2022 SSMO Accuracy Round

SMO Team

AR 1: Consider a bijective function (meaning each element in the domain maps to a distinct element in the range) $f : S \to S$, where $S = \{1, 2, 3, 4, 5\}$. What is the average of $f(1) + f(2) + f(3)$, over all f?

AR 2: Consider a cone with radius 5 and height 12, and a point P in the same plane as the base of the cone, but a distance of 10 from the center of the base of the cone. We rotate the cone 360° about P such that the plane that the base of the cone lies on stays the same. The volume of the region that the cone sweeps out can be expressed as $m\pi$. Find m.

AR 3: Let $A = (6, 3, 2), B = (2, -9, -6)$, and $O = (0, 0, 0)$. Suppose that D is a point in space such that OD bisects $\angle AOB$ and O, D, A, B are coplanar. In addition, $\angle DAO = 90^\circ$. If DO can be expressed as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is squarefree, find $a + b + c$.

AR 4: A monic polynomial f has real roots r, s, t. A monic polynomial g has roots r^3, s^3, t^3. Given that the minimum possible value of $\frac{g(1)}{f(1)}$ is $\frac{m}{n}$, for relatively prime positive integers m and n, find $m + n$.

AR 5: Find the number of ordered pairs (a, b), where $1 \le a, b \le 9$, for which the largest integer n that satisfies

$$(a - b)^k \equiv a^k - b^k \pmod{n}$$

for all $k \ge 1$ is $ab - b^2$.

AR 6: Consider an unfair 6-sided die labeled from 1 to 6, such that the probability of rolling a number m is directly proportional to $7 - m$. However, if we roll any number n, then the probability of rolling a number less than n becomes 0, and the probability of rolling any number m from n to 6 inclusive remains directly proportional to $7 - m$. The expected number of rolls until a 6 is rolled can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

AR 7: After a robber drives in a car for t (not necessarily integral) minutes, the car goes at $120 - t$ miles per hour. Whenever the car's speed drops below 60 miles per hour, the robber switches into a new car with no time loss. A police car can drive at a constant speed of 117 miles per hour. Given that the robber starts 1 hour before the police car, how many minutes will pass between when the police car starts and when the police car catches up to the robber?

AR 8: Let $ABCD$ be a trapezoid with $AB \parallel CD$. Suppose that $AD = 1$, $DC = 4$, $CB = 2$, and $AB < CD$. Let X be the midpoint of AB. If E is the intersection of AC and BD, and $\angle XEB = \angle ADC$, then $AB = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

AR 9: The graph Γ has 52 vertices labeled $A, A', B, B', \ldots, Z, Z'$, such that A is not connected to A', B is not connected to B', and so on. Suppose that all the vertices other than A' have different degrees (number of connections to the vertex). Find the sum of all possible values for the number of edges (connections) in Γ.

AR 10: Let

$$S = \sum_{k=5}^{\infty} 2^{-10k} \binom{2k}{10}.$$

Then the value of S can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the largest positive integer a such that $2^a \mid mn$.

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