

2024 SSMO Tiebreaker Round Solutions

SMO Team

Tiebreaker Round Problem 1: Compute the exact value of

$$2^2 + 0^2 + 2^2 + 4^2 + 20^2 + 22^2 + 24^2 + 40^2 + 42^2 + 202^2.$$

Answer: 45652

Solution: We have

$$\begin{aligned} 2^2 + 0^2 + 2^2 + 4^2 + 20^2 + 22^2 + 24^2 + 40^2 + 42^2 + 202^2 = \\ 4 + 0 + 4 + 16 + 400 + 484 + 576 + 1600 + 1764 + 40804 = \boxed{45652}. \end{aligned}$$

Tiebreaker Round Problem 2: Bob is attempting to shoot a 3-point throw. Bob attempts the basket 97 times. Each time, Bob has a 35% chance of making the shot. If S_1 denotes the expected number of points Bob will make and S_2 the number of points Bob is most likely to make, then $|S_1 - S_2| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 23

Solution: Firstly, the expected number of shots Bob makes is $\left(\frac{35}{100}\right) \cdot 97$. Now, the number of shots Bob is most likely to make is the largest integer a such that

$$\begin{aligned} \left(\frac{35}{100}\right)^a \left(\frac{65}{100}\right)^{97-a} \binom{97}{a} &\geq \left(\frac{35}{100}\right)^{a-1} \left(\frac{65}{100}\right)^{98-a} \binom{97}{a-1} \implies \\ 35(98-a) &\geq 65(a) \implies \\ 3430 &\geq 100a \implies \\ 34 &\geq a. \end{aligned}$$

So, Bob is most likely to make 34 shots. Therefore, the answer is

$$\left| 3 \left(\frac{35 \cdot 97}{100} - 34 \right) \right| = \frac{3}{20} \implies 20 + 3 = \boxed{23}.$$

Tiebreaker Round Problem 3: Let $A = \dots a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \dots$ be a terminating decimal. The length of A is defined to be the length of the shortest sub-sequence of consecutive digits that include all nonzero digits and at least one of a_0, a_{-1} . So, the length of 12.03 is 4 and the length of 0.123 is 3. Let $f(n)$ be the average of all numbers with a terminating decimal of length n . Find the value of $\left\lfloor \sum_{n=0}^{10} (n+1)f(n) \right\rfloor$.

Answer: 6790123448

Solution: First, we will compute the value of $f(n)$. Denote $f(n, k)$ as the expected value of a terminating decimal of length n with leading term a_k . We have

$$f(n) = \frac{\sum_{i=-1}^{n-1} f(n, i)}{n+1} \implies f(n) = \sum_{i=-1}^{n-1} f(n, i).$$

So,

$$\sum_{n=0}^{10} (n+1)f(n) = \sum_{n=0}^{10} \sum_{i=-1}^{n-1} f(n, i).$$

Now,

$$\begin{aligned} f(n, i) &= \mathbb{E}(a_i a_{i-1} \dots a_0 . a_{-1} \dots a_{i+1-n}) \\ &= \mathbb{E}\left(\sum_{k=i+1-n}^i a_k 10^k\right) \\ &= \sum_{k=i+1-n}^i \mathbb{E}(a_k 10^k) \\ &= \sum_{k=i+1-n}^i 10^k \mathbb{E}(a_k) \\ &= 10^i \mathbb{E}(a_i) + \sum_{k=i+1-n}^{i-1} 10^k \mathbb{E}(a_k) \\ &= 10^i \left(\frac{\sum_{k=1}^9 k}{9}\right) + \sum_{k=i+1-n}^{i-1} 10^k \mathbb{E}\left(\frac{\sum_{k=0}^9 k}{10}\right) \\ &= 5 \cdot 10^i + \sum_{k=i+1-n}^{i-1} (4.5 \cdot 10^k). \end{aligned}$$

Substituting, we have

$$\begin{aligned}
\sum_{n=1}^{10} (n+1)f(n) &= \sum_{n=1}^{10} \sum_{i=-1}^{n-1} f(n, i) \\
&= \sum_{n=1}^{10} \sum_{i=-1}^{n-1} \left(5 \cdot 10^i + \sum_{k=i+1-n}^{i-1} (4.5 \cdot 10^k) \right) \\
&= \sum_{n=1}^{10} \sum_{i=-1}^{n-1} (5 \cdot 10^i) + \sum_{n=1}^{10} \sum_{i=-1}^{n-1} \sum_{k=i+1-n}^{i-1} (4.5 \cdot 10^k) \\
&= \left(\sum_{i=-1}^9 ((50 - 5i) \cdot 10^i) - \frac{1}{2} \right) + \sum_{n=1}^{10} \sum_{i=-1}^{n-1} \left(4.5 \cdot \left(\frac{10^i - 10^{i+1-n}}{9} \right) \right) \\
&= \left(\sum_{i=-1}^9 ((50 - 5i) \cdot 10^i) - \frac{1}{2} \right) + \frac{1}{10} \sum_{n=1}^{10} \sum_{i=-1}^{n-1} (5 \cdot 10^i) - \sum_{n=1}^{10} \sum_{i=-1}^{n-1} \left(\frac{10^{i+1-n}}{2} \right) \\
&= \left(\sum_{i=-1}^9 ((50 - 5i) \cdot 10^i) - \frac{1}{2} \right) + \frac{1}{10} \left(\sum_{i=-1}^9 ((50 - 5i) \cdot 10^i) - \frac{1}{2} \right) \\
&\quad - \left(\sum_{i=-10}^0 \left(\frac{(i+11) \cdot 10^i}{2} \right) + \frac{1}{2} \right) \\
&= \frac{11}{10} \left(\sum_{i=-1}^9 ((50 - 5i) \cdot 10^i) \right) - \sum_{i=-10}^0 \left(\frac{(i+11) \cdot 10^i}{2} \right) - \frac{21}{20}.
\end{aligned}$$

Now, we will approximate this value. The first summation approximates to

$$\begin{array}{r}
5000000000.0 \\
1000000000.0 \\
150000000.0 \\
20000000.0 \\
2500000.0 \\
300000.0 \\
35000.0 \\
4000.0 \\
450.0 \\
+ \quad 50.0 \\
\hline
6172839505.5
\end{array}$$

Multiplying by $\frac{11}{10}$, we have

$$\begin{array}{r}
6172839505.50 \\
6790123456.05 \\
\hline
+ 6790123456.05
\end{array}$$

Now, the second summation is approximately

$$5.5 + 0.5 + 0.045 = 6.045.$$

Combining, we can approximate the expression as

$$6790123456.05 - 6.045 - 1.05 = 6790123448.955 \implies$$

$$\lfloor 6790123448.955 \rfloor = \boxed{6790123448}.$$