2021 WSMO Tiebreaker Round Solutions

SMO Team

Tiebreaker Round Problem 1: Find the number of factors of $24^6 - 20^6$.

Answer: $\boxed{208}$

Solution: Note that

$$\begin{aligned}
24^6 - 20^6 &= (24^3 - 20^3)(24^3 + 20^3) \\
&= (24 - 20)(24^2 + 24 \cdot 20 + 20^2)(24 + 20)(24^2 - 24 \cdot 20 + 20^2) \\
&= (4)(1456)(44)(496) \\
&= 2^2 \cdot (2^4 \cdot 31)(2^2 \cdot 11)(2^4 \cdot 7 \cdot 13) \\
&= 2^{12} \cdot 7 \cdot 11 \cdot 13 \cdot 31,
\end{aligned}$$

which has

$$(12 + 1)(1 + 1)(1 + 1)(1 + 1)(1 + 1) = (13)(2)(2)(2)(2) = \boxed{208}.$$

Tiebreaker Round Problem 2: Real numbers x and y satisfy $x^3 + y^3 = 76895$ and $x + y = 65$. Find xy.

Answer: $\boxed{1014}$

Solution: Note that

$$\begin{aligned}
xy &= \frac{xy(x + y)}{(x + y)} \\
&= \frac{1}{3}\left(\frac{3x^2y + 3xy^2}{x + y}\right) \\
&= \frac{1}{3}\left(\frac{(x^3 + 3x^2y + 3xy^2 + y^3) - (x^3 + y^3)}{x + y}\right) \\
&= \frac{1}{3}\left(\frac{(x + y)^3 - (x^3 + y^3)}{x + y}\right) \\
&= \frac{1}{3}\left(\frac{65^3 - 76895}{65}\right) \\
&= \frac{65^2 - 1183}{3} = \boxed{1014}.
\end{aligned}$$

Tiebreaker Round Problem 3: Let S be the set of all values of a such that the area of a triangle with side lengths 5, 7, and a is a positive integer. Find $\sum_{x \in S} \left(x^2\right)$.

Answer: $\boxed{2516}$

Solution: Letting 5 be the base of the triangle, we have a height of $\frac{2h}{5}$ for $0 < \frac{2h}{5} \leq 7 \implies h \in \{1, 2, \ldots, 17\}$. Note for each height, we have two posible values of a, if the triangle if acute or obtuse. We have

$$\begin{aligned} a_1^2 &= \left(5 - \sqrt{7^2 - \left(\frac{2h}{5}\right)^2}\right)^2 + \left(\frac{2h}{5}\right)^2 \\ &= 5^2 + \left(7^2 - \left(\frac{2h}{5}\right)^2\right) - 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} + \left(\frac{2h}{5}\right)^2 \\ &= 74 - 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \end{aligned}$$

for when the triangle is acute and

$$\begin{aligned} a_2^2 &= \left(5 + \sqrt{7^2 - \left(\frac{2h}{5}\right)^2}\right)^2 + \left(\frac{2h}{5}\right)^2 \\ &= 5^2 + \left(7^2 - \left(\frac{2h}{5}\right)^2\right) + 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} + \left(\frac{2h}{5}\right)^2 \\ &= 74 + 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \end{aligned}$$

for when the triangle is obtuse. We have

$$a_1^2 + a_2^2 = \left(74 - 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2}\right) + \left(74 + 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2}\right) = 148.$$

So,

$$\sum_{x \in S} \left(x^2\right) = \sum_{h \in \{1,2,\ldots,17\}} 148 = 148 \cdot 17 = \boxed{2516}.$$