PART VI

ca. 1943/1944

1. Proofs give propositions an order. They organize them.

2. The concept of a formal test presupposes the concept of a transformation-rule, and hence of a technique.

For only through a technique can we *grasp* a regularity.

The technique is external to the pattern of the proof. One might have a perfectly accurate view of the proof, yet not understand it as a transformation according to such-and-such rules.

One will certainly call adding up the numbers . . . to see whether they come to 1000, a formal test of the numerals. But all the same that is *only* when adding is a practised technique. For otherwise how could the procedure be called any kind of test?

It is only within a *technique* of transformation that the proof is a formal test.

When you ask what right you have to pronounce this rule, the proof is the answer to your question.

What right have you to say that? *What* right have you to say it?

How do you test a theme for a contrapuntal property? You transform it according to *this* rule, you put it together with another one in *this* way; and the like. In this way you get a definite result. You get it, as you would also get it by means of an experiment. So far what you are doing may even have been an experiment. The word "get" is here used temporally; you got the result at three o'clock.—In the mathematical proposition which I then frame the verb ("get", "yields" etc.) is used non-temporally.

The activity of testing produced such and such a result.

So up to now the testing was, so to speak, experimental. Now it is taken as a proof. And the proof is the *picture* of a test.

The proof, like the application, lies in the background of the proposition. And it hangs together with the application.

The proof is the route taken by the test.

The test is a formal one only in so far as we conceive the result as the result of a formal proposition.

3. And if this picture justifies the prediction—that is to say, if you only have to see it and you are convinced that a procedure will take such-and-such a course—then naturally this picture also justifies the rule. In this case the proof stands behind the rule as a picture that justifies the rule.

For why does the picture of the movement of the mechanism justify the belief that this kind of mechanism will always move in *this* way?—It gives our belief a particular direction.

When the proposition seems not to be right in application, the proof must surely shew me why and how it *must* be right; that is, *how* I must reconcile it with experience.

Thus the proof is a blue-print for the employment of the rule.

4. How does the proof justify the rule?—It shews how, and therefore why, the rule can be used.

The King's Bishop[1] shews us how 8 × 9 makes 72—but here the rule of counting is not acknowledged as a rule.

The King's Bishop shews us *that* 8 × 9 makes 72: Now we are acknowledging the rule.

[1] It is possible to devise a 'rule of counting' to fit the text—*e.g.* one of counting up the positions commanded from eight of the ten squares commanding nine (the colour of the bishop's diagonals being determinate). But we have not been able to find out what routine Wittgenstein did have in mind. (Eds.)

Or ought I to have said: the King's Bishop shews me how 8×9 *can* make 72; that is to say, it shews me *a* way?

The procedure shews me a How of '*making*'.

In so far as $8 \times 9 = 72$ is a rule, of course it means nothing to say that that shews me *how* $8 \times 9 = 72$; unless this were to mean: someone shews me a process through the contemplation of which one is led to this rule.

Now isn't going through any proof such a process?

Would it mean anything to say: "I want to shew you how 8×9 originally made 72"?

5. What is really queer is that the picture, not the reality, should be able to prove a proposition! As if here the picture itself took over the role of reality.—But that's not how it is: for what I derive from the picture is only a rule. And this rule does not stand to the picture as an empirical proposition stands to reality.—The picture of course does not shew that such-and-such happens. It only shews that what does happen can be taken in *this* way.

The proof shews how one proceeds according to the rule without a hitch.

And so one may even say: the procedure, the proof, shews one how far $8 \times 9 = 72$.

The picture shews one, not, of course, anything that happens, but that what ever does happen will allow of being looked at like this.

We are brought to the point of using this technique in this case. I am brought to this—and *to that extent* I am convinced of something.

See, in this way 3 and 2 make 5. Note this procedure. "In doing so you at once notice the rule."

6. The Euclidean proof of the infinity of prime numbers might be so conducted that the investigation of the numbers between p and $p! + 1$ was carried out on one or more examples, and in this way we learned a technique of investigation. The force of the proof would of course in that case not reside in the fact that a prime number $> p$ was found in *this* example. And at first sight this is queer.

It will now be said that the algebraic proof is stricter than the one by way of examples, because it is, so to speak, the extract of the effective principle of these examples. But after all, even the algebraic proof is not quite naked. Understanding—I might say—is needed for both!

The proof teaches us a technique of finding a prime number between p and $p! + 1$. And we become convinced that this technique must always lead to a prime number > *p*. Or that we have miscalculated if it doesn't.

Would one be inclined to say here that the proof shews us *how* there is an infinite series of prime numbers? Well, one might say so. And at any rate: "What there being an infinity of primes amounts to." For it could also be imagined that we had a proof that did indeed determine us to say that there were infinitely many primes, but did not teach us to find a prime number > *p*.

Now perhaps it would be said: "Nevertheless, these two proofs prove the same proposition, the same mathematical fact." There might be reason at hand for saying this, or again there might not.

7. The spectator sees the whole impressive procedure. And he becomes convinced of something; that is the special impression that he gets. He goes away from the performance convinced of something. Convinced that (for example) he will end up the same way with other numbers. He will be ready to express what he is convinced of in such-and-such a way. Convinced of what? Of a psychological fact?—

He will say that he has drawn a conclusion from what he has seen.—*Not*, however as one does from an experiment. (Think of periodic division.)

Could he say: "What I have seen was very impressive. I have drawn a conclusion from it. In future I shall . . ."?

(E.g.: In future I shall always calculate like *this*.)
He tells us: "I saw that it must be like that."

"I realised that it must be like that"—that is his report.

He will now perhaps run through the proof procedure in his mind.

But he does not say: I realised that *this* happens. Rather: that it must be like that. This "must" shews what kind of lesson he has drawn from the scene.

The "must" shews that he has gone in a circle.

I decide to see things like *this*. And so, to act in such-and-such a way.

I imagine that whoever sees the process also draws a moral from it.

'It must be so' means that this outcome has been defined to be essential to this process.

8. This *must* shews that he has adopted a concept.

This *must* signifies that he has gone in a circle.

He has read off from the process, not a proposition of natural science but, instead of that, the determination of a concept.

Let concept here mean method. In contrast to the application of the method.

9. See, 50 and 50 make 100 like *this*. One has, say, added 10 to 50 five times in succession. And one goes on with the increase of the number until it grows to 100. Here of course the observed process would be a process of calculating in some fashion (on the abacus, perhaps); a proof.

The meaning of that "like *this*" is of course not that the proposition "50 + 50 = 100" says: this takes place somewhere. So it is not as when I say: "See, a horse canters like this"—and shew him a picture.

One could however say: "See, *this* is *why* I say 50 + 50 = 100".

Or: See, *this* is how one gets 50 + 50 = 100.

But if I now say: See, *this* is how 3 + 2 make 5, laying 3 apples on the table and then 2 more, here I mean to say: 3 apples and 2 apples make 5 apples, if none are added or taken away.—Or one might even tell someone: If you put 3 apples and then 2 more on the table (as I am doing), then what you see now almost always happens—and there are now 5 apples lying there.

I want perhaps to shew him that 3 apples and 2 apples don't make 5 apples in *such* a way as they might make 6 (because e.g. one makes a sudden appearance). This is really an explanation, a definition of the operation of adding. This is indeed how one might actually explain adding with the abacus.

"If we put 3 things by 2 things, that may yield various counts of things. But we see as a *norm* the procedure that 3 things and 2 things make 5 things. See, *this* is how it looks when they make 5."

Couldn't one say to a child: "Shew me how 3 and 2 make 5." And the child would then have to calculate 3 + 2 on the abacus.

When, in teaching the child to calculate, one asks, "How do 3 + 2 make 5?"—what is he supposed to shew? Well, obviously he is supposed to move three beads up to 2 beads and then to count the beads (or something like that).

Might one not say "Shew me how this theme makes a canon." And someone asked this would have to prove that it does make a canon.—One would ask someone "*how*" if one wanted to to get him to shew that he does grasp what is in question here.

And if the child now shews how 3 and 2 make 5, then he shews a procedure that can be regarded as a ground for the rule "2 + 3 = 5."

10. But suppose one asks the pupil: "Shew me how there are infinitely many prime numbers."—Here the grammar is doubtful! But it would be appropriate to say: "Shew me in how far one may say that there are infinitely many prime numbers."

When one says "Shew me that it is . . .," then the question *whether it is* is already put and it remains only to answer "yes" or "no". But if one says "Shew me *how* it is that . . .", then here the language-game itself needs to be explained. At any rate, one has so far no *clear* concept of what one is supposed to be at with this assertion. (One is asking, so to speak; "How can such an assertion be justified at all?")

Now am I meant to give different answers to the question: "Shew me *how* . . ." and "Shew me that . . ."?

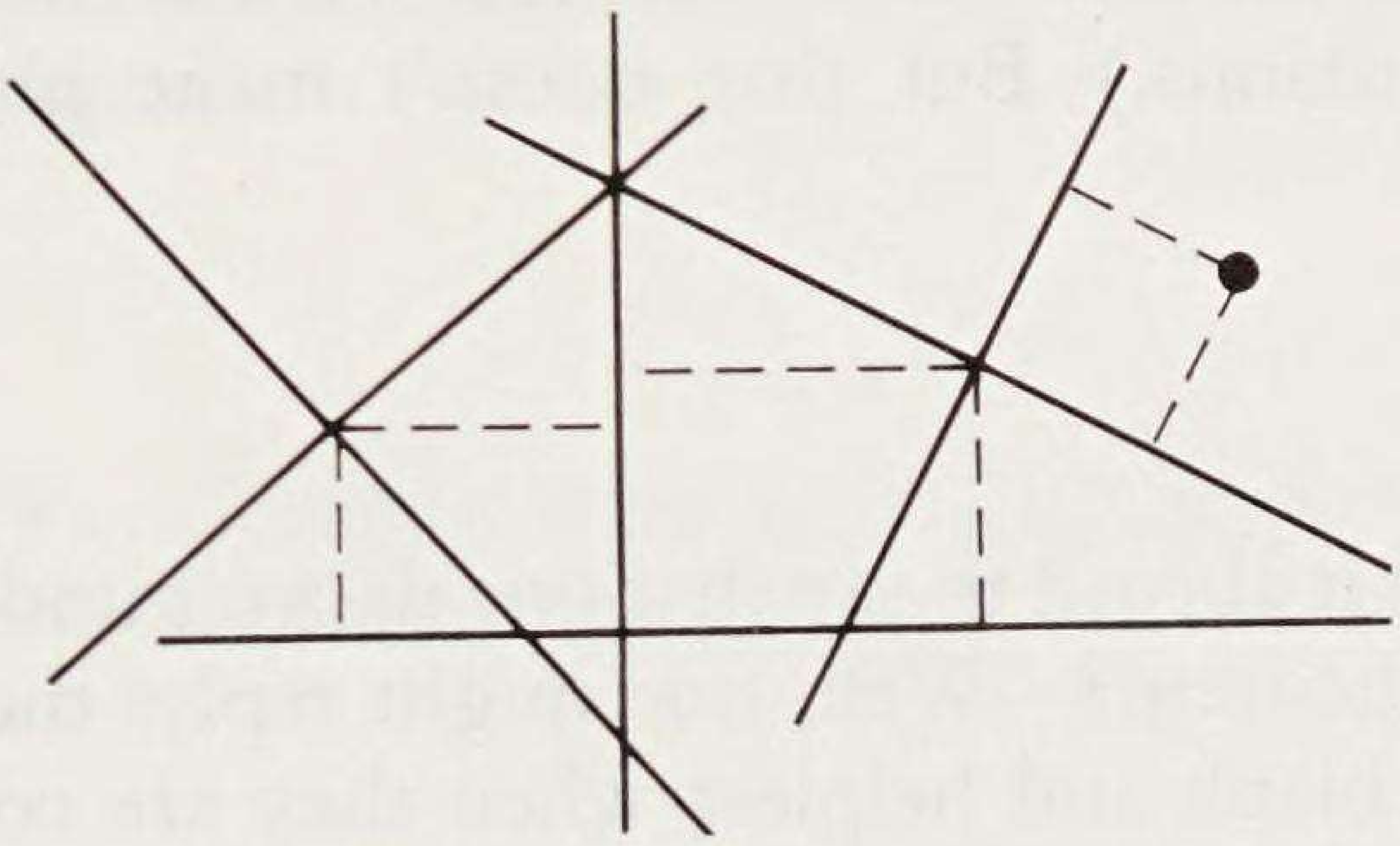
From the proof you derive a theory. If you derive a theory from the proof, then the sense of the theory must be independent of the proof; for otherwise the theory could never have been separated from the proof.

In the same way as I can remove auxiliary construction lines in a drawing and leave the rest.

Thus it is as if the proof did not determine the sense of the proposition proved; and yet as if it did determine it.

But isn't it like that with any verification of any proposition?

11. I believe this: Only in a large context can it be said at all that there are infinitely many prime numbers. That is to say: For this to be possible there must already exist an extended technique of calculating with cardinal numbers. That proposition only makes sense within this technique. A proof of the proposition locates it in the whole system of calculations. And its position therein can now be described in more than one way, as of course the whole complicated system in its background is presupposed.



If for example 3 co-ordinate systems are given a definite mutual arrangement, I can determine the position of a point for all of them by giving it for any one.

The proof of a proposition certainly does not mention, certainly does not describe, the whole system of calculation that stands behind the proposition and gives it its sense.

Assume that an adult with intelligence and experience has learnt only the first elements of calculation, say the four fundamental operations with numbers up to 20. In doing so he has also learnt the word "prime number". And suppose someone said to him "I am going to prove to you that there are infinitely many prime numbers." Now, how can he prove it to him? He has got to *teach him to calculate*. That is here part of the proof. It takes that, so to speak, to give the question "Are there infinitely many prime numbers?" any sense.

12. Philosophy has to work things out in face of the temptations to misunderstand on *this* level of knowledge. (On another level there are again new temptations.) But that doesn't make philosophising any easier!

13. Now isn't it absurd to say that one doesn't understand the sense of Fermat's last theorem?—Well, one might reply: the mathematicians are not *completely* blank and helpless when they are confronted by this proposition. After all, they try certain methods of proving it; and, so far as they try methods, *so far* do they understand the proposition.—But is that correct? Don't they *understand* it just as completely as one can possibly understand it?

Now let us assume that, quite contrary to mathematicians' expectations, its contrary were proved. So now it is shewn that it *cannot* be so at all.

But, if I am to know what a proposition like Fermat's last theorem says, must I not know what the criterion is, for the proposition to be true? And I am of course acquainted with criteria for the truth of *similar* propositions, but not with any criterion for the truth of this proposition.

'Understanding' is a vague concept.

In the first place, there is such a thing as *belief* that one understands a proposition.

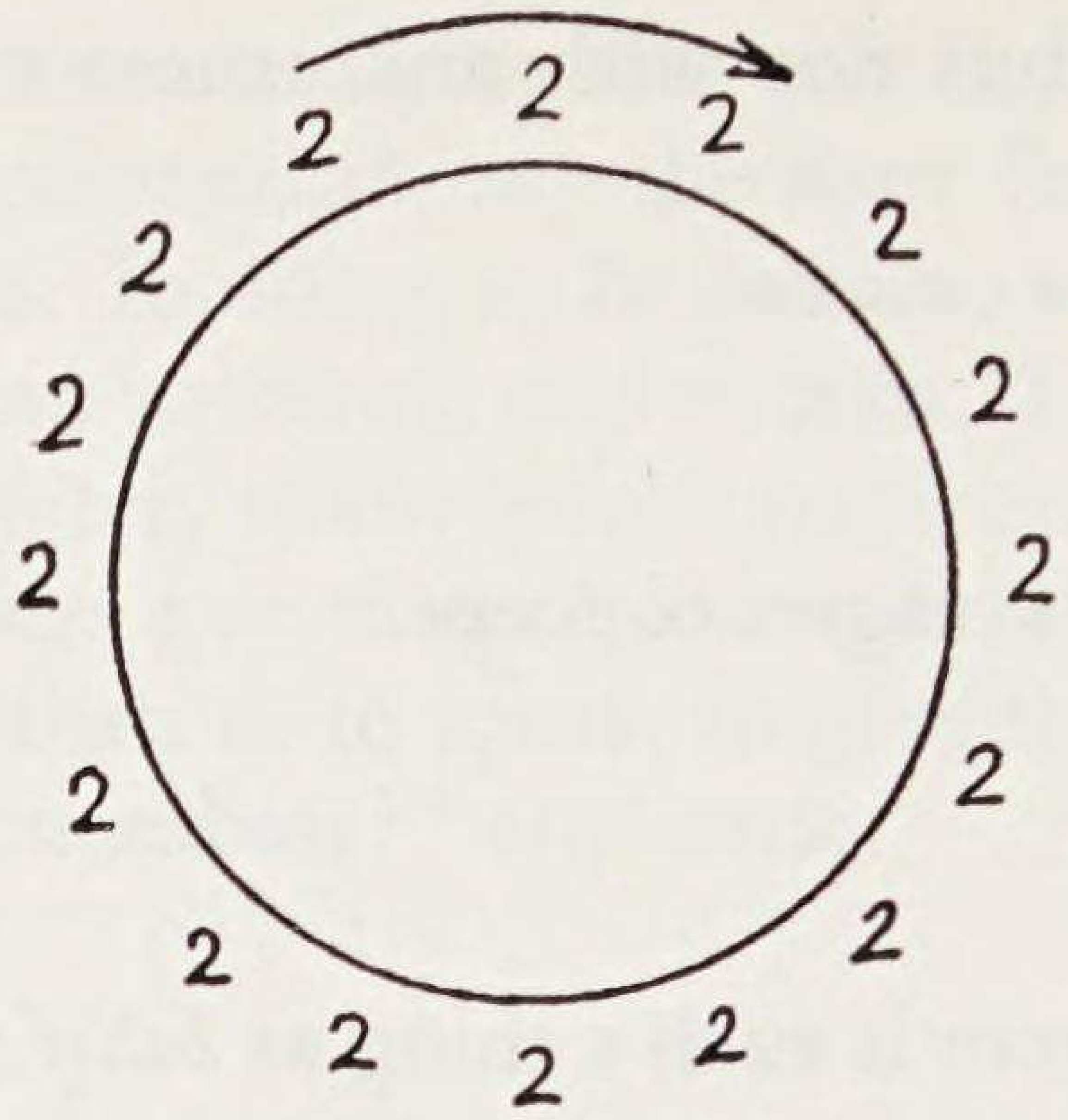
And if understanding is a psychical process—why should it interest us so much? Unless experience connects it with the capacity to make use of the proposition.

"Shew me how . . ." means: shew me the connexions in which you are using this proposition (this machine-part).

14. "I am going to shew you how there are infinitely many prime numbers" presupposes a condition in which the proposition that there are infinitely many prime numbers had no, or only the vaguest, meaning. It might have been merely a joke to him, or a paradox.

If this procedure convinces you of that, then it must be very impressive.—But is it?—Not particularly. Why is it not *more* so? I believe it would only be impressive if one were to explain it quite radically. If for example one did not merely write $p! + 1$, but first explained it and illustrated it with examples. If one did not presuppose the techniques as something obvious, but gave an account of them.

15.



We keep on copying the last figure "2" going round to the right. If we copy correctly, the last figure is in turn a copy of the first one.

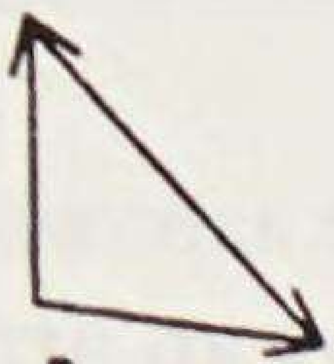
A language-game:

One person (A) predicts the result to another (B). The other follows the arrows with excitement, as it were curious how they will conduct him, and is pleased at the way they end by leading him to the predicted result. He reacts to it perhaps as one reacts to a joke.

A may have constructed the result before, or merely have guessed it. B knows nothing about it and it does not interest him.

Even if he was acquainted with the rule, still he had never followed it *thus*. He is *now* doing something new. But there is also such a thing as curiosity and surprise when one has already travelled this road. In this way one can read a story again and again, even know it by heart, and yet keep on being surprised at a particular turn that it takes.

Before I have followed the two arrows 

like this  R ,

I don't know how the route or the result will look. I do not know what face I shall see. Is it strange that I did not know it? How should I have known it? I had never seen it! I knew the rule and had mastered it and I saw the sheaf of arrows.

But why wasn't this a genuine prediction: "If you follow the rule, you will produce this"? Whereas the following is certainly a genuine prediction: "If you follow the rule as best you can, you will . . ." The answer is: the first is not a prediction because I might also have said: "If you follow the rule, you *must* produce this." It is not a prediction if the concept of *following* the rule is so determined, that the result is the criterion for whether the rule was followed.

A says: "If you follow the rule you will get *this*" or he says simply: "You will get this." At the same time he draws the resulting arrow there.

Now was what A said in this game a prediction? Well damn it, Yes—in a certain sense. Does that not become particularly clear if we make the suggestion that the prediction was *wrong*? It was only not a prediction in the case where the *condition* turned the proposition into a pleonasm.

A might have said: "If you are in agreement with each of your steps, then you will arrive *at this*."

Suppose that while B is deriving the polygon, the arrows of the sheaf were to alter their direction a little. B always draws an arrow parallel, as it is just at this moment. He is now just as surprised and excited as in the foregoing game, although here the result is not that of a calculation. So he had taken the first game in the same way as the second.

The reason why "If you follow the rule, this is where you'll get to" is not a prediction is that this proposition simply says: "The result of this calculation is . . ." and that is a true or false mathematical proposition: The allusion to the future and to yourself is mere clothing.

Now must A have a clear idea at all, of whether his prediction is meant mathematically or otherwise? He simply says "If you follow the rule . . . will result" and enjoys the game. If for example the predicted result does not come out, he does not investigate any further.

16. . . . And this series is defined by a rule. Or again by the training in proceeding according to the rule. And the inexorable proposition is that according to this rule this number is the successor of this one.[1]

[1] An amendment of and addition to the fourth sentence of § 4, Part I, which runs: "And isn't *this* series just *defined* by this sequence?" (p. 37). In a revision belonging to about the same period as the present passage there then comes "Not by the sequence; but by a rule; or by the training in the use of a rule." (Eds.)

And this proposition is not an empirical one. But why not an empirical one? A rule is surely something that we go by, and we produce one numeral out of another. Is it not matter of experience, that this rule takes someone from here to there?

And if the rule + 1 carries him one time from 4 to 5, perhaps another time it carries him from 4 to 7. Why is that impossible?

The question arises, what we take as criterion of going according to the rule. Is it for example a feeling of satisfaction that accompanies the act of going according to the rule? Or an intuition (intimation) that tells me I have gone right? Or is it certain practical consequences of proceeding that determine whether I have really followed the rule?—In that case it would be possible that 4 + 1 sometimes made 5 and sometimes something else. It would be thinkable, that is to say, that an experimental investigation would shew whether 4 + 1 always makes 5.

If it is not supposed to be an empirical proposition that the rule leads from 4 to 5, then *this*, the result, must be taken as the criterion for one's having gone by the rule.

Thus the truth of the proposition that 4 + 1 makes 5 is, so to speak, *overdetermined*. Overdetermined by this, that the result of the operation is defined to be the criterion that this operation has been carried out.

The proposition rests on one too many feet to be an empirical proposition. It will be used as a determination of the concept 'applying the operation + 1 to 4'. For we now have a new way of judging whether someone has followed the rule.

Hence 4 + 1 = 5 is now itself a rule, by which we judge proceedings.

This rule is the result of a proceeding that we assume as *decisive* for the judgment of other proceedings. The rule-grounding proceeding is the proof of the rule.

17. How does one describe the process of learning a rule?—If A claps his hands, B is always supposed to do it too.

Remember that the description of a language-game is already a description.

I can train someone in a *uniform* activity. E.g. in drawing a line like this with a pencil on paper:

— . . — . . — . . — . . — . . — . . — . .

Now I ask myself, what is it that I want him to do, then? The answer is: He is always to go on as I have shewn him. And what do I really mean by: he is always to go on in that way? The best answer to this that I can give myself, is an example like the one I have just given.

I would use this example in order to shew him, and *also* to shew myself, what I mean by uniform.

We talk and act. That is already presupposed in everything that I am saying.

I say to him "That's right," and this expression is the bearer of a tone of voice, a gesture. I leave him to it. Or I say "No!" and hold him back.

18. Does this mean that 'following a rule' is indefinable? No. I can surely define it in countless ways. Only definitions are no use to me in these considerations.

19. I might also teach him to understand an order of the form:

(– • •) → or (– • • • • –) →

(Let the reader guess what I mean.)

Now what do I mean him to do? The best answer that I can give myself to this is to carry these orders on a bit further. Or do you believe that an algebraic expression of this rule presupposes less?

And now I train him to follow the rule

– • – • • – • • • etc.

And again I don't myself know any more about what I want from him, than what the example itself shews. I can of course paraphrase the rule in all sorts of different forms, but that makes it more intelligible only for someone who can already follow these paraphrases.

20. This, then, is how I have taught someone to count and to multiply in the decimal system, for example.

"365 × 428" is an order and he complies with it by carrying out the multiplication.

Here we insist on this, that the same sum that is set always has the same multiplication-pattern in its train, and so the same result. Different patterns of multiplication for the same set sum we reject.

The situation will now arise, of a calculator making mistakes in calculation; and also of his correcting mistakes.

A further language-game is this: He gets asked "How much is '365 × 428'?" And he may act on this question in two different ways. Either he does the multiplication, or if he has already done it before, he reads off the previous result.

21. The application of the concept 'following a rule' presupposes a custom. Hence it would be nonsense to say: just once in the history of

the world someone followed a rule (or a signpost; played a game, uttered a sentence, or understood one; and so on).

Here there is nothing more difficult than to avoid pleonasms and only to say what really describes something.

For here there is an overwhelming temptation to say something more, when everything has already been described.

It is of the greatest importance that a dispute hardly ever arises between people about whether the colour of this object is the same as the colour of that, the length of this rod the same as the length of that, etc. This peaceful agreement is the characteristic surrounding of the use of the word "same".

And one must say something analogous about proceeding according to a rule.

No dispute breaks out over the question whether a proceeding was according to the rule or not. It doesn't come to blows, for example.

This belongs to the framework, out of which our language works (for example, gives a description).

22. Now someone says that in the series of cardinal numbers that obeys the rule + 1, the technique of which was taught to us in such-and-such a way, 450 succeeds 449. That is not the empirical proposition that we come from 449 to 450 when it strikes us that we have applied the operation + 1 to 449. Rather is it a stipulation that only when the result is 450 have we applied this operation.

It is as if we had hardened the empirical proposition into a rule. And now we have, not an hypothesis that gets tested by experience, but a paradigm with which experience is compared and judged. And so a new kind of judgment.

For one judgment is: "He worked out 25 × 25, was attentive and conscientious in doing so and made it 615"; and another: "He worked out 25 × 25 and got 615 out instead of 625."

But don't the two judgments come to the same thing in the end?

The arithmetical proposition is not the empirical proposition: "When I do *this*, I get *this*"—where the criterion for my doing *this* is not supposed to be what results from it.

23. Might we not imagine that the main point in multiplying was the concentration of the mind in a definite way, and that indeed one didn't always work out the same sums the same way, but for the particular practical problems that we want to solve, just these differences of result were advantageous?

Is the main thing not this: that in *calculating* the main weight would be placed on whether one has calculated right or wrong, quite prescinding from the psychical condition etc. of the person who is doing the calculation?

The justification of the proposition 25 × 25 = 625 is, naturally, that if anyone has been trained in such-and-such a way, then under normal circumstances he gets 625 as the result of multiplying 25 by 25. But the arithmetical proposition does not assert *that*. It is so to speak an empirical proposition hardened into a rule. It stipulates that the rule has been followed only when that is the result of the multiplication. It is thus withdrawn from being checked by experience, but now serves as a paradigm for judging experience.

If we want to make practical use of a calculation, we convince ourselves that it has been "worked out right", that the *correct* result has been obtained. And there can be only *one* correct result of (e.g.) the multiplication; it doesn't depend on what you get when you *apply* the calculation. Thus we judge the facts by the aid of the calculation and quite differently from the way in which we should do so, if we did not regard the result of the calculation as something determined once for all.

Not empiricism and yet realism in philosophy, that is the hardest thing. (Against Ramsey.)

You do not yourself understand any more of the rule than you can explain.

24. "I have a particular concept of the rule. If in this sense one follows it, then from that number one can only arrive at this one". That is a spontaneous decision.

But why do I say "I *must*", if it is my decision? Well, may it not be that I must decide?

Doesn't its being a spontaneous decision merely mean: that's how I act; ask for no reason!

You say you must; but cannot say what compels you.

I have a definite concept of the rule. I know what I have to do in any particular case. I know, that is I am in no doubt: it is obvious to me. I say "Of course". I can give no reason.

When I say "I decide spontaneously", naturally that does not mean: I consider which number would really be the best one here and then plump for . . .

We say: "First the calculations must be done right, and then it will be possible to pass some judgment on the facts of nature."

25. Someone has learned the rule of counting in the decimal system. Now he takes pleasure in writing down number after number in the "natural" number series.

Or he follows the rule in the language-game "Write down the successor of the number .... in the series ...."—How can I explain this language-game to anyone? Well, I can describe an example (or examples).—In order to see whether he has understood the language-game, I may make him work out examples.

Suppose someone were to verify the multiplication tables, the logarithm tables etc., because he did not trust them. If he reaches a different result, he trusts it, and says that his mind had been so concentrated on the rule that the result it gets must count as the right one. If someone points out a mistake to him he says that he would rather doubt the trustworthiness of his own understanding and his own meaning *now* than then when he first made the calculation.

We can take agreement for granted in all questions of calculation. But now, does it make any difference whether we utter the proposition used in calculating as an empirical proposition or as a rule?

26. Should we acknowledge the rule $25^2 = 625$, if we did not all arrive at this result? Well, why then should we not be able to make use of the empirical proposition instead of the rule?—Is the answer to that: Because the contrary of the empirical proposition does not correspond to the contrary of the rule?

When I write down a bit of a series for you, that you then see *this* regularity in it may be called an empirical fact, a psychological fact. But, *if* you have seen this law in it, that you then continue the series in *this* way—that is no longer an empirical fact.

But how is it not an empirical fact?—for "seeing *this* in it" was presumably not the *same* as: continuing it like this.

One can only say that it is not an empirical proposition, by *defining* the step on this level as the one that corresponds to the expression of the rule.

Thus you say: "By the rule that *I* see in this sequence, it goes on in *this* way." Not: according to experience! Rather: that just is the meaning of this rule.

I understand: You say "that is not according to experience"—but still *isn't* it according to experience?

"By this rule it goes like *this*": i.e., you *give* this rule an extension. But why can't I give it this extension today, that one tomorrow?

Well, so I can. I might for example alternately give one of two interpretations.

27. If I have once grasped a rule I am bound in what I do further. But of course that only means that I am bound in my *judgment* about

what is in accord with the rule and what not.

If I now see a rule in the sequence that is given me—can that simply consist in, for example, my seeing an algebraic expression before me? Must it not belong to a language?

Someone writes up a sequence of numbers. At length I say: "Now I understand it; I must always . . ." And this *is* the expression of a rule. *But*, only within a language!

For when do I say that I see the rule—or a rule—in this sequence? When, for example, I can talk to myself about this sequence in a particular way. But surely also when I simply can continue it? No, I give myself or someone else a general explanation of how it is to be continued. But might I not give this explanation purely in the mind, and so without any real language?

28. Someone asks me: What is the colour of this flower? I answer: "red".—Are you absolutely sure? Yes, absolutely sure! But may I not have been deceived and called the wrong colour "red"? No. The certainty with which I call the colour "red" is the rigidity of my measuring-rod, it is the rigidity from which I start. When I give descriptions, *that* is not to be brought into doubt. This simply characterizes what we call describing.

(I may of course even here assume a slip of the tongue, but nothing else.)

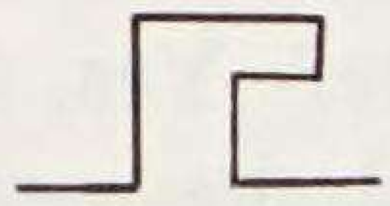
Following according to the rule is FUNDAMENTAL to our language-game. It characterizes what we call description.

This is the similarity of my treatment with relativity-theory, that it is so to speak a consideration about the clocks with which we compare events.

Is $25^2 = 625$ a fact of experience? You'd like to say: "No".—Why isn't it?—"Because, by the rules, it can't be otherwise."—And why so?—Because *that* is the meaning of the rules. Because that is the procedure on which we build all judging.

29. When we carry out a multiplication, we give a law. But what is the difference between the law and the empirical proposition that we give this law?

When I have been taught the rule of repeating the ornament



and now I have been told "Go on like that": how do I know what I have to do the next time?—Well, I do it with certainty, I shall also know how to defend what I do—that is, up to a certain point. If that does not count as a defence then there is none.

"As I understand the rule, *this* comes next."

Following a rule is a human activity.

I give the rule an extension.

Might I say: See here, if I follow the order I draw this line? Well in certain cases I shall say that. When for example I have constructed a curve according to an equation.

"See here! if I follow the order I do *this*!" That is naturally not supposed to mean: if I follow the order I follow the order. So I must have a different identification for the "this".

"So *that's* what following this order looks like!"

Can I say: "Experience teaches me: if I take the rule like *this* then *this* is how I must go on?"

Not if I make 'taking it so' one and the same with 'continuing so'.

Following a rule of transformation is not more problematic than following the rule: "keep on writing the same". For the transformation is a kind of identity.

30. It might however be asked: if all humans that are educated like this also calculate like *this*, or at least agree to *this* calculation as the right one; then what does one need the *law* for?

"$25^2 = 625$" cannot be the empirical proposition that people calculate like that, because $25^2 \neq 626$ would in that case not be the proposition that people get not this but another result; and also it could be true if people did not calculate at all.

The agreement of people in calculation is not an agreement in opinions or convictions.

Could it be said: "In calculating, the rules strike you as inexorable; you feel that you can only do that and nothing else if you want to follow the rule"?

"As I see the rule, *this* is what it requires." It does not depend on whether I am disposed this way or that.

I feel that I have given the rule an interpretation before I have followed it; and that this interpretation is enough to *determine* what I have to do in order to follow it in the particular case.

If I take the rule as I have taken it, then only doing *this* will correspond to it.

"Have you understood the rule?"—Yes, I have understood—"Then apply it now to the numbers . . . . ." If I want to follow the rule, have I now any choice left?

Assuming that he orders me to follow the rule and that I am frightened not to obey him: am I now not compelled?

But that is surely so too if he orders me: "Bring me this stone." Am I compelled less by *these* words?

31. To what extent can the function of language be described? If someone is not master of a language, I may bring him to a mastery of it by training. Someone who is master of it, I may remind of the kind of training, or I may describe it; for a particular purpose; thus already using a technique of the language.

To what extent can the function of a rule be described? Someone who is master of none, I can only train. But how can I explain the nature of a rule to myself?

The difficult thing here is not, to dig down to the ground; no, it is to recognize the ground that lies before us as the ground.

For the ground keeps on giving us the illusory image of a greater depth, and when we seek to reach this, we keep on finding ourselves on the old level.

Our disease is one of wanting to explain.

"Once you have got hold of the rule, you have the route traced for you."

32. What sort of public must there be if a game is to exist, if a game can be invented?

What surrounding is needed for someone to be able to invent, say, chess?

Of course I might invent a board-game today, which would never actually be played. I should simply describe it. But that is possible only because there already exist similar games, that is because such games *are played*.

One might also ask: is regularity possible *without* repetition?

I may give a new rule today, which has never been applied, and yet is understood. But would that be possible, if no rule had *ever* actually been applied?

And if it is now said: "Isn't it enough for there to be an imaginary application?" the answer is: No. (Possibility of a private language.)

A game, a language, a rule is an institution.

"But how often must a rule have actually been applied, in order for one to have the right to speak of a rule?" How often must a human being have added, multiplied, divided, before we can say that he has mastered the technique of these kinds of calculation? And by that I don't mean: how often must he have calculated right in order to convince *others* that he can calculate? No, I mean: in order to prove it to himself.

33. But couldn't we imagine that someone without any training should see a sum that was set to do, and straightway find himself in the mental state that in the normal course of things is only produced by training and practice? So that he knew he could calculate although he had never calculated. (One might, then, it seems, say; The training would merely be history, and merely as a matter of empirical fact would it be necessary for the production of knowledge.)—But suppose now he is in that state of certainty and he calculates wrong? What is he supposed to say himself? And suppose he then multiplied sometimes right, sometimes again quite wrong.—The training may of course be overlooked as mere history, if he now *always* calculates right. But that he *can* calculate he shews, to himself as well as to others only by this, that he *calculates* correctly.

What, in a complicated surrounding, we call "following a rule" we should certainly not call that if it stood in isolation.

34. Language, I should like to say, relates to a *way* of living.

In order to describe the phenomenon of language, one must describe a practice, not something that happens once, *no matter of what kind.*

It is very hard to realize this.

Let us imagine a god creating a country instantaneously in the middle of the wilderness, which exists for two minutes and is an exact reproduction of a part of England, with everything that is going on there in two minutes. Just like those in England, the people are pursuing a variety of occupations. Children are in school. Some people are doing mathematics. Now let us contemplate the activity of some human being during these two minutes. One of these people is doing exactly what a mathematician in England is doing, who is just doing a calculation.—Ought we to say that this two-minute-man is calculating? Could we for example not imagine a past and a continuation of these two minutes, which would make us call the processes something quite different?

Suppose that these beings did not speak English but apparently communicated with one another in a language that we are not acquainted with. What reason should we have to say that they were speaking a language? And yet *could* one not conceive what they were doing as that?

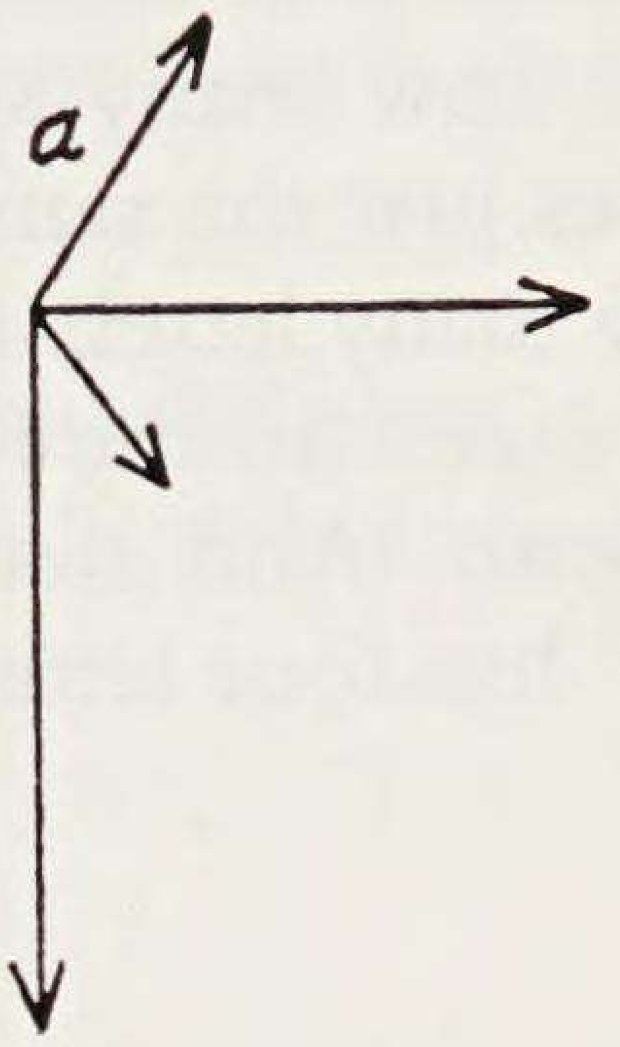
And suppose that they were doing something that we were inclined to call "calculating"; perhaps because its outward appearance was similar.—But *is* it calculating; and do (say) the people who are doing it know, though we do not?

35. How do I know that the colour that I am now seeing is called "green"? Well, to confirm it I might ask other people; but if they did not agree with me, I should become totally confused and should perhaps

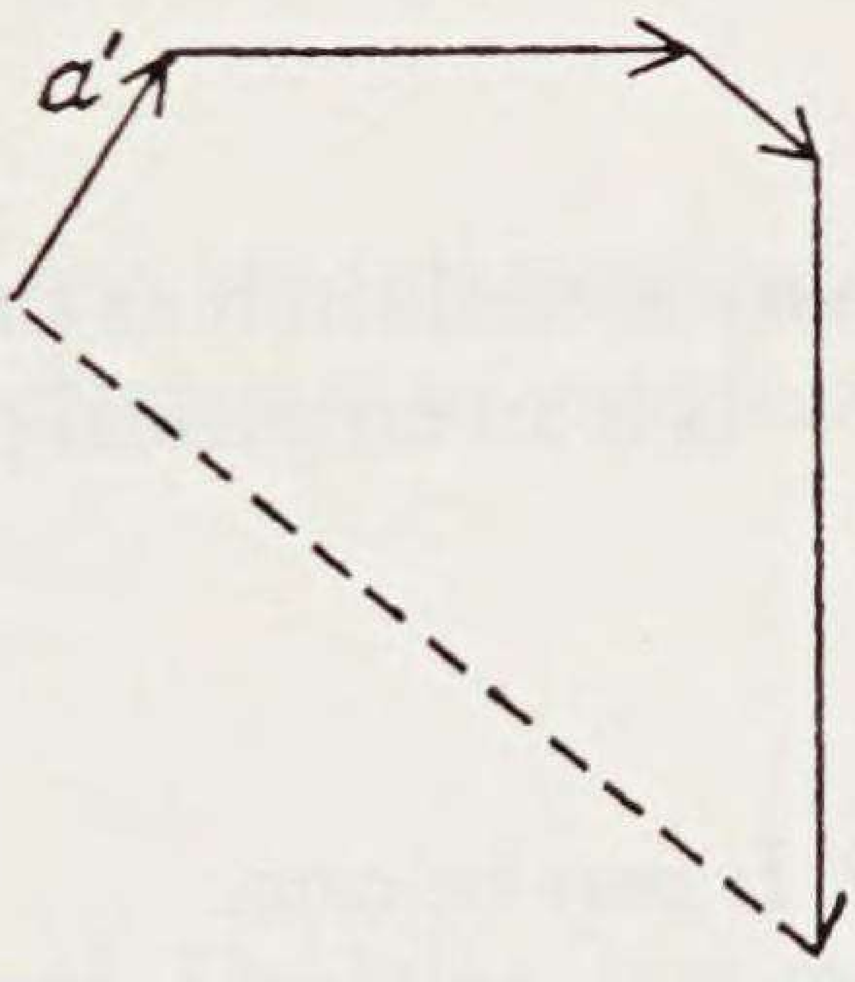
take them or myself for crazy. That is to say: I should either no longer trust myself to judge, or no longer react to what they say as to a judgement.

If I am drowning and I shout "Help!", how do I know what the word Help means? Well, that's how I react in this situation.—Now *that* is how I know what "green" means as well and also know how I have to follow the rule in the particular case.

Is it *imaginable* that the polygon of forces of



looks, not like this:



but otherwise? Well, is it imaginable that the parallel to *a* should not look to have the direction of *a'* but a different direction? That is to say: is it imaginable that I should regard not *a'* but a differently directed arrow as parallel to *a*? Well, I might for example imagine that I was somehow seeing the parallel lines in perspective and so I call ↗ ↑ parallel arrows, and that it never occurs to me that I have been using a different way of looking at them. Thus, then, it *is* imaginable that I should draw a different polygon of forces corresponding to the arrows.

36. What sort of proposition is this: "There are four sounds in the word *OBEN*"?
Is it an empirical proposition?

Before we have counted the letters, we don't know it.

Someone who counts the letters in the word 'OBEN' in order to find out how many sounds there are in a sequence that sounds like that, does just the same thing as someone who counts in order to find out how many letters there are in the word that is written in such-and-such a place. So the former is doing something that might also be an experiment. And that might be reason to call the proposition that 'OBEN' has four letters synthetic *a priori.*

The word "Plato" has as many sounds in it as the pentacle has corners. Is that a proposition of logic?—Is it an empirical proposition?

Is counting an experiment? It *may* be one.

Imagine a language-game in which someone has to count the sounds in a word. Now it might be that a word apparently always had the same sound, but that when we count its sounds we come to different numbers on different occasions. It might be, for example, that a word did seem to us to sound the same in different contexts (as it were by an acoustical illusion) but the difference emerged when we counted the sounds. In such a case we shall perhaps keep on counting the sounds of a word on different occasions, and this will perhaps be a kind of experiment.

On the other hand it may be that we count the sounds in words once for all, make a calculation, and make use of the result of this counting.

The resulting proposition will in the first case be a temporal one, in the second it will be non-temporal.

When I count the sounds in the word "Daedalus" I can regard the result in two different ways: (1) The word that is written there (or looks like this or was just now pronounced or etc.) has 7 sounds. (2) The sound-pattern "Dædalus" has 7 sounds.

The second proposition is timeless.

The employment of the two propositions must be different.

The *counting* is the same in the two cases. Only, what we reach by means of it is different.

The timelessness of the second proposition is not e.g. a result of the counting, but of the decision to employ the result of counting in a particular way.

In English the word Dædalus has 7 sounds. That is surely an empirical proposition.

Imagine that someone counted the sounds in words in order to find or test a linguistic law, say a law of development of language. He says: "'Dædalus' has 7 sounds". That's an empirical proposition. Consider here the *identity* of the word. The same word may here have now this, now that number of sounds.

Now I tell someone: "Count the sounds in these words and write down the number by each word."

I should like to say: "Through counting the sounds one may get an empirical proposition—but also one may get a rule."

To say: "The word .... has .... sounds—in the timeless sense" is a determination about the identity of the concept 'The word ....'. Hence the timelessness.

Instead of "The word .... has .... sounds—in the timeless sense," one might also say: "The word .... has *essentially* .... sounds."

37.

$$p|p \cdot | \cdot q|q = p \cdot q$$
$$p|q \cdot | \cdot p|q = p \vee q$$
$$x|y \cdot | \cdot z|u \overset{\text{Def}}{=} || (x, y, z, u)$$

Definitions would not at all need to be abbreviations; they might make new connexions in another way. Say by means of brackets or the use of different colours for the signs.

I may for example prove a proposition by using colours to indicate that it has the form of one of my axioms, lengthened by a certain substitution.

38. "I know how I have to go" means: I am in no doubt how I have to go.

"How can one follow a rule?" That is what I should like to ask.

But how does it come about that I want to ask that, when after all I find no kind of difficulty in following a rule?

Here we obviously misunderstand the facts that lie before our eyes.

How can the word "Slab" indicate what I have to do, when after all I can bring any action into accord with any interpretation?

How can I follow a rule, when after all whatever I do can be interpreted as following it?

What must I know, in order to be able to obey the order? Is there some *knowledge*, which makes the rule followable only in *this* way?

Sometimes I must *know* something, *sometimes* I must *interpret* the rule before I apply it.

Now, *how* was it possible for the rule to have been given an interpretation during instruction, an interpretation which reaches as far as to any arbitrary step?

And if this step was not named in the explanation, how then *can* we agree about what has to happen at this step, since after all whatever happens can be brought into accord with the rule and the examples?

Thus, you say, nothing definite has been said about these steps.

Interpretation comes to an end.

39. It is true that *anything* can be somehow justified. But the phenomenon of language is based on regularity, on agreement in action.

Here it is of the greatest importance that all or the enormous majority of us agree in certain things. I can, e.g., be quite sure that the colour of this object will be called 'green' by far the most of the human beings who see it.

It would be imaginable that humans of different stocks possessed languages that all had the same vocabulary, but the meanings of the words were different. The word that meant green among one tribe, meant same among another, table for a third and so on. We could even imagine that the same sentences were used by the tribes, only with entirely different senses.

Now in this case I should not say that they spoke the same language.

We say that, in order to communicate, people must agree with one another about the meanings of words. But the criterion for this agreement is not just agreement with reference to definitions, e.g., ostensive definitions—but *also* an agreement in judgments. It is essential for communication that we agree in a large number of judgments.

40. Language-game (2),[1] how can I explain it to someone, or to myself? Whenever A shouts "Slab" B brings *this* kind of object.—I might also ask: how can *I* understand it? Well, *only* as far as I can explain it.

But there is here a queer temptation which expresses itself in my inclination to say: I cannot understand it, because the interpretation of the explanation is still vague.

[1] §2 of *Philosophical Investigations*. An imaginary language 'is supposed to serve for communication between a builder A and an assistant B. A is constructing a building out of building stones; there are cubes, pillars, slabs and beams available. B has to pass him the blocks, and in the order that A needs them in. To this end they make use of a language consisting of the words "cube", "pillar", "slab" and "beam". A calls them out;—B brings the block that he has learnt to bring at this call. Conceive this as a complete primitive language.' (Eds.)

That is to say, both to you and to myself I can only give examples of the application.

41. The word "agreement" and the word "rule" are *related*, they are cousins. The phenomena of agreement and of acting according to a rule hang together.

There might be a cave-man who produced *regular* sequences of marks for himself. He amused himself, e.g., by drawing on the wall of the cave:

— · — — · — — · — — ·

or

— · — · · — · · · — · · · · —

But he is not following the general expression of a rule. And when we say that he acts in a regular way that is not because we can form such and expression.

But suppose he now developed π! (I mean without a general expression of the rule.)

Only in the practice of a language can a word have meaning.

Certainly I can give myself a rule and then follow it. But is it not a rule only for this reason, that it is analogous to what is called 'rule' in human dealings?

When a thrush always repeats the same phrase several times in its song, do we say that perhaps it gives itself a rule each time, and then follows the rule?

42. Let us consider very simple rules. Let the expression be a figure, say this one:

| - - |

and one follows the rule by drawing a straight sequence of such figures (perhaps as an ornament).

| - - || - - || - - || - - || - - |

Under what circumstances should we say: someone gives a rule by writing down such a figure? Under what circumstances: someone is following this rule when he draws that sequence? It is difficult to describe this.

If one of a pair of chimpanzees once scratched the figure | – – | in the earth and thereupon the other the series | – – || – – | etc., the first would not have given a rule nor would the other be following it, whatever else went on at the same time in the mind of the two of them.

If however there were observed, e.g., the phenomenon of a kind of instruction, of shewing how and of imitation, of lucky and misfiring attempts, of reward and punishment and the like; if at length the one who had been so trained put figures which he had never seen before one after another in sequence as in the first example, then we should probably say that the one chimpanzee was writing rules down, and the other was following them.

43. But suppose that already the first time the one chimpanzee had *purposed* to repeat this procedure? Only in a particular technique of

acting, speaking, thinking, can someone purpose something. (This 'can' is the grammatical 'can'.)

It is possible for me to invent a card-game today, which however never gets played. But it means nothing to say: in the history of mankind just once was a game invented, and that game was never played by anyone. That means nothing. Not because it contradicts psychological laws. Only in a quite definite surrounding do the words "invent a game" "play a game" make sense.

In the same way it cannot be said either that just once in the history of mankind did someone follow a sign-post. Whereas it can be said that just once in the history of mankind did some walk parallel with a board. And that first impossibility is again not a psychological one.

The words "language", "proposition", "order", "rule", "calculation", "experiment", "following a rule" relate to a technique, a custom.

A preliminary step towards acting according to a rule would be, say, pleasure in simple regularities such as the tapping out of simple rhythms or drawing or looking at simple ornaments. So one might train someone to obey the order: "draw something regular", "tap regularly". And here again one must imagine a particular technique.

You must ask yourself: under what special circumstances do we say that someone has "made a mere slip of the pen" or "he could perfectly well have gone on, but on purpose did not do so" or "he had meant to repeat the figure that he drew, but he happened not to do it".

The concept "regular tapping", "regular figure", is taught us in the same way as 'light-coloured' or 'dirty' or 'gaudy'.

44. But aren't we guided by the rule? And how can it guide us, when its expression can after all be interpreted by us both thus and otherwise? I.e. when after all various regularities correspond to it. Well, we are inclined to say that an expression of the rule guides us, i.e., we are inclined to use this metaphor.

Now what is the difference between the proceeding according to a rule (say an algebraic expression) in which one derives number after number according to the series, and the following proceeding: When we shew someone a certain sign, e.g. , a numeral occurs to him; if he looks at the numeral and the sign, another numeral occurs to him and so on. And each time we engage in this experiment the same series of numerals occurs to him. Is the difference between this proceeding and that of going on according to the rule the psychological one that in the second case we have something occurring to him? Might I not say: When he was following the rule "| – –|", then "| – – |" kept on occurring to him?

Well in our own case we surely have intuition, and people say that intuition underlies acting according to a rule.

So let us assume that that, so to speak, magical sign produces the series 123123123 etc.: is the sign *then* not the expression of a rule? No.

Acting according to a rule presupposes the recognition of a ***uniformity*** and the sign "123123123 etc." was the natural expression of a uniformity.

Now perhaps it will be said that | 22 || 22 || 22 | is indeed a uniform sequence of marks but surely

| 2 || 22 || 222 || 2222 |

is not.

Well, I might call this another kind of uniformity.

45. Suppose however there were a tribe whose people apparently had an understanding of a kind of regularity which I do not grasp. That is they would also have learning and instruction, quite analogous to that in § 42. If one watches them one would say that they follow rules, learn to follow rules. The instruction effects, e.g., agreement in actions on the part of pupil and teacher. But if we look at one of their series of figures we can see no regularity of any kind.

What should we say now? We *might* say: "They appear to be following a rule which escapes us," but also "Here we have a phenomenon of behaviour on the part of human beings, which we don't understand".

Instruction in acting according to the rule can be described without employing "and so on".

What can be described in this description is a gesture, a tone of voice, a sign which the teacher uses in a particular way in giving instruction, and which the pupils imitate. The effect of these expressions can also be described, again without calling 'and so on' to our aid, i.e. finitely. The effect of "and so on" will be to produce agreement going beyond what is done in the lessons, with the result that we all or nearly all count the same and calculate the same.

It would be possible, though, to imagine the very instruction without any "and so on" in it. But on leaving school the people would still all calculate the same beyond the examples in the instruction they had had.

Suppose one day instruction no longer produced agreement?

Could there be arithmetic without agreement on the part of calculators?

Could there be only one human being that calculated? Could there be only one that followed a rule?

Are these questions like, say, this one: "Can one man alone engage in commerce?"

It only makes sense to say "and so on" when "and so on" is *understood.* I.e., when the other is as capable of going on as I am, i.e., does go on just as I do.

Could two people engage in trade with one another?

46. When I say: "If you follow the rule, this *must* come out," that doesn't mean: it must, because it always has. Rather, that it comes out is one of my *foundations*.

What *must* come out is a foundation of judgment, which I do not touch.

On what occasion will it be said: "If you follow the rule this *must* come out"?

This may be a mathematical definition given in the train of a proof that a particular route branches. It may also be that one says it to someone in order to impress the nature of a rule upon him, in order to tell him something like: "You are *not* making an experiment here".

47. "But at every step I know absolutely what I have to do; what the rule demands of me." The rule, as I conceive it. I don't reason. The picture of the rule makes it clear how the picture of the series is to be continued.

"But I know at every step what I have to do. I see it quite clear before me. It may be boring, but there is no doubt what I have to do."

Whence this certainty? But why do I ask that question? Is it not enough that this certainty exists? What for should I look for a source of it? (And I can indeed give *causes* of it.)

When someone, whom we fear to disobey, orders us to follow the rule . . . which we understand, we shall write down number after number without any hesitation. And that is a typical kind of reaction to a rule.

"You already know how it is"; "You already know how it goes on."

I can now determine to follow the rule (–•–) →.

Like this: – • – – – – – – – – – • –

But it is remarkable that I don't lose the meaning of the rule as I do it. For how do I hold it fast?

But—how do I know that I do hold it fast, that I do not lose it?! It makes no sense at all to say I have held it fast unless there is such a thing as an outward mark of this. (If I were falling through space I might hold something, but not hold it still.)

Language just is a phenomenon of human life.

48. One person makes a bidding gesture, as if he meant to say "Go!" The other slinks off with a frightened expression. Might I not call this procedure "order and obedience", even if it happened only once?

What is this supposed to mean: "Might I not call the proceeding ——"? Against any such naming the objection could naturally be made, that among human beings other than ourselves a quite different gesture corresponds to "Go away!" and that perhaps our gesture for this order has among them the significance of our extending the hand in token of friendship. And whatever interpretation one has to give to a gesture depends on other actions, which precede and follow the gesture.

As we employ the word "order" and "obey", gestures no less than words are intertwined in a net of multifarious relationships. If I am now construing a simplified case, it is not clear whether I ought still to call the phenomenon "ordering" and "obeying".

We come to an alien tribe whose language we do not understand. Under what circumstances shall we say that they have a chief? What will occasion us to say that this man is the chief even if he is more poorly clad than others? The one whom the others obey—is he without question the chief?

What is the difference between inferring wrong and not inferring? between adding wrong and not adding? Consider this.

49. What you say seems to amount to this, that logic belongs to the natural history of man. And that is not combinable with the hardness of the logical "must".

But the logical "must" is a component part of the propositions of logic, and these are not propositions of human natural history. If what a proposition of logic said was: Human beings agree with one another in such and such ways (and that would be the form of the natural-historical proposition), then its contradictory would say that there is here a *lack* of agreement. Not, that there is an agreement of another kind.

The agreement of humans that is a presupposition of logic is not an agreement in *opinions*, much less in opinions on questions of logic.