How to think of proofs of arguments:

Think of a proof as a *path* from premises to a conclusion, i.e., as a way for getting to the conclusion from the premises. Each line in the proof (each step you take down the path) **must** be allowed by a rule.

This means that in order to offer a proof of an argument you need two skills that we have used before and one new skill: (i) recognizing that a formula might be an instance of different forms, (ii) identifying which forms a formula is an instance of, and (iii) making a plan for the proof (i.e., sketching the path).

1. For each of the following formulas, give at least two statements with metavariables[1] that the formula is an instance of:

   (a) $p \to ((\neg p \land s) \lor q)$

   (b) $(q \to r) \land (r \lor p)$

   (c) $\neg(r \lor t) \lor t$

2. See the following rules:

$$\begin{array}{l|ll} l & \varphi \land \psi & \\ n & \varphi & \text{E}\land\ 1 \end{array} \qquad \begin{array}{l|ll} l & \varphi \land \psi & \\ m & \psi & \text{E}\land\ 1 \end{array}$$

$$\begin{array}{l|ll} l & \phi & \\ m & \psi & \\ n & \phi \land \psi & \text{I}\land\ 1,\ m \end{array} \qquad \begin{array}{l|ll} l & \phi \to \psi & \\ m & \phi & \\ n & \psi & \text{E}\to\ 1,\ m \end{array}$$

$$\begin{array}{l|ll} l & \quad \begin{array}{|l} \phi \\ \hline \\ \end{array} & \\ m & \quad \begin{array}{|l} \psi \end{array} & \\ n & \phi \to \psi & \text{I}\to\ \text{l-m} \end{array} \qquad \begin{array}{l|ll} l & \varphi \lor \psi & \\ m & \varphi \to \chi & \\ n & \psi \to \chi & \\ p & \chi & \text{E}\lor\ 1,\ m,\ n \end{array}$$

$$\begin{array}{l|ll} l & \varphi & \\ m & \varphi \lor \psi & \text{I}\lor\ 1 \end{array} \qquad \begin{array}{l|ll} l & \varphi & \\ m & \psi \lor \varphi & \text{I}\lor\ 1 \end{array}$$

[1] Greek letters like ϕ, ψ and χ

Now, determine which of the following proofs have a correct instance of the rule in question (i.e., which of the following applications of rules have no mistakes).

(a) I$\wedge$

	$\vdots$	
7	p	...
8	$q \to r$	...
	$\vdots$	
13	$(q \to r) \wedge p$	I$\wedge$ 1, 13

1	$s \to p$	Ass.
2	$u \vee r$	Ass.
	$\vdots$	
23	$s \wedge u$	I$\wedge$ 1, 2

1	$s \wedge u$	Ass.
	$\vdots$	
10	$q \vee \neg r$	....
	$\vdots$	
16	$(q \vee \neg r) \wedge (s \wedge u)$	I$\wedge$ 1, 10

(b) E$\wedge$

1	$(p \wedge q) \wedge r$	Ass.
	$\vdots$	
	$\vdots$	
13	r	E$\wedge$ 1
14	$p \wedge q$	E$\wedge$ 1

	$\vdots$	
16	$(p \wedge q) \wedge r$	...
17	q	E$\wedge$ 16
18	r	E$\wedge$ 16
	$\vdots$	

1	$p \to (q \wedge u)$	Ass.
2	$(q \wedge u) \to r$	Ass.
3	p	Ass.
4	$q \wedge u$	I$\to$ 1,3
5	r	I$\to$ 2,4
6	$p \to r$	I$\to$ 3-5
7	q	E$\wedge$ 4

(c) E$\to$

1	$(q \vee t) \to p$	Ass.
2	q	Ass.
	$\vdots$	
13	p	E$\to$ 1,2

	$\vdots$	
4	$(p \wedge q)$	...
5	$p \to (q \to r)$	...
	$\vdots$	
13	r	I$\wedge$ 4,5

1	$(w \vee s) \to t$	Ass.
	$\vdots$	
8	$w \vee s$	...
	$\vdots$	
27	t	E$\to$ 1, 8

(d) I$\vee$

	$\vdots$	
16	$(p \wedge q) \wedge r$	...
17	q	E$\wedge$ 16
	$\vdots$	

	$\vdots$	
4	p	Ass.
	$\vdots$	
8	$(p \vee q) \vee r$	I$\vee$ 4

1	p	Ass.
2	$q \vee p$	I$\vee$ 1

(e) I$\rightarrow$

	$\vdots$	
6	$(p \wedge q) \wedge r$	...
7	u	Ass.
8	r	E$\wedge$ 6
9	$u \rightarrow r$	I$\rightarrow$ 7-8
	$\vdots$	

	$\vdots$	
11	$r \rightarrow s$	...
12	$(p \wedge q) \wedge r$	...
13	u	Ass.
14	r	E$\wedge$ 12
15	s	E$\rightarrow$ 11, 14
16	$u \rightarrow s$	I$\rightarrow$ 14-15
	$\vdots$	

1	p	Ass.
2	q	Ass.
3	q	Rep. 2
4	$q \rightarrow q$	I$\rightarrow$ 2-3

(f) E$\vee$

	$\vdots$	
13	$(r \vee t) \rightarrow p$	...
14	$s \rightarrow p$	...
15	$u \wedge t$	...
16	$(r \vee t) \vee s$	...
17	p	E$\vee$ 13,14,16
18	u	E$\wedge$ 15
19	$u \wedge p$	I$\wedge$ 17,18
	$\vdots$	

1	$r \rightarrow s$	Ass.
2	$(p \wedge q) \rightarrow s$	Ass.
3	t	Ass.
4	$(p \wedge q) \vee r$	Ass.
5	s	E$\vee$ 1,2,4
	$\vdots$	

3. Offer a proof for the following arguments:

   (a) $s, s \to p, (p \wedge q) \to r, q \vdash r$

   (b) $(p \vee q) \to s, p \vdash s$

   (c) $s \to t, r \to t, p \wedge (s \vee r) \vdash t$

   (d) $r \to q, s, (q \wedge s) \to u \vdash r \to q$

   (e) **Hard:** $s \to r, q \to r, u \to q, (r \vee u) \to p, s \vee u \vdash (w \to r) \wedge r$[2]

[2] Requires the rule Repetition.