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On the Interpretation of Intuitionistic Logic

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The present essay can be considered from two quite different standpoints.

1. If one does not accept the intuitionistic epistemological assumptions, then only the first section is relevant. The results of this section can be summarized approximately as follows:

In addition to theoretical logic, which systematizes the proof schemata of theoretical truths, one can systematize the schemata of the solution of problems, for example, of geometrical construction problems. For example, corresponding to the principle of syllogism the following principle occurs here: *If we can reduce the solution of b to the solution of a, and the solution of c to the solution of b, then we can also reduce the solution of c to the solution of a.*

One can introduce a corresponding symbolism and give the formal computational rules for the symbolical construction of the system of such schemata for the solution of problems. Thus in addition to theoretical logic one obtains a new *calculus of problems*. One needs here no special epistemological, for example, intuitionistic, assumptions.

Then the following remarkable fact holds: *The calculus of problems is formally identical with the Brouwerian intuitionistic logic, which has recently been formalized by Mr. Heyting.*[1]

2. In the second section intuitionistic logic is critically investigated while accepting the general intuitionistic assumptions. It will be shown that intuitionistic logic should be replaced by the calculus of problems, for its objects are in reality not theoretical propositions but rather problems.

§1

We do not define what a *problem* is; rather we explain this by some examples. Problems are:

*"Zur Deutung der intuitionistischen Logik," *Mathematische Zeitschrift* **35**, 1932, pp. 58–65. Translated from the German by Paolo Mancosu. Published by permission of Springer-Verlag GmbH & Co. KG, Heidelberg.

1. To find four whole numbers x, y, z, n for which the relations

$$x^n + y^n = z^n, \qquad n > 2$$

hold.

2. To prove the falsity of Fermat's theorem.
3. To draw a circle passing through three given points (x, y, z).[2]
4. Provided that one root of the equation $ax^2 + bx + c = 0$ is given, to find the other one.
5. Provided that the number π is expressed rationally, say, $\pi = m/n$, to find an analogous expression for the number e.

That the second problem is different from the first is clear, and this does not yet constitute a particular intuitionistic claim.[3] The fourth and the fifth problem are examples of *conventional* problems; yet the assumption in the fifth problem is impossible, and consequently the problem itself is *without content* [*inhaltslos*]. In what follows, the proof that a problem is without content will always be considered as its solution.

We believe that according to these examples and explanations the concepts "problem" and "solution of a problem" can be employed without misunderstandings in all cases that occur in the concrete areas of mathematics.[4] From now on problems will be designated with lower-case italic letters $a, b, c, \ldots$.

If a and b are two problems, then $a \wedge b$ designates the problem "to solve both problems a and b," while $a \vee b$ designates the problem "to solve at least one of the problems a and b." Furthermore, $a \supset b$ is the problem "to solve b provided that the solution for a is given" or, equivalently, "to reduce the solution of b to the solution of a."

In the preceding we did not assume that every problem is solvable. If, for example, Fermat's theorem is true, then the solution of the first problem would be contradictory. Correspondingly $\neg a$ designates the problem "to obtain a contradiction provided that the solution of a is given."[5]

If $a, b, c, d, \ldots$ are the problems, then, according to the above definitions, every formula $p(a, b, c, \ldots)$ constructed with the help of the signs $\wedge, \vee, \supset, \neg$ also designates a problem. However, if $a, b, c, \ldots$ are only symbols for indeterminate problems, then one says that $p(a, b, c, \ldots)$ is a function of the problem variables $a, b, c, \ldots$. If x is a variable (of any sort) and $a(x)$ designates a problem whose meaning depends on the value of x, then $(x)a(x)$ stands in general for the problem "to give a general method for the solution of $a(x)$ for every single value of x." One should understand this as follows: To solve the problem $(x)a(x)$ means to be able to solve for any given single value x_0 of x the problem $a(x_0)$ after a finite number of steps known in advance (prior to the choice of x_0).[6]

Furthermore, for the functions $p(a, b, c, \ldots)$ of indeterminate problems $a, b, c, \ldots$ one simply writes

$$\vdash p(a, b, c, \ldots)$$

instead of

$$(a)(b)(c)\ldots p(a, b, c, \ldots)^{6a}$$

Thus $\vdash p(a, b, c, \ldots)$ designates the problem "to give a general method for the solution of $p(a, b, c, \ldots)$ for any single choice of problems $a, b, c, \ldots$."

The problems of the form $\vdash p(a, b, c, \ldots)$, where p is expressed by means of the signs $\vee$, $\wedge$, $\supset$, and $\neg$, form the object of the *elementary calculus of problems*.[7]

The corresponding functions $p(a, b, c, \ldots)$ are *the elementary functions of problems*.

The fact that *I* have solved a problem is a purely subjective fact that in itself has as yet no general interest. However, the logical and mathematical problems possess the special property *of the general validity of their solutions*: If I have solved a logical or a mathematical problem, then I can present this solution in a way that is intelligible to all and it is *necessary* that it be recognized as a correct solution although this necessity has to a certain extent an ideal character, for it presupposes a sufficient intelligence on the part of the listener.[8]

The proper goal of the calculus of problems consists in giving a method for the solution of problems of the form $\vdash p(a, b, c, \ldots)$, where $p(a, b, c, \ldots)$ is an elementary function of problems, by means of the mechanical application of some simple computational rules. However, in order to reduce everything to these computational rules, we must assume that the solution of some elementary problems are already known. We *postulate* that we have already solved the following two groups of problems A and B. The further presentation addresses only a reader who has already solved all these problems.[9]

$$\text{A.}\quad \left\{\begin{array}{ll}
2.1. & \vdash . a \supset a \wedge a.^{10} \\
2.11. & \vdash . a \wedge b \supset b \wedge a. \\
2.12. & \vdash . a \supset b . \supset . a \wedge c \supset b \wedge c. \\
2.13. & \vdash . a \supset b . \wedge . b \supset c . \supset . a \supset c. \\
2.14. & \vdash . b \supset . a \supset b. \\
2.15. & \vdash . a \wedge . a \supset b . \supset b. \\
3.1. & \vdash . a \supset a \vee b. \\
3.11. & \vdash . a \vee b \supset b \vee a. \\
3.12. & \vdash . a \supset c . \wedge . b \supset c . \supset . a \vee b \supset c. \\
4.1. & \vdash . \neg a \supset . a \supset b. \\
4.11. & \vdash . a \supset b . \wedge . a \supset \neg b . \supset \neg a.^{a}
\end{array}\right.$$

We thus assume that the reader can solve all the problems appearing here after the sign $\vdash$ for any choice of problems a, b, c. This also presents no difficulties at all. For example, in problem (2.12) one must, under the assumption that the solution of a has already been reduced to the solution of b, reduce the solution of $b \wedge c$ to the solution of $a \wedge c$. Let the solution of $a \wedge c$ be given; this means that both the solution of a and the solution of c are given. From the solution of a we can infer, according to the assumption, the solution of b. And since the solution of c is already given, we obtain the solution of both problems b and c, and thus the solution $b \wedge c$. In this consideration is contained a general method for the solution of the problem

$$a \supset b . \supset . a \wedge c \supset b \wedge c$$

which is valid for arbitrary a, b, c. We thus have the right to consider the problem

$$2.13. \ \vdash . a \supset b . \supset . a \wedge c \supset b \wedge c.$$

(with the universal sign $\vdash$) as solved.

Regarding Problem 4.1 in particular: As soon as $\neg a$ is solved, then the solution of a is impossible and the problem $a \supset b$ is without content.

The second *group of problems B* for which we postulate the existence of solutions contains only three problems.[11] Namely, we assume that we are always able to (or that we possess a general method for) solving the following problems for arbitrary elementary function problems $p, q, r, s, \ldots$:

I. If $\vdash p \wedge q$ is solved, to solve $\vdash p$.
II. If $\vdash p$ and $\vdash p \supset q$ are solved, to solve $\vdash q$.
III. If $\vdash p(a, b, c, \ldots)$ is solved, to solve $\vdash p(q, r, s, \ldots)$.

We can now give *the rules* of our calculus of problems:

1. First of all we put the problems of the group A on the list of solved problems.
2. If $\vdash p \wedge q$ is already on our list, then $\vdash p$ can also be added to it.
3. If both formulas $\vdash p$ and $\vdash p \supset q$ are there, then $\vdash q$ may also be added.
4. If $\vdash p(a, b, c, \ldots)$ is already on the list and $q, r, s, \ldots$ are arbitrary functions of problems, then $\vdash p(q, r, s, \ldots)$ can also be added to it.

On the basis of the previously assumed postulates, one convinces oneself easily that these formal computations actually guarantee the solution of the corresponding problems.

We abstain here from further carrying out these computations, since all the above formal rules of computation and a priori written formulas coincide with the rules of computation and axioms of Heyting's first essay[12]; consequently we can interpret all the formulas of his essay as problems and regard as solved all these problems.

Among these problems we only make a note here of some especially interesting ones (which are to be considered as already solved):

$$\begin{array}{ll}
4.3. & \vdash . a \supset \neg\neg a. \\
4.2. & \vdash . a \supset b . \supset . \neg b \supset \neg a. \\
4.32. & \vdash . \neg\neg\neg a \supset \neg a.
\end{array}$$

The solution of 4.3 and 4.2 is clear without any computation. The solution of 4.32 is obtained from 4.3 and 4.2 by substituting in 4.2 the problem b by $\neg\neg a$.

If one adds to the formulas B assumed a priori the formula

$$\vdash . a \vee \neg a. \tag{1}$$

(in propositional logic the principle of the excluded middle), then one obtains a complete axiom system of classical propositional logic. In our problem interpretation the formula (1) reads as follows: to give a general method that allows, for every problem a, either to find a solution for a, or to infer a contradiction from the existence of a solution for a!

In particular, if the problem a consists in the proof of a proposition, then one must possess a general method either to prove or to reduce to a contradiction any proposition. If our reader does not consider himself to be omniscient, he will probably determine that the formula (1) cannot be found on the list of problems solved by him.

It is, however, remarkable that one can solve the problem

$$4.8 \vdash . \neg\neg . a \vee \neg a.^{13}$$

as Heyting's calculus shows.

Likewise, the formula

$$\vdash . \neg\neg a \supset a. \qquad (2)$$

(in the propositional calculus the principle of double negation) cannot appear in our calculus of problems, for the formula (1) follows from it by means of 4.8.

Therefore we see that, unlike Heyting's formulas of intuitionistic logic, already very simple formulas of classical propositional logic cannot appear in our calculus of problems.

Let us also remark that if a formula $\vdash p$ is false in the classical propositional calculus, the corresponding problem $\vdash p$ cannot be solved. Actually, from such a formula $\vdash p$ one can infer by means of the previously assumed formulas and rules of computation of the calculus of problems the obviously contradictory formula $\vdash . \neg a$.[14]

§2

The basic principle of the intuitionistic critique of logical and mathematical theories is the following: *Any proposition that is not without content should refer to one or more completely determinate states of affairs accessible to our experience.*[15]

If a is a general proposition of the form "any element of the set K possesses the property A," and if in addition the set K is infinite, then the negation of a, "a is false" does not satisfy the above principle. In order to avoid this situation Brouwer gives a new definition of negation :"a is false" should mean "a leads to a contradiction." Thus the negation of a is transformed into an *existential proposition*: "There exists a chain of logical inferences that, under the assumption of the correctness of a, leads to a contradiction." However, the existential propositions are also subjected by Brouwer to a deep critique.

Namely, from the intuitionistic standpoint it makes no sense at all simply to say: "There is among the elements of an infinite set K at least one element with the property A" without exhibiting this element.

However, Brouwer does not want to throw the existential propositions completely outside of mathematics. He only declares that one should not state an existential proposition without giving a corresponding construction. On the other hand,

an existential proposition does not, according to Brouwer, merely consist in stating that we have already found a corresponding element in K; in the latter case the existential proposition would be false *before* the invention [*Erfindung*] of the construction, and only *after that* would it be true. Thus arises this quite special type of proposition that is supposed to have a content that does not change over time but even so may only be stated under special conditions.

Of course, one can ask whether this special type of proposition is perhaps only a fiction. Actually, we are given a problem: "to find in the set K an element with the property A"; this problem has really a determinate meaning that is independent of our knowledge; if one has *solved* this problem, that is, if one has found a corresponding element x, then one obtains an empirical proposition: "Now our problem is solved." Thus what Brouwer understands by an existential proposition is completely broken down into two elements: the objective element (the problem) and the subjective element (the solution). As a consequence one finds no object left that one would have to call an existential proposition in the proper sense.

Therefore, the main result of the intuitionistic critique of negative propositions should simply be formulated as follows: *For any universal proposition it is in general meaningless to consider its negation as a determinate proposition.* However, with this disappears the object of intuitionistic logic, for now the principle of the excluded middle holds for all the propositions for which the negation is meaningful in general.[16]

Hence it follows that one should consider the solution of problems as the independent goal of mathematics (in addition to the proofs of theoretical propositions). As was shown in the first section the formulas of intuitionistic logic also receive a new meaning in the field of problems and solutions.[17]

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Notes

1. Heyting, Die formalen Regeln der intuitionistischen Logik, *Sitz. d. Preus. Akad.* **I.** 1930 p. 42; **II.** p. 57; **III.** p. 158.
2. To be very precise, in the formulation of this problem one should indicate the allowed means of construction.
3. On the other hand, the *propositions* "Fermat's theorem is false" and "four numbers satisfying (1) exist" are equivalent from the standpoint of classical logic.
4. The main concepts of propositional logic, "proposition," and "proof of a proposition" are in no better position.
5. Let us observe that $\neg a$ should not be understood as the problem "to prove the unsolvability of a." If one considers in general "the unsolvability of a" as a well-defined concept, then one only obtains the theorem that from $\neg a$ follows the unsolvability of a, but not the converse. For example, if it were proven that the well-ordering of the continuum surpasses our abilities, one could still not claim that a contradiction follows from the existence of such a well-ordering.
6. Here, too, as earlier, we hope that these definitions cannot lead to any misunderstandings in the concrete mathematical domains.

6a. This explanation of the meaning of the sign ⊢ is very different from that of Heyting, although it leads to the same computational rules.
7. This definition is analogous to the definition of the elementary propositional calculus. However, in the propositional calculus the logical functions analogous to ∧, ∨, ⊃, ¬ can be expressed by two among them. In the calculus of problems all four functions are independent.
8. All this holds word for word also for the proof of theoretical propositions. It is, however, essential that every proven proposition is *correct* [*richtig*]; for problems one has no concept corresponding to this *correctness* [*Richtigkeit*].
9. In the case of the propositional calculus, one must first convince oneself of the correctness of the axioms, if one wants to determine the correctness of the consequences.
10. On the numbering of the formulas and the use of separation marks (dots), see Heyting I.
11. They cannot, however, be expressed symbolically with the signs of the elementary calculus of problems.
12. Heyting I.
13. In propositional logic (4.8) represents the Brouwerian proposition on the consistency of the principle of the excluded middle.
14. See V. Glivenko, Acad. r. de Belgique, 5e série, **15**, 1929, p. 183.
15. See H. Weyl, Über die neue Grundlagenkrise der Mathematik, *Math. Zeitschr.* **10**, 1921, p. 39. The whole further investigation of negative and existential propositions essentially follows this work by Weyl.
16. However, a new question arises: Which logical laws are valid for propositions whose negation has no sense?
17. Remark inserted in proof. This interpretation of intuituionistic logic is closely connected with the ideas Mr. Heyting has developed in the last volume of *Erkenntnis* **2**, 1931, p. 106; yet in Heyting a clear distinction between propositions and problems is missing.

Translator's Notes

a. The original text contains a misprint.

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