2024 SSMO Team Round Solutions

SMO Team

TR 1: Find the number of ordered triples of positive integers (a, b, c) that satisfy the equation
$$2(a^b)^c + 1 = 513.$$

Answer: $\boxed{10}$

Solution: We have

$$\begin{aligned} 2(a^b)^c + 1 &= 513 \\ \implies a^{bc} = 256 &\implies a = 2^{a_1} \mid a_1 \in \mathbb{Z}_{\geq 0} \\ \implies 2^{a_1 bc} &= 2^8 \\ \implies a_1 bc = 2^3 &\implies a_1 = 2^{a_2}, b = 2^{b_1}, c = 2^{c_1} \mid a_2, b_1, c_1 \in \mathbb{Z}_{\geq 0} \\ \implies 2^{a_2+b_1+c_1} &= 2^3 \\ \implies a_2 + b_1 + c_1 &= 3. \end{aligned}$$

From the Hockey Stick Identity, it follows that this equation has $\binom{5}{2} = \boxed{10}$ solutions.

TR 2: Find the sum of the three smallest positive integers n where the last two digits of n^4 are 01.

Answer: $\boxed{51}$

Solution: From $n^4 \equiv 1 \pmod{100}$, we have $n^4 \equiv 1 \pmod 4$ and $n^4 \equiv 1 \pmod{25}$. From the first congruence, we have $n \equiv 1 \pmod 2$. Now, let $n \equiv 5a + b \pmod{25}$ for integers $0 \leq a, b < 5$. Then,

$$(5a + b)^4 = 5^4a^4 + 4 \cdot 5^3a^3b + 6 \cdot 5^2a^2b^2 + 4 \cdot 5ab^3 + b^4 \equiv 20ab^3 + b^4 \pmod{25}$$

so
$$20ab^3 + b^4 \equiv 1 \pmod{25}.$$

Now, we will proceed using casework. Clearly, b is not a multiple of 5. Now, note that

$$\begin{aligned} b \equiv 1 \pmod 5 &\implies a \equiv 0 \pmod 5, \\ b \equiv 2 \pmod 5 &\implies a \equiv 1 \pmod 5, \\ b \equiv 3 \pmod 5 &\implies a \equiv 3 \pmod 5, \text{ and} \\ b \equiv 4 \pmod 5 &\implies a \equiv 4 \pmod 5. \end{aligned}$$

So, we have $5a+b \in \{1, 7, 18, 24\}$. Combining this with $n \equiv 1 \pmod 2$, we find the following solutions as residues mod 50: $1, 7, 43$, and 49. So, the answer is $1 + 7 + 43 = \boxed{51}$.

TR 3: Consider positive integers N such that when N's units digit and leading nonzero digit are removed, what remains is a two-digit perfect square. The average of all N can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers Find $m+n$.

Answer: $\boxed{32733}$

Solution: Note that the two-digit perfect square obtained from removing the leading and units digit has to be in the set $\{16, 25, \ldots, 81\}$. Taking the expected value of N, we have

$$\mathbb{E}(N) = \mathbb{E}(i \in \{1, 2, \ldots, 9\}) \cdot 1000 + \left(\frac{4^2 + 5^2 + \cdots + 9^2}{6}\right) \cdot 10 + \mathbb{E}(i \in \{0, 1, 2 \ldots, 9\})$$
$$= 5 \cdot 1000 + \left(\frac{271}{6}\right) \cdot 10 + \frac{9}{2} = \frac{32737}{6} \implies 32737 + 6 = \boxed{32733}.$$

TR 4: Let ABC be a right triangle with circumcenter O and incenter I such that $\angle ABC = 90^\circ$ and $\frac{AB}{BC} = \frac{3}{4}$. Let D the projection of O onto AB, and let E be the projection of O onto BC. Denote ω_1 be the incenter of ADO and ω_2 as the incenter of OEC. The value of $\frac{[\omega_1 \omega_2 I]}{[ABC]}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{53}$

Solution: We will use analytic geometry. WLOG, let $B = (0,0), A = (0,6), C = (8,0)$. without loss of generality due to rigid transformations or dilations. Since the area of ABC is $\frac{1}{2} \cdot 8 \cdot 6 = 24$, the inradius is $\frac{2 \cdot 24}{6+8+10} = 2$, meaning $I = (2,2)$. Since ABC is a right triangle, the O must be the midpoint of AC. So, D and E are the midpoints of AB and BC, respectively. Now, as ADO and OEC are both similar to ABC, we can easily compute $\omega_1 = (1,4)$ and $\omega_2 = (5,1)$. From the shoelace theorem, the area of $\omega_1 \omega_2 I$ is $\frac{5}{2}$, meaning our answer is

$$\frac{\frac{5}{2}}{24} = \frac{5}{48} \implies 5 + 48 = \boxed{53}.$$

TR 5: Let ABC be a triangle with $AB = AC = 5$ and $BC = 6$. Let ω_1 be the circumcircle of ABC and let ω_2 be the circle externally tangent to ω_1 and tangent to rays AB and AC. The distance between the centers of ω_1 and ω_2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{27}$

Solution: Let D be the midpoint of BC and let E be the point at which ω_1 and ω_2 are tangent. By symmetry, O_1 and O_2, the centers of ω_1 and ω_2, are on line ADE. Now, $\triangle ADC \sim \triangle ACE$, so the radius of ω_1 is

$$r_1 = \frac{5 \cdot \frac{5}{4}}{2} = \frac{25}{8}.$$

If F is the point at which ω_2 and AC are tangent, then AO_2F is a 3-4-5 right triangle. If r_2 is the radius of ω_2, we find that

$$\frac{r_2}{r_2 + \frac{25}{4}} = \frac{3}{5}$$

so $r_2 = \frac{75}{8}$. Therefore, the final answer is

$$r_1 + r_2 = \frac{25}{2} \implies 25 + 2 = \boxed{27}.$$

SSMO
SUMMER SOLSTICE MATH OLYMPIAD

TR 6: Let α, β, and γ be the roots of the polynomial $x^3-6x^2-19x-n$. If n is an integer, what is the least possible positive value of $\alpha^3+\beta^3+\gamma^3$?

Answer: $\boxed{3}$

Solution: From Vieta's Formulas, we have

$$\alpha+\beta+\gamma=6, \alpha\beta+\alpha\gamma+\beta\gamma=-19, \text{ and } \alpha\beta\gamma=n.$$

Since $x^3-6x^2-19x-n=0$ for $x\in\{\alpha,\beta,\gamma\}$, we have $x^3=6x^2+19x+n$. So,

$$\begin{aligned}\alpha^3+\beta^3+\gamma^3 &= (6\alpha^2+19\alpha+n)+(6\beta^2+19\beta+n)\\ &+(6\gamma^2+19\gamma+n)\\ &=6(\alpha^2+\beta^2+\gamma^2)+19(\alpha+\beta+\gamma)+3n\\ &=6((\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma))\\ &+19(\alpha+\beta+\gamma)+3n\\ &=6((6^2)-2(-19))+19(6)+3n=558+3n.\end{aligned}$$

Since n is an integer and we are seeking to find the least positive value of $558+3n=3(n+186)$, we let $n=-185$, giving an answer of $\boxed{3}$.

TR 7: Let a and b be real numbers that satisfy

$$a^3+8ab^2=8b^3+4a^2b=375.$$

Find $\lfloor ab\rfloor$.

Answer: $\boxed{12}$ We can factor the first equation as such:

$$\begin{aligned}a^3+8ab^2=8b^3+4a^2b &\implies\\ a^3-8b^3=4a^2b-8ab^2 &\implies\\ (a-2b)(a^2+2ab+4b^2)=4ab(a-2b) &\implies\\ (a-2b)(a^2-2ab+4b^2)=0.&\end{aligned}$$

Consider the case where $a-2b$ is nonzero. Then, the discriminant of the quadratic factor (in a) is $(2b)^2-4(4b^2)=-12b^2$. Since a and b are reals, this means $b=0$ which is clearly not possible. So, $a=2b$. Substituting, we have

$$8b^3+16b^3=375 \implies b^3=\frac{125}{8} \implies b=\frac{5}{2}.So,$$

$\lfloor ab\rfloor=\left\lfloor(5)\left(\frac{5}{2}\right)\right\rfloor=\left\lfloor\frac{25}{2}\right\rfloor=\boxed{12}$.

TR 8: Three integers $0\le a\le b\le c<229$ satisfy the congruence $n^3\equiv 1 \pmod{229}$. Given that 71^2-3 and 107^2+1 are both multiples of 229, find the value of b.

Answer: $\boxed{94}$

Solution: Note that $n^3\equiv 1 \pmod{229} \implies (n^3-1)\equiv 0 \pmod{229}$. Consider the complex third roots of unity 1 and $\frac{-1\pm i\sqrt{3}}{2}$. They can be defined based on $\frac{1}{2},\sqrt{-1}$, and $\sqrt{3}$. From the given multiples of 229, we can $\sqrt{3}\equiv 71 \pmod{229}$ and $i\equiv 107 \pmod{229}$

since they function like square roots. In addition, 229 is odd so 2^{-1} exists. We have $i\sqrt{3} \equiv 107 \cdot 71 \equiv 269 \pmod{229}$. Thus, $\frac{-1+i\sqrt{3}}{2} \equiv 134 \pmod{229}$ and $\frac{-1-i\sqrt{3}}{2} \equiv 94 \pmod{229}$. So, $a = 1, b = \boxed{94}$, and $c = 134$.

TR 9: Let $ABCDEFGH$ be an equiangular octagon such that $AB = 6, BC = 8, CD = 10, DE = 12, EF = 6, FG = 8, GH = 10$, and $AH = 12$. The radius of the largest circle that fits inside the octagon can be expressed as $a + b\sqrt{c}$, where c is a squarefree positive integer. Find $a + b + c$.

Answer: $\boxed{10}$

Solution: Note that $AB \parallel EF, BC \parallel FG, CD \parallel GH$, and $DE \parallel HA$. Thus, the diameter of the circle is the least distance between any pair of these opposite parallel sides. Note that a side of length x oriented at a 45-degree angle relative to a pair of opposite parallel sides contributes $\frac{x}{\sqrt{2}}$ to the distance and a side of length x oriented perpendicularly contributes x to the distance. Obviously, we only have to add up the contributions of one run of three sides; for example for AB and EF we only have to consider sides from B to E or F to A. However, it doesn't matter which run we pick since opposite sides are congruent. Let the distances be a, b, c, and d respectively, with the order given in the first sentence. We have

$$\begin{aligned} a &= 10 + \frac{8}{\sqrt{2}} + \frac{12}{\sqrt{2}} = 10 + 10\sqrt{2}, \\ b &= 12 + \frac{10}{\sqrt{2}} + \frac{6}{\sqrt{2}} = 12 + 8\sqrt{2}, \\ c &= 6 + \frac{12}{\sqrt{2}} + \frac{8}{\sqrt{2}} = 6 + 10\sqrt{2}, \text{ and} \\ d &= 8 + \frac{6}{\sqrt{2}} + \frac{10}{\sqrt{2}} = 8 + 8\sqrt{2}. \end{aligned}$$

Obviously, $10+10\sqrt{2} \geq 8+8\sqrt{2}$ and $12+8\sqrt{2} \geq 8+8\sqrt{2}$. We also have $6+10\sqrt{2} \geq 8+8\sqrt{2}$ since $\sqrt{2} \geq 1$. So, the least distance between pairs of opposite parallel sides is $8 + 8\sqrt{2}$ and the radius of the largest circle that fits inside the octagon is $4 + 4\sqrt{2}$. Thus, the answer is $4 + 4 + 2 = \boxed{10}$.

TR 10: The side-lengths of a convex cyclic quadrilateral $ABCD$ are integers and satisfy

$$(AB \cdot AD + BC \cdot CD)^2 = AC^2 \cdot BD^2 - 72.$$

Find the perimeter of $ABCD$.

Answer: $\boxed{37}$ Let $AB = a, BC = b, CD = c, DA = d$. From Ptolemy's Theorem, we have $BC \cdot CD = ac + bd$. So, the equation is equivalent to

$$\begin{aligned} (ad + bc)^2 = (ac + bd)^2 - 72 &\implies \\ 72 = a^2c^2 + b^2d^2 - a^2d^2 - b^2c^2 &\implies \\ 72 = (a^2 - b^2)(c^2 - d^2). \end{aligned}$$

WLOG, assume that $a^2 - b^2 > c^2 - d^2$. Since the difference of two perfect squares cannot be $2 \pmod 4$, we have $(a^2-b^2, c^2-d^2) = (72, 1), (24, 3), (8, 9)$. Clearly, $(72, 1)$ has no solutions, as $c^2 - d^2 = 1 \implies c = 1, d = 0$. For $(24, 3)$, we have $c^2 - d^2 = 3 \implies c = 2, d = 1$

and $a^2 - b^2 = 24 \implies (a-b)(a+b) = 24$. This means $(a-b, a+b) = (2,12),(4,6) \implies (a,b) = (7,5),(5,1)$. Finally, for $(8,9)$, we have $c^2 - d^2 = 9 \implies c = 5, d = 4$ and $a^2 - b^2 = 8 \implies a = 3, b = 1$. This gives $(a,b,c,d) = (7,5,2,1),(5,1,2,1),(3,1,5,4)$. It is easy to verify that all three of these possibilities work, giving an answer of

$(7+5+2+1)+(5+1+2+1)+(3+1+5+4) = 15+9+13 = \boxed{37}$.

TR 11: Let S denote the set of positive divisors of 5400. Let

$$S_i = \{d \mid d \in S,\ d \equiv i4\}$$

and let s_i denote the sum of all elements of S_i. Find the value of

$$s_0^2 + s_1^2 + s_2^2 + s_3^2 - 2s_0s_2 - 2s_1s_3.$$

Answer: $\boxed{13020}$

Solution: Note that $5400 = 2^3 \cdot 3^3 \cdot 5^2$. Firstly, we have

s$_0^2$+s_1^2+s_2^2+s_3^2−$2s_0s_2$−$2s_1s_3 = (s_0-s_2)^2+(s_1-s_3)^2.Now, lets(n)denotethesumofpositivedivisorsof$n.$Wehav$
$s_2 = (8+4-2)s(3^3 \cdot 5^2) = 10s(3^3 \cdot 5^2) = 10s(3^3)s(5^2) = 400s(5^2).Inaddition,s_3 - s_1 = (27 - 9+3-1)s(5^2) = 20s(5^2).Itiseasytocomputes(5^2) = 31$. So, our answer is $420s(5^2) = \boxed{13020}$.

TR 12: What is the smallest positive integer n with 3 positive prime factors such that for all integers k, $k^n \equiv k \pmod{n}$?

Answer: $\boxed{561}$

Solution: Let the prime factors of n be p, q, and r. For this to be true, note that $p-1|n-1$ and similarly with q and r. Obviously n has to be squarefree, so $n = pqr$. WLOG let $p < q < r$. It is easy to see that if $p = 2$ is one of the factors we have a contradiction. We can arrive at similar contradictions with pairs of factors such as $(3,7),(3,13),(3,19)$ and $(5,11)$. This eliminates most possibilities, so we can manually check triples now. $(3,5,17)$ does not work, and neither does $(5,7,13)$. Finally, we note that $(3,11,17)$ does indeed work, and our answer is $n = 3 \cdot 11 \cdot 17 = \boxed{561}$.

TR 13: In a deck of 54 cards (2 identical jokers, 4 identical cards with $1,2,3,\ldots,13$), each card is dealt to one of 3 people, each having a $\frac{1}{3}$ chance of receiving each card. The expected sum of the number of unique cards the three of them have can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{917}$

Solution: Let $\mathbb{E}(\text{rank})$ denote the expected number of people that have the rank rank for rank $\in \{1,2,3\ldots,13\}$. Consider the cards distinguishable for probability calculation purposes. There are 3 ways for one person to have it, $3!\cdot 4+3\cdot\binom{4}{2}$ ways for two people to have it (3/1 or 2/2 split), and $4\cdot 3\cdot 3$ ways for three people to have it. There are 81 total (equally likely) distributions of rank among the 3 people, so $\mathbb{E}(\text{rank})$ is simply

$$\frac{3\cdot 1+(24+18)\cdot 2+(36)\cdot 3}{81}=\frac{65}{27}.$$

For rank = joker, using a similar counting process, we have $\mathbb{E}(\text{rank})=\frac{3\cdot 1+6\cdot 2}{9}=\frac{5}{3}$. So, our answer is

$$\frac{5}{3}+13\cdot\frac{65}{27}=\frac{890}{27}\implies 890+27=\boxed{917}.$$

TR 14: Let $a_1,a_2,\ldots,a_7$ be the roots of the polynomial

$$x^7+5x^6+9x^5+x^4+x^3+10x^2+5x+1.$$

Find the value of

$$\left|\prod_{n=1}^{7}\prod_{m=n+1}^{7}(a_na_m-1)\right|.$$

Answer: $\boxed{12}$

Solution: Let $f(x)$ denote the polynomial. We have

$$\left(\prod_{n=1}^{7}\prod_{m=n+1}^{7}(a_na_m-1)\right)^2\prod_{n=1}^{7}(a_n^2-1)=\prod_{n=1}^{7}\prod_{m=1}^{7}(a_na_m-1),$$

since $1\le n\le 7, n+1\le m\le 7$ covers the $m>n$ case of $1\le m,n\le 7$, (which is the domain of the RHS product) squaring the nested product doubles it, covering the symmetric $n>m$ case, and the third factor covers the $m=n$ case. Now, note that

$$\begin{aligned}\prod_{m=1}^{7}(a_na_m-1)&=-a_n^7\prod_{m=1}^{7}\left(\frac{1}{a_n}-a_m\right)\\&=-a_n^7f\left(\frac{1}{a_n}\right)\\&=-a_n^7-(5a_n^6+10a_n^5+a_n^4+a_n^3+9a_n^2+5a_n+1)\\&=(5a_n^6+9a_n^5+a_n^4+a_n^3+10a_n^2+5a_n+1)-(5a_n^6+10a_n^5+a_n^4+a_n^3+9a_n^2+5a_n+1)\\&=-a_n^5+a_n^2\\&=-(a_n)^2(a_n^3-1).\end{aligned}$$

So,

$$\begin{aligned}\prod_{n=1}^{7}\prod_{n=1}^{7}(a_na_m-1)&=\prod_{n=1}^{7}\left(-(a_n)^2(a_n^3-1)\right)\\&=-\left(\prod_{n=1}^{7}a_n\right)^2\prod_{n=1}^{7}(a_n-1)\prod_{n=1}(a_n-e^{\frac{2\pi i}{3}})\prod_{n=1}(a_n-e^{\frac{4\pi i}{3}})\\&=-(-1)^2(-f(1))\left(-f\left(e^{\frac{2\pi i}{3}}\right)\right)\left(-f\left(e^{\frac{4\pi i}{3}}\right)\right)\\&=f(1)f\left(e^{\frac{2\pi i}{3}}\right)f\left(e^{\frac{4\pi i}{3}}\right).\end{aligned}$$

For $\omega^3 = 1$ and $\omega \neq 1$, $f(\omega) = 19\omega^2 + 7\omega + 7 = 12\omega^2$. So,

$$f\left(e^{\frac{2\pi i}{3}}\right) = 12\left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{-12-12i\sqrt{3}}{2} = -6-6i\sqrt{3} \text{ and}$$
$$f\left(e^{\frac{4\pi i}{3}}\right) = 12\left(\frac{-1-i\sqrt{3}}{2}\right)^2 = \frac{-12+12i\sqrt{3}}{2} = -6+6i\sqrt{3} \implies$$
$$f\left(e^{\frac{2\pi i}{3}}\right) f\left(e^{\frac{4\pi i}{3}}\right) = (-6-6i\sqrt{3})(-6+6i\sqrt{3}) = 144.$$

Thus,

$$\prod_{n=1}^{7}\prod_{n=1}^{7}(a_n a_m - 1) = f(1) f\left(e^{\frac{2\pi i}{3}}\right) f\left(e^{\frac{4\pi i}{3}}\right) = 33 \cdot 144.$$

Now, we have

$$\begin{aligned}\prod_{n=1}^{7}(a_n^2 - 1) &= \prod_{n=1}^{7}(a_n - 1)\prod_{n-1}^{7}(a_n + 1) \\ &= (-f(1))(-f(-1)) \\ &= f(1)f(-1) \\ &= 33 \cdot 1 = 33.\end{aligned}$$

Substituting this into

$\left(\prod_{n=1}^{7}\prod_{m=n+1}^{7}(a_n a_m - 1)\right)^2 \prod_{n=1}^{7}(a_n^2 - 1) = \prod_{n=1}^{7}\prod_{m=1}^{7}(a_n a_m - 1), wehave \left(\prod_{n=1}^{7}\prod_{m=n+1}^{7}(a_n a_m - 1)\right)^2 = \frac{33\cdot 144}{33} = 144 \implies \left|\prod_{n=1}^{7}\prod_{m=n+1}^{7}(a_n a_m - 1)\right| = \sqrt{144} \implies \boxed{12}$.

TR 15: In triangle ABC inscribed in circle ω, let M be the midpoint of BC. Denote P as the intersection of AM with ω. If $BP = 9, CP = 13$, and $AM = 20$, find the perimeter of triangle ABC.

Answer: $\boxed{64}$

Solution: We will use barycentric coordinates with ABC as the reference triangle. Let $A = (1:0:0), B = (0:1:0), C(0:0:1)$. Note that the circumcircle of ABC can be represented as $a^2yx + b^2xz + c^2xy = 0$. Since P lies on cevian AM, with $M = \left(0 : \frac{1}{2} : \frac{1}{2}\right)$, we have $P = (x_p : y_p : z_p)$, for $y_p = z_p$. Substituting into the equation for the circumcircle of ABC, we have

$$\begin{aligned}a^2y_pz_p + b^2x_pz_p + c^2x_py_p = 0 &\implies \\ a^2y_p^2 + b^2x_py_p + c^2x_py_p = 0 &\implies \\ a^2y_p = -(b^2+c^2)x_p &\implies \\ y_p = -\frac{b^2+c^2}{a^2}x_p.\end{aligned}$$

From $x_p + y_p + z_p = 1$, we have

$$x_p + \left(-\frac{(b^2+c^2)}{a^2}\right)x_p + \left(-\frac{(b^2+c^2)}{a^2}\right)x_p = 1 \implies x_p = \frac{a^2}{2b^2+2c^2-a^2}.$$

So, $P = \left(-\frac{a^2}{2b^2+2c^2-a^2} : \frac{b^2+c^2}{2b^2+2c^2-a^2} : \frac{b^2+c^2}{2b^2+2c^2-a^2}\right)$. Now, we have $\overrightarrow{PB} = (x_{pb} : y_{pb} : z_{pb}) = \left(\frac{a^2}{2b^2+2c^2-a^2} : \frac{b^2+c^2-a^2}{2b^2+2c^2-a^2} : -\frac{b^2+c^2}{2b^2+2c^2-a^2}\right)$. So,

$$\begin{aligned}
PB^2 &= |a^2 y_{pb} z_{pb} + b^2 x_{pb} z_{pb} + c^2 x_{pb} y_{pb}| \\
&= \left| a^2 \left(\frac{b^2+c^2-a^2}{2b^2+2c^2-a^2}\right)\left(-\frac{b^2+c^2}{2b^2+2c^2-a^2}\right) \right. \\
&\quad + b^2\left(\frac{a^2}{2b^2+2c^2-a^2}\right)\left(-\frac{b^2+c^2}{2b^2+2c^2-a^2}\right) \\
&\quad \left. + c^2\left(\frac{a^2}{2b^2+2c^2-a^2}\right)\left(\frac{b^2+c^2-a^2}{2b^2+2c^2-a^2}\right) \right| \\
&= \frac{|a^4b^2 - 2a^2b^4 - 2a^2b^2c^2|}{(2b^2+2c^2-a^2)^2} \\
&= \frac{a^2b^2(2b^2+2c^2-a^2)}{(2b^2+2c^2-a^2)^2} \\
&= \frac{a^2b^2}{2b^2+2c^2-a^2} \implies \\
PB &= \frac{ab}{\sqrt{2b^2+2c^2-a^2}}
\end{aligned}$$

In the same manner, we have

$$PC = \frac{ac}{\sqrt{2b^2+2c^2-a^2}}.$$

So, we have

$$AM = 20 \implies \frac{\sqrt{2b^2+2c^2-a^2}}{2} = 20 \implies \sqrt{2b^2+2c^2-a^2} = 40.$$

This means

$$ab = PB\sqrt{2b^2+2c^2-a^2} = 9 \cdot 40$$

and

$$ac = PC\sqrt{2b^2+2c^2-a^2} = 13 \cdot 40.$$

Solving for b and c, we have $b = \frac{9\cdot 40}{a}$, $c = \frac{13\cdot 40}{a}$. Substituting, we get

$$\begin{aligned}
\sqrt{2b^2+2c^2-a^2} &= 40 \\
2b^2+2c^2-a^2 &= 40^2 \implies \\
2\left(\left(\frac{9\cdot 40}{a}\right)^2 + \left(\frac{13\cdot 40}{a}\right)^2\right) - a^2 &= 40^2 \implies \\
a^4 + 40^2a^2 - 2\left((9\cdot 40)^2 + (13\cdot 40)^2\right) &= 0 \implies \\
a^2 &= \frac{-40^2 \pm \sqrt{40^4 + 8\left((9\cdot 40)^2 + (13\cdot 40)^2\right)}}{2} \\
&= (20)\left(\pm\sqrt{40^2 + 8(9^2+13^2)} - 40\right) \\
&= (20)(\pm 60 - 40) = 400, -2000 \implies \\
a &= 20.
\end{aligned}$$

So, $b = \frac{9\cdot 40}{a} = \frac{9\cdot 40}{20} = 18$ and $c = \frac{13\cdot 40}{20} = 26$. In conclusion, the perimeter of ABC is $20 + 18 + 26 = \boxed{64}$.

SSMO
SUMMER SOLSTICE MATH OLYMPIAD