

# 2023 WSMO Tiebreaker Round Solutions

SMO Team

**Tiebreaker Round Problem 1:** Find the number of factors of  $24^6 - 20^6$ .

**Answer:** 208.

**Solution:** Note that

$$\begin{aligned} 24^6 - 20^6 &= (24^3 - 20^3)(24^3 + 20^3) \\ &= (24 - 20)(24^2 + 24 \cdot 20 + 20^2)(24 + 20)(24^2 - 24 \cdot 20 + 20^2) \\ &= (4)(1456)(44)(496) \\ &= 2^2 \cdot (2^4 \cdot 31)(2^2 \cdot 11)(2^4 \cdot 7 \cdot 13) \\ &= 2^{12} \cdot 7 \cdot 11 \cdot 13 \cdot 31, \end{aligned}$$

which has

$$(12 + 1)(1 + 1)(1 + 1)(1 + 1)(1 + 1) = (13)(2)(2)(2)(2) = \boxed{208}.$$

**Tiebreaker Round Problem 2:** If  $x^3 + y^3 = 76895$ ,  $x < y$  and  $x + y = 65$ , find  $xy$ .

**Answer:** 1014.

**Solution:** Note that

$$\begin{aligned} xy &= \frac{xy(x + y)}{(x + y)} \\ &= \frac{1}{3} \left( \frac{3x^2y + 3xy^2}{x + y} \right) \\ &= \frac{1}{3} \left( \frac{(x^3 + 3x^2y + 3xy^2 + y^3) - (x^3 + y^3)}{x + y} \right) \\ &= \frac{1}{3} \left( \frac{(x + y)^3 - (x^3 + y^3)}{x + y} \right) \\ &= \frac{1}{3} \left( \frac{65^3 - 76895}{65} \right) \\ &= \frac{65^2 - 1183}{3} = \boxed{1014}. \end{aligned}$$

**Tiebreaker Round Problem 3:** Let  $S$  be the set of all values of  $a$  such that the area of a triangle with side lengths 5, 7,  $a$  is a positive integer. Find  $\sum_{x \in S} (x^2)$ .



**Answer:** 2516.

**Solution:** Letting 5 be the base of the triangle, we have a height of  $\frac{2h}{5}$  for  $0 < \frac{2h}{5} \leq 7 \implies h \in \{1, 2, \dots, 17\}$ . Note for each height, we have two possible values of  $a$ , if the triangle is acute or obtuse. We have

$$\begin{aligned} a_1^2 &= \left( 5 - \sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \right)^2 + \left(\frac{2h}{5}\right)^2 \\ &= 5^2 + \left( 7^2 - \left(\frac{2h}{5}\right)^2 \right) - 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} + \left(\frac{2h}{5}\right)^2 \\ &= 74 - 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \end{aligned}$$

for when the triangle is acute and

$$\begin{aligned} a_2^2 &= \left( 5 + \sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \right)^2 + \left(\frac{2h}{5}\right)^2 \\ &= 5^2 + \left( 7^2 - \left(\frac{2h}{5}\right)^2 \right) + 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} + \left(\frac{2h}{5}\right)^2 \\ &= 74 + 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \end{aligned}$$

for when the triangle is obtuse. We have

$$a_1^2 + a_2^2 = \left( 74 - 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \right) + \left( 74 + 10\sqrt{7^2 - \left(\frac{2h}{5}\right)^2} \right) = 148.$$

So,

$$\sum_{x \in S} (x^2) = \sum_{h \in \{1, 2, \dots, 17\}} 148 = 148 \cdot 17 = \boxed{2516}.$$

