2021 WSMO Team Round Solutions

SMO Team

TR 1: Bob has a number of pencils. When Bob splits them into groups of 10, he has 3 left over. When he splits them into groups of 12, he has 5 left over. Find the smallest number of pencils Bob can have.

Answer: $\boxed{53}$

Solution: From the condition, possible numbers of pencils Bob could have are

$$3, 13, 23, 33, 43, \boxed{53}, 63, \ldots$$

and from the second condition we have

$$5, 17, 29, 41, \boxed{53}, 65, \ldots.$$

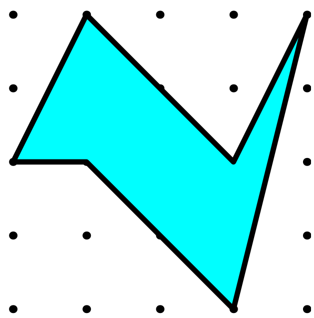
So, the smallest number of pencils Bob could have is $\boxed{53}$.

TR 2: Integers W, S, M, and O satisfy $W + S + M + O = 13$. Find the maximum possible value of $WSMO$.

Answer: $\boxed{108}$

Solution: From the AM-GM inequality, we want W, S, M, O as close to each other as possible. So, let $W = S = M = 3, O = 4 \implies WSMO = \boxed{108}$.

TR 3: In the figure below, there are 1023 total circles. The area between circles alternate between shaded and non-shaded. The area of the shaded region can be expressed as $k\pi$. Find $k\pi$.



Answer: $\boxed{776}$

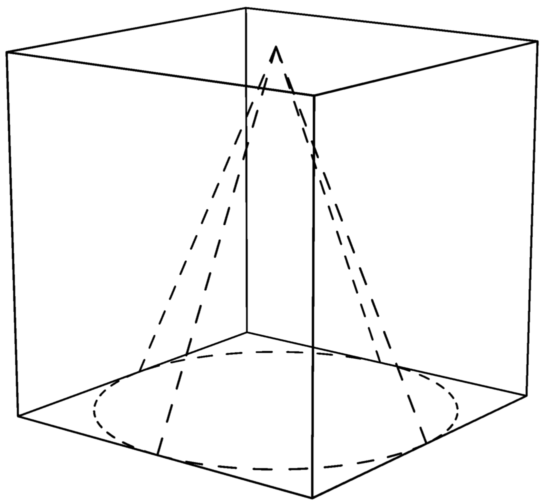
Solution: Note that

$$\begin{aligned}k &= 1023^2 - 1022^2 + 1021^2 - 1019^2 \ldots + 3^2 - 2^1 + 1^2\\ &= (1023-1022)(1023+1022) + \ldots + (3-2)(3+2) + 1^2\\ &= (1023+1022) + (1021+1019) + \ldots + (3+2) + 1\\ &= \frac{1023 \cdot 1024}{2} \implies\\ k &\equiv \boxed{776} \pmod{1000}.\end{aligned}$$

TR 4: Honko the hamster is in his cage. He wants to find the smallest distance needed to travel to reach four tennis balls. His current position is $(0,0)$. The tennis balls are located at $(1,1)$, $(2,-2)$, $(-3,-3)$, and $(-4,4)$. The length of the shortest path can be expressed as $\sum_1^n \sqrt{a_i}$, where n is minimal. Find $\sum_1^n a_i$.

Answer: $\boxed{108}$

Solution:



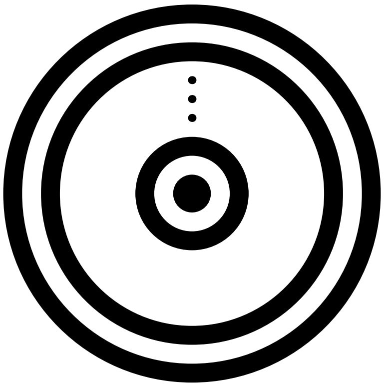
Let $O=(0,0)$, $A=(1,1)$, $B=(2,-2)$, $C=(-3,-3)$, and $D=(-4,4)$. We compute $AB=\sqrt{10}$, $BC=\sqrt{26}$, $CD=\sqrt{50}$, and $DA=\sqrt{34}$. From casework on the first tennis ball Honko reaches, we have the following possible paths:

$$\begin{gathered}O \to A \to B \to C \to D\\ O \to B \to A \to D \to C\\ O \to C \to B \to A \to D\\ O \to D \to A \to B \to C,\end{gathered}$$

of which the first one has the shortest length of

$$\sqrt{2}+\sqrt{10}+\sqrt{26}+\sqrt{50}=\sqrt{10}+\sqrt{26}+\sqrt{72} \implies 10+26+72=\boxed{108}.$$

TR 5: A monkey is throwing darts at the dart board pictured below. The dart is equally likely to land anywhere on the board. Point values for the three regions are labeled and the radii the three circles are 1, 2, and 3, respectively. The expected value of points the monkey gets from 5 dart throws is $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: $\boxed{194}$

Solution: The areas of the $3, 5, 7$ point regions are $5\pi, 3\pi, \pi$, respectively. So, the expected points of each throw is

$$\frac{5\pi}{9\pi} \cdot 3 + \frac{3\pi}{9\pi} \cdot 5 + \frac{\pi}{9\pi} \cdot 7 = \frac{37}{9}.$$

The expected points the monkey gets from 5 dart throws is

$$\frac{37}{9} = \frac{185}{9} \implies 185 + 9 = \boxed{194}.$$

TR 6: A quartic real polynomial $f(x)$ satisfying $f(3+2i) = 0$ has 3 distinct roots. If the sum of the three roots is 12, find their product.

Answer: $\boxed{468}$

Solution: From the conjugate root theorem, since $3+2i$ is a root, $3-2i$ must also be a root. Since there are only three distinct roots, let x be the value of the remaining two roots. We have

$$x + (3+2i) + (3-2i) = 12 \implies x + 6 = 12 \implies x = 6.$$

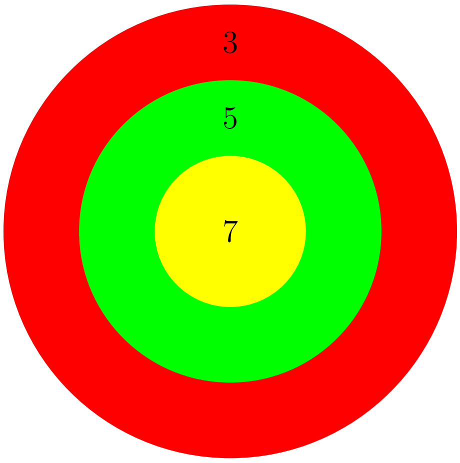
So, the product of the four roots is

$$(3+2i)(3-2i)(6)(6) = (3^2+2^2)(6)(6) = \boxed{468}.$$

TR 7: In triangle ABC with $AB = 13, AC = 14$, and $BC = 15$, a rectangle $WXYZ$ is inscribed such that the area of $WXYZ$ is maximized. The minimum possible value of $\frac{WX}{XY}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{337}$

Solution:



Note that $WXYZ$ must have two vertices on a side of ABC and the other two vertices on each of the other two sides of ABC. Assume WLOG that Z, Y lie on AB, W lies on AC, and X lies on CB. For simplicity, denote $c = AB = 13$ and $h = \frac{168}{c}$ C-height of triangle ABC. Let $WX = ZY = x$ and $XY = WZ = y$. Let D and E be the foot of the perpendicular from C to WX and AB, respectively. From similar triangles $\triangle CWX \sim \triangle CAB$, we have

$$\frac{CD}{CE} = \frac{WX}{AB} \implies \frac{h-y}{h} = \frac{x}{c} \implies \frac{\frac{168}{c} - y}{\frac{168}{c}} = \frac{x}{c} \implies x = \frac{168c - c^2y}{168}.$$

So,

$$[WXYZ] = xy = \left(\frac{168c - c^2y}{168}\right) y = \frac{168cy - c^2y^2}{168},$$

which is minimized when $y = -\frac{168c}{-2c^2} = \frac{84}{c}$. Substituting, we have

$$x = \frac{168c - c^2 \cdot \frac{84}{c}}{168} = \frac{84c}{168} = \frac{c}{2},$$

meaning

$$\frac{WX}{XY} = \frac{x}{y} = \frac{\frac{c}{2}}{\frac{84}{c}} = \frac{c^2}{168} = \frac{169}{168} \implies 169 + 168 = \boxed{337}.$$

Note that if $WYXZ$ had two vertices on sides AC, BC, then we still have $[WXYZ]$ maximized to $\frac{[ABC]}{2}$ but $\frac{WX}{XY} = \frac{14^2}{168}, \frac{15^2}{168} > \frac{169}{168}$, respectively.

TR 8: Let $f(x) = \sqrt{x - \sqrt{x - \sqrt{x - \dots}}}$. Find the modulo 1000 on the minimum integer a such $f(f(f(f(f(a)))))$ is a positive integer.

Answer: $\boxed{442}$

Solution: Note that

$$\begin{aligned} f(x) &= \sqrt{x - f(x)} \implies \\ [f(x)]^2 &= x - f(x) \implies \\ x &= [f(x)]^2 + f(x). \end{aligned}$$

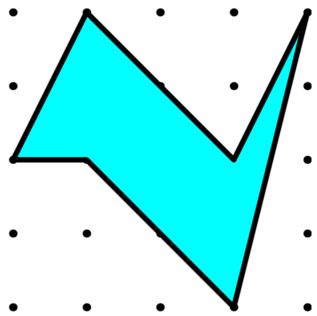
Since $[f(x)]^2 + f(x)$ is a strictly increasing function, the minimum value of a occurs when

$$\begin{aligned} f(f(f(f(f(a))))) &= 1 \implies \\ f(f(f(f(a)))) &= 1^2 + 1 = 2 \implies \\ f(f(f(a))) &= 2^2 + 2 = 6 \implies \\ f(f(a)) &= 6^2 + 6 = 42 \implies \\ f(a) &= 42^2 + 42 = 1806 \implies \\ a &= 1806^2 + 1806 \implies \\ a &\equiv \boxed{442} \pmod{1000}. \end{aligned}$$

TR 9: A circle ω has three chords of equal length, $4 + 2\sqrt{3}$ which intersect forming a triangle with side lengths 2, 2, and $2\sqrt{3}$. The square of the radius of ω can be expressed as $a - b\sqrt{c}$, where b and c are positive integers with c squarefree. Find $a + b + c$.

Answer: $\boxed{39}$

Solution:



Let the triangle formed from the intersection be ABC and the center of the circle be O. Since the three chords are of equal length, O must be equidistant to sides AB, BC, and AC, meaning O is the incenter of ABC. The inradius is equal to the quotient of the area and semiperimeter, which is

$$\frac{\sqrt{3}}{\frac{2+2+2\sqrt{3}}{2}} = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3} - 3.$$

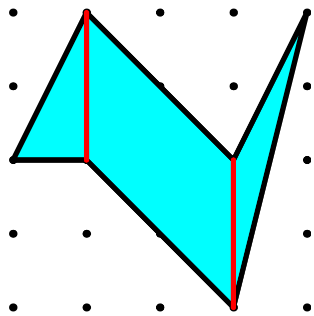
From the Pythagorean Theorem, we have

$$r^2 = (2\sqrt{3}-3)^2 + (2+\sqrt{3})^2 = (21 - 12\sqrt{3}) + (7 + 4\sqrt{3}) = 28 - 8\sqrt{3} \implies 28 + 8 + 3 = \boxed{39}.$$

TR 10: Let square $ABCD$ be a square with side length 4. Define ellipse ω as the ellipse that is able to be inscribed inside $ABCD$ such that 2 of its vertices on its minor axis and 1 of its vertices on its major axis form an equilateral triangle. The largest possible area of ω is $m\pi\sqrt{n}$, where n is a squarefree positive integer. Find $m + n$.

Answer: $\boxed{5}$

Solution:



We will use analytic geometry. Let $O = (0,0)$ be the center of the square. We have $A = (-2,2), B = (2,0), C = (2,-2)$, and $D = (-2,-2)$. Now, let the length of the equilaterial triangle be $2t$. So, the minor and major axes have length $2t$ and $2t\sqrt{3}$, respectively, meaning the focal distance is $2t\sqrt{2}$. Define points F_1 and F_2 to be the two focus points of the ellipse. Note that the major axis coincides with the diagonal of the square, meaning the angle of the ellipse with respect to the horizontal must be $45°$. This means $F_1 = (t,t)$ and $F_2 = (-t,-t)$. By the definition of an ellipse, all points P located on the ellipse must have $F_1P + F2P = 2t\sqrt{3}$. Let X be the tangency point of the ellipse with CD and F_2' be the reflection of F_2 across CD. Due to tangency, $F_1P + F_2P$ is minimized at $P = X$ over all points P on CD. By properties of a reflection, we have $F_1X + F_2X = F_1X + XF_2'$. From the triangle inequality, X must be located on F_1F_2'. So, $F_1X + XF_2' = F_1F_2'$. Since F_2' is the reflection of $F_2 = (-t,-t)$ across $CD : y = -2$, we have $F_2' = (-t, -4+t)$. So, $F_1F_2' = (2t)^2 + (-4)^2 = 4t^2 + 16$. Setting this equal to $2t\sqrt{3}$, we have $4t^2 + 16 = 12t^2 \implies t^2 = 2$. This means the major and minor axes have length $2\sqrt{6}, 2\sqrt{2}$, respectively, meaning our answer is $(\sqrt{6})(\sqrt{2})\pi = 2\pi\sqrt{3} \implies 2 + 3 = \boxed{5}$.

TR 11: When $\frac{1}{7}$ is expressed in base k, the digits are $0.a_{k,1}a_{k,2}\ldots$, where $a_{k,1}a_{k,2},\ldots$ are base-k digits. Let $f(p)$ denote the minimum positive integer $x \geq 2$ such that $a_{p,1} = a_{p,x}$, for $k \not\equiv 0 \pmod 7$, and 1 for $k \equiv 0 \pmod 7$. Find $\sum_{i=2}^{100} f(i)$.

Answer: $\boxed{396}$

Solution: We have

$$0.a_{k,1}a_{k,2}\cdots = 0.\overline{a_{k,1}a_{k,2}\ldots a_{k,f(p)-1}} = \frac{\overline{a_{k,1}a_{k,2}\ldots a_{k,f(p)-1}}_k}{k^{f(p)-1} - 1}.$$

So, $f(p) - 1$ is the order of k modulo 7. From here, we compute

$$f(p) = 1, 2, 4, 7, 4, 7, 3$$

for

$$p \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7,$$

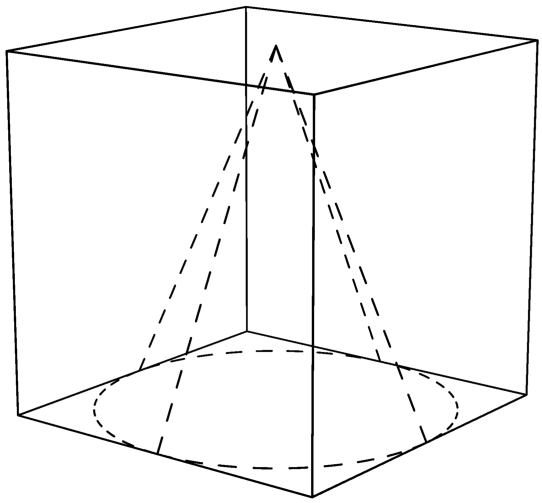
respectively. So,

$$\sum_{i=2}^{100} f(i) = (1 + 2 + 4 + 7 + 4 + 7 + 3) \cdot 14 + 4 = \boxed{396}.$$

TR 12: Consider parabola $\mathcal{P}$ pointing upwards with vertex at the origin of the Cartesian plane. Denote the focus of it as F and the directrix of it as $\mathcal{L}$. Point P with x-coordinate 4 is selected on $\mathcal{P}$. The perpendicular bisector of FP meets $\mathcal{L}$ at Q. Given the x-coordinate of Q is 3, then $FP^2 = \frac{a+b\sqrt{c}}{d}$, where c is a squarefree positive integer and a, b, d are relatively prime positive integers. Find $a+b+c+d$.

Answer: $\boxed{46}$

Solution:



Let the parabola have equation $y = ax^2$. We have

$$F = \left(0, \frac{1}{4a}\right) \quad \text{and} \quad \mathcal{L} : y = -\frac{1}{4a}.$$

Since the x-coordinate of P is 4, we have $P = (4, 16a)$. Since Q lies on $\mathcal{L}$ and has x-coordinate 3, we have $Q = \left(3, -\frac{1}{4a}\right)$. Since Q lies on the perpendicular bisector of FP, we have $FQ = PQ$. So

$$\begin{aligned}
(3-0)^2 + \left(\frac{1}{4a} - \left(-\frac{1}{4a}\right)\right)^2 &= (4-3)^2 + \left(16a + \frac{1}{4a}\right)^2 \implies \\
3^2 + \frac{1}{4a^2} &= 1 + \left(64a^2 + 8 + \frac{1}{16a^2}\right) \implies \\
9 + \frac{1}{4a^2} &= 9 + 64a^2 + \frac{1}{16a^2} \implies \\
\frac{1}{4a^2} &= 64a^2 + \frac{1}{16a^2} \implies \\
\frac{3}{16a^2} &= 64a^2 \implies a^4 = \frac{3}{64} \implies a = \frac{\sqrt[4]{3}}{8}.
\end{aligned}$$

Now,

$$\begin{aligned}
FP^2 &= 4^2 + \left(16a - \frac{1}{4a}\right)^2 \\
&= 16 + \frac{(64a^2-1)^2}{16a^2} \\
&= 16 + \frac{(\sqrt{3}-1)^2}{\frac{\sqrt{3}}{4}} \\
&= \frac{24+16\sqrt{3}}{3} \implies 24+16+3+3\boxed{46}.
\end{aligned}$$

TR 13: Gerry the Squirrel is at a corner of a 9×9 grid. An acorn A_1 is placed. There are four coins placed randomly on the 9×9 grid. Gerry will go the shortest path to A_1, and if there are multiple shortest paths, Gerry will pick one randomly. Given that Gerry does not start on top of a coin, the acorn is not on a coin, find the expected value of coins Gerry passes during his trip to A_1 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{537}$

Solution: Place Gerry at $(0,0)$. Suppose the acorn A_1 is located at (x,y), where $0 \le x,y \le 8, (x,y) \neq (0,0)$ (so 80 possible positions). Any shortest path from $(0,0)$ to (x,y) passes through $x+y-1$ lattice points. So, the expected number of points Gerry passes through is

$$\frac{\sum_{x=0}^{8}\sum_{y=0}^{8}(x+y-1)-(0+0-1)}{80} = \frac{81\cdot(4+4-1)+1}{80} = \frac{568}{80}.$$

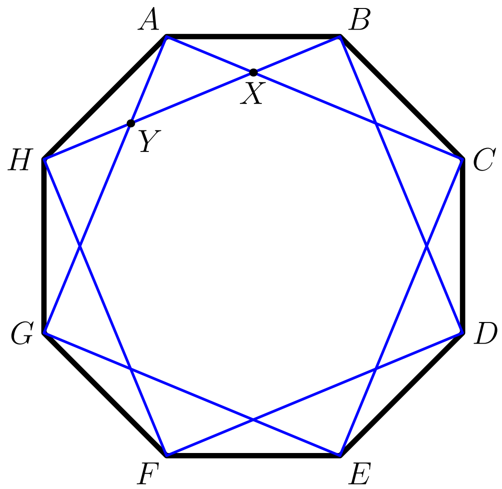
The four coins are placed randomly among the 79 remaining lattice points. Hence, each point Gerry passes has a $\frac{4}{79}$ probability of containing a coin. So, the expected number of coins Gerry passes is

$$\frac{568}{80}\cdot\frac{4}{79} = \frac{142}{395} \implies 142+395 = \boxed{537}.$$

TR 14: Consider quadrilateral $ABCD$ with circumcircle centered at O. Given $AB = 2 < CD$, the circumcircle of $\triangle ABO$ is tangent to the circumcircle of $\triangle COD$. The circumcircle of $\triangle AOD$ passes through the center of the circumcircle of $\triangle ABO$. Given the radius of the circumcircle of $ABCD$ is 6. Let A and P be the area and perimeter of $ABCD$, respectively. The value of $\frac{A}{P}$ can be expressed as $\frac{a\sqrt{b}}{c}$, where b is a squarefree positive integer and a, c are relatively prime positive integers. Find $a+b+c$.

Answer: $\boxed{3096}$

Solution:



Let O_3 and O_1 be the circumcenters of COD and ABO, respectively. Since ABO and COD share point O, they must be externally tangent at O, meaning O_3, O, and O_1 are collinear. Since $AO = BO$ and $CO = DO$, ABO and COD are isosceles, meaning $OO_1 \perp AB$ and $OO_3 \perp CD$. From the fact that that O_3, O, O_1 are collinear, this directly implies that

$AB \parallel CD$, meaning $ABCD$ is an isosceles trapezoid. Now, the O-height of ABO is of length $\sqrt{6^2 - \left(\frac{2}{3}\right)^2} = \sqrt{35}$. So,

$$OO_1 = \frac{(OA)(AB)(BO)}{4[ABO]} = \frac{6 \cdot 2 \cdot 6}{4 \cdot \frac{2 \cdot \sqrt{35}}{2}} = \frac{18}{\sqrt{35}}.$$

Let $\theta = \angle OAO_1 = \angle AOO_1$. Since OO_1A is isosceles, have

$$\cos\theta = \frac{\frac{OA}{2}}{O_1A} = \frac{3}{\frac{18}{\sqrt{35}}} = \frac{\sqrt{35}}{6} \implies \sin\theta = \sqrt{1 - \left(\frac{\sqrt{35}}{6}\right)^2} = \frac{1}{6}.$$

Now, from angle chasing, we have

$$\begin{aligned}
\angle OO_1A &= 180 - 2\theta && \text{since } OO_1A \text{ is isosceles,} \\
\angle ODA &= 180 - \angle OO_1A = 2\theta && \text{since } ODAO_1 \text{ is cyclic,} \\
\angle OAD &= \angle ODA = 2\theta && \text{since } AOD \text{ is isosceles,} \\
\angle AOB &= 2\angle AOO_1 = 2\theta && \text{from symmetry,} \\
\angle OAB &= \frac{180 - \angle AOB}{2} = 90 - \theta && \text{since } ABO \text{ is isosceles,} \\
\angle DAB = \angle DAO + \angle OAB &= 2\theta + (90 - \theta) = 90 + \theta) && \text{, and} \\
\angle CDA &= 180 - \angle DAB = 90 - \theta && \text{since } AB \parallel CD.
\end{aligned}$$

Now,

$$\sin OAD = \sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{1}{6}\right)\left(\frac{\sqrt{35}}{6}\right) = \frac{\sqrt{35}}{18} \quad \text{and}$$

$$\cos(2\theta) = \sqrt{1 - \left(\frac{\sqrt{35}}{18}\right)^2} = \frac{17}{18}.$$

We have

$$AD = 2 \cdot (AO\cos\angle OAD) = 12 \cdot \cos(2\theta) = 12 \cdot \frac{17}{18} = \frac{34}{3}.$$

Since $ABCD$ is an isosceles trapezoid, this means $CB = \frac{34}{3}$ as well. Now,

$$\begin{aligned}
CD &= AB + 2AD\cos(\angle CDA) = 2 + 2AD\cos(90 - \theta) \\
&= 2 + 2AD\sin\theta = 2 + 2\left(\frac{34}{3}\right)\left(\frac{1}{6}\right) = \frac{52}{9}.
\end{aligned}$$

This gives us a perimeter of

$$N = AB + BC + CD + DA = 2 + \frac{34}{3} + \frac{52}{9} + \frac{34}{3} = \frac{274}{9}$$

and an area of

$$\begin{aligned}M &= \left(\frac{AB+CD}{2}\right)(AD\sin\angle CDA)\\&= \left(\frac{2+\frac{52}{9}}{2}\right)\left(\frac{34}{3}\sin(90-\theta)\right)\\&= \frac{35}{9}\cdot\frac{34}{3}\cdot\cos\theta\\&= \frac{35}{9}\cdot\frac{34}{3}\cdot\frac{\sqrt{35}}{6}\\&= \frac{595\sqrt{35}}{81}.\end{aligned}$$

So,

$$\frac{M}{N} = \frac{\frac{595\sqrt{35}}{81}}{\frac{274}{9}} = \frac{595\sqrt{35}}{2466} \implies 595+35+2466 = \boxed{3096}.$$

TR 15: On a number line labeled 0, 1, 2, 3, 4, 5, and old man starts at 0 and tries to reach 5. Initially, he knows to walk right. However, he has dementia. On each move, there is a $\frac{1}{3}$ chance he forgets which direction he is supposed to go, resulting in him walking the opposite direction. If the old man tries to walk left when he is at 0, he falls off a cliff and dies. The probability the old man reaches 5 without dying is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{95}$

Solution: Let P_x be the probability the old man reaches 5. We have

$$\begin{aligned}&P_0 = \frac{2}{3}P_1 \implies \underline{P_1 = \frac{3}{2}P_0},\\&P_1 = \frac{2}{3}P_2 + \frac{1}{3}P_0 \implies \frac{3}{2}P_0 = \frac{2}{3}P_2 + \frac{1}{3}P_0 \implies\\&\quad \frac{7}{6}P_0 = \frac{2}{3}P_2 \implies \underline{P_2 = \frac{7}{4}P_0}\\&P_2 = \frac{2}{3}P_3 + \frac{1}{3}P_1 \implies \frac{7}{4}P_0 = \frac{2}{3}P_3 + \frac{1}{2}P_0 \implies\\&\quad \frac{5}{4}P_0 = \frac{2}{3}P_3 \implies \underline{P_3 = \frac{15}{8}P_0}\\&P_3 = \frac{2}{3}P_4 + \frac{1}{3}P_2 \implies \frac{15}{8}P_0 = \frac{2}{3}P_4 + \frac{7}{12}P_0 \implies\\&\quad \frac{31}{24}P_0 = \frac{2}{3}P_4 \implies \underline{P_4 = \frac{31}{16}P_0}\\&P_4 = \frac{2}{3} + \frac{1}{3}P_3 \implies \frac{31}{16}P_0 = \frac{2}{3} + \frac{5}{8}P_0 \implies\\&\quad \frac{21}{16}P_0 = \frac{2}{3} \implies \underline{P_0 = \frac{32}{63}} \implies 32+63 = \boxed{95}.\end{aligned}$$