2 t e^{-\omega_c t}$ and $\mathbf{1}(t)$ is the unit step function. Thus, the transfer function of the linear filter (i.e., the Laplace transform of $g(\omega_c, t)$) is given by

$$\hat{g}(\omega_c, s) = \frac{2\omega_c s + \omega_c^2(1 + 2e^{-2}e^{-2s/\omega_c})}{s^2 + 2\omega_c s + \omega_c^2}. \qquad (19)$$

We see that the DC gain ($s = 0$) of the filter (18) is $1 + 2e^{-2}$, independent of w_c. We may now formalize the above discussion.

Proposition 4.3. Suppose $\mathcal{H} \subset \mathbb{R}_+$ is a compact set such that $\{0, 2\} \not\subset \mathcal{H}$ and let $\alpha > 0$ be given. Then, if

$$a_z(\hat{H}) < (1 + 2e^{-2})^{-1} \qquad (20)$$

for all $\hat{H} \in \mathcal{H}$, there exists $\hat{\omega}_c > 0$ such that every periodic orbit $\gamma_{\hat{H}}$, $\hat{H} \in \mathcal{H}$, is made exponentially stable by the control law (6) with $0 < \omega_c < \hat{\omega}_c$.

Proof. Since $\{0, 2\} \not\subset \mathcal{H}$, the fundamental frequency of $\bar{y}(\cdot)$, $1/T(\hat{H})$, is bounded away from zero on $\mathcal{H}$. The result follows from the above discussion. □

Although conclusive analytical estimates are difficult to obtain, the left-hand side of (20) is easily computed numerically. Thus, Proposition 4.3 provides a practical approach for determining a suitable ω_c in the control law (6). It is important to note that (20) provides a conservative sufficient condition for stabilization by a controller of the given structure.

5. NUMERICAL TECHNIQUES

In this section, we develop specific numerical techniques to determine the critical cutoff frequency $\hat{\omega}_c$ for a compact set $\mathcal{H}$ of desired swing energies $\hat{H}$. This conservative bound is compared with the value of ω_c that makes the periodic orbit unstable. Simulation results are given to indicate the effectiveness of the approach.

To determine $\hat{\omega}_c(\mathcal{H})$, we calculate

$$\bar{\omega}_c(\hat{H}) := \sup\{\bar{\omega}_c > 0 : b_y(\hat{H}, \omega_c) < 1 \text{ for all } \omega_c < \bar{\omega}_c\}$$

for $\hat{H} \in \mathcal{H}$ and note that

$$\hat{\omega}_c(\mathcal{H}) = \min_{\hat{H}\in\mathcal{H}} \bar{\omega}_c(\hat{H}).$$

Now, the linear filter is such

$$\lim_{\omega_c\to 0} b_y(\hat{H}, \omega_c) = (1 + 2e^{-2})a_z(\hat{H}),$$

$$\lim_{\omega_c\to\infty} b_y(\hat{H}, \omega_c) = (1 + 2e^{-2})b_z(\hat{H}).$$

Thus, since $b_y(\cdot,\cdot)$ is Lipschitz continuous, if

$$a_z(\hat{H}) < (1+2e^{-2})^{-1} \tag{21}$$

then $\bar{\omega}_c(\hat{H}) > 0$ (it may be infinite). If, in addition,

$$b_z(\hat{H}) > (1+2e^{-2})^{-1} \tag{22}$$

then $\bar{\omega}_c(\hat{H})$ will be finite and is the smallest zero of the function $\omega_c \mapsto b_y(\hat{H},\omega_c) - 1$. Figure 3 shows the functions $a_z(\hat{H})$ and $b_z(\hat{H})$ as well as the constant function $(1+2e^{-2})^{-1}$ for $\mathscr{H} = [0.20, 1.95]$. In this case, both (21) and (22) are satisfied, so that $\bar{\omega}_c(\hat{H})$ is a well-defined, Lipschitz function of $\hat{H}$.

To compute the first ω_c such that $b_y(\hat{H},\omega_c) = 1$, we need an efficient way of evaluating $b_y(\hat{H},\omega_c)$ for a range of ω_c. A natural separation of variables due to the fact that $\bar{y}(\hat{H},\omega_c,\cdot)$ is given by the convolution of $g(\omega_c,\cdot)$ and $\bar{z}(\hat{H},\cdot)$ leads to such an algorithm. We first compute $\bar{e}(\hat{H},\cdot)$ as the (globally stable) periodic solution of

$$\dot{\bar{e}}(\hat{H},t) = -\alpha\beta^2(\hat{H},t)\bar{e}(\hat{H},t) + |\beta(\hat{H},t)|, \tag{23}$$

and set

$$\bar{z}(\hat{H},t) = \alpha\,|\beta(\hat{H},t)|\,\bar{e}(\hat{H},t).$$

Then, using the single function $\bar{z}(\hat{H},\cdot)$, we can compute $\bar{y}(\hat{H},\omega_c,\cdot)$ by repeated convolution with $g(\omega_c,\cdot)$ for the desired range of ω_c. Taking advantage of the periodicity of $\bar{z}(\hat{H},\cdot)$ and using the explicit formula for $\hat{g}(\omega_c,s)$, this convolution may be efficiently calculated in the frequency domain. With $\bar{y}(\hat{H},\omega_c,\cdot)$ in hand, $b_y(\hat{H},\omega_c)$ is easily determined.

Figure 4 shows the plot of $b_y(\hat{H},\omega_c)$ versus ω_c for $\hat{H} = 1.416$. The critical value $\bar{\omega}_c(\hat{H})$ making $b_y(\hat{H},\bar{w}_c) = 1$ can be computed using, for example, a bisection method or a nonsmooth version of Newton's method.

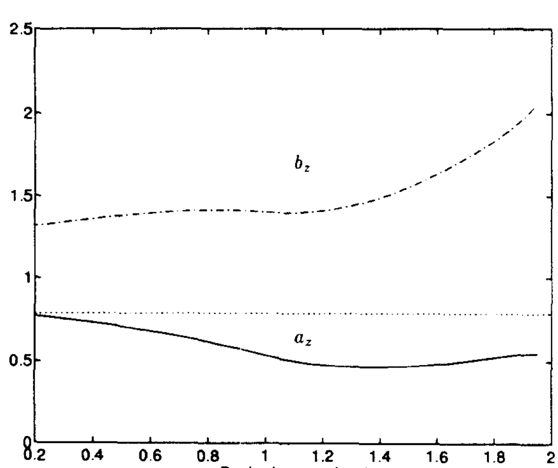

Fig. 3. The functions $a_z(\hat{H})$ and $b_z(\hat{H})$ compared with the constant function $(1+2e^{-2})^{-1}$ for $\mathscr{H} = [0.20, 1.95]$.

Since the function $\bar{\omega}_c(\cdot)$ is continuous, once $\bar{\omega}_c(\hat{H})$ has been found for an initial $\hat{H}$, further function values may be obtained using a continuation method (Rheinboldt, 1986). Figure 5 shows the function $\bar{\omega}_c(\cdot)$. Thus, for $\mathscr{H} = [0.20, 1.95]$, we see that $\hat{w}(\mathscr{H}) = 0.0225$.

Also shown in the figure is the function $\hat{H} \mapsto \tilde{\omega}_c(\hat{H})$ that gives the actual value at which the periodic orbit $\gamma_{\hat{H}}$ transitions from being stable to being unstable. This value was computed using Floquet theory. That is, $\tilde{\omega}_c(\hat{H})$ is such that all the eigenvalues of $\Phi_a^{\hat{H},\omega_c}(T(\hat{H}),0)$ have magnitude less than one for $\omega_c < \tilde{\omega}_c$ and at least one eigenvalue has magnitude greater than one for $\omega_c > \tilde{\omega}_c$.

Note the conservatism of $\bar{\omega}_c(\hat{H})$ compared with $\tilde{\omega}_c(\hat{H})$. This conservatism is due to the fact that $\omega_c < \bar{\omega}_c(\hat{H})$ guarantees that the L_∞ loop gain of (11)–(13) is less than one. In the next section, we shall provide some simulation results to contrast the behavior of the closed-loop system when ω_c is chosen near $\tilde{\omega}_c(\hat{H})$ with that obtained using $\omega_c < \bar{\omega}_c(\mathscr{H})$.

6. SIMULATION RESULTS

In this section, we provide simulation results to illustrate that the behavior of the closed-loop nonlinear system (5), (6) for a given value of $\hat{H}$ and different values of ω_c.

The first simulation illustrates the lack of robustness resulting from choosing ω_c near $\tilde{\omega}_c(\hat{H})$. Let $\hat{H} = 1.2$. Then $\tilde{\omega}_c(\hat{H}) = 1.15$ and $\bar{\omega}_c(\hat{H}) = 0.48$. Choosing $\omega_c = 1.1$, we simulate the closed-loop nonlinear system with initial values $H(0)$ of 0.9 and 1.1. For simplicity, all initial conditions for the presented simulations are chosen such that $(\theta,\omega,x_c,v_c)(0) = (\theta_0,0,0,0)$ where θ_0 is determined by $H(0)$.

Figure 6 shows the results. Note that for small initial error $H(0)-\hat{H}$, the stability of the closed-loop system guarantees that $H(t) \to \hat{H}$. However, since the closed-loop system is nearly unstable, the domain of attraction is not very large.

We now choose $\omega_c = 0.4 < \bar{\omega}_c(\mathscr{H}) = 0.48$ and redo the simulations. As Fig. 7 shows, the closed-loop nonlinear system now has a stable transient response when $H(0)$ equals 1.1 and 0.9. Furthermore, the system is stable when $H(0)$ is as large as 1.5.

We conclude that the closed-loop system with $\omega_c = 0.4$ is far more robust to initial errors than that with $\omega_c = 1.1$. Furthermore, our experience with simulations involving a broader class of

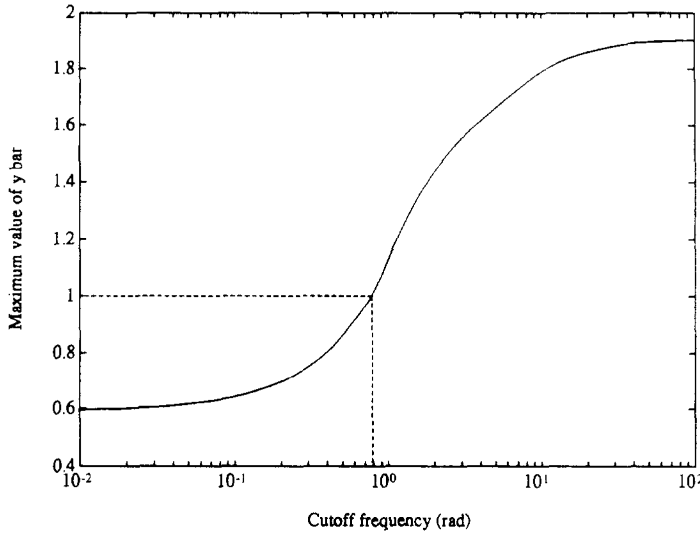

Fig. 4. The function $\omega_c \mapsto b_y(\hat{H}, \omega_c)$ for $\hat{H} = 1.416$.

initial conditions, disturbance torques and forces, and parameter variations, indicates that choosing ω_c sufficiently less than $\hat{\omega}_c(\mathcal{H})$ results in a closed-loop system possessing good robustness for a large range of $\hat{H} \in \mathcal{H}$.

7. EXPERIMENTAL RESULTS

The experimental system is depicted in Fig. 8. The 'cart' is a linear DC brushless motor (Northern Magnetics BLC04-32034-023-0, 0.85 m travel) driven by a 22 kHz PWM current amplified (ServoByte SBA-48). A linear optical encoder (0.01 mm resolution) is used to sense the position of the motor. The 'pendulum' is a (0.44 m) solid aluminum bar attached to the shaft of a (4800 ppr) rotary optical encoder that is mounted on the moving table of the linear motor.

The measured outputs are the two digital position measurements from the optical encoders and the controlled input is the analog voltage input to the current amplifier, which corresponds to a commanded force on the linear motor. The dynamics of the encoders and the internal dynamics of the motor and amplifier have been ignored in the control law design.

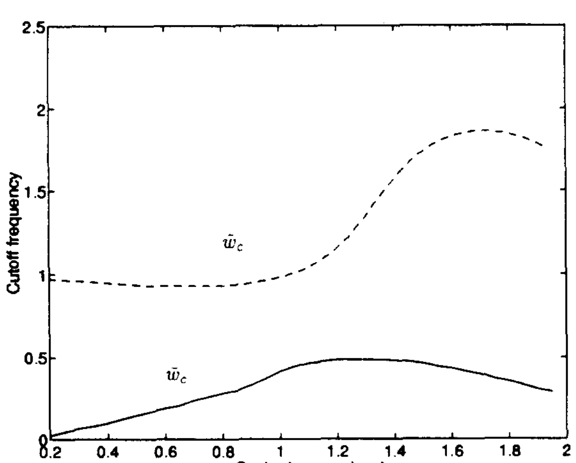

Fig. 5. Comparing the functions $\bar{\omega}_c(\cdot)$ and $\tilde{\omega}_c(\cdot)$ over $\mathcal{H}$.

The closed-loop control laws are implemented on a DSP board (Spectrum TMS320C30 Real-Time System) that resides in a personal computer (486/33 MHz PC). The actual routines have been written in the C programming language, with some lower-level details handled in assembly language.

The goal of our control problem is to do a good job of regulating the swinging energy of the pendulum. In order to achieve this, we must have a reliable measure of the swinging energy. In general, the normalized energy of the pendulum is given by

$$H(\theta, \omega) = 1 - \cos\theta + k\omega^2,$$

where k is a parameter that depends on the physical parameters (mass distribution and length) of the pendulum. Since we do not have a sensor to directly measure the swinging energy of the pendulum, we must construct one to provide a reliable estimate. In constructing this sensor, we have been guided by physical principles (e.g. conservation of energy) as well as system theoretical principles.

Although the optical encoder provides a reliable measurement of θ, we do not have a direct measurement of ω. Although a full state observer could be built to estimate $(x_c, v_c, \theta, \omega)$, its implementation would require detailed information of many system parameters, and might well be sensitive to the nonlinear effects present in the PWM amplifier/brushless motor implementation.

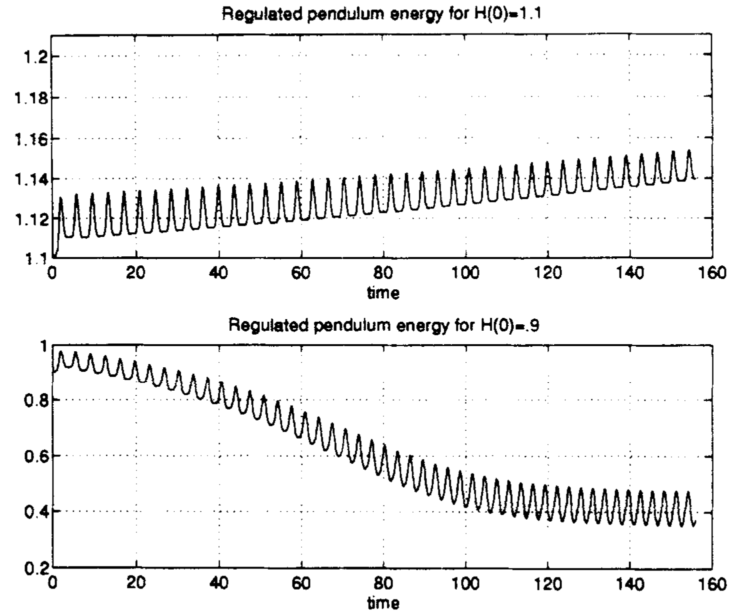

Fig. 6. Simulation results with $\omega_c = 1.1$ and initial values of H equal to 0.9 and 1.1. The desired energy is $\hat{H} = 1.2$.

We take the following, less complex, approach to estimating ω. First, there is a part of the dynamics that we know perfectly, namely

$$\dot{\theta} = \omega.$$

We also know that ω can only be changed by acceleration. Motivated by this information and observer theory, we believe that a good estimate of the angular velocity can be obtained using the linear filter

$$\omega_e(t) = \left[\frac{l_0 s}{s^2 + l_1 s + l_0}\right](\theta(t)).$$

Note that, if we ignore measurement errors (e.g., those due to encoder discretization), the estimated and actual velocities are related by

$$\omega_e(t) = \left[\frac{l_0}{s^2 + l_1 s + l_0}\right](\omega(t)).$$

As long as the frequency contents of the measurements and errors are separated, the filter gains l_0 and l_1 can be chosen such that ω_e is a reliable (phase-delayed) estimate of ω.

The swinging energy of the pendulum may now be estimated by using the estimated angular velocity and experimentally determining the best value for the parameter k. One way to determine k is to fix the position of the cart and let the

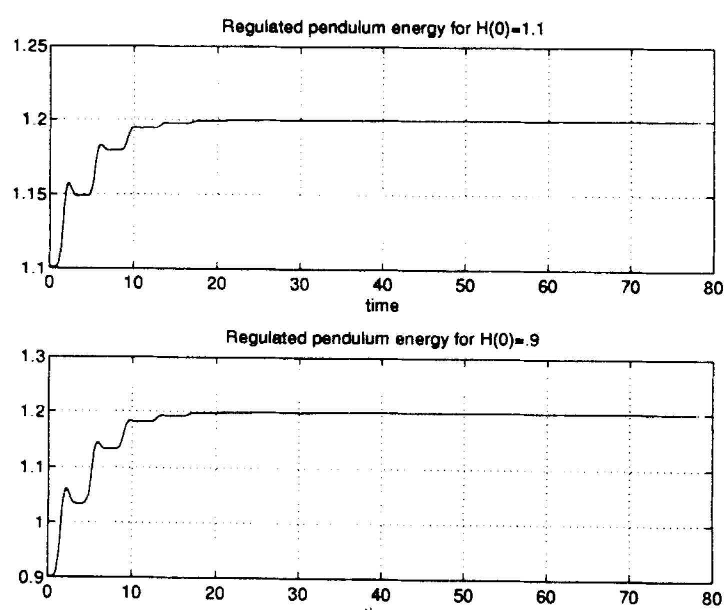

Fig. 7. Simulation results with $\omega_c = 0.4$ and initial values of H equal to 0.9 and 1.1. The desired energy is $\hat{H} = 1.2$.

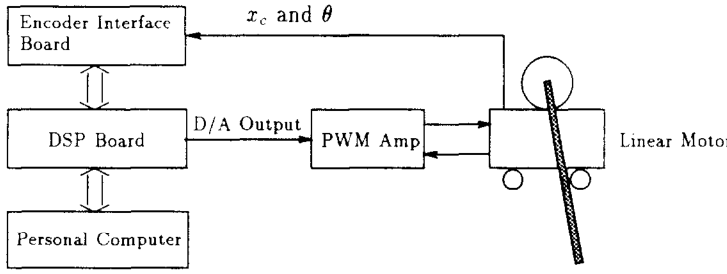

Fig. 8. System configuration for the cart and pendulum system experiment.

pendulum swing from a nonzero energy. The parameter k may then be adjusted so that the estimated energy is monotonically decreasing (since no energy is being added to the system). At this point, we noted that the small phase delay ($\approx$10 ms) in the estimate of ω prevents such a k from being found. That is, the energy estimate using the best value of k still violated the conservation of energy. Using this estimate in the closed-loop control law actually resulted in the system fighting to reduce an artificial error. This error in the estimated energy due to the small time delay is quite apparent, since the energy is nonlinear in both θ and ω. Indeed, better results might be obtained by using the estimated values of θ and ω (from the observer). However, if the filter is implemented using an observer structure with output injection, the estimates of θ and ω will possess different time delays precluding satisfaction of the conservation of energy criterion.

Fortunately, it is not difficult to resolve this difficulty. Indeed, using the fact that θ and ω have essentially the same frequency content, we can use a (filtered) estimate of θ that is delayed by the same amount as ω_e. Thus, we filter θ as

$$\theta_e(t) = \left[\frac{l_0}{s^2 + l_1 s + l_0}\right](\theta(t))$$

to give θ_e the same phase delay as ω_e and we define

$$H_e = 1 - \cos\theta_e + k\omega_e^2.$$

Then, provided the frequency content of θ is reasonably narrowband, k can be chosen such that H_e satisfies the energy criterion. Furthermore, it is easy to implement both filters using a single linear system.

Note that the choice of observer structure and the presence of phase delays is usually not important for control systems designed to stabilize an equilibrium point. Since it is easy to detect when the desired goal has been achieved (e.g., at an equilibrium point), the separation principle guarantees that any stable observer will do.

This is not the case for our system. Indeed, we use the principle of conservation of energy to help us select a filter structure that helps us to detect when the desired goal (swing energy equals desired) has been achieved. Our approach helps to ensure that the control system is not trying to correct errors that are, in fact, not present.

Using this approach, the physical closed-loop system provides much smoother regulation of the swinging energy. Using the DSP, we were able to update the nonlinear control law every 1.2 ms, providing an essentially continuous time implementation. The above linear filters were implemented in a single discrete time system.

Figure 9 shows the experimental regulation of the swinging energy about a desired level of $\hat{H} = 0.4$. Since the friction at the pendulum pivot is nonzero, the cart oscillates around its zero position in order to restore the energy lost to friction.

Figure 10 illustrates the effect of step changes in the desired energy level on the system performance. The desired energy level was changed from 1.0 to 1.3 in 0.1 step increments. Note that for energy levels of 1.0 or greater, the pendulum is swinging past the horizontal (where

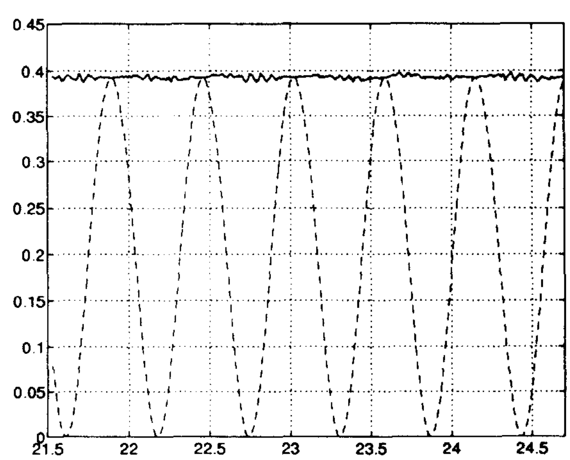

Fig. 9. Experimental steady-state response for $\hat{H} = 0.4$. The solid and dotted lines indicate total and potential energy, respectively, of the pendulum.

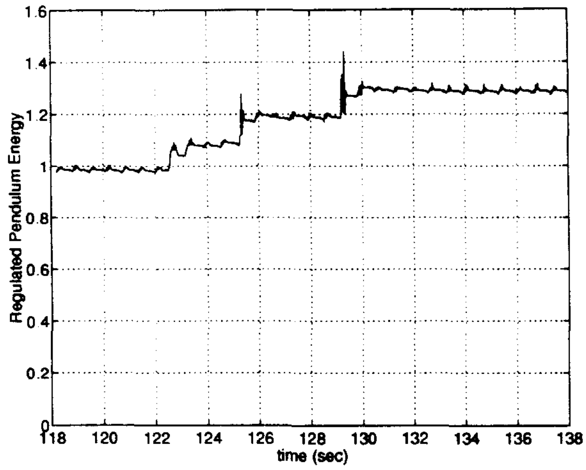

Fig. 10. Experimental response for transitions from $\hat{H} = 1.0$ to $\hat{H} = 1.3$. The desired energy level $\hat{H}$ was changed to 1.1, 1.2, and 1.3 at $t = 122.5$, 125.2, and 129.2 s, respectively.

the effect of inputs changes sign). Small increments were used in part to reduce the possibility of exceeding the limited travel of the linear motor. As it turns out, the effect of step changes in desired energy level on the translation of the cart depends greatly on the relative timing of the command within a swing cycle. This is due to that fact that the ability of the cart to add energy to the pendulum varies greatly over the course of a swing cycle. By timing the command properly, a much larger step change can be given. In any case, the system is robust to reasonable changes in the desired energy level.

We also used the normalized control law

$$u = \frac{\alpha\omega\cos\theta\, E}{\epsilon + \omega^2\cos^2\theta} - (2\zeta\omega_c v_c + \omega_c^2 x_c) \qquad (24)$$

with $\epsilon > 0$ to provide more uniform performance over a range of desired energy levels. Heuristic reasons for this normalization are as follows. Roughly speaking, when the original control law (6) is used, the E dynamics may be said to have an instantaneous gain of approximately $\alpha\omega^2\cos^2\theta$. This has the effect of making the dynamics substantially faster when ω is large (which occurs when $\hat{H}$ is commanded). This, in turn, excites some of the unmodeled dynamics, resulting in undesirable vibrations in the system. In contrast, the normalized control law (24) leads to a transverse dynamics of

$$\begin{aligned} \dot{E} &= -\frac{\alpha\omega^2\cos^2\theta}{\epsilon + \omega^2\cos^2\theta}E + \omega\cos\theta(2\zeta\omega_c v_c + \omega_c^2 x_c), \\ \dot{x}_c &= v_c, \qquad (25) \\ \dot{v}_c &= \frac{\alpha\omega\cos\theta}{\epsilon + \omega^2\cos^2\theta}E - (2\zeta\omega_c v_c + \omega_c^2 x_c), \end{aligned}$$

so that the instantaneous gain of the E dynamics is approximately $\alpha\omega^2\cos^2\theta/(\epsilon + \omega^2\cos^2\theta)$ which is limited in magnitude. This prevents the dynamics from becoming too fast (and interacting with the unmodeled dynamics), and allows the choice of α to more directly affect the speed of the E dynamics. Heuristics aside, the closed-loop stability of the system with the modified control law (24) can be analyzed using the techniques of Sections 2 and 5. In addition, the stability and uniform performance for desired energy levels from 0.1 to 1.9 (with fixed α, ϵ and ω_c) have been validated experimentally.

8. CONCLUSIONS

In this paper, we have developed a nonlinear controller to regulate the swinging energy of the pendulum for a cart and pendulum system. Roughly speaking, the controller was designed to regulate an output (the swing energy) while providing internal stability (regulating the cart position). In contrast to controllers that use a command generator and possibly a time-varying feedback, our nonlinear state feedback controller results in an autonomous closed-loop system possessing a stable periodic orbit. The stability of the periodic orbit was analyzed by studying the linearization of the transverse dynamics along a trajectory on the orbit. By arranging these dynamics as a feedback system, we were able to use the small-gain theorem to ensure the asymptotic stability of the periodic orbit. Furthermore, the structure of the feedback system was used to obtain an understanding of the effects of the design parameters. The effectiveness of the nonlinear controller has been demonstrated experimentally.

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APPENDIX—PROOFS

Proof of Proposition 4.1

Since $\xi = 0$ on $\bar{x}(t)$, we see that the perturbations $\delta\xi$ and δx are related by

$$\delta\xi = \mathrm{D}h(\bar{x}(t))\,\delta x,$$

so that the dynamics of $\delta\xi$ is given by

$$\delta\dot{\xi} = [f^{\mathrm{T}}(\bar{x}(t))\,\mathrm{D}^2h(\bar{x}(t)) + \mathrm{D}h(x)\,\mathrm{D}f(x)]\,\delta x \qquad (\mathrm{A}.1)$$

We must show that the right-hand side of (A.1) is a linear function of $\delta\xi$ ($=\mathrm{D}h(\bar{x}(t))\,\delta x$); that is, we must show that $\mathrm{D}h(\bar{x}(t))$ is a right factor of

$$f^{\mathrm{T}}(\bar{x}(t))\,\mathrm{D}^2h(\bar{x}(t)) + \mathrm{D}h(\bar{x}(t))\,\mathrm{D}f(\bar{x}(t)). \qquad (\mathrm{A}.2)$$

Differentiating

$$h(\bar{x}(t)) = 0,$$

we get

$$\mathrm{D}h(\bar{x}(t))\,f(\bar{x}(t)) = 0. \qquad (\mathrm{A}.3)$$

Since $\mathrm{D}h(\bar{x}(t))$ has rank $n-1$ and $f(\bar{x}(t))$ is nonzero for all t, it is clear that the null space of $\mathrm{D}h(\bar{x}(t))$ is spanned by $f(\bar{x}(t))$. Differentiating (A.3) with respect to time, we get

$$[f^{\mathrm{T}}(\bar{x}(t))\,\mathrm{D}^2h(\bar{x}(t)) + \mathrm{D}h(\bar{x}(t))\,\mathrm{D}f(\bar{x}(t))]f(\bar{x}(t)) = 0.$$

We conclude that $\mathrm{D}h(\bar{x}(t))$ is a right factor of (A.2), since $f(\bar{x}(t))$ lies in the null space of (A.2). □

Proof of Proposition 4.2

By Proposition 4.1, we know that the linearization of the system dynamics along a periodic trajectory $\bar{x}(t)$ in γ will have the form

$$\begin{bmatrix} \delta\dot{\xi} \\ \delta\dot{\eta} \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ p(t) & q(t) \end{bmatrix} \begin{bmatrix} \delta\xi \\ \delta\eta \end{bmatrix}, \qquad (\mathrm{A}.4)$$

where $A(t) \in \mathbb{R}^{(n-1)\times(n-1)}$, $p(t) \in \mathbb{R}^{1\times(n-1)}$, and $q(t) \in \mathbb{R}$ are all periodic. Clearly, the state transition matrix of (A.4) is given by

$$\Phi(t,s) = \begin{bmatrix} \Phi_A(t,s) & 0 \\ \displaystyle\int_s^t \Phi_q(t,\tau)p(\tau)\Phi_A(\tau,s)\,\mathrm{d}\tau & \Phi_q(t,s) \end{bmatrix}. \qquad (\mathrm{A}.5)$$

By Floquet theory (Hartman, 1964), we know that the periodic orbit of the nonlinear system $\dot{x} = f(x)$ is exponentially stable if all of the eigenvalues of $\Phi(T,0)$ except one have magnitude less than one. The eigenvalues of $\Phi(T,0)$ consist of the scalar $\Phi_q(T,0)$ and the eigenvalues of $\Phi_A(T,0)$. Now, since $\delta\eta$ is a perturbation along (i.e., tangent to) γ, it is clear that $\Phi_q(T,0) = 1$ (see Hartman, 1964). Hence, if the eigenvalues of $\Phi_A(T,0)$ have magnitude less than one, or, equivalently, if the linear system (15) is asymptotically (hence, exponentially) stable, then γ will be exponentially stable. □