Assignment-1

Course : Mathematical Foundations for Data Science
Course code : S1-22_DSECLZC416

Instructions:

- Submissions beyond 28th of December, 2022 5pm would not be graded.
- Assignments sent via email would not be accepted.
- The assignment should be hand written and single scanned pdf must me uploaded.
- This is a group assignment. There should be only one submission per group.
- The file name should be Group_Number.pdf only (Only pdf file is accepted)
- In the 1st page should contain the group number, name and bits id of each member and the percentage of contribution of each member.
- Scans must be clear and should not be out of focus. Out of focus assignments will be rejected and no marks will be given.
- Avoid photographs and images of handwritten sheets and use only scanning.

Questions

1. We have studied that during Gaussian elimination we can write $U = E_m E_{m-1} \dots . E_1 A$ where the matrices E_i are elementary transformations. Is this true for any arbitrary invertible $n \times n$ matrix? If it is true for an arbitrary invertible $n \times n$ matrix, provide a justification. If it is not true for an arbitrary $n \times n$ matrix, explain why and show what modifications you will make to the equation $U = E_m E_{m-1} \dots . E_1 A$ to make it work for any arbitrary invertible $n \times n$ matrix.

2. To solve systems like $Ax = b$ where A is an invertible $n \times n$ matrix we write a program $Solve(A, b)$ that takes a matrix A and right-hand side b as input and computes the solution to $Ax = b$. Suppose that the algorithm used by $Solve(A, b)$ is the augmented matrix method. Let us say we need to solve k systems of the type $Ax = b$ **where the right-hand side changes, but the left hand side stays the same**. We can do this by making k invocations to the procedure $Solve(A, b)$. Can you come up with a better way of solving such systems, and characterize the improvement in operation count compared with making k calls to $Solve(A, b)$?

3. Consider $n \times n$ elementary matrices where E_{ij} represents the elementary matrix where there is a non-zero value at the ith row and jth column, in addition to 1s on the diagonal. Given a particular elementary matrix E_{ij}, for which other elementary matrices E_{pq} is it the case that $E_{pq}E_{ij} = E_{ij}E_{pq}$?

4. As part of a computer application, a sub-routine needs to be written, whose input parameter p has to be used in the computation A^p where A is a 100×100 symmetric, positive-definite matrix A. Note that A is a fixed matrix, and it is only p which is the input parameter. What is the most efficient way you can come up with to perform the required computation, if the sub-routine is called millions of times for arbitrary values of p? Your solution needs to be efficient in terms of both time and space taken by the algorithm.

5. We need to send a 1000×1000 matrix of numbers across a channel, and would like to minimize the total amount of data sent on the channel for reasons having to do with both the possibility of data corruption and time taken to send the data. Can you think of a way of minimizing the amount of data to be sent across the channel, so that we can represent the most important information in the matrix?

6. Consider a linear system Ax=b. Assume that column vectors $\mathbf{a}_1,\dots,\mathbf{a}_n \in \mathrm{R}^{\mathbf{n}}$ are columns of matrix A ie. $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3 \dots \mathbf{a}_n]$. Let $C = [A|b]$ be the augmented matrix associated with this linear system. Let us consider 2 different scenarios

a) Suppose I interchange column $\mathbf{a}_i$ and $\mathbf{a}_k$ of the augmented matrix C giving a new matrix C_1 ($i,k <= n$). Now I solve the problem assuming C_1 is my augmented matrix. How are the solutions of augmented matrix C and C_1 related.

b) Suppose I scale the i^{th} column of the augmented matrix C by α giving a new matrix C_2 ($i,k <= n$). Now I solve the problem assuming C_2 is my augmented matrix. How are the solutions of augmented matrix C and C_2 related.

7. In a class, a professor informed students that M is a real 3 by 3 real matrix such that $M^3=I$. Using the given information, students were asked whether the matrix is invertible and to find the eigenvalues of M. Is M invertible and find the eigenvalues of M.

8. A student from Linear Algebra class received a matrix of the form given below.

$$A = \begin{bmatrix} 0 & \Delta & 1 & 0 \\ \Delta & 0 & \Delta & 0 \\ 1 & \Delta & 0 & \Delta \\ 0 & 0 & \Delta & 1 \end{bmatrix}, \Delta > 1$$

Help him evaluate the column rank, trace and the determinant and then using trace, rank and determinant, what conclusion can be drawn about the signs (+ or -) of eigenvalues of A.

9. Harry is the team lead in a company but is new to Linear Algebra. While working on his project, he arrived at a problem. He got three vectors $v_1, v_2, v_3 \in \boldsymbol{R}^5$

Let $S = span\{v_1, v_2, v_3\}$ and $W = \{u_1, u_2, \dots, u_r\}$.

The problem is to find all those u_i s that belong to S and if $u_i \in S$, find linear combination of u_i in terms of v_1, v_2, v_3. Explain the method to solve Harry's problem with proper justification. Is the method efficient?

Using the above method solve if $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $W = \left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

10. Consider system of equations in the matrix form as $Ax = b$, where A is a matrix of order $m \times n$.

Case 1.

If m=5, n=6, then will the system have solutions for every choice of b? Discuss with explanation.

Case 2.

If m=6, n=8 and rank of A is 6, then is it possible to make the system have no solution by changing b? Discuss with explanation.

Case 3.

If m=10, n=12, b_i =0 for all i from 1 to 12 then is it possible that all solutions are multiples of one fixed non-zero solution? Discuss with explanation.

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