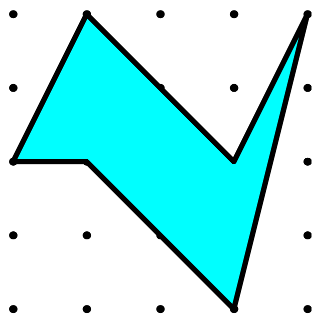
2021 WSMO Speed Round Solutions

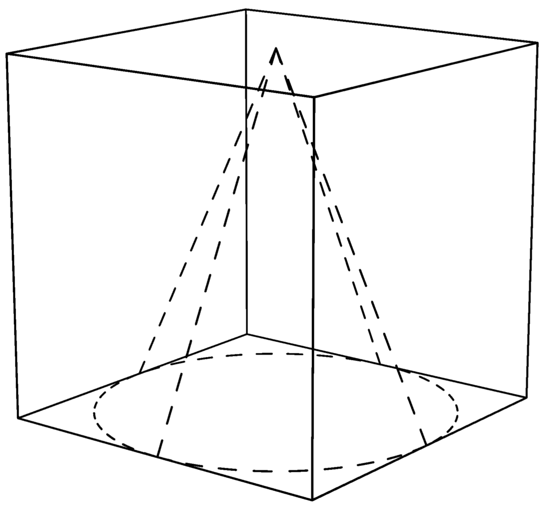
SMO Team

SR 1: Find the number of square units in the area of the shaded region.



Answer: $\boxed{6}$

Solution: We divide the shaded region into three sections as follows



The left triangle has an area of $\frac{1}{2} \cdot 1 \cdot 2 = 1$, the center parallelogram has an area of $2 \cdot 2 = 4$, and the right triangle has an area of $\frac{1}{2} \cdot 1 \cdot 2 = 1$. Thus, our answer is $1 + 4 + 1 = \boxed{6}$.

SR 2: There are 4 tables and 5 chairs at each table. Each chair seats 2 people. There are 10 people who are seated randomly. Andre and Emily are 2 of them, and are a couple. If the probability that Andre and Emily are in the same chair is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{40}$

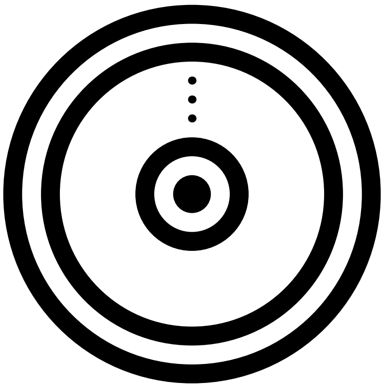
Solution: Note that there are $4 \cdot 5 \cdot 2 = 40$ possible places to seat. After Andre is assigned a seat at random, there are 39 remaining seats, only one of which is in the same chair as Emily. Thus, our answer is $\frac{1}{39} \implies 1 + 39 = \boxed{40}$.

SR 3: There are 6 pairs of socks for each color of the rainbow (red, orange, yellow, green, blue, indigo, violet) in a sock drawer. How many socks must be drawn from the drawer to guarantee that a pair of red socks have been drawn?

Answer: $\boxed{74}$

Solution: There are $6 \cdot 2 \cdot 6 = 72$ non-red socks. Thus, a pair of red socks is guaranteed after $72 + 2 = \boxed{74}$ socks have been drawn.

SR 4: A right circular cone is inscribed in a right prism as shown. If the ratio of the volume of the cone to the volume of the prism is $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: $\boxed{13}$

Solution: Let s and h denote the sidelength and height of the right prism, respectively. The ratio of the two volumes is equal to

$$\frac{\frac{1}{3} \cdot \left(\frac{s}{2}\right)^2 \cdot h\pi}{s^2 \cdot h} = \frac{\frac{\pi}{12} \cdot s^2 \cdot h}{s^2 \cdot h} = \frac{\pi}{12} \implies 1 + 12 = \boxed{13}.$$

SR 5: There exists a rational function $f(x)$ such that for all x in the range $(0, 1)$, $f(x) = \sum_{n=1}^{\infty} nx^n$. The maximum of $f(x)$ over $[6, 9]$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{31}$

Solution: We have

$$\begin{aligned} f(x) &= x + 2x^2 + 3x^3 + 4x^4 + \ldots \text{ and} \\ xf(x) &= x^2 + 2x^3 + 3x^4 + \ldots. \end{aligned}$$

So,

$$f(x) - xf(x) = x + x^2 + x^3 + \ldots = \frac{x}{1-x} \implies f(x) = \frac{x}{(1-x)^2}$$

for x in the range $(0, 1)$. For $x > 1$, $f(x)$ is strictly decreasing, meaning $f(x)$ is maximized at $x = 6$. Thus, our answer is

$$\frac{6}{(1-6)^2} = \frac{6}{25} \implies 6 + 25 = \boxed{31}.$$

SR 6: Let ABC be an equilateral triangle of side length 6. Points A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, and C_3 are chosen such that A_1, A_2, A_3 divide BC into four equal segments, B_1, B_2, B_3 divide AC into four equal segments, and C_1, C_2, C_3 divide AB into four equal segments. If i, j, k are chosen from the set $\{1, 2, 3\}$ independently and randomly, the expected area of $A_iB_jC_k$ is $\frac{a\sqrt{b}}{c}$, where b is a squarefree positive integer and a and c are relatively prime positive integers. Find $a + b + c$.

Answer: $\boxed{16}$

Solution: Let $[XYZ]$ denote the area of triangle XYZ. We have

$$\begin{aligned}
[A_iB_jC_k] &= [ABC] - [AB_jC_k] - [BC_kA_i] - [CA_iB_j] \implies \\
&= 6^2 \cdot \frac{\sqrt{3}}{4} - \frac{1}{2}\sin(60^\circ)(AB_j)(AC_k) - \frac{1}{2}\sin(60^\circ)(BC_k)(BA_i) - \frac{1}{2}\sin(60^\circ)(CA_i)(CB_j) \\
&= 9\sqrt{3} - \frac{\sqrt{3}}{4}\left((AB_j)(AC_k) + (BC_k)(BA_i) + (CA_i)(CB_j)\right).
\end{aligned}$$

Let $AB_j = c$, $BC_k = a$, and $CA_i = b$. This directly implies that $B_jC = 6 - c$, $C_kA = 6 - a$, and $A_iB = 6 - b$. So, we have

$$\begin{aligned}
[A_iB_jC_k] &= 9\sqrt{3} - \frac{\sqrt{3}}{4}\left(c(6-a) + a(6-b) + b(6-c)\right) \\
&= 9\sqrt{3} - \frac{\sqrt{3}}{4}\left(6(a+b+c) - (ab+ac+bc)\right) \implies \\
\mathbb{E}([A_iB_jC_k)) &= 9\sqrt{3} - \frac{\sqrt{3}}{4}\left(6 \cdot (\mathbb{E}(a) + \mathbb{E}(b) + \mathbb{E}(c)) - (\mathbb{E}(ab) + \mathbb{E}(ac) + \mathbb{E}(bc)\right).
\end{aligned}$$

We have

$$\mathbb{E}(a) = \mathbb{E}(b) = \mathbb{E}(c) = \frac{6}{2} = 3$$

and

$$\mathbb{E}(ab) = \mathbb{E}(ac) = \mathbb{E}(bc) = (\mathbb{E}(a))^2 = 9.$$

Substituting,

$$\begin{aligned}
\mathbb{E}([A_iB_jC_k]) &= 9\sqrt{3} - \frac{\sqrt{3}}{4}\left(6 \cdot (3+3+3) - (9+9+9)\right) \\
&= 9\sqrt{3} - \frac{\sqrt{3}}{4}(54 - 27) = 9\sqrt{3} - \frac{27\sqrt{3}}{4} \\
&= \frac{9\sqrt{3}}{4} \Rightarrow 9 + 3 + 4 = \boxed{16}.
\end{aligned}$$

SR 7: Let a, b, and c be real numbers such that $a + b + c = 1$ and $e \geq -\frac{1}{3}$, $b \geq -1$ and $c \geq -\frac{5}{3}$. Find the maximum value of $\sqrt{3a+1} + \sqrt{3b+3} + \sqrt{3c+5}$.

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Answer: $\boxed{6}$

Solution: From the Cauchy-Schwarz inequality, we have

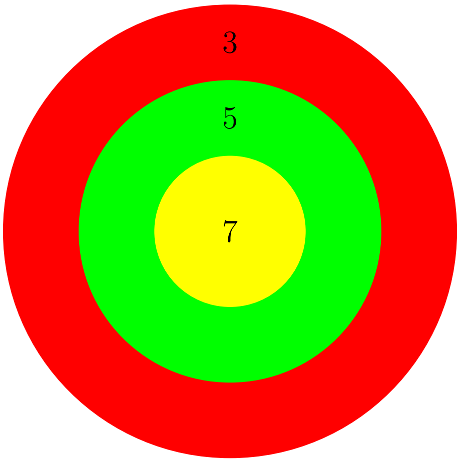
$$\left(\sqrt{3e+1}+\sqrt{3a+3}+\sqrt{3j+5}\right)^2 \leq ((3e+1)+(3a+3)+(3j+5))\,(1+1+1)$$
$$\leq (3(e+a+j)+9)\,(3) = (3(1)+9)(3) = 36 \implies$$

$\sqrt{3e+1}+\sqrt{3a+3}+\sqrt{3j+5} \leq \sqrt{36} = \boxed{6}$.

SR 8: In regular octagon $ABCDEFGH$ of sidelength 4, quadrilaterals $ACEG$ and $BDFH$ are drawn. Find the square of the area of the overlap of the two quadrilaterals.

Answer: $\boxed{2048}$

Solution:



Let $X = BH \cap AC$ and $Y = BH \cap AG$. Suppose that $BX = s$. From symmetry, we have $AX = AY = HY = s$. From the Pythagorean Theorem on AXY, we have $XY = s\sqrt{2}$. So,

$$BH = HY + YX + XB = s + s\sqrt{2} + s = s(2+\sqrt{2}),$$

meaning

$$XY = s\sqrt{2} = \frac{BH\sqrt{2}}{2+\sqrt{2}} = \frac{BH}{1+\sqrt{2}}$$

From the Law of Cosines on ABH, we have

$$BH = \sqrt{AH^2 + AB^2 - 2(AB)(AH)\cos(135°)}$$
$$= \sqrt{4^2 + 4^2 - 2(4)(4)\left(-\frac{\sqrt{2}}{2}\right)}$$
$$= \sqrt{32 + 16\sqrt{2}}.$$

Now, from the formula of the area of an octagon, we have

$$\begin{aligned}\text{area} &= (XY)^2(2+2\sqrt{2})\\ &= \frac{BH^2}{(1+\sqrt{2})^2}\cdot(2+2\sqrt{2})\\ &= \frac{32+16\sqrt{2}}{(1+\sqrt{2})^2}\cdot(2+2\sqrt{2})\\ &= 16\sqrt{2}\cdot 2 = 32\sqrt{2},\end{aligned}$$

meaning our answer is

$$(32\sqrt{2})^2 = \boxed{2048}.$$

SR 9: Suppose that b and c are the roots of the equation $x^2 - \log(16)x + \log(64)$. If $\sqrt{a+b} + \sqrt{a+c} = \sqrt{b+c}$, then 2^a can be expressed as $\frac{\sqrt{m}}{n}$, where m is a squarefree positive integer. Find $m+n$.

Answer: $\boxed{6}$

Solution: We have

$$\begin{aligned}\sqrt{a+b}+\sqrt{a+c} &= \sqrt{b+c} \implies\\ \left(\sqrt{a+b}+\sqrt{a+c}\right)^2 &= \left(\sqrt{b+c}\right)^2 \implies\\ (a+b)+(a+c)+2\sqrt{(a+b)(a+c)} &= b+c \implies\\ 2a+2\sqrt{(a+b)(a+c)} &= 0 \implies\\ 2\sqrt{(a+b)(a+c)} &= -2a \implies\\ \left(\sqrt{(a+b)(a+c)}\right)^2 &= (-a)^2 \implies\\ (a+b)(a+c) &= a^2 \implies\\ a^2+ab+ac+bc &= a^2 \implies\\ ab+ac+bc &= 0 \implies\\ a &= -\frac{bc}{b+c}.\end{aligned}$$

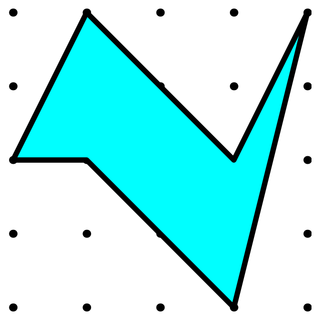
From Vieta's Formulas, we have

$$\begin{aligned}a = -\frac{\log(64)}{\log(16)} = -\log_{16}64 = -\log_{4^2}4^3 = -\frac{3}{2} \implies\\ 2^a = 2^{-\frac{3}{2}} = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4} \implies 2+4 = \boxed{6}.\end{aligned}$$

SR 10: Consider acute triangle ABC, H is the orthocenter. Extend AH to meet BC at D. The angle bisector of $\angle ABH$ meets the midpoint of AD, M. If $AB = 10, BH = 4$, then the area of ABC is $\frac{a\sqrt{b}}{c}$, where b is a squarefree positive integer and a and c are relatively prime positive integers. Find $a+b+c$.

Answer: $\boxed{476}$

Solution:



From the angle bisector theorem, we have

$$\frac{AB}{AM}=\frac{HB}{HM}\implies\frac{AM}{HM}=\frac{AB}{HB}=\frac{10}{4}=\frac{5}{2}.$$

Let $HM=2a$ and $AM=5a$. Since M is the midpoint of AD, we have

$$AM=MD\implies AM=MH+HD\implies 5a=2a+HD\implies HD=3a.$$

From the Pythagorean Theorem on triangles ABD and HBD, we have

$$BD=\sqrt{AB^2-AD^2}=\sqrt{100-100a^2}\quad\text{and}\quad BD=\sqrt{HB^2-HD^2}=\sqrt{36-9a^2}.$$

So, we have

$$100-100a^2=16-9a^2\implies a^2=\frac{12}{13},$$

meaning

$$BD=\sqrt{100-100a^2}=\sqrt{100-100\cdot\frac{12}{13}}=\sqrt{\frac{100}{13}}=\frac{10\sqrt{13}}{13}.$$

Let P be the foot of the perpendicular from B to AC. We have $\triangle HBD\sim\triangle CBP\sim\triangle CAD$. Comparing ratios, we have

$$\frac{BD}{HD}=\frac{AD}{CD}\implies\frac{\frac{10\sqrt{13}}{13}}{3a}=\frac{10a}{CD}\implies CD=\frac{30a^2}{\frac{10\sqrt{13}}{13}}=\frac{36\sqrt{13}}{13}.$$

So,

$$BC=BD+DC=\frac{10\sqrt{13}}{13}+\frac{36\sqrt{13}}{13}=\frac{46\sqrt{13}}{13}\quad\text{and}\quad AD=10a=10\sqrt{\frac{12}{13}}=\frac{20\sqrt{39}}{13}.$$

The area of ABC is

$$\frac{BC\cdot AD}{2}=\frac{\frac{46\sqrt{13}}{13}\cdot\frac{20\sqrt{39}}{13}}{2}=\frac{460\sqrt{3}}{13}\implies 460+3+13=\boxed{476}$$