PART VII

1941 and 1944

1. The role of propositions which deal with measures and are not 'empirical propositions'.—Someone tells me: "this stretch is two hundred and forty inches long". I say: "that's twenty foot, so it's roughly seven paces" and now I have got an idea of the length.—The transformation is founded on arithmetical propositions and on the proposition that 12 inches = 1 foot.

No one will ordinarily see this last proposition as an empirical proposition. It is said to express a convention. But measuring would entirely lose *its ordinary character* if, for example, putting 12 bits each one inch long end to end didn't ordinarily yield a length which can in its turn be preserved in a special way.

Does this mean that I have to say that the proposition '12 inches = 1 foot' asserts all those things which give measuring its present point?

No. The proposition *is grounded in* a technique. And, if you like, also in the physical and psychological facts that make the technique *possible*. But it doesn't follow that its sense is to express these conditions. The opposite of that proposition, 'twelve inches = one foot' does not say that rulers are not rigid enough or that we don't all count and calculate in the same way.

2. The proposition has the typical (but that doesn't mean *simple*) role of a rule.

I can use the proposition '12 inches = 1 foot' to make a prediction; namely that twelve inch-long pieces of wood laid end to end will turn out to be of the same length as one piece measured in a different way. Thus the point of that rule is, e.g., that it can be used to make certain predictions. Does it lose the character of a *rule* on that account?

Why can one make those predictions? Well,—all rulers are made alike; they don't alter much in length; nor do pieces of wood cut up into inch lengths; our memory is good enough for us not to take numbers twice in counting up to '12', and not to leave any out; and so on.

But then can the rule not be replaced by an empirical proposition saying that rulers are made in such and such ways, that people do *this* with them? One might give an ethnological account of this human institution.

Now it is evident that this account could take over the function of a rule.

If you know a mathematical proposition, that's not to say you yet know anything. If there is confusion in our operations, if everyone calculates differently, and each one differently at different times, then there isn't any calculating yet; if we agree, then we have only set our watches, but not yet measured any time.

If you know a mathematical proposition, that's not to say you yet know *anything*.

I.e., the mathematical proposition is only supposed to supply a framework for a description.

3. How can the mere transformation of an expression be of practical consequence?

The fact that I have 25×25 nuts can be verified by my counting 625 nuts, but it can also be discovered in another way which is closer to the form of expression '25×25'. And of course it is in the linking of these two ways *of determining* a number that one point of multiplying lies.

A rule *qua* rule is detached, it stands as it were alone in its glory; although what gives it importance is the facts of daily experience.

What I have to do is something like describing the office of a king;—in doing which I must never fall into the error of explaining the kingly dignity by the king's usefulness, but I must leave neither his usefulness nor his dignity out of account.

I am guided in practical work by the result of transforming an expression.

But in that case how can I still say that it means the same thing whether I say "here are 625 nuts", or "here are 25×25 nuts"?

If you verify the proposition "here are 625 . . ." then in doing that you are also verifying "here are 25 × 25 . . ."; etc. But the one form is closer to one kind of verification, the other closer to another.

How can you say that ". . . 625 . . ." and ". . . 25 × 25 . . ." say the same thing?—Only through our arithmetic do they *become one*.

I can at one time arrive at the one, and at another time at the other kind of description, e.g. by counting. That is to say, I can arrive at either of these forms in either way; but by different routes.

It might now be asked: if the proposition ". . . 625 . . ." was verified at one time in this way and at another time in a different way, then did it mean the same thing both times?

Or: what happens if one method of verification gives '625', but the other not '25 × 25'?—Is ". . . 625 . . ." true and ". . . 25 times 25 . . ." false? No.—To doubt the one means to doubt the other: that is the grammar given to these signs by our arithmetic.

If both ways of counting are supposed to justify *giving a number* then giving *one* number, even though in different forms, is all that is *provided for*. On the other hand there is no contradiction in saying: "By one method of counting I get 25 × 25 (and so 625), by the other not 625 (and so not 25 × 25)". Arithmetic has no objection to this.

For arithmetic to equate the two expressions is, one might say, a grammatical trick.

In this way arithmetic bars a particular kind of description and conducts description into other channels. (And it goes without saying that this is connected with the facts of experience.)

4. Suppose I have taught somebody to multiply; not, however, by using an explicit general rule, but only by his seeing how I work out examples for him. I can then set him a *new* question and say: "Do the same with *these* two numbers as I did with the previous ones". But I can also say: "If you do with these two what I did with the others, then you will arrive at the number . . .". What kind of proposition is that?

"You will write such-and-such" is a prediction. 'If you write such-and-such, then you will have done it as I shewed you' determines what he calls "following his example".

'The solution to this problem is . . .'.—If I read this before I have worked out the sum,—what sort of proposition is it?

"If you do with these numbers what I did with the others, you will get . . ."—that surely means: "The result of this calculation is . . ."—and that is not a prediction but a mathematical proposition. But it is none the less a prediction too—A prediction of a special kind. Just as someone who at the end finds that he really does get such-and-such when he adds up the column may be really surprised; for example may exclaim: Good Lord, it does come out!

Just think of this procedure of prediction and confirmation as a

special language-game—I mean: isolated from the rest of arithmetic and its application.

What is so singular about this game of prediction? What strikes me as singular would disappear if the prediction ran: "If you believe that you have gone by my example, then you will have produced *this*" or: "If everything seems correct to you, *this* will be the result". This game could be imagined in connexion with the administration of a particular poison and the prediction would be that the injection affects our faculties, our memory for example, in such-and-such a way.—But if we can imagine the game with the administration of a poison, then why not with the administration of a medicine? But even then the weight of the prediction may still always rest on the fact that the *healthy* man sees *this* as the result. Or perhaps: that *this* satisfies the healthy man.

"Do as I do, and this is what you will get" doesn't of course mean: "If you do as I do then you will do as I do"—nor: "Calculate like *this*, and you will calculate like *this*".—But what does "Do as I do" mean? In the language game—it can simply be an order: "Now do as I do!"

What is the difference between these predictions: "If you calculate correctly you will get *this* result"—and: "If you believe you are calculating correctly you will get *this* result"?

Now who says that the prediction in my language-game above does not mean the latter? It seems not to——but what *shews* this? Ask yourself *in what circumstances* the prediction would seem to predict the one thing and in what circumstances the other. For it is clear that it all depends on the rest of the circumstances.

If you predict that I shall get *this*, are you not simply predicting that I shall take this result as correct?—"But"—perhaps you say—"only because it really *is* correct!"—But what does it mean to say: "I take the calculation as correct because it is correct"?

And yet we can say: the person who is calculating in my language-game does not think of it as a peculiarity of *his* nature that he gets *this*; the fact does not appear to him as a psychological one.

I am imagining him as under the impression that he has only followed a thread that is already there, and accepting the How of the following as something that is a matter of course; and only knowing *one* explanation of his action, namely: how the thread runs.

He does just let himself go on when he follows the rule or the examples; however, he does not regard what he does as a peculiarity of *his* course; he says, not: "so *that's* how I went", but: "so *that's* how it goes".

But now, suppose someone did say at the end of the calculation in our language-game: "so *that's* how I went"—or: "so *this* course satisfies me"—can I say he has misunderstood the whole language-game? Certainly not! So long as he does not make some further unwelcome application of it.

5. Isn't it the *application* that elicits that conception: that it is not we, but the calculation, that takes a certain course?

The different 'conceptions' must correspond to different applications.

For there is indeed a distinction between these two things: being surprised that the figures on the paper seem to behave like *this*; and being surprised that *this* is what comes out as the result. In each case, however, I see the calculation in a different context.

I think of the feeling of its 'coming out' when for instance we add up a rather long column of numbers of various patterns, and a round number of 1,000,000 comes out, as we had been told it would before, "Yes, by Jove, another nought—" we say.
"One wouldn't guess it from looking at the numbers", I might say.

How would it be if we said—instead of '6 × 6 gives 36':—'The number 36's being given by 6 × 6'?—Replacing the *proposition* by a substantival expression. (The proof shews *the being given.*)

Why do you always want to look at mathematics under the aspect of finding and not of doing?

It must have a great influence, that we use the words "right" and "true" and "wrong" and the form of statement, in calculating. (Head-shaking and nodding.)

Why should I say that the knowledge that *this* is the way in which all human beings who have learned to calculate do calculate isn't *mathematical* knowledge? Because it seems to point in the direction of a different context.

Then is working out what someone will get out by a calculation already applied mathematics?—and hence also: working out what I myself get out?

6. There is no doubt at all that *in certain language-games* mathematical propositions play the part of rules of description, as opposed to descriptive propositions.

But that is not to say that this contrast does not shade off in all directions. And *that* in turn is not to say that the contrast is not of the greatest importance.

We feel that mathematics stands on a pedestal—this pedestal it has because of a particular role that its propositions play in our language games.

What is proved by a mathematical proof is set up as an internal relation and withdrawn from doubt.

7. What is common to a mathematical proposition and a mathematical proof, that they are both called "mathematical"?

Not, that the mathematical proposition has to be proved mathematically; not, that the mathematical proof has to prove a mathematical proposition.

What is mathematical about an unproved proposition (an axiom)? what has it in common with a mathematical *proof*?

Should I answer: "The inference rules of mathematical proof are always mathematical propositions"? Or: "Mathematical propositions and proofs are used in inference"? That would be getting closer to the truth.

8. Proof must shew an internal relation, not an external one. For we might also imagine a process of transforming a sentence by *experiment*, and a transformation which would be used to predict what would be asserted by the transformed sentence. One might imagine, e.g., signs getting shifted through adding other signs to them, in such fashion that they form a true prediction on the basis of the conditions expressed in their initial position. And if you like, you may regard the calculating human being as an apparatus for such an experiment.

For, that a human being *works out* the result, in the sense that he doesn't write down the result at once, but only after he has written down various other things—doesn't make him any the less a physical-chemical means of producing one sequence of signs from another.

Thus I should have to say: The proved proposition is not: that sequence of signs which the man who has received such-and-such schooling produces under such-and-such conditions.

When we think of proving in that way, what we see in it changes entirely. The intermediate steps become an uninteresting by-product. (Like a rattle in the insides of the automatic machine before it discharges its wares for us.)

9. We say that a proof is a picture. But this picture stands in need of ratification, and that we give it when we work over it.—

True enough; but if it got ratification from one person, but not from another, and they could not *come to any understanding*—would what we had here be calculation?

So it is not the ratification by itself that makes it calculation but the agreement of ratifications.

For another game could quite well be imagined, in which people were prompted by expressions (similar perhaps to general rules) to let sequences of signs come to them for particular practical purposes, i.e. *ad hoc*; and that this even proved to pay. And here the 'calculations' if we choose to call them that, do not have to agree with one another. (Here we might speak of 'intuition'.)

The agreement of ratifications is the pre-condition of our language-game, it is not affirmed in it.

If a calculation is an experiment and the *conditions are fulfilled*, then we must accept whatever comes, as the result; and if a calculation is an experiment then the proposition that it yields such and such a result is after all the proposition that under such conditions this kind of sign makes its appearance. And if under these conditions one result appears at one time and another at another, we have no right to say "there's something wrong here" or "both calculations cannot be all right", but we should have to say: this calculation does not always yield the same result (*why* need not be known). But although the procedure is now just as interesting, perhaps even more interesting, what we have here *now* is no longer calculation. And this is of course a grammatical remark about the use of the word "calculation". And this grammar has of course a point.

What does it mean to reach an *understanding* about a difference in the result of a calculation? It surely means to arrive at a calculation that is free of discrepancy. And if we can't reach an understanding, then the one cannot say that the other is calculating too, only with different results.

10. Now how about this—ought I to say that the same sense can only have *one* proof? Or that when a proof is found the sense alters?

Of course some people would oppose this and say: "Then the proof of a proposition cannot ever be found, for, if it has been found, it is no longer the proof of *this* proposition". But to say this is so far to say nothing at all.—

It all depends *what* settles the sense of a proposition, what we choose

to say settles its sense. The use of the signs must settle it; but what do we count as the use?—

That these proofs prove the same proposition means, e.g.: both demonstrate it as a suitable instrument for the same purpose.

And the purpose is an allusion to something outside mathematics.

I once said: 'If you want to know what a mathematical proposition says, look at what its proof proves'.[1] Now is there not both truth and falsehood in this? For is the sense, the point, of a mathematical proposition really clear as soon as we can follow the proof?

What Russell's '$\sim f(f)$' lacks above all is application, and hence meaning.

If we do apply this form, however, that is not to say that '$f(f)$' need be a proposition in any ordinary sense or '$f(\xi)$' a propositional function. For the concept of a proposition, apart from that of a proposition of logic, is only explained in Russell in its general conventional features.

Here one is looking at language without looking at the language-game.

When we say of different sequences of configuration that they shew e.g. that $25 \times 25 = 625$, it is easy enough to recognize what fixes the

[1] cf. *Philosophical Grammar*, p. 369 f; cf. also *Philosophical Remarks*, pp. 183, 184. (Eds.)

place of this proposition, which is reached by the two routes.

A new proof gives the proposition a place in a new system; here there is often a translation of one kind of operation into a quite different kind. As when we translate equations into curves. And then we realize something about curves and, by means of that, about equations. But what right have we to be convinced by lines of thought which are apparently quite remote from the object of our thought?

Well, our operations are not more remote from that object than is, say, dividing in the decimal system from sharing out nuts. Especially if one imagines (what is quite easy to imagine) that operation as originally invented for a different purpose from that of making divisions and the like.

If you ask: "What right have we?" the answer is: perhaps none.— What right have you to say that the development of this system will always run parallel with that one? (It is as if you were to fix *both* inch and foot as units, and assert that $12n$ inches will always be the same length as n feet.)

When two proofs prove the same proposition it is possible to imagine circumstances in which the whole surrounding connecting these proofs fell away, so that they stood naked and alone, and there were no cause to say that they had a common point, proved the same proposition.

One has only to imagine the proofs without the organism of applications which envelopes and connects the two of them: as it were stark naked. (Like two bones separated from the surrounding manifold

context of the organism; in which alone we are accustomed to think of them.)

11. Suppose that people calculated with numbers, and sometimes did divisions by expressions of the form $(n - n)$, and in this way occasionally got results different from the normal results of multiplying etc. But that nobody minded this.—Compare with this: lists, rolls, of people are prepared, but not alphabetically as we do it; and in this way it happens that in some lists the same name appears more than once.—— But now it can be supposed that this does not strike anyone; or that people see it, but accept it without worrying. As we could imagine people of a tribe who, when they dropped coins on the ground, did not think it worth while to pick them up. (They have, say, an idiom for these occasions: "It belongs to the others" or the like.)

But now times have changed and people (at first only a few) begin to demand exactness. Rightly, wrongly?—Were the earlier lists *not* really lists?—

Say we quite often arrived at the results of our calculations through a hidden contradiction. Does that make them illegitimate?——But suppose that we now absolutely refuse to accept such results, but still are afraid that some might slip through.—Well then, in that case we have an idea which might serve as a model for a new calculus. As one can have the idea of a new game.

The Russellian contradiction is disquieting, not because it is a contradiction, but because the whole growth culminating in it is a cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly.

Now can we say: "We want a calculus which more certainly tells us the truth"?

But you can't allow a contradiction to stand!—Why not? We do sometimes use this form in our talk, of course not often—but one could imagine a technique of language in which it was a regular instrument.

It might for example be said of an object in motion that it existed and did not exist in this place; change might be expressed by means of contradiction.

Take a theme like that of Haydn's (St. Antony Chorale), take the part of one of Brahms's variations corresponding to the first part of the theme, and set the task of constructing the second part of the variation in the style of its first part. That is a problem of the same kind as mathematical problems are. If the solution is found, say as Brahms gives it, then one has no doubt;—that is the solution.

We are agreed on this route. And yet, it is obvious here that there may easily be different routes, on each of which we can be in agreement, each of which we might call consistent.

'We take a number of steps, all legitimate—i.e. allowed by the rules—and suddenly a contradiction results. So the list of rules, as it is, is of no use, for the contradiction wrecks the whole game!' Why do you have it wreck the game?

But what I want is that one should be able to go on inferring *mechanically* according to the rule without reaching any contradictory results. Now, what kind of provision do you want? One that your present calculus does not allow? Well, that does not make that calculus a bad piece of mathematics,—or not mathematics in the fullest sense. The meaning of the word "mechanical" misleads you.

12. When, for some practical purpose, you want to avoid a contradiction mechanically, as your calculus so far cannot do, this is e.g. like looking for a construction of the . . .-gon, which you have up to now only been able to draw by trial and error; or for a solution of a third degree equation, to which you have so far only approximated.

What is done here is not to improve bad mathematics, but to create a new bit of mathematics.

Suppose I wanted to determine that the pattern '777' did not occur in the expansion of an irrational number. I might take π and settle that if that pattern occurs, we replace it by '000'. Now I am told: that is not enough, for whoever is calculating the places is prevented from looking back to the earlier ones. Now I need another calculus; one in which I can be assured in advance that it cannot yield '777'. A mathematical problem.

'So long as freedom from contradiction has not been proved I can never be quite certain that someone who calculates without thinking, but according to the rules, won't work out something wrong.' Thus so long as this provision has not been obtained the calculus is untrustworthy.—But suppose that I were to ask: "*How* untrustworthy?" —If we spoke of degrees of untrustworthiness mightn't this help us to take the metaphysical sting out of it?

Were the first rules of the calculus not good? Well, we gave them only *because* they were good.—If a contradiction results later,—have they *failed* in their office? No, they were not given for this application.

I may want to supply my calculus with a particular kind of provision. This does not make it into a *proper* piece of mathematics, but e.g. into one that is more useful for a certain purpose.

The idea of the mechanization of mathematics. The fashion of the axiomatic system.

13. But suppose the 'axioms' and 'methods of inference' are not just some kind of construction, but are absolutely convincing. Well, this means that there are cases in which a construction out of these elements is *not* convincing.

And the logical axioms are in fact not at all convincing if for the propositional variables we substitute structures which no one originally foresaw as possible values, when, that is, we began by acknowledging the truth of the axioms absolutely.

But what about saying: the axioms and methods of inference surely ought to be so chosen that they cannot prove any false proposition?

'We want, not just a fairly trustworthy, but an *absolutely* trustworthy calculus. Mathematics must be *absolute*.'

Suppose I had erected rules for a game of 'hare and hounds'—fancying it to be a nice amusing game.—Later, however, I find that the hounds can always win once one knows how.

Now, let's say, I am dissatisfied with my game. The rules which I gave brought forth a result which I did not foresee and which spoils the game for me.

14. "N. came upon the fact that in their calculations people had often reduced by expressions of the form '$(n - n)$'. He pointed out the consequent discrepancy of results and shewed how this way of calculating had led to the loss of human life."

But let us suppose that other people too had noticed these contradictions, only they had not been able to give any account of their source. They calculated as it were with a bad conscience. They had chosen *one* among contradictory results but with uncertainty, whereas N's discovery would have made them quite certain.—But did they tell themselves: "There's something wrong with our calculus"? Was their uncertainty of the same kind as ours when we do a physical calculation

but are not certain whether these formulae really give the correct result here? Or was it a doubt whether their calculating was really calculating? In this case: what did they do to get over the difficulty?

So far these people have only rarely made use of reduction by expressions of values. But some time somebody discovers that they can actually arrive at any arbitrary result in this way.—What do they do now? Well, we could imagine very different things. They may now, e.g., state that this kind of calculation has lost its point, and that in future people are not to calculate in *this* way any more.

'He believes that he is calculating'—one would like to say—'but as a matter of fact he is not calculating.'

15. If the calculation lost its point for me as soon as I knew I could work out any arbitrary result—did it have none so long as I did *not* know that?

I may of course now declare all these calculations to be null—for I have given up doing them now——but does that mean that they weren't calculations?

I at one time inferred *via* a contradiction without realizing it. Is my result then wrong, or at any rate wrongly got?

If the contradiction is so well hidden that no one notices it, why shouldn't we call what we do now proper calculation?

We say that the contradiction would *destroy* the calculus. But suppose it only occurred in tiny doses in lightning flashes as it were, not as a constant instrument of calculation, would it nullify the calculus?

Imagine people had fancied that $(a + b)^2$ must be equal to $a^2 + b^2$. (Is this a fancy of the same kind as that there must be a trisection of the angle by ruler and compass?) Is it possible, then, to fancy that two ways of calculating had to yield the same result, if it is not the same?

I add up a column, doing it in a variety of ways (e.g. I take the numbers in a different order), and I keep on getting random different results.—I shall perhaps say: "I am in a complete muddle, either I am making random mistakes in calculating, or I am making certain mistakes in particular connexions: e.g. always saying '$7 + 7 = 15$' after '$6 + 3 = 9$'."

Or I might imagine that suddenly, once in the sum, I subtract instead of adding, but don't think I am doing anything different.

Now it might be that I didn't find the mistake and thought I had lost my wits. But this would not have to be my reaction.

'Contradiction destroys the calculus'—what gives it this special position? With a little imagination, I believe, it can certainly be shaken.

To resolve these philosophical problems one has to compare things which it has never seriously occurred to anyone to compare.

In this field one can ask all sorts of things which, while they belong to the topic, still do not lead through its centre.

A particular series of questions leads through the centre and out into the open. The rest get answered incidentally.

It is enormously difficult to find the path through the centre.

It goes *via new* examples and comparisons. The hackneyed ones don't shew us it.

Let us suppose that the Russellian contradiction had never been found. Now—is it quite clear that in that case we should have possessed a false calculus? For aren't there various possibilities here?

And suppose the contradiction had been discovered but we were not excited about it, and had settled e.g. that no conclusions were to be drawn from it. (As no one does draw conclusions from the 'Liar'.) Would this have been an obvious mistake?

"But in that case it isn't a proper calculus! It loses all *strictness*!" Well, not *all*. And it is only lacking in full strictness, if one has a particular ideal of rigour, wants a particular style in mathematics.

'But a contradiction in mathematics is incompatible with its application.

'If it is consistently applied, i.e. applied to produce arbitrary results, it makes the application of mathematics into a farce, or some kind of superfluous ceremony. Its effect is e.g. that of non-rigid rulers which permit various results of measuring by being expanded and contracted.' But was measuring by pacing not measuring at all? And if people worked with rulers made of dough, would that of itself have to be called wrong?

Couldn't reasons be easily imagined, on account of which a certain elasticity in rulers might be desirable?

"But isn't it right to manufacture rulers out of ever harder, more unalterable material?" Certainly it is right; if that is what one wants!

'Then are you in favour of contradiction?' Not at all; any more than of soft rulers.

There is *one* mistake to avoid: one thinks that a contradiction *must* be senseless: that is to say, if e.g. we use the signs 'p', '$\sim$', '.' *consistently*,

then '$p. \sim p$' cannot say anything.—But think: what does it mean to continue such and such a use 'consistently'? ('A consistent continuation of this bit of a curve.')

16. What does mathematics need a foundation for? It no more needs one, I believe, than propositions about physical objects—or about sense impressions, need an *analysis*. What mathematical propositions do stand in need of is a clarification of their grammar, just as do those other propositions.

The *mathematical* problems of what is called foundations are no more the foundation of mathematics for us than the painted rock is the support of a painted tower.

'But didn't the contradiction make Frege's logic useless for giving a foundation to arithmetic?' Yes, it did. But then, who said that it had to be useful for this purpose?

One could even imagine a savage's having been given Frege's logic as an instrument with which to derive arithmetical propositions. He derived the contradiction unawares, and now he derives arbitrary true and false propositions from it.

'Up to now a good angel has preserved us from going *this* way.' Well, what more do you want? One might say, I believe: a good angel will always be necessary, whatever you do.

17. One says that calculation is an experiment, in order to shew how it is that it can be so practical. For we do know that an experiment really does have practical value. Only one forgets that it possesses this value in virtue of a technique which is a fact of natural history, but whose rules do not play the part of propositions of natural history.

"The limits of empiricism."[1]—(Do we live because it is practical to live? Do we think because thinking is practical?)

He knows that an experiment is practical; and so calculation is an experiment.

Our experimental activities have indeed a characteristic physiognomy. If I see somebody in a laboratory pouring a liquid into a test tube and heating it over a Bunsen burner, I am inclined to say he is making an experiment.

Let us suppose that people, who know how to count, want—just as we do—to know numbers for practical purposes of various kinds. And to this end they ask certain people who, having had the practical problem explained to them, shut their eyes, and let the appropriate number occur to them——here there wouldn't be any calculation, however trustworthy the numbers given might be. This way of determining numbers might be even more trustworthy in practice than any calculation.

[1] cf. p. 197 n.

A calculation—it might be said—is perhaps a part of the technique of an experiment, but is by itself not an experiment.

Do we forget that a particular *application* is part of a procedure's being an experiment? And the calculation is an instrument of the application.

For would anyone *think* of calling the translation of a cipher by means of a key an experiment?

When I doubt whether n and m multiplied yield l, my doubt isn't about whether our calculating is going to fall into confusion, and e.g. half of mankind say one thing is right and the other half another.

An action is an 'experiment' only as seen from a certain point of view. And it is *obvious* that the action of calculating can also be an experiment.

I may for example want to test what this man calculates, in such-and-such circumstances, when set this question.—But isn't that exactly what you are asking when you want to know what 52×63 is? I may very well ask that—my question may even be expressed in these words. (Compare: is the sentence "Listen, she's groaning!" a proposition about her behaviour or about her suffering?)

But suppose I *work over* his calculation?—'Well, then I am making a further experiment so as to find out with complete certainty that all normal human beings react like that.'—And if they do *not* react uniformly—which one is the mathematical result?

18. "If calculation is to be practical, then it must uncover facts. And only experiment can do that."

But what things are 'facts'? Do you believe that you can shew what fact is meant by, e.g., pointing to it with your finger? Does that of itself clarify the part played by 'establishing' a fact?—Suppose it takes mathematics to define the *character* of what you are calling a 'fact'!

'It is interesting to know *how many* vibrations this note has! But it took arithmetic to teach you this question. It taught you to see this kind of fact.

Mathematics—I want to say—teaches you, not just the answer to a question, but a whole language-game with questions and answers.

'Are we to say that *mathematics* teaches us to count?

Can mathematics be said to teach us experimental *methods of investigation*? Or to help us to discover such methods of investigation?

'To be practical, mathematics must tell us facts.'—But do these facts have to be the *mathematical* facts?—But why should not mathematics, instead of 'teaching us facts', create the forms of what we call facts?

"Yes but surely it remains an empirical fact that men calculate like this!"—Yes, but that does not make the propositions used in calculating into empirical propositions.

"Yes, but surely our calculating must be founded on empirical facts!" Certainly. But what empirical facts are you now thinking of? The psychological and physiological ones that make it possible, or those that make it a useful activity? The connexion with *the latter* consists in the fact that the calculation is the picture of an experiment as it practically always turns out. From the former it gets its point, its physiognomy; but that is certainly not to say that the propositions of mathematics have the functions of empirical propositions. (That would almost be as if someone were to believe that because only the actors appear in the play, no other people could usefully be employed upon the stage of the theatre.)

There are no causal connexions in a calculation, only the connexions of the pattern. And it makes no difference to this that we work over the proof in order to accept it. That we are therefore tempted to say that it arose as the result of a psychological experiment. For the psychical course of events is not psychologically investigated when we calculate.

'There are 60 seconds to a minute.' This proposition is very *like* a mathematical one. Does its truth depend on experience?—Well, could we talk about minutes and hours, if we had no sense of time; if there were no clocks, or could be none for physical reasons; if there did not exist all the connexions that give our measures of time meaning and importance? In that case—we should say—the measure of time would have lost its meaning (like the action of delivering check-mate if the game of chess were to disappear)—or it would have some quite different meaning. But suppose our experience were like that—then would experience make the proposition false; and the contrary experience make it true? No; *that* would not describe its function. It functions quite differently.

'Calculating, if it is to be practical, must be grounded in empirical facts.'——Why should it not rather determine what empirical facts *are*?

Consider: 'Our mathematics turns experiments into definitions'.

19. But can't we imagine a human society in which calculating quite in our sense does not exist, any more than measuring quite in our sense?—Yes.—But then why do I want to take the trouble to work out what mathematics is?

Because we have a mathematics, and a special conception of it, as it were an ideal of its position and function,—and this needs to be clearly worked out.

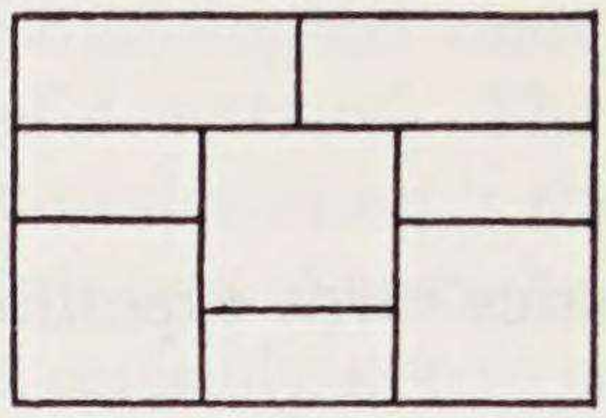
Don't demand too much, and don't be afraid that your just demand will dwindle into nothing.

It is my task, not to attack Russell's logic from within, but from without.

That is to say: not to attack it mathematically—otherwise I should be doing mathematics—but its position, its office.

My task is, not to talk about (e.g.) Gödel's proof, but to by-pass it.

20. The problem: find the number of ways in which we can trace the joins in this wall:



continuously and without repetition, will be recognized by everyone as a *mathematical* problem.—If the drawing were much bigger and more complicated, and could not be taken in at a glance, it could be supposed to change without our noticing; and then the problem of finding that number (which perhaps changes according to some law) would no longer be a mathematical one. But even if it does not change, the problem is, in this case, still not mathematical.——But even when the wall can be taken in at a glance, that cannot be said to make the question mathematical, as when we say: *this* question is now a question in embryology. Rather: *here* we need a mathematical solution. (Like: here what we need is a *model*.)

Did we 'recognize' the problem as a mathematical one because mathematics treats of making tracings from drawings?

Why, then, are we inclined to call this problem straight away a 'mathematical' one? Because we see at once that here the answer to a *mathematical* question is *practically* all we need. Although the problem could easily be seen as, for example, a psychological one.

Similarly with the task of folding a piece of paper in such-and-such a way.

It may look as if mathematics were here a science that makes experiments with *units*; experiments, that is, in which it does not matter what kind of units they are, whether for instance they are peas, glass marbles, strokes and so on.—Mathematics discovers only what holds for *all* these things. And so it does not discover anything about e.g. their melting point, but that 2 and 2 of them are 4. And the first problem of the wall is a mathematical one, i.e. can be solved by means of *this* kind of experiment.—And what does the mathematical experiment consist in? Well, in setting things out and moving them about, in drawing lines, writing down expressions, propositions, etc. And we must not be disturbed by the fact that the outward appearance of these experiments is not that of physical or chemical experiments, etc.; they just are of a different kind. Only there is a difficulty here: the procedure is easy enough to see, to describe,—but *how* is it to be looked at as an experiment? What is the head and what the tail of the experiment here? What are the conditions of the experiment, what its result? Is the result what is yielded by the calculation; or the pattern of calculation; or the assent (whatever that consists in) of the person doing the calculation?

But does it make the principles of dynamics, say, into propositions of pure mathematics if we leave their interpretation open, and then use them to produce a system of measurement?

'A mathematical proof must be perspicuous'—this is connected with the perspicuousness of that figure.

21. Do not forget that the proposition asserting of itself that it is unprovable is to be conceived as a *mathematical* assertion——for that is not *a matter of course*.

It is not a matter of course that the proposition that such-and-such

a structure cannot be constructed is to be conceived as a mathematical proposition.

That is to say: when we said: "it asserts of itself"—this has to be understood in a special way. For here it is easy for confusion to occur through the variegated use of the expression "this proposition asserts something of . . .".

In this sense the proposition '$625 = 25 \times 25$' also asserts something about itself: namely that the left-hand number is got by the multiplication of the numbers on the right.

Gödel's proposition, which asserts something about itself, does not *mention* itself.

'The proposition says that this number cannot be got from these numbers in this way.'—But are you also certain that you have translated it correctly into English? Certainly it looks as if you had.—But isn't it possible to go wrong here?

Could it be said: Gödel says that one must also be able to trust a mathematical proof when one wants to conceive it practically, as the proof that the propositional pattern can be constructed according to the rules of proof?

Or: a mathematical proposition must be capable of being conceived as a proposition of a geometry which is actually applicable to itself. And if one does this it comes out that in certain cases it is not possible to rely on a proof.

The limits of empiricism[1] are not assumptions unguaranteed, or intuitively known to be correct: they are ways in which we make comparisons and in which we act.

22. 'Let us assume that we have an arithmetical proposition saying that a particular number . . . cannot be obtained from the numbers . . ., . . ., . . ., by means of such and such operations. And let us assume that a rule of translation can be given according to which this arithmetical proposition is translatable into the figures of the first number—the axioms from which we are trying to prove it, into the figures of the other numbers—and our rules of inference into the operations mentioned in the proposition.—If we had then derived *the arithmetical proposition* from the axioms according to our rules of inference, then *by this means* we should have demonstrated its derivability, but we should also have proved a proposition which, by that translation rule, can be expressed: this arithmetical proposition (namely ours) is not derivable.'

What would have to be done here? I am supposing that we trust our *construction* of the *propositional sign*; i.e. we trust the *geometrical* proof. So we say that this 'propositional pattern' can be obtained from those in such and such ways. And, merely translated into another notation, this means: this number can be got from those by means of these operations. So far the proposition and its proof have nothing to do with any special *logic*. Here the constructed proposition was simply another way of writing the constructed number; it had the *form* of a proposition but we don't compare it with other propositions as a sign *saying* this or that, making *sense*.

But it must of course be said that that sign need not be regarded either as a propositional sign or as a number sign.—Ask yourself: what makes it into the one, and what into the other?

[1] cf. note p. 197. (Eds.)

If we now read the constructed proposition (or the figures) as a proposition of mathematical language (in English, say) then it says the opposite of what we regard as proved. Thus we have demonstrated the falsity of the real sense of the proposition and at the same time *proved* it—if, that is, we look on its construction from the admitted axioms by means of the admitted rules of inference as a proof.

If someone objects to us that we couldn't make such *assumptions*, for they would be *logical* or *mathematical* assumptions, then we reply that we need only assume that someone has made a mistake in calculating and so has reached the result we 'assume', and that for the time being he cannot find the mistake.

Here once more we come back to the expression "the proof convinces us". And what interests us about conviction here is neither its expression by voice or gesture, nor yet the feeling of satisfaction or anything of that kind; but its ratification in the use of what is proved.

It might justly be asked what importance Gödel's proof has for our work. For a piece of mathematics cannot solve problems of the sort that trouble *us*.—The answer is that the *situation*, into which such a proof brings us, is of interest to us. 'What are we to say now?'—That is our theme.

However queer it sounds, my task as far as concerns Gödel's proof seems merely to consist in making clear what such a proposition as:

"Suppose this could be proved" means in mathematics.

23. We take it much too much for granted that we ask "How many?" and thereupon count and calculate.

Do we count because it is practical to count? We count!—And in the same way we calculate.

An experiment—or whatever one likes to call it—can be what we go on, sometimes in determining the measurement of the thing measured, and sometimes even in determining the appropriate measure.

Then is the unit of measurement in this way the result of measurements? Yes and no. Not the result reached in measuring but perhaps the *consequence* of measurements.

"Has experience taught us to calculate in *this* way?" would be one question and: "Is calculation an experiment?" another.

24. But isn't it possible to derive anything from anything according to some rule or other—nay, according to *any* rule with a suitable interpretation? What does it mean for me to say, for example: this number can be got from that pair of numbers by multiplying? Ask yourself: When does one use this proposition? Well, it isn't, e.g., a psychological proposition saying what humans will do under certain conditions;

what will satisfy them; nor is it a physical proposition concerning the behaviour of marks on paper. That is, it will be applied in a surrounding other than a psychological or physical one.

Assume that human beings learn to calculate, roughly as they in fact do; but now imagine different 'surroundings' which turn the calculating, now into a psychological experiment, now into a physical experiment with the marks used in calculating, now into something else!

We assume that children learn counting and the simple kinds of sum by means of imitation, encouragement and correction. But now, from a certain point of view, the non-agreement of the one who is doing the sums (i.e., the mistakes) get treated, not as something bad, but as something psychologically interesting. "So you took that for correct then, did you? The rest of us did it like *this*."

I want to say that what we call mathematics, the *mathematical* conception of the proposition $13 \times 14 = 182$, hangs together with the special position that we assign to the activity of calculating. Or, the special position that the calculation . . . has in our life, in the rest of our activities. The language-game in which it is found.

One may learn a piece of music by heart in order to be able to play it correctly; but also as part of a psychological experiment, in order to investigate the working of musical memory. But one might also impress it on one's memory in order thereby to judge some alterations in the score.

25. A language-game: I am doing multiplication and I say to the other: if you calculate right you will get such-and-such a result; whereupon he carries out the calculation and is pleased at the correctness, and sometimes the incorrectness, of my prediction. What does this language-game presuppose? That 'mistakes in calculating' are easy to discover, and that agreement about the rightness or wrongness of the calculation is always quickly achieved.

"If you agree with each step, you will arrive at this result."

What is the criterion for a step in the calculation's being right; isn't it that the step seems right to me, and other things of the same sort?

What is the criterion for my working out the same figure twice? Isn't it things like the figures' *appearing* to me to be the same?

What is the criterion for my having followed the paradigm here?

"If you say that each step is correct, this is what will come out."

The prediction really is: where you hold what you do to be right, *this* is what you will do.

Where you hold each step to be right, you will go this way.—And so you will reach this end.

A *logical* conclusion is being drawn, when no experience can contradict the conclusion without contradicting the premises. I.e., when the inference is only a movement within the means of representation.

26. In some language-game sentences are used; reports, orders and so on. And now the people also employ calculating propositions. They say them to themselves perhaps, in between the orders and the reports.

A language-game, in which someone calculates according to a rule and places the blocks of a building according to the results of the calculation. He has learnt to operate with written signs according to rules.—Once you have described the procedure of this teaching and learning, you have said everything that can be said about acting correctly according to a rule. We can go no further. It is no use, for example, to go back to the concept of agreement, because it is no more certain that one proceeding is in agreement with another, than that it has happened in accordance with a rule. Admittedly going according to a rule is also founded on an agreement.

To repeat, what the correct following of a rule consists in cannot be described *more closely* than by describing the *learning* of 'proceeding according to the rule.' And this description is an everyday one, like that of cooking and sewing, for example. It presupposes as much as these. It distinguishes one thing from another, and so it informs a human being who is ignorant of something particular. (Cf. the remark: Philosophy doesn't use a preparatory language, etc.)

For if you give me a description of how people are trained in following a rule and how they react correctly to the training, you will yourself employ the expression of a rule in the description and will presuppose that I understand it.

We have, then, taught someone the technique of multiplying. So we employ expressions of acquiescence and rejection. We shall also sometimes write down the goal of the multiplication: "You must get this, if it is to be right," we may say to him.

But now, can the pupil contradict and say: "How do you know that? And what do you want?—Do you want me to follow the rule, or to get this result? For there's no need for the two to coincide." Well, we do not assume that the pupil can say that; we assume that he accepts the rule as valid when approached from either side. That he conceives both the individual step *and* the multiplication-pattern—and therefore the result of the multiplication—as criteria of correctness, and if these are not in accord with one another, he believes there is some confusion of his senses.

27. Now is it imaginable for someone to follow the rule right and nevertheless to work out different results at different times in multiplying 15×13? It all depends on what criteria one allows to count for correct following of the rule. In mathematics the result itself is also a criterion for correct calculation. Here then it is unthinkable that one should follow the rule right and should produce different patterns of multiplication.

Not letting a contradiction stand is something that characterises the technique of our employment of our truth-functions. If we do let the contradiction stand in our language-games, we alter that technique—as, if we departed from regarding a double negative as an affirmative. And this alteration would be significant, because the technique of our logic is connected in its character with the conception of the truth-functions.

"The rules compel me to . . ."—this can be said if only for the reason that it is not all a matter of my own will what seems to me to agree with the rule. And that is why it can even happen that I memorize the rules of a board-game and subsequently find out that in this game whoever starts *must* win. And it is something like this, when I discover that the rules lead to a contradiction.

I am now compelled to acknowledge that this is not a proper game.

'The rules of multiplication, once adopted, compel me to acknowledge that . . . × . . . = ...' Suppose it were disagreeable for me to acknowledge this proposition. Am I to say: "Well, this arises from that type of training. Human beings who are so trained, so conditioned, then get into this kind of difficulty"?

'How does one count in the decimal system?'—"We write 2 after 1, 3 after 2, . . . 14 after 13 . . . 124 after 123 *and so on.*"—That is an explanation for someone who, while there is indeed something he doesn't know, does understand 'and so on'. And understanding it

means not understanding it as an abbreviation: it does *not* mean that he now sees a much longer series in his mind than that of my examples. That he understands it comes out in his now making certain applications, in his saying *this* and acting *so* in particular cases.

"How do we count in the decimal system?"—. . . . . — Now is that not an answer?—But it isn't one for someone who did not understand the 'and so on'.—But may our explanation not have made it intelligible to him? May he not, through it, have got hold of the idea of the rule?—Ask yourself what are the criteria for his having got hold of the idea now.

What is it that compels me?—the expression of the rule?—Yes, once I have been educated in this way. But can I say it compels me to follow it? Yes: if here one thinks of the rule, not as a line that I trace, but rather as a spell that holds us in thrall.

(("plain nonsense, and bumps . . .'"))

28. Why shouldn't it be said that such a contradiction as: 'heterological' ϵ heterological $\equiv \sim$ ('heterological' ϵ heterological), shews a logical property of the concept 'heterological'?

" 'Two-syllabled' is heterological", or " 'Four-syllabled' is not heterological" are empirical propositions. It might be important in some contexts to find out whether adjectives possess the properties they stand for or not. The word "heterological" would in that case

be used in a language-game. But now, is the proposition " 'h' ϵ h" supposed to be an empirical proposition? It obviously is not one, nor should we admit it as a proposition in our language-game even if we had not discovered the contradiction.

'h' ϵ $h \equiv \sim$ ('h' ϵ h) might be called 'a true contradiction'.—But this contradiction is not a significant proposition! Agreed, but the tautologies of logic aren't either.

"The contradiction is true" means: it is proved; derived from the rules for the word "h". Its employment is, to shew that " 'h' " is one of those words which do not yield a proposition when inserted into 'ξ ϵ h'.

"The contradiction is true" means: this really is a contradiction, and so you cannot use the word " 'h' " as an argument in 'ξ ϵ h'.

29. I am defining a game and I say: "If you move like this, then I move like *this*, and if you do that, then I do *this*.—Now play." And now he makes a move, or something that I have to accept as a move and when I want to reply according to my rules, whatever I do proves to conflict with the rules. How can this have come about? When I set the rules up, I *said* something: I was following a certain use. I did not foresee what we should go on to do, or I saw only a particular possibility. It was just as if I had said to somebody: "Give up the game; you can't mate with these pieces" and had overlooked an existing possibility of mating.

The various half joking guises of logical paradox are only of interest in so far as they remind anyone of the fact that a serious form of the paradox is indispensable if we are to understand its function properly. The question is: what part can such a logical mistake play in a language-game?

You may instruct someone what to do in such-and-such a case; and these instructions later prove *nonsensical*.

30. Logical inference is part of a language-game. And someone who carries out logical inferences in the language-game follows certain instructions which were given him in the actual learning of the language-game. If, say, a builder's mate is building a house in accordance with certain orders, he has to interrupt his cartage of materials etc. from time to time and carry out certain operations with signs on paper; and then he takes up his work again in conformity with the result.

Imagine a procedure in which someone who is pushing a wheelbarrow comes to realize that he must clean the axle of the wheel when the wheelbarrow gets too difficult to push. I don't mean that he says to himself: "Whenever the wheelbarrow can't be pushed . . .", but he simply *acts* in this way. And he happens to shout to someone else: "The wheelbarrow won't push; clean the axle", or again: "The wheelbarrow won't push. So the axle needs cleaning." Now this is an inference. Not a logical one, of course.

Can I now say: "Non-logical inference can prove wrong; but logical inference not"?

Is logical inference correct when it has been made according to rules; or when it is made according to *correct* rules? Would it be wrong, for example, if it were said that p should always be inferred from $\sim p$? But why should one not rather say: such a rule would not give the signs '$\sim p$' and 'p' their usual meaning?

We can conceive the rules of inference—I want to say—as giving the signs their meaning, because they are rules for the use of these signs. So that the rules of inference are involved in the determination of the meaning of the signs. In this sense rules of inference cannot be right or wrong.

In the course of building A has measured the length and breadth of an area and gives B the order: "bring 15×18 slabs". B is trained to multiply and to count out a number of slabs in conformity with the result.[1]

The sentence '$15 \times 18 = 270$' need of course never be uttered.

It might be said: experiment—calculation are poles between which human activities move.

31. We condition a man in such-and-such ways; then bring a question to bear on him; and get a number-sign. We go on to use

[1] cf. *Philosophical Investigations*, § 2, § 8. (Eds.)

this for our purposes and it proves practical. That is calculating.—No, it isn't enough! It might be an eminently *sensible* procedure—but need not be what we call 'calculating'. As one could imagine sounds being emitted for purposes now served by language, which sounds yet did not form a language.

It is essential to calculating that everyone who calculates right produces the same pattern of calculation. And 'calculating right' does not mean calculating with a clear understanding or smoothly; it means calculating *like this*.

Every mathematical proof gives the mathematical edifice a new leg to stand on. (I was thinking of the legs of a table.)

32. I have asked myself: if mathematics has a purely fanciful application, isn't it still mathematics?—But the question arises: don't we call it 'mathematics' only because e.g. there are transitions, bridges from the fanciful to non-fanciful applications? That is to say: should we say that people possessed a mathematics if they used calculating, operating with signs, *merely* for occult purposes?

33. But in that case isn't it incorrect to say: the *essential* thing about mathematics is that it forms concepts?—For mathematics is after all an anthropological phenomenon. Thus we can recognize it as the essential thing about a great part of mathematics (of what is called 'mathematics') and yet say that it plays no part in other regions. This insight by itself will of course have some influence on people once they learn to see mathematics in this way. Mathematics is, then, a family; but that is not to say that we shall not mind what is incorporated into it.

We might say: if you did not understand *any* mathematical proposition better than you understand the Multiplicative Axiom,[1] then you would *not* understand mathematics.

34. —There is a contradiction here. But we don't see it and we draw conclusions from it. E.g. we infer mathematical propositions; and wrong ones. But we accept these inferences.—And now if a bridge collapses, which we built on the basis of these calculations, we find some other cause for it, or we call it an Act of God. Now was our calculation wrong; or was it not a calculation?

Certainly, if we are explorers observing the people who do this we shall perhaps say: these people don't calculate at all. Or: there is an element of arbitrariness in their calculations, which distinguishes the nature of their mathematics from ours. And yet we should not be able to deny that these people have a mathematics.

What kind of rules must the king[2] give so as to escape henceforward from the awkward position, which his prisoner has put him in?—What sort of problem is this?—It is surely like the following one: how must I change the rules of this game, so that such-and-such a situation cannot occur? And that is a mathematical problem.

But can it be a mathematical problem to make mathematics into mathematics?

Can one say: "After this mathematical problem was solved, human beings began really to calculate"?

[1] I.e. the Axiom of Choice. (Eds.)

[2] Presumably the king who made the law that all who came to his city must state their business and be hanged if they lied. A sophist said he came to be hanged under that law.—(Eds.)

35. What sort of certainty is it that is based on the fact that in general there *won't* actually be a run on the banks by all their customers; though they would break if it did happen?! Well, it is a *different* kind of certainty from the more primitive one, but it is a kind of certainty all the same.

I mean: if a contradiction were now actually found in arithmetic—that would only prove that an arithmetic with *such* a contradiction in it could render very good service; and it will be better for us to modify our concept of the certainty required, than to say that it would really not yet have been a proper arithmetic.

"But surely this isn't ideal certainty!"—Ideal for what purpose?

The rules of logical inference are rules of the *language-game*.

36. What *sort* of proposition is: "The class of lions is not a lion, but the class of classes is a class"? How is it verified? How could it be *used*?—So far as I can see, only as a grammatical proposition. To draw someone's attention to the fact that the word "lion" is used in a fundamentally different way from the name of a lion; whereas the class word "class" is used like the designation of one of the classes, say the class *lion*.

One may say that the word "class" is used reflexively, even if for instance one accepts Russell's theory of types. For it is used reflexively there too.

Of course to say in this sense that the class of lions is not a lion etc. is like saying one has taken an "e" for an "a" when one has taken a ball for a bell.

The sudden change of aspect in the picture of a cube and the impossibility of seeing 'lion' and 'class' as comparable concepts.

The contradiction says: "Look out . . . .".

But suppose that one gives a particular lion (the king of lions) the name "Lion"? Now you will say: But it is clear that in the sentence "Lion is a lion" the word "lion" is being used in two different ways. (*Tractatus Logico-philosophicus.*[1]) But can't I count them as *one* kind of use?

But if the sentence "Lion is a lion" is used in this way: shouldn't I be drawing your attention to anything, if I drew your attention to the difference of employment of the two "lion"s?

One can examine an animal to see if it is a cat. But at any rate the concept cat cannot be examined in this way.

Even though "the class of lions is not a lion" seems like nonsense,

[1] Cf. *Tractatus* 3.323. (Eds.)

to which one can only ascribe a sense out of politeness; still I do not want to take it like that, but as a proper sentence, if only it is taken right. (And so not as in the *Tractatus*.) Thus my conception is a different one here. Now this means that I am saying: there is a language-game with this sentence too.

"The class of cats is not a cat."—How do you know?

The fable says: "The lion went for a walk with the fox", not a lion with a fox; nor yet the lion so-and-so with the fox so-and-so. And here it actually is as if the species lion came to be seen as a lion. (It isn't as Lessing[1] says, as if a particular lion were put in the place of some lion or other. "Reynard the Fox" does not mean: a fox of the name "Reynard".)

Imagine a language in which the class of lions is called "the lion of all lions", the class of trees "the tree of all trees", etc.—Because people imagine all lions as forming *one* big lion. (We say: "God created man".)

Then it would be possible to set up the paradox that there isn't a definite number of all lions. And so on.

[1] *Abbandlungen über die Fabeln*, in: G. E. Lessing, *Fabeln*, 1759. (Eds.)

But would it be impossible to count and calculate in such a language?

37. We might ask: What role can a sentence like "I always lie" have in human life? And here we can imagine a variety of things.

38. Is turning inches into centimetres logical inference? "The cylinder is 2 inches long.—So it is about 5 cm. long." Is that a *logical* inference?[1]

But isn't a rule something arbitrary? Something that I *lay down*? And could I lay it down that the multiplication 18×15 shall *not* yield 270?—Why not?—But then it just hasn't taken place according to the rule which I first laid down, and whose use I have practised.

Is something that follows from a rule itself in turn a rule? And if not,—what kind of proposition am I to call it?

"It is . . . impossible for human beings to recognize an object as different from itself." Well, if only I had an inkling how it is done,—I should try at once!—But, if it is impossible for us to recognize an object as different from itself, is it quite possible to recognize two objects as different from one another? I have e.g. two chairs before me and I recognize that they are *two*. But here I may sometimes believe that they are only *one*; and in *that* sense I can also take one for two.—

[1] Cf. Part I, § 9. (Eds.)

But that doesn't mean that I recognize the chair as different from itself! Very well; but then neither have I recognized the two as different from one another. If you think you can do this and you are playing a kind of psychological game, then translate it into a game with gestures. When you have two objects before you, point with each hand at one of them; as if, as it were, you wanted to indicate that they were independent. If you only have one object before you then you point to it with both hands in order to indicate that no difference between it and itself can be made.—But now, why should one not play the game the opposite way?

39. The words "right" and "wrong" are used when giving instruction in proceeding according to a rule. The word "right" makes the pupil go on, the word "wrong" holds him back. Now could one explain these words to a pupil by saying instead: "this agrees with the rule—that not"? Well yes, if he has a concept of agreement. But what if this has yet to be formed? (The point is how he reacts to the word "agree".)

One does not learn to obey a rule by first learning the use of the word "agreement".

Rather, one learns the meaning of "agreement" by learning to follow a rule.

If you want to understand what it means "to follow a rule", you have already to be able to follow a rule.

"If you accept this rule you *must* do this."—This may mean: the rule doesn't leave two paths open to you here. (A mathematical proposition.) But I mean: the rule conducts you like a gangway with rigid walls. But against this one can surely object that the rule could be interpreted in all sorts of ways.—Here is the rule, like an order! And like an order too in its *effect*.

40. A language-game: to bring something *else*; to bring the *same*. Now, we can imagine how it is played.—But how can I explain it to anyone? I can give him this training.—But then how does he know what he is to bring the next time as 'the same'—with what justice can I say that he has brought the right thing or the wrong?—Of course I know very well that in certain cases people would turn on me with signs of opposition.

And does this mean e.g. that the definition of "same" would be this: same is what all or most human beings with one voice take for the same?—Of course not.

For of course I don't make use of the agreement of human beings to affirm identity. What criterion do you use, then? None at all.

To use the word without a justification does not mean to use it wrongfully.

The problem of the preceding language-game exists also here: Bring me something red. For what shews me that something is red? The

agreement of the colour with a sample?—What right have I to say: "Yes, that's red"? Well, I say it; and it cannot be justified. And it is characteristic of this language-game as of the other that all men consent in it without question.

An undecided proposition of mathematics is something that is accepted neither as a rule nor as the opposite of a rule, and which has the form of a *mathematical* statement.—But is this form a sharply circumscribed concept?

Imagine $\lim_{n \to \infty} \phi n = e$ as a property of a piece of music (say). But of course not as if the piece went on endlessly, but as a property that can be recognized by the ear (as it were an *algebraic* property) of the piece.

Imagine equations used as ornaments (wallpaper patterns), and now a test of these ornaments with a view to discovering what kind of curves they correspond to. The test would be analogous to that of the contrapuntal properties of a piece of music.

41. A proof that shews that the pattern '777' occurs in the expansion of π, but does not shew *where*.[1] Well, proved in this way this 'existential proposition' would, for certain purposes, not be *a rule*. But might it not serve e.g. as a means of classifying expansion rules? It would perhaps be proved in an analogous way that '777' does not occur in π^2 but it does occur in $\pi \times e$ etc. The question would simply be: is it reasonable to say of the proof concerned: it proves the existence

[1] Cf. Part V, § 27. (Eds.)

of '777' in this expansion? This can be simply misleading. It is in fact the curse of prose, and particularly of Russell's prose, in mathematics.

What harm is done e.g. by saying that God knows *all* irrational numbers? Or: that they are already all there, even though we only know certain of them? Why are these pictures not harmless?
For one thing, they hide certain problems.—

Suppose that people go on and on calculating the expansion of π. So God, who knows everything, knows whether they will have reached '777' by the end of the world. But can his *omniscience* decide whether they *would* have reached it after the end of the world? It cannot. I want to say: Even God can determine something mathematical only by mathematics. Even for him the mere rule of expansion cannot decide anything that it does not decide for us.

We might put it like this: if the rule for the expansion has been given us, a *calculation* can tell us that there is a '2' at the fifth place. Could God have known this, without the calculation, purely from the rule of expansion? I want to say: No.

42. When I said that the propositions of mathematics determine concepts, that is *vague*; for '$2 + 2 = 4$' forms a concept in a different sense from '$p \supset p$', '$(x).\ fx \supset fa$', or Dedekind's Theorem. The point is, there is a family of cases.

The concept of the rule for the formation of an infinite decimal is—of course—not a specifically mathematical one. It is a concept connected with a rigidly determined *activity* in human life. The concept of this rule is not more mathematical than that of: following the rule. Or again: this latter is not less sharply defined than the concept of such a rule itself.—For the expression of the rule and its sense is only a part of the language-game: following the rule.

One has the *same* right to speak of such rules in general, as of the activities of following them.

Of course, we say: "all this is involved in the concept itself", of the rule for example—but what that means is that we incline to *these* determinations of the concept. For what have we in our heads, which of itself contains all these determinations?

A number is, as Frege says, a property of a concept——but in mathematics it is a mark of a mathematical concept. $\aleph_0$ is a *mark* of the concept of a cardinal number; and the *property* of a technique. $2^{\aleph_0}$ is a mark of the concept of an infinite decimal, but what is this number a property of? That is to say: of what kind of concept can one assert it empirically?

43. The proof of a proposition shews me what I am prepared to stake on its truth. And different proofs can perfectly well cause me to stake the same thing.

Something surprising, a paradox, is a paradox only in a particular, as it were defective, surrounding. One needs to complete this surrounding in such a way that what looked like a paradox no longer seems one.

If I have proved that $18 \times 15 = 270$, I have thereby also proved the geometrical proposition that we get the sign '270' by applying certain transformation rules to the sign '18×15'.—Now suppose that people, having their vision or memory impaired (as we now put it) by some harmful drug, did not get '270' when they did this calculation.—If we cannot use it to make a correct prediction of the result anyone is going to get under normal circumstances, isn't the calculation useless? Well, even if it is, that does not shew that the proposition '$18 \times 15 = 270$' is the empirical proposition: people in general calculate like *this*.

On the other hand it is not clear that the general agreement of people doing calculations is a characteristic mark of all that is called "calculating". I could imagine that people who had learned to calculate might in particular circumstances, say under the influence of opium, begin to calculate differently from one another, and might make use of these calculations; and that they were not said not to be calculating at all and to be deranged—but that their calculations were accepted as a reasonable procedure.

But must they not at least be trained to do the same calculations? Doesn't *this* belong essentially to the concept of calculating? I believe that we could imagine deviations here too.

44. Can we say that mathematics teaches an experimental method of investigation, teaches us to formulate empirical questions (cf. p. 381).

[1] cf. p. 381.

Can't it be said to teach me e.g. to ask whether a particular body moves according to the equation of a parabola?—What does mathematics do in this case? Without it, or without the mathematicians, we should of course not have arrived at the definition of this curve. But was defining this curve itself a piece of mathematics? Would it for instance imply mathematics for people to investigate the movement of a body so as to see whether its path can be represented by the construction of an ellipse with two pegs and a string? Was whoever invented this inquiry doing mathematics?

He did create a new *concept*. But was it in the same way as mathematics does? Was it like the way the multiplication $18 \times 15 = 270$ gives us a new concept?

45. Then *can't* one say that mathematics teaches us to count? But if it teaches us to count, then why doesn't it also teach us to compare colours?

It is clear that if someone teaches us the equation of an ellipse he is teaching us a new concept. But if someone proves to us that *this* ellipse and *this* straight line intersect at these points—he too is giving us a new concept.

Teaching us the equation of an ellipse is like teaching us to count. But it is also like teaching us to ask the question: "Are there a hundred times as many marbles here as there?".

Now if I had taught someone this question in a language-game, and a method of answering it, should I have taught him mathematics? Or

would it have been that only if he operated with signs?

(Would that be like asking: "Would it be geometry, even if it *only* consisted of the Euclidian axioms?")

If arithmetic teaches us the question "how many?", then why doesn't it also teach the question "how dark?"?

Is a new conceptual connexion a new concept? And does mathematics create conceptual connexions?

But the question "are there a hundred times as many marbles here as there?" is surely not a mathematical question. And the answer to it is not a mathematical proposition. A mathematical question would be: "are 170 marbles a hundred times as many as 3 marbles?" (And this is a question of pure, not of applied mathematics.)

Now ought I to say that whoever teaches us to count etc. gives us new concepts; and *also* whoever uses such concepts to teach us pure mathematics?

The word "concept" is too vague by far.

Mathematics teaches us to operate with concepts in a new way. And hence it can be said to change the way we work with concepts.

But only a mathematical proposition that has been proved or that is assumed as a postulate does this, not a problematic proposition.

46. But can we not experiment mathematically? for instance, try whether a square bit of paper can be folded into a cat's head, where the *physical* properties of the paper, such as stiffness or elasticity, don't come into the question? Now certainly we speak of trying here. And why not of experimenting too? This case is like one in which we substitute pairs of numbers in the equation $x^2 + y^2 = 25$ in order to find by trial and error one that satisfies the equation. And if one finally arrives at $3^2 + 4^2 = 25$, is this proposition now the result of an experiment? For why did we call our procedure "trying"? Should we also have called it that if someone always solved such problems first time off with complete certainty (giving the signs of certainty) but without calculating? What did the experiment consist in here? Suppose that before he gives the solution, he has a vision of it.—

47. If a rule does not compel you, then you aren't *following* a rule.

But how am I supposed to be following it; if I can after all follow it as I like?

How am I supposed to follow a sign-post, if whatever I do is a way of following it?

But, that everything can (also) be *interpreted* as following, doesn't mean that everything is following.

But how then does the teacher interpret the rule for the pupil? (For he is certainly supposed to give it a particular interpretation.)—Well, how but by means of words and training?

And if the pupil reacts to it thus and thus; he possesses the rule inwardly.

But *this* is important, namely that this reaction, which is our guarantee of understanding, presupposes as a surrounding particular circumstances, particular forms of life and speech. (As there is no such thing as a facial expression without a face.)

(This is an important movement of thought.)

48. Does a line compel me to trace it?—No; but if I have decided to use it as a model in *this* way, then it compels me.—No; then *I* compel myself to use it in this way. I as it were cleave to it.—But here it is surely important that I can form the decision with the (general) interpretation so to speak once for all, and can hold by it, and do not *interpret* afresh at every step.

The line, it might be said, intimates to me how I am to go. But that is of course only a picture. And if I judge that it intimates this or that to me as it were irresponsibly, then I would not say that I was following it *as a rule*.

"The line intimates to me how I am to go": that is merely a paraphrase for:—it is my *last* court of appeal for how I am to go.

49. Imagine someone was following a line as a rule in this way: he holds a pair of compasses, one point of which he carries along the rule, while the other point draws the line that follows the rule. And as he goes along the rule-line in this fashion, he opens and closes the compasses, to all appearances with great exactness; as he does this, he keeps on looking at the rule, as if *it* determined what he was doing. Now we, who are watching him, can see no regularity of any kind in this opening and shutting. Hence we cannot learn his way of following the rule from him either. But we believe him when he tells us that the line intimated to him to do what he did.

Here we should perhaps really say: "The model seems to *intimate* to him how he has to go. But it isn't a rule."

50. Suppose someone follows the series "1, 3, 5, 7, . . . in writing the series $2x + 1$; and he asked himself "But am I always doing the same thing, or something different every time?"

If from one day to the next someone promises: "Tomorrow I will give up smoking," does he say the same thing every day, or every day something different?

How is it to be decided whether he always does the same, when the line intimates to him how he is to go?

51. Didn't I want to say: Only the total picture of the use of the word "same" as it is interwoven with the uses of the other words, can determine whether he does use the word as we do?

Doesn't he always do the same, namely, let the line intimate to him how he is to go? But suppose he says that the line intimates now this to him and now that? Couldn't he now say: in *one* sense he is always doing the same thing, but still he isn't following a rule? And cannot the one who is following a rule nevertheless also say that in a certain sense he does something different every time? Thus whether he does the same thing or keeps on doing something different does not determine whether he is following a rule.

The procedure of following a rule can be described only like this: by describing in a different way what we do in the course of it.

Would it make sense to say: "If he did something *different* every time, we should not say he was following a rule"? That does *not* make sense.

52. Following a rule is a particular language-game. How can it be described? When do we say he has understood the description?—We do this and that; if he now reacts in such-and-such a way, he understood the game. And this "this and that," "in such-and-such a way" doesn't contain an "and so on."—Or: if I used an "and so on" in the description, and were to be asked what that meant, I should have to explain that in turn by the narration of examples; or perhaps by means of a gesture. And I should then regard it as a sign of understanding, if

he, say, repeated the gesture with an intelligent expression; and in special cases acted in such-and-such a way.

"But then doesn't the understanding reach beyond all the examples?" A very remarkable expression, and one that is entirely natural.

When one recounts examples and then says "and so on", this latter expression does not get explained in the same way as the examples.

For the "and so on" might on the one hand be replaced by an arrow, which indicates that the end of the series of examples is not supposed to signify an end of their application. On the other hand "and so on" also means: that's enough, you've understood; we don't need any more examples.

If we replace the expression by a gesture, it might easily be that people only took our series of examples as they were supposed to (only followed it correctly) when we made this gesture at the end. Thus it would be quite analogous to pointing to an object or a place.

53. Let us imagine a line intimating to me how I am to follow it; that is, as my eye travels along the line a voice within me says: "*This* way!"—What is the difference between this process of obeying a kind of inspiration and that of obeying a rule? For they are surely not

the same. In the case of inspiration I *await* direction. I shall not be able to teach anyone else my 'technique' of following the line. Unless, indeed, I teach him some way of hearkening, some kind of receptivity. But then, of course, I cannot require him to follow the line in the same way as I do.

It would also be possible to imagine such instruction in a sort of calculating. The children can calculate, each in his own way—as long as they listen to their inner voice and obey it. Calculating in this way would be something like composing.

For doesn't the technique (the *possibility*) of training someone else in following it belong to the following of a rule? To be sure, by means of examples. And the criterion of his understanding must be the agreement of their individual actions. Hence it is not as it is with instruction in receptivity.

54. How do you follow the rule?—"I do it like *this*; . . ." and now there follow general explanations and examples.—How do you follow the voice of the line?—"I look at it, exclude all thoughts, etc., etc."

"I wouldn't say that it kept on intimating something else to me—if I were following it as a rule". Can one say that? "Doing the same" is tied up with "following the rule."

55. Can you imagine having absolute pitch, if you don't have it? Can you imagine it, *if* you do?—Can a blind man imagine the seeing of red? Can *I* imagine it? Can I imagine spontaneously reacting in such-and-such a way if I don't do so? Can I imagine it better, if I do do so?

But can I play the language-game, if I don't react in this way?

56. One does not feel that one must always be awaiting the tip-off of the rule. On the contrary. We are not excited about what it will tell us to do next, rather it always tells us the same thing, and we do what it says.

It might be said: we look at what we do in following according to the rule from the point of view: *always the same*.

You might say to someone you were beginning to train: "See, I always do the same: . . ."

57. When do we say: "the line intimates this to me *as* a *rule*—always the same." And on the other hand: "It keeps on intimating to me what I have to do—it is not a rule."

In the first case the story is: I have no further court of appeal for what I have to do. The rule does it all by itself; I need only follow it (and following just is *one thing*). I don't feel for example that it's queer that the line always tells me something.—

The other proposition says: I don't know what I shall do; the line will tell me.

Calculating prodigies, who reach the right result, but cannot tell how. Are we to say: they don't calculate? (A family of cases.)

These things are finer spun than crude hands have any inkling of.

58. May I not *believe* I am following a rule? Doesn't this case exist? And if so, then may I not also believe I am *not* following a rule and yet be following a rule? Isn't there something that we should call *that*, too?

59. How can I explain the word "same"?—Well, by means of examples.—But is that *all*? Isn't there a still deeper explanation; or must not the *understanding* of the explanation be deeper?—Well, have I myself a deeper understanding? *Have* I more than I give in the explanation?

But whence arises the feeling, as if I had more than I can say?

Is it that I interpret the not-limited as length which reaches further than any given length? (The permission that is not limited, as a permission for something limitless.)

The image that goes with the limitless, is of something so big that we can't see its end.

The employment of the word "rule" is interwoven with the employment of the word "same".

Consider: Under what circumstances will the explorer say: The word ". . ." of this tribe means the same as our "and so on"? Imagine the details of their life and their language, which would justify him in this.

"But I know what 'same' means!"—I have no doubt of that; I know it too.

60. "The line intimates to me . . ." Here the emphasis is on the *impalpability* of the intimating. On this: that *nothing* stands between the rule and my action.

One could however imagine that someone multiplied, multiplied correctly, with such feelings; kept on saying: "I don't know—now suddenly the rules intimates *this* to me!" and that we reply: "Of course; for you are going ahead perfectly in accordance with the rule."

Following a rule: this can be contrasted with various things. Among other things the explorer will also describe the circumstances under which someone of these people doesn't want to say he is following a rule. Even when in this or that respect it looks as if he were.

But might we not also calculate as we do calculate (all agreeing), etc., and yet at every step have the feeling of being guided by the rule as if by a spell; astonished maybe at agreeing with one another? (Thanking the deity perhaps for this agreement.)

From this you can just see how much there is to the physiognomy of what we call "following a rule" in everyday life!

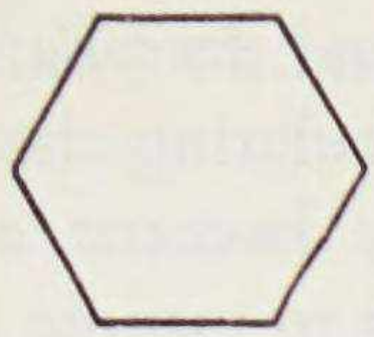
One follows the rule *mechanically*. Hence one compares it with a mechanism.

"Mechanical"—that means: without thinking. But *entirely* without thinking? Without *reflecting*.

The explorer might say: "they follow rules, but it looks different from the way it looks among us."

"It—for no reason—intimates this or that to me" means: I can't teach you *how* I follow the line. I make no presumption that you will follow it as I do, even if you do follow it.

61. An addition of shapes together, so that some of the edges fuse, plays a very small part in our life.—As when



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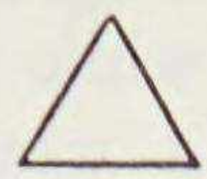

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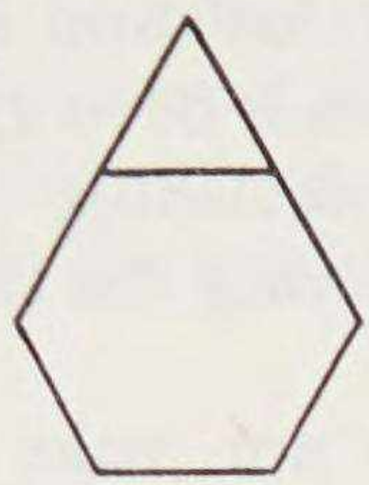


But if this were an *important* operation, our ordinary concept of arithmetical addition would perhaps be different.

It is natural for us to regard it as a geometrical fact, not as a fact of physics, that a square piece of paper can be folded into a boat or hat. But is not geometry, so understood, part of physics? No; we split geometry off from physics. The geometrical possibility from the physical one. But what if we left them together? If we simply said: "If you do this and this and this with the piece of paper then *this* will be the result"? What has to be done might be told in a rhyme. For might it not be that someone did not distinguish at all between the two possibilities? As e.g. a child who learns this technique does not. It does not know and does not consider whether these results of folding are possible only because the paper stretches, is pulled out of shape, when it is folded in such-and-such a way, or because it is *not* pulled out of shape.

And now isn't it like this in arithmetic too? Why shouldn't it be possible for people to learn to calculate without having the concepts of a mathematical and a physical fact? They merely know that this

is always the result when they take care and do what they have learnt.

Let us imagine that while we were calculating the figures on paper altered erratically. A 1 would suddenly become a 6 and then a 5 and then again a 1 and so on. And I want to assume that this does not make any difference to the calculation because, as soon as I read a figure in order to calculate with it or to apply it, it once more becomes the one that we have in *our* calculating. At the same time, one would see how the figures change during the calculation; but we are trained not to worry about this.

Of course, even if we do not make the above assumption, this calculation could lead to useful results.

Here we calculate strictly according to rules, yet this result does not *have* to come out.—I am assuming that we see no sort of regularity in the alteration of the figures.

I want to say: this calculating could really be conceived as an experiment, and we might for example say: "Let's try what will come out now if I apply this rule".

Or again: "Let us make the following experiment: we'll write the figures with ink of such-and-such a composition . . . and calculate according to the rule. . . ."

Now you might of course say: "In this case the manipulation of figures according to rules is not calculation."

"We are calculating only when there is a *must* behind the result."—But suppose we don't know this *must*,—is it contained in the calcula-

tion all the same? Or are we not calculating, if we do it quite naïvely?

How about the following: You aren't calculating if, when you get now this, now that result, and cannot find a mistake, you accept this and say: this simply shews that certain circumstances which are still unknown have an influence on the result.

This might be expressed: if calculation reveals a causal connexion to you, then you are not calculating.

Our children are not only given practice in calculation but are also trained to adopt a particular attitude towards a mistake in calculating.[1]

What I am saying comes to this, that mathematics is *normative*. But "norm" does not mean the same thing as "ideal".

62. The introduction of a new rule of inference can be conceived as a transition to a new language-game. I can imagine one in which for example one person pronounces: '$p \supset q$', another 'p' and a third draws the conclusion.

63. Is it possible to observe that a surface is coloured red and blue;

[1] [Variant]: . . . towards a departure from the norm. (Ed.)

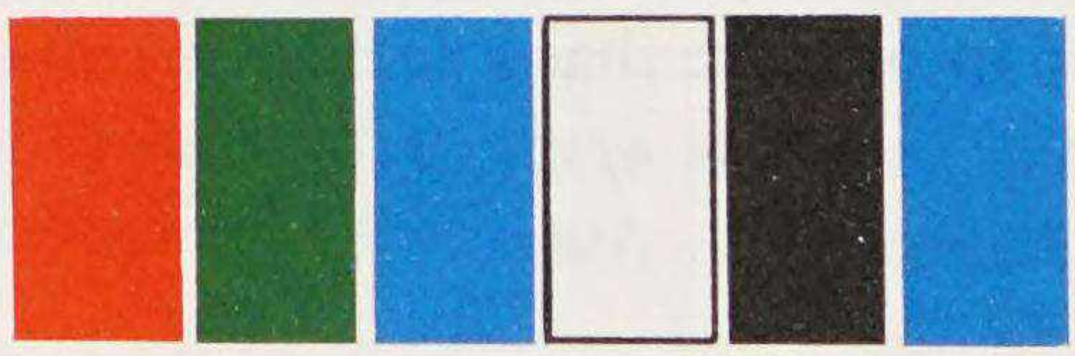
and not to observe that it is red? Imagine that a kind of colour adjective were used for things that are half red and half blue: they are said to be 'bu'. Now might not someone be trained to observe whether something is bu; and not to observe whether it is also red? Such a man would then only know how to report: "bu", or "not bu". And from the first report we could draw the conclusion that the thing was partly red.

I am imagining that the observation happens by means of a psychological sieve, which for example only lets through the fact that the surface is blue-white-red (the French tricolour) or that it is not.

Now if it is a special observation that the surface is partly red, how can this follow logically from the preceding? Surely logic cannot tell us what to observe.

Someone is counting apples in a box; he counts up to 100. Someone else says: "so there are at any rate 50 apples in the box" (that is all that interests him). This is surely a logical conclusion; but isn't it also a special piece of experience?

64. A surface which is divided into a number of strips is observed by several people. The colours of the strips change every minute, all at the same time.



Now the colours are: red, green, blue, white, black, blue.

It is observed:

$$\text{red} \,.\, \text{blue} \supset \text{black} \,.\supset.\, \text{white}.$$

It is also observed:

$$\sim\text{green} \supset \sim\text{white}$$

and someone draws the conclusion:

$$\sim\text{green} \supset : \text{red} \,.\, \text{blue} \,.\, \sim\text{black}.$$

And these implications are 'material implications' in Russell's sense.

But then is it possible to *observe* that

$$\text{red} \,.\, \text{blue} \supset \text{black} \,.\supset.\, \text{white?}$$

Isn't one observing *arrangements* of colours, and so for example that red.blue.black.white; and then deducing that proposition?

But may not someone who is observing a surface be quite preoccupied with the question whether it is going to turn green or not green; and if he now sees: ~green, need he be attentive to the particular colour that the surface is?

And might not someone be preoccupied with the aspect red.blue ⊃ black .⊃. white? If, for example, he has been taught to forget everything else, and only to look at the surface from this point of view. (In particular circumstances it might be all one to people whether objects were red or green but important whether they had one of these colours or some third one. And in this case there might be a colour word for "red or green".)

But if one can observe that

$$\text{red} \,.\, \text{blue} \supset \text{black} \,.\supset.\, \text{white}$$

and

$$\sim\text{green} \supset \sim\text{white}$$

then one can also observe, and not merely infer, that

$$\sim\text{green} \supset : \text{red} \,.\, \text{blue} \,.\, \sim\text{black}.$$

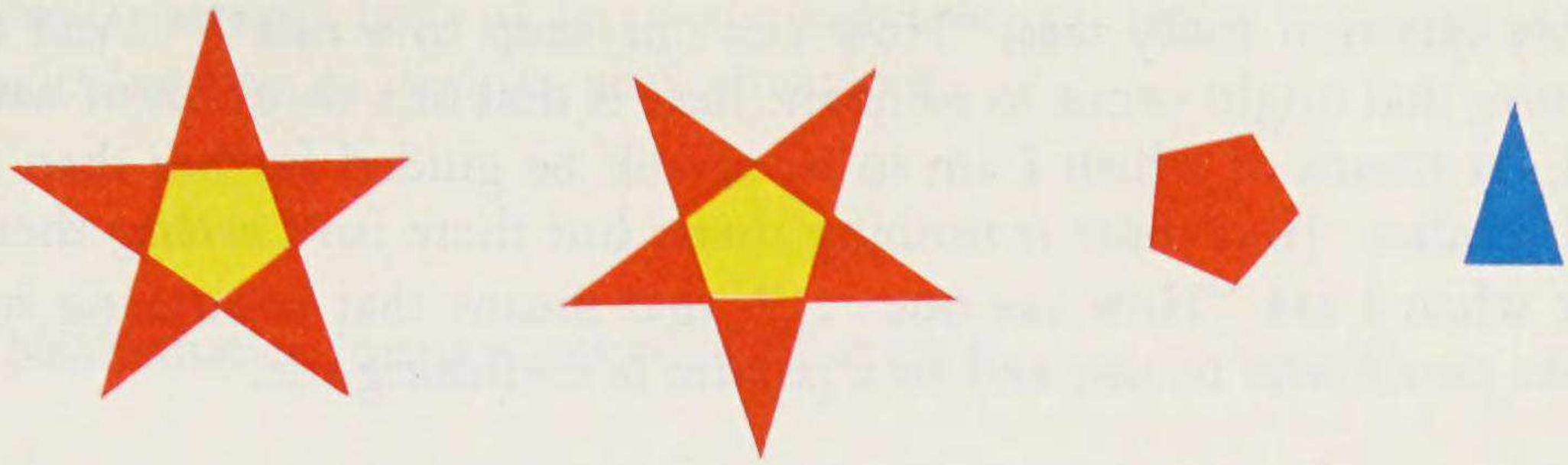
If these are three observations then it must also be possible for the third observation not to agree with the logical conclusion from the first two.

Then is it imaginable that someone observing a surface should see the combination red-black (say as a flag), but if he now sets himself to see *one* of the two halves, he sees blue instead of red? Well, you have just described it.—It would perhaps be as if someone were to look at a group of apples and always see it as two groups of two apples each, but as soon as he tried to take the whole lot in at a glance, they seemed to him to be five. This would be a very remarkable phenomenon. And it is not one of whose possibility we take any notice.

Remember that a rhombus, seen as a diamond, does not look like a parallelogram. Not that the opposite sides seem *not* to be parallel, only the parallelism does not strike us.

65. I could imagine someone saying that he saw a red and yellow star but did not see anything yellow—because he sees the star as, so to speak, a *conjunction* of coloured parts, which he cannot separate.

For example he had figures like these before him:



Asked whether he sees a red pentagon he would say "yes"; asked whether he sees a yellow one, "no". In the same way he says that he sees a blue triangle but not a red one.—When his attention was drawn to it perhaps he said: "Yes, now I see it; I had not taken the star like that."

And it might seem to him that you can't separate the colours in the star, because you can't separate the shapes.

You cannot learn to view the geography of a landscape as a whole, if you move on in it so slowly that you have already forgotten one bit when you come to another.

66. Why do I always speak of being compelled by a rule; why not of the fact that I can *choose* to follow it? For that is equally important.

But I don't want to say, either, that the rule compels me to act like this; but that it makes it possible for me to hold by it and let it compel me.

And if e.g. you play a game, you keep to its rules. And it is an interesting fact that people set up rules for the fun of it, and then keep to them.

My question really was: "How can one keep to a rule[1]?" And the picture that might occur to someone here is that of a short bit of hand-rail, by means of which I am to let myself be guided further than the rail reaches. [But there *is* nothing there; but there isn't *nothing* there!] For when I ask "How *can* one . . .", that means that something here looks *paradoxical* to me; and so a picture is confusing me.

"I never thought of its being red too; I only saw it as part of a multi-coloured ornament."

Logical inference is a transition that is justified if it follows a particular paradigm and its rightness is not dependent on anything else.

67. We say: "If you really follow the rule in multiplying, you *must* all get the same result." Now if this is only the somewhat hysterical way of putting things that you get in university talk, it need not interest us overmuch.

It is however the expression of an attitude towards the technique of calculation, which comes out everywhere in our life. The emphasis of the *must* corresponds only to the inexorableness of this attitude both to the technique of calculating and to a host of related techniques.

The mathematical Must is only another expression of the fact that mathematics forms concepts.

[1] Cf. Part VI, § 47. (Eds.)

And concepts help us to comprehend things. They correspond to a particular way of dealing with situations.

Mathematics forms a network of norms.

68. It is possible to see the complex formed of A and B, without seeing A or B. It is even possible to call the complex a "complex of A and B" and to think that this name points to some kind of kinship of this whole with A and with B. Thus it is possible to say that one is seeing a complex formed from A and B but neither A nor B. As for example one might say that there is a reddish yellow here but neither red nor yellow.

Now can I have A and B before me and also see them both, but only observe A ∨ B? Well, in a certain sense this is surely possible. I was thinking of it like this: the observer is preoccupied with a particular aspect; for example, he has a special kind of paradigm before him; he is engaged in a particular routine of application.—And just as he can be adjusted to A ∨ B, so he can also be adjusted to A.B. Thus only A.B strikes him, and not, for example, A. To be adjusted to A ∨ B might be said to mean: to react to such-and-such a situation with the concept 'A ∨ B'. And one can of course do exactly the same thing with A.B too.

Say someone is interested only in A.B, and so whatever happens he judges merely either "A.B", or "∼(A.B)"; then I can imagine his judging "A.B" and saying "No, I see A.B" when he is asked "Do you see B?" As for example some people who see A.B will not concede that they see A ∨ B.

69. But 'seeing' the surface 'blue all over' and 'seeing' it 'red all over' are surely 'genuine' experiences, and yet we say that a man could not have them at the same time.

Now suppose he assured us that he saw this surface really red all over and blue all over at the same time? We should have to say: "You aren't making yourself intelligible to us."

With us the proposition "1 foot = . . . cm." is timeless. But we could imagine the case in which the foot and the metre gradually altered somewhat, and kept on having to be compared anew in order for us to calculate their translations into one another.

But have we not determined the relative length of foot and metre experimentally? Yes; but the result was given the character of a rule.

70. In what sense can a proposition of arithmetic be said to give us a concept? Well let us interpret it, not as a proposition, as something that decides a question, but as a—somehow accepted—connexion of concepts.

The equating of 25^2 and 625 could be said to give me a new concept. And the proof shews what the position is regarding this equality.—"To give a new concept" can only mean to introduce a new employment of a concept, a new practice.

"How can the proposition be separated from its proof?" This question betrays a false conception.

The proof is part of the *surroundings* of the proposition.

'Concept' is a vague concept.

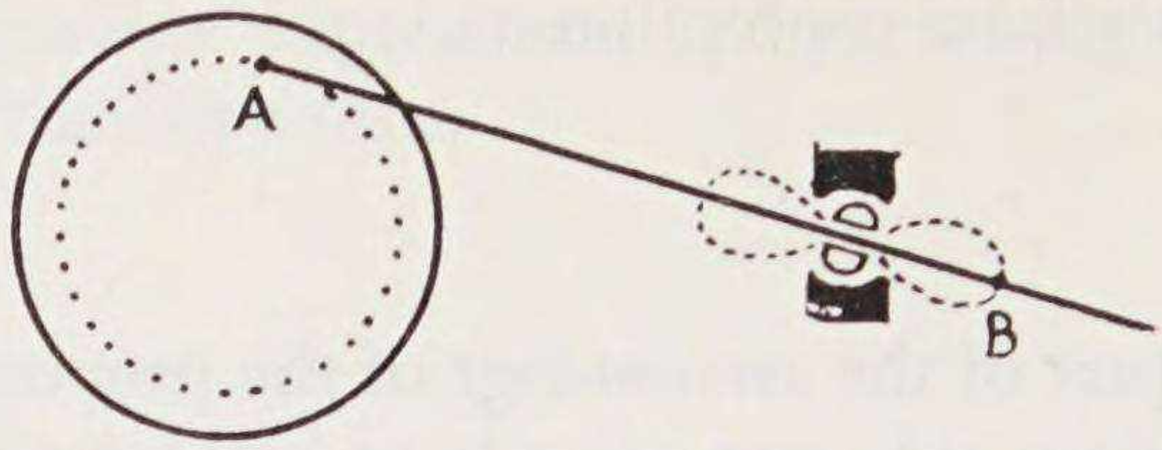
71. It is not in every language-game that there occurs something that one would call a concept.

Concept is something like a picture with which one compares objects.

Are there concepts in language-game (2)[1]? Still, it would be easy to add to it in such a way that "slab", "block" etc. became concepts. For example, by means of a technique of describing or portraying those objects. There is of course no sharp dividing line between language-games which work with concepts and others. What is important is that the word "concept" refers to one kind of expedient in the mechanism of language-games.

[1] *Philosophical Investigations*, § 2; here, above, p. 343n. (Eds.)

72. Consider a mechanism. For example this one:



While the point A describes a circle, B describes a figure eight. Now we write this down as a proposition of kinematics.

When I work the mechanism its movement proves the proposition to me; as would a construction on paper. The proposition corresponds e.g. to a picture of the mechanism with the paths of the points A and B drawn in. Thus it is in a certain respect a picture of that movement. It holds fast what the *proof* shews me. Or—what it persuades me of.

If the proof registers the procedure according to the rule, then by doing this it produces a new concept.

In producing a new concept it convinces me of something. For it is essential to this conviction that the procedure according to these rules must always produce the same configuration. ('Same', that is, by our ordinary rules of comparison and copying.)

With this is connected the fact that we can say that proof must shew the existence of an internal relation. For the internal relation is the operation producing one structure from another, seen as equivalent to the picture of the transition itself—so that now the transition

according to this series of configurations is *eo ipso* a transition according to those rules for operating.

In producing a concept, the proof convinces me of something: what it convinces me of is expressed in the proposition that it has proved.

Problem: Does the adjective "mathematical" always mean the same: when we speak of "mathematical" concepts, of "mathematical" propositions and of mathematical proofs?

Now what has the proved proposition got to do with the concept created by the proof? Again: what has the proved proposition got to do with the internal relations demonstrated by the proof?

The picture (proof-picture) is an instrument producing conviction.

It is clear that one can also apply an unproved mathematical proposition; even a false one.

The mathematical proposition says to me: Proceed like this!

73. "If the proof convinces us, then we must also be convinced of the axioms." Not as by empirical propositions, that is not their role. In the language-game of verification by experience they are excluded. Are, not empirical propositions, but principles of judgement.

A language-game: How have I to imagine one in which axioms, proofs and proved propositions occur?

Someone who hears a bit of logic for the first time at school is straightway convinced when he is told that a proposition implies itself, or when he hears the law of contradiction, or of excluded middle.—Why is he immediately convinced? Well, these laws fit entirely into the use of language that he is so familiar with. Then he learns perhaps to prove more complicated propositions of logic. The proofs are exhibited to him, and he is again convinced; or he invents proofs himself.

In this way he learns new techniques of inference. And also, what account to lay it to, if now errors appear.

The proof convinces him that he must hold fast to the proposition, to the technique that it prescribed; but it also shews him how he can hold fast to the proposition without running any risk of getting into conflict with experience.

74. Any proof in applied mathematics may be conceived as a proof in pure mathematics which proves that *this* proposition follows from *these* propositions, or can be got from them by means of such and such operations; etc.

The proof is a particular *path*. When we describe it, we do not mention causes.

I act on the proof.—But how?—I act according to the proposition that got proved.

The proof taught me e.g., a technique of approximation. But still it proved *something*, convinced me of something. *That* is expressed by the proposition: It says what I shall now do on the strength of the proof.

The proof belongs to the background of the proposition. To the system in which the proposition has an effect.

See, *this* is how 3 and 2 yield 5. Note this proceeding.

Every empirical proposition may serve as a rule if it is fixed, like a machine part, made immovable, so that now the whole representation turns around it and it becomes part of the coordinate system, independent of facts.

"This is how it is, if this proposition is derived from these ones. That you have to admit."—What I admit is, *this* is what I call such a procedure.