EE 505 HW 4

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3.14 Consider the system

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2)\\ \dot{x}_2 &= -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)\end{aligned}$$

where λ and τ are positive constants.

(a) Derive the sensitivity equations as λ and τ vary from their nominal values λ_0 and τ_0.

(b) Show that $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality

$$\dot{r} \leq -\frac{1}{\tau}r + 2\sqrt{2}$$

(c) Using the comparison lemma, show that the solution of the state equation satisfies the inequality

$$\|x(t)\|_2 \leq e^{-t/\tau}\|x(0)\|_2 + 2\sqrt{2}\tau(1 - e^{-t/\tau})$$

a) The sensitivity equation in given by

$$\dot{S} = AS + B \qquad S(0) = 0$$

where A and B are given by

$$A = \frac{\partial f(x,\lambda)}{\partial x}\bigg|_{nominal} = \begin{bmatrix} -\frac{1}{\tau_0} + \lambda_0 \operatorname{sech}^2(\lambda_0 x_1) & -\lambda_0 \operatorname{sech}^2(\lambda_0 x_2) \\ \lambda_0 \operatorname{sech}^2(\lambda_0 x_1) & -\frac{1}{\tau_0} + \lambda_0 \operatorname{sech}^2(\lambda_0 x_2) \end{bmatrix}$$

$$B = \frac{\partial f}{\partial \epsilon}\bigg|_{nominal} = \begin{bmatrix} x_1 \operatorname{sech}^2(\lambda_0 x_1) - x_2 \operatorname{sech}^2(\lambda_0 x_2) & \frac{1}{\tau_0^2}x_1 \\ x_1 \operatorname{sech}^2(\lambda_0 x_1) + x_2 \operatorname{sech}^2(\lambda_0 x_2) & \frac{1}{\tau_0^2}x_2 \end{bmatrix}$$

where $\epsilon = [\lambda, \tau]^T$.

b) Notice that

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$$

Substituting the expressions for $\dot{x}_1$ and $\dot{x}_2$ give

$$r\dot{r} \leq -\frac{1}{\tau}(x_1^2 + x_2^2) + x_1(\tanh(\lambda x_1) - \tanh(\lambda x_2)) + x_2(\tanh(\lambda x_1) + \tanh(\lambda x_2))$$

Substitute $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Since $|\tanh(x)| \leq 1$ and $|\sin\theta + \cos\theta| \leq \sqrt{2}$ we have

$$r\dot{r} \leq -\frac{1}{\tau}r^2 + 2r(\sin\theta + \cos\theta) \leq -\frac{1}{\tau}r^2 + 2\sqrt{2}r$$

c) Let $r(t) = ||x(t)||_2$ and $r(0) = u(0) = ||x(0)||_2$. Also, let $u(t)$ satisfy

$$\dot{u} = -\frac{1}{\tau}u + 2\sqrt{2} \qquad u(0) = ||x(0)||_2$$

Using Laplace transforms, we solve for $u(t)$:

$$u(t) = 2\sqrt{2}\tau(1 - e^{-t/\tau}) + e^{-t/\tau}u(0)$$

Since

$$\dot{r} \leq -\frac{1}{\tau}r^2 + 2\sqrt{2}r$$

we have by the comparison lemma $r(t) \leq u(t)$ and

$$||x(t)||_2 \leq 2\sqrt{2}\tau(1 - e^{-t/\tau}) + e^{-t/\tau}||x(0)||_2$$

3.15 Using the comparison lemma, show that the solution of the state equation

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1 + x_2^2}, \qquad \dot{x}_2 = -x_2 + \frac{2x_1}{1 + x_1^2}$$

satisfies the inequality

$$\|x(t)\|_2 \leq e^{-t}\|x(0)\|_2 + \sqrt{2}\left(1 - e^{-t}\right)$$

Let $r = \sqrt{x_1^2 + x_2^2}$ Then,

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2 = -r^2 + 2x_1x_2\left(\frac{1}{1 + x_1^2} + \frac{1}{1 + x_2^2}\right)$$

But

$$\max\left|\frac{2x}{1 + x^2}\right| = 1$$

Using this fact and substituting in $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, we obtain

$$r\dot{r} \leq -r^2 + r(\cos\theta + \sin\theta) \leq -r^2 + \sqrt{2}r$$

And similarly

$$\dot{r} \leq -r + \sqrt{2}$$

Now we use the comparison lemma. Consider $r(t) \leq u(t)$ where $u(t)$ satisfies

$$\dot{u} = -u + \sqrt{2} \qquad u(0) = r(0) = ||x(0)||_2$$

Solving for $u(t)$ we obtain

$$u(t) = e^{-t}u(0) + \sqrt{2}(1 - e^{-t})$$

and so by the comparison lemma we obtain

$$||x(t)||_2 \leq e^{-t}||x(0)||_2 + \sqrt{2}(1 - e^{-t})$$

3.16 Using the comparison lemma, find an upper bound on the solution of the scalar equation

$$\dot{x} = -x + \frac{\sin t}{1 + x^2}, \quad x(0) = 2$$

There are many different achievable upper bounds for this problem. Some are better than others. We consider the case when $v(t) = x(t)$ for simplicity:

$$\dot{v} = -v + \frac{\sin t}{1 + v^2} \leq -v + \frac{1}{1 + v^2} \leq -v + 1$$

Let $u(t)$ satisfy

$$\dot{u} = -u + 1 \qquad u(0) = x(0) = 2$$

Solving for $u(t)$ we obtain

$$u(t) = 1 + e^{-t}$$

By the comparison lemma,

$$x(t) \leq 1 + e^{-t}$$

3.17 Consider the initial-value problem (3.1) and let $D \subset R^n$ be a domain that contains $x = 0$. Suppose $x(t)$, the solution of (3.1), belongs to D for all $t \geq t_0$ and $\|f(t, x)\|_2 \leq L\|x\|_2$ on $[t_0, \infty) \times D$. Show that

(a)

$$\left| \frac{d}{dt} \left[x^T(t)x(t) \right] \right| \leq 2L\|x(t)\|_2^2$$

(b)

$$\|x_0\|_2 \exp[-L(t - t_0)] \leq \|x(t)\|_2 \leq \|x_0\|_2 \exp[L(t - t_0)]$$

a) This part is easy to show:

$$\left| \frac{d}{dt}[x^T x] \right| = \left| \frac{d}{dt}[x_1^2 + \cdots + x_n^2] \right| = \left| 2[x_1\dot{x}_1 + \cdots + x_n\dot{x}_n] \right| =$$

$$= \left| 2x^T\dot{x} \right| = \left| 2x^T f(t, x) \right| \leq 2||x||_2||f(t, x)||_2 \leq 2L||x(t)||_2^2$$

b) The trick to this problem is the careful choice of $V(t) = x^T(t)x(t) = ||x(t)||_2^2$ and $V(0) = ||x(0)||_2^2$. From part a) we know that:

$$
\begin{aligned}
-2L||x||_2^2 \leq \frac{d}{dt}[x^T x] \leq 2L||x||_2^2 \\
-2LV \leq \dot{V} \leq 2LV \\
\int_{t_0}^{t} -2L\,dt \leq \int_{V_0}^{V} \frac{dV}{V} \leq \int_{t_0}^{t} 2L\,dt \\
-2L(t-t_0) \leq \ln\frac{V}{V_0} \leq 2L(t-t_0) \\
V_0 e^{-2L(t-t_0)} \leq V \leq V_0 e^{2L(t-t_0)} \\
||x(0)||_2^2 e^{-2L(t-t_0)} \leq ||x||_2^2 \leq ||x(0)||_2^2 e^{2L(t-t_0)} \\
||x(0)||_2 e^{-L(t-t_0)} \leq ||x||_2 \leq ||x(0)||_2 e^{L(t-t_0)}
\end{aligned}
$$

3.18 Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]\, d\tau$$

where k_1, k_2, and k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall–Bellman inequality, show that

$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

This problem is a lot of tedious algebra and calculus. For this solution, we provide a very high level outline. The critical step is to use the hint in the problem statement and write

$$z(t) = y(t)e^{\alpha(t-t_0)} \leq k_1 + \int_{t_0}^{t} e^{\alpha(\tau-t_0)}[k_2 y(\tau) + k_3]\, d\tau$$

From here, we split the integral into two parts and perform one integration.

$$z(t) \leq k_1 + \frac{k_3}{\alpha}[e^{\alpha(t-t_0)} - 1] + \int_{t_0}^{t} k_2 z(\tau)\, d\tau$$

We notice that the expression is a form suitable for the Gronwall-Bellman inequality:

$$z(t) \leq \lambda(t) + \int_{t_0}^{t} k_2 z(\tau)\, d\tau$$

So by Gronwall-Bellman we have

$$z(t) \leq \lambda(t) + \int_{t_0}^{t} k_2 \lambda(s) e^{k_2(t-s)}\, ds$$

This integral is actually doable, but extremely tedious. I employed the use of MATLAB to perform the integral and simplify the resulting algebra:

```
syms k1 k2 k3 a t t0 s
lambda = @(x) k1 + k3/a*(exp(a*(x-t0))-1);
f = @(s) k2*lambda(s)*exp(k2*(t-s));
expr = lambda(t) + int(f, s, t0, t)
```

Now, we multiply both sides by $e^{-\alpha(t-t_0)}$ to make the left hand side equal to $y(t)$ and simplify. Eventually, we arrive at the answer. Because the focus of this problem is on applying the comparison principle and the Gronwall-Bellman inequality and not on tedious algebraic simplifications, we omit the remainder of this problem. I was able to reach the answer by hand, but it took multiple pages of calculations.

3.19 Let $f : R^n \to R^n$ be locally Lipschitz in a domain $D \subset R^n$. Let $S \subset D$ be a compact set. Show that there is a positive constant L such that for all $x, y \in S$,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Hint: The set S can be covered by a finite number of neighborhoods; that is,

$$S \subset N(a_1, r_1) \cup N(a_2, r_2) \cup \cdots \cup N(a_k, r_k)$$

Consider the following two cases separately:

- $x, y \in S \cap N(a_i, r_i)$ for some i.
- $x, y \notin S \cap N(a_i, r_i)$ for any i; in this case, $\|x - y\| \geq \min_i r_i$.

In the second case, use the fact that $f(x)$ is uniformly bounded on S.

From definition of LL, we know that for any two points $x, y \in D$ we have

$$||f(x) - f(y)|| \leq L'||x - y||$$

We seek to show that the Lipschitz condition can be strengthened from LL on D to Lipschitz on S. We use the fact that any compact subset can be covered by a finite number of neighborhoods. In the first case, we have $x, y \in S \cap N(a_i, r_i)$. From LL, we have

$$||f(x) - f(y)|| \leq L_i||x - y||$$

so if x, y are in the same neighborhood then just choose $L' = L_i$. In the second case, $x, y \notin S \cap N(a_i, r_i)$ for any i so we cannot use the same argument. But, we know that $f(x)$ is uniformly bounded. That is,

$$||f(x)|| \leq R$$

where R is not a function of x. This implies that,

$$||f(x) - f(y)|| \leq ||f(x)|| + ||f(y)|| \leq 2R$$

is bounded. We are given that in the second case,

$$||x - y|| \geq \min_i r_i$$

This means that

$$||f(x) - f(y)|| \leq 2R \leq \frac{2R}{\min_i r_i}||x - y||$$

and we can choose $L' = \frac{2R}{\min_i r_i}$. Putting the two cases together, we just need to find the largest possible value of L that will ensure the Lipschitz condition holds over S. Obviously,

$$L = \max\left\{L_1, L_2, \ldots, L_k, \frac{2R}{\min_i r_i}\right\}$$

ensures that

$$||f(x) - f(y)|| \leq L||x - y||$$

regardless of which case we find ourselves in.

Although not rigorous, a good way to justify this result is to imagine we compute L' for every possible pairs of points $x, y \in S$. This is possible because S is compact (closed and bounded). Then, choose the largest L' as the Lipschitz constant such that the LL condition always holds.

3.20 Show that if $f : R^n \to R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

Since f is Lipschitz on W

$$||f(x) - f(y)|| \leq L||x - y||$$

Uniform continuous means that

$$\forall \varepsilon > 0, \exists \delta > 0 \ni ||x - y|| \leq \delta \implies ||f(x) - f(y)|| \leq \varepsilon$$

Now, given any $\varepsilon > 0$ choose $\delta = \varepsilon/L$. Because of the Lipschitz condition,

$$||f(x) - f(y)|| \leq L\delta = \varepsilon$$

therefore $f(x)$ is uniformly continuous on W.