2024 SSMO Speed Round Solutions

SMO Team

SR 1: Find the sum of the distinct prime factors of $2024^2 - 1$.

Answer: $\boxed{32}$

Solution: Note that $2024^2 - 1$ can be factored as $(2024-1)(2024+1) = (2023)(2025)$. Then, 2023 can be factored as $7 \cdot 17^2$ and 2025 can be factored as $45^2 = 3^4 \cdot 5^2$. Thus,

$$2024^1 - 1 = 2023 \cdot 2025 = 3^4 \cdot 5^2 \cdot 7 \cdot 17^2,$$

meaning the sum of the distinct prime factors is $3 + 5 + 7 + 17 = \boxed{32}$.

SR 2: Gracie's students play with some toys. When 4 or 5 students are present, the toys can be equally distributed to everyone. However, when there are only 3 students, there is one toy leftover after giving everyone the same number of toys. What is the least possible number of toys that Gracie could have?

Answer: $\boxed{40}$

Solution: Suppose that there x toys. Since x toys can be split equally when 4 of 5 students are present, x must be a multiple of $\text{lcm}(4,5) = 20$. So, x can be $20, 40, 60, \ldots$. Now, since there is one toy left over when the toys are split when 3 students are present, x must be one greater than a multiple of 3. As $20 = 3 \cdot 6 + 2$ and $40 = 3 \cdot 13 + 1$, we conclude that the answer is $\boxed{40}$.

SR 3: The polynomial $x^3 - 15x^2 + 4x + 4$ has distinct real roots r, s, and t. Find the value of

$$\left|(r^2 + s^2 + t^2)(rst)\right|.$$

$$\left|(r^2 + s^2 + t^2)(rst)\right|.$$

Answer: $\boxed{868}$

Solution: From Vieta's formulas, we have $r + s + t = 15, rs + rt + st = 4$, and $rst = -4$. Now, note that

$$\begin{aligned}\left|\left(r^2+s^2+t^2\right)(rst)\right| &= \left|\Big((r+s+t)^2 - 2(rs+rt+st)\Big)\right| \\ &= \left((15^2 - 2 \cdot 4\right)(-4) \\ &= |217 \cdot (-4)| = \boxed{868}.\end{aligned}$$

SR 4: Sam wants to read the Harry Potter and Warriors books. There are 7 Harry Potter books that must be read in a specific order, and there are 6 Warriors books that also must be read in a specific order; however, he can read the two series at the same time. For example, he could read the first three Harry Potter books, then the first five Warriors books, then the remaining Harry Potter books, and finally the last Warriors book. In how many unique orders can Sam read the books?

Answer: $\boxed{1716}$

Solution: Note that there are 13! ways to order the books. However, out of the 7! possible ways to order the 7 Harry Potter books we included, only one actually works. Similarly, out of the 6! possible ways to order the 6 Warrior books we included, only one actually works. So, the answer is

$$\frac{13!}{7!6!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{6! = 720} = 13 \cdot 12 \cdot 11 = \boxed{1716}.$$

SR 5: Let $\triangle ABC$ and $\triangle ADC$ be right triangles, such that $\angle ABC = \angle ADC = 90^\circ$. Given that $\angle ACB = 30^\circ$ and $BC = 3\sqrt{3}$, find the maximum possible length of BD.

Answer: $\boxed{6}$

Solution: Note that quadrilateral $ABCD$ is a cyclic quadrilateral with diameter AC. Since $\angle ACB = 30^\circ$ and $BC = 3\sqrt{3}$, we have $AC = \frac{BC}{\cos\angle ACB} = \frac{3\sqrt{3}}{\frac{\sqrt{3}}{2}} = 6$. So, the maximum possible length of BD is 6, as the maximum length of any chord on a circle is the diameter.

SR 6: There are 4 people and 4 houses. Each person independently randomly chooses a house to live in. The expected number of inhabited houses can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{239}$

Solution: For each house, the probability that it is uninhabited is $\left(\frac{3}{4}\right)^4$. So, the probability that it is inhabited is $1 - \left(\frac{3}{4}\right)^4 = \frac{175}{256}$. Thus, the expected number of inhabited houses is

$$4 \cdot \left(\frac{175}{256}\right) = \frac{175}{64} \implies 175 + 64 = \boxed{239}.$$

SR 7: Let S denote the set of all ellipses centered at the origin and with axes AB and CD where $A = (-x, 0), B = (x, 0), C = (0, -y)$, and $D = (0, y)$, for $2 \mid x + y$ and $0 \leq x, y \leq 10$. Let T denote the set of similar ellipses centered at the origin and passing through (x, y) for $2 \nmid x + y$ and $0 \leq x, y, \leq 10$. The positive difference between the sum of the areas of all ellipses in T and the sum of the areas of all the ellipses in S can be expressed as $m\pi$. Find m.

Answer: $\boxed{3025}$

Solution: The answer is

$$\left(\left(\sum_{i=0}^{10}(-1)^i i^2\right)\left(\sum_{i=0}^{10}(-1)^i i^2\right)\right) = \left(\sum_{i=0}^{10} i\right) = \boxed{3025}.$$

SSMO
SUMMER SOLSTICE MATH OLYMPIAD

SR 8: Bob has two coins; one is fair, and one lands on heads with a probability of $\frac{2}{3}$. Bob chooses a random coin and flips it twice. Alice watches the two coin flips and guesses whether Bob flipped the fair or rigged coin. Given that Alice is a good mathematician and guesses the more likely option (guessing randomly when they are equally likely), the probability she guesses right can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{115}$

Solution: Let $P(f,k)$ denote the fair coin shows k heads and $P(r,k)$ denote the probability the rigged coin shows k heads. Note that the probability k heads are shown when Bob flips the coin is $\frac{P(f,k)+P(r,k)}{2}$. Now, if k heads are showing, the probability that Alice guesses correct is $\frac{\max(P(f,k),P(r,k))}{P(f,k)+P(r,k))}$. So, the answer is

$$\sum_{k=0}^{2}\left(\left(\frac{P(f,k)+P(r,k)}{2}\right)\left(\frac{\max(P(f,k),P(r,k))}{P(f,k)+P(r,k)}\right)\right).$$

It is easy to compute

$$P(f,0)=\frac{1}{4}, P(f,1)=\frac{1}{2}, P(f,2)=\frac{1}{4}, P(r,0)=\frac{4}{9}, P(r,1)=\frac{4}{9}, \text{and } P(r,2)=\frac{1}{9}.$$

Therefore, the answer is

$$\left(\frac{\frac{4}{9}+\frac{1}{4}}{2}\right)\left(\frac{\frac{4}{9}}{\frac{4}{9}+\frac{1}{4}}\right)+\left(\frac{\frac{4}{9}+\frac{1}{2}}{2}\right)\left(\frac{\frac{1}{2}}{\frac{4}{9}+\frac{1}{2}}\right)+\left(\frac{\frac{1}{9}+\frac{1}{4}}{2}\right)\left(\frac{\frac{1}{4}}{\frac{1}{9}+\frac{1}{4}}\right)=$$
$$\frac{2}{9}+\frac{1}{4}+\frac{1}{8}=\frac{43}{72}\implies\boxed{115}.$$

SR 9: Let $a,b,c,$ and d be positive integers such that $abcd=a+b+c+d$. Find the maximum possible value of a.

Answer: $\boxed{4}$

Solution: Note that $(a,b,c,d)=(4,2,1,1)$ satisfies the equation. Now, assume for the sake of contradiction that $a\geq 5$ has a solution. We have

$$a(b+1)cd-a-(b+1)-c-d>abcd-a-b-c-d\implies$$
$$acd>1,$$

clearly true for $a\geq 5$ and $c,d,\geq 1$. Clearly, we $b=c=d=1$ doesn't satisfy the equation. So, the smallest value of $abcd-a-b-c-d$ for $a\geq 5$ occurs when $b=2, c=d=1$. This gives $abcd-a-b-c-d=a-4\geq 1$, for all $a\geq 5$, a contradiction. In conclusion, the maximum possible value of a is $\boxed{4}$.

SR 10: Let $a_1,a_2,\ldots a_{14}$ be the roots of $(x^7-x^3+2)^2=0$. Find the value of $\prod_{i=1}^{14}(a_i^7+1)$.

Answer: $\boxed{4}$

Solution: Let $a_1,a_2,\ldots,a_7$ be the roots of

$$f(x)=x^7-x^3+2=\prod_{i=1}^{7}(x-a_i).$$

It is easy to see that
$$\prod_{i=1}^{14}(a_i^7+1)=\left(\prod_{i=1}^{7}(a_i^7+1)\right)^2.$$
Now, we have $a_i^7-a_i^3+2=0 \implies a_i^7+1=a_i^3-1$. Denote $1, \omega, \frac{1}{\omega}$ as the roots of x^3-1, with $\omega^3=1$. So,

$$\begin{aligned}
\prod_{i=1}^{7}(a_i^7+1) &= \prod_{i=1}^{7}(a_i^3-1) \\
&= \prod_{i=1}^{7}(a_i-1)\prod_{i=1}^{7}(a_i-\omega)\prod_{i=1}^{7}\left(a_i-\frac{1}{\omega}\right) \\
&= -\prod_{i=1}^{7}(1-a_i)\prod_{i=1}^{7}(\omega-a_i)\prod_{i=1}^{7}\left(\frac{1}{\omega}-a_i\right) \\
&= -f(1)f(\omega)f\left(\frac{1}{\omega}\right) \\
&= -2(\omega^7-\omega^3+2)\left(\left(\frac{1}{\omega}\right)^7-\left(\frac{1}{\omega}\right)^3+2\right) \\
&= -2(\omega-1+2)\left(\frac{1}{\omega}-1+2\right) \\
&= -2(\omega+1)\left(\frac{1}{\omega}+1\right) \\
&= -2\left(1+\left(\omega+\frac{1}{\omega}\right)+1\right) \\
&= -2\,(1-1+1)=-2.
\end{aligned}$$

Thus, our answer is $(-2)^2=\boxed{4}$.