# Categorical Logic and Constructive Mathematics

University of Verona, Department of Computer Science
Wednesday, October 23rd, 2024

## Schedule

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| 9:15-11:30 | **Morning Session – Aula M** |
| 09:15 | *welcome* |
| 09:30 | Iosif Petrakis (University of Verona)<br>TBA |
| 10:30 | Giulio Fellin (University of Brescia)<br>*“Conservation for nuclei over substructural entailment relations”* |
| 11:30-12:00 | ***coffee break*** |
| 12:00-14:00 | **Midday Session – Sala riunioni** |
| 12:00 | Francesco Ciraulo (University of Padua)<br>*“The Kuratowski's Problem in Pointfree Topology”* |
| 13:00 | Marco Benini (University of Insubria)<br>*“Strong normalization in the full simple theory of types”* |
| 14:00-16:00 | ***lunch break*** |
| 16:00-18:30 | **Afternoon Session – Sala verde** |
| 16:00 | <u>*LOGIC SEMINAR:*</u><br>Hajime Ishihara (Toho University)<br>*“A constructive theory of uniform spaces and its application to integration theory”* |
| 17:00 | <u>*DEPARTMENT SEMINAR:*</u><br>Tarmo Uustalu (Reykjavik University / Tallinn University of Technology)<br>*“The proof theory of skew logics”* |
| 20:00 | **Social dinner** |

# Abstracts

## Iosif Petrakis (University of Verona), TBA

TBA

## Giulio Fellin (University of Brescia),
## *“Conservation for nuclei over substructural entailment relations”*

Glivenko’s theorem has been studied in an abstract setting, where double negation is replaced by arbitrary nuclei, and classical and intuitionistic propositional logics are generalised to abstract entailment relations. This paper extends that framework to substructural logics, focusing on cases where structural rules such as weakening, contraction, and exchange are restricted. We show that while the absence of weakening or contraction has minimal impact, the absence of the exchange rule introduces significant and nontrivial distinctions.

This is a joint work with Tarmo Uustalu and Cheng-Syuan Wan.

## Francesco Ciraulo (University of Padua),
## *“The Kuratowski's Problem in Pointfree Topology”*

A classic result of Kuratowski states that there are at most 7 combinations of the operators of interior i and closure c on a topological space, which become 14 if also the set-theoretic complement - is considered. These operators form an ordered monoid w.r.t. composition and pointwise ordering, the so-called Kuratowski's monoid.

Special classes of spaces can be characterized by the fact that two or more of these operators coincide [3]; for instance, a space whose open sets form a complete Boolean algebra satisfies the equation ici=i.

What happens to this picture if it is looked at from a constructive point of view?

And what about the pointfree (i.e. localic) version of the Kuratowski's problem?

We answer both of these questions and we explain why they are related to each other.

In a constructive setting, the collection of subsets of a given set is only a frame (a.k.a. a complete Heyting algebra), instead of a complete Boolean algebra, and the set-theoretic “complement” is only a pseudocomplement. This naturally poses a general version of the

Kuratowski's problem on an arbitrary frame. However, the presence of the pseudocomplement greatly increases the number of possible combinations [1].

To simplify the matter, we restrict to the case in which c=-i- (this equation is constructively true in all topological spaces, although its dual i=-c- is not): this we call the interior-pseudocomplement problem on a frame. Contrary to the Boolean case, we get 31 possible combinations (instead of 14) that apparently are all different [2], in general.

This constructive result can be applied to solve the Kuratowski's problem in a pointfree framework, that is, for locales. Indeed, it is well known that the sublocales of a given locale form a co-frame; in particular, every sublocale has a co-pseudocomplement, a.k.a. a supplement. Moreover, the usual notion of an open (closed) sublocale gives rise to an interior (a closure) operator on the co-frame of sublocales which are suitably linked by supplement.

So the Kuratowki's problem for sublocales becomes a special case, although in a dual way, of the constructive interior-pseudocomplement problem discussed above. We can therefore apply the previous result and, thanks to some specific properties of sublocales, we can lower the number of possible combinations of interior, closure and supplement to 21. Showing that this picture cannot be further simplified is still an open problem [2].

*References*

[1] O. Al-Hassani, Q.-A. Mahesar, C. Sacerdoti Coen, and V. Sorge. A term rewriting system for Kuratowski's closure-complement problem. In 23rd International Conference on Rewriting Techniques and Applications, volume 15 of LIPIcs. Leibniz Int. Proc. Inform., pages 38--52. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2012.

[2] F. Ciraulo. Kuratowski's problem in constructive topology. Journal of Logic and Analysis, (to appear).

[3] B. J. Gardner and M. Jackson. The Kuratowski closure-complement theorem. New Zealand J. Math., 38:9--44, 2008.

## Marco Benini (University of Insubria),

## *"Strong normalization in the full simple theory of types"*

Proving normalization in the simple theory of types is not difficult, although not elementary, even. Indeed, many different proofs are available in the literature: they differ in the set of type constructors, in the allowed contractions, in aiming at weak or strong normalization, not to mention the variety of techniques of interest in their own. However, there is a significant gap: no proof considers the full language of intuitionistic propositional logic and a rich enough set of contractions (computation, uniqueness, and permutation) to satisfy the logical needs. This contribution illustrates a proof that fills the gap.

## Hajime Ishihara (Toho University),
## *"A constructive theory of uniform spaces and its application to integration theory"*

We introduce the constructive notion of a uniform space with the spirit of Sambin's notion of a basic pair, and construct a completion of a uniform space and a product of uniform spaces. We show some natural properties of the completion and the product. Then we define topological linear spaces and topological vector lattices as linear spaces and vector lattices equipped with uniform structures, and show that these algebraic and topological structures are preserved under the completion. We introduce the notion of an abstract integration space consisting of a vector lattice and a positive linear functional. By defining two uniform structures on an abstract integration space, we define corresponding topological vector lattices, and spaces of integration and measurable functions as the completion of these topological vector lattices. Finally, we show some convergence theorems on these spaces such as Lebesgue's monotone and dominated convergence theorems and Fatou's lemma.

## Tarmo Uustalu (Reykjavik University / Tallinn University of Technology),
## *"The proof theory of skew logics"*

I will talk about skew logics. By skew logics, I mean substructural logics defined by categories with various types of skew structure: skew monoidal categories and variations like skew closed or prounital closed categories, skew monoidal closed categories, partially normal skew monoidal categories, symmetric skew monoidal categories, etc.

Skew monoidal categories (due to K. Szláchanyi) differ from normal (ordinary) monoidal categories in that the structural laws of unitality and associativity of the tensor are not required to be natural isomorphisms; they are only natural transformations in a specific direction. Skew closed categories are a similar relaxation of closed categories. Partially normal skew monoidal categories are between skew and normal: they have some structural laws invertible, e.g., in left-normal skew monoidal categories, the left unitality law is a natural isomorphism.

In a series of papers, I, Niccolò Veltri, Cheng-Syuan Wan, and Noam Zeilberger have studied skew logics using a particular proof-theoretical methodology.

For each logic we have considered, we have devised a sequent calculus whose derivations represent maps of the free category with the relevant skew structure. Furthermore, we have identified a subcalculus of normal-form derivations oriented at root-first proof search, with the property that every map is represented by exactly one derivation. This has given us for each logic a method to decide equality of two maps and a method to enumerate without duplicates all maps between a pair of objects (this is a finite set).

We have found that, because they are so weak (generally dropping commutativity and some directions of unitality and associativity), skew logics provide excellent insights into the fine anatomy of substructural logics. In particular, they explain which ingredients contribute to which characteristics of the stronger more familiar substructural logics.

We started our project with purely multiplicative logics, but Niccolò Veltri and Cheng-Syuan Wan have also analyzed a combination of multiplicative connectives with additives.