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# ENERGY BASED CONTROL OF A CLASS OF UNDERACTUATED MECHANICAL SYSTEMS

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**Abstract.** In this paper we discuss some design methods for the control of a class of underactuated mechanical systems. This class of systems includes gymnastic robots, such as the Acrobot, as well the classical cart–pole system. The systems considered here are nonminimum phase which greatly complicates the control design issues. Our design techniques combine nonlinear partial feedback linearization with Lyapunov methods based on saturation functions, switching and energy shaping.

**Keywords.** control theory, robotics, mechanical systems, feedback linearization, saturation.

## 1. INTRODUCTION

Underactuated mechanical systems are mechanical systems with fewer actuators than degrees–of–freedom and arise in several ways, from intentional design as in the the Acrobot (Bortoff, 1992), in mobile robot systems when a manipulator arm is attached to a mobile platform, a space platform, or an undersea vehicle, or because of the mathematical model used for control design as when joint flexibility is included in the model. In the latter sense, then, all mechanical systems are underactuated if one wishes to control flexible modes that are not directly actuated (the noncollocation problem), or even to include such things as actuator dynamics in the model description.

### 1.1 *Upper and Lower Actuated Systems*

We consider an $n$–degree–of–freedom system with generalized coordinates $q^1, \ldots, q^n$, and $m < n$ actuators, each of which directly actuates a single degree of freedom. Each actuated degree of freedom is called an *active joint*. The remaining $\ell = n - m$ unactuated degrees of freedom are called *passive joints*, as shown in Figure 1.

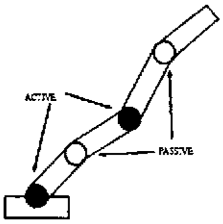


Fig. 1. A General Underactuated System

A so–called *Upper Actuated System* is one in which the upper arm, or the first $m$–joints are actuated while a *Lower Actuated System* is one in which the lower arm, or the last $m$–joints are actuated (see Figure 2). By suitably numbering and partitioning the vector $q$ of generalized coordinates we may write $q^T = (q_1^T, q_2^T)$ where $q_1 \in R^\ell$ corresponds to the passive joints and $q_2 \in R^m$ corresponds to the active joints. Thus all systems will be considered as though they are lower actuated without loss of generality.

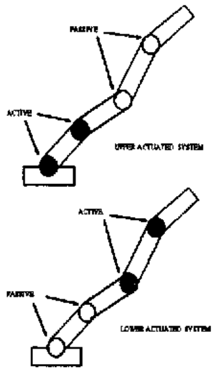


Fig. 2. Upper and Lower Actuated Systems

### 1.2 *Dynamics*

With the vector $q \in R^n$ of generalized coordinates partitioned as above with $q_1 \in R^\ell$ and $q_2 \in R^m$, we may write the dynamic equations of the $n$ degree of freedom system as

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \tag{1}$$

$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau \tag{2}$$

where

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{3}$$

is the symmetric, positive definite inertia matrix, the vector functions $h_1(q,\dot{q}) \in R^\ell$ and $h_2(q,\dot{q}) \in R^m$ contain Coriolis and centrifugal terms, the vector functions $\phi_1(q) \in R^\ell$ and $\phi_2(q) \in R^m$ contain gravitational terms, and $\tau \in R^m$ represents the input generalized force produced by the $m$ actuators at the active joints. For notational simplicity we will henceforth not write the explicit dependence on $q$ of these coefficients. The equations (1)–(2) represent the standard dynamics of $n$ link robots except that there is no control input to the first $\ell$ equations (Spong and Vidyasagar, 1989).

### 1.3 *Collocated Partial Feedback Linearization*

Consider the first equation (1)

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \tag{4}$$

The term $M_{11}$ is an invertible $\ell \times \ell$ matrix as a consequence of the uniform positive definiteness of the robot inertia matrix $M$ in (3). Therefore we may solve for $\ddot{q}_1$ in equation (4) as

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12}\ddot{q}_2 + h_1 + \phi_1) \tag{5}$$

and substitute the resulting expression (5) into (2) to obtain

$$\bar{M}_{22}\ddot{q}_2 + \bar{h}_2 + \bar{\phi}_2 = \tau \tag{6}$$

where the terms $\bar{M}_{22}$, $\bar{h}_2$, $\bar{\phi}_2$ are given by

$$\begin{aligned} \bar{M}_{22} &= M_{22} - M_{21}M_{11}^{-1}M_{12} \\ \bar{h}_2 &= h_2 - M_{21}M_{11}^{-1}h_1 \\ \bar{\phi}_2 &= \phi_2 - M_{21}M_{11}^{-1}\phi_1 \end{aligned}$$

As shown in (Gu and Loh, 1993) the $m \times m$ matrix $\bar{M}_{22}$ is itself symmetric and positive definite. To see this we note that a simple calculation (Gu and Loh, 1993) yields

$$\bar{M}_{22} = T^T M T \tag{7}$$

where $T$ is an $n \times m$ matrix defined by

$$T = \begin{bmatrix} -M_{11}^{-1}M_{12} \\ I_{m\times m} \end{bmatrix} \tag{8}$$

with $I_{m\times m}$ the $m \times m$ identity matrix. Since $T$ has rank $m$ for all $q$ and $M$ is symmetric and positive definite, it follows that $\bar{M}_{22}$ is symmetric and positive definite.

A partial feedback linearizing controller can therefore be defined for equation (6) according to

$$\tau = \bar{M}_{22}u + \bar{h}_2 + \bar{\phi}_2 \tag{9}$$

where $u \in R^m$ is an additional control input yet to be defined. The complete system up to this point may be written as

$$M_{11}\ddot{q}_1 + h_1 + \phi_1 = -M_{12}u \tag{10}$$

$$\ddot{q}_2 = u \tag{11}$$

We see that the input/output system from $u$ to $q_2$ is linear and second order. The complete system therefore has $m$–vector relative degree $(2,\ldots,2)^T$ (Isidori, 1989) with respect to the output $q_2$ and the equation (10) represents the internal dynamics.

We first apply a feedback control $u$ according to

$$u = -k_1 q_2 - k_2\dot{q}_2 + k_3\bar{u} \tag{12}$$

so that the linear subsystem (11) is asymptotically stable for $\bar{u} = 0$. The remaining design problem then is the choice of the additional term $\bar{u}$. We will address this in the next section.

## 2. ENERGY BASED CONTROL

In the case of exact feedback linearization the control design problem is largely complete once the system is linearized. For partial, or input/output, linearization with asymptotically stable zero dynamics local stabilization is easily achieved, but global or semi-global stabilization requires consideration of issues such as peaking, etc. (Sussman and Kokotović, 1991; Teel, 1992). For the class of systems considered here, the collocated linearization approach above, generally result in systems having unstable zero dynamics. Thus the second stage control, i.e. the design of the outer loop terms $u$ in (11), more to the point, the choice of $u$ in (12), is nontrivial. To deal with this we will outline methods for the design of $\bar{u}$ that combines the high gain and/or saturating controls typically considered with methods based on passivity and energy concepts. By exploiting the physical structure of the system in this way, we can achieve closed loop stability without requiring stable zero dynamics. In general, though, the stability achieved is not asymptotic to a fixed point, but only to a manifold. For this reason, for problems such as swing up control of inverted pendula or the Acrobot, the control must eventually switch to separate controller that achieves local asymptotic stability to the desired equilibrium configuration. We will illustrate these designs on several examples, without proof.

### 2.1 *Swing Up Control of The Acrobot*

The Acrobot is a two-link planar robot with a single actuator at the elbow.

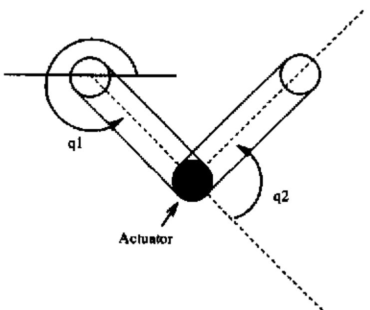


Fig. 3. The Acrobot

The equations of motion are given by (Spong and Vidyasagar, 1989)

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \qquad (13)$$

$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau_2 \qquad (14)$$

where

$$\begin{aligned}
m_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos(q_2)) + I_1 + I_2 \\
m_{22} &= m_2\ell_{c2}^2 + I_2 \\
m_{12} &= m_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos(q_2)) + I_2 \\
h_1 &= -m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2^2 - 2m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2\dot{q}_1 \\
h_2 &= m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_1^2 \\
\phi_1 &= (m_1\ell_{c1} + m_2\ell_1)g\cos(q_1) + m_2\ell_{c2}g\cos(q_1+q_2) \\
\phi_2 &= m_2\ell_{c2}g\cos(q_1+q_2)
\end{aligned}$$

The parameters $m_i$, $\ell_i$, $\ell_{ci}$, and $I_i$ are masses, link lengths, centers of masses, and moments of inertia, respectively. The zero configuration, $q_i = 0$, in this model corresponds to the arm extended horizontally. Therefore, the swing up task is move the robot from the vertically downward configuration $q_1 = -\pi/2$, $q_2 = 0$ to the inverted configuration $q_1 = +\pi/2$, $q_2 = 0$. Our strategy is as follows: We first apply the partial feedback linearization control (9) with the outer loop term given by (12). The resulting system can be written as

$$m_{11}\ddot{q}_1 + h_1 + \phi_1 = -m_{12}(\bar{u} - k_2\dot{q}_2 - k_1 q_2) \qquad (15)$$

$$\ddot{q}_2 + k_2\dot{q}_2 + k_1 q_2 = \bar{u} \qquad (16)$$

We then choose the additional control $\bar{u}$ to swing the second link "in phase" with the motion of the first link in such a way that the amplitude of the swing of the first link increases with each swing. To accomplish this let $E$ represent the total energy of the Acrobot given by

$$E = \frac{1}{2}m_{11}\dot{q}_1^2 + m_{12}\dot{q}_1\dot{q}_2 + \frac{1}{2}m_{22}\dot{q}_2^2 + P(q_1, q_2)$$

where $P(q_1, q_2)$ is the potential Energy. Let $E_c$ represent the energy of the Acrobot in its balancing position. Then $E_c$ represents the Potential Energy at the configuration $q_1 = \pi/2$, $q_2 = 0$. Set $\tilde{E} = E - E_c$, and choose the outer loop control $\bar{u}$ as

$$\bar{u} = \text{sat}(\tilde{E}\dot{q}_1) \qquad (17)$$

where sat() is the saturation function.

The behavior of the resulting closed loop system is quite interesting and we can discuss its qualitative behavior. We are attempting to make rigorous the following qualitative statements using Lyapunov functions based on the total energy in conjunction with LaSalle's Theorem. In some cases, local proofs are available; we conjecture that at least semi-global results are possible using these techniques. The total energy $E$ should converge to $E_c$ while $q_2$ and $\dot{q}_2$ converge to zero. The "steady–state" behavior of the Acrobot under this control is therefore nearly identical to the behavior of a single simple pendulum converging to its homoclinic orbit. The state trajectory therefore converges to an invariant set consisting of the union of the two equilibrium configurations $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (\pm\pi/2, 0, 0, 0)$ with the set $(E, \dot{E}, q_2, \dot{q}_2) = (E_c, 0, 0, 0)$.

The final step is to switch to a "balancing controller" when the "swingup controller" $\bar{u}$ brings the state into the basin of attraction of the balancing controller. We have investigated various methods for designing balancing controllers, chiefly pseudo-linearization [bor92] and Linear–Quadratic methods. The behavior of the swingup controller followed by the balancing controller is illustrated below in Figure(4).

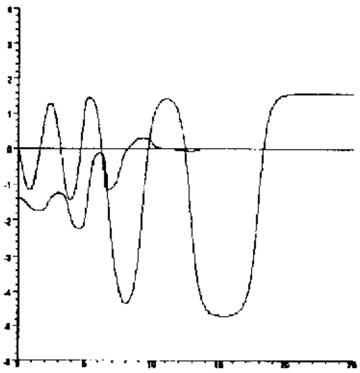

Fig. 4. Swingup and Balance of The Acrobot

## 2.2 *Three–Link Gymnast Robot*

Next we apply the partial feedback linearization control to execute a so–called *giant swing* maneuver and balance for a three–link planar gymnast robot with two actuators shown below.

The equations of motion are of the form

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 + h_1 + \phi_1 = 0 \tag{18}$$

$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 + h_2 + \phi_2 = \tau_2 \tag{19}$$

$$m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 + h_3 + \phi_3 = \tau_3 \tag{20}$$

In this case we can linearize two of the three degrees of freedom to obtain

$$m_{11}\ddot{q}_1 + h_1 + \phi_1 = -m_{12}u_2 + m_{13}u_3 \tag{21}$$

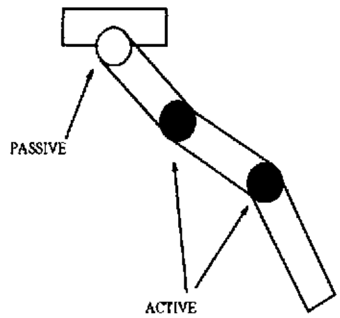


Fig. 5. Three Link Gymnast Robot

$$\ddot{q}_2 = u_2 \tag{22}$$

$$\ddot{q}_3 = u_3 \tag{23}$$

Using our insight gained from the Acrobot, we specify the outer loop controls according to

$$u_2 = -k_{21}q_2 - k_{22}\dot{q}_2 + \bar{u}_2 \tag{24}$$

$$u_3 = -k_{32}q_3 - k_{32}\dot{q}_3 + \bar{u}_3 \tag{25}$$

with

$$\bar{u}_2 = k_{23}\ \mathrm{sat}(\tilde{E}\dot{q}_1) \tag{26}$$

$$\bar{u}_3 = k_{33}\ \mathrm{sat}(\tilde{E}\dot{q}_2) \tag{27}$$

where $\tilde{E} = E - E_c$, $E$ is the total energy of the three link system and $E_c$ is the potential energy at the inverted position. Figure 6 shows a plot of the resulting giant swing maneuver including a switch to a linear controller at the end to balance the robot in the inverted position.

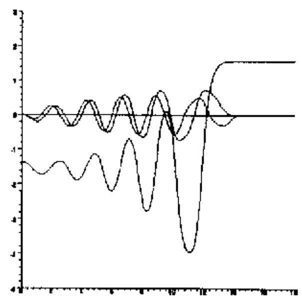

Fig. 6. Three Link Gymnast Robot: Swingup and Balance

## 2.3 *The Cart–Pole System*

In this section we treat the familiar cart–pole system and show how the same design ideas as above can be applied. The Euler-Lagrange equations for this system are

$$(M+m)\ddot{x} + ml\cos(\theta)\ddot{\theta} = ml\dot{\theta}^2\sin(\theta) + F \quad (28)$$

$$\ddot{x}\cos(\theta) + l\ddot{\theta} = g\sin(\theta) \quad (29)$$

where $x$ is the cart position and $\theta$ is the pendulum angle measured from the vertical. Applying the partial feedback linearization control, it is easy to show that the cart–pole system may be written as:

$$\dot{x} = v \quad (30)$$

$$\dot{v} = u \quad (31)$$

$$\dot{\theta} = \omega \quad (32)$$

$$\dot{\omega} = sin(\theta) - cos(\theta)u \quad (33)$$

where we have used the more descriptive notation, $x$, $v$, $\theta$ and $\omega$ instead of $z_i$ and $\eta_i$ to represent the cart position and velocity, and pendulum angle and angular velocity, respectively, and have normalized the parameter values. Using our above strategy, the simple control

$$u = k_3 \operatorname{sat}(\tilde{E}\cos(\theta)\omega - k_1 x - k_2 v) \quad (34)$$

can be used both to swing up the pendulum by controlling its swinging energy and to regulate the cart position and velocity, where $E$ is the total energy of the pendulum and is given by

$$E = \frac{1}{2}\omega^2 + \cos(\theta) \quad (35)$$

and $E = 1$ corresponds to the upright position of the pendulum.

The earliest use of such energy ideas to swing up a pendulum was reported in (Wicklund, et.al., 1993) without consideration of the cart position or velocity. The later controller of (Chung and Hauser, 1995) is identical to our controller (34) if the saturation function is removed, i.e. if

$$u = -k_1 x - k_2 v + k_3\tilde{E}\cos(\cos(\theta)\omega \quad (36)$$

(Chung and Hauser, 1995) proved local exponential covergence of the cart position and velocity to zero and of the pendulum energy to unity. Figure (7) shows the response of the cart–pole using the control (34).

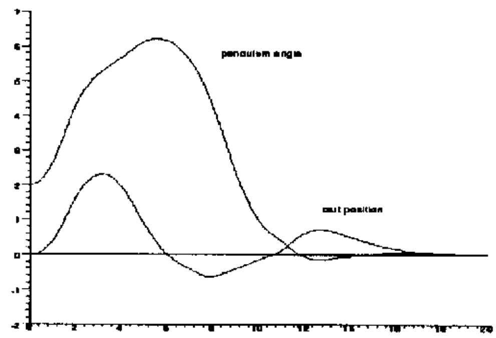


Fig. 7. Cart–Pole Response Using (34)

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