EE 505 HW 6

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4.11 Using Theorem 4.3, prove Lyapunov's first instability theorem:
For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Let $V_1 = V$. We have $V_1(0) = 0$ and $\dot{V}_1(x) > 0$ when $x \neq 0$ and $x \subset D$. Since V_1 is not positive semidefinite near the origin, there exist some arbitrarily small x_0 such that $V(x_0) > 0$. Define $U = \{x \in B_r \mid V_1(x) > 0\}$ for some $B_r \subset D$. We have met the conditions for Theorem 4.3, so the origin is unstable.

4.12 Using Theorem 4.3, prove Lyapunov's second instability theorem:
For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Let $V = V_1$. We have $V_1(0) = 0$. Also, there exists some point x_0 arbitrarily close to the origin such that $V_1(x_0) > 0$. Choose $U = \{x \in B_r \mid V_1(x) > 0\}$ and $U \subset D$. Since $W(x) > 0$ in D, we have $\dot{V}_1(x) > 0$ in U. The conditions for Theorem 4.3 are met, so the origin is unstable.

4.13 For each of the following systems, show that the origin is unstable:

$$(1) \qquad \dot{x}_1 = x_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

$$(2) \qquad \dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

1) Choose the function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. We plot $V(x) > 0$ in green, $\dot{V}(x) > 0$ in red, and define $U = \{x \in D \mid V(x) > 0\}$, with $D = B_r$ where $r^2 = \frac{1}{2}$. We see that $\dot{V}(x) > 0$ for all $x \in U$. By Theorem 4.3, the origin is unstable.

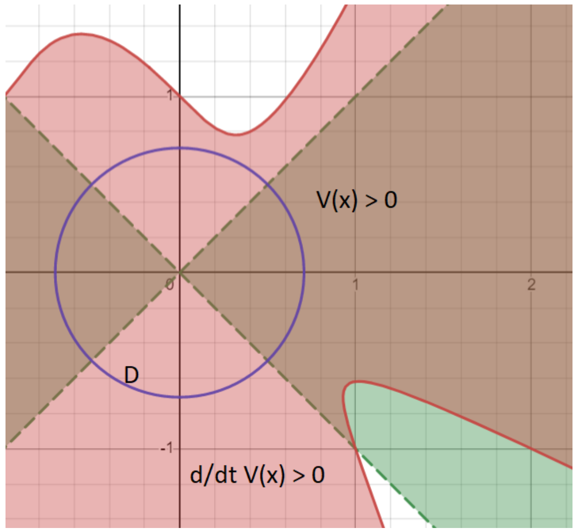

Figure 1: Problem 4.13 (1)

2) Γ is a nonempty positively invariant set which contains the equilibrium points $(0,0)$ and $(1,1)$. We know that Γ is positively invariant because the vector field at the boundary $x_2 = x_1^3$ points directly to the right, and the vector field at the boundary $x_2 = x_1^2$ points vertically, while inside Γ but excluding the boundary and equilibrium points, both $\dot{x}_1$ and $\dot{x}_2$ are positive. This means trajectories which start at $x_0 \neq 0 \in \Gamma$ will eventually reach the $(1,1)$ equilibrium point. Consider any point $x_0 \in \Gamma$ arbitrarily close to the origin with $||x_0|| > 0$. This trajectory will move away from the origin, so the origin is unstable.

4.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} y g(y)\, dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

First we show that $V(x)$ is positive definite and radially unbounded. The minimum value that $g(y)$ can take on is 1, therefore,

$$V(x) \geq \int_0^{x_1} y\, dy + x_1 x_2 + x^2 = \frac{1}{2}x_1^2 + x_1 x_2 + x_2^2$$

We express the right hand side in a matrix quadratic form

$$V(x) \geq \frac{1}{2} x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x$$

and see that it is positive definite. Therefore $V(x)$ is positive definite and radially unbounded. Next, we show that $\dot{V} \leq 0$. Taking the integral and employing Leibniz rule,

$$\dot{V} = x_2^2 - g(x_1)(x_1^2 + 2x_1 x_2 + 2x_2^2)$$

Since the term multiplying $-g(x_1)$ is positive definite, the largest value $\dot{V}$ could take will occur when $g(x_1) = 1$. Therefore,

$$\dot{V} \leq x_2^2 - (x_1^2 + 2x_1x_2 + 2x_2^2) = -(x_1 + x_2)^2 \leq 0$$

so $\dot{V}$ is negative semi-definite. Now we employ the invariance principle. Notice that if $\dot{V} = 0$, then $x_1 + x_2 = 0$ so $\dot{x}_2 = 0$ so $\dot{x}_1 = 0$ so $x_2 = 0$ so $x_1 = 0$. Thus, the only trajectory where $\dot{V} = 0$ for all time is at the origin, so the equilibrium point $x = 0$ is AS.

4.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)\ dy + x_2^2/2 + \int_0^{x_3} h_2(y)\ dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

a) The origin is the only equilibrium point, since $\dot{x}_1 = 0 \implies x_2 = 0 \implies x_3 = 0 \implies h(x_1) = 0 \implies x_1 = 0$.

b) We note that $\int_0^z h_i(y)\, dy > 0$ for $z \neq 0$, since $h_i(y)$ is positive for $y > 0$ and negative for $y < 0$. Therefore, $V(x)$ is the sum of three positive terms for $x \neq 0$, and $V(x)$ is positive definite.

c) Using Leibniz rule, we see that

$$\dot{V}(x) = -(x_2^2 + x_3 h_2(x_3)) \leq 0$$

and $\dot{V}(x) = 0$ only when $x_2 = 0$ and $x_3 = 0$. This implies that $\dot{x}_2 = 0$ so $x_1 = 0$. Thus, the only trajectory which satisfies $\dot{V} = 0$ is the origin, so $x = 0$ is AS.

d) For GAS, we require that $V(x)$ be radially unbounded. This occurs when

$$\lim_{z \to \infty} \int_0^z h_i(y)\, dy = \infty$$

4.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

Choose $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ which is clearly positive definite. Then, $\dot{V} = -x_2^4 \leq 0$. Using the invariance principle, if $x_2 = 0$ then $\dot{x}_2 = 0 \implies x_1 = 0$. Thus, the origin is GAS.

4.17 ([77]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\, dy + (1/2)x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat part (b) using $V(x) = (1/2)\left[x_2 + \int_0^{x_1} h(y)\, dy\right]^2 + \int_0^{x_1} g(y)\, dy$.

a) The system can be written is state space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1)x_2 - g(x_1). \end{aligned}$$

In order for the origin to be an isolated equilibrium point, we require $g(y)$ to have an isolated root at the origin.

b) With $V(x) = \int_0^{x_1} g(y)\, dy + \frac{1}{2}x_2^2$, we have $\dot{V} = -h(x_1)x_2^2$. By enforcing that $x_1 g(x_1) > 0$ and $h(x_1) > 0$ over some region containing the origin (and no other equilibrium points), then $\dot{V} \leq 0$. Applying the invariance principle, we see that equality holds only at the origin.

c) With $V(x) = \frac{1}{2}[x_2 + \int_0^{x_1} h(y)\, dy]^2 + \int_0^{x_1} g(y)\, dy$, we have $\dot{V} = -g(x_1)\int_0^{x_1} h(y)\, dy$. For AS, we require that $g(x_1)\int_0^{x_1} h(y)\, dy \geq 0$ and $\int_0^{x_1} h(y)\, dy \neq 0$. Then, by the invariance principal $g(x_1) = 0 \implies x_1 = 0 \implies \dot{x}_1 = 0 \implies x_2 = 0 \implies \dot{x}_2 = 0$ so the system is AS.

4.18 The mass–spring system of Exercise 1.12 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

The only equilibrium point occurs at $y = \frac{Mg}{k}, \dot{y} = 0$. Take $x_1 = y - \frac{Mg}{k}$ and $x_2 = \dot{y}$. Then, the system can be represented as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}(-kx_1 - c_1x_2 - c_2x_2|x_2|) \end{aligned}$$

with the equilibrium point located at the origin. We choose $V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2 > 0$ where a and b are positive constants that will be determined later. Thus,

$$\dot{V} = ax_1x_2 - \frac{bk}{M}x_1x_2 - \frac{bc_1}{M}x_2^2 - \frac{bc_2}{M}x_2^2|x_2|$$

Choose $a = k$ and $b = M$. Then,

$$\dot{V} = -c_1x_2^2 - c_2x_2^2|x_2| \leq 0$$

with equality only at the origin. To see this, $x_2 = 0 \implies \dot{x}_1 = 0 \implies \dot{x}_2 = 0 \implies x_1 = 0$. All of the conditions for Corollary 4.2 have been met, so the equilibrium point $x = 0$ is GAS.

4.19 Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q) = 0$ has an isolated root at $q = 0$.

(a) With $u = 0$, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin $(q = 0,\ \dot{q} = 0)$ is stable.

(b) With $u = -K_d\dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m, show that the point $(q = q^*,\ \dot{q} = 0)$ is an asymptotically stable equilibrium point.

a) The mechanical system can be written in state-manipulator form with

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

So, $\ddot{q} = M^{-1}(u - C\dot{q} - D\dot{q} - g)$. Using the Lyapunov candidate function $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) > 0$, we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}(\ddot{q})^T M\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}\dot{q} + \frac{1}{2}\dot{q}^T M(\ddot{q}) + \underbrace{\frac{\partial P}{\partial q}}_{g^T}\dot{q} \\
&= \frac{1}{2}\dot{q}^T \underbrace{(\dot{M} - 2C)}_{\text{Anti-symmetric}}\dot{q} - \dot{q}^T D\dot{q} \\
&\leq -\dot{q}^T D\dot{q} \leq 0
\end{aligned}$$

This proves the origin is stable.

b) With $u = -K_d\dot{q}$ and using the same V function, we have

$$\dot{V} \leq -\dot{q}^T K_d\dot{q} - \dot{q}^T D\dot{q} < 0$$

so the origin is AS.

c) With the new u, the state space model can be written in the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + K_p(q - q^*) + K_d\dot{q} = 0.$$

We move the equilibrium point to the origin by defining $y = q - q^*$ and $\dot{y} = 0$. The new system can be written in the form

$$M\ddot{y} + C\dot{y} + D\dot{y} + K_p y + K_d\dot{y} = 0.$$

As a start, try using $V = \frac{1}{2}\dot{y}^T M\dot{y}$. Then,

$$\dot{V} = -\dot{y}^T K_p y - \dot{y}^T(K_d + D)\dot{q} + \frac{1}{2}\dot{y}^T(M - 2C)\dot{q} < -\dot{y}^T K_p y$$

Unfortunately, our Lyapunov candidate gave us this weird cross term. Now, consider $V = \frac{1}{2}\dot{y}^T M\dot{y} + \frac{1}{2}y^T K_p y$. When we take a time derivative, the $-\dot{y}^T K_p y$ term cancels and we are left with $\dot{V} < 0$. Therefore, the equilibrium point $(q = q^*, \dot{q} = 0)$ is AS.

4.20 Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

From LaSalle's theorem, we know that $x(t)$ approaches M as $t \to \infty$. By definition, $\forall\varepsilon > 0, \exists T > 0 \ni d(x(t), M) < \varepsilon$. Choose $\varepsilon = \frac{1}{2}\min ||p - q||$ where $p \neq q \in M$ are isolated equilibrium points. After T seconds, $d(x(t), M) < \varepsilon$, so the trajectory is trapped in a ball with radius ε around some equilibrium point $p \in M$. As $\varepsilon \to 0$, the trajectory is trapped in a ball with arbitrarily small radius. Therefore, $x(t)$ will eventually reach an equilibrium point after some finite time T. Since the equilibrium point is positively invariant, the trajectory will never leave the equilibrium point. Therefore, $\lim_{t\to\infty} x(t) = p \in M$.

4.21 (**[81]**) A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Take $D = R^n$. Suppose the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is compact for every $c \in R$. Show that every solution of the system is defined for all $t \geq 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \ldots, p_r$.

a) First we show that $\dot{V}(x) \leq 0$:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = (\nabla V)^T(-\nabla V) = -(\nabla V)^T(\nabla V) \leq 0$$

with equality only when $\nabla V = 0$. Now, we show that $\dot{V}(x) = 0$ if and only if x is an equilibrium point. Forward proof:

$$\dot{V}(x) = 0 \implies -(\nabla V)^T(\nabla V) = 0 \implies \nabla V = 0 \implies \dot{x} = 0$$

Backwards proof:

$$\dot{x} = 0 \implies \nabla V = 0 \implies -(\nabla V)^T(\nabla V) = \dot{V}(x) = 0$$

b) Since $\dot{V} \leq 0$, we see that Ω_c is positively invariant (and we are given that it is bounded). By theorem 3.3, a unique solution exists $\forall\ t \geq 0$.

c) We know that ∇V corresponds to the equilibrium points of the gradient system. Therefore, by problem 4.20, we see that the limit $\lim_{t\to\infty} x(t)$ exists and that it must approach one of the points $p_1, \ldots, p_r$.