2021 WSMO Accuracy Round Solutions

SMO Team

AR 1: Let $x = \sqrt{69 + \sqrt{69 + \sqrt{69 \ldots}}}$. Find the value of $(2x-1)^2$.

Answer: $\boxed{277}$

Solution: We have

$$\begin{aligned} x &= \sqrt{69+x} \implies \\ x^2 &= 69 + x \implies \\ x^2 - x - 69 &= 0 \implies \\ x &= \frac{1 \pm \sqrt{277}}{2} \implies \\ (2x-1)^2 &= \left(2\left(\frac{1 \pm \sqrt{277}}{2}\right) - 1\right)^2 \\ &= \left(\pm\sqrt{277}\right)^2 = \boxed{277}. \end{aligned}$$

AR 2: When Bob is in precalculus, he gets bored and writes all the permutations in "precal". Since he is not very smart, it takes him 5 seconds to write each permutation. When Bob advances to calculus, he gets bored and writes all the permutations in "calculus". He is smart and can now write each permutation in 2 seconds. Find the positive difference in minutes between the time it takes for him to write the permutations of "precal" and "calculus".

Answer: $\boxed{108}$

Solution: There are 6! permuations of "precal", meaning it takes Bob

$$6! \cdot 5 = 720 \cdot 5 = 3600 \text{ seconds} \implies \frac{3600}{60} = 60 \text{ minutes}$$

to write all the permutations. There are $\frac{8!}{2!2!2!} = 5040$ permutations of "calculus", meaning it takes Bob

$$5040 \cdot 2 = 10080 \text{ seconds} \implies \frac{10080}{60} = 168 \text{ minutes}$$

to write all the permutations. Thus, our answer is

$$168 - 60 = \boxed{108}.$$

AR 3: $f(x) = x^3 - 8x^2 + 10x - 4$ has complex roots a, b, c. Denote $P(n) = a^n + b^n + c^n$. Find $P(-1)P(0)P(1)$.

Answer: $\boxed{60}$

Solution: We have

$$\begin{aligned}
P(-1)P(0)P(1) &= \left(a^{-1}+b^{-1}+c^{-1}\right)\left(a^{0}+b^{0}+c^{0}\right)\left(a^{1}+b^{1}+c^{1}\right) \\
&= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1+1+1)(a+b+c) \\
&= \frac{3(ab+ac+bc)(a+b+c)}{abc} \\
&= \frac{3(10)(8)}{4} = \boxed{60}.
\end{aligned}$$

AR 4: Bob and his 3 friends are standing in a line of 10 people. Given that Bob is not on either end of the line, then the probability the person in front and behind Bob are both his friends is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{13}$

Solution: The probability the person in front of Bob is his friend is $\frac{3}{9}$. The probability that the person behind Bob is his friend given that the persin in front of Bob is his friend $\frac{2}{8}$. So, our answer is

$$\left(\frac{3}{9}\right)\left(\frac{2}{8}\right) = \frac{6}{72} = \frac{1}{12} \implies 1+12 = \boxed{13}.$$

AR 5: Bob flips $x+1$ coins and Bobby flips x coins, where x is a random integer chosen between the range of $[27, 100]$. The expected probability that Bob gets more heads than Bobby is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{3}$

Solution: We will count the number of outcomes in which Bob and Bobby flip coins such that Bob gets more heads than Bobby, and divide that by the total number of possible outcomes, $2^{x+1}\cdot 2^{x} = 2^{2x+1}$. Consider Bob's and Bobby's first x flips. There are a total of $2^{x}\cdot 2^{x} = 2^{2x}$ possible outcomes. Among these,

$$\sum_{i=0}^{x}\binom{x}{i}\binom{x}{i} = \sum_{i=0}^{x}\binom{x}{i}\binom{x}{x-i} = \binom{2x}{x}$$

are the outcomes in which Bob and Bobby have the same number of heads. Now, we consider Bob's final (last) coin flip. For these $\binom{2x}{x}$ outcomes, Bob must flip a head to end up with more heads than Bobby. For the remaining $2^{2x}-\binom{2x}{x}$ outcomes, in exactly half of them Bob already has more heads than Bobby, and in the other half, Bobby has more. If Bob already has more heads, then regardless of the final flip, he will still have more heads. If he has fewer heads, then the final flip won't change that. So, the number of favorable outcomes is

$$\binom{2x}{x} + \frac{2^{2x}-\binom{2x}{x}}{2}\cdot 2 = 2^{2x}.$$

Therefore, the probability that Bob ends up with more heads than Bobby is

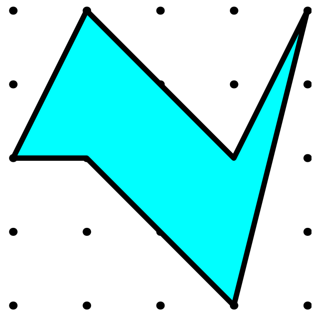
$$\frac{2^{2x}}{2^{2x+1}} = \frac{1}{2}.$$

So, the expected probability over x from $[27, 100]$ is $\frac{1}{2} \implies 1+2 = 3$.

AR 6: In quadrilateral $ABCD$, there exists a point O such that $AO = BO = CO = DO$ and $\angle(AOB) + \angle(COD) = 120^\circ$. Let K, L, M, N be the foot of the perpendiculars from A to BD, B to AC, C to BD, and D to AC. If $[ABCD] = 20$, find $([KLMN])^2$.

Answer: $\boxed{25}$

Solution:



The existence of point O implies that $ABCD$ is a cyclic quadrilateral. Now, we have

$$\begin{aligned}
\angle(AXB) &= 180 - \angle(AXD) \\
&= 180 - (180 - \angle(XAD) - \angle(XDA)) \\
&= \angle(XAD) + \angle(XDA) \\
&= \angle(CAD) + \angle(BDA) \\
&= \frac{\overset{\frown}{CD}}{2} + \frac{\overset{\frown}{AB}}{2} \\
&= \frac{\angle(COD)}{2} + \frac{\angle(AOB)}{2} \\
&= \frac{120}{2} = 60^\circ.
\end{aligned}$$

So,

$$\begin{aligned}
KM &= KX + XM \\
&= AX\cos 60^\circ + XC\cos^\circ \\
&= \frac{AX}{2} + \frac{XC}{2} \\
&= \frac{AC}{2}.
\end{aligned}$$

In the same manner, we have $LN = \frac{BD}{2}$. We have

$$\begin{aligned}
[KLMN] &= \frac{1}{2}(KM)(LN)\sin\angle(LXK) \\
&= \frac{1}{2}\left(\frac{AC}{2}\right)\left(\frac{BD}{2}\right)\sin\angle(AXB) \\
&= \frac{1}{4}\left(\frac{1}{2}(AC)(BD)\sin\angle(AXB)\right) \\
&= \frac{[ABCD]}{4} \implies \\
[KLMN] &= \frac{20}{4} = 5 \implies \\
([KLMN]) &= 5^2 = \boxed{25}.
\end{aligned}$$

AR 7: How many ordered triplets of integers (a, b, c) satisfy $a^2 + 2ab + b^2 = c^2 - 6c + 9$ and $-2 \le a, b, c \le 7$?

Answer: $\boxed{95}$

Solution: Taking the square root, we have

$$|a + b| = |c - 3| = d.$$

For each of the 9 values of $c \neq 3$, there are 10 ordered pairs (a, b) such that $|a + b| = d \neq 0$. For $c = 3$, there are only 5 ordered pairs (a, b) such that $|a + b| = d = 0$. So, there are a total of

$$9 \cdot 10 + 5 = \boxed{95}$$

such ordered triplets.

AR 8: Let $f(x) = x^3 - 3x^2 + 4x + 5$ have complex roots a, b, c. Then,

$$\frac{1}{a^2 + b^2} + \frac{1}{a^2 + c^2} + \frac{1}{b^2 + c^2}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{68}$

Solution: Note that

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc) = 3^2 - 2 \cdot 4 = 1.$$

So,

$$\frac{1}{a^2 + b^2} + \frac{1}{a^2 + c^2} + \frac{1}{b^2 + c^2} = \frac{1}{1 - c^2} + \frac{1}{1 - b^2} + \frac{1}{1 - a^2} = \frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{c_1}$$

where

$$a_1 = 1 - a^2, \quad b_1 = 1 - b^2, \quad c_1 = 1 - c^2.$$

Note that a_1, b_1, c_1 are the roots of $f(\sqrt{1 - x})$. Solving, we have

$$\begin{aligned}
\left(\sqrt{1-x}\right)^3 - 3\left(\sqrt{1-x}\right)^2 + 4\left(\sqrt{1-x}\right) + 5 = 0 \\
(1 - x)\sqrt{1 - x} + 4\sqrt{1 - x} &= 3(1 - x) - 5 \\
(5 - x)\sqrt{1 - x} &= -3x - 2 \\
(5 - x)^2(1 - x)^2 &= (-3x - 2)^2 \\
-x^3 + 11x^2 - 35x + 25 &= 9x^2 + 12x + 4 \\
x^3 - 2x^2 + 47x - 21 &= 0.
\end{aligned}$$

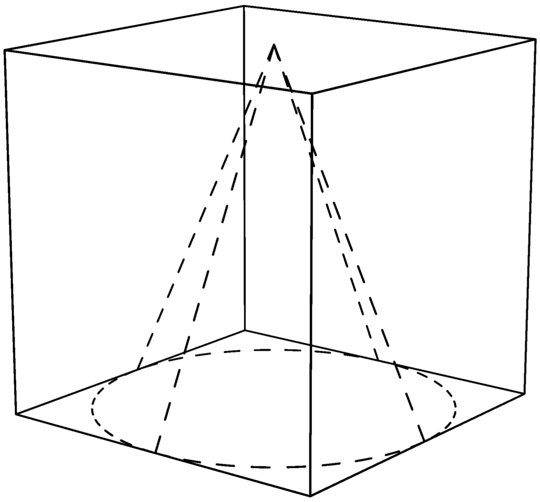
So,

$$\frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{c_1} = \frac{b_1c_1 + a_1c_1 + a_1b_1}{a_1b_1c_1} = \frac{-47}{-21} = \frac{47}{21} \implies 47 + 21 = \boxed{68}.$$

AR 9: Given circles ω_1, ω_2 with radius $1, 4$ respectively, they are externally tangent to each other. The diameters of ω_1 , ω_2 are AB, CD respectively, satisfying $AB \parallel CD$ and BD is an external tangent of the circles. The third circle ω_3 passes through A, C and is tangent to BD. The minimum possible value of the radius of ω_3 can be expressed as $\frac{a+b\sqrt{c}}{d}$, where c is a squarefree positive integer and a, b, d are relatively prime positive integers. Find $a+b+c+d$.

Answer: $\boxed{79}$

Solution:



We have

$$BD = \sqrt{(4+1)^2 - (4-1)^2} = 4.$$

Let $B = (0,0)$ and $D = (4,0)$. This means $A = (0,2)$ and $C = (4,8)$. Suppose the center of ω_3 is (a,b). Since ω_3 is tangent to BD, the radius of ω_3 must be b. Since A, C lie on ω_3, we have

$$(a^2 + (b-2)^2 = (a-4)^2 + (b-8)^2 = b^2.$$

From

$$a^2 + (b-2)62 = (a-4)^2 + (b-8)^2$$

we have

$$8a + 12b = 76 \implies a = \frac{19}{2} - \frac{3}{2}b.$$

Substituting into

$$a^2 + (b-2)^2 = b^2$$

we have

$$\frac{9}{4}b^2 - \frac{65}{2}b + \frac{377}{4} \implies b = \frac{65 - 8\sqrt{13}}{9} \implies 65 - 8 + 13 + 9 = \boxed{79}.$$

AR 10: In tetrahedron T of side length 12, let S_1 be the sphere inscribed in T and let S_2 be the sphere circumscribed around T. Let R be a rectangular prism such that all points on S_1 lie strictly inside or are touching R and all points on R lie strictly inside or are touching S_2. The minimum possible volume of R is $m\sqrt{n}$, where n is a squarefree positive integer. Find $m + n$.

Answer: $\boxed{54}$

Solution: The height of T is $4\sqrt{6}$, and the inradius is a fourth of that, which is $\sqrt{6}$. This means each dimension of R has to be at least $2\sqrt{6}$. Take a cube of side $2\sqrt{6}$ concentric with S_1. This cube completely contains S_1 and lies inside S_2. Thus, our answer is

$$2\sqrt{6} \cdot 2\sqrt{6} \cdot 2\sqrt{6} = 48\sqrt{6} \implies 48 + 6 = \boxed{54}.$$