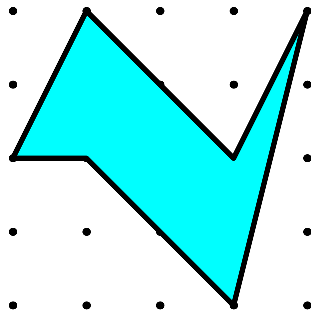
2023 WSMO Speed Round Solutions

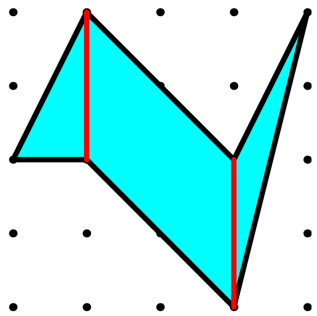
SMO Team

SR 1: Find the number of square units in the area of the shaded region.



Answer: $\boxed{6}$.

Solution: We divide the shaded region into three sections as follows



The left triangle has an area of $\frac{1}{2} \cdot 1 \cdot 2 = 1$, the center parallelogram has an area of $2 \cdot 2 = 4$, and the right triangle has an area of $\frac{1}{2} \cdot 1 \cdot 2 = 1$. Thus, our answer is $1 + 4 + 1 = \boxed{6}$.

SR 2: There are 4 tables and 5 chairs at each table. Each chair seats 2 people. There are 10 people who are seated randomly. Andre and Emily are 2 of them, and are a couple. If the probability that Andre and Emily are in the same chair is $\frac{m}{n}$, for relatively prime positive integers m and n, find $m + n$.

Answer: $\boxed{40}$.

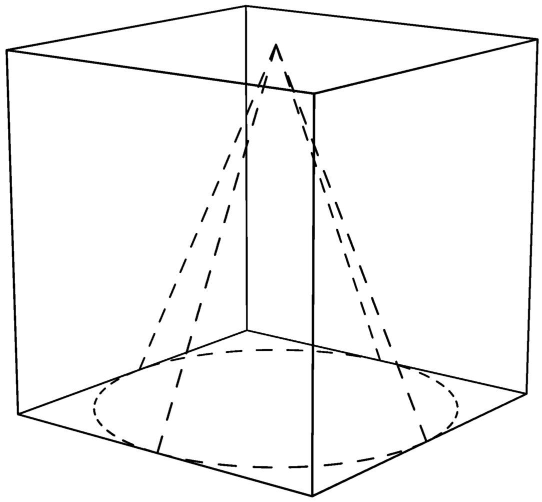
Solution: Note that there are $4 \cdot 5 \cdot 2 = 40$ possible places to seat. After Andre is assigned a seat at random, there are 39 remaining seats, only one of which is in the same chair as Emily. Thus, our answer is $\frac{1}{39} \implies 1 + 39 = \boxed{40}$.

SR 3: There are 6 pairs of socks for each color of the rainbow (red, orange, yellow, green, blue, indigo, violet) in a sock drawer. How many socks must be drawn from the drawer to guarantee that a pair of red socks have been drawn?

Answer: $\boxed{74}$.

Solution: There are $6 \cdot 2 \cdot 6 = 72$ non-red socks. Thus, a pair of red socks is guaranteed after $72 + 2 = \boxed{74}$ socks have been drawn.

SR 4: A right circular cone is inscribed in a right prism as shown. If the ratio of the volume of the cone to the volume of the prism is $\frac{m}{n}\pi$, for relatively prime positive integers m and n, find $m + n$.



Answer: $\boxed{13}$.

Solution: Let s and h denote the sidelength and height of the right prism, respectively. The ratio of the two volumes is equal to

$$\frac{\frac{1}{3} \cdot \left(\frac{s}{2}\right)^2 \cdot h\pi}{s^2 \cdot h} = \frac{\frac{\pi}{12} \cdot s^2 \cdot h}{s^2 \cdot h} = \frac{\pi}{12} \implies 1 + 12 = \boxed{13}.$$

SR 5: There exists a rational function $f(x)$ such that for all x in the range $(0, 1)$, $f(x) = \sum_{n=1}^{\infty} nx^n$. If the maximum of $f(x)$ over $[6, 9]$ is $\frac{m}{n}$, for relatively prime positive integers m and n, find $m + n$.

Answer: $\boxed{31}$.

Solution: We have

$$f(x) = x + 2x^2 + 3x^3 + 4x^4 + \ldots \text{ and}$$
$$xf(x) = x^2 + 2x^3 + 3x^4 + \ldots.$$

So,

$$f(x) - xf(x) = x + x^2 + x^3 + \ldots = \frac{x}{1-x} \implies f(x) = \frac{x}{(1-x)^2}$$

for x in the range $(0,1)$. For $x>1$, $f(x)$ is strictly decreasing, meaning $f(x)$ is maximized at $x=6$. Thus, our answer is

$$\frac{6}{(1-6)^2}=\frac{6}{25} \implies 6+25=\boxed{31}.$$

SR 6: Let ABC be an equilateral triangle of side length 6. Points $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ are chosen such that A_1, A_2, A_3 divide BC into four equal segments, B_1, B_2, B_3 divide AC into four equal segments, and C_1, C_2, C_3 divide AB into four equal segments. If i, j, k are chosen from the set $1, 2, 3$ independently and randomly, the expected area of $A_iB_jC_k$ is $\frac{a\sqrt{b}}{c}$, for squarefree b and relatively prime positive integers a and c. Find $a+b+c$.

Answer: $\boxed{16}$.

Solution: Let $[XYZ]$ denote the area of triangle XYZ. We have

$$\begin{aligned}
[A_iB_jC_k] &= [ABC]-[AB_jC_k]-[BC_kA_i]-[CA_iB_j] \implies \\
&= 6^2\cdot\frac{\sqrt{3}}{4}-\frac{1}{2}\sin(60^\circ)(AB_j)(AC_k)-\frac{1}{2}\sin(60^\circ)(BC_k)(BA_i)-\frac{1}{2}\sin(60^\circ)(CA_i)(CB_j) \\
&= 9\sqrt{3}-\frac{\sqrt{3}}{4}\left((AB_j)(AC_k)+(BC_k)(BA_i)+(CA_i)(CB_j)\right).
\end{aligned}$$

Let $AB_j=c$, $BC_k=a$, and $CA_i=b$. This directly implies that $B_jC=6-c$, $C_kA=6-a$, and $A_iB=6-b$. So, we have

$$\begin{aligned}
[A_iB_jC_k] &= 9\sqrt{3}-\frac{\sqrt{3}}{4}\left(c(6-a)+a(6-b)+b(6-c)\right) \\
&= 9\sqrt{3}-\frac{\sqrt{3}}{4}\left(6(a+b+c)-(ab+ac+bc)\right) \implies \\
\mathbb{E}([A_iB_jC_k)) &= 9\sqrt{3}-\frac{\sqrt{3}}{4}\left(6\cdot(\mathbb{E}(a)+\mathbb{E}(b)+\mathbb{E}(c))-(\mathbb{E}(ab)+\mathbb{E}(ac)+\mathbb{E}(bc)\right).
\end{aligned}$$

We have

$$\mathbb{E}(a)=\mathbb{E}(b)=\mathbb{E}(c)=\frac{6}{2}=3$$

and

$$\mathbb{E}(ab)=\mathbb{E}(ac)=\mathbb{E}(bc)=(\mathbb{E}(a))^2=9.$$

Substituting,

$$\begin{aligned}
\mathbb{E}([A_iB_jC_k]) &= 9\sqrt{3}-\frac{\sqrt{3}}{4}\left(6\cdot(3+3+3)-(9+9+9)\right) \\
&= 9\sqrt{3}-\frac{\sqrt{3}}{4}(54-27)=9\sqrt{3}-\frac{27\sqrt{3}}{4} \\
&= \frac{9\sqrt{3}}{4}\Rightarrow 9+3+4=\boxed{16}.
\end{aligned}$$

SR 7: Let e, a, j be real numbers such that $e+a+j=1$ and $e\geq-\frac{1}{3}$, $a\geq1$ and $j\geq-\frac{5}{3}$. Find the maximum value of $\sqrt{3e+1}+\sqrt{3a+3}+\sqrt{3j+5}$.

Answer: $\boxed{6}$.

Solution: From the Cauchy-Schwarz inequality, we have

$$\left(\sqrt{3e+1}+\sqrt{3a+3}+\sqrt{3j+5}\right)^2 \le \left((3e+1)+(3a+3)+(3j+5)\right)(1+1+1)$$
$$\le \left(3(e+a+j)+9\right)(3) = (3(1)+9)(3) = 36 \implies$$

$\sqrt{3e+1}+\sqrt{3a+3}+\sqrt{3j+5} \le \sqrt{36} = \boxed{6}$.

SR 8: In regular octagon $ABCDEFGH$ of sidelength 4, quadrilaterals $ACEG$ and $BDFH$ are drawn. Find the square of the area of the overlap of the two quadrilaterals.

Answer: $\boxed{2048}$.

Solution:

Solution: Let $X = BH \cap AC$ and $Y = BH \cap AG$. Suppose that $BX = s$. From symmetry, we have $AX = AY = HY = s$. From the Pythagorean Theorem on AXY, we have $XY = s\sqrt{2}$. So,

$$BH = HY + YX + XB = s + s\sqrt{2} + s = s(2+\sqrt{2}),$$

meaning

$$XY = s\sqrt{2} = \frac{BH\sqrt{2}}{2+\sqrt{2}} = \frac{BH}{1+\sqrt{2}}$$

From the Law of Cosines on ABH, we have

$$\begin{aligned} BH &= \sqrt{AH^2 + AB^2 - 2(AB)(AH)\cos(135^\circ)} \\ &= \sqrt{4^2 + 4^2 - 2(4)(4)\left(-\frac{\sqrt{2}}{2}\right)} \\ &= \sqrt{32 + 16\sqrt{2}}. \end{aligned}$$

Now, from the formula of the area of an octagon, we have

$$\begin{aligned} \text{area} &= (XY)^2(2+2\sqrt{2}) \\ &= \frac{BH^2}{(1+\sqrt{2})^2}\cdot(2+2\sqrt{2}) \\ &= \frac{32+16\sqrt{2}}{(1+\sqrt{2})^2}\cdot(2+2\sqrt{2}) \\ &= 16\sqrt{2}\cdot 2 = 32\sqrt{2}, \end{aligned}$$

meaning our answer is

$$(32\sqrt{2})^2 = \boxed{2048}.$$

SR 9: Suppose that b and c are the roots of the equation $x^2 - \log(16)x + \log(64)$. If $\sqrt{a+b} + \sqrt{a+c} = \sqrt{b+c}$, then $2^a = \frac{\sqrt{m}}{n}$, find $m+n$.

Answer: $\boxed{6}$.

Solution: We have

$$\begin{aligned}
\sqrt{a+b} + \sqrt{a+c} &= \sqrt{b+c} \implies \\
\left(\sqrt{a+b} + \sqrt{a+c}\right)^2 &= \left(\sqrt{b+c}\right)^2 \implies \\
(a+b) + (a+c) + 2\sqrt{(a+b)(a+c)} &= b+c \implies \\
2a + 2\sqrt{(a+b)(a+c)} &= 0 \implies \\
2\sqrt{(a+b)(a+c)} &= -2a \implies \\
\left(\sqrt{(a+b)(a+c)}\right)^2 &= (-a)^2 \implies \\
(a+b)(a+c) &= a^2 \implies \\
a^2 + ab + ac + bc &= a^2 \implies \\
ab + ac + bc &= 0 \implies \\
a &= -\frac{bc}{b+c}.
\end{aligned}$$

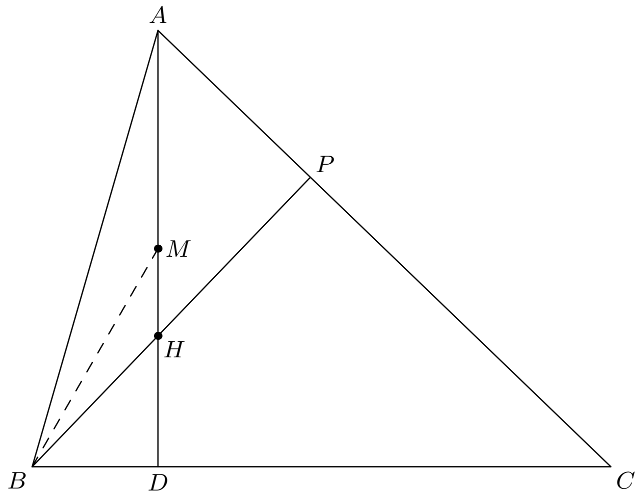
From Vieta's Formulas, we have

$$\begin{aligned}
a = -\frac{\log(64)}{\log(16)} = -\log_{16} 64 = -\log_{4^2} 4^3 = -\frac{3}{2} \implies \\
2^a = 2^{-\frac{3}{2}} = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4} \implies 2 + 4 = \boxed{6}.
\end{aligned}$$

SR 10: Consider acute triangle ABC, H is the orthocenter. Extend AH to meet BC at D. The angle bisector of $\angle ABH$ meets the midpoint of AD, M. If $AB = 10, BH = 4$, then the area of ABC is $\frac{a\sqrt{b}}{c}$, for squarefree b and relatively prime positive integers a and c. Find $a+b+c$.

Answer: $\boxed{476}$.

Solution:



From the angle bisector theorem, we have

$$\frac{AB}{AM} = \frac{HB}{HM} \implies \frac{AM}{HM} = \frac{AB}{HB} = \frac{10}{4} = \frac{5}{2}.$$

Let $HM = 2a$ and $AM = 5a$. Since M is the midpoint of AD, we have

$$AM = MD \implies AM = MH + HD \implies 5a = 2a + HD \implies HD = 3a.$$

From the Pythagorean Theorem on triangles ABD and HBD, we have

$$BD = \sqrt{AB^2 - AD^2} = \sqrt{100 - 100a^2} \quad \text{and} \quad BD = \sqrt{HB^2 - HD^2} = \sqrt{36 - 9a^2}.$$

So, we have

$$100 - 100a^2 = 16 - 9a^2 \implies a^2 = \frac{12}{13},$$

meaning

$$BD = \sqrt{100 - 100a^2} = \sqrt{100 - 100 \cdot \frac{12}{13}} = \sqrt{\frac{100}{13}} = \frac{10\sqrt{13}}{13}.$$

Let P be the foot of the perpendicular from B to AC. We have $\triangle HBD \sim \triangle CBP \sim \triangle CAD$. Comparing ratios, we have

$$\frac{BD}{HD} = \frac{AD}{CD} \implies \frac{\frac{10\sqrt{13}}{13}}{3a} = \frac{10a}{CD} \implies CD = \frac{30a^2}{\frac{10\sqrt{13}}{13}} = \frac{36\sqrt{13}}{13}.$$

So,

$$BC = BD + DC = \frac{10\sqrt{13}}{13} + \frac{36\sqrt{13}}{13} = \frac{46\sqrt{13}}{13} \quad \text{and} \quad AD = 10a = 10\sqrt{\frac{12}{13}} = \frac{20\sqrt{39}}{13}.$$

The area of ABC is

$$\frac{BC \cdot AD}{2} = \frac{\frac{46\sqrt{13}}{13} \cdot \frac{20\sqrt{39}}{13}}{2} = \frac{460\sqrt{3}}{13} \implies 460 + 3 + 13 = \boxed{476}$$

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