EE 505 HW 2

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Problem 2.2

a) Find all equilibrium points and determine the type of each isolated equilibrium:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{1}{16}x_1^5 - x_2\end{aligned}$$

Setting the state equations to zero, we see that $x_2 = 0$ and

$$\begin{aligned}0 &= -x_1 + \frac{1}{16}x_1^5 \\ &= -x_1(1 + \frac{1}{4}x_1^2)(1 + \frac{1}{2}x_1)(1 - \frac{1}{2}x_1)\end{aligned}$$

so the equilibrium points are $(x_1, x_2) = (0, 0)$, $(2, 0)$, and $(-2, 0)$. The Jacobian is given by

$$J = \frac{\partial f_i}{\partial x_j} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{5}{16}x_1^4 & -1 \end{bmatrix}$$

Evaluating at $(0, 0)$,

$$J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

The eigenvalues are found by solving $Jx = \lambda x$ which gives

$$\lambda_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

so the $(0, 0)$ equilibrium point is a stable focus. Evaluating J at $(\pm 2, 0)$,

$$J = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$$

with eigenvalues

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$$

so these equilibrium points are saddle points.

b) Find all equilibrium points and determine the type of each isolated equilibrium:

$$\begin{aligned}\dot{x}_1 &= 2x_1 - x_1 x_2 \\ \dot{x}_2 &= 2x_1^2 - x_2\end{aligned}$$

The equilibrium points, eigenvalues, and equilibrium types are summarized in the table below:

Equilibrium Point	Eigenvalues	Equilibrium Type
$(0,0)$	$\lambda_1 = 2, \lambda_2 = -1$	Saddle Point
$(1,2)$	$\lambda_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$	Stable Focus
$(-1,2)$	$\lambda_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$	Stable Focus

c) Find all equilibrium points and determine the type of each isolated equilibrium:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \psi(x_1 - x_2)\end{aligned}$$

where

$$\psi(y) = \begin{cases} y^3 + \frac{1}{2}y & |y| \leq 1 \\ 2y - \frac{1}{2}\operatorname{sgn}(y) & |y| > 1 \end{cases}$$

When $x_2 = 0$, we see that

$$\psi(x_1) = 0$$

so the only equilibrium point is located at the origin. Near the equilibrium point, the system is equivalent to

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - (x_1 - x_2)^3 - \frac{1}{2}(x_1 - x_2)\end{aligned}$$

so the Jacobian evaluated at equilibrium is

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

with eigenvalues

$$\lambda_{1,2} = -\frac{1}{4} \pm j\frac{\sqrt{3}}{4}$$

so the origin is a stable focus.

Problem 2.3

The vector fields are provided on following pages.

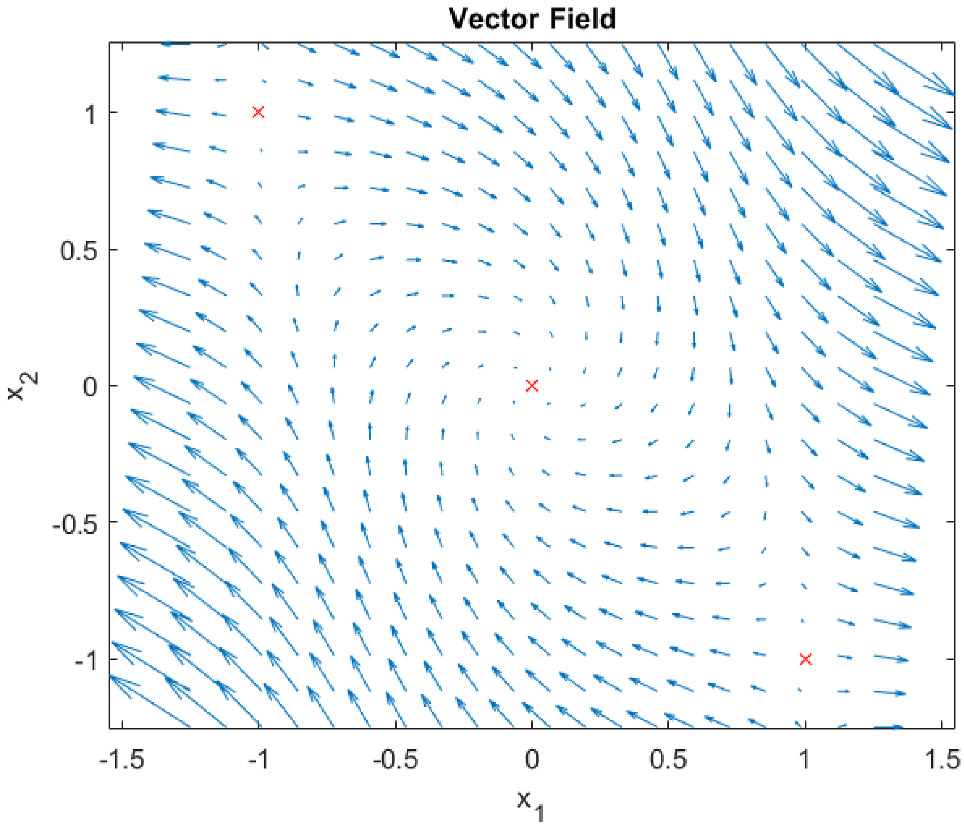

Figure 1: Vector Field 2.3.1 - The equilibrium point at the origin is a stable focus, while the other equilibrium points are unstable. Trajectories near the bottom left and upper right will not converge to an equilibrium point.

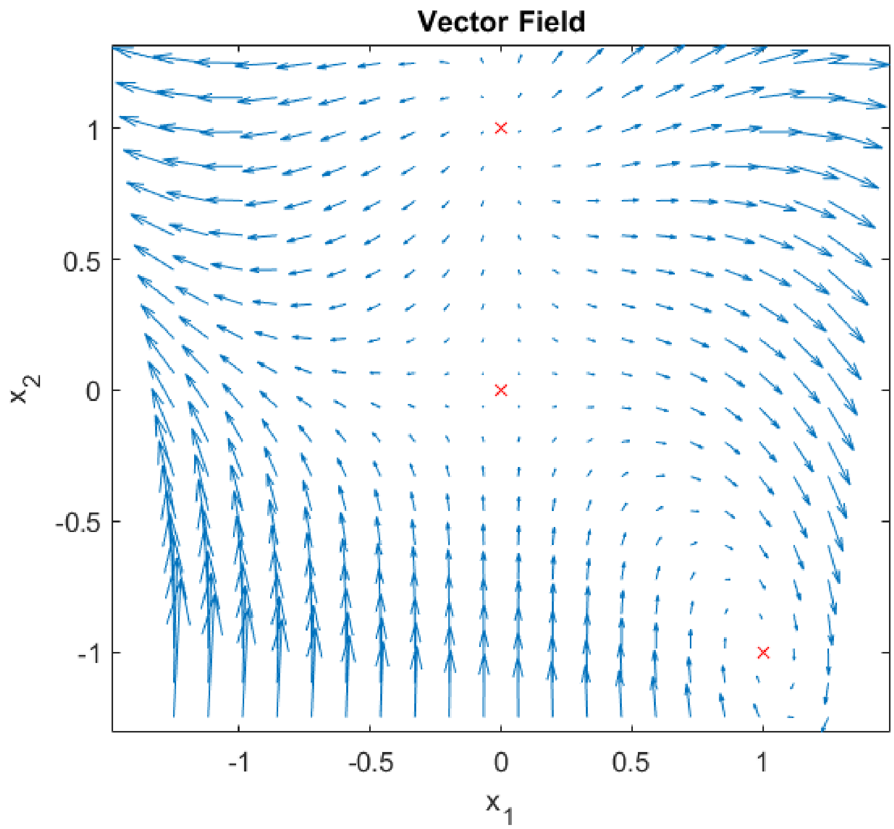

Figure 2: Vector Field 2.3.2 - The equilibrium point in the bottom right is a stable focus, while the others are unstable. Trajectories near the left side of the vector field never converge.

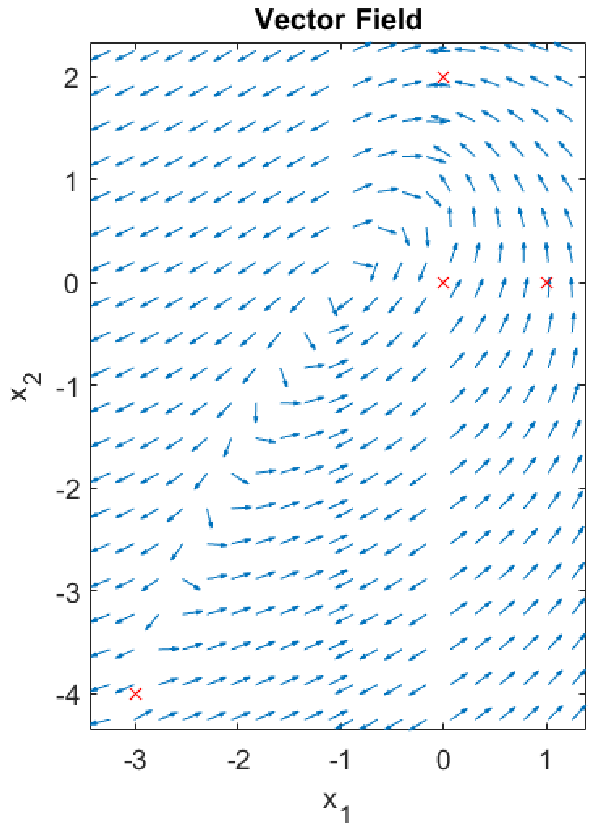

Figure 3: Vector Field 2.3.3 - Trajectories on the top left escape, while trajectories in the top right spiral into an equilibrium point. Four total equilibrium points exist.

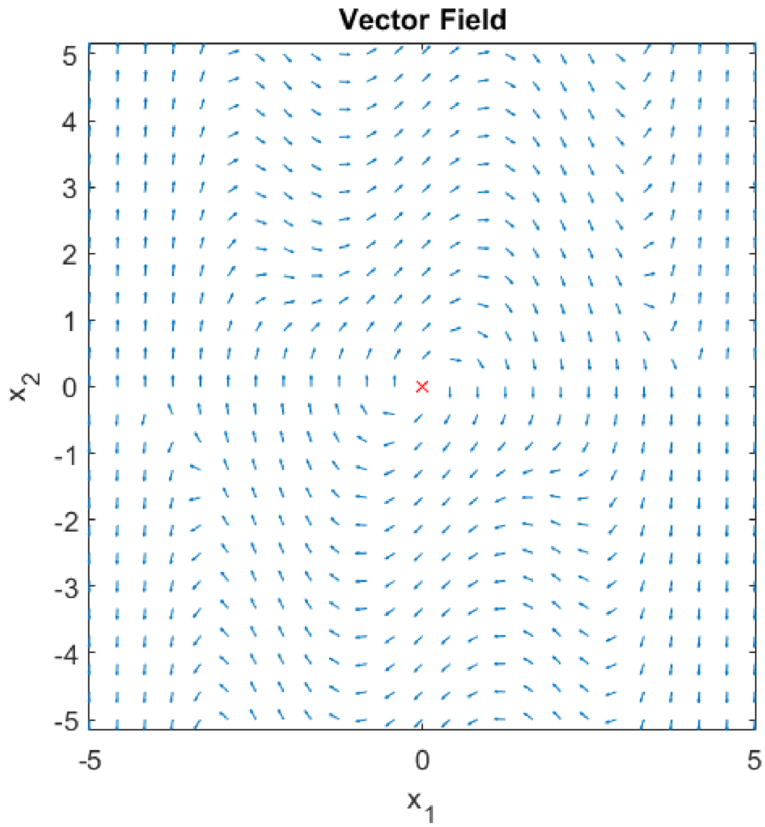

Figure 4: Vector Field 2.3.4 - The only equilibrium point is located at the origin, and it is an unstable focus. Some limit cycles appear to exist encircling the origin, but the near the left and right edges of the vector field, trajectories escape.

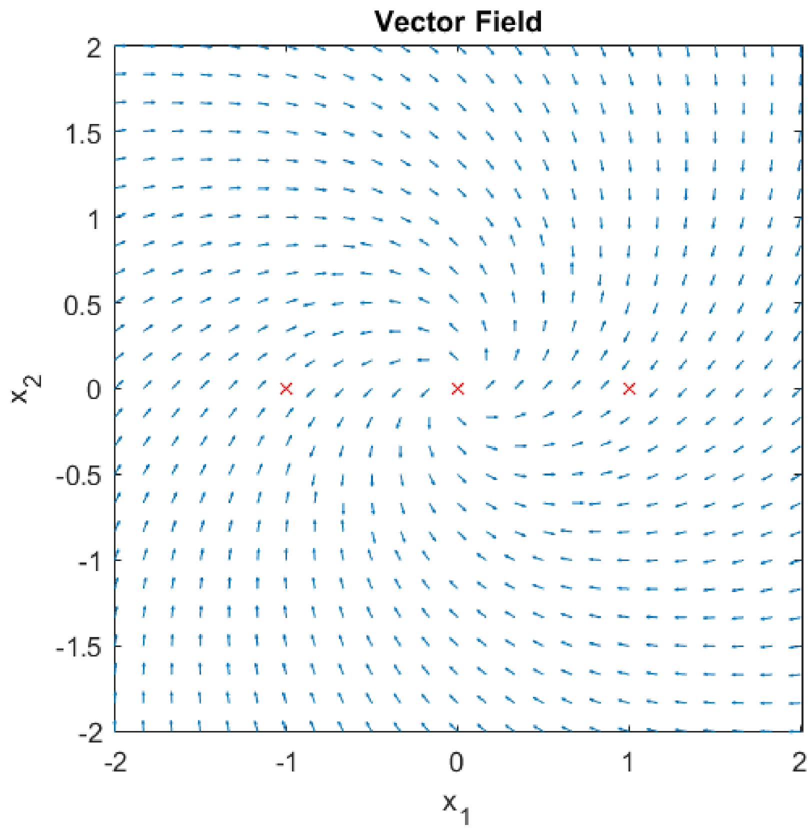

Figure 5: Vector Field 2.3.5 - The equilibrium point at the origin is an unstable focus, while the other equilibrium points on the $x_2 = 0$ axis are stable foci. It appears that every trajectory will eventually settle at an equilibrium point.

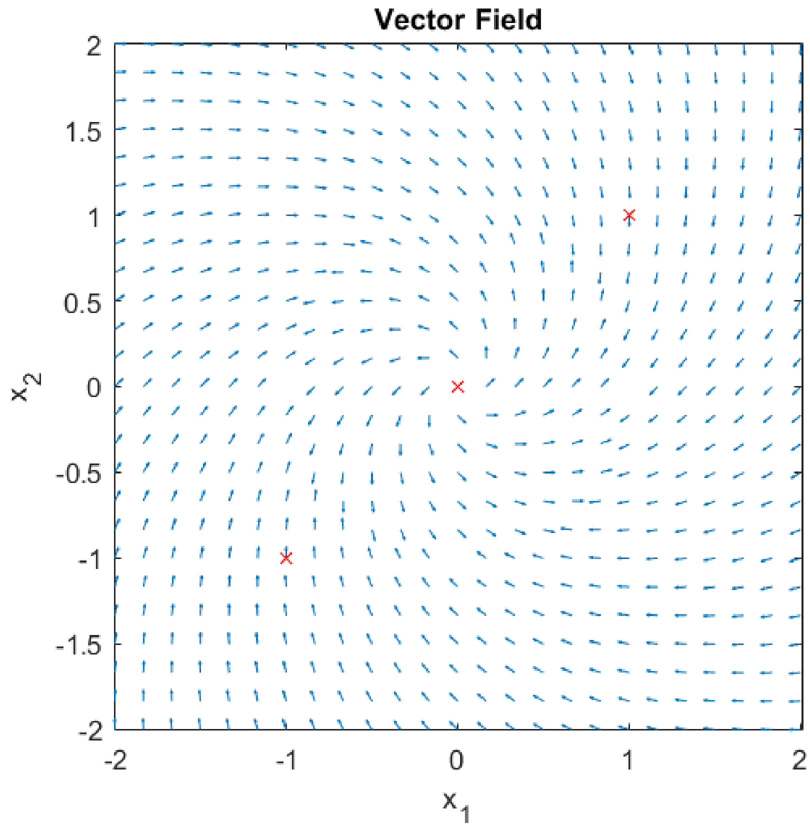

Figure 6: Vector Field 2.3.6 - The equilibrium point at the origin is an unstable focus, and the other equilibrium points are also unstable. It appears a limit cycle exists centered around the origin but with radius inside the outside equilibrium points.

Problem 2.4

The phase portraits are provided on following pages.

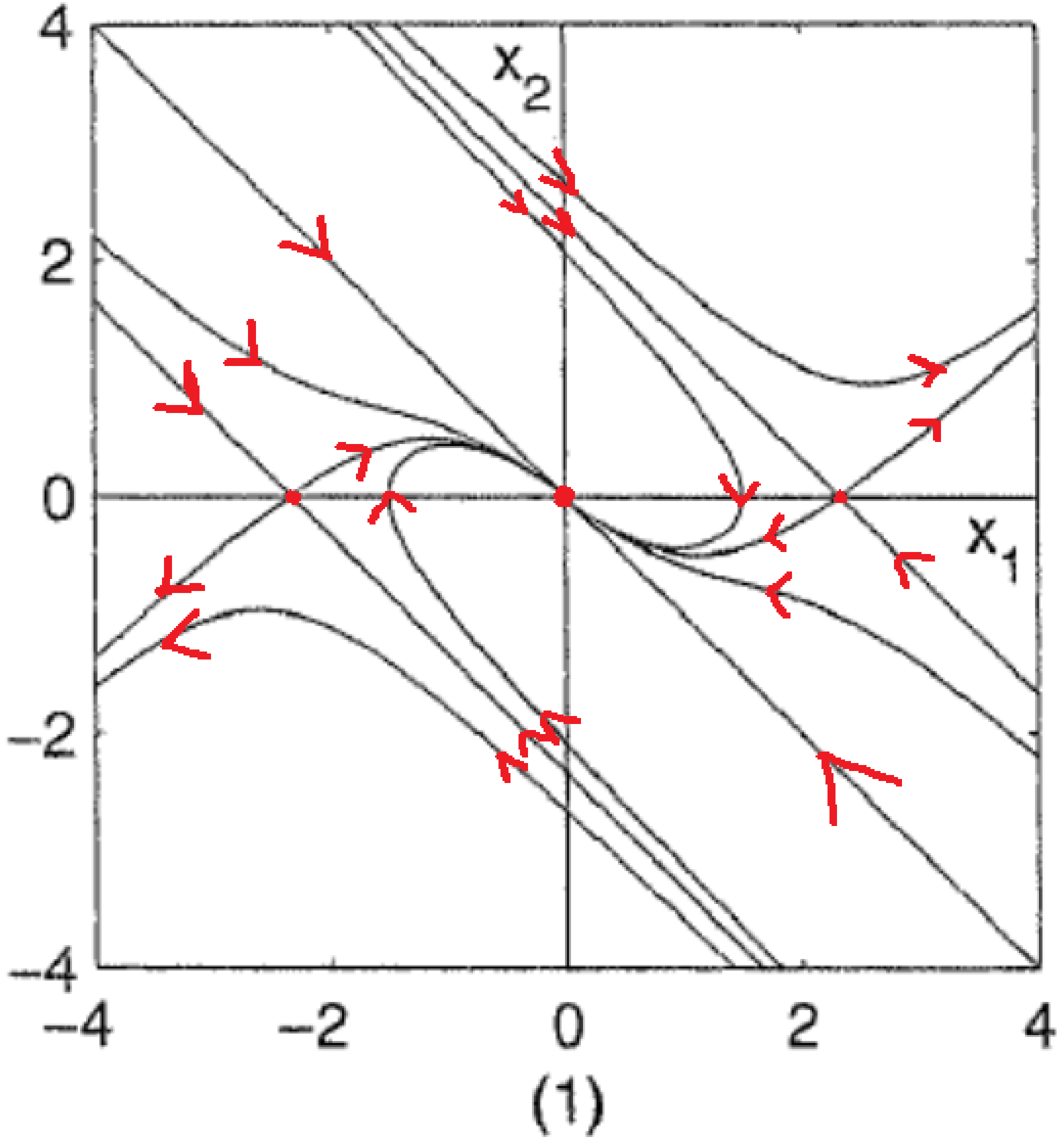

Figure 7: Phase Portrait 2.4.1 - The equilibrium point at the origin is stable, and trajectories in the band near the line $x_2 = -x_1$ converge to the origin. Outside this regions, the trajectories escape.

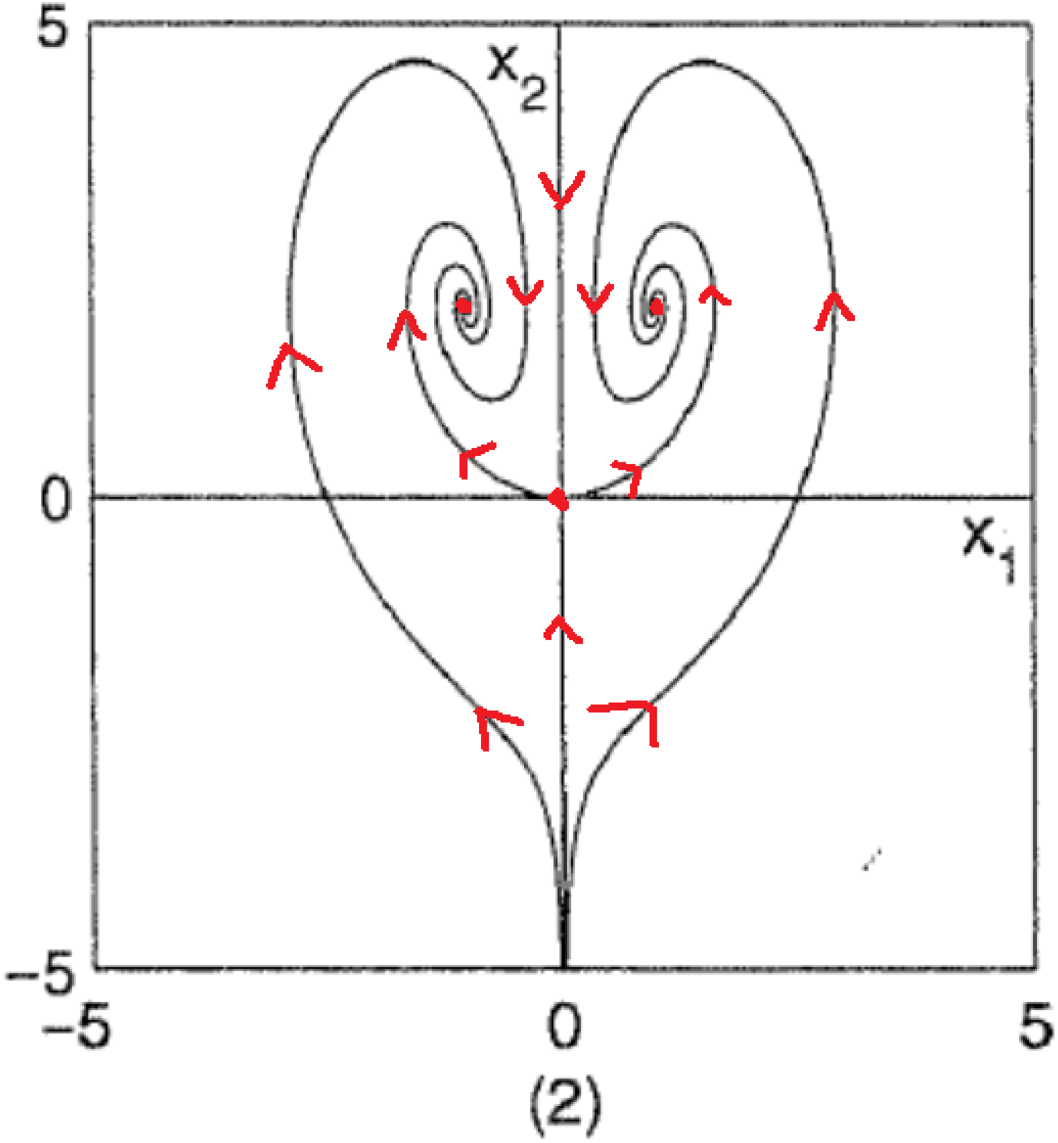

Figure 8: Phase Portrait 2.4.2 - The equilibrium point at the origin is unstable, while all other trajectories spiral into one of the other equilibrium points, shown with the red dots. These other equilibrium points are stable nodes.

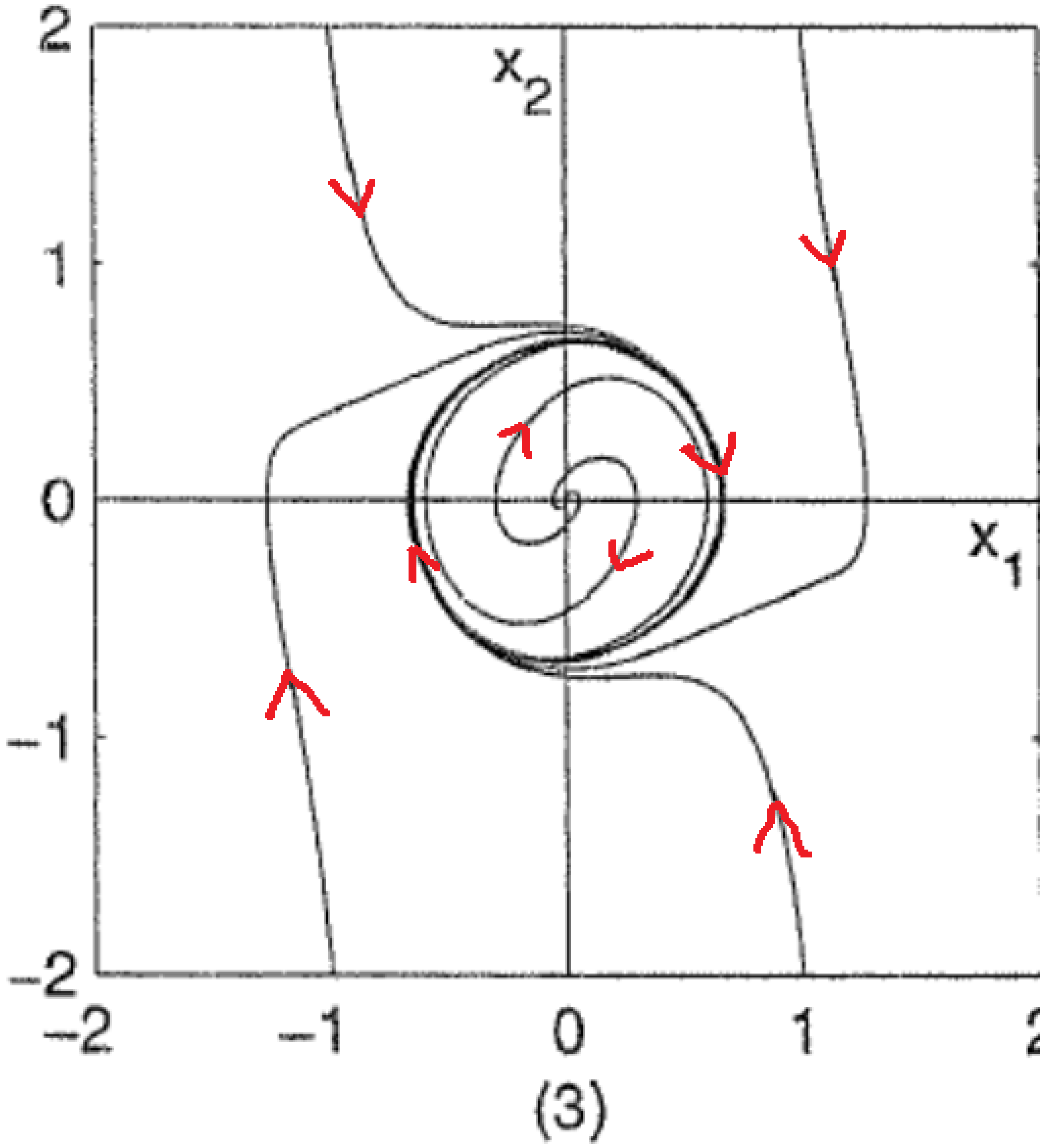

Figure 9: Phase Portrait 2.4.3 - The equilibrium point at the origin is unstable. A stable limit cycle encircles the origin for all nonzero trajectories.

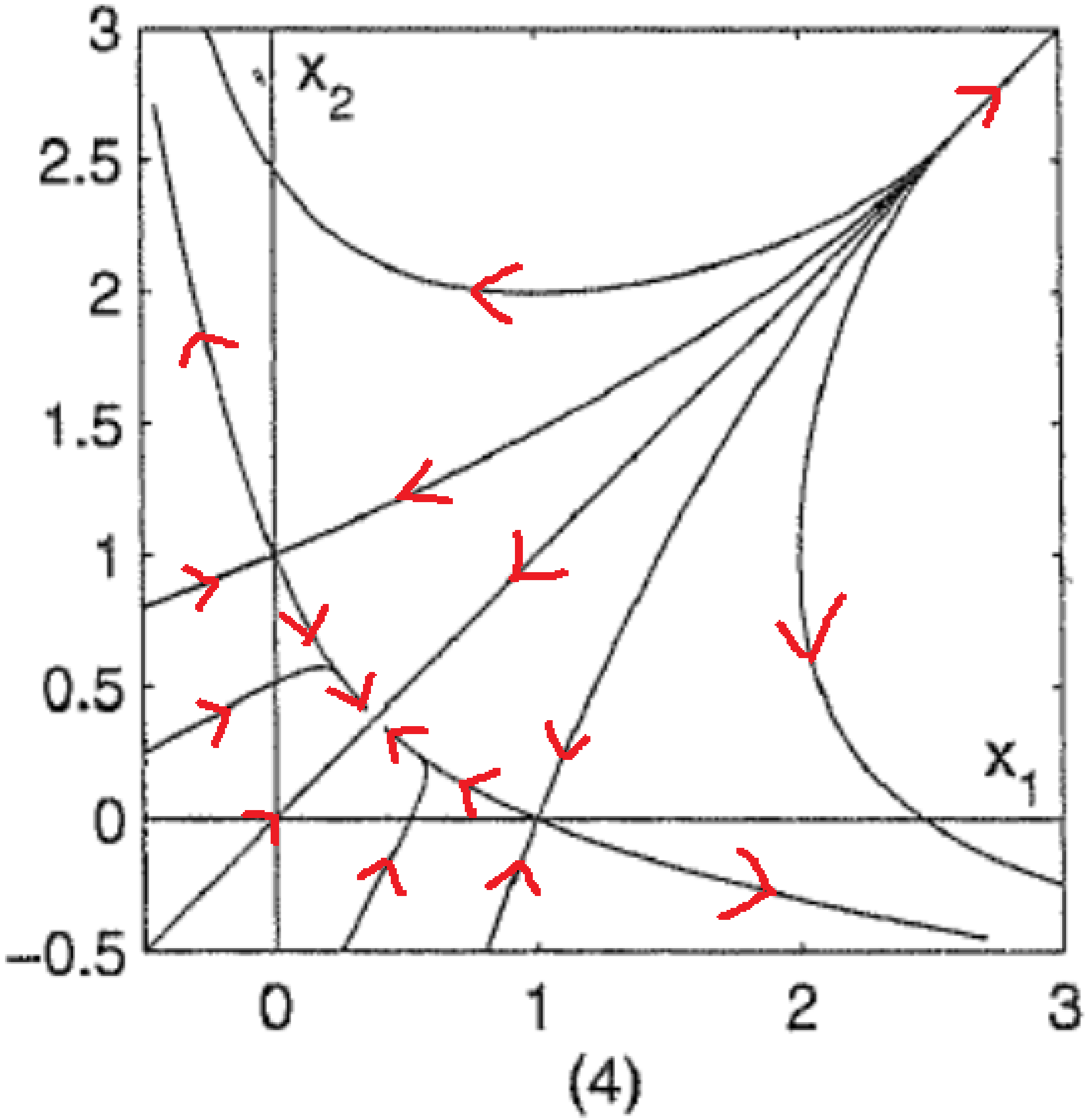

Figure 10: Phase Portrait 2.4.4 - There is no equilibrium point at the origin. The phase portrait is symmetric with respect to the line $x_1 = x_2$. There is one unstable node, two saddle points, and one stable node shown in the phase portrait. The equilibrium points are shown in blue. Many trajectories escape.

Problem 2.6

a) The three equilibrium points (x_1, x_2) are located at:

$$\left\{(0,0),\quad (0,-a),\quad \left(\frac{ab+b^2}{1+b^2}, -\frac{a+b}{1+b^2}\right)\right\}$$

b) The Jacobian is given by

$$J = \begin{bmatrix} -1 - bx_2 & a - bx_1 + 2x_2 \\ -(a+b) + 2bx_1 - x_2 & -x_1 \end{bmatrix}$$

For the equilibrium point $(0,0)$, the eigenvalues are located at

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{-4a^2 - 4ab + 1}}{2}$$

which is stable focus when the discriminant is negative, a stable node when the discriminant is positive and less than 1, and a saddle when the discriminant is positive and greater than 1.
For the equilibrium point $(0,-a)$, the eigenvalues are located at

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= ab \end{aligned}$$

which is a stable node when $ab < 0$ and a saddle when $ab > 0$.

For the equilibrium point $(\frac{ab+b^2}{1+b^2}, -\frac{a+b}{1+b^2})$, the eigenvalues are located at

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{-4b^2 - 4ab + 1}}{2}$$

which is stable focus when the discriminant is negative, a stable node when the discriminant is positive and less than 1, and a saddle when the discriminant is positive and greater than 1.

c) The phase portraits are provided below. The equilibrium points are shown with red $\mathbf{x}$'s, which move as functions of a and b. Notice how the general heart-shaped swirl behavior in the phase portraits changes direction with a and b.

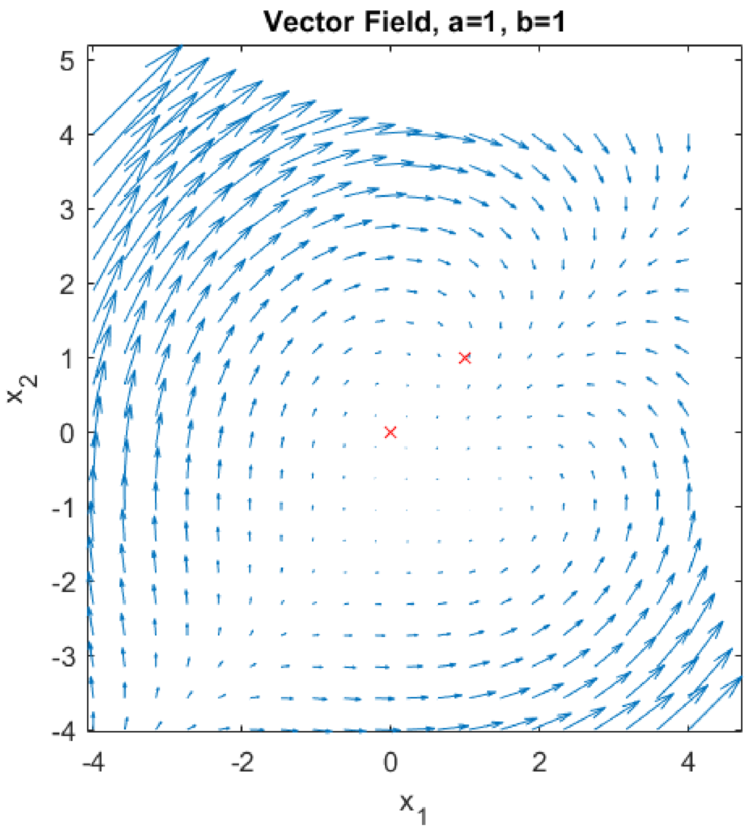

Figure 11: Vector Field 2.6.1 - Trajectories from the bottom left move along the line $x_2 = x_1$ before breaking away and returning to one of the equilibrium points. The vector field is stronger along the line $x_2 = -x_1$.

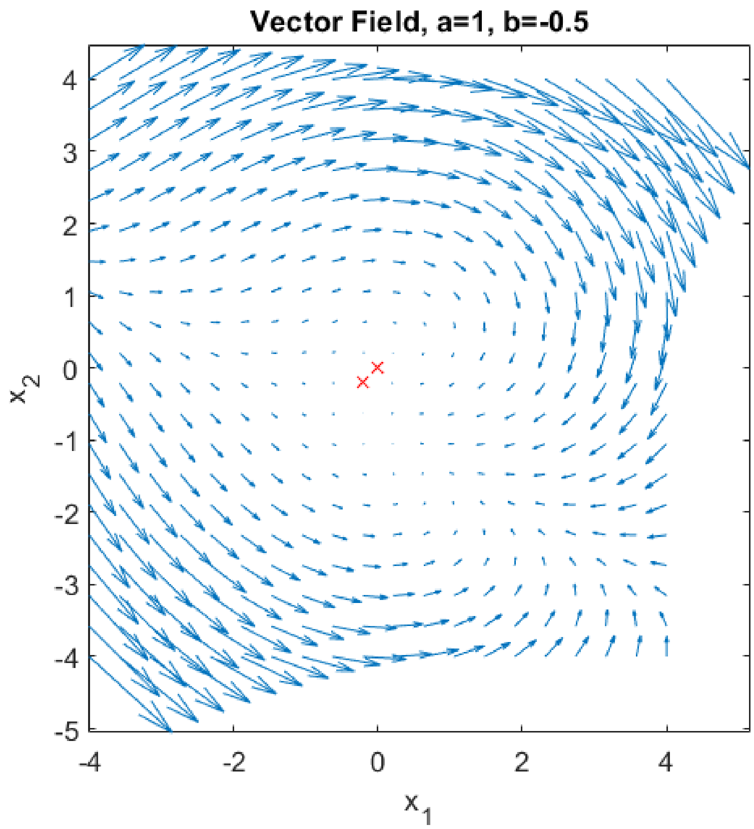

Figure 12: Vector Field 2.6.2 - Comparing this to the previous vector field, the field appears to have rotated, and the equilibrium points are closer together. Notice how most trajectories tend to come to rest at an equilibrium point, and no limit cycles appear to exist.

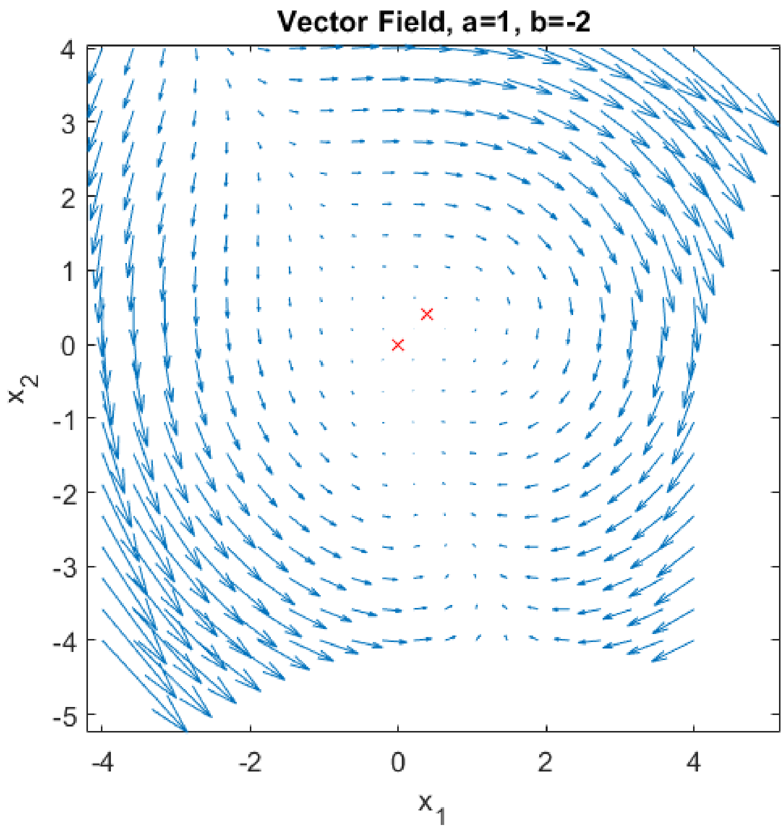

Figure 13: Vector Field 2.6.3 - Again, the vector field appears to have rotated. No limit cycles appear to exist, and all trajectories seem to end at an equilibrium point for the region plotted.

Problem 2.9

a) The state model of the system is given by

$$\begin{aligned}\dot{x}_1 &= -x_2 + v_d \\ m\dot{x}_2 &= K_i x_1 + K_p(v_d - x_2) - K_c\,\mathrm{sgn}(x_2) - K_f x_2 - K_a x_2^2\end{aligned}$$

where v_d is the input to the system.

b) The equilibrium point is located at

$$\begin{aligned}x_1 &= \frac{K_c + K_f v_d + K_a v_d^2}{K_i} \\ x_2 &= v_d\end{aligned}$$

Using the constants provided in the problem, the Jacobian expressed as a function of K_i is

$$J_{x_1,x_2} = \begin{bmatrix} 0 & -1 \\ \frac{K_i}{1500} & -\frac{3}{8} \end{bmatrix}$$

with eigenvalues

$$\lambda_{1,2} = -\frac{3}{16} \pm \frac{\sqrt{\frac{9}{64} - \frac{K_i}{375}}}{2}$$

so the equilibrium point is a stable node when $K_i = 15$ and a stable focus when $K_i = 150$. (The cutoff is located at $K_i = 52.7$)

Parts c), d), and e) are provided on the next pages.

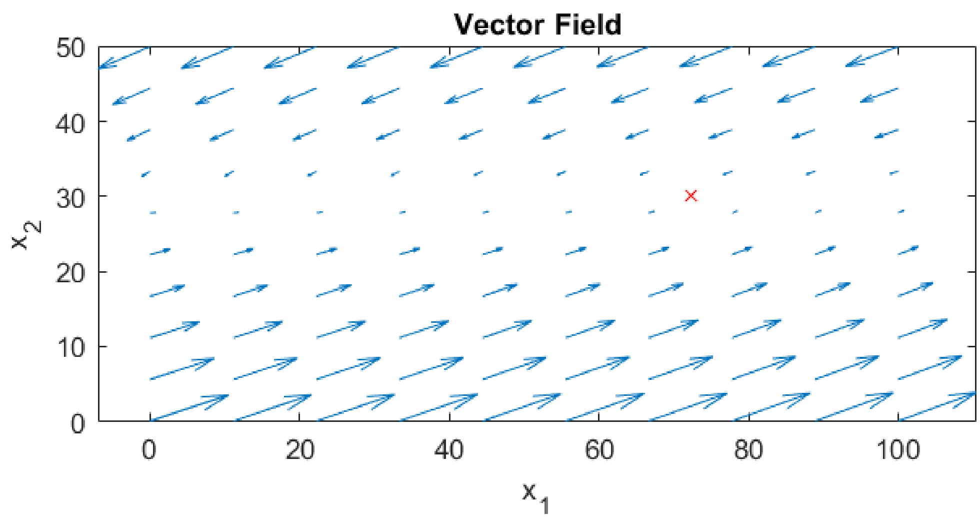

Figure 14: 2.9.c - The equilibrium point is a stable node located at $x_2 = 30$ which implies zero steady state error. This is to be expected for a PI controller with a constant reference signal $v_d = 30$.

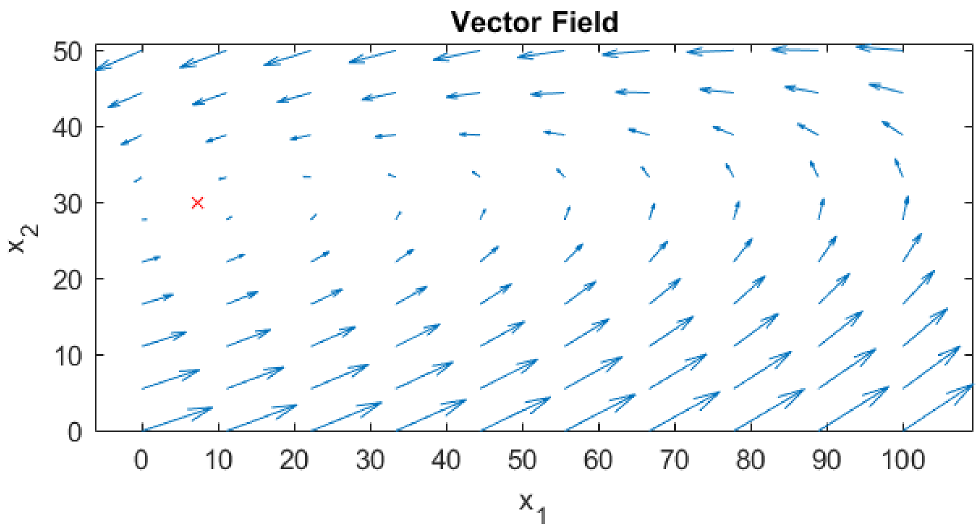

Figure 15: 2.9.d - The equilibrium point is a stable node located at $x_2 = 30$ which implies zero steady state error. However, now x_1 is smaller, which means the integrated error is reduced when steady state is reached. This also implies that the car reaches the cruising speed much faster than when $K_i = 15$ in 2.9.c.

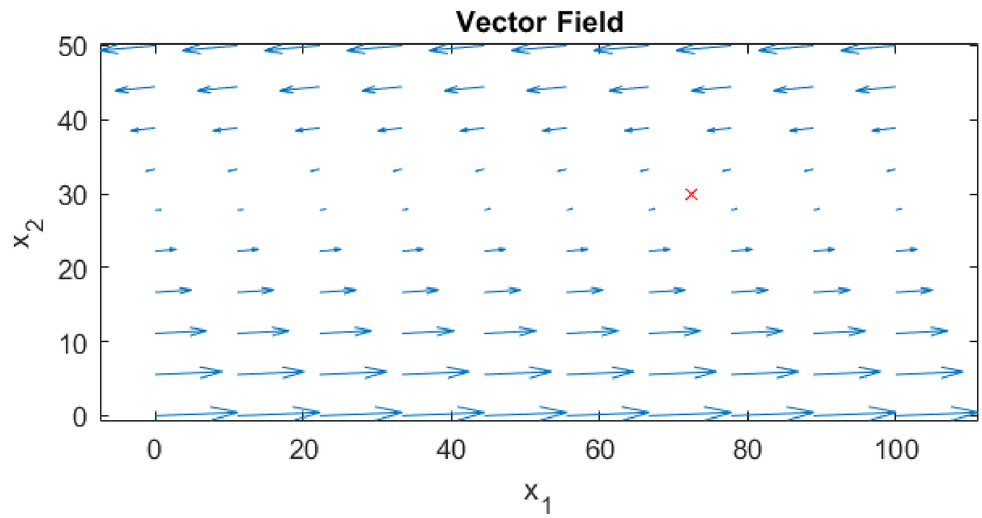

Figure 16: 2.9.e - The saturation on the plant input (actuator) causes the system to reach the same equilibrium point as 2.9.c, but much slower. This is most noticeable for small values of x_2, where the slope of the vector field is smaller, implying that the system takes longer to reach steady state.

Problem 2.10

a) There are five total equilibrium points, however the only real equilibrium point occurs at

$$(x_1, x_2) = (0.057, 0.72)$$

The eigenvalues associated with this point are

$$\lambda_1 = -0.065$$
$$\lambda_2 = -4.0$$

so this the equilibrium point is a stable node.

b) The phase portrait is shown below.

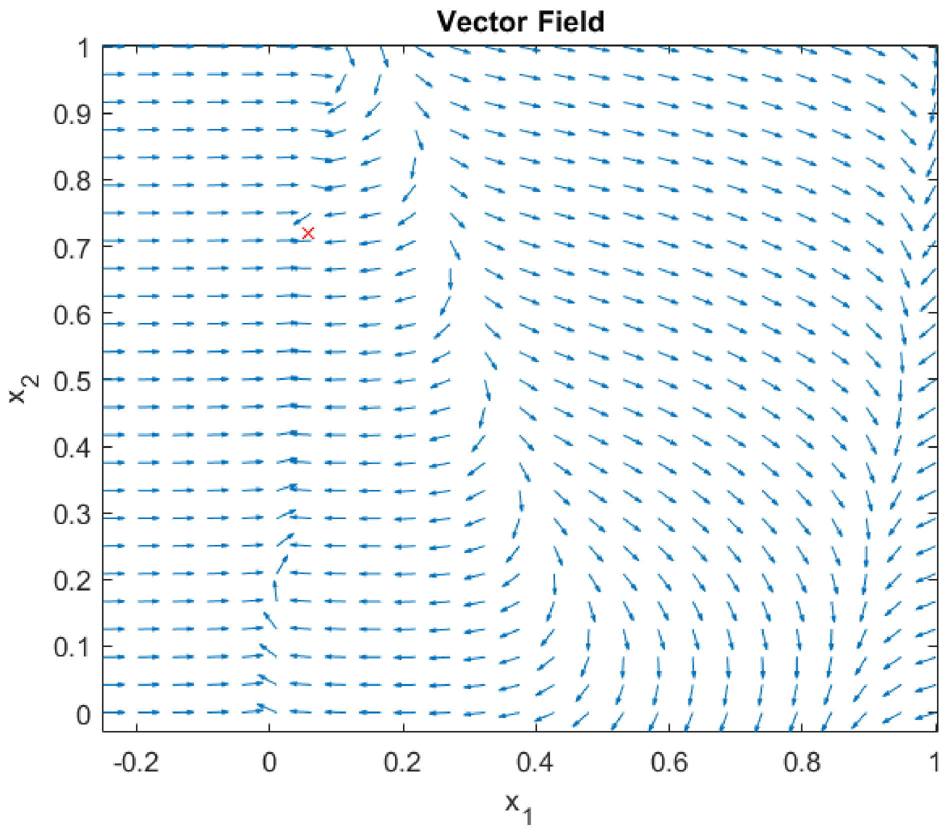

Figure 17: 2.10.b - The only equilibrium point is located near what we called Q_1 in the example found in the textbook. This is a stable node, so trajectories in the neighborhood will eventually settle at this point. Trajectories to the left will move to the right, trajectories immediately to the right will move left, but trajectories too far to the right will keep moving right before eventually circling around and approaching the equilibrium point from the bottom.

Problem 2.16

a) The three equilibrium points belong to the set

$$(x_1, x_2) \in \left\{(0,0), (1,0), \left(\frac{1}{a+1}, \frac{1}{a+1}\right)\right\}$$

The equilibrium point at the origin has eigenvalues

$$\lambda_1 = 0 \quad \lambda_2 = 1$$

which is a saddle point. The equilibrium point $(1,0)$ has eigenvalues

$$\lambda_1 = -1 \quad \lambda_2 = b$$

which is a saddle point because $b > 0$. The equilibrium point $(\frac{1}{a+1}, \frac{1}{a+1})$ has eigenvalues

$$\lambda_{1,2} = -\frac{b+1}{2(a+1)} \pm \frac{\sqrt{b^2 - 4ab - 2b + 1}}{2(a+1)}$$

which can describe a stable focus, a stable node, or a saddle point depending on the value of the discriminant $b^2 - 4ab - 2b + 1$. For the case $a = 1, b = 0.5$, the eigenvalues are

$$\lambda_{1,2} = -\frac{3}{8} \pm j\frac{\sqrt{7}}{8}$$

which is a stable focus.

b) The vector field for $a = 1, b = 0.5$ is shown below.

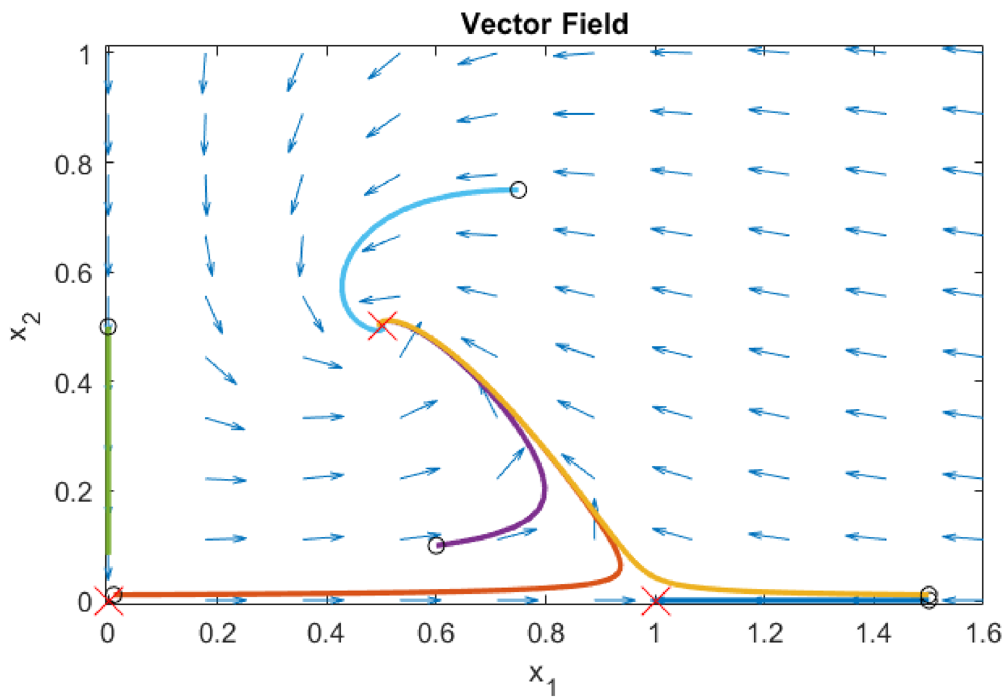

Figure 18: 2.16.b - The two saddle points are shown on the $x_2 = 0$ axis. As shown, trajectories near, but not along the $x_2 = 0$ axis will diverge from the saddle points and eventually spiral into the equilibrium point located at $(0.5, 0.5)$. However, trajectories which start on the x_1 or x_2 axis will ride along the saddle points until reaching these precarious equilibrium points. Note that the equilibrium points are denoted with red x's while the start of each trajectory is denoted with a blue circle.

Problem 2.17

a) The system can be written in state space form with $y = x_1$ and $\dot{y} = x_2$ as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varepsilon x_2(1 - x_1^2 - x_2^2) - x_1\end{aligned}$$

The only equilibrium point occurs at the origin. The Jacobian evaluated at this point is

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$$

and the corresponding characteristic polynomial has solutions at

$$\lambda_{1,2} = \frac{\varepsilon}{2} \pm \frac{\sqrt{\varepsilon^2 - 4}}{2}$$

For all $\varepsilon > 0$, the eigenvalues have positive real parts. Now, consider the surface $V(x) = x_1^2 + x_2^2 = c$ and closed region $M = \{x \mid V(x) \leq c\}$. Then,

$$f \cdot \nabla V(x) = 2x_2^2(1 - (x_1^2 + x_2^2)) = 2x_2^2(1 - c) \leq 0$$

for any $c \geq 1$. By the Poincaré-Bendixson lemma, the system has a periodic orbit. Using MATLAB, we can verify that the limit cycle exists with radius $r = 1$.

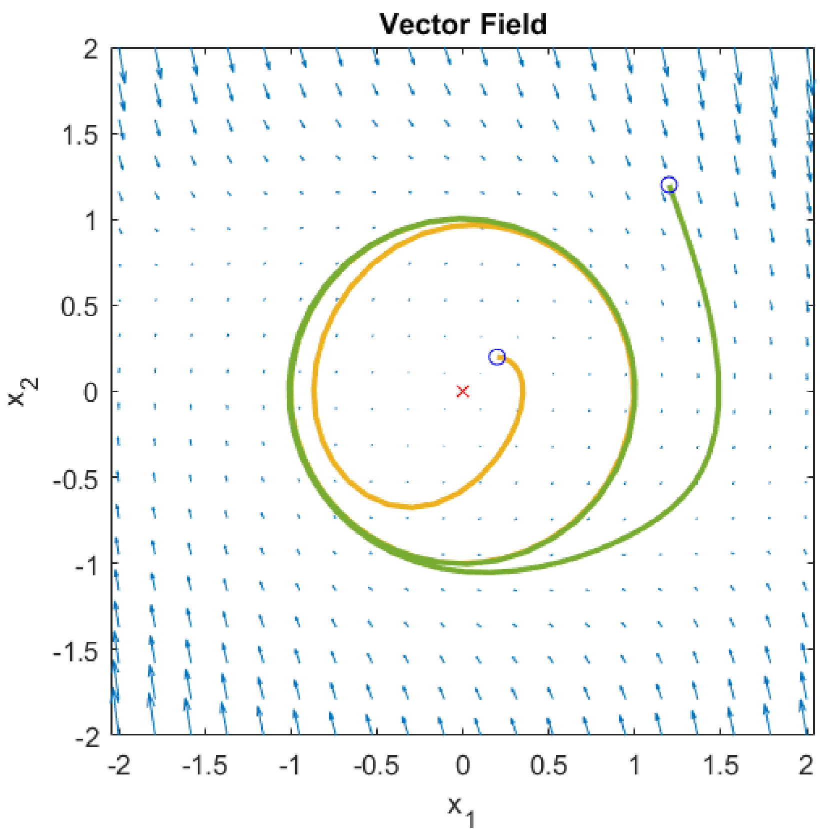

Figure 19: 2.17.a - A periodic orbit exists with radius 1 regardless of the parameter $\varepsilon > 0$. Larger values of $\varepsilon > 0$ cause the trajectories to converge to the periodic orbit faster.

b) The only equilibrium point occurs at the origin and the eigenvalues have positive real parts $\Re(\lambda_{1,2}) = 0.5$. Consider the surface $V(x) = x_1^2 + x_2^2 = c$ and closed region $M = \{x \mid V(x) \leq c\}$.

Then,

$$f \cdot \nabla V(x) = 2x_2^2(2 - 2c - x_1^2) \leq 0$$

for any $c \geq 1$ because $x_1^2 \geq 0$ and $x_2^2 \geq 0$. By the Poincaré-Bendixson lemma, the system has a periodic orbit.

c) The only equilibrium point occurs at the origin and the eigenvalues have positive real parts $\Re(\lambda_{1,2}) = 0.5$. Consider the surface $V(x) = x_1^2 + x_2^2 = c$ and closed region $M = \{x \mid V(x) \leq c\}$. Then,

$$f \cdot \nabla V(x) = 1 - 2x_1x_2 - 4x^2$$

Now, $2x_1x_2 \leq x_1^2 + x_2^2$ so

$$1 - 2x_1x_2 - 4x^2 \leq 1 - (x_1^2 + x_2^2) - 4x^2 \leq 1 - c - 4x^2 \leq 0$$

for any $c \geq 1$ because $x_2^2 \geq 0$. By the Poincaré-Bendixson lemma, the system has a periodic orbit.

d) The only equilibrium point occurs at the origin and has eigenvalues $\lambda_{1,2} = 1 \pm j\sqrt{2}$, so the equilibrium point is an unstable focus. Consider the surface $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. We see that

$$f \cdot \nabla V(x) = (x_1^2 + x_2^2)(1 - h(x)) - x_1x_2$$

Now we find when $f \cdot \nabla V(x) \leq 0$. We know that

$$x_1x_2 \leq |x_1x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

So,

$$\begin{aligned}
(x_1^2 + x_2^2)(1 - h(x)) &\leq \frac{1}{2}(x_1^2 + x_2^2) \\
1 - h(x) &\leq \frac{1}{2} \\
h(x) &\geq \frac{1}{2}
\end{aligned}$$

Let M be the square region enclosed by the curve $h(x) = \frac{1}{2}$. Every trajectory starting in M at $t = 0$ will remain inside for all $t \geq 0$. By the Poincaré-Bendixson lemma, the system has a periodic orbit.

Problem 2.20

a) Observe that for all $a \neq 1$,

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 + a \neq 0$$

and is a constant value. By the Bendixson Criterion, no limit cycle exists.

b) Observe that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2) - 2$$

In the regions $M_1 = \{x \mid x_1^2 + x_2^2 > \frac{1}{2}\}$ and $M_2 = \{x \mid x_1^2 + x_2^2 < \frac{1}{2}\}$, no limit cycle can exist. A periodic orbit could still exist at $x_1^2 + x_2^2 = \frac{1}{2}$ if the vector field is tangent to this closed curve. However, this is not the case. Consider the point $(x_1, x_2) = (1/2, 0)$. We have $f_2/f_1 = 0$ and $\theta = \pi$. We needed $\theta = \pi/2$ or $\theta = 3\pi/2$ for the vector field to be tangent to the closed curve at this point.

c) Observe that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -x_2^2$$

so no limit cycles can exist when $x_2 \neq 0$. The equation $x_2 = 0$ describes a line, and a limit cycle cannot exist on a line, so no limit cycles exist.

d) Observe that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_2 + 1$$

No limit cycles can exist when $x_2 \neq 1$. The equation $x_2 = 1$ describes a line, and a limit cycle cannot exist on a line, so no limit cycles exist.

e) Observe that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -x_2 \sin x_1$$

We divide the plane into regions by drawing the lines $x_2 = 0$ and $x_1 = 0, \pm\pi, \pm 2\pi, \cdots$. In each of the closed U-shaped (or upside down U-shaped) regions which touches exactly 3 lines, $-x_2 \sin x_1$ is not identically zero and does not change sign. Since the lines where $-x_2 \sin x_1 = 0$ do not form any closed loops, no limit cycle can exist on the lines. Therefore, no limit cycles exist anywhere.

Problem 2.29

a) First, we show that there is an equilibrium point located at $(x_1, x_2) = (a/5, 1 + a^2/25)$. Second, we will show that the equilibrium point changes from an unstable focus to a stable focus at the point $b = 3a/5 - 25/a$. Third, we will show that the there exists a region inside a closed curve where all trajectories cannot escape.

i) By direct substitution, it is easy to verify that the solution to the state equation is given by $(x_1, x_2) = (a/5, 1 + a^2/25)$.

ii) The Jacobian evaluated at equilibrium is given by

$$J = \frac{25}{1 + a^2/25} \begin{bmatrix} 3a^2 - 125 & -20a \\ 2a^2 b & -5ab \end{bmatrix}$$

Solving $\det(J - \lambda I) = 0$ over the region given in iii) shows that the real part of the eigenvalues is proportional to

$$\Re(\lambda_{1,2}) = 3a^2 - 5ab - 125$$

Equating this to zero and solving for b gives the relationship

$$\Re(\lambda_{1,2}) > 0 \text{ when } b < \frac{3a}{5} - \frac{25}{a}$$

iii) Consider the rectangular region between points $(0,0)$ and $(a, 1+a^2)$ which sits on the x_1 axis. We will show that along this closed curve, the vector field points inwards.

Along the $x_1 = 0$ axis, the state equation is

$$\begin{aligned}\dot{x}_1 &= a > 0 \\ \dot{x}_2 &= 0\end{aligned}$$

so the vector field points inwards (i.e. $\dot{x}_1 \geq 0$) for all x_2.

Along the $x_2 = 0$ axis, the state equation is

$$\begin{aligned}\dot{x}_1 &= a - x_1 \\ \dot{x}_2 &= bx_1 \geq 0\end{aligned}$$

so the vector field points inwards (i.e. $\dot{x}_2 \geq 0$) for $0 \leq x_1 \leq a$.

Along the segment from $(a, 0)$ to $(a, 1+a^2)$ with $x_1 = a$ fixed, the state equation is

$$\begin{aligned}\dot{x}_1 &= -\frac{4ax_2}{1+a^2} \leq 0 \\ \dot{x}_2 &= ba\left(1 - \frac{x_2}{1+a^2}\right)\end{aligned}$$

It is easy to see that the vector field points inwards (i.e. $\dot{x}_1 \leq 0$) when $0 \leq x_2 \leq 1+a^2$.

Along the segment from $(0, 1+a^2)$ to $(a, 1+a^2)$ with $x_2 = 1+a^2$ fixed, the state equation is

$$\begin{aligned}\dot{x}_1 &= a - x_1 - \frac{4x_1(1+a^2)}{1+x_1^2} \\ \dot{x}_2 &= bx_1\left(1 - \frac{1+a^2}{1+x_1^2}\right)\end{aligned}$$

It is easy to see that the vector field points inwards (i.e. $\dot{x}_2 \leq 0$) when $0 \leq x_1 \leq a$.

We have shown that all trajectories inside the region will remain in that region, and found the condition under which the equilibrium point has eigenvalues with positive real parts. By the Poincaré-Bendixson, the system has a limit cycle if and only if

$$b < \frac{3a}{5} - \frac{25}{a}$$

b) The vector field for $a = 10, b = 2$ is shown below. c) The vector field for $a = 10, b = 4$ is shown below.

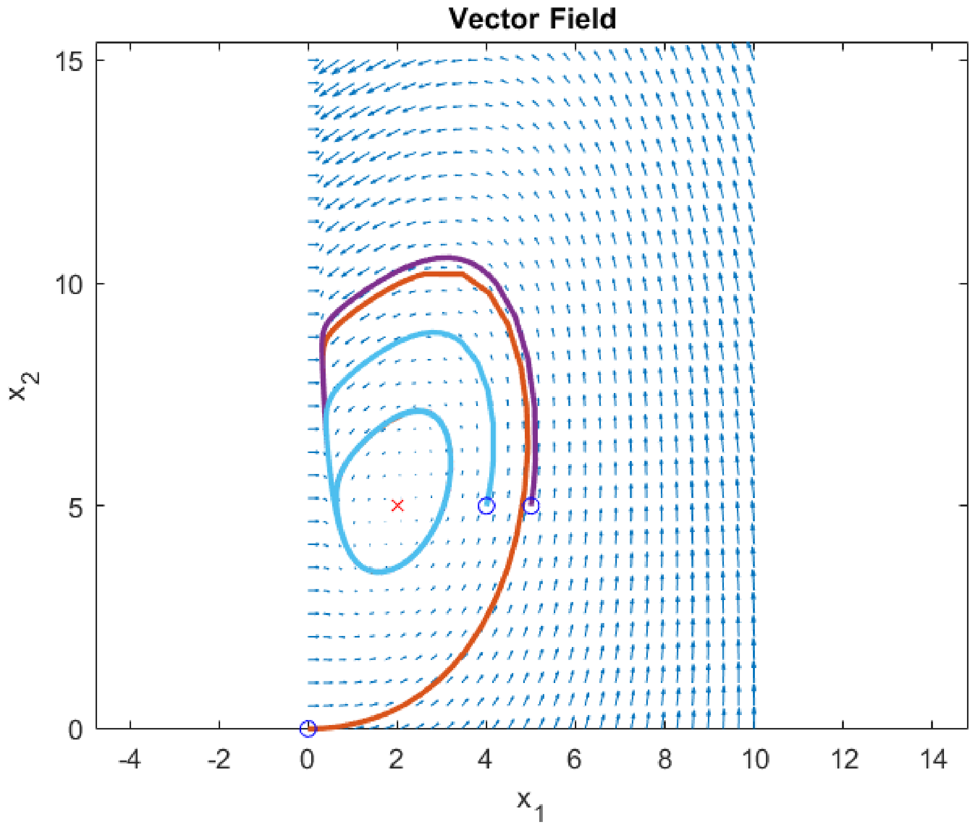

Figure 20: 2.29.b - With $a = 10, b = 2$, a limit cycle exists because the equilibrium point is an unstable focus. Any trajectory in the rectangular region $0 \leq x_1 \leq a, 0 \leq x_2 \leq 1+a^2$ will eventually reach the stable limit cycle. The equilibrium point is shown with a red x.

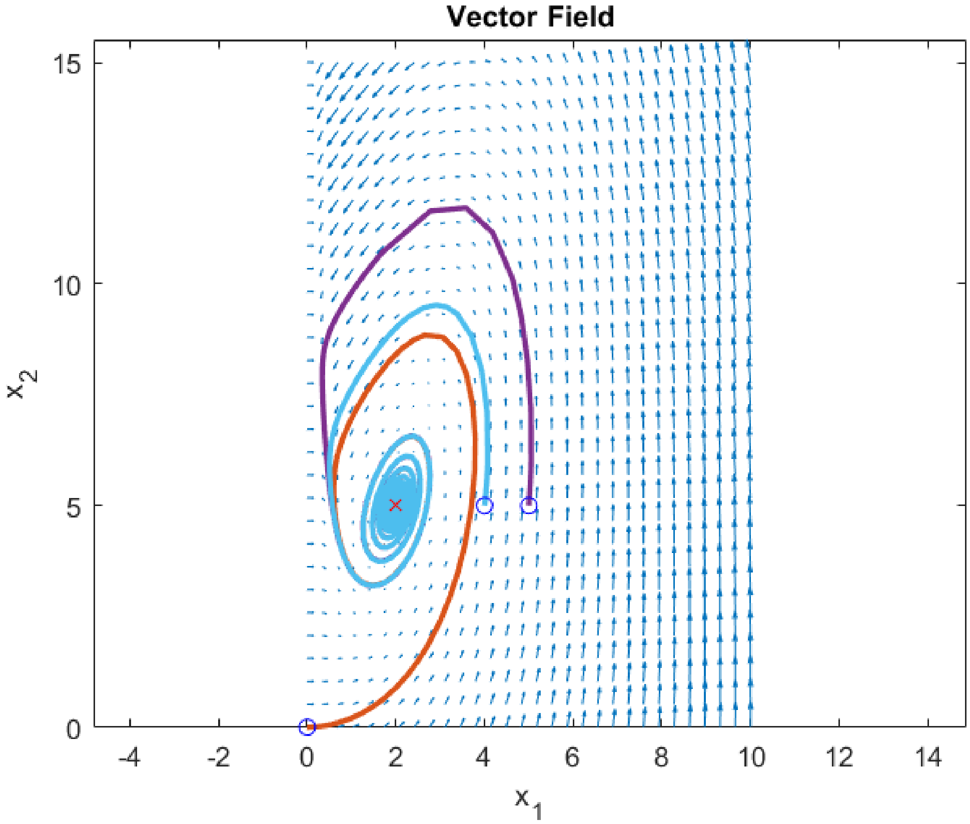

Figure 21: 2.29.c - With $a = 10, b = 4$, a limit cycle no longer exists because the equilibrium point is a stable focus. Any trajectory in the rectangular region $0 \leq x_1 \leq a, 0 \leq x_2 \leq 1 + a^2$ will eventually reach the equilibrium point, which is shown with a red x.

d) As the parameter b is increased, the eigenvalues move from the right half plane to the left half plane, crossing through the $j\omega$ axis. This is known as a supercritical Hopf bifurcation.

Problem 2.30

a) There are two equilibrium points, located at

$$(x_1, x_2) = \left\{(0, 4), \left(\frac{82d - 40}{50d - 25}, \frac{d}{5 - 10d}\right)\right\}$$

The second equilibrium point is a function of d. The distance from this equilibrium point to the origin is plotted below in a bifurcation diagram.

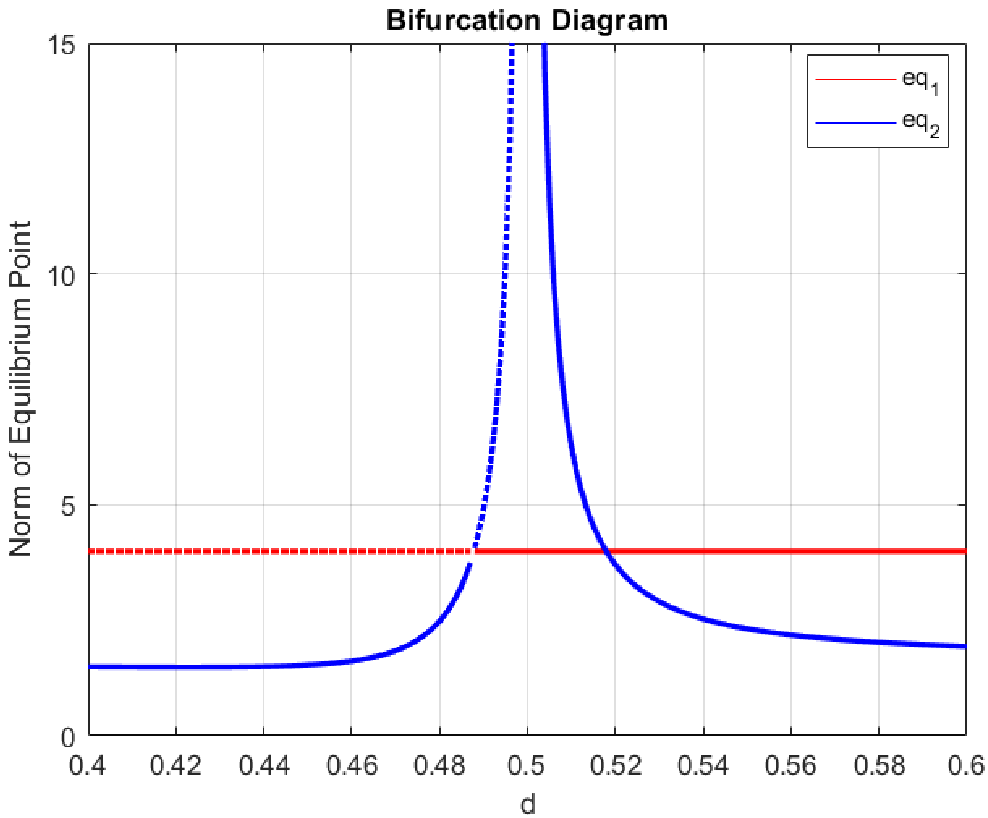

Figure 22: 2.30.a - The second equilibrium point varies with d. When $d = 0.5$, the equilibrium points diverge to $\pm\infty$. For $d > 0.5$, the equilibrium point approaches $82/50, -1/10)$. The dotted line corresponds to regions where the equilibrium point is a saddle, and the solid line corresponds to regions where the equilibrium point is stable node.

For the first equilibrium point, the eigenvalues are located at

$$\lambda_1 = \frac{20}{41} - d$$
$$\lambda_2 = -d$$

When $0 < d < 40/82 = 0.4878$, this is a saddle. When $d > 40/82$, this is a stable node.

For the second equilibrium point, the eigenvalues are located at

$$\lambda_1 = -d$$
$$\lambda_2 = -82d^2 + 81d - 20$$

When $40/82 < d < 41/82$, this is a saddle. Otherwise, this is a stable node.

b) The eigenvalues are plotted as a function of d. Each equilibrium point is associated with two real eigenvalues, shown in the figure below.

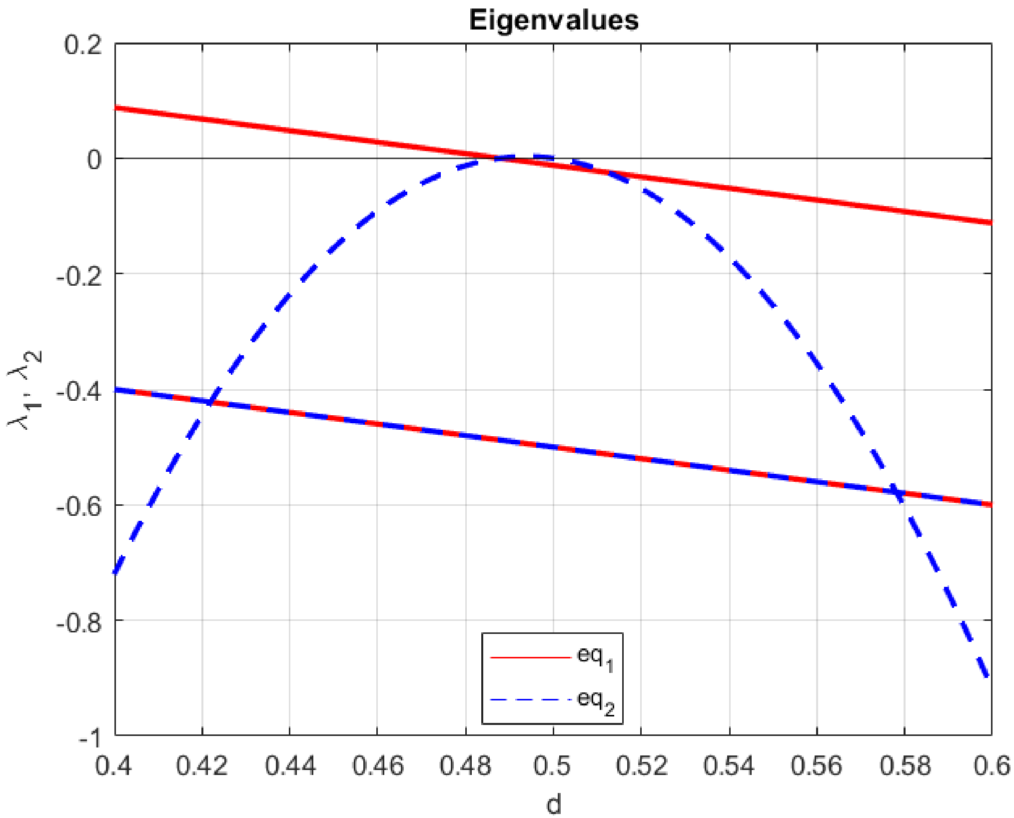

Figure 23: 2.30.b - The eigenvalues for the first equilibrium point are shown in red, while the eigenvalues for the second equilibrium point are shown in blue. Notice that the eigenvalues are positive only for a very small region, and over no continuous interval are both equilibrium points a saddle.

c) The vector field for $d = 0.4$ is shown below. The first equilibrium point $(0, 4)$ is a saddle while the second is a stable node.

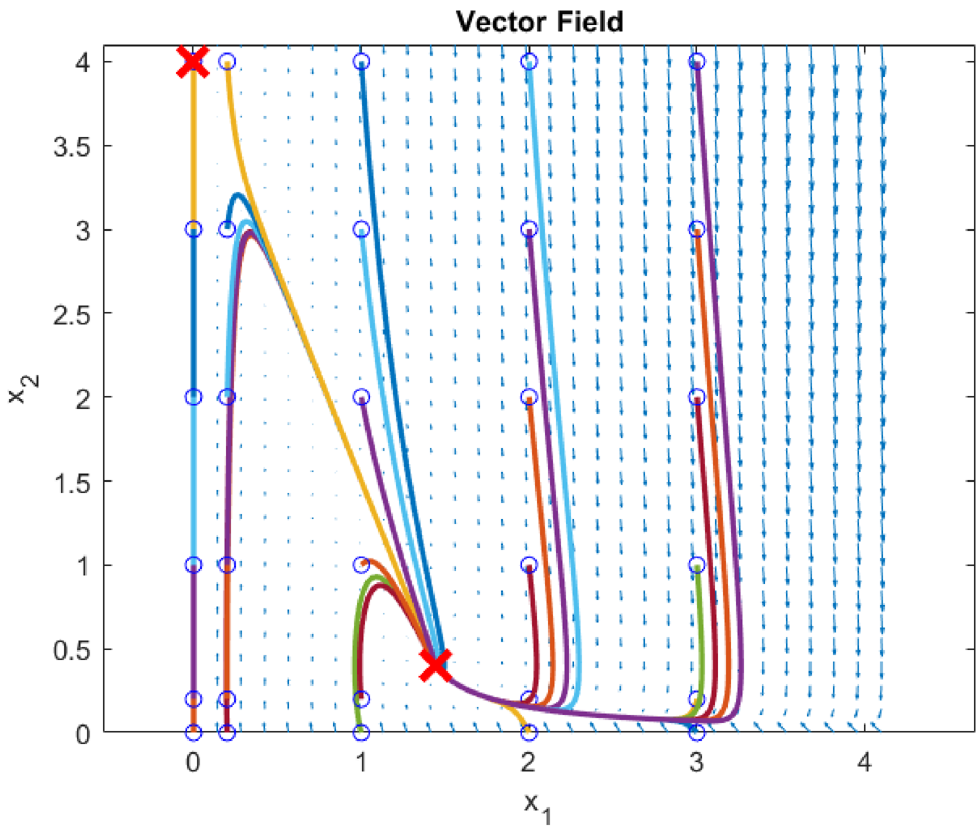

Figure 24: 2.30.c - The first equilibrium point $(0,4)$ is a saddle while the second is a stable node. As expected, most of the trajectories are attracted to the stable node. The equilibrium points are designated with red x's, while the initial conditions for each trajectory are designated with blue circles. We can conclude that under

Problem 2.31

a) There are either one or three equilibrium points depending on the value of d. They are

$$(0,4)$$
$$\left(\frac{8}{5} - \frac{\sqrt{5}\,\sqrt{16\,d^2 - 20\,d + 5} - 10\,d + 5}{25\,d}, \frac{\sqrt{5}\,\sqrt{16\,d^2 - 20\,d + 5} - 10\,d + 5}{10\,d}\right)$$
$$\left(\frac{10\,d + \sqrt{5}\,\sqrt{16\,d^2 - 20\,d + 5} - 5}{25\,d} + \frac{8}{5}, -\frac{10\,d + \sqrt{5}\,\sqrt{16\,d^2 - 20\,d + 5} - 5}{10\,d}\right)$$

The first equilibrium point exists over the domain $d > 0$, while the second two equilibrium points exist over the domain

$$0 < d < \frac{5 - \sqrt{5}}{8} \quad \vee \quad d > \frac{5 + \sqrt{5}}{8}$$

b) The bifurcation diagram is shown below. The y axis is the p-2 norm of the equilibrium point.

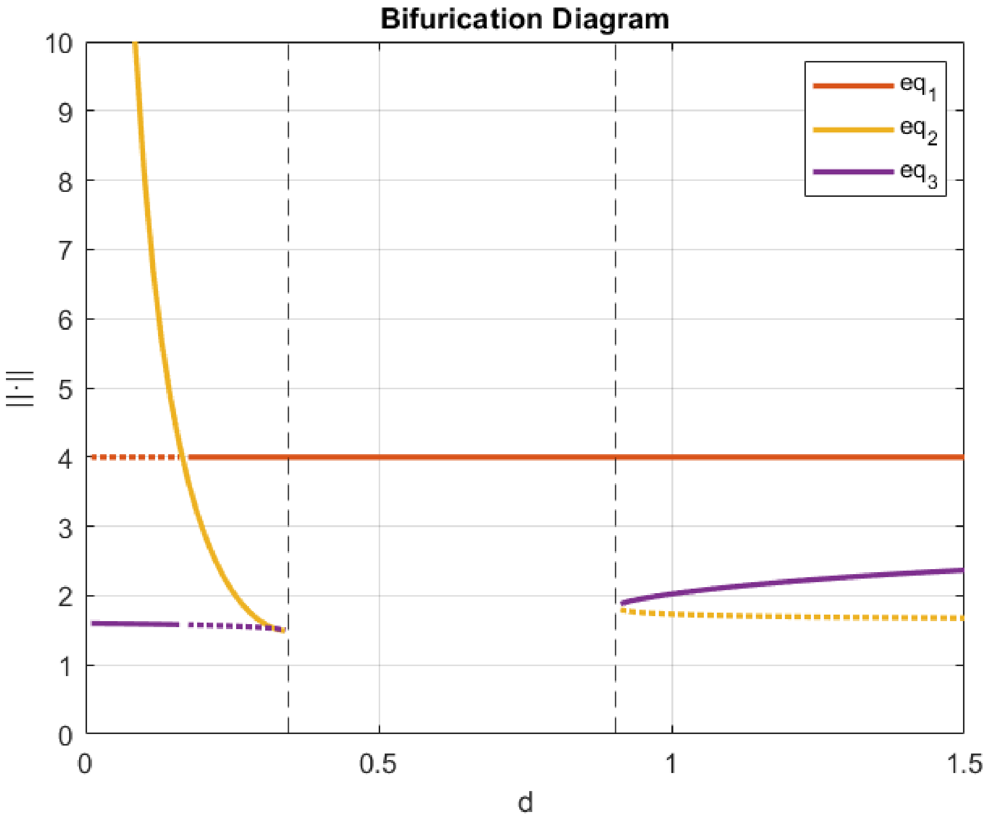

Figure 25: 2.31.b - The first equilibrium point transitions from a saddle to a stable node at $d = 0.165$. This is a saddle-node bifurcation. The second equilibrium point is a stable node for $0 < d < \frac{5-\sqrt{5}}{8}$, before it disappears. When $d > \frac{5+\sqrt{5}}{8}$, the second equilibrium point returns as a saddle. The third equilibrium point transitions from a stable node to a saddle at $d = 0.165$, before it disappears. When $d > \frac{5+\sqrt{5}}{8}$, the third equilibrium returns as a stable node. Note that the system undergoes a sub-critical pitchfork bifurcation at $d = \frac{5-\sqrt{5}}{8}$, and a supercritical pitchfork bifurcation at $d = \frac{5+\sqrt{5}}{8}$.

c) d) e) See figures below.

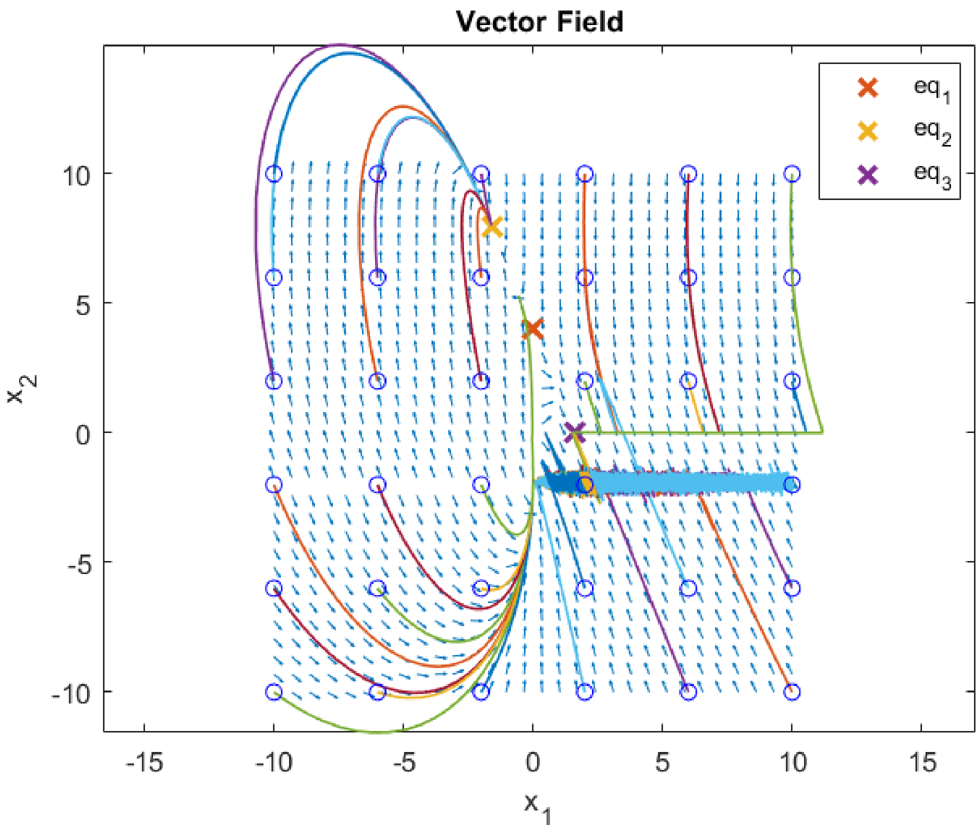

Figure 26: 2.31.c - $d = 0.1$. There are three equilibrium points, two of which are stable (yellow and purple). Notice that all trajectories in this phase portrait eventually settle one of the equilibrium points instead of escaping to infinity. There is a peculiar region where $x_1 > 0$ and $x_2 = -2$ where the trajectories seem to "get stuck" and slow down significantly, before working their way to the third equilibrium point. Each quadrant of the vector field behaves in a different way.

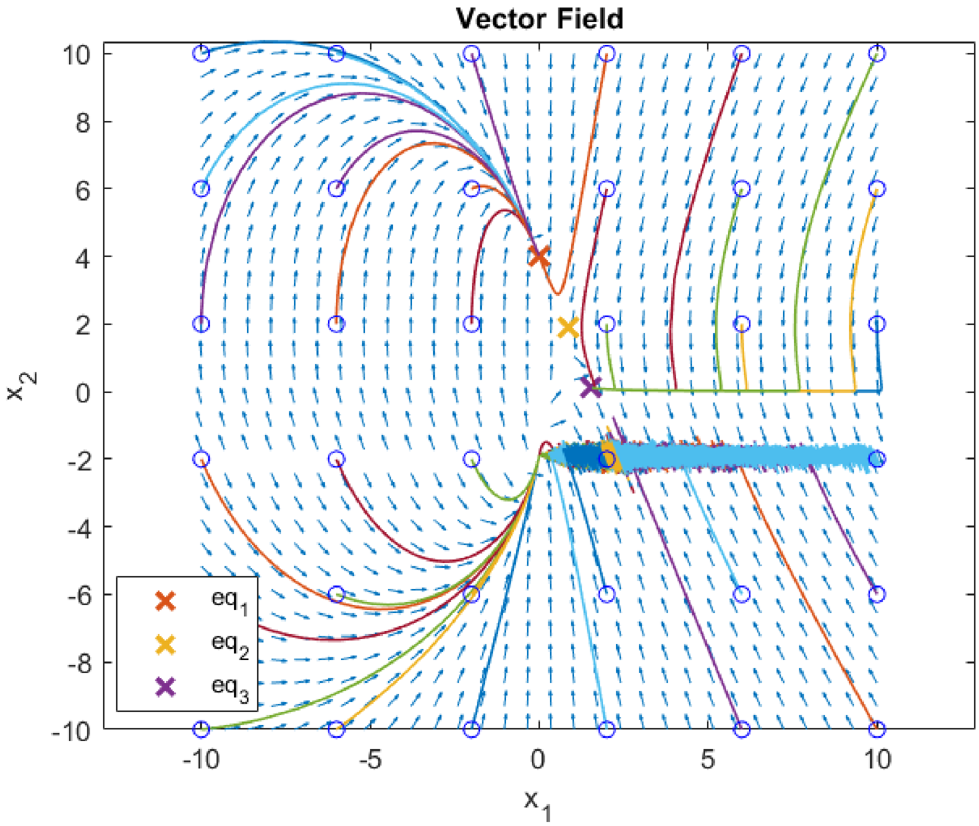

Figure 27: 2.31.d - $d = 0.25$. There are three equilibrium points, two of which are stable (but now they are red and yellow). All trajectories in this phase portrait eventually settle one of the equilibrium points instead of escaping to infinity. The difference between this and the $d = 0.1$ phase portrait is that the the trajectories converge to different equilibrium points, and these points have moved in the plane.

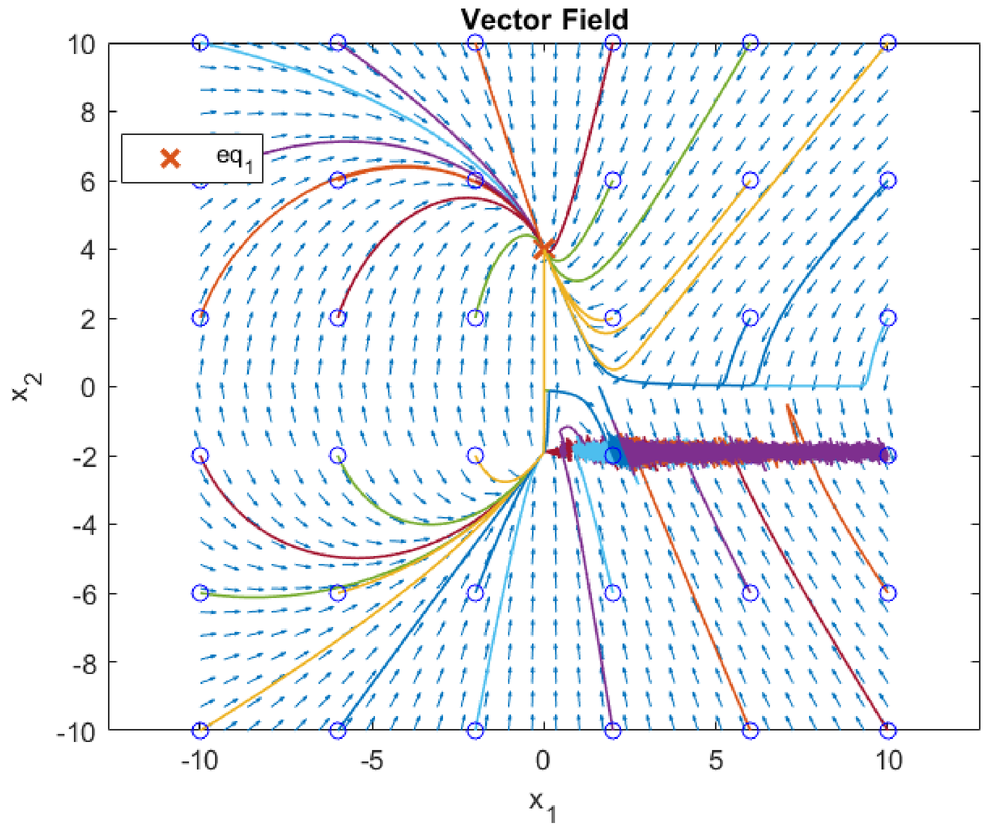

Figure 28: 2.31.d - $d = 0.5$. There is only one equilibrium point (red) located at $(0, 4)$, which is a stable node. The phase portrait is similar to the $d = 0.25$ case in terms of the general shape of the trajectories, but now all trajectories converge to the only equilibrium point.