2024 SSMO Accuracy Round Solutions

SMO Team

AR 1: Let a time of day be three-full if exactly three of its digits are 3s when displayed on a 12-hour clock in the $hh : mm : ss$ format. How many seconds of the day are three-full?

Answer: $\boxed{1188}$

Solution: Note that hh is a number from 01 to 12, mm and ss are both integers from 00 to 59. There are $45, 14, 1$ two-digit integers from 00 to 59 that have $0, 1, 2$ $3's$, respectively. If hh contains a 3, then there are $(45)(1) + (14)(14) + (1)(45) = 286$ possibilities. Otherwise, there are $(11)((1)(14) + (14)(1)) = 308$ possibilities. Since the all times on the 12-hour clock appear twice each day, our answer is $2(286 + 308) = \boxed{1188}$.

AR 2: Equilateral triangle N is inscribed within circle O. A smaller equilateral triangle P is inscribed within N such that the vertices of P lie on the midpoints of N. The ratio of the areas between O and P can be expressed as $\frac{a\pi\sqrt{b}}{c}$, where b is a squarefree positive integer and a and c are relatively prime positive integers. Find $a + b + c$.

Answer: $\boxed{28}$

Solution: WLOG, let the radius of O be 1. Then, the side length of N is $\sqrt{3}$, meaning the side length of P is $\frac{\sqrt{3}}{2}$. Thus, the answer is

$$\frac{\pi}{\left(\frac{\sqrt{3}}{2}\right)^2 \cdot \left(\frac{\sqrt{3}}{4}\right)} = \frac{\pi}{\frac{3\sqrt{3}}{16}} = \frac{16\sqrt{3}\pi}{9} \implies 16 + 3 + 9 = \boxed{28}.$$

AR 3: Three distinct random integers a, b, and c are selected so that $1 \leq a, b, c \leq 10$. The probability that $5|a! + b! + c!$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{19}$

Solution: Note that $1!, 2!, \ldots, 10!$ have residues $1, 2, 1, 4, 0, 0, \ldots$ when taken modulo 5. We have 2 cases for residues modulo 5: either one is 0, one is 1, and one is 4, or they are all 0. For the first case, we have 2 ways to choose a 1, 1 way to choose a 4, 6 ways to choose a 0, and 6 ways to permute them, giving $2 \cdot 6 \cdot 6 = 72$ solutions. For the other case, we have $6 \cdot 5 \cdot 4 = 120$ ways to choose the 3 multiples of 5. Thus, the total number of solutions is $120 + 72 = 192$, meaning the answer is

$$\frac{192}{10 \cdot 9 \cdot 8} = \frac{4}{15} \implies 4 + 15 = \boxed{19}.$$

AR 4: Right triangle ABC has a right angle at C and hypotenuse 1. Let points D and E lie on AC such that $\angle BDC = \angle BEC = 45°$. A, D, C, and E are collinear in that order.

Given that $EA = 13DA$, the area of $\triangle ABC$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{106}$

Solution: Since BDC and BEC are isosceles right triangles, we have $DC = BC = EC$. Since $AD+DC+CE = EA = 13AD$, we have $DC+CE = 12AD \implies DC = CE = 6AD$. From here, we see that $AC = AD + DC = 7AD$. By the Pythagorean Theorem,

$$AC^2 + BC^2 = AB^2 = 1 \implies (7AD)^2 + (6AD)^2 = 1 \implies AD^2 = \frac{1}{85}.$$

The area of triangle ABC is

$$\frac{AC \cdot BC}{2} = 21AD^2 = \frac{21}{85} \implies 21 + 85 = \boxed{106}.$$

AR 5: Let $ABCD$ be a convex quadrilateral such that $\angle ABC = 120^\circ$ and $\angle ADC = 60^\circ$. If $AB = BC = CD = 5$, the area of $ABCD$ can be expressed as $a + b\sqrt{c}$, where a, b, c, are positive integers with c squarefree. Find $a + b + c$.

Answer: $\boxed{82}$

Solution: From the Law of Cosines on triangle ABC, we have

$$AC^2 = 5^2 + 5^2 - 2\left(-\frac{1}{2}\right) \cdot 5 \cdot 5 \implies AC = 5\sqrt{3}.$$

Let $AD = x$. From the Law of Cosines on triangle ADC, we have

$$x^2 + 5^2 - 2\left(\frac{1}{2}\right) 5 \cdot x = \left(5\sqrt{3}\right)^2 \implies x^2 - 5x - 50 = 0 \implies x = 10.$$

Finally, we use the sine area formula:

$$[ABCD] = [ABC]+[ACD] = \frac{1}{2}\left(5 \cdot 5 \cdot \left(\frac{\sqrt{3}}{2}\right) + 10 \cdot 5\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{75\sqrt{3}}{4} \implies 75+3+4 = \boxed{82}.$$

AR 6: Three six-sided dice are rolled. Then, the product of the three numbers on the top faces in calculated. The probability that the product is not a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: $\boxed{197}$

Solution: For simplicity, we will count the number of ways of getting a product that is a perfect square. We will do casework on the number of perfect squares rolled. Firstly, there are $2^3 = 8$ ways to roll 3 perfect squares. Next, if two perfects squares rolled and the other isn't a perfect square, it is impossible for the final product to be a perfect square. If 1 perfect square is rolled, then the other two numbers rolled must be equal. There are 3 ways to permute the rolls, 2 perfect squares, and 4 non-perfect squares, giving $3 \cdot 2 \cdot 4 = 24$ possibilities. Finally, if no perfect squares are rolled, we must have 236, giving another 6 possibilities. In conclusion, the answer is

$$1 - \frac{8 + 24 + 6}{6^3} = 1 - \frac{19}{108} = \frac{89}{108} \implies 89 + 108 = \boxed{197}.$$

AR 7: Find the value of $a+b+c+d$ given

$$\begin{aligned} &a^2+d^2=b^2+c^2=361, \\ &ac+bd=247, \text{ and} \\ &ab+cd=570. \end{aligned}$$

Answer: $\boxed{56}$

AR 8: $ABCD$ is a convex cyclic quadrilateral with $AB=2, BC=5, CD=10$, and $AD=11$. Let W, Y, X, and Z be the midpoints of sides AB, BC, CD, and DA respectively. The value of $|(WX^2-YZ^2)|$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{209}$

Solution: We will use complex numbers. For all points P, let p denote it's complex number representation. Since $AB^2+AD^2=BC^2+CD^2=125$, the diameter of the circumcircle of $ABCD$ is BD. From $AB=2, BC=5, CD=10$, and $AD=11$, we have

$$|a-b|=2 \implies |a-b|^2=2^2 \implies (a-b)\left(\frac{1}{a}-\frac{1}{b}\right)=\frac{4}{\frac{125}{4}} \implies \frac{a}{b}+\frac{b}{a}=2-\frac{4}{\frac{125}{4}},$$

$$|b-c|=2 \implies |b-c|^2=5^2 \implies (b-c)\left(\frac{1}{b}-\frac{1}{c}\right)=\frac{25}{\frac{125}{4}} \implies \frac{b}{c}+\frac{b}{c}=2-\frac{25}{\frac{125}{4}},$$

$$|c-d|=2 \implies |c-d|^2=10^2 \implies (c-d)\left(\frac{1}{c}-\frac{1}{d}\right)=\frac{100}{\frac{125}{4}} \implies \frac{c}{d}+\frac{c}{d}=2-\frac{100}{\frac{125}{4}}, \text{ and}$$

$$|d-a|=2 \implies |d-a|^2=11^2 \implies (d-a)\left(\frac{1}{d}-\frac{1}{a}\right)=\frac{121}{\frac{125}{4}} \implies \frac{d}{a}+\frac{d}{a}=2-\frac{121}{\frac{125}{4}}.$$

Now,

$$\begin{aligned} WX^2 &= \left|\frac{a+b-c-d}{2}\right|^2 = \left(\frac{a+b-c-d}{2}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}-\frac{1}{d}}{2}\right) \implies \\ 4WX^2 &= 4+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{a}{c}+\frac{c}{a}\right)-\left(\frac{a}{d}+\frac{d}{a}\right) \\ &\quad -\left(\frac{b}{c}+\frac{c}{b}\right)-\left(\frac{b}{d}+\frac{d}{b}\right). \end{aligned}$$

In the same manner,

$$\begin{aligned} YZ^2 &= \left|\frac{-a+b+c-d}{2}\right|^2 = \left(\frac{-a+b+c-d}{2}\right)\left(\frac{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{d}}{2}\right) \implies \\ 4YZ^2 &= 4+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{d}+\frac{d}{a}\right)-\left(\frac{a}{c}+\frac{c}{a}\right)-\left(\frac{c}{d}+\frac{d}{c}\right) \\ &\quad -\left(\frac{b}{a}+\frac{a}{b}\right)-\left(\frac{b}{d}+\frac{d}{b}\right). \end{aligned}$$

So,

$$4(WX^2 - YZ^2) = 2\left(\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{b}{c}+\frac{c}{b}\right)-\left(\frac{d}{a}+\frac{a}{d}\right)\right)$$
$$= 2\left(\frac{4+100-25-121}{\frac{125}{4}}\right) = \frac{-336}{125} \implies$$
$$|WX^2 - YZ^2| = \frac{84}{125} \implies 84+125 = \boxed{209}.$$

AR 9: In the game of S-set, there are 3^{12} unique cards, each card containing a 12 traits of variants A, B, and C. A full house in the game of Sset is defined to be a hand of cards in which for each trait, all cards in the hand either share the same variant or have all different variants. The number of hands that can be considered a full house in the game of set can be expressed as $a^b + c^d + e^f$, where a, b, c, d, e, and f are positive integers and $a+c+e$ is minimized. Find $a+b+c+d+e+f$.

Answer: $\boxed{66}$

Solution: Note that all full houses can contain at most 3 cards. Now, 2 cards uniquely determine a hand. So, there are $\binom{3^{12}}{2}$ full house hands of 2 cards. As there are 6 permutations for 3 card hands, there are $\frac{(3^{12})(3^{12}-1)}{6}$ full house hands of 3 cards. Lastly, note that all one-card hands are considered full houses. Thus, the answer is

$$\frac{3^{24}-3^{12}}{2}+\frac{3^{24}-3^{12}}{6}+3^{12} = \frac{2}{3}(3^{24})+\frac{1}{3}(3^{12}) =$$
$$3^{23}+3^{23}+3^{11} \implies 3+23+3+23+3+11 = \boxed{66}.$$

AR 10: Bobby is spinning a rigged wheel with three sections labeled A, B, and C. Two integers $0 \le x, y \le 50$ are chosen randomly and independently, such that there is a $x\%$ chance the wheel lands on A and a $y\%$ chance the wheel lands on B. Given that the wheel lands on C the first time, the probability that it will land on C if Bobby spins it again can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: $\boxed{119}$

Solution: For each x, y, the probability of the first spin being C is $\left(1-\frac{x+y}{100}\right)$ and the probability of the first and second spins both land on C is $\left(1-\frac{x+y}{100}\right)^2$. So, the answer is

$$\frac{\sum_{0\le x,y,\le 50}\left(1-\frac{x+y}{100}\right)^2}{\sum_{0\le x,y,\le 50}\left(1-\frac{x+y}{100}\right)}.$$

First,

$$\sum_{0\le x,y,\le 50}\left(1-\frac{x+y}{100}\right) = 51^2\cdot\mathbb{E}\left(1-\frac{x+y}{100}\right) = 51^2\left(1-\mathbb{E}\left(\frac{x}{100}\right)-\mathbb{E}\left(\frac{y}{100}\right)\right) =$$

$$51^2\left(1-\frac{1}{4}-\frac{1}{4}\right) = \frac{51^2}{2}.$$

Now,

$$\sum_{0\le x,y,\le 50}\left(1-\frac{x+y}{100}\right)^2=\sum_{0\le x,y,\le 50}1-2\sum_{0\le x,y,\le 50}\left(\frac{x+y}{100}\right)+\sum_{0\le x,y,\le 50}\left(\frac{x+y}{100}\right)^2.$$

We have

$$\sum_{0\le x,y,\le 50}1=51^2$$

and

$$2\sum_{0\le x,y,\le 50}\left(\frac{x+y}{100}\right)=2\cdot 51^2\cdot\mathbb{E}\left(\frac{x+y}{100}\right)=2\cdot 51^2\cdot\left(\mathbb{E}\left(\frac{x}{100}\right)+\left(\frac{y}{100}\right)\right)$$
$$=51^2\cdot\left(\frac{1}{4}+\frac{1}{4}\right)=2\cdot\left(\frac{51^2}{2}\right)=51^2.$$

So,

$$\sum_{0\le x,y,\le 50}\left(1-\frac{x+y}{100}\right)^2=51^2-51^2+\sum_{0\le x,y,\le 50}\left(\frac{x+y}{100}\right)^2=\sum_{0\le x,y,\le 50}\left(\frac{x+y}{100}\right)^2.$$

We compute

$$\begin{aligned}\sum_{0\le x,y,\le 50}(x+y)^2&=\sum_{0\le x,y,\le 50}(x^2+y^2)+2\sum_{0\le x,y,\le 50}xy\\&=2\cdot 51^2\left(\mathbb{E}(x^2)\right)+2\left(\sum_{0\le x\le 50}x\right)^2\\&=2\cdot 51^2\left(\frac{\sum_{i=0}^{50}i^2}{51}\right)+2\left(\frac{50\cdot 51}{2}\right)^2\\&=\frac{51\cdot 50\cdot 51\cdot 101}{3}+\frac{50\cdot 51\cdot 50\cdot 51}{2}\\&=50\cdot 51\cdot 51\left(\frac{101}{3}+\frac{50}{2}\right)\\&=\frac{50\cdot 51\cdot 51\cdot 176}{3}.\end{aligned}$$

So, the answer is

$$\frac{\frac{\frac{50\cdot 51\cdot 51\cdot 176}{3}}{100\cdot 100}}{\frac{51\cdot 51}{2}}=\frac{2\cdot 50\cdot 176}{3\cdot 100\cdot 100}=\frac{44}{75}\implies 44+75=\boxed{119}.$$