SOLUTIONS TO *CALCULUS* VOLUME 1 BY TOM APOSTOL.

ERNEST YEUNG

Fund Science! & Help Ernest finish his Physics Research! : quantum super-A-polynomials - a thesis by Ernest Yeung

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SOLUTIONS TO VOLUME 1 *One-Variable Calculus, with an Introduction to Linear Algebra*

I 2.5 Exercises - Introduction to set theory, Notations for designating sets, Subsets, Unions, intersections, complements.

Exercise 10. *Distributive laws*

$$
\begin{aligned}
&\text{Let } X = A \cap (B \cup C), Y = (A \cap B) \cup (A \cap C) \\
&\quad \text{Suppose } x \in X \\
&\qquad x \in A \text{ and } x \in (B \cup C) \Longrightarrow x \in A \text{ and } x \text{ is in at least } B \text{ or in } C \\
&\qquad \text{then } x \text{ is in at least either } (A \cap B) \text{ or } (A \cap C) \\
&\qquad x \in Y, X \subseteq Y \\
&\quad \text{Suppose } y \in Y \\
&\qquad y \text{ is at least in either } (A \cap B) \text{ or } A \cap C \\
&\qquad \text{then } y \in A \text{ and either in } B \text{ or } C \\
&\qquad y \in X, Y \subseteq X \\
&\quad X = Y
\end{aligned}
$$

$$
\begin{aligned}
&\text{Let } X = A \cup (B \cap C), Y = (A \cup B) \cap (A \cup C) \\
&\quad \text{Suppose } x \in X \\
&\qquad \text{then } x \text{ is at least either in } A \text{ or in } (B \cap C) \\
&\qquad \text{if } x \in A, x \in Y \\
&\qquad \text{if } x \in (B \cap C), x \in Y \qquad\qquad x \in Y, X \subseteq Y \\
&\quad \text{Suppose } y \in Y \\
&\qquad \text{then } y \text{ is at least in } A \text{ or in } B \text{ and } y \text{ is at least in } A \text{ or in } C \\
&\qquad \text{if } y \in A, \text{ then } y \in X \\
&\qquad \text{if } y \in A \cap B \text{ or } y \in A \cup C, y \in X \text{ (various carvings out of } A\text{, simply)} \\
&\qquad \text{if } y \in (B \cap C), y \in X \qquad\qquad y \in X, Y \subseteq X \\
&\quad X = Y
\end{aligned}
$$

Exercise 11. If $x \in A \cup A$, then x is at least in A or in A. Then $x \in A$. So $A \cup A \subseteq A$. Of course $A \subseteq A \cup A$.

If $x \in A \cap A$, then x is in A and in A. Then $x \in A$. So $A \cap A \subseteq A$. Of course $A \subseteq A \cap A$.

Exercise 12. Let $x \in A$. $y \in A \cup B$ if y is at least in A or in B. x is in A so $x \in A \cup B$. $\Longrightarrow A \subseteq A \cup B$.

Suppose $\exists b \in B$ and $b \notin A$. $b \in A \cup B$ but $b \notin A$. so $A \subseteq A \cup B$.

Exercise 13. Let $x \in A \cup \emptyset$, then x is at least in A or in $\emptyset$. If $x \in \emptyset$, then x is a null element (not an element at all). Then actual elements must be in A. $\Longrightarrow A \cup \emptyset \subseteq A$.

Let $x \in A$. Then $x \in A \cup \emptyset$. $A \subseteq A \cup \emptyset$. $\Longrightarrow A = A \cup \emptyset$.

Exercise 14. From distributivity, $A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B)$.

If $x \in A \cap (A \cup B)$, $x \in A$ and $x \in A \cup B$, i.e. $x \in A$ and x is at least in A or in B.
$\Longrightarrow$ x is in A and is in B or is not in B. Then $x \in A$. $\Longrightarrow A \cap (A \cup B) \subseteq A$. Of course, $A \subseteq A \cap (A \cup B)$. $\Longrightarrow A \cap (A \cup B) = A \cup (A \cap B) = A$.

Exercise 15. $\forall a \in A, a \in C$ and $\forall b \in B, b \in C$. Consider $x \in A \cup B$. x is at least in A or in B. In either case, $x \in C$. $\Longrightarrow A \cup B \subseteq C$.

Exercise 16.

$$\begin{aligned}
&\text{if } C \subseteq A \text{ and } C \subseteq B, \text{ then } C \subseteq A \cap B \\
&\forall c \in C, c \in A \text{ and } c \in B \\
&x \in A \cap B, x \in A \text{ and } x \in B. \text{ Then } \forall c \in C, c \in A \cap B. \quad C \subseteq A \cap B
\end{aligned}$$

Exercise 17.

(1)
$$\begin{aligned}
&\text{if } A \subset B \text{ and } B \subset C \text{ then} \\
&\forall a \in A, a \in B. \forall b \in B, b \in C. \\
&\text{then since } a \in B, a \in C, \exists c \in C \text{ such that } c \notin B. \\
&\forall a \in A, a \in B \text{ so } a \neq c \forall a. \Longrightarrow A \subset C
\end{aligned}$$

(2) If $A \subseteq B, B \subseteq C, A \subseteq C$ since, $\forall a \in A, a \in B, \forall b \in B, b \in C$. Then since $a \in B, a \in C$. $A \subseteq C$
(3) $A \subset B$ and $B \subseteq C$. $B \subset C$ or $B = C$. $A \subset B$ only. Then $A \subset C$.
(4) Yes, since $\forall a \in A, a \in B$.
(5) No, since $x \neq A$ (sets as elements are different from elements)

Exercise 18. $A - (B \cap C) = (A - B) \cup (A - C)$

Suppose $x \in A - (B \cap C)$
then $x \in A$ and $x \notin B \cap C \Longrightarrow x \notin B \cap C$
then x is not in even at least one B or C
$\Longrightarrow x \in (A - B) \cup (A - C)$
Suppose $x \in (A - B) \cup (A - C)$
then x is at least in $(A - B)$ or in $(A - C) \Longrightarrow x$ is at least in A and not in B or in A and not in C
then consider when one of the cases is true and when both cases are true $\Longrightarrow x \in A - (B \cap C)$

Exercise 19.

$$\begin{aligned}
&\text{Suppose } x \in B - \bigcup_{A \in \mathcal{F}} A \\
&\text{then } x \in B, x \notin \bigcup_{A \in \mathcal{F}} A \\
&x \notin \bigcup_{A \in \mathcal{F}} A \Longrightarrow x \notin A, \forall A \in \mathcal{F} \\
&\text{since } \forall A \in \mathcal{F}, x \in B, x \notin A, \text{ then } x \in \bigcap_{A \in \mathcal{F}} (B - A)
\end{aligned}$$

$$\text{Suppose } x \in \bigcap_{A\in\mathcal{F}} (B-A)$$

then $x \in B - A_1$ and $x \in B - A_2$ and $\ldots$

then $\forall A \in \mathcal{F}, x \in B, x \notin A$

then $x \notin$ even at least one $A \in \mathcal{F}$

$$\implies x \in B - \bigcup_{A\in\mathcal{F}} A$$

$$\text{Suppose } x \in B - \bigcap_{A\in\mathcal{F}} A$$

$$\text{then } x \notin \bigcap_{A\in\mathcal{F}} A$$

then at most $x \in A$ for $\forall A \in \mathcal{F}$ but one

then x is at least in one $B - A$

$$\implies x \in \bigcup_{A\in\mathcal{F}} (B-A)$$

$$\text{Suppose } x \in \bigcup_{A\in\mathcal{F}} (B-A)$$

then x is at least in one $B - A$

then for $A \in \mathcal{F}, x \in B$ and $x \notin A$

Consider $\forall A \in \mathcal{F}$

$$\implies \text{ then } x \in B - \bigcap_{A\in\mathcal{F}} A$$

Exercise 20.

(1) (ii) is correct.

Suppose $x \in (A - B) - C$

then $x \in A - B, x \notin C$

then $x \in A$ and $x \notin B$ and $x \notin C$

$x \notin B$ and $x \notin C \implies x \notin$ even at least B or C

$x \in A - (B \cup C)$

Suppose $x \in A - (B \cup C)$

then $x \in A, x \notin (B \cup C)$

then $x \in A$ and $x \notin B$ and $x \notin C$

$\implies x \in (A - B) - C$

To show that (i) is sometimes wrong,

Suppose $y \in A - (B - C)$

$y \in A$ and $y \notin B - C$

$y \notin B - C$

then $y \notin B$ or $y \in C$ or $y \notin C$

(where does this lead to?)

Consider directly,

Suppose $x \in (A - B) \cup C$

then x is at least in $A - B$ or in C

then x is at least in A and $\notin B$ or in C

Suppose $x = c \in C$ and $c \notin A$

(2)

$$\text{If } C \subseteq A,$$
$$A - (B - C) = (A - B) \cup C$$

I 3.3 Exercises - The field axioms. The goal seems to be to abstract these so-called real numbers into just x's and y's that are purely built upon these axioms.

Exercise 1. Thm. I.5. $a(b-c) = ab - ac$.

Let $y = ab - ac; x = a(b-c)$
Want: $x = y$
$ac + y = ab$ (by Thm. I.2, possibility of subtraction)
Note that by Thm. I.3, $a(b-c) = a(b+(-c)) = ab + a(-c)$ (by distributivity axiom)
$ac + x = ac + ab + a(-c) = a(c + (-c)) + ab = a(0+b) = ab$

But there exists exactly one y or x by Thm. I.2. $x = y$.

Thm. I.6. $0 \cdot a = a \cdot 0 = 0$.

$$0(a) = a(0) \text{ (by commutativity axiom)}$$
$$\text{Given } b \in \mathcal{R} \text{ and } 0 \in \mathcal{R}, \exists \text{ exactly one } -b \text{ s.t. } b - a = 0$$
$$0(a) = (b + (-b))a = ab - ab = 0 \text{ (by Thm. I.5. and Thm. I.2)}$$

Thm. I.7.

$$ab = ac$$
$$\text{By Axiom 4}, \exists y \in \mathcal{R} \text{ s.t. } ay = 1$$
$$\text{since products are uniquely determined, } yab = yac \Longrightarrow (ya)b = (ya)c \Longrightarrow 1(b) = 1(c)$$
$$\Longrightarrow b = c$$

Thm. I.8. Possibility of Division.
Given $a, b, a \neq 0$, choose y such that $ay = 1$.
Let $x = yb$.

$$ax = ayb = 1(b) = b$$

Therefore, there exists at least one x such that $ax = b$. But by Thm. I.7, there exists only one x (since if $az - b$, and so $x = z$).

Thm. I.9. If $a \neq 0$, then $b/a = b(a^{-1})$.

$$\text{Let } x = \frac{b}{a} \text{ for } ax = b$$
$$y = a^{-1} \text{ for } ay = 1$$
$$\text{Want: } x = by$$
$$\text{Now } b(1) = b, \text{ so } ax = b = b(ay) = a(by)$$
$$\Longrightarrow x = by \text{ (by Thm. I.7)}$$

Thm. I.10. If $a \neq 0$, then $(a^{-1})^{-1} = a$.
Now $ab = 1$ for $b = a^{-1}$. But since $b \in \mathcal{R}$ and $b \neq 0$ (otherwise $1 = 0$, contradiction), then using Thm. I.8 on b, $ab = b(a) = 1; \quad a = b^{-1}$.

Thm.I.11. If $ab = 0, a = 0$ or $b = 0$.
$ab = 0 = a(0) \Longrightarrow b = 0$ or $ab = ba = b(0) \Longrightarrow a = 0$. (we used Thm. I.7, cancellation law for multiplication)

Thm. I.12. Want: $x = y$ if $x = (-a)b$ and $y = -(ab)$.

$$ab + y = 0$$
$$ab + x = ab + (-a)b = b(a + (-a)) = b(a - a) = b(0) = 0$$
$$0 \text{ is unique, so } ab + y = ab + x \text{ implies } x = y(\text{ by Thm. I.1 })$$

Thm. I.13. Want: $x + y = z$, if $a = bx, c = dy, (ad + bc) = (bd)z$.

$$(bd)(x + y) = bdx + bdy = ad + bc = (bd)z$$

So using $b, d \neq 0$, which is given, and Thm. I.7, then $x + y = z$.

Thm. I.14. Want: $xy = z$ for $bx = a, dy = c, ac = (bd)z$.

$$(bd)(xy) = (bx)(dy) = ac = (bd)z$$

$b, d \neq 0$, so by Thm. I.7, $xy = z$.

Thm.I.15. Want: $x = yz$, if $bx = a, dy = c, (bc)z = ad$

$$(bc)z = b(dy)z = d(byz) = da$$
$$d \neq 0 \text{ so by Thm. I.7, by } z = a, byz = abx$$
$$b \neq 0 \text{ so by Thm. I.7}, yz = x$$

Exercise 2. Consider $0 + z = 0$. By Thm. I.2, there exists exactly one z, $z = -0$. By Axiom 4, $z = 0$. $0 = -0$.

Exercise 3. Consider $1(z)z(1) = 1$. Then $z = 1^{-1}$. But by Axiom 4, there exists distinct 1 such that $z(1) = 1$, so $z = 1$.

Exercise 4. Suppose there exists x such that $0x = 1$, but $0x = 0$ and 0 and 1 are distinct, so *zero has no reciprocal* .

Exercise 5. $a + (-a) = 0, 0 + 0 = 0$. Then

$$a + (-a) + b + (-b) = (a + b) + (-a) + (-b) = 0$$
$$-(a + b) = -a + (-b) = -a - b$$

Exercise 6. $a + (-a) = 0, b + (-b) = 0$, so

$$a + (-a) + b + (-b) = a + (-b) + (-a) + b = (a - b) + (-a) + b = 0 + 0 = 0$$

$-(a - b) = -a + b$.

Exercise 7.

$$(a - b) + (b - c) = a + (-b) + b + (-c) = a + (b + (-b)) + (-c) = a - c$$

Exercise 8.

$$\begin{aligned} (ab)x &= 1 & (ab)^{-1} &= x \\ a(bx) &= 1 & a^{-1} &= bx \\ b(ax) &= 1 & b^{-1} &- ax \end{aligned}$$
$$a^{-1}b^{-1} = (abx)x = 1(x) = (ab)^{-1}$$

Exercise 9. Want: $x = y = z$, if

$$\begin{aligned} z &= \frac{a}{-b} & a &= zt \qquad b + t = 0 \\ y &= \frac{(-a)}{b} & by &= u \qquad a + u = 0 \\ x &= -\left(\frac{a}{b}\right) & \left(\frac{a}{b}\right) + x &= v + x = 0 \qquad vb = a \end{aligned}$$

$$a + (-a) = vb + by = b(v + y) = 0$$
$$\text{if } b \neq 0, v + y = 0, \text{ but } v + x = 0$$
$$\text{by Thm. I.1 }, x = y$$
$$b + t = 0, \text{ then } z(b + t) = zb + zt = zb + a = z(0) = 0$$
$$a + zb = 0 \Longrightarrow -a = zb = by$$
$$\text{since } b \neq 0, z = y \text{ so } x = y = z$$

Exercise 10. Since $b, d \neq 0$, Let

$$\begin{aligned} z &= \frac{ad - bc}{bd} & (bd)z &= ad - bc \text{ by previous exercise or Thm. I.8, the possibility of division} \\ x &= \frac{a}{b} & bx &= a \\ t &= \frac{-c}{d} & dt &= -c \text{ (By Thm. I.3, we know that } b - a = b + (-a) \text{)} \end{aligned}$$

$$dbx + bdt = (bd)(x+y) = ad - bc = (bd)z$$
$$b, d \neq 0, \text{ so } x + y = z$$

I 3.5 Exercises - The order axioms.

Theorem 1 (I.18). *If $a < b$ and $c > 0$ then $ac < bc$*

Theorem 2 (I.19). *If $a < b$ and $c > 0$, then $ac < bc$*

Theorem 3 (I.20). *If $a \neq 0$, then $a^2 > 0$*

Theorem 4 (I.21). $1 > 0$

Theorem 5 (I.22). *If $a < b$ and $c < 0$, then $ac > bc$.*

Theorem 6 (I.23). *If $a < b$ and $-a > -b$. In particular, if $a < 0$, then $-a > 0$.*

Theorem 7 (I.24). *If $ab > 0$, then both a and b are positive or both are negative.*

Theorem 8 (I.25). *If $a < c$ and $b < d$, then $a + b < c + d$.*

Exercise 1.

(1) By Thm. I.19, $-c > 0$

$$a(-c) < b(-c) \to -ac < -bc$$
$$-bc - (-ac) = ac - bc > 0. \text{ Then } ac > bc \text{ (by definition of } > \text{)}$$

(2)

$$a < b \to a + 0 < b + 0 \to a + b + (-b) < b + a + (-a) \to (a+b) - b < (a+b) + (-a)$$
$$\text{By Thm.I.18 } (a+b) + -(a+b) + (-b) < (a+b) - (a+b) + (-a)$$
$$-b < -a$$

(3)

If $a = 0$ or $b = 0$, $ab = 0$, but $0 \not> 0$

If $a > 0$, then if $b > 0, ab > 0(b) = 0$. If $b < 0, ab < 0(b) = 0$. So if $a > 0$, then $b > 0$.

If $a < 0$, then if $b > 0, ab < 0(b) = 0$. If $b < 0, ab > 0(b) = 0$. So if $a < 0$, then $b < 0$.

(4)

$$a < c \text{ so } a + b < c + b = b + c$$
$$b < d \text{ so } b + c < d + c$$
$$\text{By Transitive Law }, a + b < d + c$$

Exercise 2. If $x = 0, x^2 = 0$. $0 + 1 = 1 \neq 0$. So $x \neq 0$.

$$\text{If } x \neq 0, x^2 > 0, \text{ and by Thm. I.21 }, 1 > 0$$
$$x^2 + 1 > 0 + 0 = 0 \to x^2 + 1 \neq 0$$
$$\implies \nexists x \in \mathbb{R} \text{ such that } x^2 + 1 = 0$$

Exercise 3.

$$a < 0, b < 0, a + b < 0 + 0 = 0 \text{ (By Thm. I.25)}$$

Exercise 4. Consider $ax = 1$.

$$ax = 1 > 0. \text{ By Thm. I.24 }, a, x \text{ are both positive or } a, x \text{ are both negative}$$

Exercise 5. Define x, y such that $ax = 1, by = 1$. We want $x > y$ when $b > a$.

$$xb - ax = xb - 1 > 0 \implies bx > 1 = by$$
$$b > 0 \text{ so } x > y$$

Exercise 6.

$$
\begin{aligned}
&\text{If } a = b \text{ and } b = c, \text{ then } a = c\\
&\text{If } a = b \text{ and } b < c, \text{ then } a < c\\
&\text{If } a < b \text{ and } b = c, \text{ then } a < c\\
&\text{If } a < b \text{ and } b < c, \text{ then } a < c \text{ (by transitivity of the inequality)}\\
&\Longrightarrow a \leq c
\end{aligned}
$$

Exercise 7. If $a \leq b$ and $b \leq c$, then $a \leq c$. If $a = c$, then by previous proof, $a = b$.

Exercise 8. If $a \leq b$ and $b \leq c$, then $a \leq c$. If $a = c$, then by previous proof, $a = b$.

Exercise 8. If a or b is zero, a^2 or $b^2 = 0$. By Thm. I.20, $b^2 \geq 0$ or $a^2 \geq 0$, respectively.

Otherwise, if neither are zero, by transitivity, $a^2 + b^2 > 0$.

Exercise 9. Suppose $a \geq x$. Then $a - x \geq 0$.

If $a \in \mathbb{R}$ so $\exists y \in \mathbb{R}$, such that $a - y = 0$.
Consider $y + 1 \in \mathbb{R}$ (by closure under addition).

$$a - (y+1) = a - y - 1 = 0 - 1 < 0 \text{ Contradiction that } a \geq y + 1$$

Exercise 10.

$$
\begin{aligned}
&\text{If } x = 0, \text{ done.}\\
&\text{If } x > 0, x \text{ is a positive real number. Let } h = \frac{x}{2}.\\
&\Longrightarrow \frac{x}{2} > x \text{ Contradiction.}
\end{aligned}
$$

I 3.12 Exercises - Integers and rational numbers, Geometric interpretation of real numbers as points on a line, Upper bound of a set, maximum element, least upper bound (supremum), The least-upper-bound axiom (completeness axiom), The Archimedean property of the real-number system, Fundamental properties of the suprenum and infimum. We use Thm I.30, the Archimedean property of real numbers, alot.

Theorem 9 (I.30)**.** *If $x > 0$ and if y is an arbitrary real number, there exists a positive integer n such that $nx > y$.*

We will use the least upper-bound axiom (completeness axiom) alot for continuity and differentiation theorems later. Apostol states it as an axiom; in real analysis, the existence of a sup for nonempty, bounded sets can be shown with an algorithm to zoom into a sup with monotonically increasing and monotonically decreasing sequence of "guesses" and showing its difference is a Cauchy sequence.

Axiom 1 (Least upper-bound axiom)**.** *Every nonempty set S of real numbers which is bounded above has a suprenum; that is, there's a real number B s.t. $B = supS$.*

Exercise 1. $0 < y - x$.

$$
\begin{gathered}
\Longrightarrow n(y-x) > h > 0, n \in \mathbb{Z}^+, h \text{ arbitrary}\\
y - x > h/n \Longrightarrow y > x + h/n > x\\
\text{so let } z = x + h/n \text{ Done.}
\end{gathered}
$$

Exercise 2. $x \in \mathbb{R}$ so $\exists n \in \mathbb{Z}^+$ such that $n > x$ (Thm. I.29).

Set of negative integers is unbounded below because
If $\forall m \in \mathbb{Z}^-, -x > -m$, then $-x$ is an upper bound on $\mathbb{Z}^+$. Contradiction of Thm. I.29. $\Longrightarrow \exists m \in \mathbb{Z}$ such that $m < x < n$

Exercise 3. Use Archimedian property.

$x > 0$ so for $1, \exists n \in \mathbb{Z}^+$ such that $nx > 1, x > \frac{1}{n}$.

Exercise 4. x is an arbitrary real number. By Thm. I.29 and well-ordering principle, there exists a smallest $n+1$ positive integer such that $x < n + 1$ (consider the set of all $m + 1 > x$ and so by well-ordering principle, there must be a smallest element of this specific set of positive integers).

If $x = n$ for some positive integer n, done.
Otherwise, note that if $x < n$, then $n + 1$ couldn't have been the smallest element such that $m > x$. $x > n$.

Exercise 5. If $x = n$, done. Otherwise, consider all $m > x$. By well-ordering principle, there exists a smallest element n such that $n > x$.

If $x+1 < n$, then $x < n-1$, contradicting the fact that n is the smallest element such that $x < n$. Thus $x+1 > n$.

Exercise 6. $y - x > 0$.

$$n(y-x) > h,\ h \text{ arbitrary },n \in \mathbb{Z}^+$$
$$y > x + h/n = z > x$$

Since h was arbitrary, there are infinitely many numbers in between x, y.

Exercise 7. $x = \frac{a}{b} \in \mathbb{Q}, y \notin \mathbb{Q}$.

$$x \pm y = \frac{a \pm by}{b}$$

If $a \pm by$ was an integer, say m, then $y = \pm\left(\frac{a - mb}{b}\right)$ which is rational. Contradiction.

$$xy = \frac{a}{b}\frac{y}{1} = \frac{ay}{b}$$

If ay was an integer, $ay = n$, $y = \dfrac{n}{a}$, but y is irrational. $\implies xy$ is irrational.

$$\frac{x}{y}$$

y is not an integer

Exercise 8. Proof by counterexamples. We want that the sum or product of 2 irrational numbers is not always irrational. If y is irrational, $y+1$ is irrational, otherwise, if $y+1 \in Q$, $y \in Q$ by closure under addition.
$\implies y + 1 - y = 1$

Likewise, $y\frac{1}{y} = 1$.

Exercise 9.

$$y - x > 0 \implies n(y-x) > k, n \in \mathbb{Z}^+, k \text{ arbitrary. Choose } k \text{ to be irrational. Then } k/n \text{ irrational.}$$
$$y > \frac{k}{n} + x > x. \text{ Let } z = x + \frac{k}{n}, z \text{ irrational .}$$

Exercise 10.

(1) Suppose $n = 2m_1$ and $n + 1 = 2m_2$.

$$2m_1 + 1 = 2m_2 \quad 2(m_1 - m_2) = 1 \quad m_1 - m_2 = \frac{1}{2}. \text{ But } m_1 - m_2 \text{ can only be an integer.}$$

(2) By the well-ordering principle, if $x \in \mathbb{Z}^+$ is neither even and odd, consider the set of all x. There must exist a smallest element x_0 of this set. But since $x_0 \in \mathbb{Z}^+$, then there must exist a $n < x$ such that $n + 1 = x_0$. n is even or odd since it doesn't belong in the above set. So x_0 must be odd or even. Contradiction.

(3)

$$(2m_1)(2m_2) = 2(2m_1m_2) \text{ even}$$
$$2m_1 + 2m_2 = 2(m_1 + m_2) \text{ even}$$
$$(2m_1 + 1) + (2m_2 + 1) = 2(m_1 + m_2 + 1) \implies \text{ sum of two odd numbers is even}$$
$$(n_1 + 1)(n_2 + 1) = n_1n_2 + n_1 + n + 2 + 1 = 2(2m_1m_2)$$
$$2(2m_1m_2) - (n_1 + n_2) - 1 \text{ odd, the product of two odd numbers } n_1, n_2 \text{ is odd}$$

(4) If n^2 even, n is even, since for $n = 2m$, $(2m)^2 = 4m^2 = 2(2m^2)$ is even.

$a^2 = 2b^2$. $2(b^2)$ even. a^2 even, so a even.

$$\text{If } a \text{ even } a = 2n.a^2 = 4n^2$$
$$\text{If } b \text{ odd , } b^2 \text{ odd. } b \text{ has no factors of 2 } b^2 \neq 4n^2$$

Thus b is even.

(5) For $\frac{p}{q}$, If p or q or both are odd, then we're done.
Else, when p, q are both even, $p = 2^l m, q = 2^n p, m, p$ odd.

$$\frac{p}{q} = \frac{2^l m}{2^n p} = \frac{2^{l-n} m}{p} \text{ and at least } m \text{ or } p \text{ odd}$$

Exercise 11. $\frac{a}{b}$ can be put into a form such that a or b at least is odd by the previous exercise.

However, $a^2 = 2b^2$, so a even, b even, by the previous exercise, part (d) or 4th part. Thus $\frac{a}{b}$ cannot be rational.

Exercise 12. *The set of rational numbers satisfies the Archimedean property but not the least-upper-bound property.*

Since $\frac{p}{q} \in \mathbb{Q} \subseteq \mathbb{R}$, $n\frac{p_1}{q_1} > \frac{p_2}{q_2}$ since if $q_1, q_2 > 0$,

$$\frac{np_1q_2}{q_1q_2} > \frac{q_1p_2}{q_1q_2} \quad np_1q_2 > q_1p_2$$

n exists since $(p_1q_2), (q_1p_2) \in \mathbb{R}$.
The set of rational numbers does not satisfy the least-upper-bound property.
Consider a nonempty set of rational numbers S bounded above so that $\forall x = \frac{r}{s} \in S, x < b$.

Suppose $x < b_1, x < b_2 \forall x \in S$.

$$\frac{r}{s} < b_2 < nb_1 \text{ but likewise } \frac{r}{s} < b_1 < mb_2,\ n, m \in \mathbb{Z}^+$$

So it's possible that $b_1 > b_2$, but also $b_2 > b_1$.

I 4.4 Exercises - An example of a proof by mathematical induction, The principle of mathematical induction, The well-ordering principle. Consider these 2 proofs.

$$N + N + \cdots + N = N^2$$

$$(N-1) + (N-2) + \cdots + (N-(N-1)) + (N-N) = N^2 - \sum_{j=1}^N j = \sum_{j=1}^{N-1} j$$

$$N^2 + N = 2\sum_{j=1}^N j \quad \Longrightarrow \sum_{j=1}^N j = \frac{N(N+1)}{2}$$

An interesting property is that

$$S = \sum_{j=m}^n j = \sum_{j=m}^n (n+m-j)$$

So that

$$\sum_{j=1}^N j = \sum_{j=m}^N j + \sum_{j=1}^m j = \sum_{j=m}^N j + \frac{m(m+1)}{2} = \frac{N(N+1)}{2}$$

$$\sum_{j=m}^N j = \frac{N(N+1) - m(m+1)}{2} = \frac{(N-m)(N+m+1)}{2}$$

Another way to show this is the following.

$$\begin{array}{rcccccc} S = & 1+ & 2+ & \cdots + & (N-2)+ & (N-1)+ & N \\ \text{but } S = & N+ & N-1+ & \cdots + & 3+ & 2+ & 1 \end{array}$$

$$2S = (N+1)N \quad S = \frac{N(N+1)}{2}$$

Telescoping series will let you get $\sum_{j=1}^{N} j^2$ and other powers of j.

$$\sum_{j=1}^{N}(2j-1) = 2\frac{N(N+1)}{2} - N = N^2$$

$$\sum_{j=1}^{N}(j^2-(j-1)^2) = \sum_{j=1}^{N}(j^2-(j^2-2j+1)) = \sum_{j=1}^{N}(2j-1) = 2\left(\frac{N(N+1)}{2}\right) - N = N^2$$

$$\sum_{j=1}^{N}(j^3-(j-1)^3) = N^3 = \sum_{j=1}^{N}(j^3-(j^3-3j^2+3j-1)) = \sum_{j=1}^{N}(3j^2-3j+1)$$

$$\implies 3\sum_{j=1}^{N} j^2 = -3\frac{N(N+1)}{2} + N = N^3 \implies \frac{2N^3+2N-3N^2-3N}{2} = \frac{N(N+1)(2N+1)}{6} = \sum_{j=1}^{N} j^2$$

$$\sum_{j=1}^{N} j^4-(j-1)^4 = N^4 = \sum_{j=1}^{N} j^4-(j^4-4j^3+6j^2-4j+1) = \sum_{j=1}^{N} 4j^3-6j^2+4j-1 =$$

$$= 4\sum_{j=1}^{N} j^3 - 6\frac{N(N+1)(2N+1)}{6} + 4\frac{N(N+1)}{2} - N = N^4$$

$$\implies \sum_{j=1}^{N} j^3 = \frac{1}{4}(N^4 + N(N+1)(2N+1) - 2N(N+1) + N) = \frac{1}{4}(N^4 + (2N)N(N+1) - N(N+1) + N)$$

$$= \frac{1}{4}(N^4+2N^3+2N^2-N^2-N+N) = \frac{1}{4}N^2(N^2+2N+1) = \frac{1}{4}\frac{(N(N+1))^2}{2}$$

Exercise 1. Induction proof.

$$\frac{1(1+1)}{2} \qquad \sum_{j=1}^{N+1} j = \sum_{j=1}^{n} j + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

Exercise 6.

(1)

$$A(k+1) = A(k) + k + 1 = \frac{1}{8}(2k+1)^2 + k + 1 = \frac{1}{8}(4k^2+4k+1) + \frac{8k+8}{8} = \frac{(2k+3)^2}{8}$$

(2) The $n=1$ case isn't true.

(3)

$$1+2+\cdots+n = \frac{(n+1)n}{2} = \frac{n^2+n}{2} < \frac{n^2+n+\frac{1}{4}}{2}$$
$$\text{and } \left(\frac{2n+1}{2}\right)^2\frac{1}{2} = \frac{(n+1/2)^2}{2} = \frac{n^2+n+1/4}{2}$$

Exercise 7.

$$(1+x)^2 > 1+2x+2x^2$$
$$1+2x+x^2 > 1+2x+2x^2$$
$$0 > x^2 \implies \text{ Impossible}$$
$$(1+x)^3 = 1+3x+3x^2+x^3 > 1+3x+3x^2$$
$$\implies x^3 > 0$$

By well-ordering principle, we could argue that $n=3$ must be the smallest number such that $(1+x)^n > 1+2x+2x^2$. Or we could find, explicitly

$$(1+x)^n = \sum_{j=0}^{n}\binom{n}{j}x^j = 1 + nx + \frac{n(n-1)}{2}x^2 + \sum_{j=3}^{n}\binom{n}{j}x^j$$

and

$$\begin{aligned}\frac{n(n-1)}{2} &> n \\ n^2 - n &> 2n \\ n^2 &> 3n \\ n &> 3\end{aligned}$$

Exercise 8.

$$\begin{aligned}&a_2 \leq ca_1, a_3 \leq ca_2 \leq c^2 a_1 \\ &a_{n+1} \leq ca_n \leq ca_1 c^{n-1} = a_1 c^n\end{aligned}$$

Exercise 9.

$$\begin{aligned}&n = 1, \ \sqrt{1} = 1 \\ &\sqrt{1^2 + 1^2} = \sqrt{2} \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3} \\ &\sqrt{(\sqrt{n})^2 + 1^2} = \sqrt{n+1}\end{aligned}$$

Exercise 10.

$$\begin{aligned}&1 = qb + r \\ &\quad q = 0, b = 1, r = 1 \\ &2 = qb + r, q = 0, r = 2, b = 1, 2 \text{ or } r = 0, q = 2; q = 1, r = 0 \\ &\quad \text{Assume } n = qb + r; 0 \leq r < b; b \in \mathbb{Z}^+, b \text{ fixed} \\ &\quad n + 1 = qb + r + 1 = qb + 1 + r = qb + 1 + b - 1 = (q+1)b + 0\end{aligned}$$

Exercise 11. For $n > 1$, $n = 2, 3$ are prime. $n = 4 = 2(2)$, a product of primes.

Assume the $k-1$th case. Consider $\frac{k}{j}$, $1 \leq j \leq k$.
If $\frac{k}{j} \in \mathbb{Z}^+$, only for $j = 1, j = k$, then k prime.
If $\frac{k}{j} \in \mathbb{Z}^+$, for some $1 < j < k$, $\frac{k}{j} = c \in \mathbb{Z}^+$. $c, j < k$.
Thus $k = cj$. c, j are products of primes or are primes, by induction hypothesis. Thus k is a product of primes.

Exercise 12. $n = 2$. G_1, G_2 are blonde. G_1 has blue eyes. Consider G_2. G_2 may not have blue eyes. Then G_1, G_2 are not all blue-eyed.

I 4.7 Exercises - Proof of the well-ordering principle, The summation notation. Exercise 1.

(1) $\frac{n(n+1)}{2} = \sum_{k=1}^4 k = 10$
(2) $\sum_{n=2}^5 2^{n-2} = \sum_{n=0}^3 2^n = 1 + 14 = 15$
(3) $2\sum_{r=0}^3 2^{2r} = 2\sum_{r=0}^3 4^r = 170$
(4) $\sum_{j=1}^4 j^j = 1 + 4 + 27 + 4^4 = 288$
(5) $\sum_{j=0}^5 (2j+1) = 2\frac{5(6)}{2} + 6(1) = 36$
(6) $\sum \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Exercise 9.

$$\begin{aligned}&n = 1(-1)(3) + 5 = 2 = 2n \\ &n = 2(-1)(3) + 5 + (-1)7 + 9 = 4 = 2n \\ &\quad n \sum_{k=1}^{2n} (-1)^k (2k+1) = 2n \\ &n + 1 \sum_{k=1}^{2(n+1)} (-1)^k (2k+1) = \sum_{k=1}^{2n} (-1)^k (2k+1) + (-1)^{2n+1}(4n+3) + (-1)^{2n+2}(4n+5) = \\ &\quad = 2n + 2 = 2(n+1)\end{aligned}$$

Exercise 10.

(1) $a_m + a_{m+1} + \cdots + a_{m+n}$
(2)

$$n = 1 \frac{1}{2} = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$

$$n+1 \sum_{k=n+2}^{2(n+1)} \frac{1}{k} = \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m} - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} = \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m} + - \frac{1}{2(n+1)} + \frac{(-1)^{2n+1+1}}{(2n+1)}$$

$$= \sum_{m=1}^{2(n+1)} \frac{(-1)^{m+1}}{m}$$

Exercise 13.

$$n = 12(\sqrt{2} - 1) < 1 < 2 \text{ since } \frac{1}{2} > \sqrt{2} - 1$$

$$n \text{ case } (\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = n + 1 - n = 1 < \frac{\sqrt{n+1} + \sqrt{n}}{2\sqrt{n}} = \frac{1}{2}(\sqrt{1 + \frac{1}{n}} + 1)$$

$$n+1 \text{ case } (\sqrt{n+2} - \sqrt{n+1})(\sqrt{n+2} + \sqrt{n+1}) = n + 2 - (n+1) = 1$$

$$\frac{\sqrt{n+2} + \sqrt{n+1}}{2\sqrt{n+1}} = \frac{1 + \sqrt{1 + \frac{1}{n+1}}}{2} > 1$$

So then, using the telescoping property,

$$\sum_{n=1}^{n-1} 2(\sqrt{n+1} - \sqrt{n}) = 2(\sqrt{m} - 1) < \sum_{n=1}^{m} \frac{1}{\sqrt{n}} < \sum_{n=1}^{m} 2(\sqrt{n} - \sqrt{n-1}) = 2(\sqrt{m} - 1) < 2\sqrt{m} - 1$$

I 4.9 Exercises - Absolute values and the triangle inequality. Exercise 1.

(1) $|x| = 0 iff x = 0$
If $x = 0, x = 0, -x = -0 = 0$. If $|x| = 0, x = 0, -x = 0$.
(2)

$$|-x| = \begin{cases} -x & \text{if } -x \geq 0 \\ x & \text{if } -x \leq 0 \end{cases} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

(3) $|x - y| = |y - x|$ by previous exercise and $(-1)(x - y) = y - x$ (by distributivity)
(4) $|x|^2 = \begin{cases} (x)^2 & \text{if } x \geq 0 \\ (-x)^2 & \text{if } x \leq 0 \end{cases} = x^2$
(5) $\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases} = |x|$
(6) We want to show that $|xy| = |x||y|$

$$|xy| = \begin{cases} xy & \text{if } xy \geq 0 \\ -xy & \text{if } xy \leq 0 \end{cases} = \begin{cases} xy & \text{if } x, y \geq 0 \text{ or } x, y \leq 0 \\ -xy & \text{if } x, -y \geq 0 \text{ or } -x, y \leq 0 \end{cases}$$

$$|x||y| = \begin{cases} x|y| & \text{if } x \geq 0 \\ -x|y| & \text{if } x \leq 0 \end{cases} = \begin{cases} xy & \text{if } x, y \geq 0 \\ -xy & \text{if } x, -y \geq 0 \\ -xy & \text{if } -x, y \geq 0 \\ xy & \text{if } -x, -y \geq 0 \end{cases}$$

(7) By previous exercise, since

$$\left| \frac{x}{y} \right| = |xy^{-1}| = |x||y^{-1}|$$

$$\left| \frac{1}{y} \right| = \begin{cases} \frac{1}{y} & \text{if } \frac{1}{y} \geq 0 \\ \frac{-1}{y} & \text{if } \frac{1}{y} \leq 0 \end{cases} \quad \frac{1}{|y|} = \begin{cases} \frac{1}{y} & \text{if } \frac{1}{y} \geq 0 \\ \frac{-1}{y} & \text{if } \frac{1}{y} \leq 0 \end{cases}$$

(8) We know that $|a-b| \leq |a-c| + |b-c|$.

$$\text{Let } c = 0 \Longrightarrow |x-y| \leq |x| + |y|$$

(9) $x = a-b, b-c = -y$.

$$|x| \leq |x-y| + |-y| \quad |x| - |y| \leq |x-y|$$

(10)

$$||x|-|y|| = \begin{cases} |x|-|y| & \text{if } |x|-|y| \geq 0 \\ |y|-|x| & \text{if } |x|-|y| \leq 0 \end{cases}$$
$$|x| \leq |x-y| + |-y| \Longrightarrow |x|-|y| \leq |x-y|$$
$$|y| \leq |y-x| + |-x| \Longrightarrow |y|-|x| \leq |y-x| = |x-y|$$

Exercise 4.

$$\Rightarrow$$

$$\text{If } \forall k = 1 \dots n;\ a_k x + b_k = 0$$
$$\left(\sum_{k=1}^n a_k(-xa_k)\right)^2 = \left(x\sum_{k=1}^n a_k^2\right)^2 = \left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n (-xa_k)^2\right) = \left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n b_k^2\right)$$
$$\Leftarrow \text{ Proving } a_k x + b_k = 0 \text{ means } x = -\frac{b_k}{a_k},\ a_k \neq 0$$
$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 = \sum_{j=1}^n a_j^2 b_j^2 + \sum_{j \neq q}^n a_j a_k b_j b_k == \sum_{j=1}^n a_j^2 b_j^2 + \sum_{j \neq k}^n a_j^2 b_k^2$$
$$\Longrightarrow a_j^2 b_k^2 - a_j a_k b_j b_k = a_j b_k (a_j b_k - a_k b_j) = 0$$
$$\text{if } a_j, b_k \neq 0,\ a_j b_k - a_k b_j = 0 \Longrightarrow a_k \left(\frac{b_j}{-a_j}\right) + b_k = 0$$

Exercise 8. The trick of this exercise is the following algebraic trick ("multiplication by conjugate") and using telescoping property of products:

$$(1 - x^{2^j})(1 + x^{2^j}) = 1 - x^{2^j + 2^j} = 1 - x^{2^{j+1}}$$
$$\prod_{j=1}^{1} 1 + x^{2^{j-1}} = \prod_{j=1}^{1} \frac{1 - x^{2^j}}{1 - x^{2^{j-1}}} = \frac{1 - x^{2^n}}{1 - x}$$
$$\text{if } x = 1, 2^n$$

Exercise 10.

$$\begin{aligned} & x > 1 \\ & x^2 > x \qquad x^{n+1} = x^n x > x^2 > x \\ & x^3 > x^2 > x \end{aligned}$$
$$\begin{aligned} & 0 < x < 1 \\ & x^2 < x \qquad x^{n+1} = x^n x < x^2 < x \Longrightarrow x^{n+1} < x \\ & X^3 < x^2 < x \end{aligned}$$

Exercise 11. Let $S = \{n \in \mathbb{Z}^+ | 2^n < n!\}$.

By well-ordering principle, $\exists$ smallest $n_0 \in S$. Now
$2^4 = 16,\ 4! = 24$. So S starts at $n = 4$.

Exercise 12.

(1)

$$\left(1+\frac{1}{n}\right)^n = \sum_{j=0}^n \binom{n}{k}\left(\frac{1}{n}\right)^j = \sum_{k=0}^n \frac{n!}{(n-k)!k!}\left(\frac{1}{n}\right)^k$$

$$\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right) = \prod_{r=0}^{k-1}\left(\frac{n-r}{n}\right) = \left(\frac{1}{n^k}\right)\frac{n!}{(n-k)!}$$

$$\sum_{k=1}^n \frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right) = \left(\frac{1}{n^k}\right)\frac{n!}{(n-k)!}$$

(2)

$$(1+\frac{1}{n})^n = 1+\sum_{k=1}^n\left(\frac{1}{k!}\prod_{r=0}^{k-1}(1-\frac{r}{n})\right) < 1+\sum_{k=1}^n\frac{1}{k!} < 1+\sum_{k=1}^n\frac{1}{2^k} = 1+\frac{\frac{1}{2}-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 1+(1-\left(\frac{1}{2}\right)^n)$$
$$< 3$$

The first inequality obtained from the fact that if $0 < x < 1, x^n < x < 1$. The second inequality came from the previous exercise, that $\frac{1}{k!} < \frac{1}{2^k}$.

$$(1+\frac{1}{n})^n = \sum_{k=0}^n\binom{n}{k}\left(\frac{1}{n}\right)^k = 1+\frac{1}{n}+\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{1}{n}\right)^k = 1+\frac{1}{n}+\sum_{k=2}^{n-1}\binom{n}{k}\left(\frac{1}{n}\right)^k+\frac{n}{1}\left(\frac{1}{n}\right) > > 2$$

Exercise 13.

(1)

$$S = \sum_{k=0}^{p-1}\left(\frac{b}{a}\right)^k = \frac{1-\left(\frac{b}{a}\right)^p}{1-\frac{b}{a}}$$

$$\sum_{k=0}^{p-1} b^k a^{p-1-k} = a^{p-1}\frac{1-\left(\frac{b}{a}\right)^p}{1-\frac{b}{a}} = \frac{b^p-a^p}{b-a}$$

(2)

(3) Given

$$n^p < \frac{(n+1)^{p+1}-n^{p+1}}{p+1} < (n+1)^p$$

We want

$$\sum_{k=1}^{n-1}k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^n k^p$$

$$n = 21^p < \frac{2^{p+1}}{p+1} < 1^p+2^p$$
$$p = 1$$
$$1 < 2^2/2 = 2.2 < 1+2 = 3$$
$$p-2$$
$$1 < 8/3 < 1+4 = 5$$

I 4.10 Miscellaneous exercises involving induction. Exercise 13.

(1)
(2)
(3) Let $n = 2$.

$$\sum_{k=1}^{2-1}k^p = 1^p = 1,\ \frac{n^{p+1}}{p+1} = \frac{2^{p+1}}{p+1}\ \sum_{k=1}^2 k^p = 1+2^p$$

What makes this exercise hard is that **we have to use induction on p itself**. Let $p = 1$.

$$1 < \frac{2^{1+1}}{1+2} = 2 < 1+2^1 = 3$$

Now assume pth case. Test the $p+1$ case.

$$\frac{2^{p+2}}{p+2} = \frac{2(p+1)}{p+2}\left(\frac{2^{p+1}}{p+1}\right) > 1$$
since $p+2 < 2p+2 = 2(p+1)$ for $p \in \mathbb{Z}^+$

For the right-hand inequality, we will use the fact just proven, that $2^p - (p) > 0$ and pth case rewritten in this manner

$$(1+2^p) > \frac{2^{p+1}}{p+1} \Longrightarrow (1+2^p)(p+1) > 2^{p+1}$$

So

$$\begin{aligned}(p+2)(1+2^{p+1}) = (p+2) + ((p+1)+1)2^p(2) = (p+2) + 2(p+1)2^p + 2^p(2) > \\ > (p+2) + 2(2^{p+1} - (p+1)) + 2^p(2) = -p + 2^{p+2} + 2^{p+1} > 2^{p+2}\end{aligned}$$

So the $n=2$ case is true for all $p \in \mathbb{Z}^+$.

Assume nth case is true. We now prove the $n+1$ case.

$$\begin{aligned}\sum_{k=1}^{n} k^p = \sum_{k=1}^{n-1} k^p + n^p < \frac{n^{p+1}}{p+1} + n^p < \frac{n^{p+1}}{p+1} + \frac{(n+1)^{p+1} - n^{p+1}}{p+1} = \frac{(n+1)^{p+1}}{p+1} \\ \sum_{k=1}^{n+1} k^p = \sum_{k=1}^{n} k^p + (n+1)^p > \frac{n^{p+1}}{p+1} + \frac{(n+1)^{p+1} - n^{p+1}}{p+1} = \frac{(n+1)^{p+1}}{p+1}\end{aligned}$$

We had used the inequality proven in part b, $n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$.

Exercise 14. Use induction to prove a general form of Bernoulli's inequality.

$$\begin{aligned}1 + a_1 = 1 + a_1 \\ (1+a_1)(1+a_2) = 1 + a_2 + a_1 + a_1 a_2 \geq 1 + a_1 + a + 2\end{aligned}$$

Test the $n+1$ case

$$\begin{aligned}(1+a_1)(1+a_2)\dots(1+a_{n+1}) \geq (1 + a_1 + a_2 + \dots + a_n)(1+a_{n+1}) = \\ = 1 + a_1 + a_2 + \dots + a_n + a_{n+1} + a_{n+1}(a_1 + a_2 + \dots a_n) \geq \\ \geq 1 + a_1 + a_2 + \dots + a_n + a_{n+1}\end{aligned}$$

Note that the last step depended upon the given fact that all the numbers were of the same sign.

For $a_1 = a_2 = \dots = a_n = x$, then we have $(1+x)^n \geq 1 + nx$.

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j = 1 + nx$$

Since x and n are arbitrary, we can compare terms of x^j's. Then $x = 0$.

Exercise 15. $\frac{2!}{2^2} = \frac{1}{2}\ \frac{3!}{3^3} = \frac{2}{9} < 1$.

So we've shown the $n = 2, n = 3$ cases. Assume the nth case, that $\frac{n!}{n^n} \leq \left(\frac{1}{2}\right)^k$, where k is the greatest integer $\leq \frac{n}{2}$.

$$\frac{(n+1)!}{(n+1)^{n+1}} \geq \frac{(n+1)n^n \left(\frac{1}{2}\right)^k}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \left(\frac{1}{2}\right)^k = \left(1 - \frac{1}{n+1}\right)^n \left(\frac{1}{2}\right)^k < \frac{1}{2}\left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1}$$

where in the second to the last step, we had made this important observation:

$$k \leq \frac{n}{2} \Longrightarrow k + \frac{1}{2} \leq \frac{n+1}{2} \Longrightarrow \frac{1}{n+1} \leq \frac{1}{2k+1} < \frac{1}{2}$$

Exercise 16.

$$
\begin{aligned}
a_1 = 1 &< \frac{1+\sqrt{5}}{2} \\
a_2 = 2 &< \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} \\
a_{n+1} = a_n + a_{n-1} &< \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^n \left(1 + \frac{2}{1+\sqrt{5}}\right) = \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^n \left(\frac{2(1-\sqrt{5})}{1-5} + \frac{4}{4}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}
\end{aligned}
$$

Exercise 17. Use Cauchy-Schwarz, which says

$$\left(\sum a_k b_k\right)^2 \leq \left(\sum a_k^2\right)\left(\sum b_k^2\right)$$

Let $a_k = x_k^p$ and $b_k = 1$. Then Cauchy-Schwarz says

$$\left(\sum x_k^p\right)^2 \leq \left(\sum x_k^{2p}\right) n \Longrightarrow \sum (x_k^{2p}) \geq \frac{(\sum x_k^p)^2}{n}$$

We define M_p as follows:

$$M_p = \left(\frac{\sum_{k=1}^n x_k^p}{n}\right)^{1/p}$$

So then

$$nM_p^p = \sum_{k=1}^n x_k^p$$

$$
\begin{aligned}
M_{2p} &= \left(\frac{\sum_{k=1}^n x_k^{2p}}{n}\right)^{1/2p} \\
nM_{2p}^{2p} &= \sum_{k=1}^n x_k^{2p}
\end{aligned}
\qquad \sum x_k^{2p} = nM_{2p}^{2p} \geq \frac{(nM_p^p)^2}{n} = nM_p^{2p}
$$

$$M_{2p}^{2p} \geq M_p^{2p} \Longrightarrow M_{2p} \geq M_p$$

Exercise 18.

$$\left(\frac{a^4+b^4+c^4}{3}\right)^{1/4} \geq \left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} = \frac{2^{3/2}}{3^{1/2}} \text{ since}$$

$$a^4+b^4+c^4 \geq \frac{64}{3}$$

Exercise 19. $a_k = 1$, $\sum_{k=1}^n 1 = n$

Now consider the case of when not all $a_k = 1$.

$$a_1 = 1$$

$a_1 a_2 = 1$ and suppose, without loss of generality $a_1 > 1$. Then $1 > a_2$.

$(a_1 - 1)(a_2 - 1) < 0$

$a_1 a_2 - a_1 - a_2 + 1 < 0 \Longrightarrow a_1 + a_2 > 2$

(consider $n+1$ case) If $a_1 a_2 \dots a_{n+1} = 1$, then suppose $a_1 > 1$, $a_{n+1} < 1$ without loss of generality

$b_1 = a_1 a_{n+1}$

$b_1 a_2 \dots a_n = 1 \Longrightarrow b_1 + a_2 + \dots + a_n \geq n$ (by the induction hypothesis)

$(a_1 - 1)(a_{n+1} - 1) = a_1 a_{n+1} - a_1 - a_{n+1} + 1 < 0$, $b_1 < a_1 + a_{n+1} - 1$

$\Longrightarrow a_1 + a_{n+1} - 1 + a_2 + \dots + a_n > b_1 + a_2 + \dots + a_n \geq n$

$\Longrightarrow a_1 + a_2 + \dots + a_{n+1} \geq n + 1$

1.7 Exercises - The concept of area as a set function. We will use the following axioms:

Assume a class $\mathcal{M}$ of measurable sets (i.e. sets that can be assigned an area), set function a, $a : \mathcal{M} \to \mathbb{R}$.

- **Axiom 2** (Nonnegative property)**.**
$$\forall S \in \mathcal{M}, a(S) \geq 0 \tag{1}$$
- **Axiom 3** (Additive property)**.** *If $S, T \in \mathcal{M}$, then $S \cup T, S \cap T \in \mathcal{M}$ and*
$$a(S \cup T) = a(S) + a(T) - a(S \cap T) \tag{2}$$
- **Axiom 4** (Difference property)**.** *If $S, T \in \mathcal{M}, S \subseteq T$ then $T - S \in \mathcal{M}$ and*
$$a(T - S) = a(T) - a(S) \tag{3}$$
- **Axiom 5** (Invariance under congruence)**.** *If $S \in \mathcal{M}, T = S$, then $T \in \mathcal{M}$, $a(T) = a(S)$*
- **Axiom 6** (Choice of scale)**.** *$\forall$ rectangle $R \in \mathcal{M}$, if R has edge lengths h, k then $a(R) = hk$*
- **Axiom 7** (Exhaustion property)**.** *Let Q such that*
$$S \subseteq Q \subseteq T \tag{4}$$
If $\exists$ only one c such that $a(S) \leq c \leq a(T), \forall S, T$ such that they satisfy Eqn. (**??**)
then Q measurable and $a(Q) = c$

Exercise 1.

(1) We need to say that we consider a line segment or a point to be a special case of a rectangle allowing h or k (or both) to be zero.
Let $T_l = \{$ line segment containing x_0 $\}, Q = \{x_0\}$.
For Q, only $\emptyset \subset Q$
By Axiom 3, let $T = S$.
$$\begin{gathered} a(T - S) = a(\emptyset) = a(T) - a(T) = 0 \\ \emptyset \subset Q \subseteq T_l \implies a(\emptyset) \leq a(Q) \leq a(T_l) \implies 0 \leq a(Q) \leq 0 \\ \implies a(Q) = 0 \end{gathered}$$

(2)
$$a\left(\bigcup_{j=1}^N Q_j\right) = \sum_{j=1}^N a(Q_j)$$
if Q_j's disjoint. Let $Q_j = \{x_j\}$.

Since $a(Q_j) = 0$. By previous part, $a\left(\bigcup_{j=1}^N Q_j\right) = 0$

Exercise 2. Let A, B be rectangles. By Axiom 5, A, B are measurable. By Axiom 2, $A \cap B$ measurable.
$$a(A \cap B) = \sqrt{a^2 + b^2}d + ab - (\frac{1}{2}ab + \sqrt{a^2 + b^2}d) = \frac{1}{2}ab$$

Exercise 3. Prove that every trapezoid and every parallelogram is measurable and derive the usual formulas for their areas.

A trapezoid is simply a rectangle with a right triangle attached to each end of it. $T_r = R + T_1 + T_2$. T_1, T_2 are right triangles and so by the previous problem, T_1, T_2 are measurable. Then T_r is measurable by the Additive property axiom (note that the triangles and the rectangle don't overlap).

We can compute the area of a trapezoid:

$$T_r = R + T_1 + T_2 \Longrightarrow a(T_r) = a(R) + a(T_1) + a(T_2)$$
$$a(T_r) = hb_1 + \frac{1}{2}h(b_2 - b_1)/2 + \frac{1}{2}h(b_2 - b_1)/2 = \frac{1}{2}h(b_1 + h_2)$$

$P = R$ (a parallelogram consists of a right triangle rotated by π and attached to the other side of the same right triangle; the two triangles do not overlap). Since two right triangles are measurable, the parallelogram, P is measurable.
Using the Additive Axiom, $a(P) = 2a(T) = 2\frac{1}{2}bh = bh$

Exercise 4. A point (x, y) in the plane is called a lattice point if both coordinates x and y are integers. Let P be a polygon whose vertices are lattice points. The area of P is $I + \frac{1}{2}B - 1$, where I denotes the number of lattice points inside the polygon and B denotes the number on the boundary.

(1) Consider one side of the rectangle lying on a coordinate axis with one end on the origin. If the rectangle side has length l, then $l + 1$ lattice points lie on this side (you have to count one more point at the 0 point. Then consider the same number of lattice points on the opposite side. We have $2(l + 1)$ lattice points so far, for the *boundary*.

The other pair of sides will contribute $2(h-1)$ lattice points, the -1 to avoid double counting. Thus $2(l+h) = B$.

$I = (h - 1)(l - 1)$ by simply considering multiplication of $(h - 1)$ rows and $(l - 1)$ columns of lattice points inside the rectangle.

$$I + \tfrac{1}{2}B - 1 = hl - h - l + 1 + (l + h) - 1 = hl = a(R)$$

(2)

(3)

Exercise 5. Prove that a triangle whose vertices are lattice points cannot be equilateral.

My way: I will take, for granted, that we know an equilateral triangle has angles of $\pi/3$ for all its angles.

Even if we place two of the vertices on lattice points, so that its length is $2L$, and put the midpoint and an intersecting perpendicular bisector on a coordinate axis (a picture would help), but the ratio of the perpendicular bisector to the third vertex to half the length of the triangle is $\cot \pi/3 = \frac{1}{\sqrt{3}}$. Even if we go down by an integer number L, L steps down, we go "out" to the third vertex by an irrational number $\sqrt{3}L$. Thus, the third vertex cannot lie on a lattice point.

Exercise 6. Let $A = \{1, 2, 3, 4, 5\}$ and let $\mathcal{M}$ denote the class of all subsets of A. (There are 32 altogether counting A itself and the empty set $\emptyset$). (My Note: the set of all subsets, in this case, $\mathcal{M}$, is called a *power set* and is denoted 2^A. This is because the way to get the total number of elements of this power set, $|2^A|$, or the size, think of assigning to each element a "yes," if it's in some subset, or "no", if it's not. This is a great way of accounting for all possible subsets and we correctly get all possible subsets.) For each set S in $\mathcal{M}$, let $n(S)$ denote the number of distinct elements in S. If $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$,

$$n(S \bigcup T) = 5$$
$$n(S \bigcap T) = 2$$
$$n(S - T) = n(\{1, 2\}) = 2$$
$$n(T - S) = n(\{5\}) = 1$$

n satisfies nonnegative property because by definition, there's no such thing as a negative number of elements. If S, T are subsets of A, so are $S \bigcup T$, $S \bigcap T$ since every element in $S \bigcup T$, $S \bigcap T$ is in S. Thus n could be assigned to it, so that it's *measurable*. Since n counts only distinct elements, then $n(S \bigcup T) = n(S) + a(T) - a(S \bigcap T)$, where $-a(S \bigcap T)$ ensures there is no double counting of distinct elements. Thus, the Additive Property Axiom is satisfied.

For $S \subseteq T$, then $\forall x \in T - S$, $x \in T$, $x \notin S$ Now $S \subseteq T$, so $\forall x \in S$, $x \in T$. Thus $T - S$ is complementary to S "with respect to" T. $n(S) + n(T - S) = n(T)$, since n counts up distinct elements.

1.11 Exercises - Intervals and ordinate sets, Partitions and step functions, Sum and product of step function. Exercise 4.

(1)

$$[x+n] = y \leq x+n, y \in \mathbb{Z}; y - n \leq x$$
$$[x] + n = z + n \leq x + n$$

If $y - n < z,$ then $y < z + n \leq x + n$. then y wouldn't be the greatest integer less than $x + n$

$$\implies y = z + n$$

(2)

$$= y_2 \leq x \; - [x] = -y_2 \geq -x \; - y_2 - 1 \leq x$$
$$-x \geq y_1 = [-x] = -y_2 - 1 = -[x] - 1; \text{ (and } y_1 = -y_2 - 1 \text{ since } -y_2 > -x \text{)}$$
$$\text{If } x \text{ is an integer } -[x] = [-x]$$

(3) Let $x = q_1 + r_1, y = q_2 + r_2; 0 \leq r_1, r_2 < 1$.

$$= [q_1 + q_2 + r_1 + r_2] = \begin{cases} q_1 + q_2 \\ q_1 + q_2 + 1 \text{ if } r_1 + r_2 \geq 1 \end{cases}$$
$$[x] + [y] = q_1 + q_2 \quad [x] + [y] + 1 = q_1 + q_2 + 1$$

(4)

$$\text{If } x \text{ is an integer }, [2x] = 2x = [x] + [x + \frac{1}{2}] = [x] + [x] = 2x$$
$$[x] + [x + \frac{1}{2}] = q + \begin{cases} q & \text{if } r < \frac{1}{2} \\ 2q + 1 \text{ if } r > \frac{1}{2} \end{cases}$$
$$[2x] = [2(q+r)] = [2q + 2r] = \begin{cases} 2q & \text{if } r < \frac{1}{2} \\ 2q+1 & \text{if } r > \frac{1}{2} \end{cases}$$

(5)

$$[x] + [x + \frac{1}{3}] + [x + \frac{2}{3}] = q + \begin{cases} q & \text{if } r < \frac{2}{3} \\ q + 1 \text{ if } r > \frac{2}{3} \end{cases} + \begin{cases} q & \text{if } r < \frac{1}{3} \\ q+1 & \text{if } r > \frac{1}{3} \end{cases} = \begin{cases} 3q & \text{if } r < \frac{1}{3} \\ 3q+1 & \text{if } \frac{1}{3} < r < \frac{2}{3} \\ 3q+2 & \text{if } r > \frac{2}{3} \end{cases}$$
$$[3x] = [3(q+r)] = [3q + 3r] = \begin{cases} 3q & \text{if } r < \frac{1}{3} \\ 3q+1 & \text{if } \frac{1}{3} < r < \frac{2}{3} \\ 3q+2 & \text{if } r > \frac{2}{3} \end{cases}$$

Exercise 5. Direct proof.

$$[nx] = [n(q+r)] = \begin{cases} nq & \text{if } r < \frac{1}{n} \\ nq+1 & \text{if } \frac{1}{n} < r < \frac{2}{n} \\ nq+n-1 & \text{if } r > \frac{n-1}{n} \end{cases}$$

Exercise 6.

$$a(R) = hk = I_R + \frac{1}{2} B_R - 1$$
$$\sum_{n=a}^{b} [f(n)] = [f(a)] + [f(a+1)] + \dots + [f(b)]$$

$[f(n)] = g \leq f(n), g \in \mathbb{Z}$, so that if $f(n)$ is an integer,$g = f(n)$, and if $f(n)$ is not an integer, g is the largest integer such that $g < f(n)$, so that all lattice points included and less than g are included.

Exercise 7.

(1) Consider a right triangle with lattice points as vertices. Consider $b+1$ lattice points as the base with b length.

Start from the vertex and move across the base by increments of 1.

The main insight is that the slope of the hypotenuse of the right triangle is $\frac{a}{b}$ so as we move 1 along the base, the hypotenuse (or the y-value, if you will) goes up by $\frac{a}{b}$. Now

$$\left[\frac{na}{b}\right] = \text{number of interior points at } x=n \text{ and below the hypotenuse line of the right triangle of sides } a,b, \text{ including points on the hypotenuse} \qquad (5)$$

$$\begin{gathered} \sum_{n=1}^{b-1}\left[\frac{na}{b}\right] + \frac{1}{2}((a+1)+b) - 1 = \frac{ab}{2} \\ \text{Now } \frac{(a-1)(b-1)}{2} = \frac{ab}{2} - \frac{a}{2} - \frac{b}{2} + \frac{1}{2} \\ \Longrightarrow \sum_{n=1}^{b-1}\left[\frac{na}{b}\right] = \frac{(a-1)(b-1)}{2} \end{gathered}$$

(2) $a,b\in\mathbb{Z}^+$

$$\sum_{n=1}^{b-1}\left[\frac{na}{b}\right] = \sum_{n=1}^{b-1}\left[\frac{a(b-n)}{b}\right] \quad \text{(reverses order of summation)}$$

$$\begin{aligned} \sum_{n=1}^{b-1}\left[a - \frac{an}{b}\right] &= \begin{cases} -\sum_{n=1}^{b-1}\left[\frac{an}{b} - a\right] & \text{if } \frac{an}{b} - a4 \text{ is an integer (but } a\left(\frac{n}{b}-1\right) \text{ can't be!)} \\ -\sum_{n=1}^{b-1}\left(\left[\frac{an}{b}-a\right]-1\right) & \text{otherwise} \end{cases} \\ &= -\sum_{n=1}^{b-1}\left(\left[\frac{an}{b}-a\right]-1\right) = -\sum_{n=1}^{b-1}\left(\left[\frac{an}{b}\right]-a\right) - (b-1) = \\ &= -\sum_{n=1}^{b-1}\left[\frac{an}{b}\right] + a(b-1) - (b-1) \\ \sum_{n=1}^{b-1}\left[\frac{na}{b}\right] &= \frac{(a-1)(b-1)}{2} \end{aligned}$$

Exercise 8. Recall that for the step function $f = f(x)$, there's a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that $f(x) = c_k$ if $x \in I_k$.

Given that $\chi_s(x) = \begin{cases} 1 & \forall x \in S \\ 0 & \forall x \notin S \end{cases}$.

If $x\in[a,b]$, then x must only lie in one open subinterval I_j, since real numbers obey transitivity.

$$\sum_{k=1}^n c_k \chi_{I_k}(x) = c_j \text{ for } x \in I_j \Longrightarrow \sum_{k=1}^n c_k\chi_{I_k}(x) = f(x) \forall x \in [a,b]$$

1.15 Exercises - The definition of the integral for step functions, Properties of the integral of a step function, Other notations for integrals.

Exercise 1.

(1) $\int_1^3 [x]dx = (-1) + 1 + (2) = 2$
(2) $\int_{-1}^3 [x+\frac{1}{2}]dx = \int_{-1/2}^{7/2}[x]dx = (-1)\frac{1}{2} + (1)(1) + (2)(1) + \frac{1}{2}3 = 4$
(3) $\int_{-1}^3 ([x] + [x+\frac{1}{2}])dx = 6$
(4) $\int_{-1}^3 2[x]dx = 4$
(5) $\int_{-1}^3 [2x]dx = \frac{1}{2}\int_{-2}^6 [x]dx = \frac{1}{2}((-2)1 + (-1) + (1) + 2 + 3 + 4 + 5) = 6$
(6) $\int_{-1}^3 [-x]dx = -\int_1^{-3}[x]dx = \int_{-3}^1 [x]dx = -3 + -2 + -1 = -6$

Exercise 2.

$$s = \begin{cases} 5/2 & \text{if } 0 < x < 2 \\ -1 & \text{if } 2 < x < 5 \end{cases}$$

Exercise 3. $[x] = y \leq x$ so $-y \geq -x$.

$-y-1 \leq -x$, otherwise if $-y-1 \geq -x$, $y+1 \leq x$ and so y wouldn't be the largest integer $\leq x$.

$\Longrightarrow [x] + [-x] = y - y - 1 = -1$
Or use Exercise 4(c), pp. 64.

$$\int_a^b ([x] + [-x])dx = \int_a^b [x-x]dx = \int_a^b (-1)dx = a - b$$

Exercise 4.

(1) $n \in \mathbb{Z}^+$, $\int_0^n [t]dt = \sum_{t=0}^{n-1} t = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}$
(2)

Exercise 5.

(1) $\int_0^2 [t^2]dt = \int_1^2 [t^2]dt = 1(\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5 - \sqrt{2} - \sqrt{3}$
(2) $\int_{-3}^3 [t^2]dt = \int_0^3 [t^2]dt + \int_{-3}^0 [t^2]dt = \int_0^3 [t^2]dt + -\int_3^0 [t^2]dt = 2\int_0^3 [t^2]dt$

$$\int_2^3 [t^2]dt = 4(\sqrt{5}-2) + 5(\sqrt{6}-\sqrt{5}) + 6(\sqrt{7}-\sqrt{6}) + 7(\sqrt{8}-\sqrt{7}) + 8(3-\sqrt{8})$$
$$16 - \sqrt{5} - \sqrt{6} - \sqrt{7} - \sqrt{8}$$
$$\int_0^2 [t^2]dt + \int_2^3 [t^2]dt = 21 - 3\sqrt{2} - \sqrt{3} - \sqrt{5} - \sqrt{6} - \sqrt{7} - \sqrt{8}$$
$$\Longrightarrow \int_{-3}^3 [t^2]dt = 42 - 2(3\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7})$$

Exercise 6.

(1) $\int_0^n [t]^2 dt = \int_1^n [t]^2 dt = \sum_{j=1}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6}$
(2) $\int_0^x [t]^2 dt = \sum_{j=1}^{[x-1]} j^2 + q^2 r$ where $x = q + r$, $q \in \mathbb{Z}^+, 0 \leq r < 1$.

$$\int_0^x [t]^2 dt = \frac{q(q-1)(2q-1)}{6} + q^2 r = 2(x-1) = 2(q+r-1)$$
$$\Longrightarrow q(q-1)(2q-1) + 6q^2 r = 12q + 12r - 12$$
$$\Longrightarrow x = 1, x = 5/2$$

Exercise 7.

(1)

$$\int_0^9 [\sqrt{t}]dt = \int_1^9 [\sqrt{t}]dt = 3(1) + 5(2) = 13$$
$$\int_0^1 6[\sqrt{t}]dt = 3(1) + 5(2) + 7(3) = 34 = \frac{(4)(3)(17)}{6}$$

Assume $\int_0^{n^2} [\sqrt{t}]dt = \frac{n(n-1)(4n+1)}{6}$

$$\int_0^{(n+1)^2} [\sqrt{t}]dt = \int_0^{n^2} [\sqrt{t}]dt + \int_{n^2}^{(n+1)^2} [\sqrt{t}]dt = \frac{n(n-1)(4n+1)}{6} + n((n+1)^2 - n^2) =$$
$$= \frac{(n^2-n)(4n+1) + 6n(2n+1)}{6} = \frac{4n^3 + n^2 - 4n^2 - n + 12n^2 + 6n}{6} = \frac{4n^3 + 9n^2 + 5n}{6}$$

indeed ,

$$\frac{(n+1)(n)(4(n+1)+1)}{6} = \frac{(n^2+n)(4n+5)}{6} = \frac{4n^3 + 5n^2 + 4n^2 + 5n}{6}$$

Exercise 8. $\int_{a+c}^{b+c} f(x)dx = \int_{a+c-c}^{b+c-c} f(x-(-c))dx = \int_a^b f(x+c)dx$

Exercise 9. $\int_{ka}^{kb} f(x)dx = \frac{1}{\frac{1}{k}} \int_{(ka)/k}^{(kb)/k} f\left(\frac{x}{1/k}\right) dx = k\int_a^b f(kx)dx$

Exercise 10. Given $s(x) = (-1)^n n$ if $n \leq x < n+1; n = 0, 1, 2, \dots p-1; s(p) = 0,\ p \in \mathbb{Z}^+.\ f(p) = \int_0^p s(x)dx.$

So for $f(3) = \int_0^3 s(x)dx$, we need to consider $n = 0, 1, 2$.

$$\begin{aligned} & s(0 \leq x < 1) = 0 \\ & s(1 \leq x < 2) = (-1)(1) \\ & s(2 \leq x < 3) = 2; \\ & s(3 \leq x < 4) = -3 \end{aligned}$$

So then

$$\begin{aligned} & f(3) = (-1)(1) + 2(1) = 1 \\ & f(4) = 1 + (-3)(1) = -2 \\ & f(f(3)) = f(1) = 0 \end{aligned}$$

We obtain this formula

$$f(p) = \begin{cases} \frac{p}{2}(-1)^{p+1} & p \text{ even} \\ \frac{p-1}{2}(-1)^{p+1} & p \text{ even} \end{cases} \quad \text{since}$$

$$\begin{aligned} f(p+1) = f(p) + \int_p^{p+1} s(x)dx & = \left\{ \frac{p-1}{2}(-1)^{p+1} \quad p \text{ even} \right. + (-1)^p p \\ & = \begin{cases} \frac{-p}{2} & p \text{ even} \\ \frac{p-1}{2} & p \text{ odd} \end{cases} + \begin{cases} p \\ -p \end{cases} = \begin{cases} \frac{p}{2} \\ \frac{-p-1}{2} \end{cases} = \\ & = \begin{cases} -\frac{(p+1)}{2} & \text{if } p+1 \text{ even} \\ \frac{p}{2} & \text{if } p+1 \text{ odd} \end{cases} \end{aligned}$$

Thus, $p = 14, p = 15$.

Exercise 11.

(1)
$$\int_a^b s(x)dx = \sum_{k=1}^n s_k^3(x_k - x_{k-1})$$
$$\int_a^b s + \int_b^c s = \sum_{k=1}^{n_1} s_k^2(x_k - x_{k-1}) + \sum_{k=n_1}^{n_2} s_k^3(x_k - x_{k-1}) = \sum_{k=1}^{n_2} s_k^3(x_k - x_{k-1}) = \int_a^c s(x)dx$$

(2) $\int_a^b (s+t) = \sum_{k=1}^{n_3} (s+t)_k^3 (x_k - x_{k-1}) \neq \int_a^b s + \int_a^b t$

(3) $\int_a^b cs = \sum_{k=1}^n (cs)^3 (x_k - x_{k-1}) \neq c\int_a^b s$

(4) Consider these facts that are true, that $x_{k-1} < x < x_k,\ s(x) = s_k;\ x_0 = a+c,\ x_n = b+c,$ $x_{k-1} - c < x - c < x_l - c \Longrightarrow y_{k-1} < y < y_k$ so then $s(y+c) = s_k$.

$$\begin{aligned} \sum_{k=1}^n s_k^3(x_k - x_{k-1}) & = \sum_{k=1}^k s_k^3(x_k - c - (x_{k-1} - c)) = \\ & = \sum_{k=1}^n s_k^3(y_k - y_{k-1}) = \int_a^b s(y+c)dy \end{aligned}$$

(5) $s < t, \int_a^b s = \sum_{k=1}^n s_k^3(x_k - x_{k-1})$.

$$\begin{aligned} & \text{if } 0 < s, s^3 < s^2 t < st^2 < t^3 \\ & \text{if } s < 0t, s^3 < \text{ and } t^3 > 0 \\ & \text{if } s < t < 0,\ s^3 < s^2 t,\ s(st) < t(ts) = t^2 s \ \begin{matrix} ts > t^2 \\ t^2 s < t^3 \end{matrix} \end{aligned}$$

$$s^3 < s^2 t < t^2 s < t^3$$

Then $\int_a^b s < \int_a^b t$.

Exercise 12.

(1) $\int_a^b s + \int_b^c s = \sum_{k=1}^{n_1} s_k(x_k^2 - x_{k-1}^2) + \sum_{k=n_1}^{n_2} s_k(x_k^2 - x_{k-1}^2) = \sum_{k=1}^{n_3} s_k(x_k^2 - x_{k-1}^2) = \int_a^c s$

(2) $\int_a^b (s+t) = \sum_{k=1}^{n_3}(s+t)_k(x_k^2 - x_{k-1}^2) = \sum_{k=1}^{n_3}(s_k+t_k)(x_k^2-x_{k-1}^2) = \sum_{k=1}^{n_3} s_k(x_k^2-x_{k-1}^2)+\sum_{k=1}^{n_3} t_k(x_k^2-x_{k-1}^2)$

since $P_3 = \{x_k\}$ is a finer partition than the partition for s, P_1, t, P_2, then consider
$s_k(y_j^2 - y_{j-1}^2) = s_k((x_{k+1}^2 - x_k^2) + (x_k^2 - x_{k-1}^2))$, so

$$\sum_{k=1}^{n_3} s_k(x_k^2 - x - k - 1^2) + \sum_{k=1}^{n_3} t_k(x_k^2 - x - k - 1^2) = \sum_{j=1}^{n_1} s_j(x_j^2 - x - j - 1^2) + \sum_{j=1}^{n_2} t_j(x_j^2 - x - j - 1^2) =$$
$$= \int_a^b s + \int_a^b t$$

(3) $\int_a^b cs = \sum_{k=1}^n cs_k(x_k^2 - x_{k-1}^2) = c\sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2) = c\int_a^b s$

(4) $\int_{a+c}^{b+c} s(x)dx = \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2)$ where

$$s(x) = s_k \text{ if } x_{k-1} < x < x_k$$
$$x(y+c) = s_k \text{ if } x_{k-1} < y + c < x_k \Longrightarrow x_{k-1} - c < y < x_k - c \Longrightarrow y_{k-1} < y < y_k$$

where $P' = \{y_k\}$ is a partition on $[a, b]$

$$\int_a^b s(y+c)dy = \sum_{k=1}^n s_k(y_k^2 - y_{k-1}^2) =$$
$$= \sum_{k=1}^n s_k((x_k - c)^2 - (x_{k-1} - c)^2) = \sum_{k=1}^n s_k(x_k^2 - 2x_kc + c^2 - (x_{k-1}^2 - 2x_{k-1}c + c^2)) =$$
$$= \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2 - 2c(x_k - x_{k-1})) \neq \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2)$$

(5) Since $x_k^2 - x_{k-1}^2 > 0$, $\int_a^b sdx = \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2) < \sum_{k=1}^n t_k(x_k^2 - x_{k-1}^2) = \int_a^b tdx$
Note that we had shown previously that the integral doesn't change under finer partition.

Exercise 13.

$$\int_a^b s(x)dx \sum_{k=1}^n s_k(x_k - x_{k-1}); \int_a^b t(x)dx = \sum_{k=1}^{n_2} t_k(y_k - y_{k-1})$$
$$P = \{x_0, x_1, \dots, x_n\}, Q = \{y_0, y_1, \dots, y_n\}$$

Note that $x_0 = y_0 = a; x_n = y_{n_2} = b$.

Consider $P \bigcup Q = R$. R consists of n_3 elements, (since $n_3 \leq n + n_2$ some elements of P and Q may be the same. R is another partition on $[a, b]$ (by partition definition) since $x_k, y_k \in \mathbb{R}$ and since real numbers obey transitivity, $\{x_k, y_k\}$ can be arranged such that $a < z_1 < z_2 < \dots < z_{n_3-2} < b$ where $z_k = x_k$ or y_k.

$$(s+t)(x) = s(x) + t(x) = s_j + t_k \text{ if } x_{j-1} < x < x_j; y_{j-1} < x < y_j$$
$$\text{If } x_{j-1} \lesseqgtr y_{j-1}, \text{ let } z_{l-1} = y_{j-1}, x_{j-1} \text{ and}$$
$$\text{If } x_j \lesseqgtr y_j, \text{ let } z_l = x_j, y_j$$
$$\text{Let } s_j = s_l; t_k = t_l$$
$$(s+t)(x) = s(x) + t(x) = s_l + t_l, \text{ if } z_{l-1} < x < z_l$$

$$\int_a^b (s(x) + t(x))dx = \int_a^b ((s+t)(x))dx = \sum_{l=1}^{n_3}(s_l + t)l)(z_l - z_{l-1}) = \sum_{l=1}^{n_3} s_l(z_l - z_{l-1}) + \sum_{l=1}^{n_3} t_l(z_l - z_{l-1})$$

In general, it was shown (Apostol I, pp. 66) that any finer partition doesn't change the integral R is a finer partition. So

$$\sum_{l=1}^{n_3} s_l(z_l - z_{l-1}) + \sum_{l=1}^{n} t_l(z_l - z_{l-1}) = \sum_{k=1}^{n} s_k(x_k - x_{k-1}) + \sum_{k=1}^{n_2} t_k(y_k - y_{k-1}) = \int_a^b s(x)dx + \int_a^b t(x)dx$$

Exercise 14. Prove Theorem 1.4 (the linearity property).

$$\begin{aligned} c_1 \int_a^b s(x)dx + c_2 \int_a^b t(x)dx &= c_1 \sum_{k=1}^{n} s_k(x_k - x_{k-1}) + c_2 \sum_{k=1}^{n_2} t_k(x_k - x_{k-1}) = \\ &= \sum_{l=1}^{n_3} c_1 s_l(z_l - z_{l-1}) + \sum_{l=1}^{n_3} c_2 t_l(z_l - z_{l-1}) = \sum_{l=1}^{n_3} (c_1 s_l + c_2 t_l)(z_l - z_{l-1}) = \\ &= \int_a^b (c_1 s + c_2 t)(x)dx \end{aligned}$$

We relied on the fact that we could define a finer partition from two partitions of the same interval.

Exercise 15. Prove Theorem 1.5 (the comparison theorem).

$$s(x) < t(x) \, \forall x \in [a,b]; \; s(x)(z_l - z_{l-1}) < t(x)(z_l - z_{l-1}) \, (z_l - z_{l-1} > 0)$$

$$\begin{aligned} \int_a^b s(x)dx = \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = \sum_{l=1}^{n_3} s_l(z_l - z_{l-1}) < \sum_{l=1}^{n_3} t_l(z_l - z_{l-1}) = \sum_{k=1}^{n2} t_k(y_l - y_{k-1}) = \\ = \int_a^b t(x)dx \end{aligned}$$

$$\Longrightarrow \int_a^b s(x)dx < \int_a^b t(x)dx$$

Exercise 16. Prove Theorem 1.6 (additivity with respect to the interval).

Use the hint: P_1 is a partition of $[a,c]$, P_2 is a partition of $[c,b]$, then the points of P_1 along with those of P_2 form a partition of $[a,b]$.

$$\int_a^c s(x)dx + \int_a^b s(x)dx = \sum_{k=1}^{n_1} s_l(x_k - x_{k-1}) + \sum_{k=1}^{n_2} s_k(x_k - x_{k-1}) = \sum_{k=1}^{n_3} s_k(x_k - x_{k-1}) = \int_a^b s(x)dx$$

Exercise 17. Prove Theorem 1.7 (invariance under translation).

$$\begin{gathered} P' = \{y_0, y_1, \dots, y_n\}; \; y_k = x_k + c; \\ \Longrightarrow x_{k-1} + c < y < x_k + c \\ x_{k-1} < y - c < x_k \\ y_k - y_{k-1} = x_k + c - (x_{k-1} + c) = x_k - x_{k-1} \end{gathered}$$

$$s(y - c) = s_k \text{ if } x_{k-1} < y - c < x_k \; , \; k = 1, 2, \dots n$$

$$\int_a^b s(x)dx = \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = \sum_{k=1}^{n} s_k(y_k = y_{k-1}) = \int_{y_0}^{y_n} s(y-c)dy = \int_{a+c}^{b+c} s(x-c)dx$$

1.26 Exercises - The integral of more general functions, Upper and lower integrals, The area of an ordinate set expressed as an integral, Informal remarks on the theory and technique of integration, Monotonic and piecewise monotonic functions. Definitions and examples, Integrability of bounded monotonic functions, Calculation of the integral of a bounded monotonic function, Calculation of the integral $\int_0^b x^p dx$ when p is a positive integer, The basic properties of the integral, Integration of polynomials.

Exercise 16. $\int_0^2 |(x-1)(3x-1)|dx =$

$$\int_1^2 (x-1)(3x-1)dx = \int_1^2 (3x^2-4x+1)dx = \left.(x^3-2x^2+x)\right|_1^2 = 2$$
$$\int_{1/3}^1 (1-x)(3x-1)dx = -\left.(x^3-2x^2+x)\right|_{1/3}^1 = \frac{4}{27}$$
$$\int_0^{1/3} (x-1)(3x-1)dx = \frac{4}{27}$$
So the final answer for the integral is $62/27$.

Exercise 17. $\int_0^3 (2x-5)^3 dx = 8\int_0^3 (x-\frac{5}{2})^3 dx = 8\int_{-5/2}^{3-5/2} x^3 dx = 8\ \frac{1}{4}x^4\Big|_{-5/2}^{1/2} = \frac{39}{2}$

Exercise 18. $\int_{-3}^3 (x^2-3)^3 dx = \int_0^3 (x^2-3)^3 + \int_{-3}^x (x^2-3)^3 = \int_0^3 (x^2-3)^2 + - \int_0^3 (x^2-3)^3 = 0$

2.4 Exercises - Introduction, The area of a region between two graphs expressed as an integral, Worked examples.

Exercise 15. $f = x^2, g = cx^3, c > 0$

For $0 < x < \frac{1}{c}$, $cx < 1$ (since $c > 0$). So $cx^3 < x^2$ (since $x^2 > 0$).

$$\int f - g = \int x^2 - cx^3 = \left.\left(\frac{1}{3}x^3 - \frac{c}{4}x^4\right)\right|_0^{1/c} = \frac{1}{12c^3}$$
$$\int f - g = \frac{2}{3} = \frac{1}{12c^3};\ \boxed{c = \frac{1}{2\sqrt{2}}}$$

Exercise 16. $f = x(1-x), g = ax.$

$$\int f - g = \int_0^{1-a} x - x^2 - ax = \left.\left((1-a)\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\right|_0^{1-a} = (1-a)^3\frac{1}{6} = 9/2 \Longrightarrow \boxed{a = -2}$$

Exercise 17. $\pi = 2\int_{-1}^1 \sqrt{1-x^2}dx$

(1)
$$\int_{-3}^3 \sqrt{9-x^2}dx = 3\int_{-3}^3 \sqrt{1-\left(\frac{x}{3}\right)^2} = 3(3)\int_{-1}^1 \sqrt{1-x^2} = \boxed{\frac{9\pi}{2}}$$

Now
$$\int_{ka}^{kb} f\left(\frac{x}{k}\right)dx = k\int_a^b f dx$$

(2)
$$\int_0^2 \sqrt{1-\frac{1}{4}x^2}dx = 2\int_0^1 \sqrt{1-x^2}dx = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

(3) $\int_{-2}^2 (x-3)\sqrt{4-x^2}dx$

$$\int_{-2}^2 x\sqrt{4-x^2}dx = (-1)\int_2^{-2} -x\sqrt{4-x^2} \Longrightarrow 2\int_{-2}^2 x\sqrt{4-x^2} = 0$$
$$-3\int_{-2}^2 2\sqrt{1-\left(\frac{x}{2}\right)^2}dx = (-6)(2)\int_{-1}^1 \sqrt{1-x^2} = \boxed{-6\pi}$$

Exercise 18. Consider a circle of radius 1 and a twelve-sided dodecagon inscribed in it. Divide the dodecagon by isosceles triangle pie slices. The interior angle that is the vertex angle of these triangles is $360/12 = 30$ degrees.

Then the length of the bottom side of each triangle is given by the law of cosines:

$$c^2 = 1 + 1 - 2(1)(1)\cos 30^\circ = 2\left(1 - \frac{\sqrt{3}}{2}\right) \Longrightarrow c = \sqrt{2}\sqrt{1-\frac{\sqrt{3}}{2}}$$

The height is given also by the law of cosines

$$h = 1\cos 15^\circ = \sqrt{\frac{1+\cos 30^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

The area of the dodecagon is given by adding up twelve of those isosceles triangles

$$(12)\frac{1}{2}\left(\sqrt{1+\frac{\sqrt{3}}{2}}\left(\frac{1}{\sqrt{2}}\right)\right)\left(\sqrt{2}\sqrt{1-\frac{\sqrt{3}}{2}}\right) = 3$$

So $3 < \pi$.

Now consider a dodecagon that's circumscribing the circle of radius 1.

$$(12)\frac{1}{2}\left(2\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}}\right)(1) = \boxed{12\left(2-\frac{\sqrt{3}}{2}\right)} > \pi$$

Exercise 19.

(1) $(x, y) \in E$ if $x = ax_1, y = by_1$ such that $x_1^2 + y_1^2 \leq 1$
$\implies \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

(2)

$$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$2\int_{-a}^{a} b\sqrt{1-\left(\frac{x}{a}\right)^2} = 2ba\int_{-1}^{1}\sqrt{1-x^2} = ba\frac{\pi}{2}(2) = \pi ba$$

Exercise 20. Let f be nonnegative and integrable on $[a, b]$ and let S be its ordinate set.

Suppose x and y coordinates of S were expanded in different ways $x = k_1x_1, y = k_2y_1$.

If $f(x_1) = y_1, g(x) = k_2 f\left(\frac{x}{k_1}\right) = k_2y_1 = y$.

integrating g on $[k_1a, k_1b]$,

$$\int_{k_1a}^{k_1b} g(x)dx = \int_{k_1a}^{k_1b} k_2 f\left(\frac{x}{k_1}\right)dx = k_2k_1\int_a^b f(x)dx = k_2k_1A$$

2.8 Exercises - The trigonometric functions, Integration formulas for the sine and cosine, A geometric description of the sine and cosine functions. **Exercise 1.**

(1) $\sin\pi = \sin 0 = 0$. sine is periodic by 2π, so by induction, $\sin n\pi = 0$.

$$\sin 2(n+1)\pi = \sin 2\pi n + 2\pi = \sin 2\pi n = 0$$
$$\sin(2(n+1)+1)\pi = \sin(2n+3)\pi = \sin((2n+1)\pi + 2\pi) = \sin(2n+1)\pi = 0$$

(2) $\cos\pi/2 = \cos -\pi/2 = 0$

$$\text{by induction, } \cos\pi/2 + 2\pi j = \cos\pi/2(1+4j)$$
$$\cos -\pi/2 + 2\pi j = \cos(4j-1)\pi/2, j \in \mathbb{Z}^+$$

Exercise 2.

(1) $\sin\pi/2 = 1, \sin\pi/2(1+4j) = 1, j \in \mathbb{Z}^+$.
(2) $\cos x = 1, \cos 0 = 1, \cos 2\pi j = 1$

Exercise 3.

$$\sin x + \pi = -\sin x + \pi/2 + \pi/2 = \cos x + \pi/2 = -\sin x$$
$$\cos x + \pi = \cos x + \pi/2 + \pi/2 = -\sin x + \pi/2 = -\cos x$$

Exercise 4.

$$\sin 3x = \sin 2x \cos x + \sin x \cos 2x = 2 \sin x \cos^2 x + \sin x(\cos^2 x - \sin^2 x) = 3\cos^2 x \sin x - \sin^3 x =$$
$$= 3(1 - \sin^2 x)\sin x - \sin^3 x = 3 \sin x - 4 \sin^3 x$$

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x = (\cos^2 x - \sin^2 x)\cos x - (2 \sin x \cos x) \sin x = \cos x - 4\sin^2 x \cos x$$
$$\cos 3x = -3\cos x + 4\cos^3 x$$

Exercise 5.

(1) This is the most direct solution. Using results from Exercise 4 (and it really helps to choose the cosine relationship, not the sine relationship),

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$x = \pi/6$$
$$\cos 3\pi/6 = 0 = 4\cos^3 \pi/6 - 3\cos \pi/6 = \cos \pi/6(4\cos^2 \pi/6 - 3) = 0$$
$$\implies \cos \pi/6 = \sqrt{3}/2, \sin \pi/6 = 1/2(\text{ by Pythagorean theorem })$$

(2) $\sin 2\pi/6 = 2\cos \pi/6 \sin \pi/6 = \sqrt{3}2, \cos \pi/3 = 1/2$ (by Pythagorean theorem)
(3) $cos 2\pi/4 = 0 = 2\cos \pi/4 - 1, \cos \pi/4 = 1/\sqrt{2} = \sin \pi/4$

Note that the most general way to solve a cubic is to use this formula. For $x^3 + bx^2 + cx + d = 0$,

$$R = \frac{9bc - 27d - 2b^3}{54} \qquad S = (R + \sqrt{Q^3 + R^2})^{1/3}$$
$$Q = \frac{3c - b^2}{9} \qquad T = (R - \sqrt{Q^3 + R^2})^{1/3}$$
$$x_1 = S + T - b/3$$
$$x_2 = -1/2(S + T) - b/3 + 1/2\sqrt{-3}(S - T)$$
$$x_3 = -1/2(S + T) - b/3 - 1/2\sqrt{-3}(S - T)$$

Exercise 6.

$$\tan x - y = \frac{\sin x - y}{\cos x - y} = \frac{\sin x \cos y - \sin y \cos x}{\cos x \cos y + \sin x \sin y}\left(\frac{\frac{1}{\cos x \cos y}}{\frac{1}{\cos x \cos y}}\right) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

if $\tan x \tan y \neq -1$

Similarly,

$$\tan x + y = \frac{\sin x + y}{\cos x + y} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y} = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \tan x \tan y \neq 1$$
$$\cot x + y = \frac{\cos x + y}{\sin x + y} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \sin y \cos x} = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

Exercise 7. $3\sin x + \pi/3 = A \sin x + B \cos x = 3(\sin x\frac{1}{2} + \frac{\sqrt{3}}{2}\cos x) = \frac{3}{2}\sin x + \frac{3\sqrt{3}}{2}\cos x$

Exercise 8.

$$C \sin x + \alpha = C(\sin x \cos \alpha + \cos x \sin \alpha) = C \cos \alpha \sin x + C \sin \alpha \cos x$$
$$A = C\cos \alpha, B = C \sin \alpha$$

Exercise 9. If $A = 0$, $B \cos x = B \sin \pi/2 + x = C \sin x + \alpha$ so $C = B, \alpha = \pi/2$ if $A = 0$.

If $A \neq 0$,

$$A \sin x + B \cos x = A(\sin x + \frac{B}{A}\cos x) == A(\sin x + \tan \alpha \cos x)$$
$$= \frac{A}{\cos \alpha}(\cos \alpha \sin x + \sin \alpha \cos x) = \frac{A}{\cos \alpha}(\sin x + \alpha)$$

where $-\pi/2 < \alpha < \pi/4$, $B/A = \tan \alpha$, $C = \frac{A}{\cos \alpha}$

Exercise 10. $C \sin x + \alpha = C \sin x \cos \alpha + C \cos x \sin \alpha$.

$$C \cos \alpha = -2, C \sin \alpha = -2, C = -2\sqrt{2}, \alpha = \pi/4$$

Exercise 11. If $A = 0, C = B, \alpha = 0$. If $B = 0, A = -C, \alpha = \pi/2$. Otherwise,

$$A\sin x + B\cos x = B(\cos x + \frac{A}{B}\sin x) = \frac{B}{\cos\beta}(\cos x\cos\beta + \sin\beta\sin x) = C\cos x + \alpha$$

where $\frac{A}{B} = \tan\beta$, $\alpha = -\beta$, $C = \frac{B}{\cos\beta}$.

Exercise 12.

$$\sin x = \cos x = \sqrt{1-\cos^2 x} \Longrightarrow \cos x = 1/\sqrt{2} \Longrightarrow x = \frac{\pi}{4}$$

Try $5\pi/4$. $\sin 5\pi/4 = \cos 3\pi/4 = -\sin\pi/4 = -1/\sqrt{2}$.
$\cos 5\pi/4 = -\sin 3\pi/4 = -\cos\pi/4 = -1/\sqrt{2}$. So $\sin 5\pi/4 = \cos 5\pi/4$. $x = 5\pi/4$ must be the other root.

So $\theta = \pi/4 + \pi n$ (by periodicity of sine and cosine).

Exercise 13.

$$\sin x - \cos x = 1 = \sqrt{1-\cos^2 x} = 1 + \cos x$$
$$\Longrightarrow 1 - \cos^2 x = 1 + 2\cos x + \cos^2 x \Longrightarrow 0 = 2\cos x(1+\cos x)$$
$$\cos x = -1, x = \pi/2 + 2\pi n$$

Exercise 14.

$$\cos x - y + \cos x + y = \cos x\cos y + \sin x\sin y + \cos x\cos y - \sin x\sin y = 2\cos x\cos y$$
$$\cos x - y - \cos x + y = \sin x\cos y - \sin y\cos x + \sin x\cos y + \sin y\cos x = 2\sin x\cos y$$
$$\sin x - y + \sin x + y = \sin x\cos y - \sin y\cos x + \sin x\cos y + \sin y\cos x = 2\sin x\cos y$$

Exercise 15.

$$\frac{\sin x + h - \sin x}{h} = \frac{\sin(x+h/2)\cos h/2 + \cos(x+h/2)\sin h/2 - \sin(x+h)\cos h/2 - \cos x + h/2\sin h/2}{h}$$
$$= \frac{\sin h/2}{h/2}\cos(x+h/2)$$

$$\frac{\cos x + h - \cos x}{h} = \frac{\cos(x+h/2)\cos h/2 - \sin(x+h/2)\sin h/2 - (\cos(x+h/2)\cos h/2 + \sin(x+h/2)\sin h/2)}{h}$$
$$= -\frac{\sin h/2}{h/2}\sin(x+h/2)$$

Exercise 16.

(1)

$$\sin 2x = 2\sin x\cos x$$

$$\text{if } \sin 2x = 2\sin x \text{ and } x \neq 0, x \neq \pi n, \cos x = 1 \text{ but } x \neq \pi n \Longrightarrow \boxed{x = 2\pi n}$$

(2) $\cos x + y = \cos x\cos y - \sin x\sin y = \cos x + \cos y$.

$$\cos x\cos y - \cos x - \cos y = \sin y\sqrt{1-\cos^2 x}$$
$$\text{Letting } A = \cos x, B = \cos y,$$
$$A^2B^2 + A^2 + B^2 - 2A^2B - 2AB^2 + 2AB = 1 - A^2 - B^2 + A^2B^2$$
$$A^2 + B^2 - A^2B - AB^2 + AB = 1/2$$
$$B^2(1-A) + B(A - A^2) + A^2 - 1/2 = 0$$

$$B = \frac{A(1-A) \pm \sqrt{A^2(1-A)^2 - 4(1-A)(A^2-1/2)}}{1-A} = A \pm \frac{1}{\sqrt{1-A}}(A^2(1-A) - 4(A^2-1/2))^{1/2} =$$
$$= A \pm \frac{1}{\sqrt{1-A}}(-3A^2 - A^3 + 2)^{1/2}$$

Note that $-1 \leq B \leq 1$, but for $|A| \leq 1$.

Solve for the roots of $-3A^2 - A^3 + 2$, $A_0 = -1, -1+\sqrt{3}, -1-\sqrt{3}$. So suppose $\cos x = 9/10$. Then there is no real number for y such that $\cos y$ would be real and satisfy the above equation.

(3) $\sin x + y = \sin x\cos y + \sin y\cos x = \sin x + \sin y$

$$\Longrightarrow \sin y(1-\cos x) + \sin y + -\cos x\sin y = 0, \Longrightarrow y = 2\pi n$$

Checking our result, we find that $\sin(2\pi n + y) = \sin 2\pi n + \sin y(1)$

(4)

$$\int_0^y \sin x dx = -\cos x|_0^y = -(\cos y - 1) = 1 - \cos y = \sin y$$
$$\implies 1 - \cos y = \sqrt{1-\cos^2 y}$$
$$1 - 2\cos y + \cos^2 y = 1 - \cos^2 y \implies \cos y(\cos y - 1) = 0; y = \frac{2(j+1)\pi}{2}, 2\pi n$$

Exercise 17. $\int_a^b \sin x dx = -\cos x|_a^b = -\cos b + \cos a$

(1) $-\frac{\sqrt{3}}{2} + 1$
(2) $-\frac{\sqrt{2}}{2} + 1$
(3) $\frac{1}{2}$
(4) 1
(5) 2
(6) 0 We were integrating over one period, over one positive semicircle and over one negative semicircle.
(7) 0 We had integrated over two equal parts, though it only shaded in up to $x = 1$.
(8) $-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}$

Exercise 18. $\int_0^\pi (x + \sin x)dx = \left(\frac{1}{2}x^2 - \cos x\right)\Big|_0^\pi = \frac{\pi^2}{2} - (-1-1) = \frac{\pi^2}{2} + 2$

Exercise 19. $\int_0^{\pi/2}(x^2 + \cos x)dx = \left(\frac{1}{3}x^3 + \sin x\right)\Big|_0^{\pi/2} = \frac{1}{3}(\pi/2)^3 + 1$

Exercise 20. $\int_0^{\pi/2}(\sin x - \cos x)dx = (-\cos x - \sin x)|_0^{\pi/2} = -1 - (-1) = 0$

Exercise 21. $\int_0^{\pi/2} |\sin x - \cos x| dx =$ (by symmetry)$2\int_0^{\pi/4}(\cos x - \sin x)dx = 2(\sin x + \cos x)|_0^{\pi/4} = 2(\sqrt{2} - 1)$

Exercise 22. $\int_0^\pi (\frac{1}{2} + \cos t)dt = \left(\frac{1}{2}t + \sin t\right)\Big|_0^\pi = \frac{\pi}{2}$

Exercise 23.

$$\int_0^{2\pi/3} (\frac{1}{2} + \cos t)dt + \int_{2\pi/3}^\pi -(\frac{1}{2} + \cos t)dt = \left(\frac{t}{2} + \sin t\right)\Big|_0^{2\pi/3} + \left(\frac{t}{2} + \sin t\right)\Big|_\pi^{2\pi/3}$$
$$= 2(\frac{\pi}{3} + \frac{\sqrt{3}}{2}) - \frac{\pi}{2} = \frac{\pi}{6} + \sqrt{3}$$

Exercise 24. If $-\pi < x \leq -\frac{2\pi}{3}$,

$$\int_{-\pi}^x -(\frac{1}{2} + \cos t)dt = \int_x^{-\pi} (\frac{1}{2} + \cos t)dt = \left(\frac{t}{2} + \sin t\right)\Big|_x^{-\pi} = -\frac{\pi}{2} - \frac{x}{2} - \sin x$$

If $-2\pi/3 \leq x \leq 2\pi/3$,

$$\int_{-\pi}^{-2\pi/3} -(\frac{1}{2} + \cos t)dt + \int_{-2\pi/3}^x (\frac{1}{2} + \cos t)dt = \frac{-\pi/6}{+}\sqrt{3}/2 + (t/2 + \sin t)|_{-2\pi/3}^x$$
$$= x/2 + \sin x - \pi/3 - \sqrt{3}/2 + \sqrt{3}/2 - \pi/6 = \frac{x}{2} + \sin x - \pi/3$$

If $2\pi/3 \leq x \leq \pi$,

$$\sqrt{3}/2 + \int_{2\pi/3}^x -(1/2 + \cos t)dt = \sqrt{3}/2 + (t/2 + \sin t)|_x^{2\pi/3} = \pi/3 + \sqrt{3} - x/2 - \sin x$$

Exercise 25. $\int_x^{x^2} (t^2 + \sin t)dt = (\frac{1}{3}t^3 + -\cos t) = \frac{x^6 - x^3}{3} + \cos x - \cos x^2$

Exercise 26. $\int_0^{\pi/2} \sin 2x dx = \left(\frac{-\cos(2x)}{2}\right)\Big|_0^{\pi/2} = (-1/2)(-1-1) = 1$

Exercise 27. $\int_0^{\pi/3} \cos x/2 dx = 2\sin x/2|_0^{\pi/3} = 2\frac{1}{2} = 1$

Exercise 28.

$$\int_0^x \cos{(a+bt)}dt = \int_0^x (\cos{a}\cos{bt} - \sin{a}\sin{bt})dt = \left(\frac{\cos{a}}{b}\sin{bt} - \sin{a}(-\cos{bt}/b) \right)\Big|_0^x =$$
$$= \frac{\cos{a}}{b}\sin{bx} + \frac{\sin{a}}{b}(\cos{bx} - 1) = \frac{1}{b}\sin{a+bx} - \sin{a}/b$$
$$\int_0^x \sin{(a+bt)}dt = \int_0^x (\sin{a}\cos{bt} + \sin{bt}\cos{a})dt = \left(\frac{\sin{a}}{b}\sin{bt} - \frac{\cos{a}}{b}\cos{bt} \right)\Big|_0^x =$$
$$= \frac{1}{b}(\cos{bx+a} + \cos{a})$$

Exercise 29.

(1)
$$\int_0^x \sin^3{t}dt = \int_0^x \frac{3\sin{t} - \sin{3t}}{4}dt = \left(-\frac{3}{4}\cos{t} + \cos{3t}/12 \right)\Big|_0^x = -3/4(\cos{x} - 1) + \frac{\cos{3x} - 1}{12} =$$
$$= \frac{1}{3} - \frac{3}{4}\cos{x} + \frac{1}{12}(\cos{2x}\cos{x} - \sin{2x}\sin{x}) = 2/3 - 1/3\cos{x}(2 + \sin^2{x})$$

(2)
$$\int_0^x \cos^3{t}dt = \int_0^x \frac{1}{4}(\cos{3t} + 3\cos{t})dt = \left(\frac{1}{4}\frac{\sin{3t}}{3} + \frac{3}{4}\sin{t} \right)\Big|_0^x =$$
$$= \frac{1}{12}(\sin{2x}\cos{x} + \sin{x}\cos{2x}) + \frac{3}{4}\sin{x} = \frac{1}{12}(2\sin{x}\cos{x} + \sin{x}(2\cos^2{x} - 1)) =$$
$$= \frac{\sin{x}\cos^2{x} + 2\sin{x}}{3}$$

Exercise 30. Now using the definition of a periodic function,

$$f(x) = f(x+p); f(x + (n+1)p) = f(x + np + p) = f(x + np) = f(x)$$

and knowing that we could write any real number in the following form,

$$a = np + r; 0 \leq < p, r \in \mathbb{R}; n \in \mathbb{Z}$$

then
$$\int_a^{a+p} f(x)dx = \int_r^{r+p} f(x+np)dx = \int_r^{r+p} f(x)dx = \int_r^p f + \int_p^{r+p} f(x)dx =$$
$$= \int_r^p f + \int_0^r f(x-p)dx = \int_r^p f + \int_0^r f = \int_0^p f$$

Exercise 31.

(1)
$$\int_0^{2\pi} \sin{nx}dx = \frac{1}{n}\int_0^{2\pi n} \sin{x}dx = \frac{1}{n}(-\cos{x})\Big|_0^{2\pi n} = -\frac{1}{n}(1-1) = 0$$
$$\int_0^{2\pi} \cos{nx}dx = \frac{1}{n}\int_0^{2\pi n} \cos{x}dx = \frac{1}{n}\sin{x}\Big|_0^{2\pi n} = 0$$

(2)
$$\int_0^{2\pi} \sin{nx}\cos{mx}dx = \int_0^{2\pi} \frac{1}{2}(\sin{(n+m)x} + \sin{(n-m)x})dx = 0 + 0 = 0$$
$$\int_0^{2\pi} \sin{nx}\sin{mx}dx = \int_0^{2\pi} \frac{1}{2}(\cos{(n-m)x} + \cos{(n+m)x})dx = 0 + 0 = 0$$
$$\int_0^{2\pi} \cos{nx}\cos{mx}dx = \int_0^{2\pi} \frac{1}{2}(\cos{(n-m)x} + \cos{(n+m)x})dx = 0 + 0 = 0$$

While
$$\int_0^{2\pi} \sin^2{nx}dx = \int_0^{2\pi} \frac{1 - \cos{2nx}}{2}dx = \pi$$
$$\int_0^{2\pi} \cos^2{nx}dx = \int_0^{2\pi} \frac{1 + \cos{2nx}}{2}dx = \pi$$

Exercise 32. Given that $x \neq 2\pi n; \sin x/2 \neq 0$,

$$\begin{aligned}\sum_{k=1}^n 2\sin x/2 \cos kx = 2\sin x/2 \sum_{k=1}^n \cos kx = \sum_{k=1}^n \sin{(2k+1)}\frac{x}{2} - \sin{(2k-1)}\frac{x}{2} = \sin{(2n+1)}\frac{x}{2} - \sin x/2 \\ = \sin nx \cos x/2 + \sin x/2 \cos nx - \sin x/2 = \\ = 2\sin nx/2 \cos nx/2 \cos x/2 + \sin x/2 (1 - 2\sin^2 nx/2) - \sin x/2 = \\ = 2(\sin nx/2)(\cos{(n+1)}x/2)\end{aligned}$$

Exercise 33. Recall that

$$\begin{aligned}\cos{(2k+1)}x/2 - \cos{(2k-1)}x/2 &= \cos kx + x/2 - \cos kx - x/2 = \\ &= \cos kx \cos x/2 - \sin kx \sin x/2 - (\cos kx \cos x/2 + \sin kx \sin x/2) = \\ &= -2\sin kx \sin x/2\end{aligned}$$

$$\begin{aligned}-2\sin x/2 \sum_{k=1}^n \sin kx = \sum_{k=1}^n (\cos{(2k+1)}x/2 - \cos{(2k-1)}x/2) = \cos{(2n+1)}x/2 - \cos x/2 = \\ = \cos nx + x/2 - \cos x/2\end{aligned}$$

Now

$$\begin{aligned}\sin nx/2 \sin nx/2 + x/2 &= \sin nx/2(\sin nx/2 \cos x/2 + \sin x/2 \cos nx/2) = \\ &= \sin^2 nx/2 \cos x/2 + \sin x/2 \cos nx/2 \sin nx/2 = \\ &= \left(\frac{1-\cos nx}{2} \right) \cos x/2 + \frac{\sin nx}{2} \sin x/2 = \\ &= \frac{1}{2} (\cos x/2 - \cos x/2 \cos nx + \sin nx \sin x/2) = \frac{1}{2}(\cos x/2 - \cos{(nx_x/2)}\end{aligned}$$

Then

$$\begin{aligned}-2\sin x/2 \sum_{k=1}^n \sin kx = -2\sin nx/2 \sin \frac{1}{2}(n+1)x \\ \sum_{k=1}^n \sin kx = \frac{\sin nx/2 \sin \frac{1}{2}(n+1)x}{\sin x/2}\end{aligned}$$

Exercise 34. Using triangle OAP, not the right triangle, if $0 < x < \pi/2$

$$\begin{aligned}\frac{1}{2}\cos x \sin x < \frac{1}{2} \sin x < \frac{x}{2} \\ \Longrightarrow \sin x < x\end{aligned}$$

Now if $0 > x > -\pi/2$, $\sin x < 0$,

$$|\sin x| = -\sin x = \sin -x = \sin |x| < |x|$$

2.17 Exercises - Average value of a function.

Exercise 1. $\frac{1}{b-a}\int x^2 dx = \frac{1}{3}(b^2 + ab + a^2)$

Exercise 2. $\frac{1}{1-0}\int x^2 + x^3 = \frac{7}{12}$

Exercise 3. $\frac{1}{4-0}\int x^{1/2} = \frac{4}{3}$

Exercise 4. $\frac{1}{8-1}\int x^{1/3} = \frac{45}{28}$

Exercise 5. $\frac{1}{\pi/2-0}\int_0^{\pi/2} \sin x = \frac{2}{\pi}$

Exercise 6. $\frac{1}{\pi/2 - -\pi/2}\int \cos x = 2/\pi$

Exercise 7. $\frac{1}{\pi/2-0}\int \sin 2x = -1/\pi(-1-1) = 2/\pi$

Exercise 8. $\frac{1}{\pi/4-0}\int \sin x \cos x = \frac{1}{\pi}$

Exercise 9. $\frac{1}{\pi/2-0}\int \sin^2 x = \frac{1}{\pi}(x-\sin 2x/2)\Big|_0^{\pi} = \frac{1}{2}$

Exercise 10. $\frac{1}{\pi-0}\int \cos^2 x = \frac{1}{2}$

Exercise 11.

(1) $\frac{1}{a-0}\int x^2 = a^2/3 = c^2 \Longrightarrow c = a/\sqrt{3}$

(2) $\frac{1}{a-0}\int x^n = \frac{1}{a}\frac{1}{n+1}x^{n+1}\Big|_0^a = \frac{a^n}{n+1} = c^n \Longrightarrow c = \frac{a}{(n+1)^{1/n}}$

Exercise 12.

$$\begin{aligned}
&A = \int wf / \int w \quad \int wx^2 = k\int x \\
&\int x^3 = \frac{1}{4}x^4 = k\frac{1}{2}x^2; k = \frac{1}{2}, w = x \\
&\int x^4 = \frac{1}{5}x^5 = k\frac{1}{3}x^3; k = \frac{3}{5}, w = x^2 \\
&\int x^5 = \frac{1}{6}x^6 = k\frac{1}{4}x^4; k = \frac{2}{3}, w = x^3
\end{aligned}$$

Exercise 13.

$$\begin{aligned}
&A(f+g) = \frac{1}{b-a}\int f+g = \frac{1}{b-a}\int f + \frac{1}{b-a}\int g = A(f) + A(g) \\
&A(cf) = \frac{1}{b-a}\int cf = c\left(\frac{1}{b-a}\right)\int f \\
&A(f) = \frac{1}{b-a}\int f \leq \frac{1}{b-a}\int g = A(g)
\end{aligned}$$

Exercise 14.

$$\begin{aligned}
A(c_1f + c_2g) &= \frac{\int w(c_1f+c_2g)}{\int w} = \frac{c_1\int wf}{\int w} + \frac{c_2\int wg}{\int w} \\
&= c_1A(f) + c_2A(g) \\
f \leq g\, w > 0(&\text{ nonnegative }), \Longrightarrow wf \leq wg
\end{aligned}$$

Exercise 15.

$$A_a^b(f) = \frac{1}{b-a}\int_a^b f = \frac{1}{b-a}\left(\int_a^c f + \int_c^b f\right) = \left(\frac{c-a}{b-a}\right)\left(\frac{\int_a^c f}{c-a}\right) + \frac{b-a-(c-a)}{b-a}\frac{\int_a^b f}{b-c}$$

$$a < c < b$$

$$0 < \frac{c-a}{b-a} < 1 \text{ Let } t = \frac{c-a}{b-a}$$

$$\Longrightarrow A_a^b(f) = tA_a^c(f) + (1-t)A_c^b(f)$$

$$A_a^b(f) = \frac{\int_a^b wf}{\int_a^b w} = \frac{\int_a^c w}{\int_a^b w}\frac{\int_a^c wf}{\int_a^c w} + \left(\frac{\int_a^b w - \int_a^c w}{\int_a^b w}\right)\frac{\int_c^b wf}{\int_c^b w}$$

$$0 < \frac{\int_a^c w}{\int_a^b w} < 1 \text{ since } w \text{ is a nonnegative function. Let } t = \frac{\int_a^c w}{\int_a^b w}$$

$$\Longrightarrow A_a^b(f) = tA_a^c(f) + (1-t)A_c^b(f)$$

Exercise 16. Recall that $x_{cm} = \frac{\int x\rho}{\int \rho}$ or $r_{cm} = \frac{\int r dm}{M}$.

$$
\begin{aligned}
x_{cm} &= \frac{\int_0^L x}{\int_0^L 1} = \frac{L}{2} \\
I_{cm} &= \int r^2 dm = \int x^2(1) = L^3/3 \\
r^2 &= \frac{I_{cm}}{\int_0^L 1} = L^2/3 \Longrightarrow r = \frac{L}{\sqrt{3}}
\end{aligned}
$$

Exercise 17.

$$
\begin{aligned}
x_{cm} &= \frac{\int_0^{l/2} x + \int_{L/2}^{L} 2x dx}{\frac{L}{2} + 2(L - L/2)} = \frac{yL^2}{12} \\
I_{cm} &= \int_0^{L/2} x^2 + \int_{L/2}^{L} 2x^2 = 5L^3/8 \quad r^2 = \frac{5L^3/8}{3L/2} = \frac{5L^2}{12} \Longrightarrow r = \frac{\sqrt{5}L}{2\sqrt{3}}
\end{aligned}
$$

Exercise 18. $\rho(x) = x$ for $0 \leq x \leq L$

$$
\begin{aligned}
x_{cm} &= \frac{\int xxdx}{\int xdx} = \frac{\frac{1}{3}x^3\big|_0^L}{\frac{1}{2}x^2\big|_0^L} = \frac{2}{3}L \\
I_{cm} &= \int x^2 xdx = L^4/4 \\
r^2 &= \frac{L^4/4}{L^2/2} = L^2/2 \quad r = \frac{L}{\sqrt{2}}
\end{aligned}
$$

Exercise 19.

$$
\begin{aligned}
x_{cm} &= \frac{\int xxdx + \int x\frac{L}{2}dx}{\int xdx + \int L/2} = \frac{\frac{1}{3}x^3\big|_0^{L/2} + \frac{L}{2}(x^2/2)\big|_{L/2}^{L}}{\frac{1}{2}x^2\big|_0^{L/2} + \frac{L}{2}(L - L/2)} = 11L/18 \\
I_{cm} &= \int x^2 xdx + \int x^2 L/2dx = L^4 31/192 \\
r^2 &= I_{cm}/(L^2 3/8) = L^2 31/72 \quad r = \frac{\sqrt{31}L}{6\sqrt{2}}
\end{aligned}
$$

Exercise 20. $\rho(x) = x^2$ for $0 \leq x \leq L$

$$
\begin{aligned}
x_{cm} &= \frac{\int xx^2dx}{\int x^2} = 3L/4 \\
I_{cm} &= \int x^2x^2dx = L^5/5 \\
r^2 &= \frac{I_{cm}}{\frac{1}{3}L^3} = \frac{3}{5}L^2 \quad r = \sqrt{\frac{3}{5}}L
\end{aligned}
$$

Exercise 21.

$$
\begin{aligned}
x_{cm} &= \frac{\int_0^{L/2} xx^2dx + \int_{L/2}^{L} x\frac{L^2}{4}dx}{\int_0^{L/2} x^2dx + \int_{L/2}^{L} \frac{L^2}{4}dx} = 21L/32 \\
I_{cm} &= int_0^{L/2} x^2x^2dx + \int_{L/2}^{L} x^2\frac{L^2}{4}dx = 19L^5/240 \\
r^2 &= \frac{I_{cm}}{L^3/6} = 19L^2/40 \Longrightarrow r = \sqrt{19}L/2\sqrt{10}
\end{aligned}
$$

Exercise 22. Be flexible about how you can choose a convenient origin to evaluate the center-of-mass from

Let $\rho = cx^n$

$$c\int_0^L x^n dx = \frac{1}{n+1}L^{n+1}c = M$$
$$\Longrightarrow c = \frac{(n+1)M}{L^{n+1}}$$
$$c\int_0^L xx^n dx = c\frac{1}{n+2}L^{n+2} = \frac{n+1}{n+2}ML = \frac{3ML}{4}$$
$$x_{cm} = \frac{\int x\rho}{M} = \frac{3L}{4} \Longrightarrow \int x\rho = \frac{n+1}{n+2} = \frac{3}{4} \Longrightarrow n = 2$$
$$\boxed{\rho = \frac{3M}{L^3}x^2}$$

Exercise 23.

(1)
$$\frac{1}{\pi/2 - 0}\int 3\sin 2t = \frac{6}{\pi}$$

(2)
$$\frac{1}{\pi/2 - 0}\int 9\sin^2 2t = 9/2 \Longrightarrow v_{rms} = 3\sqrt{2}/2$$

Exercise 24. $T = 2\pi$ (just look at the functions themselves)

$$\frac{1}{2\pi}\int_0^{2\pi} 160\sin t 2\sin(t-\pi/6) = 80\sqrt{3}$$

2.19 Exercises - The integral as a function of the upper limit. Indefinite integrals. **Exercise 1.** $\int_0^x (1+t+t^2)dt = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$

Exercise 2. $2y + 2y^2 + 8y^3/3$

Exercise 3. $2x + 2x^2 + 8x^3/3 - (-1 + 1/2 + -1/3) = 2(x + x^2 + 4x^3/3) + 5/6$

Exercise 4. $\int_1^{1-x}(1-2t+3t^2)dt = (t - t^2 + t^3)\big|_1^{1-x} = -2x + 2x^2 - x^3$

Exercise 5. $\int_{-2}^x t^4 + t^2 = \frac{1}{5}t^5 + \frac{1}{3}t^3\big|_{-2}^x = \frac{x^5}{5} + \frac{x^3}{3} + \frac{40}{3}$

Exercise 6. $\int_x^{x^2} t^4 + 2t^2 + 1 = \left(\frac{t^5}{5} + \frac{2}{3}t^3 + t\right)\Big|_x^{x^2} = \frac{1}{5}(x^{10} - x^5) + \frac{2}{3}(x^6 - x^3) + x^2 - x$

Exercise 7. $\left(\frac{2}{3}t^{3/2} + t\right)\big|_1^x = \frac{2}{3}(x^{3/2} - 1) + (x - 1)$

Exercise 8. $\left(\frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/4}\right)\big|_x^{x^2} = \frac{2}{3}(x^3 - x^{3/2}) + \frac{4}{5}(x^{5/2} - x^{5/4})$

Exercise 9. $\sin t\big|_{i\pi}^x = \sin x$

Exercise 10. $\left(\frac{t}{2} + \sin t\right)\big|_0^{x^2} = \frac{x^2}{2} + \sin x^2$

Exercise 11. $\left(\frac{1}{2}t + \cos t\right)\big|_x^{x^2} = \frac{x^2 - x}{2} + \cos x^2 - \cos x$

Exercise 12. $\left(\frac{1}{3}u^3 + -\frac{1}{3}\cos 3u\right)\big|_0^x = \frac{x^3}{3} + -\frac{1}{3}(\cos 3x - 1)$

Exercise 13. $\left(\frac{1}{3}v^3 + \frac{\cos 3v}{-3}\right)\Big|_x^{x^2} = \frac{x^6 - x^3}{3} + \frac{-1}{3}(\cos 3x^2 - \cos 3x)$

Exercise 14. $\int \frac{1-\cos 2x}{2} + x = \left(\frac{1}{2}x - \frac{\sin 2x}{4} + \frac{1}{2}x^2\right)\big|_0^y = \boxed{\frac{y}{2} - \frac{\sin 2y}{4} + \frac{y^2}{2}}$

Exercise 15. $\left(\frac{-\cos 2w}{2} + 2\sin\frac{w}{2}\right)\Big|_0^x = \boxed{-\frac{(\cos 2x - 1)}{2} + 2\sin\frac{x}{2}}$

Exercise 16. $\int_{-\pi}^x (\frac{1}{2} + \cos t)^2 dt = \int_{-\pi}^x \frac{1}{4} + \cos t + \cos^2 t = \frac{1}{4}(x+\pi) + \sin x + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_{-\pi}^x = \frac{3}{4}(x+\pi) + \sin x + \frac{1}{4}\sin 2x$

Exercise 17. $\int_0^x (t^3 - t)dt = \frac{1}{3}\int_{\sqrt{2}}^x (t - t^3)dt$

Note that $t^3 - t < 0$ for $0 < t \leq 1$ and $t^3 - t > 0$ for $t > 1$. $t - t^3 < 0$ for $t > \sqrt{2}$.

$$\frac{1}{4}x^4 - \frac{1}{2}x^2 = \frac{1}{3}\left(\frac{1}{2}t^2 - \frac{1}{4}t^4\right)\Big|_{\sqrt{2}}^x = \frac{1}{6}x^2 - \frac{1}{12}x^4$$
$$\Longrightarrow \frac{1}{3}x^4 - \frac{2}{3}x^2 = 0 \Longrightarrow x = 0, \boxed{x = \sqrt{2}}$$

$\int_0^1 (t^3 - t)dt + \int_1^{\sqrt{2}} (t^3 - t)dt$ "cancel" each other out.

Exercise 18. $f(x) = x - [x] - \frac{1}{2}$ if x is not an integer; $f(x) = 0$ if $x \in \mathbb{Z}$.

For any real number, $x = q + r, 0 \leq r < 1, q \in \mathbb{Z}$. So then

$$x - [x] = r$$
$$f(x) = r - \frac{1}{2}$$

(1) To show the periodicity, consider

$$f(x+1) = x + 1 - [x+1] - \frac{1}{2} = r - \frac{1}{2} = f(x) \text{ since} x + 1 = q + 1 + r,\ [x+1] = q + 1$$
$$x + 1 - [x+1] = r - \frac{1}{2}$$

(2) $P(x) = \int_0^x f(t)dt = \int_0^x (t - \frac{1}{2}) = \frac{1}{2}x^2 - \frac{1}{2}x$ because given $0 < x \leq 1$, then $q = 0$ for x, so we can use $r = t$.
To show periodicity,

$$P(x+1) = \int_0^{x+1} f(t)dt = \int_0^1 f(t)dt + \int_1^{x+1} f(t)dt = 0 + \int_0^x f(t+1)dt = \int_0^x f(t)dt = P(x)$$
$$\text{since } \int_0^1 f(t)dt = \frac{1}{2}(x^2 - x)\Big|_0^1 = 0$$

(3) Since P itself is periodic by 1, then we can consider $0 \leq x < 1$ only. Now $x - [x] = r$ and $P(x) = \frac{1}{2}(r^2 - r)$. So $P(x) = \frac{1}{2}((x - [x])^2 - (x - [x]))$.

(4)

$$\int_0^1 (P(t) + c)dt = 0 \Longrightarrow \int_0^1 P(t)dt = -c$$
$$0 \leq t \leq 1 \text{ so } P(t) = \frac{1}{2}(t^2 - t)$$
$$\Longrightarrow \int_0^1 P(t)dt = \frac{1}{2}\left(\frac{1}{3}t^3 - \frac{1}{2}t^2\right)\Big|_0^1 = \frac{1}{2}\frac{-1}{6} \Longrightarrow \boxed{c = \frac{1}{12}}$$

(5) $Q(x) = \int_0^x (P(t) + c)dt$

$$Q(x+1) = \int_0^{x+1} (P(t) + c)dt = \int_0^1 (P(t) + c)dt + \int_1^{x+1} (P(t) + c)dt =$$
$$= 0 + \int_0^x (P(t+1) + c)dt = \int_0^x (P(t) + c)dt = Q(x)$$

so without loss of generality, consider $0 \leq x < 1$

$$\Longrightarrow Q(x) \int_0^x \frac{1}{2}(t^2 - t) + \frac{1}{12} = \boxed{\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{x}{12}}$$

Exercise 19. $g(2n) = \int_0^{2\pi} f(t)dt$

Consider

$$\int_{-1}^1 f(t)dt = \int_0^1 f(t)dt + \int_{-1}^0 f(t)dt = \int_0^1 f(t)dt + \frac{1}{-1}\int_1^0 f(-1t)dt =$$
$$= \int_0^1 f + \int_1^0 f(t)dt = 0$$

Consider that $\int_1^3 f(t)dt = \int_{-1}^1 f(t+2)dt = \int_{-1}^1 f(t)dt = 0$. Then, by induction,

$$\int_1^{2n+1} f = \int_1^{2n-1} f + \int_{2n-1}^{2n+1} f(t)dt = 0 + \int_{-1}^1 f(t+2n)dt = \int_{-1}^1 f(t)dt = 0$$

(1)

$$g(2n) = \int_0^1 f + \int_1^{2n-1} f + \int_{2n-1}^{2n} f = \int_0^1 f + \int_{-1}^0 f(t)dt = \int_0^1 f + - \int_1^0 f(-t)dt$$
$$= \int_0^1 f + \int_1^0 f = 0$$

(2)

$$g(-x) = \int_0^{-x} f = -\int_0^x f(-t)dt = \int_0^x f(t)dt = g(x)$$
$$g(x+2) = \int_0^{x+2} f(t)dt = \int_0^2 f + \int_2^{x+2} f = \int_0^x f(t+2)dt = \int_0^x f(t)dt = g(x)$$

Exercise 20.

(1) g is odd since

$$g(-x) = \int_0^{-x} f(t)dt = -\int_0^x f(-t)dt = -\int_0^x f(t)dt = -g(x)$$

Now

$$g(x+2) = \int_0^{x+2} f = \int_0^2 f + \int_2^{x+2} f = g(2) + \int_0^x f(t+2)dt = g(2) + \int_0^x f(t)dt = g(2) + g(x)$$
$$\implies g(x+2) - g(x) = g(2)$$

(2)

$$g(2) = \int_0^2 f = \int_1^2 f + \int_0^1 f = \int_1^2 f + A = \int_{-1}^0 f(t+2)dt + A = \int_{-1}^0 f(t)dt + A =$$
$$-\int_1^0 f(-t)dt + A = 2A$$
$$g(5) - g(3) = g(2)$$
$$g(3) = g(2) + \int_2^3 f(t)dt = 2A + \int_0^1 f(t+2)dt = 2A + A = 3A$$
$$\implies g(5) = 3A + 2A = 5A$$

(3) The key observation is to see that g must repeat itself by a change of 2 in the argument. To make $g(1) = g(3) = g(5)$, they're different, **unless** $A = 0$!

Exercise 21. From the given, we can derive

$$g(x) = f(x+5), f(x) = \int_0^x g(t)dt$$
$$\implies f(5) = \int_0^5 g(t)dt = g(0) = 7$$

(1) The **key insight** I uncovered was, when stuck, one of the things you can do, is to **think geometrically** and **draw a picture**.

$$g(-x) = f(-x+5) = g(x) = -f(x-5)$$
$$\implies -g(x) = f(x-5)$$

(2)

$$\int_0^5 f(t)dt = \int_{-5}^0 f(t+5)dt = \int_{-5}^0 g(t)dt = -\int_0^{-5} g(t)dt = \int_0^5 g(-t)dt = \int_0^5 g(t)dt = f(5) = 7$$

(3)

$$\int_0^x f(t)dt = \int_{-5}^{x-5} f(t+5)dt = \int_{-5}^{x-5} g(t)dt = \int_0^{x-5} g + \int_{-5}^0 g = f(x-5) + -\int_0^{-5} g(t)dt =$$

$$f(x-5) + \int_0^5 g(-t)dt = f(x-5) + f(5) = -g(x) + g(0)$$

where we've used $f(x-5) = -g(x)$ in the second and third to the last step.

3.6 Exercises - Informal description of continuity, The definition of the limit of a function, The definition of continuity of a function, The basic limit theorems. More examples of continuous functions, Proofs of the basic limit theorems. Polynomials are continuous.

Exercise 1. $\lim_{x\to 2}\frac{1}{x^2} = \frac{1}{\lim_{x\to 2} x^2} = \frac{1}{4}$

Exercise 2. $\frac{\lim_{x\to 0}(25x^3+2)}{\lim_{x\to 0}(75x^7-2)} = -1$

Exercise 3. $\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)} = 4$

Exercise 4. $\lim_{x\to 1}\frac{(2x-1)(x-1)}{x-1} = 1$

Exercise 5. $\lim_{h\to 0}\frac{t^2+2th+h^2-t^2}{h} = 2t$

Exercise 6. $\lim_{x\to 0}\frac{(x-a)(x+a)}{(x+a)^2} = -1$

Exercise 7. $\lim_{a\to 0}\frac{(x-a)(x+a)}{(x+a)^2} = 1$

Exercise 8. $\lim_{x\to a}\frac{(x-a)(x+a)}{(x+a)^2} = 0$

Exercise 9. $\lim_{t\to 0}\tan t = \frac{\lim_{t\to 0}\sin t}{\lim_{x\to 0}\cos t} = \frac{0}{1} = 0$

Exercise 10. $\lim_{t\to 0}(\sin 2t + t^2\cos 5t) = \lim_{t\to 0}\sin 2t + \lim_{t\to 0} t^2 \lim_{t\to 0}\cos 5t = 0 + 0 = 0$

Exercise 11. $\lim_{x\to 0^+}\frac{|x|}{x} = 1$

Exercise 12. $\lim_{x\to 0^-}\frac{|x|}{x} = -1$

Exercise 13. $\lim_{x\to 0^+}\frac{\sqrt{x^2}}{x} = +1$

Exercise 14. $\lim_{x\to 0^-}\frac{\sqrt{x^2}}{x} = -1$

Exercise 15. $\lim_{x\to 0}\frac{2\sin x\cos x}{x} = 2$

Exercise 16. $\lim_{x\to 0}\frac{2\sin x\cos x}{\cos 2x\sin x} = 2$

Exercise 17. $\lim_{x\to 0}\frac{\sin x\cos 4x + \sin 4x\cos x}{\sin x} = 1 + \lim_{x\to 0}\frac{2\sin 2x\cos 2x}{\sin x} = 1 + 2\left(\lim_{x\to 0}\frac{2\sin x\cos x\cos 2x}{\sin x}\right) = 5$ **Exercise 18.**

$\lim_{x\to 0}\frac{5\sin 5x}{5x} - \lim_{x\to 0}\frac{3\sin 3x}{3x} = 5 - 3 = 2$ **Exercise 19.**

$$\lim_{x\to 0}\frac{\sin\left(\frac{x+a}{2}+\frac{x-a}{2}\right)-\sin\left(\frac{x+a}{2}-\left(\frac{x-a}{2}\right)\right)}{x-a}=$$

$$=\lim_{x\to 0}\left(\frac{\sin\frac{x+a}{2}\cos\frac{x-a}{2}+\sin\frac{x-a}{2}\cos\frac{x+a}{2}-\left(\sin\frac{x+a}{2}\cos\frac{x-a}{2}-\sin\frac{x-a}{2}\cos\frac{x+a}{2}\right)}{x-a}\right)=$$

$$=\lim_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a}=\cos a$$

Exercise 20. $\lim_{x\to 0}\frac{2\sin^2 x/2}{4(x/2)^2}=\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin x/2}{x/2}\right)=\frac{1}{2}$

Exercise 21. $\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}\left(\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)=\lim_{x\to 0}\frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})}=\boxed{\frac{1}{2}}$

Exercise 22. b, c are given.

$\sin c = ac+b, a=\frac{\sin c-b}{c}$, $c\neq 0$.
if $c=0$, then $b=0, a\in\mathbb{R}$.

Exercise 23. b, c are given.

$2\cos c = ac^2+b, a=\frac{2\cos c-b}{c^2}$, $c\neq 0$.
If $c=0$, then $b=2, a\in\mathbb{R}$.

Exercise 24.

tangent is continuous for $x\notin(2n+1)\pi/2$

cotangent is continuous for $x\notin 2n\pi$

Exercise 25. $\lim_{x\to 0} f(x)=\infty$. No $f(0)$ cannot be defined.

Exercise 26.

(1) $|\sin x-0|=|\sin x|<|x|$. Choose $\delta=\epsilon$ for a given ϵ.
Then $\forall\epsilon>0, \exists\delta>0$ such that $|\sin x-0|<\epsilon$ when $|x|<\delta$.

(2)
$$|\cos x-1|=|-2\sin^2 x/2|=2|\sin x/2|^2<2|\frac{x}{2}|^2=\frac{|x|^2}{2}<2\epsilon/2=\epsilon$$
If we had chosen $\delta_0=\sqrt{2\epsilon}$ for a given ϵ. $|x-0|<\delta=\sqrt{2\epsilon}$.

(3)
$$|\sin x(\cos h-1)+\cos x\sin h|\leq|\sin x||\cos h-1|+|\cos h||\sin h|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$
$$|\cos x+h-\cos x|=|\cos x\cos h-\sin x\sin h-\cos x|=|\cos x(\cos h-1)-\sin x\sin h|\leq$$
$$\leq|\cos x||\cos h-1|+|\sin x||\sin h|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$
$$\text{since }\forall\epsilon>0\exists\delta_1,\delta_2>0\text{ such that }|\cos h-1|<\epsilon_0;|\sin h|<\epsilon\text{ whenever }|h|<\min(\delta_1,\delta_2)$$
$$\text{Choose }\delta_3\text{ such that if }|h|<\delta_3;|\cos h-1|<\frac{\epsilon}{2};|\sin h|<\frac{\epsilon}{2}$$

Exercise 27. $f(x)-A=\sin\frac{1}{x}-A$.

Let $x=\frac{1}{n\pi}$.
$$|f(x)-A|=|\sin n\pi-A|>||\sin n\pi|-|A||>|1-|A||$$
Consider $|x-0|=|x|=\frac{1}{n\pi}\leq\delta(n)$. Consider $\epsilon_0=\frac{|1-|A||}{2}$. Then suppose a $\delta(n)\geq|x-0|$ but $|f(x)-A|>\epsilon_0$. Thus, contradiction.

Exercise 28. Consider $x\leq\frac{1}{n}, n\in\mathbb{Z}^+$, $n>M(n)$ (n is a given constant)
$$f(x)=\left[\frac{1}{x}\right]=[n]=n,\text{ for }m>M(n), x=\frac{1}{m}f(x)>M(n)$$
so $\forall\epsilon>0$, we cannot find $\delta=\frac{1}{n}$ such that $|f(x)-A|<\epsilon$ for $x<\delta$.
So $f(x)\to\infty$ as $x\to 0^+$.

Consider $\frac{1}{n} \geq x > 0$, $n \in \mathbb{Z}^-$; $-n > M(n)$.

$$f(x) = \left[\frac{1}{x}\right] = [n] = n < -M(n)$$

Since integers are unbounded, we can consider $n < A$, so that

$$|f(x) - A| > ||f| - |A|| = -n - |A| > M(n) - |A|. \text{ Choose } n \text{ such that } M(n) - |A| > 0$$

Exercise 29.

$$|f - A| = |(-1)^{[1/x]} - A| \geq ||(-1)^{[1/x]}| - |A|| = |1 - |A||$$

Choose $\epsilon < |1 - |A||$. Then $\forall \delta > 0$ (such that $|x| < \delta$), $|f - A| > \epsilon$. Thus there's no value for $f(0)$ we could choose to make this function continuous at 0.

Exercise 30. Since

$$|f(x)| = |x||(-1)^{[1/x]}| = |x|$$

So $\forall \epsilon$, let $\delta = \epsilon$.

Exercise 31. f continuous at x_0.

Choose some $\epsilon_0, 0 < \epsilon_0 < \min(b - x_0, x_0 - a)$. Then $\exists \delta_0 = \delta(x_0, \epsilon_0)$.

Consider $\epsilon_1 = \frac{\epsilon_0}{2}$ and $\delta_1 = \delta(x_0, \epsilon_1)$
Consider $x_1 \in (x_0 - \delta_1, x_0 + \delta_1)$, so that $|f(x_1) - f(x_0)| < \epsilon_1$.
Proceed to construct a δ for x_1, some $\delta(x_1; \epsilon_0)$

$$|x - x_1| = |x - x_0 + x_0 - x_1| < |x - x_0| + |x_0 - x_1|$$

Without loss of generality, we can specify x_1 such that $|x_0 - x_1| < \frac{\delta_1}{2}$. Also, "pick" only the x's such that

$$|x - x_0| < \frac{\delta_1}{2} < \delta_1$$
$$\Longrightarrow |x - x_1| < \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1$$

Thus, "for these x's"

$$|f(x) - f(x_1)| = |f(x) - f(x_0) + f(x_0) - f(x_1)| < |f(x) - f(x_0)| + |f(x_1) - f(x_0)| < \epsilon_1 + \epsilon_1 = \epsilon_0$$

So $\forall \epsilon_0, \exists \delta_1$ for x_1. f is continuous at $x_1 \in (a, b)$. Thus, there must be infinitely many points that are continuous in (a, b), and at the very least, some or all are "clustered" around some neighborhood about the one point given to make f continuous.

Exercise 32. Given $\epsilon = \frac{1}{n}$, $|f(x)| = |x \sin \frac{1}{x}| = |x||\sin 1/x| < |x|(1)$.

Let $\delta = \delta(n) = \frac{1}{n}$, so that $|x| < \frac{1}{n}$.
$\Longrightarrow |f(x)| < \frac{1}{n}$

Exercise 33.

(1) Consider $x_0 \in [a, b]$.

Choose some $\epsilon_0, 0 < \epsilon_0 < \min(b - x_0, x_0 - a) \neq 0$, (x_0 could be a or b)
Consider, without loss of generality, only "x's" such that $x \in [a, b]$.

$$|f(x) - f(x_0)| \leq |x - x_0|$$

Let $\delta_0 = \delta(\epsilon_0, x_0) = \epsilon_0 \Longrightarrow |f(x) - f(x_0)| < \epsilon_0$.

Since we didn't specify $x_0, \forall x_0 \in [a, b], f$ is continuous at x_0.

(2)

$$\left| \int_a^b f(x)dx - (b - a)f(a) \right| = \left| \int_a^b (f(x) - f(a))dx \right| \leq \int_a^b |f(x) - f(a)|dx \leq$$
$$\leq \int_a^b |x - a|dx = \left(\frac{1}{2}x^2 - ax\right)\Big|_a^b = \frac{1}{2}(b - a)(b + a) - a(b - a) = \frac{(b - a)^2}{2}$$

(3)

$$\left|\int_a^b f(x)dx - (b-a)f(c)\right| = \left|\int_a^b (f(x)-f(c))dx\right| \le \int_a^b |f(x)-f(c)|dx \le \int_a^b |x-c|dx =$$
$$= \int_a^c (c-x)dx + \int_c^b (x-c)dx = c(c-a) - \frac{1}{2}(c-a)(c+a) + \frac{1}{2}(b-c)(b+c) - c(b-c) =$$
$$= \frac{1}{2}((c-a)^2 + (b-c)^2)$$

Draw a figure for clear, geometric reasoning.

Consider a square of length $(b-a)$ and a $45-45$ right triangle inside. From the figure, it's obvious that right triangles of $c-a$ length and $(b-c)$ length lie within the $(b-a)$ right triangle.

Compare the trapezoid of $c-a, b-a$ bases with the $b-a$ right triangle.

$$\frac{1}{2}(b-c)(b-a+c-a) = \frac{1}{2}(b-c)(b-c+2(c-a)) > \frac{1}{2}(b-c)^2$$

Indeed, the trapezoid and $c-a$ right triangle equals the $b-a$ trapezoid since

$$\frac{1}{2}(b-c)(b-a+c-a) + \frac{1}{2}(c-a)^2 = \frac{1}{2}(b^2-c^2-2ab+2ac+c^2-2ca+a^2) = \frac{1}{2}(b-a)^2$$
$$\implies \frac{1}{2}(b-a)^2 > \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2$$
$$\text{so then } \left|\int_a^b f(x)dx - (b-a)f(c)\right| \le \frac{(b-a)^2}{2}$$

3.11 Exercises - Bolzano's theorem for continuous functions, The intermediate-value theorem for continuous functions. These theorems form the foundation for continuity and will be valuable for differentiation later.

Theorem 10 (Bolzano's Theorem)**.**
Let f be cont. at $\forall x \in [a,b]$.
Assume $f(a)$, $f(b)$ have opposite signs.
Then $\exists$ at least one $c \in (a,b)$ s.t. $f(c)=0$.

Proof. Let $f(a) < 0,\ f(b) > 0$.
Want: Fine one value $c \in (a,b)$ s.t. $f(c) = 0$
Strategy: find the largest c.
Let $S = \{$ all $x \in [a,b]$ s.t. $f(x) \le 0 \}$.
S is nonempty since $f(a) < 0$. S is bounded since all $S \subseteq [a,b]$.
$\implies S$ has a suprenum.
Let $c = supS$.
If $f(c) > 0$, $\exists(c-\delta, c+\delta)$ s.t. $f > 0$
$c-\delta$ is an upper bound on S
but c is a least upper bound on S. Contradiction.
If $f(c) < 0$, $\exists(c-\delta, c+\delta)$ s.t. $f < 0$
$c+\delta$ is an upper bound on S
but c is an upper bound on S. Contradiction. □

Theorem 11 (Sign-preserving Property of Continuous functions)**.**
Let f be cont. at c and suppose that $f(c) \neq 0$.
then $\exists(c-\delta, c+\delta)$ s.t. f be on $(c-\delta, c+\delta)$ has the same sign as $f(c)$.

Proof. Suppose $f(c) > 0$.
$\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $f(c) - \epsilon < f(x) < f(c) + \epsilon$ if $c - \delta < x < c + \delta$ (by continuity).
Choose δ for $\epsilon = \frac{f(c)}{2}$. Then

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2} \quad \forall x \in (c-\delta, c+\delta)$$

Then f has the same sign as $f(c)$. □

Theorem 12 (Intermediate value theorem)**.**
Let f be cont. at each pt. on $[a,b]$.
Choose any $x_1, x_2 \in [a,b]$ s.t. $x_1 < x_2$. s.t. $f(x_1) \neq f(x_2)$.
Then f takes on every value between $f(x_1)$ and $f(x_2)$ somewhere in (x_1, x_2).

Proof. Suppose $f(x_1) < f(x_2)$
Let k be any value between $f(x_1)$ and $f(x_2)$
Let $g = f - k$

$$g(x_1) = f(x_1) - k < 0$$
$$g(x_2) = f(x_2) - k > 0$$

By Bolzano, $\exists c \in (x_1, x_2)$ s.t. $g(c) = 0 \quad \implies f(c) = k$ ☐

Exercise 1. $f(0) = c_0$. $f(0) \gtrless 0$.

Since $\lim_{x\to\infty} \frac{c_k x^k}{c_{k-1}x^{k-1}} = \lim_{x\to\infty} \frac{c_k}{c_{k-1}} x = \infty$ $\exists M > 0$ such that $|c_n M^n| > |\sum_{k=0}^{n-1} c_k M^k$. So then

$$f(M) = c_n M^n + \sum_{k=0}^{n-1} c_k M^k \lesseqgtr c_n$$

By Bolzano's theorem $\exists b \in (0, M)$ such that $f(b) = 0$.

Exercise 2. Try alot of values systematically. I also cheated by taking the derivatives and feeling out where the function changed direction.

(1) If $P(x) = 3x^4 - 2x^3 - 36x^2 + 36x - 8$, $P(-4) = 168, P(-3) = -143$, $P(0) = -8, P(\frac{1}{2}) = \frac{15}{16}$, $P(1) = -7$, $P(-3) = -35, P(4) = 200$
(2) If $P(x) = 2x^4 - 14x^2 + 14x - 1$, $P(-4) = 231, P(-3) = -7$, $P(0) = -1, P(\frac{1}{2}) = \frac{1}{8}, P(\frac{3}{2}) = -\frac{11}{8}, P(2) = 2$
(3) If $P(x) = x^4 + 4x^3 + x^2 - 6x + 2$, $P(-3) = 2, P(-\frac{5}{2}) = -\frac{3}{16}, P(-2) = 2$, $P(\frac{1}{3}) = \frac{22}{81}, P(\frac{1}{2}) = -\frac{3}{16}, P(\frac{2}{3}) = -\frac{14}{81}, P(1) = 2$.

Exercise 3. . Consider $f(x) = x^{2j+1} - a$. $f(0) = -a > 0$.

Since a is a constant, choose $M < 0$ such that $M^{2j+1} - a < 0$. $f(M) < 0$.
By Bolzano's theorem, there is at least one $b \in (M, 0)$ such that $f(b) = b^{2j+1} - a = 0$.

Since $x^{2j+1} - a$ is monotonically increasing, there is exactly one b.

Exercise 4. $\tan x$ is not continuous at $x = \pi/2$.

Exercise 5. Consider $g(x) = f(x) - x$. Then $g(x)$ is continuous on $[0,1]$ since f is.

Since $0 \leq f(x) \leq 1$ for each $x \in [0,1]$, consider $g(1) = f(1) - 1$, so that $-1 \leq g(1) \leq 0$. Likewise $0 \leq g(0) \leq 1$.
If $g(1) = 0$ or $g(0) = 0$, we're done ($g(0) = f(0) - 0 = 0$. $f(0) = 0$. Or $g(1) = f(1) - 1 = 0$, $f(1) = 1$).

Otherwise, if $-1 \leq g(1) < 0$ and $0 < g(0) \leq 1$, then by Bolzano's theorem, $\exists$ at least one c such that $g(c) = 0$ ($g(c) = f(c) - c = 0$. $f(c) = c$).

Exercise 6. Given $f(a) \leq a$, $f(b) \geq b$,

Consider $g(x) = f(x) - x \leq 0$. Then $g(a) = f(a) - a \leq 0$, $g(b) = f(b) - b \geq 0$.
Since f is continuous on $[a,b]$ (so is g) and since $g(a), g(b)$ are of opposite signs, by Bolzano's theorem, $\exists$ at least one c such that $g(c) = 0$, so that $f(c) = c$.

3.15 Exercises - The process of inversion, Properties of functions preserved by inversion, Inverses of piecewise monotonic functions. **Exercise 1.** $D = \mathbb{R}$, $g(y) = y - 1$

Exercise 2. $D = \mathbb{R}$, $g(y) = \frac{1}{2}(y - 5)$

Exercise 3. $D = \mathbb{R}$, $g(y) = 1 - y$

Exercise 4. $D = \mathbb{R}$, $g(y) = y^{1/3}$

Exercise 5. $D = \mathbb{R}$,

$$g(y) = \begin{cases} y & \text{if } y < 1 \\ \sqrt{y} & \text{if } 1 \leq y \leq 16 \\ \left(\frac{y}{8}\right)^2 & \text{if } y > 16 \end{cases}$$

Exercise 6. $f(M_f) = f(f^{-1}\left(\frac{1}{n}\sum_{i=1}^n f(a_i)\right)) = \frac{1}{n}\sum_{i=1}^n f(a_i)$

Exercise 7. $f(a_1) \lesseqgtr \frac{1}{n}\sum_{i=1}^n f(a_i) \lesseqgtr f(a_n)$. Since f is strictly monotonic.

g preserves monotonicity.

$$\Longrightarrow a_1 \lesseqgtr M_f \lesseqgtr a_n$$

Exercise 8. $h(x) = af(x) + b$, $a \neq 0$

$$M_h = H\left(\frac{1}{n}\sum_{i=1}^n h(a_i)\right) = H\left(\frac{1}{n}\sum_{i=1}^n (af(a_i) + b)\right) = H\left(a\frac{1}{n}\sum_{i=1}^n f(a_i) + b\right)$$

The inverse for h is $g\left(\frac{h-b}{a}\right) = H(h) = h^{-1}$. So then

$$M_h = g\left(\frac{1}{n}\sum_{i=1}^n f(a_i)\right) = M_f$$

The average is invariant under translation and expansion in ordinate values.

3.20 Exercises - The extreme-value theorem for continuous functions, The small-span theorem for continuous functions (uniform continuity), The integrability theorem for continuous functions.

Since for $c \in [a,b]$, $m = \min_{x\in[a,b]} f \leq f(c) \leq \max_{x\in[a,b]} f = M$

and $\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} = f(c)$

Exercise 1.

$$g = x^9 > 0 \text{ for } x \in [0,1]; f = \frac{1}{\sqrt{1+x}} \, m = \frac{1}{\sqrt{2}}, M = 1$$

$$\int_0^1 x^9 = \frac{1}{10}x^{10}\Big|_0^1 = \frac{1}{10}$$

$$\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}}dx \leq \frac{1}{10}$$

Exercise 2.

$$\sqrt{1-x^2} = \frac{1-x^2}{\sqrt{1-x^2}}.\, f = \frac{1}{\sqrt{1-x^2}} \, g = (1-x^2) \, M = \frac{2}{\sqrt{3}}, m = 1$$

$$\int_0^{1/2}(1-x^2)dx = (x - \frac{1}{3}x^3)\Big|_0^{1/2} = \frac{11}{24}$$

$$\frac{11}{24} \leq \int_0^{1/2}\sqrt{1-x^2}dx \leq \frac{11}{24}\sqrt{\frac{4}{3}}$$

Exercise 3.

$$f = \frac{1}{1+x^6}\, g = 1 - x^2 + x^4 \int_0^a 1 - x^2 + x^4 = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \right)\Big|_0^a = a - \frac{a^3}{3} + \frac{a^5}{5}$$

$$m = \frac{1}{1+a^6}\, M = 1$$

$$\frac{1}{1+a^6}\left(a - \frac{a^3}{3} + \frac{a^5}{5} \right) \leq \int_0^a \frac{1}{1+x^2}dx \leq \left(a - \frac{a^3}{3} + \frac{a^5}{5} \right)$$

So if $a = \frac{1}{10}$, $(a - a^3/3 + a^5/5) = a - 0.333\dots a^3 + 0.2a^5 = 0.099669$

Exercise 4. (b) is wrong, since it had chosen $g = \sin t$, but g needed to be nonnegative.

Exercise 5. At worst, we could have utilized the fundamental theorem of calculus.

$$\int \sin t^2 dt = \int \left(\frac{1}{2t} \right)(2t \sin t^2)dt = \frac{1}{2c}\,(-\cos t^2)\big|_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} =$$

$$= \frac{-1}{2c}((-1)^{n+1} - (-1)^n) = \boxed{\frac{1}{c}(-1)^n}$$

Exercise 6. $\int_a^b (f)(1) = f(c)\int_a^b 1 = f(c)(b-a)$. Then $f(c) = \frac{\int_a^b f}{b-a} = 0$ for some $c \in [a,b]$ by Mean-value theorem for integrals.

Exercise 7. f nonnegative. Consider f at a point of continuity c, and suppose $f(c) > 0$. Then $\frac{1}{2}f(c) > 0$.

$$|f(x) - f(c)| < \epsilon \Longrightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$$

$$\text{Let } \epsilon = \frac{1}{2}f(c)\, \exists \delta > 0 \text{ for } \epsilon = \frac{1}{2}f(c)$$

$$\int_{c-\delta}^{c+\delta} f(x)dx > \frac{1}{2}f(c)(2\delta) = f(c)\delta > 0$$

But $\int_a^b f(x)dx = 0$ and f is nonnegative. $f(c) = 0$.

Exercise 8.

$$m\int g \leq \int fg \leq M \int g \Longrightarrow m \int g \leq 0 \leq M \int g\, \forall g$$

$$m \leq 0 \leq M \text{ for } \int g = 1 \text{ but also}$$

$$-m \leq 0 \leq -M \Longrightarrow m \geq 0\, M \leq 0 \text{ for } \int g = -1$$

So because of this contradiction, $m = M = 0$. By intermediate value theorem, $f = 0$, $\forall x \in [a,b]$.

4.6 Exercises - Historical introduction, A problem involving velocity, The derivative of a function, Examples of derivatives, The algebra of derivatives. **Exercise 1.** $f' = 1 - 2x$, $f'(0) = 1$, $f'(1/2) = 0$, $f'(1) = -1$, $f'(10) = -19$

Exercise 2. $f' = x^2 + x - 2$

(1) $f' = 0, x = 1, -2$
(2) $f'(x) = -2, x = 0, -1$
(3) $f' = 10,\ x = -4, 3$

Exercise 3. $f' = 2x + 3$

Exercise 4. $f' = 4x^3 + \cos x$

Exercise 5. $f' = 4x^3 \sin x + x^4 \cos x$

Exercise 6. $f' = \frac{-1}{(x+1)^2}$

Exercise 7. $f' = \frac{-1}{(x^2+1)^2}(2x) + 5x^4 \cos x + x^5(-\sin x)$

Exercise 8. $f' = \frac{x-1-(x)}{(x-1)^2} = \frac{-1}{(x-1)^2}$

Exercise 9. $f' = \frac{-1}{(2+\cos x)^2}(-\sin x) = \frac{\sin x}{(2+\cos x)^2}$

Exercise 10.

$$\frac{(2x+3)(x^4+x^2+1)-(4x^3+2x)(x^2+3x+2)}{(x^4+x^2+1)^2} = \frac{-2x^5-9x^4+12x^3-3x^2-2x+3}{(x^4+x^2+1)^2}$$

Exercise 11.

$$f' = \frac{(-\cos x)(2-\cos x)-(\sin x)(2-\sin x)}{(2-\cos x)^2} = \frac{-2\cos x - 2\sin x + 1}{(2-\cos x)^2}$$

Exercise 12.

$$f' = \frac{(\sin x + x\cos x)(1+x^2) - 2x(x\sin x)}{(1+x^2)^2} = \frac{\sin x + x\cos x + x^3\cos x - x^2 \sin x}{(1+x^2)^2}$$

Exercise 13.

(1)

$$\frac{f(t+h)-f(t)}{h} = \frac{v_0 h - 32th - 16h^2}{h} = v_0 + 32t - 16h$$
$$f'(t) = v_0 - 32t$$

(2) $t = \frac{v_0}{32}$
(3) $-v_0$
(4) $T = \frac{v_0}{16}$, $v_0 = 16$ for $1sec.$ $v_0 = 160$ for $10sec.$ $\frac{v_0}{16}$ for $Tsec.$
(5) $f'' = -32$
(6) $h = -20t^2$

Exercise 14. $V = s^3,\ \frac{dV}{dS} = 3s^2$

Exercise 15.

(1) $\frac{dA}{dr} = 2\pi r = C$
(2) $\frac{dV}{dr} = 4\pi r^2 = A$

Exercise 16. $f' = \frac{1}{2\sqrt{x}}$

Exercise 17. $f' = \frac{-1}{(1+\sqrt{x})^2}\left(\frac{1}{2\sqrt{x}}\right)$

Exercise 18. $f' = \frac{3}{2}x^{1/2}$

Exercise 19. $\frac{-3}{2}x^{-5/2}$

Exercise 20. $f' = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4}\ x > 0$

Exercise 21. $f' = -\frac{1}{2}x^{-3/2} + -\frac{1}{3}x^{-4/3} - \frac{1}{4}x^{-5/4}$

Exercise 22. $f' = \frac{\frac{1}{2}x^{-1/2}(1+x)-\sqrt{x}}{(1+x)^2} = \frac{1}{2\sqrt{x}(1+x)^2}$

Exercise 23. $f' = \frac{(1+\sqrt{x})-x\frac{1}{2}\frac{1}{\sqrt{x}}}{(1+\sqrt{x})^2} = \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})^2}$

Exercise 24.

$$g = f_1 f_2$$
$$g' = f_1' f_2 + f_1 f_2' \quad \frac{g'}{g} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$
$$g = f_1 f_2 \dots f_n f_{n+1}$$
$$\begin{aligned} g' &= (f_1 f_2 \dots f_n)' f_{n+1} + (f_1 f_2 \dots f_n) f_{n+1}'; \\ \frac{g'}{g} &= \frac{(f_1 f_2 \dots f_n)'}{f_1 f_2 \dots f_n} + \frac{f_{n+1}'}{f_{n+1}} \\ &= \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \dots + \frac{f_n'}{f_n} + \frac{f_{n+1}'}{f_{n+1}} \end{aligned}$$

Exercise 25.

$$\begin{aligned} (\tan x)' &= \left(\frac{\cos x}{\sin x} \right)' = \frac{\cos^2 x - (-\sin x) \sin x}{\cos^2 x} = \sec^2 x \\ (\cot x)' &= \left(\frac{\cos x}{\sin x} \right)' = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\csc^2 x \\ (\sec x)' &= \frac{-1}{\cos^2 x}(-\sin x) = \tan x \sec x \\ (\csc x)' &= \frac{-1}{\sin^2 x} \cos x = -\cot x \csc x \end{aligned}$$

Exercise 35.

$$\begin{aligned} f' &= \frac{(2ax+b)(\sin x + \cos x) - (\cos x - \sin x)(ax^2 + bx + c)}{(\sin x + \cos x)^2} = \\ &= \frac{(2ax+b)(\sin x + \cos x) - (\cos x - \sin x)(ax^2 + bx + c)}{(\sin x + \cos x)^2} \end{aligned}$$

Exercise 36.

$$f' = a \sin x + (ax+b) \cos x + c \cos x + (cx+d)(-\sin x) = ax \cos x + (b+c) \cos x + (a-d) \sin x - cx \sin x$$

So then $a = 1, d = 1, b = d, c = 0$.

Exercise 37.

$$\begin{aligned} g' &= (2ax+b) \sin x + (ax^2 + bx + c) \cos x + (2dx + e) \cos x + (dx^2 + ex + f)(-\sin x) = \\ &= ax^2 \cos x - dx^2 \sin x + (2a - e) x \sin x + (b + 2d) x \cos x + (b - f) \sin x + (c + e) \cos x \end{aligned}$$

$g = x^2 \sin x$. So $d = -1, b = 2, f = 2, a = 0, e = 0, c = 0$.

Exercise 38. $1 + x + x^2 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1}$

(1)

$$\begin{aligned} (1 + x + x^2 + \dots + x^n)' = 1 + 2x + \dots + nx^{n-1} &= \frac{(n+1)x^n (x-1) - (1)(x^{n+1} - 1)}{(x-1)^2} \\ &= \frac{(n+1)(x^{n+1} - x^n) - x^{n+1} + 1}{(x-1)^2} = \frac{nx^{n+1} - (n_1) x^n + 1}{(x-1)^2} \end{aligned}$$
$$x(1 + 2x + \dots + nx^{n-1}) = x + 2x^2 + \dots + nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

(2)

$$\begin{aligned} (x + 2x^2 + \dots + nx^n)' &= (1 + 2^2 x^1 + \dots + n^2 x^{n-1}) = \\ &= \frac{(n(n+2)x^{n+1} - (n+1)^2 x^n + 1)(x-1)^2 - 2(x-1)(nx^{n+2} - (n+1)x^{n+1} + x)}{(x-1)^4} \end{aligned}$$
$$\begin{aligned} x + 2^2 x^2 + \dots + n^2 x^n &= \frac{(n(n+2)x^{n+2} - (n+1)^2 x^{n+1} + x)(x-1) - 2(nx^{n+3} - (n+1)x^{n+2} + x^2)}{(x-1)^3} = \\ &= \frac{n^2 x^{n+3} + (-2n^2 - 2n + 1)x^{n+2} + (n+1)^2 x^{n+1} - x^2 - x}{(x-1)^3} \end{aligned}$$

Exercise 39.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$
$$(x+h)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j}h^j$$
$$\frac{(x+h)^n - x^n}{h} = \frac{\sum_{j=1}^n \binom{n}{j} x^{n-j}h^j}{h} = \sum_{j=1}^n x^{n-j}h^{j-1}\binom{n}{j}$$
$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \binom{n}{1}x^{n-1} = \boxed{nx^{n-1}}$$

4.9 Exercises - Geometric interpretation of the derivative as a slope, Other notations for derivatives. **Exercise 6.**

(1)

$$\begin{aligned} f &= x^2 + ax + b \\ f(x_1) &= x_1^2 + ax_1 + b \\ f(x_2) &= x_2^2 + ax_2 + b \end{aligned} \qquad \begin{aligned} \frac{f(x_2)-f(x_1)}{x_2 - x_1} &= \frac{x_2^2 - x_1^2 + a(x_2 - x_1)}{x_2 - x_1} \\ &= \boxed{x_2 + x_1 + a} \end{aligned}$$

(2)

$$f' = 2x + a$$
$$m = x_2 + x_1 + a = 2x + a \boxed{x = \frac{x_2 + x_1}{2}}$$

Exercise 7. The line $y = -x$ as slope -1.

$$y = x^3 - 6x^2 + 8x \, y' = 3x^2 - 12x + 9$$
$$3x^2 - 12x + 8 = -1 \Longrightarrow x = 3, 1$$

The line and the curve meet under the condition

$$-x = x^3 - 6x^2 + 8x \Longrightarrow x = 3; f(3) = -3$$

At $x = 0$, the line and the curve also meet.

Exercise 8. $f = x(1 - x^2)$. $f' = 1 - 3x^2$.

$$f'(-1) = -2 \Longrightarrow \boxed{y = -2x - 2}$$

For the other line,

$$f'(a) = 1 - 3a^2$$
$$\Longrightarrow y(-1) = 0 = (1 - 3a^2)(-1) + b \Longrightarrow b = 1 - 3a^2$$

Now $f(a) = a(1 - a^2) = a - a^3$ at this point. The line and the curve must meet at this point.

$$\begin{aligned} y(a) = (1 - 3a^2)a + (1 - 3a^2) = \\ = a - 3a^3 + 1 - 3a^2 = a - a^3 \end{aligned}$$
$$\Longrightarrow -2a^3 + 1 - 3a^2 = 0 = a^3 - \frac{1}{2} + \frac{3}{2}a^2$$

The answer could probably be guessed at, but let's review some tricks for solving cubics.

First, do a translation in the x direction to center the origin on the point of inflection. Find the point of inflection by taking the second derivative.

$$f'' = 6a + 3 \Longrightarrow a = -\frac{1}{2}$$

So

$$a = x - \frac{1}{2}$$
$$\Longrightarrow (x - \frac{1}{2})^3 + \frac{3}{2}(x - \frac{1}{2})^2 - \frac{1}{2} = x^3 = \frac{3}{4}x - \frac{1}{4} = 0$$

Then recall this neat trigonometric fact:

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x = 4\cos^3 x - 3\cos x$$
$$\implies \cos^3 x = \frac{3}{4}\cos x - \frac{\cos 3x}{4} = 0$$

Particularly for this problem, we have $\cos 3x = 1$. So $x = 0, 2\pi/3, 4\pi/3$. $\cos x = 1, -\frac{1}{2}$. Plugging $\cos x \to x$ back into what we have for a, $a = -1$, which we already have in the previous part, and $a = \frac{1}{2}$. So

$$\boxed{f\left(\frac{1}{2}\right) = \frac{3}{8}}$$
$$\boxed{y(x) = \left(\frac{1}{4}x\right) + \frac{1}{4}}$$

Exercise 9.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq c \\ ax + b & \text{if } x > c \end{cases}$$
$$f'(x) = \begin{cases} 2x & \text{if } x \leq c \\ a & \text{if } x > c \end{cases}$$
$$\boxed{a = 2c;\ b = -c^2}$$

Exercise 10.

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > c \\ a + bx^2 & \text{if } |x| \leq c \end{cases}$$

Note that $c \not< 0$ since $|x| \leq c$, for the second condition.

$$f'(x) = \begin{cases} -\frac{1}{x^2} & \text{if } x > c \\ \frac{1}{x^2} & \text{if } x < c \\ 2bx & \text{if } |x| \leq c \end{cases}$$

So $\boxed{b = -\frac{1}{2c^3},\ a = \frac{3}{2c}}$.

Exercise 11.

$$f' = \begin{cases} \cos x & \text{if } x \leq c \\ a & \text{if } a > c \end{cases}$$

Exercise 12. $f(x) = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right) = \frac{1-A}{1+A}$

$$\mathbf{A} = \sqrt{\mathbf{x}}\, A' = a = \frac{1}{2}x^{-1/2} = \frac{1}{2A}; A'' = -\frac{1}{4}x^{-3/2} = -\frac{1}{4A^3}$$
$$f' = \frac{-A'(1+A) - A'(1-A)}{(1+A)^2} = \frac{-2A'}{(1+A)^2} = \frac{-1}{\sqrt{x}(1+\sqrt{x})^2}$$
$$f'' = \frac{1}{(A(1+A^2))^2}(A'(1+A)^2 + A(2)(1+A)A') = \frac{3\sqrt{x}+1}{2x^{3/2}(1+\sqrt{x})^3}$$
$$f''' = \frac{1}{2}\left(\frac{\frac{-1}{A^2}A'(A^2(1+A)^3) - (2AA'(1+A)^3 + 3A^2(1+A)^2A')(3+\frac{1}{A})}{(A^2(1+A)^3)^2}\right)$$
$$= \frac{-3}{4}\left(\frac{\frac{1}{A} + 4 + 5A}{A^4(1+A^4)}\right) = -\frac{3}{4}\frac{(1+4\sqrt{x}+5x)}{\sqrt{x}(x+\sqrt{x})^4}$$

Exercise 13.

$$P = ax^3 + bx^2 + cx + d \quad P''(0) = 2b = 10 \Longrightarrow b = 5$$
$$P' = 3ax^2 + 2bx + c \quad P'(0) = c = -1 \quad P(0) = d = -2$$
$$P'' = 6ax + 2b$$
$$P(1) = a + 5 + -1 + -2 = a + 2 = -2 \Longrightarrow a = -4$$

Exercise 14.

$$fg = 2,\ \frac{f'}{g'} = 2\ \frac{f'}{g} = 4\ \frac{g'}{g} = 2\ f = \frac{1}{2},\ g = 4$$

(1)

$$h' = \frac{f'g - g'f}{g^2} = \frac{f'}{g} - \frac{g'}{g}\frac{f}{g} = 4 - 2(\frac{1}{8}) = \frac{15}{4}$$

(2)

$$k' = f'g + fg' = 4g^2 + f2g = 64 + 4 = 68$$

(3)

$$\lim_{x\to 0} \frac{g'(x)}{f'(x)} = \frac{\lim_{x\to 0} g'(x)}{\lim_{x\to 0} f'(x)} = \boxed{\frac{1}{2}}$$

Exercise 15.

(1) True, by definition of $f'(a)$.

(2)

$$\lim_{h\to 0} \frac{f(a) - f(a-h)}{h} = -\lim_{h\to 0} \frac{f(a-h) - f(a)}{h} = \lim_{-h\to 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$$

True, by definition of $f'(a)$.

(3)

$$\lim_{t\to 0} \frac{f(a+2t) - f(a)}{t} = 2\lim_{2t\to 0} \frac{f(a+2t) - f(a)}{2t} = 2f'(a)$$

False.

(4)

$$\lim_{t\to 0} \frac{f(a+2t) - f(a) + f(a) - f(a+t)}{2t} =$$
$$\lim_{2t\to 0} \frac{f(a+2t) - f(a)}{2t} + -\frac{1}{2}\lim_{t\to 0} \frac{f(a+t) - f(a)}{t} =$$
$$f'(a) - \frac{1}{2}f'(a) = \frac{1}{2}f'(a)$$

False.

Exercise 16.

(1)

$$D^*(f+g) = \lim_{h\to 0} \frac{(f(x+h) + g(x+h))^2 - (f(x) + g(x))^2}{h} = \lim_{h\to 0} \frac{(F+G)^2 - (f+g)^2}{h} =$$
$$= D^*f + D^*g + \lim_{h\to 0} \frac{2FG - 2fg}{h}$$
$$\lim_{h\to 0} \frac{2FG - 2fg}{h} = \lim_{h\to 0} \frac{(2(FG) - 2fG)(F+f)}{(F+f)h} + \frac{(2fG - 2fg)(G+g)}{(g+G)h} =$$
$$= \lim_{h\to 0} \frac{2g}{F+h} \lim_{h\to 0} \frac{F^2 - f^2}{h} + \lim_{h\to 0} \frac{2f}{G+g} \lim_{h\to 0} \frac{G^2 - g^2}{h} =$$
$$= \frac{g}{f}D^*f + \frac{f}{g}D^*g$$

$$
\begin{aligned}
D^*(f-g) &= \lim_{h\to 0} \frac{(f(x+h)-g(x+h))^2-(f(x)-g(x))^2}{h} = \\
&= \lim_{h\to 0} \frac{(F-G)^2-(f-g)^2}{h} = \\
&= D^*f + D^*g + - \lim_{h\to 0} \frac{2FG-2fg}{h} \\
&= D^*f + D^*g - \frac{g}{f}D^*f + \frac{f}{g}D^*g
\end{aligned}
$$

$$
\begin{aligned}
D^*(fg) &= \lim_{h\to 0} \frac{((fg)(x+h))^2-((fg)(x))^2}{h} = \\
&= \lim_{h\to 0} \frac{(f^2(x+h))(g^2(x+h)) - f^2(x)g^2(x+h) + (g^2(x+h)-g^2(x))f^2(x)}{h} = \\
&= g^2D^*f + f^2D^*g
\end{aligned}
$$

$$
\begin{aligned}
D^*(f/g) &= \lim_{h\to 0} \frac{\frac{f^2(x+h)}{g^2(x+h)} - \frac{f^2(x)}{g^2(x)}}{h} = \lim_{h\to 0} \frac{\frac{f^2(x+h)-f^2(x)}{g^2(x+h)} + \frac{f^2(x)}{g^2(x+h)} - \frac{f^2(x)}{g^2(x)}}{h} = \\
&= \frac{D^*f}{g^2} + \frac{f^2}{g^4}(-D^*g) \text{ when } g(x)\neq 0
\end{aligned}
$$

(2)
(3)

4.12 Exercises - The chain rule for differentiating composite functions, Applications of the chain rule. Related rates and implicit differentiation. **Exercise 1.** $-2\sin 2x - 2\cos x$

Exercise 2. $\frac{x}{\sqrt{1+x^2}}$

Exercise 3. $-2x\cos x^2 + 2x(x^2-2)\sin x^2 + 2\sin x^3 + 6x^3\cos x^3$

Exercise 4.

$$
\begin{aligned}
f' &= \cos(\cos^2 x)(-2\cos x \sin x)\cos(\sin^2 x) + \sin(\cos^2 x)\sin(\sin^2 x)(2\sin x\cos x) = \\
&= -\sin 2x(\cos(\cos 2x))
\end{aligned}
$$

Exercise 5.

$$f' = n\sin^{n-1} x\cos x\cos nx + -n\sin nx\sin^n x$$

Exercise 6.

$$f' = \cos(\sin(\sin x))(\cos(\sin x))(\cos x)$$

Exercise 7.

$$f' = \frac{2\sin x\cos x\sin x^2 - 2x\cos x^2\sin^2 x}{\sin^2 x^2} = \frac{\sin 2x\sin x^2 - 2x\sin^2 x\cos x^2}{\sin^2 x^2}$$

Exercise 8. $f' = \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\csc^2\frac{x}{2}$

Exercise 9. $f' = 2\sec^2 x\tan x + -2\csc^2 x\cot x$

Exercise 10. $f' = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} = \frac{1+2x^2}{\sqrt{1+x^2}}$

Exercise 11. $f' = \frac{4}{(4-x^2)^{3/2}}$

Exercise 12.

$$f' = \frac{1}{3}\left(\frac{1+x^3}{1-x^3}\right)^{-2/3}\left(\frac{3x^2(2)}{(1-x^3)^2}\right) = \frac{2x^2}{(1-x^3)^2}\left(\frac{1+x^3}{1-x^3}\right)^{-2/3}$$

Exercise 13. **This exercise is important**. It shows a neat integration trick.

$$f(x) = \frac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})} = \frac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})}\left(\frac{x-\sqrt{1+x^2}}{x-\sqrt{1+x^2}}\right) =$$
$$= \frac{x-\sqrt{1+x^2}}{-\sqrt{1+x^2}} = 1 - \frac{x}{\sqrt{1+x^2}}$$
$$f' = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

Exercise 14.

$$\frac{1}{2}(x+\sqrt{x+\sqrt{x}})^{-1/2}(1+\frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\frac{1}{2\sqrt{x}}\right))$$

Exercise 15.

$$f' = (2+x^2)^{1/2}(3+x^3)^{1/3} + (1+x)x(2+x^2)^{-1/2}(3+x^2)^{1/3} + (1+x)(2+x^2)^{1/2}(3+x^3)^{-2/3}x^2$$

Exercise 16.

$$f' = \frac{-1}{(1+\frac{1}{x})^2}\left(\frac{-1}{x^2}\right) = \frac{1}{(x+1)^2}\, g' = \frac{-1}{\left(1+\frac{1}{f}\right)^2}\left(\frac{-1}{f^2}\right) f' = \frac{f'}{(f+1)^2}$$
$$g' = \frac{(x+1)^{-2}}{\left(\frac{x}{x+1}+1\right)^2} = \frac{1}{(2x+1)^2}$$

Exercise 17. $h' = f'g'$

x	h	h'	k	k'
0	$f(2)=0$	$2(-5)=-10$	$g(1)=0$	$1(5)=5$
1	$f(0)=1$	$5(1)=5$	$g(3)=1$	$-6(-2)=12$
2	$f(3)=2$	$4(1)=4$	$g(0)=2$	$-5(2)=-10$
3	$f(1)=3$	$-2(-6)=12$	$g(2)=3$	$1(4)=4$

Exercise 18.

$$g(x) = xf(x^2)$$
$$g'(x) = f(x^2) + x(2x)f'(x^2) = f(x^2) + 2x^2f'(x^2)$$
$$g''(x) = 2xf'(x^2) + 4xf'(x^2) + 2x^2(2x)f''(x^2) = 6xf'(x^2) + 4x^3f''(x^2)$$

x	$g(x)$	$g'(x)$	$g''(x)$
0	0	0	0
1	1	3	10
2	12	$6+8(3)=30$	$12(3)+32(0)=36$

Exercise 19.

(1)

$$g' = \frac{df(x^2)}{dx^2}2x = 2xf'$$

(2)

$$g' = 2\sin x\cos xf' - 2\cos x\sin xf' = (\sin 2x)(f'(\sin^2 x) - f'(\cos^2 x))$$

(3)

$$g' = \frac{df(f(x))}{d(f(x))}f'$$

(4)

$$g' = \frac{df(f(f(x)))}{d(f(f(x)))}\frac{d(f(f(x)))}{d(f(x))}\frac{df}{dx}$$

Exercise 20. $V = s^3$, $s = s(t)$ $\frac{dV}{dt} = 3s^2\frac{ds}{dt}$.

$$\begin{aligned} s &= 5cm\, 75cm^3/sec \\ s &= 10cm\, 300cm^3/sec \\ s &= xcm\, 3x^2cm^3/sec \end{aligned}$$

Exercise 21.

$$l = \sqrt{x^2+h^2}\, \frac{dl}{dt} = \frac{1}{l}x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{l}{x}\frac{dl}{dt} = \frac{10mi}{-\sqrt{10^2-8^2}}(-4mi/sec) = \frac{20}{3}\frac{mi}{sec}\left(\frac{3600sec}{1hr}\right)$$

Exercise 22.

$$\begin{aligned} l^2 &= x^2+s^2 \\ 2l\frac{dl}{dt} &= 2x\frac{dx}{dt}\, \frac{dl}{dt} = \frac{x}{l}\frac{dx}{dt} \\ \frac{dl}{dt}\left(x=\frac{s}{2}\right) &= 20\sqrt{5} \\ \frac{dl}{dt}(x=s) &= 50\sqrt{2} \end{aligned}$$

Exercise 23.

$$\frac{dl}{dt} = \frac{x}{l}\frac{dx}{dt} = \frac{3}{5}12 = \boxed{\frac{36}{5}mi/hr}$$

Exercise 24. Given the preliminary information

$$\frac{r}{h} = \frac{2}{5} = \alpha,\, V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\alpha^2 h^3$$

(1)

$$\begin{aligned} V &= \frac{\pi r^2}{h^2}(h^2 y - hy^2 + \frac{1}{3}y^3) \\ \frac{dV}{dt} &= \frac{\pi r^2}{h^2}(h^2 - 2hy + y^2)\frac{dy}{dt} \\ \frac{dy}{dt} = \frac{h^2}{\pi r^2}\left(\frac{1}{h^2-2hy+y^2}\right)\frac{dV}{dt} &= \frac{10^2}{\pi 4^2}\left(\frac{1}{10^2-2(10)5+25}\right)5 = \frac{5}{4\pi} \end{aligned}$$

(2)

$$\frac{dV}{dt} = \pi\alpha^2 h^2\frac{dh}{dt},\, \frac{dh}{dt} = \frac{1}{\pi\alpha^2h^2}\frac{dV}{dt} = \frac{5}{4\pi}$$

Exercise 25.

$$\begin{aligned} \alpha &= \frac{r}{h} = \frac{3}{2} \\ \frac{dV}{dt} &= \pi\alpha^2h^2\frac{dh}{dt} \\ c - 1\pi\frac{9}{4}(2^2)4 &= 36\pi \Longrightarrow \boxed{c = 36\pi+1} \end{aligned}$$

Exercise 26. The constraint equation, using Pythagorean theorem on the geometry of a bottom hemisphere, is

$$r^2 = R^2 - (R-h)^2 = 2Rh - h^2$$

So then

$$\begin{aligned} r\frac{dr}{dt} &= (R-h)\frac{dh}{dt} \\ V = \int \pi r^2 dh \Longrightarrow \frac{dV}{dh} &= \pi r^2 = \pi(2Rh-h^2) \\ \Longrightarrow \frac{dV}{dh} &= \pi(2(10(5)-25)) = 50\pi \end{aligned}$$

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}, \Longrightarrow \frac{dh}{dt} = \frac{dV}{dt}\left(\frac{1}{\pi(2Rh - h^2)}\right)$$

$$\left(\frac{r}{R-h}\right)\frac{dr}{dt} = \frac{dV}{dt}\left(\frac{1}{\pi(2Rh-h^2)}\right)$$

$$\Longrightarrow \frac{dr}{dt} = \frac{dV}{dt}\left(\frac{R-h}{r\pi(2Rh-h^2)}\right) =$$

$$= (5\sqrt{3})\left(\frac{10-5}{\pi(2(10)5-25)^{3/2}}\right) = \frac{1}{15\pi}$$

Exercise 27. **I suppose the area of the triangle is 0 at $t=0$.**

Now the point on vertex B moves up along the y axis according to $y = 1+2t$. $y\left(\frac{7}{2}\right) = 8$.

$$A = \frac{1}{2}\sqrt{(y-1)\frac{36}{7}y}$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\sqrt{\frac{36}{y}}\frac{1}{2}\frac{1}{\sqrt{y-1}}y + \sqrt{(y-1)\frac{36}{7}}\right)\frac{dy}{dt} =$$

$$= \frac{1}{2}\left(\frac{6}{2(7)}8 + 6\right)(2) = \frac{66}{7}$$

Exercise 28. From the given information, $h = 3r+3$. The volume formula is $V = \frac{\pi R^2}{3}H$. So then

$$V = \pi/3r^2(3r+3) = \pi r^3 + \pi r^2$$

$$\frac{dV}{dr} = \pi r(3r+2)\frac{dr}{dt}$$

With the given information, we get

$$\frac{dr}{dt} = \frac{1}{\pi(6)(20)}$$

Using this, we can plug this back in for the different case:

$$\frac{dV}{dt} = n = \pi(36)(110)/(120\pi) = \boxed{33}$$

Exercise 29.

(1) $\frac{dy}{dt} = 2x\frac{dx}{dt}$; when $x = \frac{1}{2}, y = \frac{1}{4}, \frac{dy}{dt} = \frac{dx}{dt}$

(2) $\boxed{t = \frac{\pi}{6}}$

Exercise 30.

(1) $3x^2 + 3y^2y' = 0 \Longrightarrow x^2 + y^2y' = 0$

(2)

$$2x + 2yy'^2 + y^2y'' = 0 \Longrightarrow y^2y'' = -2(x+yy'^2)$$

$$\Longrightarrow y'' = -2\left(\frac{xy^4 + yx^4}{y^6}\right) = -2xy^{-5}$$

Exercise 31.

$$\frac{1}{2}\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0\, y' = \frac{-\sqrt{y}}{\sqrt{x}} < 0$$

Exercise 32.

$$\pm\sqrt{\frac{12-3x^2}{4}}$$

$$6x + 8yy' = 0 \Longrightarrow y' = \frac{-3x}{4y}$$
$$3 + 4(y'^2 + yy'') = 0$$
$$y'' = \left(-\frac{3}{4} - y'^2\right)\frac{1}{y} = \frac{-9}{4y^3}$$

Exercise 33.

$$\sin xy + x\cos^2 xy(y + xy') + 4x = 0$$
$$y'x^2 \cos xy + xy\cos xy + \sin xy + 4x = 0$$

Exercise 34. $y = x^4$. $y^n = x^m$.

$$y^n = x^m,\ y'ny^{n-1} = mx^{m-1};\ y' = \frac{mx^{m-1}}{ny^{n-1}} = \frac{m}{n}\frac{x^{m-1}}{x^{m(1-1/n)}} =$$
$$y' = \frac{m}{n}x^{m/n-1}$$

4.15 Exercises - Applications of differentiation to extreme values of functions, The mean-value theorem for derivatives.

Let's recap what was shown in the past two sections:

Theorem 13 (Theorem 4.3)**.**
Let f be defined on I.
Assume f has a rel. extrema at an int. pt. $c \in I$.
If $\exists\, f'(c),\ f'(c) = 0$; the converse is not true.

Proof. $Q(x) = \frac{f(x)-f(c)}{x-c}$ if $x \neq c$, $\quad Q(c) = f'(c)$
$\exists f'(c)$, so $Q(x) \to Q(c)$ as $x \to c$ so Q is continuous at c.
If $Q(c) > 0$, $\quad \frac{f(x)-f(c)}{x-c} > 0$. For $x - c \gtrless 0$, $\quad f(x) \gtrless f(c)$, thus contradicting the rel. max or rel. min. (no neighborhood about c exists for one!)
If $Q(c) < 0$, $\quad \frac{f(x)-f(c)}{x-c} < 0$. For $x - c \gtrless 0$, $\quad f(x) \lessgtr f(c)$, thus contradicting the rel. max or rel. min. (no neighborhood about c exists for one!)

Converse is not true: e.g. saddle points. □

Theorem 14 (Rolle's Theorem)**.**
Let f be cont. on $[a,b]$, $\quad \exists f'(x) \quad \forall x \in (a,b)$ and let

$$f(a) = f(b)$$

then $\exists$ at least one $c \in (a,b)$, such that $f'(c) = 0$.

Proof. Suppose $f'(x) \neq 0 \quad \forall x \in (a,b)$.
By extreme value theorem, $\exists$ abs. max (min) M, m somewhere on $[a,b]$.
M, m on endpoints a, b (Thm 4.3).
$F(a) = f(b)$, so $m = M$. f constant on $[a,b]$. Contradict $f'(x) \neq 0$ □

Theorem 15 (Mean-value theorem for Derivatives)**.** *Assume f is cont. everywhere on $[a,b]$, $\exists f'(x) \quad \forall x \in (a,b)$.*
$\exists$ at least one $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a) \tag{6}$$

Proof.

$$h(x) = f(x)(b-a) - x(f(b) - f(a))$$
$$h(a) = f(a)b - f(a)a - af(b) + af(a)$$
$$h(b) = f(b)(b-a) - b(f(b) - f(a)) = bf(a) - af(b) = h(a)$$
$$\Longrightarrow \exists c \in (a,b),\ \text{such that } h'(c) = 0 = f'(c)(b-a) - (f(b) - f(a))$$

□

Theorem 16 (Cauchy's Mean-Value Formula)**.** *Let f, g cont. on $[a,b]$, $\exists f',\ g' \quad \forall x \in (a,b)$*
Then $\exists\, c \in (a,b)$. x

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)) \quad \textit{(note how it's symmetrical)} \tag{7}$$

Proof.

$$\begin{aligned}
h(x) &= f(x)(g(b)-g(a)) - g(x)(f(b)-f(a))\\
h(a) &= f(a)(g(b)-g(a)) - g(a)(f(b)-f(a)) = f(a)g(b) - g(a)f(b)\\
h(b) &= f(b)(g(b)-g(a)) - g(b)(f(b)-f(a))\\
\Longrightarrow h'(c) &= f'(c)(g(b)-g(a)) - g'(c)(f(b)-f(a)) = 0 \quad \text{(by Rolle's Thm.)}
\end{aligned}$$

□

Exercise 1. For any quadratic polynomial $y = y(x) = Ax^2 + Bx + C$,

$$\begin{gathered}
y(a) = Aa^2 + Ba + C\\
y(b) = Ab^2 + Bb + C\\
\frac{y(b)-y(a)}{b-a} = \frac{A(b-a)(b+a) + B(b-a)}{b-a} = A(b+a) + B\\
y' = 2Ax + B\\
y'\left(\frac{a+b}{2}\right) = A(a+b) + B
\end{gathered}$$

Thus the chord joining a and b has the same slope as the tangent line at the midpt.

Exercise 2. *The contrapositive of a theorem is always true.* So the contrapositive of Rolle's Theorem is

If $\nexists$ at least one $c \in (a,b)$ s.t. $f'(c) = 0$,

then $f(a) \neq f(b)$.

$$\begin{gathered}
g' = 3x^2 - 3 = 3(x^2-1) \Longrightarrow g'(\pm 1) = 0\\
\text{Suppose } g(B) = 0, \qquad B \in (-1,1)\\
\text{then } \forall\, x \in (-1,1), \quad x \neq B, \qquad g(x) \neq g(B), \quad \text{so } g(x) \neq 0 \text{ for } x \neq B\\
\boxed{\text{so only at most one } B \in (-1,1) \text{ s.t. } g(B) = 0}
\end{gathered}$$

Exercise 3. $f(x) = \frac{3-x^2}{2}$ if $x \leq 1$, $\quad f(x) = \frac{1}{x}$ if $x \geq 1$.

(1) See sketch.

(2)

$$f(x) = \begin{cases} \frac{3-x^2}{2} & \text{if } x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases} \qquad f(1) = 1 = f(1) = 1/1$$

$$f'(x) = \begin{cases} -x; \quad f'(1) = -1 & \text{for } x \leq 1 \\ -1/x^2; \quad f'(1) = -1 & \text{for } x > 1 \end{cases}$$

Then $f(x)$ is cont. and diff. on $[0,2]$.

For $0 \leq a < b \leq 1$

$$\frac{\frac{3-b^2}{2} - \left(\frac{3-a^2}{2}\right)}{b-a} = \frac{-(a+b)}{2} = -c$$

Note that $-1 \leq f' \leq 0$ for $0 \leq x \leq 1$

For $1 \leq a < b \leq 2$

$$\frac{\frac{1}{b} - \frac{1}{a}}{b-a} = \frac{-1}{ab} = \frac{-1}{c^2} \Longrightarrow c = \sqrt{ab}$$

Note that $-1 \leq f' \leq -1/4$

For $0 \leq a \leq 1$, $\qquad 1 \leq b \leq 2$

$$\frac{\frac{1}{b} - \left(\frac{3-a^2}{2}\right)}{b-a} = \frac{2-(3-a^2)b}{2b(b-a)} = -c \text{ or } \frac{-1}{c^2}$$

depending upon if $0 \leq c \leq 1$ or $1 \leq c \leq 2$, respectively

For instance, for $a = 0,\ b = 2$, then $\frac{f(b)-f(a)}{b-a} = -1/2$, so $c = 1/2$ or $c = \sqrt{2}$

Exercise 4.

$$f(1) = 1 - 1^{2/3} = 0 = f(-1) = 1 - ((-1)^2)^3 = 0$$

$$f' = \frac{-2}{3}x^{-1/3} \neq 0 \quad \text{for } |x| \leq 1$$

This is possible since f is not differentiable at $x = 0$.

Exercise 5. $x^2 = x \sin x + \cos x$. $g = xS + C - x^2$. $g' = S + xC - S - 2x = xC - 2x = x(C-2)$. Since $|C| \leq 1$ then $(C-2)$ is negative for all x. Then for $x \gtrless 0$, $g' \lessgtr 0$. Since $g(0) = 1$ and for $x \to \pm\infty$, $g \to \mp\infty$, then we could conclude that g must become zero between 0 and ∞ and $-\infty$ and 0.

Exercise 6.

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\begin{aligned} b &= x + h \\ b - a &= h \\ a &= x \\ x < x + &\theta h < x + h \end{aligned} \qquad \Longrightarrow f(x+h) - f(x) = hf'(x + \theta h)$$

(1) $f(x) = x^2, \quad f' = 2x.$

$$(x+h)^2 - x^2 = 2xh + h^2 = h(2(x + \theta h))$$

$$\frac{2x+h}{2} - x = \theta h \Longrightarrow \boxed{\theta = \frac{1}{2}} \quad \text{so then } \lim_{h \to 0} \theta = \frac{1}{2}$$

(2) $f(x) = x^3, \quad f' = 3x^2.$

$$(x+h)^3 - x^3 = 3x^2 h + 3xh^2 + h^3 = h3(x + \theta h)^2 \Longrightarrow \left(\sqrt{\frac{3x^2 + 3xh + h^2}{3}} - x \right) / h = \theta$$

$$\theta = \sqrt{\frac{3x^2 + 3xh + h^2}{3h^2}} - \frac{x}{h} = \frac{\sqrt{x^2 + xh + \frac{h^2}{3}} - x}{h} \left(\frac{\sqrt{x^2 + xh + \frac{h^2}{3}} + x}{\sqrt{x^2 + xh + \frac{h^2}{3}} + x} \right) =$$

$$= \frac{x + \frac{h}{3}}{x + \sqrt{x^2 + hx + \frac{h^2}{3}}}$$

$$\Longrightarrow \lim_{h \to 0} \boxed{\theta = \frac{1}{2}}$$

Notice the trick of multiplying by the conjugate on top and bottom to get a way to evaluate the limit.

Exercise 7. $f(x) = (x - a_1)(x - a_2) \dots (x - a_r) g(x)$.

(1) $a_1 < a_2 < \dots < a_r$.
Since $f(a_1) = f(a_2) = 0. \qquad f'(c) = 0$ for $c_1 \in (a_1, a_2)$.
Consider that $f(a_2) = f(a_3) = 0$ as well as $f'(c_2) = 0$ for $c \in (a_2, a_3)$.
Indeed, since $f(a_j) = f(a_{j+1}) = 0$, $f'(c) = 0$ for $c \in (a_j, a_{j+1})$.
Thus, $\exists r - 1$ zero's.
$f^{(k)}$ has $r - k$ zeros in $[a, b]$.
$f^{(k)} = (x - a_1)(x - a_2) \dots (x - a_{r-k}) g_k(x)$
Since $f(a_1) = f(a_2) = 0, \qquad f^{(k+1)}(c_1) = 0$ for $c_1 \in (a_1, a_2)$.

$$f^{(k)}(a_j) = f^{(k)}(a_{j+1}) = 0, \quad f^{(k+1)}(c_j) = 0 \text{ for } c_j \in (a_j, a_{j+1})$$

$$\Longrightarrow f^{(k)}(x) \text{ has at least } r - k \text{ zeros in } [a, b]$$

We had shown the above by induction.

(2) We can conclude that there's at most $r + k$ zeros for f (since $f^{(k)}$ has exactly r zeros, the intervals containing the r zeros are definite).

Exercise 8. Using the mean value theorem

(1)

$$\frac{\sin x - \sin y}{x-y} = \cos c \quad \Longrightarrow \left|\frac{\sin x - \sin y}{x-y}\right| = |\cos c| \le 1$$
$$\Longrightarrow |\sin x - \sin y| \le |x-y|$$

(2) $x \ge y > 0$.
$f(z) = z^n$ is monotonically increasing for $n \in \mathbb{Z}$.
By mean-value theorem,

$$\frac{x^n - y^n}{x-y} = nc^{n-1} \text{ for } y < c < x$$

Since $0 < y < c < x$; $\quad ny^{n-1} \le \frac{x^n - y^n}{x-y} \le nx^{n-1}$.

Exercise 9. Let $g(x) = \left(\frac{f(b)-f(a)}{b-a}\right)x + \left(\frac{bf(a)-af(b)}{b-a}\right)$.

$$f - g = h \qquad \begin{matrix} h(a) = h(c) \\ h(c) = h(b) \end{matrix}$$

so $\exists c_1 \in (a,c)$, $c_2 \in (c,b)$ s. t. $h'(c_1) = h'(c_2) = 0$ by Rolle's Thm.

Let $h' = H$

since $H(c_1) = H(c_2) = 0$ and H is cont. diff. on (c_1, c_2). then

$\exists c_3 \in (c_1, c_2)$ s.t. $H'(c_3) = h''(c_3) = 0$

Now $h'' = (f-g)'' = f''$ so $f''(c_3) = 0$

We've shown one exists; that's enough.

Exercise 10. Assume f has a derivative everywhere on an open interval I.

$$g(x) = \frac{f(x) - f(a)}{x-a} \text{ if } x \ne a; \quad g(a) = f'(a)$$

(1) $g = \left(\frac{1}{x-a}\right) f - \frac{1}{x-a} f(a)$. f is cont. on $(a,b]$ since $\exists f' \quad \forall x \in (a,b)$.
$\frac{1}{x-a}$ is cont. on $(a,b]$. Then g is cont. on $(a,b]$ (remember, you can add, subtract, multiply, and divide cont. functions to get cont. functions because the rules for taking limits allow so).
g is cont. at a since $\lim_{x\to a} g = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$.

By mean value theorem,

$$\left(\frac{f(x) - f(a)}{x-a}\right) = f'(c) = g(x) \quad \forall c \in (a,x) \quad \forall x \in (a,b]$$

Then $\forall c \in [a,b]$, $f'(c)$ ranges from $f'(a)$ to $g(b)$ since $f'(c) = g(x)$ so whatever $g(x)$ ranges from and to, so does $f'(c)$.

(2) Let $h(x) = \frac{f(x)-f(b)}{x-b}$ if $x \ne b$; $\quad h(b) = f'(b)$.
h is cont. on $[a,b)$ since $\frac{1}{x-b}$ is cont., $f(x)$ is cont.
$\lim_{x\to b} h = \lim_{x\to b} \frac{f(x)-f(b)}{x-b} = f'(b)$ so h is cont. at b.

h is cont. on $[a,b] \to h$ takes all values from $h(a)$ to $f'(b)$ on $[a,b]$ (by intermediate value theorem).

By mean value theorem,

$$h(x) = \frac{f(b) - f(x)}{b-x} = f'(c_2) \quad \text{ for } c_2 \in (x,b) \quad \forall x \in [a,b]$$

So then f' ranges from $h(a)$ to $f'(b)$ just like h.

$h(a) = g(b)$. So then f' must range from $f'(a)$ to $f'(b)$

4.19 Exercises - Applications of the mean-value theorem to geometric properties of functions, Second-derivative test for extrema, Curve sketching.

Exercise 1. $f(x) = x^2 - 3x + 2$

(1) $f'(x) = 2x - 3 \quad x_0 = \frac{3}{2}$.
(2) $f'(x) \gtrless 0$ for $x \gtrless \frac{3}{2}$
(3) $f'' = 2 > 0$ for $\forall\, x \in \mathbb{R}$
(4) See sketch.

Exercise 2. $f(x) = x^3 - 4x$

(1) $f' = 3x^2 - 4 \qquad x_c = \pm\frac{2}{\sqrt{3}}$
(2) $f' \gtrless 0$ when $|x| \gtrless \frac{2}{\sqrt{3}}$
(3) $f'' = 6x$ $f'' \gtrless 0$ when $x \gtrless 0$
(4) See sketch.

Exercise 3. $f(x) = (x-1)^2(x+2)$

(1) $f' = 3(x-1)(x+1) \qquad f'(x) = 0$ when $x = \pm 1$
(2) $f' \gtrless$ when $|x| \gtrless 1$
(3) $f'' = 3(2x) = 6x \qquad f'' \gtrless 0$ when $x \gtrless 0$
(4) See sketch.

Exercise 4. $f(x) = x^3 - 6x^2 + 9x + 5$

(1) $f' = 3x^2 - 12x + 9 = 3(x-3)(x-1) \qquad f'(x) = 0$ when $x = 3, 1$
(2)
$$f'(x) > 0 \text{ when } x < 1,\ x > 3$$
$$f'(x) < 0 \text{ when } 1 < x < 3$$
(3) $f'' = 6x - 12 = 6(x-2) \qquad f'' \gtrless 0$ when $x \gtrless 2$
(4) See sketch.

Exercise 5. $f(x) = 2 + (x-1)^4$

(1) $f'(x) = 4(x-1)^3$. $f'(0) = 0$ when $x = 1$
(2) $f'(x) \gtrless 0$ when $|x| \gtrless 1$
(3) $f''(x) = 12(x-1)^2 > 0 \quad \forall\, x \neq 1$
(4) See sketch.

Exercise 6. $f(x) = 1/x^2$

(1) $f' = \frac{-2}{x^3}$ $f'(x) = 0$ for no x
(2) $f' \gtrless 0$ when $x \lessgtr 0$
(3) $f'' = \frac{6}{x^4} > 0 \qquad \forall x \neq 0$
(4) See sketch.

Exercise 7. $f(x) = x + 1/x^2$

(1) $f' = 1 + \frac{-2}{x^3}$ $f'(x) = 0 = 1 - \frac{2}{x^3} \Longrightarrow x_c = 2^{1/3}$
(2)
$$f'(x) > 0 \text{ when } x < 0,\ 0 < x < 2^{1/3}$$
$$f'(x) < 0 \text{ when } x > 2^{1/3}$$
(3) $f'' = \frac{6}{x^4} > 0 \quad \forall x \neq 0$
(4) See sketch.

Exercise 8. $f(x) = \frac{1}{(x-1)(x-3)}$

(1) $f' = \frac{-1}{(x-1)^2(x-3)^2}((x-3) + x - 1) = \frac{(-2)(x-2)}{(x-1)^2(x-3)^2}$
$f'(x) = 0$ when $x = 2$
(2) $f' \gtrless 0$ when $x \lessgtr 2$

(3)

$$f'' = (-2)\left(\frac{(x-1)^2(x-3)^2 - (x-2)(2(x-1)(x-3)^2 + 2(x-3)(x-1)^2)}{(x-1)^4(x-3)^4}\right) =$$

$$= (6)\left(\frac{x^2 - 4x + \frac{13}{3}}{(x-1)^3(x-3)^3}\right)$$

$$x^2 - 4x + \frac{13}{3} > 0 \text{ since } 144 - 4(-3)(-13) = 144 + 12(-13) < 0 \text{ so}$$

$$f'' > 0 \text{ if } x > 3,\ x < 1$$

$$f'' < 0 \text{ if } 1 < x < 3$$

(4) See sketch.

Exercise 9. $f(x) = x/(1+x^2)$

(1)

$$f' = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$f'(x) = 0$ when $x = \pm 1$

(2) $f' \gtrless 0$ when $|x| \lessgtr 1$

(3)

$$f'' = \frac{-2x(1+x^2)^2 - 2(1+x^2)(2x)(1-x^2)}{(1+x^2)^4} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

$$f'' > 0 \text{ when } x > \sqrt{3}$$

$$f'' < 0 \text{ when } 0 < x < \sqrt{3}$$

$$f'' > 0 \text{ when } -\sqrt{3} < x < 0$$

$$f'' < 0 \text{ when } x < -\sqrt{3}$$

(4) See sketch.

Exercise 10. $f(x) = (x^2-4)/(x^2-9)$

(1)

$$f' = \frac{2x(x^2-9) - (x^2-4)(2x)}{(x^2-9)^2} = \frac{-10x}{(x^2-9)^2}$$

$f'(0) = 0$

(2) $f' \gtrless 0$ when $x \lessgtr 0, \qquad x \neq \pm 3$

(3)

$$f'' = (-10)\left(\frac{(x^2-9)^2 - 2(x^2-9)(2x)x}{(x^2-9)^2}\right) = (30)\frac{(x^2+3)}{(x^2-9)^3}$$

$f'' \gtrless 0$ when $|x| \gtrless 3$

(4) See the sketch.

Exercise 11. $f(x) = \sin^2 x$

(1) $f' = \sin 2x$ So then $f' = 0$ when $x = \frac{\pi}{2}n$

(2)

$$f' > 0 \text{ when } \begin{array}{c} 0 < x < \frac{\pi}{2} \\ \pi n < x < \frac{\pi}{2} + \pi n \end{array}$$

$$f' < 0 \text{ when } \frac{\pi}{2} + \pi n < x < \pi(n+1)$$

(3)

$$f'' = 2\cos 2x \qquad \begin{array}{l} f'' > 0 \text{ when } \frac{-\pi}{4} + \pi n < x < \frac{\pi}{4} + \pi n \\ f'' < 0 \text{ when } \frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n \end{array}$$

(4) See sketch.

Exercise 12. $f(x) = x - \sin x$

(1) $f' = 1 - \cos x$ $\quad\quad f' = 0$ when $x = 2\pi n$
(2) $f' > 0$ if $x \neq 2\pi n$
(3)

$$f'' = \sin x \qquad \begin{array}{l} f'' > 0 \text{ when } 2\pi n < x < 2\pi n + \pi \\ f'' < 0 \text{ when } 2\pi n + \pi < x < 2\pi(n+1) \end{array}$$

(4) See sketch.

Exercise 13. $f(x) = x + \cos x$

(1) $f' = 1 - \sin x$ $\quad\quad x = \frac{\pi}{2} + 2\pi n$ $\quad\quad f'(x) = 0$
(2) $f' > 0$ if $x \neq \frac{\pi}{2} + 2\pi n$
(3)

$$f'' = -\cos x \qquad \begin{array}{l} f'' > 0 \text{ when } \dfrac{-\pi}{2} + 2\pi n < x < \dfrac{\pi}{2} + 2\pi n \\ f'' < 0 \text{ when } \dfrac{\pi}{2} + 2\pi n < x < \dfrac{3\pi}{2} + 2\pi n \end{array}$$

(4) See sketch.

Exercise 14. $f(x) = \frac{1}{6}x^2 + \frac{1}{12}\cos 2x$

(1) $f' = \frac{1}{3}x + \frac{-\sin 2x}{6}$ $\quad\quad f'(0) = 0$
(2) $f' \gtrless 0$ when $x \gtrless 0$
(3) $f'' = \frac{1}{3} - \frac{\cos 2x}{3} = \frac{1 - \cos 2x}{3}$
$x = \pi n$ for $f'' = 0$. Otherwise $f'' > 0$ for $x \neq \pi n$
(4) See sketch.

4.21 Exercises - Worked examples of extremum problems.

Exercise 1.

$$A = xy$$
$$P = 2(x+y) = 2(x + \frac{A}{x})$$
$$P' = 2(1 - \frac{A}{x^2}) = 0$$
$$x = \sqrt{A}$$
$$P'' = \frac{4A}{x^3} > 0 \text{ for } x > 0 \text{ so } x = \sqrt{A} \text{ minimizes } P$$

Exercise 2.

$$A = xy \qquad L = 2x + y$$
$$A = x(L - 2x) = Lx - 2x^2 \implies \frac{dA}{dx} = L - 4x = 0 \text{ when } x = \frac{L}{4} \qquad y = \frac{L}{2}$$
$$A'' = -4 \text{ so } x = \frac{L}{4} \text{ maximizes } A$$

Exercise 3.

$$A = xy \qquad L = 2x + y = 2x + \frac{A}{x} \qquad \frac{dL}{dx} = 2 + \frac{-A}{x^2} = 0 \text{ when } x = \frac{\sqrt{A}}{\sqrt{2}} \qquad y = \sqrt{2}\sqrt{A}$$
$$L'' = \frac{2A}{x^3} > 0 \text{ for } x = \sqrt{\frac{A}{2}} \qquad \text{so } x \text{ minimizes } L$$

Exercise 4. $f = x^2 + y^2 = x^2 + (S - x)^2$

$f' = 2x + 2(S - x)(-1) = -2S + 4x$ $\quad\quad \implies x = \frac{S}{2}$
$f'' = 4 > 0$ so $x = \frac{S}{2}$ minimizes f

Exercise 5. $x^2 + y^2 = R > 0$

$f = x + y$

$$f' = 1 + y' = 0 = 1 + \frac{-x}{y} = 0 \qquad \Longrightarrow \boxed{y = x}$$

$$f'' = y'' = \frac{-1 - y'^2}{y} \text{ for } y > 0,\ f'' < 0 \text{ so that } f \text{ is max. when } y = x$$

$$\begin{aligned} & 2x + 2yy' = 0 \\ \text{Note that} \qquad & y' = \frac{-x}{y} \qquad 1 + y'^2 + yy'' = 0 \Longrightarrow yy'' = -1 - y'^2 \\ & x + yy' = 0 \end{aligned}$$

Exercise 6.

$$\begin{gathered} l^2 = (L - x)^2 + x^2 = L^2 - 2Lx + 2x^2 = A \\ \frac{dA}{dx} = -2L + 4x = 0 \Longrightarrow x = \frac{L}{2} \\ \frac{d^2A}{dx^2} = 4 > 0 \Longrightarrow A \text{ minimized} \\ l(x = \frac{L}{2}) = \frac{L\sqrt{2}}{2} \end{gathered}$$

Exercise 7.

$$\begin{gathered} (x + \sqrt{L^2 - x^2})^2 = A \\ A' = 2(x + \sqrt{L^2 - x^2})(1 + \frac{-x}{\sqrt{L^2 - x^2}}) = 0 \text{ when } L^2 - x^2 = x^2 \text{ or } x = \frac{L}{\sqrt{2}} \\ \text{so then the side of the circumscribing and area-maximized square is } \frac{L}{\sqrt{2}} + \sqrt{L^2 - \frac{L^2}{2}} = \frac{2L}{\sqrt{2}} \end{gathered}$$

Exercise 8.

$$\begin{gathered} A = (2x)(2\sqrt{R^2 - x^2}) = 4x\sqrt{R^2 - x^2} \\ A' = 4(\sqrt{R^2 - x^2} + \frac{-x^2}{\sqrt{R^2 - x^2}}) = 4\left(\frac{R^2 - 2x^2}{\sqrt{R^2 - x^2}} \right) \Longrightarrow x = \frac{R}{\sqrt{2}} \\ \text{since } A' \gtrless 0 \text{ when } x \lessgtr \frac{R}{\sqrt{2}}, \text{ so } A \text{ is maximized at } x = \frac{R}{\sqrt{2}} \end{gathered}$$

$2x = \frac{2R}{\sqrt{2}}$; $\qquad 2\sqrt{R^2 - x^2} = \frac{2R}{\sqrt{2}}$ so then the rectangle that has maximum size is a square.

Exercise 9. Prove that among all rectangles of a given area, the square has the smallest circumscribed circle.

$A_0 = (2x)(2\sqrt{r^2 - x^2}) = 4x\sqrt{r^2 - x^2}$ (fix the area to be A_0)
$\left(\frac{A_0}{4x}\right)^2 = r^2 - x^2 \Longrightarrow x^4 - x^2r^2 + \frac{A_0^2}{16} = 0$

$$\begin{gathered} \Longrightarrow 0 = 2xr^2 + x^2 2r\frac{dr}{dx} - 4x^3 \\ \frac{dr}{dx} = 0 \text{ (for extrema) } \Longrightarrow x = \frac{r}{\sqrt{2}} \text{ and } \sqrt{r^2 - x^2} = \frac{r}{\sqrt{2}} \end{gathered}$$

We could argue that we had found a minimum because at the "infinity" boundaries, the circumscribing circle would be infinitely large.

Exercise 10. Given a sphere of radius R, find the radius r and altitude h of the right circular cylinder with the largest lateral surface area $2\pi rh$ that can be inscribed in the sphere.

$$R^2 = \left(\frac{h}{2}\right)^2 + r^2$$

$$A = 2\pi rh = 2\pi r\sqrt{4(R^2 - r^2)} = 4\pi r\sqrt{R^2 - r^2}$$

$$\frac{dA}{dr} = 4\pi\left(\sqrt{R^2 - r^2} + \frac{-r^2}{\sqrt{R^2 - r^2}}\right) = 4\pi\left(\frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}}\right) \Longrightarrow \boxed{r = \frac{R}{\sqrt{2}}}$$

$$\Longrightarrow \boxed{h = \sqrt{2}R}$$

Exercise 11. Among all right circular cylinders of given lateral surface area, prove that the smallest circumscribed sphere has radius $\sqrt{2}$ times that of the cylinder.

$$A_0 = 2\pi RH \qquad (A_0 \text{ is the total lateral area of the cylinder})$$

$$r^2 = R^2 + \left(\frac{H}{2}\right)^2 = R^2 + \left(\frac{A_0}{4\pi R}\right)^2 = R^2 + \frac{A_0^2}{16\pi^2 R^2}$$

$$2r\frac{dr}{dR} = 2R + \frac{A_0^2}{16\pi^2}\left(\frac{-2}{R^3}\right) \Longrightarrow \frac{dr}{dR} = 0 \Longrightarrow R = \frac{\sqrt{A_0}}{2\sqrt{\pi}} \Longrightarrow \frac{H}{2} = R$$

$$r^2 = R^2 + R^2 = 2R^2 \Longrightarrow \boxed{r = \sqrt{2}R}$$

Exercise 12. Given a right circular cone with radius R and altitude H. Find the radius and altitude of the right circular cylinder of largest lateral surface area that can be inscribed in the cone.

$\frac{h}{R-r} = \frac{H}{R} = \alpha$ is the constraint (look, directly at the side, at the similar triangles formed)

$$A = 2\pi rh = 2\pi r\alpha(R - r) = 2\pi\alpha(rR - r^2)$$

$$\frac{dA}{dr} = 2\pi\alpha(R - 2r) = 0 \Longrightarrow \boxed{r = \frac{R}{2}}; \quad \boxed{h = \frac{H}{2}}$$

$$A'' = 2\pi\alpha(R - 2r)$$

$$\text{since } \frac{dA}{dr} \gtrless 0 \text{ when } r \lessgtr \frac{R}{2}, \quad r = \frac{R}{2} \text{ maximizes lateral surface area}$$

Exercise 13. Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius R and altitude H.

Constraint: $\frac{h}{R-r} = \frac{H}{R} = \alpha$

$$V = \pi r^2 h = \pi r^2\alpha(R - r) = \pi r^2\alpha(R - r) = \pi\alpha(Rr^2 - r^3)$$

$$\frac{dV}{dr} = \pi\alpha(2Rr - 3r^2) = r\pi\alpha(2R - 3r) \qquad \boxed{r = \frac{2R}{3}}$$

$$\text{since } \frac{dV}{dr} \gtrless 0 \text{ when } r \lessgtr \frac{2R}{3}, \quad r = \frac{2R}{3} \text{ maximizes volume}$$

$$\boxed{h = \frac{1}{3}H}$$

Exercise 14. Given a sphere of radius R. Compute, in terms of R, the radius r and the altitude h of the right circular cone of maximum volume that can be inscribed in this sphere.

$$V = \frac{\pi r^2}{3}\left(R + \sqrt{R^2 - r^2}\right)$$

$$\frac{dV}{dr} = \frac{\pi}{3}\left(2rR + 2r\sqrt{R^2 - r^2} + \frac{r^2(-r)}{\sqrt{R^2 - r^2}}\right) = \frac{\pi}{3}r\frac{(2R\sqrt{R^2 - r^2} + 2R^2 - 3r^2)}{\sqrt{R^2 - r^2}} = 0$$

$$\Longrightarrow \boxed{r = \frac{2\sqrt{2}R}{3}; \quad h = \frac{4R}{3}}$$

Considering the geometric or physical constraints, since $\lim_{V\to\infty} V = \lim_{h\to\infty} V = 0$, so then $r = \frac{2\sqrt{2}R}{3}$ must maximize V.

Exercise 15. Find the rectangle of largest area that can be inscribed in a semicircle, the lower base being on the diameter.

$$A = \sqrt{R^2 - x^2}x$$

$$A' = \sqrt{R^2 - x^2} + \frac{-x^2}{\sqrt{R^2 - x^2}} = 0 \Longrightarrow \boxed{x = \frac{R}{\sqrt{2}}; \qquad h = \frac{R}{\sqrt{2}}}$$

Exercise 16. Find the trapezoid of largest area that can be inscribed in a semicircle, the lower base being on the diameter.

$$\begin{aligned} A &= \frac{1}{2}h(2\sqrt{R^2 - h^2} + 2R) \\ \frac{dA}{dh} &= \sqrt{R^2 - h^2} + R + h\left(\frac{-h}{\sqrt{R^2 - h^2}}\right) \\ \frac{dA}{dh} &= 0 \Longrightarrow \boxed{h = \frac{\sqrt{3}R}{2}} \\ \Longrightarrow A = \frac{5\sqrt{3}R^2}{8} \qquad & \sqrt{R^2 - h^2} = 2\sqrt{R^2 - \frac{3}{4}R^2} = 2\frac{R}{2} = \boxed{R} \end{aligned}$$

Exercise 17. An open box is made from a rectangular piece of material by removing equal squares at each corner and turning up the sides. Find the dimensions of the box of largest volume that can be made in this manner if the material has sides (a) 10 and 10; (b) 12 and 18

(1)

$$\begin{aligned} (x - 2r)(Y - 2r)r = (xy - 2rx - 2ry + 4r^2)r &= xyr - 2r^2x - 2r^2y + 4r^3 = V \\ \frac{dV}{dr} &= xy - 4rx - 4ry + 12r^2 = 0 \\ \Longrightarrow r = \frac{4(x+y) \pm \sqrt{16(x+y)^2 - 4(12)xy}}{2(12)} &= \frac{(x+y) \pm \sqrt{x^2 + y^2 - xy}}{6} \\ \frac{d^2V}{dr^2} &= -4x - 4y + 24r = -4(x+y) + 24r \end{aligned}$$

We can plug in our expression for r into the second derivative of V, the volume of the box, to find out that we want to pick the "negative" root from r, in order to maximize the box volume.

Then for $x = 10$; $y = 10$, we have $r = \frac{5}{3}$, so that the box dimensions are $\frac{5}{3} \times \frac{20}{3} \times \frac{20}{3}$.

(2) 12 and 18

$\Longrightarrow 5 - \sqrt{7} \times 2 + 2\sqrt{7} \times 8 + 2\sqrt{7}$

Exercise 18. If a and b are the legs of a right triangle whose hypotenuse is 1, find the largest value of $2a + b$.

$$L = 2a + b = 2a + \sqrt{1 - a^2} \qquad L' = 0 \Longrightarrow \left(\frac{a}{2}\right)^2 = 1 - a^2 \Longrightarrow a = \frac{2}{\sqrt{5}}$$

$$L' = 2 + \frac{-a}{\sqrt{1 - a^2}}$$

$$L'' = (-1)\left(\frac{\sqrt{1 - a^2} - \frac{-a}{\sqrt{1-a^2}}a}{1 - a^2}\right) = (-1)\left(\frac{1}{(1 - a^2)^{3/2}}\right) < 0 \text{ (so } a = \frac{2}{\sqrt{5}} \text{ maximizes } L \text{)}$$

Exercise 19. $2 + \frac{x^2}{600}$ gallons per hour. $l_0 = 300\, mi$ $x =$ constant speed. $\frac{l_0}{x} =$ time spent. $K =$ gas cost $= 0.30$.

$C =$ gas cost $+$ driver labor cost $= l_0\left(\frac{2K}{x} + \frac{Kx}{600} + \frac{D}{x}\right)$

$$\frac{dC}{dx} = l_0\left(\frac{-2K}{x^2} + \frac{K}{600} - \frac{D}{x^2}\right) = 0 \xrightarrow{\frac{dC}{dx}=0} x = \sqrt{\frac{2K+D}{K}}10\sqrt{6}$$
$$\frac{d^2C}{dx^2} = l_0\left((-2K-D)\left(\frac{-2}{x^3}\right)\right) = l_0\left(\frac{2(2K+D)}{x^3}\right) > 0$$
$$\text{Thus, } C \text{ is minimized if } x = \sqrt{\frac{2K+D}{K}}10\sqrt{6}$$
$$\Longrightarrow C_{min} = (300)\left(\frac{\sqrt{2K+D}\sqrt{K}}{10\sqrt{6}} + \frac{\sqrt{K}\sqrt{2K+D}10\sqrt{6}}{600}\right) = \boxed{3\sqrt{2}\sqrt{6+10D}}$$

Remember that there is a **speed limit** of $60\, mi/hr$.

(1) $D = 0$, $x = 20\sqrt{3}$ $C = 6\sqrt{3} \approx 10.39$
(2) $D = 1$, $x = 40\sqrt{2}$ $C = 12\sqrt{2} \approx 16.97$
(3) $D = 2$, $x = 60$ (because of the speed limit) $C = 300\left(\frac{2K}{60} + \frac{K(60)}{600} + \frac{D}{60}\right) = 5(2.4 + D) = 22.00$
(4) $D = 3$, $x = 60$ $C = 27.00$
(5) $D = 4$, $x = 60$ $C = 32.00$

Exercise 20. $y = \frac{x}{x^2+1}$ Suppose the rectangle starts at x_0 on the x axis. Then its y coordinate intersecting the curve, and thus the height of rectangle, must be $y_0 = \frac{x_0}{x_0^2+1}$

$$\Longrightarrow x_0 = \frac{1}{2y_0} \pm \sqrt{\frac{1}{(2y_0)^2} - 1}$$
$$x_2 - x_1 = \sqrt{\frac{1}{y_0^2} - 4}$$

where $x_2 - x_1$ is going to be the base of the rectangle. The volume of the cylinder, V, which is obtained from revolving the rectangle about the x axis, is going to be

$$V = \pi y_0^2(x_2 - x_1) = \pi y_0^2\left(\sqrt{\frac{1}{y_0^2} - 4}\right) = \pi y_0\sqrt{1 - 4y_0^2}$$
$$\frac{dV}{dy_0} = \pi\left(\sqrt{1-4y_0^2} + \frac{y_0}{2\sqrt{1-4y_0^2}}(-8y_0)\right) = \pi\left(\frac{1-8y_0^2}{\sqrt{1-4y_0^2}}\right) \qquad \Longrightarrow y_0 = \frac{1}{2\sqrt{2}}$$

We could argue that V is maximized, since the "infinite" boundaries would yield a volume of 0 (imagine stretching and squeezing the rectangle inside the curve).

Then $V_{max} = \pi\frac{1}{8}2 = \boxed{\frac{\pi}{4}}$

Exercise 21. Draw a **good diagram**. Note how the right triangle that you folded is now *reflected backwards*, so that this triangle's right angle is on the left-hand side now.

The constraint is that the crease touches the left edge.

$$\begin{aligned} w_0 &= l\sin\alpha + l\sin\alpha\cos(2\alpha) = l\sin\alpha(1+\cos(2\alpha)) = \\ &= 2l\sin\alpha\cos^2\alpha \end{aligned}$$

Note that we will obtain a minimum crease because by considering the "physical infinite" boundary, we could make a big crease along the vertical half of the paper or the horizontal half of the paper.

So, isolating l, the length of the crease, and then taking the derivative,

$$l = \frac{w_0}{2\sin{(\alpha)}\cos^2{\alpha}} = \frac{w_0}{2}\csc{(\alpha)}\sec^2{(\alpha)}$$

$$\begin{aligned} \frac{dl}{d\alpha} &= \frac{w_0}{2}\left(-\cot{\alpha}\csc{\alpha}\sec^2{\alpha} + \csc{\alpha}2\sec{\alpha}\sec{\alpha}\tan{\alpha}\right) = \\ &= \frac{w_0}{2}\left(\frac{-C}{S}\left(\frac{1}{S}\right)\left(\frac{1}{C^2}\right) + \frac{1}{S}2\left(\frac{1}{C^2}\right)\left(\frac{S}{C}\right)\right) = \frac{w_0}{2}\left(\frac{-1}{S^2C} + \frac{2}{C^3}\right) \\ &\xrightarrow{\frac{dl}{d\alpha}=0} \boxed{\sin{\alpha} = \frac{1}{\sqrt{3}}} \text{ or } \tan{\alpha} = \frac{1}{\sqrt{2}} \end{aligned}$$

where α is the angle of the crease. The corresponding minimum length of the crease will be

$$l = \frac{w_0}{2}\frac{1}{\frac{1}{\sqrt{3}}\frac{2}{3}} = \boxed{\frac{9\sqrt{3}}{2}}$$

Exercise 22.

(1) Consider the center of the circle O, the apex of the isosceles triangle that makes an angle 2α, A, and one of its other vertices, B. Draw a line segment from O to B and simply consider the two triangles making up one half of the isosceles triangle. Find all the angles.

Angle AOB is $\pi - 2\alpha$ by the geometry or i.e. inspection of the figure. The complement of that angle is 2α. Beforehand, we can get the length of the isosceles triangle leg from the law of cosines.

$$\cos{(\pi - 2\alpha)} = -\cos{(2\alpha)}$$

$$\begin{aligned} s^2 &= R^2 + R^2 - 2R^2\cos{(\pi - 2\alpha)} + 2R^2(1+\cos{(2\alpha)}) = 2R^2(2\cos^2{\alpha}) = 4R^2\cos^2{\alpha} \\ s &= 2R\cos{\alpha} \end{aligned}$$

The constraint equation is

(8) $$P = 4R\cos{\alpha} + 2R\sin{(2\alpha)}$$

So then

$$P' = 4R(-\sin{\alpha}) + 4R\cos{(2\alpha)} = 0 \Longrightarrow \cos{2\alpha} = \sin{\alpha}$$

$$\sin{\alpha} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4(1)(-\frac{1}{2})}}{2(1)} = \frac{1}{2} > 0$$

$$\Longrightarrow \boxed{P = 3\sqrt{3}R}$$

$P = 3\sqrt{3}$ is a max because

Look at the "boundary conditions" imposed on P by the physical-geometry. $\alpha = 0$, triangle is completely flattened, $\alpha = \pi$, triangle "completely disappears."

(2) I had originally thought to Reuse the constraint equation, Eqn. (**??**). *This is wrong!*

Think about the problem directly and for what it actually is; less wishful thinking.
Consider a fixed perimeter L and imagine L to be a string that can be stretched into an isoceles triangle. A "trivial" isoceles triangle is a collapsed triangle with two sides of length $L/2$ only. Then the radius of the disk needs to be $L/4$.

Consider a general isosceles triangle with 2α as the vertex angle and isosceles sides of h. The perimeter for this triangle, P, is then

$$P = 2h + 2h\sin{\alpha} = 2h(1+\sin{\alpha})$$

$$\Longrightarrow h = \frac{P}{2(1+\sin{\alpha})}$$

$$\frac{h}{2} = R\cos{\alpha}$$

We could try to extremize this equation.

$$\frac{dR}{d\alpha}(4\cos\alpha + 2\sin(2\alpha)) + R(-4\sin\alpha + 4\cos(2\alpha)) = 0$$

$$\frac{dR}{d\alpha} = 0 \Longrightarrow \cos(2\alpha) = \sin\alpha \Longrightarrow \sin\alpha = \frac{1}{\sqrt{3}} \qquad \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$$

$$R = \frac{3P}{4\sqrt{2}(\sqrt{3}+1)}$$

However, this is the *minimized* R, *minimized* radius for the smallest circle fitting a particular isosceles triangle of a fixed perimeter. We want to smallest circle with a radius big enough to fit all the possible triangles. Thus $R = \frac{L}{4}$

Exercise 23. The constraint equation on perimeter is

$$P = 2h + W + \pi\left(\frac{W}{2}\right) = 2h + W(1 + \frac{\pi}{2})$$

Then intensity function, "normalized" is given by

$$I = Wh + \frac{\pi}{2}\left(\frac{W}{2}\right)^2\left(\frac{1}{2}\right) = \frac{Wp}{2} - \frac{W^2}{2} - \frac{3\pi}{16}W^2$$

So then

$$\frac{dI}{dW} = \frac{P}{2} - W - \frac{3\pi}{8}W = 0 \Longrightarrow W = \boxed{\frac{P}{2+\frac{3\pi}{4}}}$$

The height of the rectangle is

$$h = \frac{P - \frac{P}{2+\frac{3\pi}{4}}(1+\frac{\pi}{2})}{2} = P\left(\frac{4+\pi}{16+6\pi}\right)$$

Exercise 24. A log 12 feet long has the shape of a frustum of a right circular cone with diameters 4 feet and $(4+h)$ feet at its ends, where $h \geq 0$. Determine, as a function of h, the volume of the largest right circular cylinder that can be cut from the log, if its axis coincides with that of the log.

Remember to **label your diagram carefully**.

$$\frac{y}{x} = \frac{\frac{h}{2} - y}{l_0 - x} = \frac{h/2}{l_0} \Longrightarrow \frac{h/2 - \left(\frac{xh}{2l_0}\right)}{l_0 - x}$$

$$V = \pi(H+y)^2(l_0 - x) = \pi(H + \frac{xh}{2l_0})^2(l_0 - x)$$

Note that $V(x=0) = \pi H^2 l_0 = \pi 4(12)$

$$\begin{aligned}\frac{dV}{dx} &= \pi(2(H + \frac{hx}{2l_0})\frac{h}{2l_0}(l_0 - x) + (H + \frac{xh}{2l_0})^2(-1)) = \\ &= \pi(H + \frac{hx}{2l_0})(h - H - \frac{3xh}{2l_0})\end{aligned} \qquad \xrightarrow{\frac{dV}{dx}=0} x = \frac{(h-H)2l_0}{3h}$$

$$V(x = \frac{(h-H)2l_0}{3h}) = \pi\left(l_0 - \frac{(h-H)2l_0}{3h}\right)\left(H + \frac{h}{2l_0}\frac{(h-H)2l_0}{3h}\right)^2 = \boxed{\pi l_0 \frac{(h+2H)^3}{27h}}$$

where $H = 2$, $l_0 = 12$

Exercise 25.

$$S = \sum_{k=1}^n (x - a_k)^2 \Longrightarrow \frac{dS}{dx} = \sum_{k=1}^n 2(x - a_k) = 0$$

$$\Longrightarrow nx = \sum_{k=1}^n a_k \Longrightarrow x = \frac{\sum_{k=1}^n a_k}{n}$$

Since $\lim_{x\to\pm\infty} S = +\infty$, $x = \frac{\sum_{k=1}^n a_k}{n}$ minimizes S.

Exercise 26. **Hint: draw a picture** . Then observe that for $f(x) \geq 24$, A must be greater than 0 (we'll show that explicitly soon) and that f must have a *minimum* somewhere.

If $A < 0$, then consider $f(x) = \frac{5x^7+A}{x^5}$. Consider $x = \frac{-A^{1/7}}{6^{1/7}} > 0$.

$$f(x = \frac{-A^{1/7}}{6^{1/7}} = \frac{A}{6x^5} < 0$$

Thus, $A > 0$.

$$\begin{aligned} \frac{df}{dx} &= 10x - 5Ax^{-6} = 5\left(\frac{2x^7 - A}{x^6}\right) = 0 \\ x &= \left(\frac{A}{2}\right)^{1/7} \end{aligned}$$

$$\frac{d^2 f}{dx^2} = 10 + 30Ax^{-7} = 10 + 3 - A\left(\frac{2}{A}\right) = 70 > 0$$

Thus $x = \left(\frac{A}{2}\right)^{1/7}$ minimizes f for $A > 0$.

$$f(x = \left(\frac{A}{2}\right)^{1/7}) = \frac{5\left(\frac{A}{2}\right) + A}{\left(\frac{A}{2}\right)^{5/7}} = 24$$

$$\implies \boxed{A = 2\left(\frac{24}{7}\right)^{7/2}}$$

Exercise 27. Consider $f(x) = -\frac{x^3}{3} + t^2 x$ over $0 \leq x \leq 1$.

$$\begin{aligned} f(0) = 0, \ f(1) = -\frac{1}{3} + t^2 &\implies f(1) \gtrless 0 \text{ if } t^2 \gtrless \frac{1}{3} \\ f'(x) = -x^2 + t^2 = 0 & \\ &\implies x^2 = t^2 \text{ but } x \geq 0, \ \text{ so } x = |t| \\ f(x^2 = t^2) = -\frac{1}{3}t^2(x) + t^2 x &= \frac{2}{3}t^2 x > 0 \text{ for } 1 \geq x \geq 0 \end{aligned}$$

So the minimum isn't in the interior of $[0,1]$. It's on the end points.

$$\boxed{m(t) = 0 \text{ for } |t| > \frac{1}{3} \, m(t) = \frac{-1}{3} + t^2 \text{ for } |t| < \frac{1}{3}}$$

Exercise 28.

(1)

$$\begin{aligned} E(x,t) &= \frac{|t-x|}{x} \\ M(t) = \max \frac{|t-x|}{x} & \text{ as } x = a \to x = b \\ \frac{|t-x|}{x} &= \begin{cases} \frac{t-x}{x} & \text{if } t \geq x \\ \frac{x-t}{x} & \text{if } t < x \end{cases} \\ \frac{d}{dt}\left(\frac{|t-x|}{x}\right) &= \begin{cases} -\frac{t}{x^2} & \text{if } t \geq x \\ \frac{t}{x^2} & \text{if } t < x \end{cases} \end{aligned}$$

Now $t, x \geq a > 0$ (this is an important, given, fact). So $x = t$ should be a relative minimum.

So the maximum occurs at either endpoints

$$\frac{t-a}{a} = E(a,t), \ \frac{b-t}{b} = E(b,t)$$

By monotonicity on $[a,t)\,,(t,b]$, and having shown the relative minimum of $\frac{|t-x|}{x}$ at $x = t$, the maximum occurs at $x = a$ or $x = b$, depending upon the relationship $E(a,t) \gtrless E(b,t)$.

(2)

$$M(t) = \begin{cases} \frac{t-a}{a} & \text{if } \frac{t-a}{a} > \frac{b-t}{b} \text{ i.e. } t\left(\frac{b+a}{ab}\right) > 2 \\ \frac{b-t}{b} & \text{if } \frac{b-t}{b} > \frac{t-a}{a} \end{cases} \quad \frac{dM}{dt} = \begin{cases} \frac{1}{a} & \text{if } t > 2\frac{ab}{a+b} \\ \frac{-1}{b} & \text{if } t < \left(\frac{ab}{a+b}\right)2 \end{cases}$$

Since $\frac{dM}{dt} \gtrless 0$ when $t \gtrless \frac{2ab}{a+b}$, M is minimized for $\boxed{t = \frac{2ab}{a+b}}$

4.23 Exercises - Partial Derivatives.

Exercise 8. $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$.

$$\begin{aligned} f_x &= \frac{1}{\sqrt{x^2+y^2}} + \frac{-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}} \qquad f_{xx} = \frac{-3y^2x}{(x^2+y^2)^{5/2}} \\ f_y &= \frac{-xy}{(x^2+y^2)^{3/2}} \qquad f_{yy} = (-x)\left(\frac{x^2-2y^2}{(x^2+y^2)^{5/2}}\right) \end{aligned}$$

$$\begin{aligned} f_{xy} &= (-y)\left(\frac{(x^2+y^2)^{3/2} - x\frac{3}{2}(x^2+y^2)^{1/2}(2x)}{(x^2+y^2)^3}\right) = \\ &= (-y)\left(\frac{-2x^2+y^2}{(x^2+y^2)^{5/2}}\right) \end{aligned} \qquad f_{yx} = \frac{2y}{(x^2+y^2)^{3/2}} + \frac{-3y^2y}{(x^2+y^2)^{5/2}} = \frac{(-y)(-2x^2+y^2)}{(x^2+y^2)^{5/2}}$$

Exercise 9.

(1)

$$\begin{aligned} z &= (x-2y)^2 \\ z_x &= 2(x-2y) = 2\sqrt{z} \qquad x(2z) - 4zy = (x-2y)2\sqrt{z} = 2z \\ z_y &= 2(x-2y)(-2) = -4\sqrt{z} \end{aligned}$$

(2)

$$\begin{aligned} z &= (x^4+y^4)^{1/2} \\ z_x &= \frac{1}{z}2x^3 \qquad x(2z) - 4zy = (x-2y)2\sqrt{z} = 2z \\ z_y &= \frac{2y^3}{z} \end{aligned}$$

Exercise 10.

$$f = \frac{xy}{(x^2+y^2)^2}, \; f_x = \frac{y}{(x^2+y^2)^2} + \frac{-4x^2y}{(x^2+y^2)^3} = \frac{y^3-3x^2y}{(x^2+y^2)^3}$$

So

$$\begin{aligned} f_{xx} &= \frac{-6xy(x^2+y^2)^3 - 3(x^2+y^2)^2(2x)(y^3-3x^2y)}{(x^2+y^2)^6} = \\ &= \frac{12xy(x^2-y^2)}{(x^2+y^2)^4} \end{aligned}$$

By label symmetry,

$$f_{xx} + f_{yy} = \frac{12xy(x^2-y^2)}{(x^2+y^2)^4} + \frac{12yx(y^2-x^2)}{(x^2+y^2)^4} = 0$$

5.5 Exercises - The derivative of an indefinite integral. The first fundamental theorem of calculus, The zero-derivative theorem, Primitive functions and the second fundamental theorem of calculus, Properties of a function deduced from properties of its derivatve.

Review the fundamental theorems of calculus, Thm. 5.1 and Thm. 5.3. Note the **differences** between the two.

Theorem 17 (First fundamental theorem of calculus)**.**
Let f be integrable on $[a, x] \quad \forall x \in [a, b]$
Let $c \in [a, b]$ and

$$A(x) = \int_c^x f(t)dt \quad \text{if } a \leq x \leq b \tag{9}$$

Then $\exists\, A'(x) \quad \forall x \in (a, b)$ where f is continuous at x and

$$A'(x) = f(x) \tag{10}$$

Theorem 18 (Second fundamental theorem of calculus)**.**
Assume f continuous on open interval I
Let P be any primitive of f on I, i.e. $P' = f \quad \forall x \in I$
Then $\forall c,\ x \in I$

$$P(x) = P(c) + \int_c^x f(t)dt \tag{11}$$

Exercise 6. $\sqrt{2}\frac{2}{3}x^{3/2} + \sqrt{\frac{1}{2}}\frac{2}{3}x^{3/2} = \sqrt{2}x^{3/2}$

$\int_a^b f = \sqrt{2}(b^{3/2} - a^{3/2})$

Exercise 7. $f = x^{3/2} - 3x^{1/2} + \frac{7}{2}x^{-1/2}$;

$P = \frac{2}{5}x^{5/2} - 2x^{3/2} + 7x^{1/2}$

$\int_a^b f = \frac{2}{5}(b^{5/2} - a^{5/2}) - 2(b^{3/2} - a^{3/2}) + 7(b^{1/2} - a^{1/2})$

Exercise 8. $P = \frac{3}{2}x^{4/3} - \frac{3}{2}x^{2/3}; \qquad x > 0$

Exercise 9. $P = -3\cos x + \frac{x^6}{3}$

Exercise 10. $P = \frac{3}{7}x^{7/3} - 5\sin x$

Exercise 11. $f'(x) = \frac{1}{x}$

$$f = \sum_{k=-\infty}^{\infty} a_k x^k$$
$$f' = \sum_{k=-\infty}^{\infty} k a_k x^{k-1} = \frac{1}{x}$$

Comparing terms, only $k = 0$ would work, but the coefficient is unequivocally 0

Exercise 12.

$$\int_0^x |t|dt = \begin{cases} \int_0^x tdt & \text{if } x \geq 0 \\ \int_0^x -tdt & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ \frac{-1}{2}x^2 & \text{if } x < 0 \end{cases} = \frac{1}{2}x|x|$$

Exercise 13.

$$\int_0^x (t + |t|)^2 dt = \begin{cases} \int_0^x (2t)^2 dt & \text{if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases} = \begin{cases} \frac{4}{3}x^3 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = \frac{2x^2}{3}(x + |x|)$$

Exercise 14. Using 1st. fund. thm. of calc.

$$\int_0^x f(t)dt = A(x) - A(0)$$
$$A'(x) = f(x)$$
$$\Longrightarrow 2x + \sin 2x + 2x\cos 2x + -\sin 2x$$

$$f' = 2 + 4\cos 2x + -4x\sin 2x - 2\cos 2x$$
$$= 2 + 2\cos 2x - 4x\sin 2x$$

$$\boxed{\begin{aligned} f\left(\frac{\pi}{4}\right) &= \frac{\pi}{2} \\ f'\left(\frac{\pi}{4}\right) &= 2-\pi \end{aligned}}$$

Exercise 15. $\int_c^x f(t)dt = \cos x - \frac{1}{2} \quad f(x) = -\sin x \quad c = -\frac{\pi}{6}$.

Exercise 16. Suppose $f(x) = \sin x - 1$ and $c = 0$.

$$\int_0^x t\sin t - t = \left(-t\cos t + \sin t - \frac{1}{2}t^2\right)\Big|_0^x = \sin x - x\cos x - \frac{1}{2}x^2$$

So $c = 0$.

Exercise 17. For $f(x) = -x^2 f(x) + 2x^15 + 2x^17$ (found by taking the derivative of $\int_0^x f = \int_x^1 t^2 f + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C$,) Suppose that $f = 2x^{15}$.

$$\Longrightarrow \frac{x^{16}}{8} = -\frac{x^{18}}{9} + -\frac{1}{9} + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C$$
$$\Longrightarrow C = \frac{1}{9}$$

Exercise 18. By plugging in $x = 0$ into the defined $f(x)$, $f(x) = 3 + \int_0^x \frac{1+\sin t}{2+t^2}dt$, we get for $p(x) = a + bx + cx^2$,

$$a = 3$$

Continuing on,

$$f' = \frac{1+\sin x}{2+x^2}; \quad f'' = \frac{(\cos x)(2+x^2) - 2x(1+\sin x)}{(2+x^2)^2}$$
$$f'(0) = \frac{1}{2} = b \qquad f''(0) = \frac{1}{2} + 2c; c = \frac{1}{4}$$

Exercise 19.

$$f(x) = \frac{1}{2}\int_0^x (x-t)^2 g(t)dt = \frac{1}{2}\int_0^x (x^2 - 2xt + t^2)g(t)dt =$$
$$= \frac{1}{2}\left(x^2\int_0^x g - 2x\int_0^x tg + \int_0^x t^2 g\right)$$
$$f' = x\int_0^x g + \frac{x^2}{2}g(x) - \int_0^x tg - x(xg(x)) + \frac{1}{2}x^2 g(x) =$$
$$= x\int_0^x g - \int_0^x tg$$
$$f'' = \int_0^x g + xg - xg = \int_0^x g \quad \boxed{f''(1) = 2}$$
$$f''' = g \quad \boxed{f'''(1) = 5}$$

Exercise 20.

(1) $(\int_0^x (1+t^2)^{-3}dt)' = (1+x^2)^{-3}$
(2) $(\int_0^{x^2} (1+t^2)^{-3}dt)' = (1+x^4)^{-3}(2x) = \frac{2x}{(1+x^4)^3}$
(3) $(\int_{x^3}^{x^2} (1+t^2)^{-3}dt)' = (1+x^4)^{-3}(2x) - (1+x^6)^{-3}(3x^2) = \frac{2x}{(1+x^4)^3} - \frac{3x^2}{(1+x^6)^3}$

Exercise 21.

$$f'(x) = \left(\int_{x^3}^{x^2} \frac{t^6}{1+t^4} dt \right)' = \left(\frac{x^{12}}{1+x^8} \right)(2x) - \left(\frac{x^18}{1+x^{12}} \right) 3x^2$$

Exercise 22.

(1)

$$\begin{aligned} f(x) &= 2x(1+x) + x^2 = 2x + 3x^2 \\ f(2) &= 16 \end{aligned}$$

(2) $\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)dt \right) = f(b)b' - f(a)a'$

$$\begin{aligned} 2x + 3x^2 &= f(x^2)(2x) \\ f(x^2) &= (1 + \frac{3x}{2}) \\ f(2) = 1 + \frac{3\sqrt{2}}{2} \end{aligned}$$

(3) $\int_0^{f(x)} t^2 dt = x^2(1+x)$

$$\begin{aligned} 2x + 3x^2 &= (f(x))^2 f'(x) \\ \implies f^3(2) &= 3(4)(3) = 9(4) = 36 \\ f(2) &= 36^{1/3} \end{aligned}$$

(4)

$$\frac{d}{dx}\left(\int_0^{x^2(1+x)} f(t)dt \right) = 1 = f(x^2(1+x))(2x(1+x) + x^2)$$

$$x = 1 \quad (f(2))(5) = 1 \implies f(2) = \frac{1}{5}$$

Exercise 23.

$$a^3 - 2a\cos a + (2 - a^2)\sin a = \int_0^a f^2(t)dt$$

$$3x^2 - 2\cos x + 2x\sin x + -2x\sin x + (2 - x^2)\cos x = 3x^2 - x^2\cos x = f^2(x)$$

$$f(x) = x\sqrt{3 - \cos x}$$

$$\boxed{f(a) = a\sqrt{3 - \cos a}}$$

Exercise 24. $f(t) = \frac{t^2}{2} + 2t\sin t$

(1)

$$f' = 2t + 2\sin t + 2t\cos t \quad f'(\pi) = 2\pi - 2\pi = 0$$

(2)

$$f'' = 2 + 2\cos t + 2\cos t + -2t\sin t = 2 + 4\cos t - 2t\sin t$$

$$f''\left(\frac{\pi}{2}\right) = 2 - \pi$$

(3) $f''\left(\frac{3\pi}{2}\right) = 0$
(4) $f\left(\frac{5\pi}{2}\right) = \frac{25\pi^2}{8} + 5\pi$
(5) $f(\pi) = \frac{\pi^2}{2}$

Exercise 25.

(1)

$$\frac{df}{dt} = \frac{1 + 2\sin \pi t \cos \pi t}{1 + t^2} = v(t)$$
$$a(t) = \frac{2\pi(\cos(2\pi t))(1 + t^2) - 2t(\sin(2\pi t))}{(1 + t^2)^2}$$
$$a(t = 2) = a(t = 1) = \frac{4\pi}{4} = \pi$$

(2) $v(t = 1)\frac{1}{2}$
(3) $v(t) = \pi(t - 1) + \frac{1}{2};\ t > 1$
(4)

$$f(t) - f(1) = \int_1^t v(t)dt = \int_1^t \pi(t - 1) + \frac{1}{2} = \left(\frac{\pi t^2}{2} - \pi t + \frac{1}{2}t\right)\bigg|_1^t = \frac{\pi t^2}{2} + -\pi t + \frac{t}{2} + \frac{\pi}{2} - \frac{1}{2}$$

Exercise 26.

(1)

$$f''(x) > 0\,\forall x \quad f'(0) = 1; f'(1) = 0$$
$$\int_0^1 f''(t)dt = f'(1) - f'(0) = 0 - 1 < 0$$

Thus, it's impossible, since $f''(x) > 0$, so $\int_0^1 f''(t)dt > 0$
(2)

$$\int_0^1 \left(3 - \frac{\pi}{2}\sin\frac{\pi x}{2}\right)dx = \left(3x + \cos\frac{\pi x}{2}\right)\bigg|_0^1 = 3 - 1 = 2$$
$$f(x) = \frac{3x^2}{2} + \frac{2}{\pi}\sin\frac{\pi x}{2} + C$$

(3) $f''(0) > 0\,\forall x \quad f'(0) = 1;\ f(x) \leq 100\,\forall x > 0$

$$\int_a^b f''(t)dt = f'(b) - f'(a); \quad \int_c^k f'(t)dt = f(k) - f(c)$$
$$\int_0^b f'' = f'(b) - f'(0) = f'(b) - 1 \gtreqless 0 \text{ if } b \gtreqless 0$$
$$\int_c^k (f'(b) - 1)db = f(k) - f(c) - (k - c) > 0 \text{ if } k > c > 0$$
$$f(k) - f(c) > k - c$$
$$f(k) - f(0) > k - 0$$
$$f(100) - f(0) > 100$$

$f(x) \leq 100$ is untrue for all $x > 0$

(4) $f''(x) = e^x > 0 \quad f'(x) = e^x; \quad f'(0) = 1 \quad f(x) = e^x \quad \forall x < 0, e^x < 1$

Exercise 27. $f''(t) \geq 6. \qquad b - a = \frac{1}{2}. \qquad f'(0) = 0$

$$\int_a^b f'' = f'(b) - f'(a) \geq 6(b - a) = 3 \quad \text{since } b - a = \frac{1}{2}$$
$$\int_0^a f'' = f'(a) - f'(0) = f'(a) \geq 6(a - 0) = 6a$$
$$\text{If } a = \frac{1}{2},\ f'(1/2) \geq 3$$

Then by intermediate value theorem, with f being continuous and $f'(0) = 0,\ f'(1/2) \geq 3,\ \ f'$ must take on the value of 3 somewhere between 0 and 3. Thus there is an interval $[a, b]$ of length $1/2$ where $f' \geq 3$.

5.8 Exercises - The Leibniz notation for primitives, Integration by substitution.

Exercise 1. $\int \sqrt{2x+1}dx = \frac{1}{3}(2x+1)^{3/2}$.

Exercise 2. $\int x\sqrt{1+3x} = \frac{2x}{9}(1+3x)^{3/2} + -\frac{4}{135}(1+3x)^{5/2}$

Exercise 3.

$$\int x^2\sqrt{x+1} = \frac{2x^2(x+1)^{3/2}}{3} - \frac{8x(x+1)^{5/2}}{15} + \frac{16(x+1)^{7/2}}{105}$$

since

$$\left(\frac{2x^2(x+1)^{3/2}}{3}\right)' = x^2(x+1)^{1/2} + \frac{4x(x+1)^{3/2}}{3}$$

$$\left(\frac{8x(x+1)^{5/2}}{15}\right)' = \frac{4}{3}x(x+1)^{3/2} + \frac{8(x+1)^{5/2}}{15}$$

$$\left(\frac{16(x+1)^{7/2}}{105}\right)' = \frac{8(x+1)^{5/2}}{15}$$

Exercise 4.

$$\int \frac{xdx}{\sqrt{2-3x}} = \frac{2x(2-3x)^{1/2}}{-3} + \frac{4(2-3x)^{3/2}}{-27}$$

$$\int_{-2/3}^{1/3} \frac{xdx}{\sqrt{2-3x}} = -2/9 - 4/27 - (8/9 - 32/27) = -2/27$$

Exercise 5.

$$\int \frac{(x+1)dx}{((x+1)^2+1)^3} = \frac{((x+1)^2+1)^{-2}}{-4}$$

Exercise 6.

$$\int \sin^3 x = \int \sin x(1-\cos^2 x) = -\cos x + \frac{1}{3}\cos^3 x$$

Exercise 7.

$$\int x^{1/3}(1+x) = \frac{3}{4}x^{4/3} + \frac{3}{7}x^{7/3} = \frac{3}{4}(z-1)^{4/3} + \frac{3}{7}(z-1)^{7/3}$$

Exercise 8. $\frac{\sin^{-2} x}{-2}$

Exercise 9. $\frac{(4-\sin 2x)^{3/2}}{-3}\Big|_0^{\pi/4} = \frac{3^{3/2}-8}{-3}$

Exercise 10. $(3+\cos x)^{-1}$

Exercise 11. $2\cos^{-1/2} x$

Exercise 12. $2\cos\sqrt{x+1}\Big|_3^8 = 2(\cos 3 - \cos 2)$

Exercise 13. $-\frac{\cos x^n}{n}$

Exercise 14. $\frac{(1-x^6)^{1/2}}{-3}$

Exercise 15.

$$\int t(1+t)^{1/4}dt = \int (x-1)x^{1/4}dx = \frac{4}{9}x^{9/4} - \frac{4}{5}x^{5/4} = \frac{4}{9}(1+t)^{9/4} - \frac{4}{5}(1+t)^{5/4}$$

Exercise 16. $\int (x^2+1)^{-3/2}dx = ?$

$$\left(\frac{x}{\sqrt{x^2+1}}\right)' = \frac{\sqrt{x^2+1} - x^2/\sqrt{x^2+1}}{x^2+1} = \frac{1}{(x^2+1)^{3/2}}$$

Exercise 17.

$$(8x^3+27)^{5/3}\left(\frac{3}{5}\right)\left(\frac{1}{24}\right)=\frac{1}{40}(8x^3+27)^{5/3}$$

Exercise 18. $\frac{3}{2}(\sin x-\cos x)^{2/3}$

Exercise 19.

$$\begin{aligned}\int\frac{xdx}{\sqrt{1+x^2+(1+x^2)^{3/2}}}=\int\frac{\frac{1}{2}du}{\sqrt{u+u^{3/2}}}&=\\=\frac{1}{2}\int\frac{du}{\sqrt{u}\sqrt{1+u^{1/2}}}&=2(1+u^{1/2})^{1/2}=\\&=2(1+\sqrt{1+x^2})^{1/2}+C\end{aligned}$$

Exercise 20.

$$\int\frac{(x^2-2x+1)^{1/5}dx}{1-x}=\int\frac{-(x-1)^{2/5}}{x-1}dx=-\int(x-1)^{-3/5}dx=-5/2(x-1)^{2/5}$$

Exercise 21. Thm. 1.18. invariance under translation. $\int_a^b f(x)dx=\int_{a+c}^{b+c}f(x-c)dx$.

Thm. 1.19. expansion or contraction of the interval of integration.

$$\int_a^b f(x)dx=\frac{1}{k}\int_{ka}^{kb}f\left(\frac{x}{k}\right)dx$$

$$\begin{aligned}&yx+c\\&dy=dx\end{aligned}\qquad\int_a^b f(x)dx=\int_{a+c}^{b+c}f(y-c)dy$$

$$\begin{aligned}&y=kx\\&dy=kdx\end{aligned}\qquad\int_a^b f(x)dx=\frac{1}{k}\int_{ka}^{kb}f\left(\frac{y}{k}\right)dy$$

Exercise 22.

$$F\left(\frac{x}{a},1\right)=\int_0^{x/a}\frac{u^p}{(u^2+1^2)^q}du\qquad\begin{aligned}u&=\frac{t}{a}\\du&=\frac{dt}{a}\end{aligned}$$

$$\begin{aligned}F\left(\frac{x}{a},1\right)&=\frac{1}{a}\int_0^x\frac{(t/a)^pdt}{\left(\left(\frac{t}{a}\right)^2+1^2\right)^q}=\\&=a^{-p-1+2q}\int_0^x\frac{t^p}{(t^2+1^2)^q}dt=a^{-p-1+2q}F(x,a)\end{aligned}$$

Exercise 23.

$$\begin{aligned}\int_x^1\frac{dt}{1+t^2}&=F(1)-F(x)\\\int_1^x\frac{dt}{1+t^2}&=F(x)-F(1)\\\int_1^{1/x}\frac{dt}{1+t^2}&=F\left(\frac{1}{x}\right)-F(1)\end{aligned}$$

$$u=\frac{1}{t}$$

$$du=\frac{-1}{t^2}dt,\ \frac{-1}{u^2}du=dt$$

$$\begin{aligned}\int_1^x\frac{dt}{1+t^2}&=\int_1^{1/x}\frac{-du}{u^2\left(1+\frac{1}{u^2}\right)}=\\&=-\int_1^{1/x}\frac{du}{u^2+1}=\int_{1/x}^1\frac{dt}{t^2+1}\end{aligned}$$

Exercise 24.

$$\int_0^1 x^m(1-x)^ndx=-\int_1^0(1-u)^m(u^n)du=\int_0^1(1-x)^mx^ndx\ \text{using}\ \begin{aligned}u&=1-x\\x&=1-u\end{aligned}$$

Exercise 25.

$$\cos^m x \sin^m x = \left(\frac{\sin 2x}{2}\right)^m = 2^{-m}\sin^m 2x$$

$$\int_0^{\frac{\pi}{2}} \cos^m x \sin^m x dx = 2^{-m}\int_0^{\pi/2} \sin^m 2x dx = 2^{-m}\int_0^{\pi} \frac{1}{2}\sin^m x dx = 2^{-m-1}\int_0^{\pi} \sin^m x dx =$$
$$= -2^{-m-1}\int_{\pi/2}^{-\pi/2} \sin^m\left(\frac{\pi}{2} - x\right) dx = 2^{-m-1}\int_{-\pi/2}^{\pi/2} \cos^m x dx = 2^{-m}\int_0^{\pi/2} \cos^m x dx$$

Exercise 26.

(1)

$$\begin{aligned} u &= \pi - x \\ x &= \pi - u \end{aligned} \qquad \int_0^{\pi} x f(\sin x) dx = \int_{\pi}^{0} (\pi - u) f(\sin(\pi - u))(-du) = \int_0^{\pi} (\pi - u) f(\sin u) du =$$
$$= \pi \int_0^{\pi} f(\sin x) dx - \int_0^{\pi} x f(\sin x) dx$$
$$\implies \int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x) dx$$

(2)

$$\begin{aligned} u &= \cos x \\ du &= -\sin x dx \end{aligned} \qquad \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{x \sin x}{2 - \sin^2 x} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{2 - \sin^2 x} dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx =$$
$$= -\frac{\pi}{2}\int_1^{-1} \frac{du}{1 + u^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{du}{1 + u^2} = \pi \int_{-1}^{1} \frac{dx}{1 + x^2}$$

Exercise 27.

$$\begin{aligned} x &= \sin u \\ dx &= \cos u \end{aligned} \qquad \int_0^1 (1 - x^2)^{n - \frac{1}{2}} dx = \int_0^{\pi/2} (\cos^2 u)^{n - \frac{1}{2}} \cos u du = \int_0^{\pi/2} \cos^{2n} u du$$

5.10 Exercises - Integration by Parts.

Exercise 1. $\int x \sin x = -x \cos x + \sin x$

Exercise 2. $\int x^2 \sin x = -x^2 \cos x + 2x \sin x + 2 \cos x$

Exercise 3. $\int x^3 \cos x = x^3 \sin x + 3x^2 \cos x - 6x \sin x + -6 \cos x$

Exercise 4. $\int x^3 \sin x = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

Exercise 5. $\int \sin x \cos x = -\frac{1}{4}\cos 2x = -\frac{1}{4}(\cos^2 x - \sin^2 x)$

Exercise 6. $\int x \sin x \cos x dx = \int \frac{x}{2}\sin 2x = -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8}$

Exercise 7. $\int \sin^2 x = \int \sin x \sin x = -\sin x \cos x + \int \cos^2 x$

$\int \sin^2 x dx = \frac{-1}{4}\sin 2x + \frac{x}{2}$

Exercise 8.

$$\int \sin^n x dx = -\cos x \sin^{n-1} x + \int (n-1)\sin^{n-2} x \cos^2 x \qquad \begin{aligned} u &= \sin^{n-1} x \\ dv &= \sin x dx \end{aligned}$$
$$\int \sin^n x = -\cos x \sin^{n-1} x + \int (n-1)\sin^{n-2} x (1 - \sin^2 x) =$$
$$= -\cos x \sin^{n-1} x + \int (n-1)\sin^{n-2} x - (n-1)\int \sin^n x$$
$$\int \sin^n x = \frac{-1}{n}\sin^{n-1} x \cos x + \frac{(n-1)}{n}\int \sin^{n-2} x$$

Exercise 9.

(1)

$$\int \sin^2 x = \frac{-1}{2}\sin x\cos x + \frac{1}{2}\int 1 = \frac{-1}{2}\sin x\cos x + \frac{1}{2}x$$

$$\int_0^{\pi/2}\sin^2 x dx = \frac{\pi}{4}$$

(2) $\int_0^{\pi/2}\sin^4 x = \frac{-1}{4}\sin^3 x\cos x\big|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\sin^2 x = \frac{3\pi}{16}$

(3) $\int_0^{\pi/2}\sin^6 x = \frac{5}{6}\int_0^{\pi/2}\sin^4 x = \boxed{\frac{5\pi}{32}}$

Exercise 10.

(1)

$$\int \sin^3 x dx = \frac{-1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x = -\frac{1}{6}\sin 2x\cos x - \frac{2}{3}\cos x = \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x \quad \text{since}$$

$$\frac{-3}{4}\cos x + \frac{1}{12}\cos 3x = \frac{-3}{4}\cos x + \frac{1}{12}(\cos x\cos 2x - \sin 2x\sin x) =$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}(\cos x(1-2\sin^2 x) + -2\sin^2 x\cos x) = \boxed{\frac{-2}{3}\cos x - \frac{1}{3}\sin^2 x\cos x}$$

(2)

$$\int \sin^4 x dx = \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x = \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}(\frac{x}{2} - \frac{\sin 2x}{4}) = \frac{-1}{4}\sin^3 x\cos x + \frac{3x}{8} - \frac{3\sin 2x}{16}$$

$$\text{Now } \frac{1}{32}\sin 4x = \frac{1}{32}(2\sin 2x\cos 2x) = \frac{1}{8}(\sin x\cos x(1-2\sin^2 x) = \frac{\sin 2x}{16} - \frac{1}{4}\sin^3 x\cos x$$

$$\implies \frac{-1}{4}\sin^3 x\cos x + \frac{3x}{8} - \frac{3\sin 2x}{16} = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$

(3)

$$\int \sin^5 x dx = \int \sin^4 x\sin x dx = -\cos x\sin^4 x + \int \cos^2 x 4\sin^3 x =$$

$$= -\cos x\sin^4 x + 4(\int \sin^3 x - \int \sin^5 x) = -\cos x\sin^4 x + 4\int \sin^3 x - 4\int \sin^5 x$$

$$5\int \sin^5 dx = -\cos x\sin^4 x + 4\int \sin^3 x$$

$$5\int \sin^5 dx = -\cos x(1-\cos^2 x)^2 + 4(\frac{-3}{4}\cos x + \frac{1}{12}\cos 3x)$$

$$= -\cos x(1 - 2\cos^2 x + \cos^4 x) + -3\cos x + \frac{1}{3}\cos 3x$$

$$= -\cos x + 2\cos^3 x - \cos^5 x - 3\cos x + \frac{1}{3}(\cos x\cos 2x - \sin x\sin 2x) =$$

$$= -4\cos x + 2\cos^3 x - \cos^5 x + \frac{1}{3}(4\cos^3 x - 3\cos x) = -5\cos x + \frac{10\cos^3 x}{3} - \cos^5 x$$

$$\int \sin^5 x dx = -\cos x + \frac{2\cos 3x}{3} - \frac{1}{5}\cos^5 x$$

My solution to the last part of this exercise **conflicts** with what's stated in the book.

Exercise 11.

(1)

$$\int x\sin^2 x dx = (\int \sin^2 x)x - \int(\sin^2 t) = \frac{x^2}{2} - \frac{x\sin 2x}{4} - \left(\frac{x^2}{4} + \frac{\cos 2x}{8}\right) =$$

$$= \frac{x^2}{8} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8}$$

$$\text{we had used } \int \sin^2 x = \frac{x}{2} - \frac{\sin 2x}{4}$$

(2)

$$\int x\sin^3 x = \frac{-3x}{4}\cos x + \frac{x}{12}\cos 3x - \int -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x =$$
$$= \frac{-3x}{4}\cos x + \frac{x}{12}\cos 3x + \frac{3}{4}\sin x + \frac{-\sin 3x}{36}$$
$$\int \sin^3 x = \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x$$

(3)

$$\int x^2\sin^2 x dx = x^2\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) - \int 2x\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = \frac{x^3}{2} - \frac{x^2\sin 2x}{4} - \frac{1}{3}x^3 + \int \frac{x\sin 2x}{2} =$$
$$= \frac{x^3}{6} - \frac{x^2\sin 2x}{4} + \frac{1}{2}\left(\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right) =$$
$$= \boxed{\frac{x^3}{6} - \frac{x^2\sin 2x}{4} - \frac{x\cos 2x}{4} + \frac{\sin 2x}{8}}$$

Exercise 12.

$$\int \cos^n x dx = \int \cos^{n-1} x\cos x dx = \cos^{n-1} x\sin x + \int (n-1)\cos^{n-2} x\sin^2 x =$$
$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x - \int (n-1)\cos^n x$$
$$\Longrightarrow \int \cos^n x = \frac{\cos^{n-1} x\sin x}{n} + \left(\frac{n-1}{n}\right)\int \cos^{n-2} x$$

Exercise 13.

(1) $\int \cos^2 x = \frac{\sin 2x}{5} + \frac{1}{2}x$
(2) $\int \cos^3 x = \frac{\cos^2 x\sin x}{3} + \frac{2}{3}\sin x = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$ since

$$\frac{1}{12}\sin 3x = \frac{1}{12}(\sin 2x\cos x + \sin x\cos 2x) = \frac{1}{6}\sin x\cos^2 x + \frac{1}{12}\sin x(2\cos^2 x - 1) =$$
$$= \frac{1}{3}\sin x\cos^2 x - \frac{1}{12}\sin x$$

(3)

$$\int \cos^4 x dx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) = \frac{3}{8}x + \frac{3}{16}\sin 2x + \frac{\cos^3 x\sin x}{4}$$
$$\sin 4x = 2\sin 2x\cos 2x = 4\sin x\cos x(2\cos^2 x - 1) = 8\sin x\cos^3 x - 2\sin 2x \quad \text{then}$$
$$\int \cos^4 x dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$

Exercise 14.

$$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}dx$$
$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{x^2 - +1}{\sqrt{1-x^2+1-1}} = -\int \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$$
$$x^2 = x^2 - 1 + 1$$
$$\Longrightarrow \int \sqrt{1-x^2}dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}$$

Exercise 15.

(1)

$$\int (a^2-x^2)^n dx = x(a^2-x^2)^n - \int n(a^2-x^2)^{n-1}(-2x)x dx = x(a^2-x^2)^n + 2n\int x^2(a^2-x^2)^{n-1}dx$$

$$\int x^2(a^2-x^2)^{n-1}dx = \int ((x^2-a^2)+a^2)(a^2-x^2)^{n-1}dx = \int -(a^2-x^2)^n + a^2(a^2-x^2)^{n-1}dx$$

$$\implies \int (a^2-x^2)^n dx = \frac{x(a^2-x^2)^n}{2n+1} + \frac{2a^2 n}{2n+1}\int (a^2-x^2)^{n-1}dx$$

(2)

$$\int (a^2-x^2)dx = \frac{x(a^2-x^2)}{3} + \frac{2a^2}{3}x = \frac{-x^3}{3} + a^2 x$$

$$\int (a^2-x^2)^{5/2}dx = \frac{x(a^2-x^2)^{5/2}}{6} + \frac{a^2 5}{6}\int (a^2-x^2)^{3/2}dx$$

$$\int (a^2-x^2)^{3/2}dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2}{4}\int (a^2-x^2)^{1/2}dx$$

$$\int (a^2-x^2)^{1/2} = a\int \sqrt{1-\left(\frac{x}{a}\right)^2}dx = a^2\int \cos^2\theta d\theta = \qquad \sin\theta = \frac{x}{a}$$

$$= a^2\int \frac{1+\cos 2\theta}{2} = a^2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) = a^2\left(\arcsin\frac{x}{a} + \frac{1}{2}\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) \qquad \cos\theta d\theta = \frac{dx}{a}$$

$$\int_0^a \sqrt{a^2-x^2} = a^2\left(\frac{\pi}{2}-0\right) = \frac{\pi a^2}{2}$$

$$\int_0^a (a^2-x^2)^{3/2}dx = \frac{3a^2}{4}\left(\frac{\pi a^2}{2}\right) = \frac{3\pi a^4}{8}$$

$$\boxed{\int_0^a (a^2-x^2)^{5/2}dx = \frac{5a^2}{6}\left(\frac{3\pi a^4}{8}\right) = \boxed{\frac{5}{16}\pi a^6}}$$

Exercise 16. $I_n(x) = \int_0^x t^n(t^2+a^2)^{-1/2}dt$

(1)

$$I_n(x) = (t^2+a^2)^{1/2}t^{n-1} - \int (n-1)t^{n-2}(t^2+a^2)^{1/2} = t^{n-1}(t^2+a^2)^{1/2} - (n-1)\int \frac{t^{n-2}(t^2+a^2)}{(t^2+a^2)^{1/2}}$$

$$(n)I_n = x^{n-1}(x^2+a^2)^{1/2} - a^2(n-1)\int \frac{t^{n-2}}{(t^2+a^2)^{1/2}} = x^{n-1}\sqrt{x^2+a^2} - (n-1)a^2 I_{n-2}$$

(2) $n=5$; $x=2$; $a=\sqrt{5}$.

$$I_1(2) = \int_0^2 x(x^2+5)^{-1/2}dx = (x^2+5)^{1/2}\Big|_0^2 = 3-\sqrt{5}$$

$$5I_5(2) = \int_0^2 t^5(t^2+5)^{-1/2}dt = 2^{5-1}(4+5)^{1/2} - 5(5-1)I_3(2) = 48 - 20I_3(2)$$

$$3I_3(2) = 2^2\sqrt{4+5} - 5(3-1)I_1(2) = 12 - 10(3-\sqrt{5})$$

$$I_5(2) = \frac{1}{5}(48 - 20(-6+\frac{10\sqrt{5}}{3})) = \boxed{\frac{168}{5} - \frac{40\sqrt{5}}{3}}$$

Exercise 17.

$$\int t^3(c+t^3)^{-1/2}dt = \frac{t2(c+t^3)^{1/2}}{3} - \int \frac{2(c+t^3)^{1/2}}{3}$$

$$\int_{-1}^3 t^3(4+t^3)^{-1/2}dt = \frac{t2(4+t^3)^{1/2}}{3}\Big|_{-1}^3 - \frac{2}{3}\int_{-1}^3 (4+t^3)^{1/2} = \boxed{2\sqrt{31} + \frac{2\sqrt{3}}{3} - \frac{2}{3}(11.35)}$$

Exercise 18.

$$\int \frac{\sin^{n+1} x}{\cos^{m+1} x} dx = \int \sin^n x \left(\frac{\sin x}{\cos^{n+1} x} \right) dx = \frac{\sin^n x}{m \cos^m x} - \int \frac{n \sin^{n-1} x}{m \cos^{m-1} x}$$
$$\implies \int \frac{\sin^{n+1} x}{\cos^{m+1} x} dx = \frac{\sin^n x}{m \cos^m x} - \frac{n}{m} \int \frac{\sin^{n-1} x}{\cos^{m-1} x}$$

Exercise 19.

$$\int \frac{\cos^{m+1} x}{\sin^{n+1} x} dx = \int \cos^m x \left(\frac{\cos x dx}{\sin^{n+1} x} \right) = \cos^m x \frac{1}{-n \sin^n x} - \int \frac{m \cos^{m-1} x}{-n \sin^m x} =$$
$$= \frac{\cos^m x}{-n \sin^n x} + \frac{-m}{n} \int \frac{\cos^{m-1} x}{\sin^{n-1} x}$$
$$\int \cot^2 x = \int \frac{\cos^{1+1} x}{\sin^{1+1} x} = \frac{-1}{1} \frac{\cos^1 x}{\sin^1 x} - \frac{1}{1} \int dx = -\cot x - x$$
$$\int \cot^4 x dx = \int \frac{\cos^{3+1} x}{\sin^{3+1} x} = -\frac{1}{3} \frac{\cos^3 x}{\sin^3 x} - \int \frac{\cos^{3-1} x}{\sin^{3-1} x} = \frac{-1}{3} \cot^3 x - (-\cot x - x) = \boxed{\frac{-1}{3} \cot^3 x + \cot x + x}$$

Exercise 20.

(1)
$$\int_0^2 t f''(t) dt = 2 \int_0^1 t f''(2t) dt \qquad n = 2$$

(2)
$$\int_0^1 x f''(2x) dx = \frac{1}{2} \int_0^2 t f''(t) dt = \frac{1}{2} \left(t f'(t)|_0^2 - \int_0^2 f'(t) dt \right) = \frac{1}{2} \left(2f'(2) - (f(2) - f(0)) \right) = \boxed{4}$$

Exercise 21.

(1) Recall Theorem 5.5, the second mean-value theorem for integrals:
$$\int_a^b f(x) g(x) dx = f(a) \int_a^c g(x) dx + f(b) \int_c^b g(x) dx$$
$$\begin{aligned} \int_a^b \sin \phi(t) \left(\frac{\phi'(t)}{\phi'(t)} \right) dt &= \frac{1}{\phi'(a)} \int_a^c \phi'(t) \sin \phi(t) + \frac{1}{\phi'(b)} \int_a^b \phi'(t) \sin \phi(t) = \\ &= \frac{1}{\phi'(a)} \cos \phi(t)|_a^c + \frac{1}{\phi'(b)} (\cos \phi(b) - \cos \phi(a)) \leq \frac{4}{m} \end{aligned} \qquad \text{where } \frac{1}{m} \geq \frac{1}{\phi'(t)} \quad \forall t \in [a, b]$$

(2) $\phi(t) = t^2$; $\qquad \phi'(t) = 2t > 2a$ if $t > a$
$$\left| \int_a^x \sin t^2 dt \right| \leq \frac{4}{2a} = 2a$$

5.11 Miscellaneous review exercises.

Exercise 1. $g(x) = x^n f(x)$; $\qquad f(0) = 1$

$g'(x) = nx^{n-1} f(x) + x^n f'(x)$; $\quad g'(0) = 0$ especially if $n \in \mathbb{Z}^+$ (just note that if negative integer values are included, $g'(0)$ easily blows up)

$g^j(x) = \sum_{k=0}^h \binom{j}{k} \frac{n!}{(n-k)!} x^{n-k} f^{j-k}(x)$

If $j < n$, then $g^j(0)$, since each term contains some power of x

If $j \geq n$,
$$g^j(x) = \sum_{k=0}^n \binom{j}{k} \frac{n!}{(n-k)!} x^{n-k} f^{(j-k)}(x)$$
$$g^j(0) = \frac{j!}{(j-n)!} f^{(j-n)}(0)$$

If $j = n$, $g^n(0) = n!$

Exercise 2.

$$P(x) = \sum_{j=0}^{5} a_j x^j$$

$$P'(x) = \sum_{j=1}^{5} j a_j x^{j-1} \qquad P(0) = 1 = a_0 \quad P'(0) = 0 = a_1 \quad P''(0) = 0 = 2(1)a_2 \qquad a_1 = a_2 = 0$$

$$P''(x) = \sum_{j=2}^{5} j(j-1) a_j x^{j-2}$$

$$\begin{aligned} P(x) &= a_5 x^5 + a_4 x^4 + a_3 x^3 + 1 & P(1) &= a_5 + a_4 + a_3 + 1 = 2 \\ \Longrightarrow P'(x) &= 5a_5 x^4 + 4a_4 x^3 + 3a_3 x^2 & P'(1) &= 5a_5 + 4a_4 + 3a_3 = 0 \\ P''(x) &= 20 a_5 x^3 + 12 a_4 x^2 + 6 a_3 x & P''(1) &= 20a_5 + 12a_4 + 6a_3 = 0 \end{aligned}$$

Solve for the undetermined coefficients by Gauss-Jordan elimination process

$$\begin{bmatrix} 5 & 4 & 3 \\ 20 & 12 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 5 & 4 & 3 & 0 \\ 20 & 12 & 6 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & 1 & 0 & -15 \\ & 0 & 1 & 10 \\ 1 & 0 & 0 & 6 \end{array}\right]$$

$$\Longrightarrow a_5 = 6 \quad a_4 = -15 \quad a_3 = 10 \qquad \boxed{P(x) = 6x^5 - 15x^4 + 10x^3 + 1}$$

Exercise 3. If $f(x) = \cos x$ and $g(x) = \sin x$, Prove that $f^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$ and $g^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$

$$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right) = \begin{cases} \sin x(-1)^{j+1} & \text{if } n = 2j+1 \\ \cos x(-1)^j & \text{if } n = 2j \end{cases}$$

$$g^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right) = \begin{cases} \cos x(-1)^{j} & \text{if } n = 2j+1 \\ \sin x(-1)^j & \text{if } n = 2j \end{cases}$$

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f''''(x) &= \cos x \end{aligned} \qquad \begin{aligned} & f^{(2j)}(x) = \cos x(-1)^j \\ & f^{(2(j+1))}(x) = (\cos x(-1)^j)' = \cos x(-1)^{j+1} \\ & f^{(2j+1)}(x) = \sin x(-1)^{j+1} \\ & f^{(2j+3)}(x) = (\sin x(-1)^{j+1}) = \sin x(-1)^{j+2} \end{aligned}$$

$$\begin{aligned} g(x) &= \sin x \\ g'(x) &= \cos x \\ g''(x) &= -\sin x \\ g'''(x) &= -\cos x \\ g''''(x) &= \sin x \end{aligned} \qquad \begin{aligned} & g^{(2j)}(x) = \sin x(-1)^j \\ & g^{(2(j+1))}(x) = (\sin x(-1)^j)'' = \sin x(-1)^{j+1} \\ & g^{(2j+1)}(x) = \cos x(-1)^j \\ & g^{(2j+3)}(x) = (\cos x(-1)^{j+1}) \end{aligned}$$

Exercise 4.

$$h'(x) = f'g + fg' \qquad h^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$
$$h''(x) = f''g + 2f'g' + fg''$$

$$\begin{aligned} h^{(n+1)} &= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)} \right) = \\ &= f^{(1)} g^{(n)} + f g^{(n+1)} + \sum_{k=1}^{n-1} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)} \right) + f^{(n+1)} g + f^{(n)} g^{(1)} \\ &= f^{(1)} g^{(n)} + f g^{(n+1)} + \sum_{k=2}^{n} \frac{n!}{(n-k+1)!(k-1)!} \left(f^{(k)} g^{(n-k+1)} + \sum_{k=1}^{n-1} \frac{n!}{(n-k)!k!} f^{(k)} g^{(n-k+1)} \right) + \\ &+ f^{(n+1)} g + f^{(n)} g^{(1)} \end{aligned}$$

Now $\frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} = (k + (n-k+1)) \left(\frac{n!}{(n+1-k)!(k)!} \right) = \binom{n+1}{k}$ so then

$$h^{(n+1)} = f g^{(n+1)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} + f^{(n+1)} g = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}$$

By induction, this formula is true.

Exercise 5.

(1)

$$\begin{gathered} f^2 + g^2 = f(-g') + gf' \\ Y = f^2 + g^2 \qquad Y' = 2ff' + 2gg' = 2(gf + 2g(-f)) = 0 \Longrightarrow Y = C \\ Y = C = f^2 + g^2 \qquad f(0) = 0; \quad g(0) = 1 \Longrightarrow \boxed{C = 1} \end{gathered}$$

(2)

$$\begin{aligned} h &= (F-f)^2 + (G-g)^2 \\ &= f'(x) = g(x), \quad g'(x) = -f(x); \quad f(0) = 0; \quad g(0) = 1 \\ h' &= 2(F-f)(F'-f') + 2(G-g)(G'-g'); \quad h'(0) = 2(0) + 2(0) = 0 \\ &= 2(F-f)(G-g) + 2(G-g)(-F+f) = 0 \quad \forall x \end{aligned}$$
$$\begin{gathered} h(x) = C \Longrightarrow h(x) = (F(x) - f(x))^2 + (G(x) - g(x))^2 \\ h(0) = 0 \quad \text{ so } C = 0 \\ \Longrightarrow F = f; \, G = g \end{gathered}$$

Exercise 6. $\frac{df}{du} 2x = 3x^2 \qquad f'(4) = \frac{3x}{2} = 3$ where we had used the substitution

$u = x^2 \qquad u = 4; x = 2$

Exercise 7. $\frac{dg}{du} = u^{3/2}; \quad g(u) = \frac{2}{5} u^{5/2} \qquad g(4) = \frac{2}{5} 2^5 = \boxed{\frac{64}{5}}$

Exercise 8.

$$\begin{aligned} \int_0^x \frac{\sin t}{t+1} dt &= \frac{1}{0+1} \int_0^c \sin t dt + \frac{1}{x+1} \int_c^x \sin t dt = \\ &= -\cos t|_0^c + \frac{1}{x+1} - \cos t|_c^x = -\cos c + 1 + \frac{-1}{x+1} (\cos x - \cos c) = \\ &= \frac{-x\cos c + x - \cos c + 1 - \cos x + \cos c}{x+1} = \frac{x(1-\cos c) + (1-\cos c)}{x+1} > 0 \end{aligned}$$

Exercise 9. $y = x^2$ is the curve for C. $y = \frac{1}{2} x^2$ is the curve for C_1.

$$\int_0^b (x^2 - \frac{1}{2}x^2) = \frac{1}{6}b^3 \qquad P:\ (b, b^2)$$

Assume C_2 is of the form kx^2

$$\int_0^c (kx^2 - x^2) + \int_c^b (b^2 - x^2) = \frac{(k-1)}{3}c^3 + b^2(b-c) + -\frac{1}{3}(b^3 - c^3) \qquad \begin{aligned} kc^2 &= b^2 \\ k &= \frac{b^2}{c^2} \end{aligned}$$

$$\implies \frac{(b^2 - c^2)c}{3} + b^3 - cb^2 - \frac{b^3}{3} + \frac{c^3}{3} = \frac{2b^3}{3} - \frac{2}{3}cb^2 = \frac{2}{3}b^2(b-c)$$

Now $A(A) = A(B)$

$$\implies \frac{2}{3}b^2(b-c) = \frac{1}{6}b^3 \implies b = \frac{4}{3}c$$

so then $\boxed{kx^2 = \frac{16}{9}x^2}$

Exercise 10.

(1) $|Q(h) - 0| = \left|\frac{f(h)}{h}\right| = \begin{cases} \frac{h^2}{|h|} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

For now, consider $0 < h < \delta$; let $\delta(\epsilon; h = 0) = \epsilon$

$$|Q(h) - 0| = \left|\frac{f(h)}{h}\right| = \begin{cases} h & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} < \epsilon$$

(2)

$$\left|\frac{f(h) - f(0)}{h} - 0\right| = \epsilon \implies f'(0) = 0$$

Exercise 11. $\int (2 + 3x)\sin bx dx = \frac{-2}{5}\cos 5x + -\frac{3x}{5}\cos 5x + \frac{3}{25}\sin 5x$

Exercise 12. $\frac{4}{3}(1 + x^2)^{3/2}$

Exercise 13. $\frac{(x^2-1)^{10}}{20}\Big|_{-2}^{1} = \frac{-3^{10}}{20}$

Exercise 14.

$$\frac{1}{3}\int_0^1 \frac{6x + 7 + 2}{(6x+7)^3}dx = \frac{1}{3}\left(\frac{-(6x+7)^{-1}}{6} + \frac{-(6x+7)^{-2}}{6}\right)\Bigg|_0^1 = \frac{1}{3}\left(-\left(\frac{1}{13}\right)\left(\frac{1}{6}\right) + \frac{1}{42} + \frac{-1}{(6)13^2} + \frac{1}{(6)49}\right) =$$

$$= \boxed{\frac{37}{8281}}$$

Exercise 15. $\int x^4(1 + x^5)^5 dx = \frac{(1+x^5)^6}{30}$

Exercise 16.

$$\int_0^1 x^4(1-x)^{20}dx = \qquad \begin{aligned} u &= 1 - x \\ x &= 1 - u \end{aligned}$$

$$= -\int_1^0 (1-u)^4 u^{20} du = \int_0^1 (1-u)^4 u^{20} du = \int_0^1 (1 + 4(-u) + 6u^2 + -4u^3 + u^4)u^{20} du =$$

$$= \frac{1^{21}}{21} + \frac{-41^22}{22} + \frac{61^{23}}{23} + \frac{-41^{24}}{24} + \frac{1^{25}}{25} =$$

$$= \boxed{\frac{1}{265650}}$$

Make sure to check your arithmetic.

Exercise 17. $\int_1^2 x^{-2}\sin\frac{1}{x}dx = \left(\cos\frac{1}{x}\right)\Big|_1^2 = \cos\frac{1}{2} - \cos 1$

Exercise 18. $\int \sin{(x-1)^{1/4}}dx$

$$\begin{aligned} u &= (x-1)^{1/4} \\ du &= \frac{1}{4}(x-1)^{-3/4}dx = \frac{1}{4}\frac{1}{u^3}dx \end{aligned} \implies \int \sin{(x-1)^{1/4}}dx = \int (\sin{u})4u^3du = 4\int u^3\sin{u}du$$

$$\begin{aligned} \int u^3 \sin{u}du &= -u^3\cos{u} + 3u^2\sin{u} + 6u\cos{u} + 06\sin{u} \\ &= -(x-1)^{3/4}\cos{(x-1)^{1/4}} + 3(x-1)^{1/2}\sin{(x-1)^{1/4}} + 6(x-1)^{1/4}\cos{(x-1)^{1/4}} - 6\sin{(x-1)^{1/4}} \end{aligned}$$

$$\sin{(x-1)^{1/4}}dx =$$

$$= \boxed{-4(x-1)^{3/4}\cos{(x-1)^{1/4}} + 24(x-1)^{1/4}\cos{(x-1)^{1/4}} + 12(x-1)^{1/4}\sin{(x-1)^{1/4}} - 24\sin{(x-1)^{1/4}}}$$

Exercise 19. $\int x\sin{x^2}\cos{x^2}dx = (1/4)\sin^2{x^2} + C$

Exercise 20. $\int \sqrt{1+3\cos^2{x}}\sin{2x}dx = \int\sqrt{1+3\cos^2{x}}2\sin{x}\cos{x}dx = \int u^{1/2}\frac{du}{-3} = \frac{2u^{3/2}}{-9} = \boxed{\frac{-2}{9}(1+3\cos^2{x})^{3/2}}$,

where we had used this substitution:

$$u = 1+3\cos^2{x} \qquad \begin{aligned} du &= -6\cos{x}\sin{x}dx \\ \frac{du}{-3} &= 2\cos{x}\sin{x}dx \end{aligned}$$

Exercise 21. $\int_0^2 375x^5(x^2+1)^{-4}dx$

$$\begin{aligned} u &= x^2+1 \\ du &= 2xdx \\ (u-1)^2 &= x^4 \end{aligned} \implies \begin{aligned} \int_0^2 \frac{375}{2}du(u-1)^2u^{-4} &= \frac{375}{2}\int_1^5 du\frac{(u^2-2u+1)}{u^4} = \frac{375}{2}\int_1^5 du\left(\frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4}\right) = \\ &= \frac{375}{2}\left(\frac{-1}{u} + \frac{1}{u^2} + \frac{1}{-3u^3}\right)\Big|_1^5 = \boxed{64 = 2^6} \end{aligned}$$

Exercise 22. $\int_0^1(ax+b)(x^2+3x+2)^{-2}dx = \frac{3}{2}$

Since $\left(\frac{-1}{x^2+3x+2}\right)\Big|_0^1 = \frac{-1}{6} + \frac{1}{2} = \frac{2}{3}$,

then if $a = 9/2$, $b = \frac{27}{2}$, we'll obtain $3/2$

Exercise 23. $I_n = \int_0^1(1-x^2)^n dx$

$$\begin{aligned} I_n = \int_0^1 (1-x^2)^n dx &= \boxed{x(1-x^2)^n\big|_0^1 - \int_0^1 xn(1-x^2)^{n-1}(-2x)dx} = \\ &= 2\int_0^1 x^2n(1-x^2)^{n-1}dx = 2n\int_0^1((x^2-1)+1)(1-x^2)^{n-1}dx = \\ &= 2nI_{n-1} - 2nI_n \end{aligned} \implies I_n = \left(\frac{2n}{2n+1}\right)I_{n-1}$$

$$\begin{aligned} I_2 &= \int_0^1 (1-x^2)^2dx = \int_0^1 dx(1-2x^2+x^4) = \left(x - \frac{2x^3}{3} + \frac{1}{5}x^5\right)\Big|_0^1 = \boxed{\frac{8}{15}} \\ I_3 &= \frac{6}{7}\frac{8}{15}\boxed{\frac{16}{35}} \\ I_4 &= \frac{8}{9}I_3 = \boxed{\frac{128}{315}} \\ I_5 &= \frac{10}{11}I_4 = \boxed{\frac{256}{693}} \end{aligned}$$

Exercise 24. $F(m,n) = \int_0^x t^m(1+t)^ndt; \qquad m>0, \quad n>0$

$$F(m,n) = \frac{t^{m+1}}{m+1}(1+t)^n\Big|_0^x - \int_0^x \frac{t^{m+1}}{m+1}n(1+t)^{n-1}dt = \frac{x^{m+1}(1+x)^n}{m+1} - \frac{n}{m+1}F(m+1,n-1)$$

$$\boxed{(m+1)F(m,n) + nF(m+1,n-1) = x^{m+1}(1+x)^n}$$

$$F(11,1) = \int_0^x t^{11}(1+t)^1 dt = \int_0^x t^1 1 + t^{12} = \left(\frac{t^{12}}{12} + \frac{t^{13}}{13}\right)\Big|_0^x = \frac{x^{12}}{12} + \frac{x^{13}}{13}$$

$$11F(10,2) + 2\left(\frac{x^{12}}{12} + \frac{x^{13}}{13}\right) = x^{11}(1+x)^2$$

$$\boxed{F(10,2) = \frac{x^{13}}{13} + \frac{x^{12}}{6} + \frac{x^{11}}{11}}$$

Exercise 25. $f(n) = \int_0^{\pi/4} \tan^n x dx$

(1)

$$\text{Use this extremely } \textbf{important} \text{ fact: } \boxed{\int_a^b fg = f(b)\int_c^b g + f(a)\int_a^c g}$$

$$\boxed{f(n+1) = \int_0^{\pi/4} \tan^n x \tan x = \int_c^{\pi/4} \tan^n x < \int_0^{\pi/4} \tan^n x = f(n)}$$

(2)

$$f(n+2) + f(n) = \int_0^{\pi/4} \tan^n x \tan^2 x + \int_0^{\pi/4} \tan^n x = \int_0^{\pi/4} \tan^n x (\sec^2 x) =$$

$$= \frac{\tan^{n+1} x}{n+1}\Big|_0^{\pi/4} = \boxed{\frac{1}{n+1}}$$

(3)

$$f(n+2) + f(n) = \frac{1}{n+1} < f(n+1) + f(n) < 2f(n)$$

$$\frac{1}{n-1} = f(n-2) + f(n) > f(n-1) + f(n) > 2f(n)$$

$$\implies \frac{1}{n+1} < 2f(n) < \frac{1}{n-1}$$

Exercise 26. $f(0)$, $\quad f(\pi) = 2 \qquad \int_0^\pi (f(x) + f''(x))\sin x dx = 5$

$$\int_0^\pi f'' \sin x = f' \sin x - \int f' \cos x = -\int f' \cos x = -f\cos x - \int f \sin x$$

$$\int_0^\pi (f + f'')\sin x dx = \int f \sin x + -(f(\pi)(-1) - f(0)) - \int f \sin x = 2 + f(0) = 5 \qquad \implies \boxed{f(0) = 3}$$

Exercise 27.

$$\int_0^{\pi/2} \frac{\sin x \cos x}{x+1} dx = \int_0^{\pi/2} \frac{\sin 2x}{2x+2} dx = \frac{1}{2}\int_0^\pi \frac{\sin x}{x+2} dx = \frac{1}{2}\left(\frac{-\cos x}{x+2} - \int \frac{\cos x}{(x+2)^2}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{\pi+2} + \frac{1}{2} - A\right) = \frac{4+\pi}{4(\pi+2)} - \frac{A}{2}$$

Exercise 28.

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{bx} + \frac{2}{b}\int \frac{\sqrt{a+bx}}{x^2}$$

$$\int \frac{\sqrt{a+bx}}{x} = \frac{\frac{2}{3b}(a+bx)^{3/2}}{x} + \int \frac{\frac{2}{3b}(a+bx)^{3/2}}{x^2} = \frac{2}{3b}\sqrt{a+bx}\left(\frac{a}{x} + b\right) + \frac{2}{3b}\int \sqrt{a+bx}\left(\frac{a}{x^2} + \frac{b}{x}\right)$$

$$\implies \int \frac{\sqrt{a+bx}}{x} = a\int \frac{dx}{x\sqrt{a+bx}} + 2\sqrt{a+bx}$$

Exercise 29.

$$\int x^n\sqrt{(ax+b)}dx = \frac{2x^n(ax+b)^{3/2}}{3a} - \int \frac{nx^{n-1}2(ax+b)^{3/2}}{3a} = \frac{2x^n(ax+b)^{3/2}}{3a} - \frac{2n}{3a}\int x^{n-1}(ax+b)\sqrt{ax+b} =$$
$$= \frac{2x^n(ax+b)\sqrt{ax+b}}{3a} - \frac{2n}{3}\int x^n\sqrt{ax+b} - \frac{2nb}{3a}\int x^{n-1}\sqrt{ax+b}$$
$$\boxed{\int x^n\sqrt{ax+b}dx = \frac{2}{(2n+3)a}\left(x^n(ax+b)^{3/2} - nb\int x^{n-1}\sqrt{ax+b}\right) + C} \qquad n \neq \frac{-3}{2}$$

Exercise 30.

$$\int \frac{x^m}{\sqrt{a+bx}}dx = \frac{2x^m(a+bx)^{1/2}}{b} - \int \frac{mx^{m-1}2(a+bx)^{1/2}}{b}$$
$$= \frac{2}{b}x^m(a+bx)^{1/2} - \frac{2m}{b}\int \frac{x^{m-1}(a+bx)}{\sqrt{a+bx}} = \frac{2}{b}x^m(a+bx)^{1/2} - 2m\int \frac{x^m}{\sqrt{a+bx}} - \frac{2ma}{b}\int \frac{x^{m-1}}{\sqrt{a+bx}}$$
$$\boxed{\int \frac{x^m}{\sqrt{a+bx}}dx = \frac{1}{2m+1}\frac{2}{b}x^m(a+bx)^{1/2} - \frac{2ma}{b(2m+1)}\int \frac{x^{m-1}}{\sqrt{a+bx}}}$$

Exercise 31.

$$\int \frac{dx}{x^n\sqrt{ax+b}} = 2\frac{\sqrt{ax+b}}{ax^n} + \int \frac{n2\sqrt{ax+b}}{ax^{n+1}}\sqrt{\frac{ax+b}{ax+b}} = \frac{2\sqrt{ax+b}}{ax^n} + \frac{2n}{a}\int \frac{ax+b}{x^{n+1}\sqrt{ax+b}} =$$
$$= \frac{2\sqrt{ax+b}}{ax^n} + 2n\int \frac{1}{x^n\sqrt{ax+b}} + \frac{2nb}{a}\int \frac{1}{x^{n+1}\sqrt{ax+b}}$$
$$\Longrightarrow (1-2n)\int \frac{dx}{x^n\sqrt{ax+b}} - \frac{2\sqrt{ax+b}}{ax^n} = \int \frac{b\left(\frac{2n}{a}\right)}{x^{n+1}\sqrt{ax+b}}$$
$$\boxed{\int \frac{1}{x^n\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{1}{x^{n-1}\sqrt{ax+b}}}$$

Exercise 32. I derived the formulas for this and Exercise 33 by doing the **following trick**.

$$\boxed{\begin{aligned}(C^{m+1}S^{1-n})' &= (m+1)C^m(-S^{2-n}) + C^{m+2}(1-n)S^{-n} = -(m+1)C^mS^{2-n} + (1-n)C^mS^{-n}(1-S^2) = \\ &= -(m+1+1-n)C^mS^{2-n} + (1-n)C^mS^{-n} = -(m-n+2)C^mS^{2-n} + (1-n)C^mS^{-n}\end{aligned}}$$
$$\Longrightarrow \int C^mS^{-n} = \frac{-(C^{m+1}S^{1-n})}{n-1} - \frac{(m-n+2)}{n-1}\int C^mS^{2-n}$$

Exercise 33.

$$\begin{aligned}(C^{m-1}S^{1-n})' &= (m-1)C^{m-2}(-S^{2-n}) + (1-n)C^mS^{-n} \\ &= -(m-1)C^{m-2}(S^{-n})(1-C^2) + (1-n)C^mS^{-n} = \\ &= -(m-1)C^{m-2}S^{-n} + (m-1)C^mS^{-n} + (1-n)C^mS^{-n}\end{aligned}$$
$$C^{m-1}S^{1-n} = -(m-1)\int C^{m-2}S^{-n} + (m-n)\int C^mS^{-n}$$
$$\frac{m-1}{m-n}\int C^{m-2}S^{-n} + \frac{C^{m-1}S^{1-n}}{m-n} = \int C^mS^{-n}$$

Exercise 34.

(1) $P'(x) - 3P(x) = 4 - 5x + 3x^2$

$$P = \sum_{j=0}^{n} a_jx^j$$
$$P' = \sum_{j=1}^{n} a_jjx^{j-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$
$$\Longrightarrow \sum_{j=0}^{n-1}(a_{j+1}(j+1) - 3a_j)x^j = 4 - 5x + 3x^2$$

Generally, we can say

$$a_{j+1} = \frac{3}{j+1} a_j \quad \text{if } j \geq 3$$

We also have

$$\begin{aligned} &a_1(1) - 3a_0 = 4 \\ &a_2(2) - 3a_1 = -5 \\ &a_3(3) - 3a_2 = 3 \quad \Longrightarrow a_3 - a_2 = 1 \end{aligned}$$

Then let $a_2 = -1$ and $a_3 = 0$. So we have $a_1 = 1$ and $a_0 = -1$. $P(x) = -1 + x + -x^2$ is one possible polynomial and we were only asked for one.

Suppose Q s.t. $Q' - 3Q = 4 - 5x + 3x^2$ (another solution). Then

$$(P-Q)' - 3(P-Q) = 0 \quad \forall x$$

$$\Longrightarrow \frac{(P-Q)'}{P-Q} = 3 \qquad \Longrightarrow \ln(P-Q) = 3x \quad \Longrightarrow ke^{3x} = P - Q = k\sum_{j=0}^{\infty} \frac{(3x)^j}{j!}$$

$$Q = -k\sum_{j=0}^{\infty} \frac{(3x)^j}{j!} + P$$

Since we didn't specify what Q has to be, we find that, in general, any Q is P plus some "amount" of the homogeneous solution, ke^{3x}.

(2) If $Q(x)$ is a given polynomial, and suppose P is a polynomial solution to $P'(x) - 3P(x) = Q(x)$. Suppose R is another polynomial solution such that $R'(x) - 3R(x) = Q(x)$. Then just like above, $P - R = ke^{3x}$. If we wanted polynomial answers of finite terms, then k must be zero. Thus, there's at most only one polynomial solution P.

Exercise 35. Bernoulli Polynomials.

(1) $P_1(x) = 1$; $P_n'(x) = nP_{n-1}(x)$; $\int_0^1 P_n(x)dx = 0, \quad$ if $n \geq 1$

$n = 1$ $\quad (1)(1) = P_1' \quad \int_0^1 (x+c) = (\frac{1}{2}x^2 + Cx)\Big|_0^1 = \frac{1}{2} + C = 0 \quad C = -1/2$

$P_1 = x - 1/2$

$n = 2$ $\quad 2(x - 1/2) = P_2' \quad \int_0^1 (x^2 - x + C) = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + Cx\right)\Big|_0^1 = \frac{-1}{6} + C = 0 \quad C = 1/6$

$P_2 = x^2 - x + 1/6$

$n = 3$ $\quad 3(x^2 - x + \frac{1}{6}) = P_3' \quad \int_0^1 (x^3 - \frac{3x^2}{2} + \frac{x}{2} + C) = \frac{1}{4} - \frac{1^3}{2} + \frac{1}{4} + C = 0$

$P_3 = x^3 - \frac{3x^2}{2} + \frac{x}{2}$

$n = 4$ $\quad 4(x^3 - \frac{3}{2}x^2 + \frac{x}{2}) = P_4' \quad \int_0^1 (x^4 - 2x^3 + x^2 + C) = \frac{1}{5} - \frac{1^4}{2} + \frac{1}{3}1^3 + C = 0 \quad C = \frac{-1}{30}$

$P_4 = x^4 - 2x^3 + x^2 + \frac{-1}{30}$

$n = 5$ $\quad 5(x^4 - 2x^3 + x^2 - \frac{1}{30}) = P_5' \quad \int_0^1 (x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6} + C) = \frac{1}{6} - \frac{(1)^5}{2} + \frac{5(1)^4}{12} + \frac{-(1)^2}{12} + C = 0; \quad C = 0$

$P_5 = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6}$

(2) The first, second, and up to fifth case has already been proven.
Assume the nth case, that $P_n(t) = t^n + \sum_{j=0}^{n-1} a_j t^j$ (the general form of a polynomial of degree n).

$$P_{n+1}' = (n+1)(t^n + \sum_{j=0}^{n-1} a_j t^j) \Longrightarrow P_{n+1} = t^{n+1} + (n+1)\sum_{j=0}^{n-1} \frac{a_j t^{j+1}}{j+1} + C$$

(3) The first, second, and up to fifth case has already been proven.
Assume the nth case, that $P_n(0) = P_n(1)$.

$$\int P'_{n+1} = P_{n+1}(1) - P_{n+1}(0) \quad \text{(by the second fundamental theorem of calculus)}$$
$$P'_{n+1} = (n+1)P_n$$
$$\int_0^1 (n+1)P_n(t) = 0 \quad \text{(by the given properties of Bernoulli polynomials)}$$
$$\Longrightarrow P_{n+1}(1) = P_{n+1}(0)$$

(4) $P_n(x+1) - = P_n(x) = nx^{n-1}$ is true for $n = 1, 2$, by quick inspection (and doing some algebra mentally).

$$P'_{n+1} = (n+1)P_n \Longrightarrow \int P'_{n+1} = (n+1)\int P_n$$
$$P_{n+1}(x+1) - P_{n+1}(x) = P_{n+1}(a_1) + (n+1)\int_{a_1}^{x+1} P_n(t) - (P_{n+1}(a_2) + (n+1)\int_{a_2}^{x} P_n(t))$$
$$a_1 = 1;\ a_2 = 0;\ \text{so } P_{n+1}(1) - P_{n+1}(0) = 0 \quad \text{(from previous problems)}$$
$$\Longrightarrow \begin{aligned} P_{n+1}(x+1) - P_{n+1}(x) &= (n+1)(\int_1^{x+1} P_n(t) - \int_0^x P_n(t)) = \\ &= (n+1)\int_0^x P_n(t+1) - P_n(t) = (n+1)\int_0^x nt^{n-1} = (n+1)x^n \end{aligned}$$

(5)

$$\int_0^k P_n = \int_0^k \frac{P'_{n+1}}{n+1} = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}$$
$$P_n(x+1) - P_n(x) = nx^{n-1}$$
$$\frac{P_n(x+1) - P_n(x)}{n} = x^{n-1}$$
$$\Longrightarrow \begin{aligned} \sum_{x=1}^{k-1} \frac{P_{n+1}(x+1) - P_{n+1}(x)}{n+1} &= \sum_{x=1}^{k-1} x^n = \sum_{r=1}^{k-1} \frac{P_{n+1}(r+1) - P_{n+1}(r)}{n+1} = \\ &= \sum_{r=1}^{k-1} r^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1} \quad \text{(telescoping series and } P_{n+1}(1) = P_{n+1}(0)\text{)} \end{aligned}$$

(6) **This part was fairly tricky**. A horrible clue was that this part will rely directly on the last part (because of the way this question is asked), which gave us $\sum_{j=1}^{x-1} j^n = \int_0^x P_n(t)dt = \frac{P_{n+1}(x) - P_{n+1}(0)}{n+1}$.

Use induction. It can be easily verified, plugging in, that $P_n(1-x) = (-1)^n P_n(x)$ is true for $n = 0 \dots 5$. Assume the nth case is true.

$$\int_0^x P_n(t)dt = \int_1^{1-x} -P_n(1-u)du = -\int_1^{1-x} P_n(u)(-1)^n du$$
$$\text{(since } P_n(1-x) = (-1)^n P_n(x)\text{, assumed } n\text{th case is true)}$$
$$u = 1-t \qquad du = -dt$$
$$= (-1)^{n+1}\int_1^{1-x} P_n(t)dt =$$
$$= (-1)^{n+1}\int_0^{1-x} P_n(t)dt \quad \text{(since } \int_0^1 P_n = 0\text{)}$$
$$\Longrightarrow = (-1)^{n+1}\left(\frac{P_{n+1}(1-x) - P_{n+1}(0)}{n+1}\right) = \frac{P_{n+1}(x) - P_{n+1}(0)}{n+1}$$
$$\Longrightarrow P_{n+1}(1-x) = (-1)^{n+1} P_{n+1}(x)$$

In the second to last and last step, we had used $(-1)^{n+1} P_{n+1}(0) = P_{n+1}(0)$. For $n+1$ even, this is definitely true. If $n+1$ was odd,

Doing some algebra for the first five cases, we can show that $P_{2j-1}(0) = 0$ for $j = 2, 3$. Assume the jth case is true. Since P_{2j-1} is a polynomial and $P_{2j-1}(0) = 0$, then the form of P_{2j-1} must be $P_{2j-1} = \sum_{k=1}^{2j-1} a_k x^j$. Using $P'_{n+1} = (n+1)P_n$,

$$\begin{aligned} P_{2j}(x) - P_{2j}(0) &= 2j \int_0^x P_{2j-1} = 2j \sum_{k=1}^{2j-1} \int_0^x a_k t^k = 2j \sum_{k=1}^{2j-1} a_k \left. \frac{1}{k+1} t^{k+1} \right|_0^x = \\ &= 2j \sum_{k=1}^{2j-1} \frac{a_k}{k+1} x^{k+1} = 2j \sum_{k=2}^{2j} \frac{a_{k-1}x^k}{k} + P_{2j}(0) \end{aligned}$$

$$P_{2j+1}(x) - P_{2j+1}(0) = (2j+1) \int_0^x \left(\sum_{k=2} (2j) \frac{a_{k-1}}{k} t^k + P_{2j}(0) \right) = (2j+1) \sum_{k=3}^{2j+1} \frac{2ja_{k-2}x^k}{k(k-1)} + (2j+1)P_{2j}(0)x$$

If we take the integral from 0 to 1, then we find that $P_{2j+1}(0) = 0$

(7) Using $P_n(1-x) = (-1)^n P_n(x)$, derived above,

$$\begin{aligned} &P_{2j+1}(0) = (-1)^{2j+1} P_{2j+1}(1) = (-1) P_{2j+1}(0) \\ &\Longrightarrow P_{2j+1}(0) = 0 \\ &P_{2j-1}(1 - \frac{1}{2}) = (-1)^{2j-1} (P_{2j-1}(\frac{1}{2})) \\ &\Longrightarrow P_{2j-1}(\frac{1}{2}) = 0 \end{aligned}$$

Exercise 36. There's a maximum at c for f, so $f'(c) = 0$

$$\begin{aligned} &\int_a^x f''(t)dt = f'(x) - f'(a) \\ &\int_0^c f''(t)dt = f'(c) - f'(0) = -f'(0) \\ &\int_c^a f''(t)dt = f'(a) - f'(c) = f'(a) \end{aligned} \qquad \begin{aligned} &|f'(0)| = |\int_c^0 f''(t)dt| \leq \int_0^c |f''|dt \leq mc \\ &|f'(a)| \leq m(a-c) \end{aligned}$$

$$\boxed{|f'(0)| + |f'(a)| \leq ma}$$

6.9 Exercises - Introduction, Motivation for the definition of the natural logarithm as an integral, The definition of the logarithm. Basic properties; The graph of the natural logarithm; Consequences of the functional equation $L(ab) = L(a) + L(b)$; Logarithm referred to any positive base $b \neq 1$; Differentiation and integration formulas involving logarithms;Logarithmic differentiation.

Exercise 1.

(1)

$$\log x = c + \left. (\ln |t|) \right|_e^x = c + \ln |x| - 1 \qquad \Longrightarrow \ln \left(\frac{x}{|x|} \right) = c - 1$$

$\boxed{c = 1}$

(2)

$$\begin{aligned} f(x) = \ln \frac{1+a}{1-a} + \ln \frac{1+b}{1-b} = \ln \left(\frac{(1+a)(1+b)}{(1-a)(1-b)} \right) = \ln \left(\frac{1+x}{1-x} \right) \\ \Longrightarrow \frac{1+x}{1-x} = \frac{(1+a)(1+b)}{(1-a)(1-b)} = \frac{1+a+b+ab}{1-b-a+ab} \end{aligned}$$

$\boxed{x = \frac{a+b}{1+ab}}$

Exercise 2.

(1) $\log(1+x) = \log(1-x)$

$\Longrightarrow \boxed{x = 0}$

(2) $\log(1+x) = 1 + \log(1-x)$

$$\ln(1+x) = 1 + \ln(1-x) = \ln(e) + \ln(1-x)\ln e(1-x)$$

$$\Longrightarrow 1 + x = e - ex \qquad \Longrightarrow \boxed{x = \frac{e-1}{1+e}}$$

(3) $2\log x = x \log 2$

$$\ln x^2 + \ln 2^{-x} = 0 = \ln 1 = \ln x^2 2^{-x} \qquad \Longrightarrow \boxed{x = 2}$$

(4) $\log(\sqrt{x} + \sqrt{x+1}) = 1$

$$\sqrt{x+1} = e - \sqrt{x} \Longrightarrow x + 1 = e^2 - 2\sqrt{x}e + x$$

$$2\sqrt{x}e = e^2 - 1 \qquad \Longrightarrow \boxed{x = \left(\frac{e^2-1}{2e}\right)^2}$$

Exercise 3.

$$f = \frac{\ln x}{x}$$

$$f' = \frac{1 - \ln x}{x^2}$$

$$f'' = \frac{-2}{x^3} - \left(\frac{\frac{1}{x}x^2 - 2x\ln x}{x^4}\right) = \frac{2\ln x - 3}{x^3}$$

$f' < 0$ when $x > e$ $\qquad f' > 0$ when $0 < x < e$

for $x^3 > 0$, $\quad 2\ln x - 3 > 0 \Longrightarrow f''(x) < 0$ (concave), when $0 < x < e^{3/2}$;

$f''(x) > 0$ (convex), when $x > e^{3/2}$

Exercise 4. $f(x) = \log(1 + x^2)$

$$f' = \frac{2x}{1+x^2}$$

Exercise 5. $f(x) = \log\sqrt{1+x^2}$

$$f' = \frac{x}{1+x^2}$$

Exercise 6. $f(x) = \log\sqrt{4-x^2}$

$$f' = \frac{1}{2}\frac{-2x}{4-x^2} = \frac{-x}{4-x^2}$$

Exercise 7. $f(x) = \log(\log x)$

$$f' = \frac{1}{\ln x}\left(\frac{1}{x}\right) = \frac{1}{x\ln x}$$

Exercise 8. $f(x) = \log x^2 \log x$

$$f' = (2\log x + \log\log x)' = \frac{2}{x} + \frac{1}{x\log x}$$

Exercise 9. $f(x) = \frac{1}{4}\log\frac{x^2-1}{x^2+1}$

$$f' = \frac{1}{4}\left(\log x^2 - 1 - \log x^2 + 1\right)' = \frac{1}{4}\left(\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right) = x\left(\frac{1}{x^4-1}\right)$$

Exercise 10. $f(x) = (x + \sqrt{1+x^2})^n$

$$\ln f = n\ln(x + \sqrt{1+x^2})$$

$$\frac{f'}{f} = n\left(\frac{1}{x+\sqrt{1+x^2}}\right)\left(1 + \frac{x}{\sqrt{1+x^2}}\right) = \frac{n}{\sqrt{1+x^2}}$$

$$f' = (x + \sqrt{1+x^2})^n\frac{n}{\sqrt{1+x^2}}$$

Exercise 11. $f(x) = \sqrt{x+1} - \log(1 + \sqrt{x+1})$

$$f' = \frac{1}{2\sqrt{x+1}} - \frac{1}{1+\sqrt{x+1}}\left(\frac{1}{2\sqrt{x+1}}\right) = \boxed{\frac{1}{2(1+\sqrt{x+1})}}$$

Exercise 12. $f(x) = x\log(x+\sqrt{1+x^2}) - \sqrt{1+x^2}$

$$f' = \log(x+\sqrt{1+x^2}) + \frac{x}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right) - \frac{x}{\sqrt{1+x^2}}$$

Note to self: *Notice* how this had made some of the square root terms disappear.

Exercise 13. $f(x) = \frac{1}{2\sqrt{ab}}\log\frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}$

$$f = \frac{1}{2\sqrt{ab}}(\ln(\sqrt{a}+x\sqrt{b}) - \ln(\sqrt{a}-x\sqrt{b}))$$

$$f' = frac12\sqrt{ab}\left(\frac{1}{\sqrt{a}+x\sqrt{b}}\sqrt{b} - \frac{1}{\sqrt{a}-x\sqrt{b}}(-\sqrt{b})\right) = \boxed{\frac{-x\sqrt{b}}{\sqrt{a}(a-bx^2)}}$$

Exercise 14. $f(x) = x(\sin(\log x) - \cos\log x)$

$$f' = \sin(\ln x) - \cos(\ln x) + (\cos(\ln x) + \sin(\ln x)) = 2\sin(\ln x)$$

Exercise 15. $f(x) = \log^{-1} x$

$$f' = \frac{-1}{x(\ln x)^2}$$

Exercise 16. $\int \frac{dx}{2+3x} = \frac{1}{3}\ln(2+3x)$

Exercise 17. $\int log^2 x dx$

$$\begin{aligned} (x\ln^2 x)' &= \ln^2 x + 2\ln x \\ (x\ln x - x)' &= \ln x + 1 - 1 = \ln x \end{aligned} \qquad \Longrightarrow xln^2 x - 2(x\ln 2 - x) = x\ln^2 x - 2x\ln x + 2x$$

Exercise 18. $\int x\log x dx$

$$\left(\frac{x^2\ln x}{2}\right)' = x\ln x + \frac{x}{2}$$

$$\int x\ln x = \frac{x^2\ln x}{2} - \frac{x^2}{4}$$

Exercise 19. $\int x\log^2 x dx$

$$\left(\frac{x^2\ln^2 x}{2}\right)' = x\ln^2 x + x^2\ln x\left(\frac{1}{x}\right)$$

$$\Longrightarrow \int x\log^2 x = \frac{x^2\ln^2 x}{2} - \frac{x^2\ln x}{2} - \frac{x^2}{4}$$

Exercise 20. $\int_0^{e^2-1}\frac{dt}{1+t}$

$$\int_0^{e^3-1}\frac{dt}{1+t} = \ln(1+t)|_0^{e^3-1} = 3$$

Exercise 21. $\int \cot x dx$

$$\int \frac{\cos x}{\sin x}dx = \ln|\sin x|$$

Exercise 22. $\int x^n \log(ax)dx$ *Solve the problem directly.*

$$\int x^n \log ax = \int x^n \log a + \int x^n \log x = \frac{x^{n+1}}{n+1}\log a + \int x^n \log x$$
$$\int x^n \ln x = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\frac{1}{x} = \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$$
$$\Longrightarrow \int x^n \log ax = \frac{x^{n+1}}{n+1}\log a + \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$$

Exercise 23. $\int x^2 \log^2 x dx$

$$\int x^2 \log^2 x = \frac{1}{3}x^3 \ln^2 x - \int \frac{x^2}{3} 2\ln x = \frac{x^3 \ln^2 x}{3} - \frac{2}{3}\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9} \right) = \boxed{\frac{x^3 \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27}}$$

Exercise 24. $\int \frac{dx}{x \log x}$

$$\int \frac{dx}{x \ln x} = \ln(\ln x) + C$$

Exercise 25. $\int_1^{1-e^{-2}} \frac{\log(1-t)}{1-t} dt$

$$\int_0^{1-e^2} \frac{\ln(1-t)}{1-t}dt = -\frac{1}{2}(\ln(1-t))^2\Big|_0^{1-e^{-2}} = \boxed{-2}$$

Exercise 26. $\int \frac{\log|x|}{x\sqrt{1+\log|x|}}dx$

$$\int \frac{\log x}{x\sqrt{1+\log x}} = \int (2(1+\log x)^{1/2})' \log x = 2(1+\log x)^{1/2}\log x - \int \frac{2(1+\log x)^{1/2}}{x} =$$
$$= \boxed{2\log x(1+\log x)^{1/2} - \frac{4}{3}(1+\log x)^{3/2}}$$

Exercise 27. Derive

$$\int x^m \log^n x dx = \frac{x^{m+1}}{m+1}\ln^n x - \frac{n}{m+1}\int x^m \ln^{n-1} x$$

By inspection, we just needed integration by parts.

$$\int x^3 \ln^3 x = \frac{x^4}{4}\ln^3 x - \frac{3}{4}\int x^3 \ln^2 x = \frac{x^4}{4}\ln^3 x - \frac{3}{4}\left(\frac{x^4}{4}\ln^2 x - \frac{2}{4}\int x^3 \ln x \right) = \frac{x^4 \ln^3 x}{4} - \frac{3x^4 \ln^2 x}{16} + \frac{3x^4 \ln x}{32} - \frac{3x^4}{128}$$

Exercise 28. Given $x > 0$, $\quad f(x) = x - 1 - \ln x; \quad g(x) = \ln x - 1 + \frac{1}{x}$

(1)

$$f' = 1 - \frac{1}{x} \qquad\qquad xg' = f'$$
$$g' = \frac{1}{x} - \frac{1}{x^2} = \frac{1}{x}f'$$

so then if $f' > 0,\ g' > 0; \quad f' < 0, \quad g' < 0$

For $f' < 0 \quad 0 < x < 1 \qquad f' > 0 \quad x > 1 \qquad f'(1) = g'(1) = 0$

$$f(1) = g(1) = 0$$

$x - 1 - \ln x > 0$ since $f(1) = 0$ is a rel. min.

$0 < \ln x - 1 + \frac{1}{x}$ since $g(0)$ is a rel. min.

(2) See sketch.

Exercise 29. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(1) $L(x) = \int_1^x \frac{1}{t}dt\,; \qquad L'(x) = \frac{1}{x}; \qquad L'(1) = 1$
(2) Use this theorem.

Theorem 19 (Theorem I.31).
If 3 real numbers a, x, and y satisfy the inequalities

$$a \leq x \leq a + \frac{y}{n}$$
$$\forall n \geq 1, \quad n \in \mathbb{Z}, \textit{ then } x = a$$

$$1 - \frac{1}{x} < \ln x < x - 1 \Longrightarrow 1 - \frac{1}{x+1} < \ln x + 1 < x$$
$$1 - \frac{x}{1+x} < \frac{\ln x + 1}{x} < 1 \qquad \Longrightarrow \boxed{\frac{\ln(1+x)}{x} = 1}$$

Exercise 30. Using $f(xy) = f(x) + f(y)$,

$$r = \frac{p}{q} \qquad \Longrightarrow f(a^{p/q}) = f((a^{1/q})^p) = pf(a^{1/q}) = \frac{p}{q}f(a)$$

since

$$f(a) = f(a); \qquad f(a^2) = f(aa) = 2f(a) \qquad \text{If } f(a) = f((a^{1/q})q) = qf(a^{1/q})$$
$$f(a^{p+1}) = f(a^p) + f(a) = (p+1)f(a) \qquad \text{then } f(a^{1/q}) = \frac{1}{q}f(a)$$

Exercise 31. $\ln x = \int_1^{|x|} \frac{1}{t} dt$

(1) $\ln x = \int_1^{|x|} \frac{1}{t} dt$ From this definition, then for n partitions, $a_0 = 1$, $a_1 = 1 + \frac{b-1}{n}, \dots, a_n = b = x$
$\frac{b-a}{n} = \frac{b-1}{n}$; so if $a_k = 1 + k\left(\frac{b-1}{n}\right)$

$$\sum_{k=1}^n \left(\frac{a_k - a_{k-1}}{a_k}\right) < \log x < \sum_{k=1}^n \left(\frac{a_k - a_{k-1}}{a_{k-1}}\right)$$

(2) $\log x$ is greater than the step function integral consisting of rectangular strips within $\frac{1}{x}$ and less than rectangular strips covering over $\frac{1}{x}$

(3) $a_k = 1 + k \qquad \Longrightarrow \frac{a_k - a_{k-1}}{a_{k-1}} = \frac{1}{k}$

$$\sum_{k=1}^n \frac{1}{1+k} < \ln(n+1) < \sum_{k=1}^n \frac{1}{k} \Longrightarrow \sum_{k=2}^{n+1} \frac{1}{k} < \ln(n+1) < \sum_{k=1}^n \frac{1}{k}$$
$$\Longrightarrow \sum_{k=2}^n \frac{1}{k} < \ln(n) < \sum_{k=1}^{n-1} \frac{1}{k}$$

Exercise 32.

(1) $L(x) = \frac{1}{\ln b} \ln x = \log_b x$

$$\log_a x = c \log_b x \qquad \Longrightarrow \log_a a = 1 \qquad \text{(There must be a unique real number s.t. } L(a) = 1\text{)}$$
$$\log_a x = \frac{1}{\log_b a} \log_b x \qquad \Longrightarrow \log_b x = \log_b a \log_a x$$

(2) Changing labels for a, b: $\log_b x = \frac{\log_a x}{\log_a b}$

Exercise 33. $\log_e 10 = 2.302585$

$$\log_{10} e = \frac{\log_e e}{\log_e 10} \qquad \Longrightarrow \log_e 10 = \frac{1}{\log_{10} e} = \frac{1}{2.302585} \simeq 0.43429$$

Exercise 34. Given $\int_x^{xy} f(t)dt = B(y) \qquad f(2) = 2,$, We Want $A(x) = \int_1^x f(t)dt$

$$\int_x^{xy} f(t)dt = B(xy) - B(x)$$
$$\frac{d}{dx}\int_x^{xy} f(t)dt = \frac{d}{dx}\left(B(xy) - B(x)\right) = \frac{dB(xy)}{d(xy)}y - \frac{dB(x)}{dx} = f(xy)y - f(x) = 0$$
$$\Longrightarrow f(xy) = \frac{f(x)}{y}$$
$$A(x) = \int_1^x f(t)dt = \int_1^x f((2)\left(\frac{t}{2}\right))dt = \int_1^x \frac{f(2)}{(t/2)}dt = \boxed{4\ln x}$$

Exercise 35. Given $\int_1^{xy} f(t)dt = y\int_1^x f(t)dt + x\int_1^y f(t)dt$, and letting F be the antiderivative of f,

$$F(xy) - F(1) = y(F(x) - F(1)) + x(F(y) - F(1))$$
$$\xrightarrow{d/dx} f(xy)(y) = y(f(x)) + \int_0^y f(t)dt \xrightarrow{d/dx} f'(xy)y^2 = y(f'(x))$$
$$\xrightarrow{x=1} f'(y) = \frac{(f'(1))}{y} \xrightarrow{\int} f(y) = k\ln y + C \xrightarrow{y=1} f(1) = 0 + C = 3$$

$\int_1^{xy}(k\ln t + 3) = y\int_1^x(k\ln t + 3) + x\int_1^y(k\ln t+3)$

$k\left((xy)\ln xy - (xy) + 1\right) + 3(xy - 1) = y\left(k\left((x)\ln x - x + 1\right) + 3(x-1)\right) + x\left(k\left(y\ln y - y + 1\right) + 3(y-1)\right)$

$$\Longrightarrow k - 3 = ky - 3y - kxy + kx + 3xy - 3x$$
$$y(3-k) + xy(k-3) + k - 3 + (3-k)x = 0 \Longrightarrow k = 3$$
$$\Longrightarrow f(x) = 3\ln x + 3$$

Exercise 36.

6.11 Exercises - Polynomial approximations to the logarithm.

Exercise 1.

Theorem 20 (Theorem 6.5). *If $0 < x < 1$ and if $m \geq 1$,*

$$\ln\frac{1+x}{1-x} = 2(x + \frac{x^3}{3} + \dots + \frac{x^{2m-1}}{2m-1}) + R_m(x)$$

where

$$\frac{x^{2m+1}}{2m+1} < R_m(x) \leq \frac{2-x}{1-x}\frac{x^{2m+1}}{2m+1}$$
$$R_m(x) = E_{2m}(x) - E_{2m}(-x)$$

where $E_{2m}(x)$ is the error term for $\log 1 - x$

$m = 5, \quad x = \frac{1}{3}$

$$\ln\left(\frac{4/3}{2/3}\right) = \ln 2 \simeq 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \frac{\left(\frac{1}{3}\right)^9}{9}\right) \simeq 0.693146047$$

The error for $m = 5, \quad x = \frac{1}{3}$ is

$$\frac{\left(\frac{1}{3}\right)^{11}}{11} \leq R_5(x) \leq \frac{5}{2}\frac{\left(\frac{1}{3}\right)^{11}}{11}$$

Exercise 2. $\ln\left(\frac{1+x}{1-x}\right) = \ln\frac{3}{2} = \ln 3 - \ln 2$

$$x = \frac{1}{5}$$
$$m = 5$$
$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(\frac{1}{5} + \frac{(1/5)^3}{3} + \frac{(1/5)^7}{7} + \frac{(1/5)^9}{9}\right) = 0.405465104$$
$$\frac{(1/5)^{11}}{11} \simeq 0.000000002 < R_5(x) \leq \frac{9}{4}\frac{\left(\frac{1}{5}\right)^{11}}{11} \simeq 0.000000004$$
$$\implies \boxed{\begin{array}{c} \log 3 \simeq 1.098611 \\ 1.098611666 < \log 3 < 1.098612438 \end{array}}$$

Exercise 3. $x = \frac{1}{9}$ $\quad \ln\left(\frac{1+\frac{1}{9}}{1-\frac{1}{9}}\right) = \ln\left(\frac{10}{8}\right) = \ln\frac{5}{4} = \ln 5 - 2\ln 2$

For $m = 2$

$$2\left(\frac{1}{9} + \frac{\left(\frac{1}{9}\right)^5}{3} + \frac{\left(\frac{1}{9}\right)^5}{5}\right) = 0.2231435$$
$$\frac{\left(\frac{1}{9}\right)^7}{7} \simeq 0.00000003 < R_3(x) \leq \frac{\frac{17}{9}}{\frac{8}{9}}\frac{\left(\frac{1}{9}\right)^7}{7} \simeq 0.000000063$$
$$1.609437 < \log 5 < 1.609438$$

Exercise 4. $x = \frac{1}{6}$ $\quad \ln\left(\frac{1+\frac{1}{6}}{1-\frac{1}{6}}\right) = \ln\left(\frac{7}{5}\right) = \ln 7 - \ln 5.$

For $m = 3$,

$$\ln\frac{1+x}{1-x} \simeq 2\left(\frac{1}{6} + \frac{\left(\frac{1}{6}\right)^3}{3} + \frac{\left(\frac{1}{6}\right)^5}{5}\right) = 0.336471193$$

The error bounds are
$$\frac{\left(\frac{1}{6}\right)^{2(3)+1}}{2(3)+1} = 0.00000051$$
$$\frac{\left(\frac{1}{6}\right)^{2(3)+1}}{2(3)+1}\left(\frac{2-\frac{1}{6}}{1-\frac{1}{6}}\right) = 0.000001123$$
$$1.945908703 < \ln 7 < 1.945910316$$

Exercise 5.

$$0.6931460 < \ln 2 < 0.6931476$$

$$\ln 3 = 1.098614$$
$$\ln 4 = 1.386293$$
$$\ln 5 = 1.609436 \ln 6 \quad = \ln 2 + \ln 3 = 1.791700$$
$$\ln 7 = 1.945909$$
$$\ln 8 = 3\ln 2 = 2.0794404$$
$$\ln 9 = 2\ln 3 = 2.197228$$
$$\ln 10 = \ln 5 + \ln 2 = 1.302577$$

6.17 Exercises - The exponential function, Exponentials expressed as powers of e, The definition of e^x for arbitrary real x, The definition of a^x for $a > 0$ and x real, Differentiation and integration formulas involving exponentials.

Exercise 1. $f' = 3e^{3x-1}$

Exercise 2. $8xe^{4x^2}$

Exercise 3. $-2xe^{-x^2}$

Exercise 4. $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

Exercise 5. $\frac{-1}{x^2}e^{-1/x}$

Exercise 6. $\ln 22^x$

Exercise 7. $(2x\ln 2)2^{x^2}$

Exercise 8. $\cos x e^{\sin x}$

Exercise 9. $-2\cos x \sin x e^{\cos^2 x}$

Exercise 10. $\frac{1}{x}e^{\log x}$

Exercise 11. $e^x e^{e^x}$

Exercise 12. $e^{e^{e^x}}(e^x e^{e^x})$

Exercise 13. $\int xe^x dx = xe^x - e^x$

Exercise 14. $\int xe^{-x} dx = -xe^{-x} + -e^{-x}$

Exercise 15. $\int x^2 e^x = x^2 e^x - 2xe^x + 2e^x$

Exercise 16. $\int x^2 e^{-2x} dx = \frac{x^2 e^{-2x}}{-2} + \frac{xe^{-2x}}{-2} + \frac{-e^{-2x}}{4}$

Exercise 17. $\int e^{\sqrt{x}} = e^{\sqrt{x}}(2\sqrt{x}) - \int \frac{1}{\sqrt{x}} e^{\sqrt{x}} = e^{\sqrt{x}}(2\sqrt{x}) - 2e^{\sqrt{x}} = \boxed{2(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}})}$

Exercise 18. $\int x^3 e^{-x^2}$.

$$\begin{aligned}
x^2 e^{-x^2} &= -2x^3 e^{-x^2} + 2xe^{-x^2} \\
e^{-x^2} &= -2xe^{-x^2} \\
\frac{1}{-2}(x^2 e^{-x^2} + e^{-x^2})' = \frac{1}{-2}\left(2xe^{-x^2} + -2x^3 e^{-x^2} + -2xe^{-x^2}\right) &= \boxed{\frac{x^2 e^{-x^2} + e^{-x^2}}{-2}}
\end{aligned}$$

Exercise 19. $e^x = b + \int_a^b e^t dt = b + e^x - e^a, \quad e^a = b.$

Exercise 20.

$$\begin{aligned}
A &= \int e^{ax} \cos bx dx \\
B &= \int e^{ax} \sin bx dx \\
A &= \frac{e^{ax}}{a}\cos bx - \int -\frac{b \sin bx e^{ax}}{a} = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}B \quad \Longrightarrow aA + bB = e^{ax}\cos bx + C \\
B &= \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a} b \cos bx \Longrightarrow aB + bA = e^{ax}\sin bx + C \\
A &= \frac{1}{a^2+b^2}\left(ae^{ax}\cos bx + be^{ax}\sin bx\right) \\
B &= \frac{-be^{ax}\cos bx + ae^{ax}\sin bx}{a^2+b^2}
\end{aligned}$$

Exercise 21.

$$\ln f = x \ln x; \quad \frac{f'}{f} = \ln x + 1; \quad f' = x^x(\ln x + 1)$$

Exercise 22.

$$\ln \frac{f'}{f} = \frac{1}{1+x} + \frac{1}{1+e^{x^2}}(2xe^{x^2}), \quad f' = 1 + e^{x^2} + 2(1+x)xe^{x^2}$$

Exercise 23. $f = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$$\ln f = \ln\left(e^x - e^{-x}\right) - \ln\left(e^x + e^{-x}\right)$$
$$\frac{f'}{f} = \frac{1}{e^x - e^{-x}}\left(e^x + e^{-x}\right) - \frac{1}{e^x + e^{-x}}\left(e^x - e^{-x}\right)$$
$$f' = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

Exercise 24. $f' = (x^{a^a})' + (a^{x^a})' + (a^{a^x})'$.

$$\begin{aligned} f_1 &= x^{a^a} & f_2 &= a^{x^a} & f_3 &= a^{a^x} \\ \ln f_1 &= a^a \ln x & \ln f_2 &= x^a \ln a & \ln f_3 &= a^x \ln a \\ \frac{f_1'}{f_1} &= \frac{a^a}{x} & \frac{f_2'}{f_2} &= ax^{a-1}\ln a & \frac{f_3'}{f_3} &= (\ln a)^2 a^x \\ f_1' &= x^{a^a - 1}a^a & f_2' &= a^{x^a+1}x^{a-1}\ln a & f_3' &= (\ln a)^2 a^{x+a^x} \end{aligned}$$

$\Longrightarrow f' = x^{a^a-1}a^a + a^{x^a+1}x^{a-1}\ln a + (\ln a)^2 a^{x+a^x}$

Exercise 25.

$$e^f = \ln(\ln x);$$
$$f'e^f = \frac{1}{\ln x}\left(\frac{1}{x}\right)$$
$$f' = \frac{1}{\ln(\ln x)}\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$$

Exercise 26. $e^f = e^x + \sqrt{1+e^{2x}}$

$$(e^f)f' = e^x + \frac{e^{2x}}{\sqrt{1+e^{2x}}}$$
$$f' = \frac{\sqrt{1+e^{2x}}e^x + e^{2x}}{\sqrt{1+e^{2x}}(e^x + \sqrt{1+e^{2x}})} = \frac{e^x}{\sqrt{1+e^{2x}}}$$

Exercise 27. $\ln f = x^x \ln x$

$$\frac{f'}{f} = (x^x)'\ln x + \frac{x^x}{x} = (x^x(\ln x + 1))\ln x + x^{x-1}$$
$$f' = \boxed{x^{x+x^x}(\ln x + 1)\ln x + x^{x^{x^x}+x-1}}$$

Exercise 28. $\ln f = x\ln(\ln x)$

$$\frac{f'}{f} = \ln(\ln x) + \frac{1}{\ln x}$$
$$f' = (\ln x)^x(\ln(\ln x) + \frac{1}{\ln x})$$

Exercise 29. $\ln f = (\ln x)\ln x$

$$\frac{f'}{f} = \frac{2\ln x}{x}$$
$$\boxed{f' = 2x^{\log x - 1}\ln x}$$

Exercise 30. $\ln f = x\ln(\ln x) - \ln x \ln x = x\ln\ln x - (\ln x)^2$

$$\frac{f'}{f} = \ln(\ln x) + \frac{1}{\ln x} - 2\frac{\ln x}{x}$$
$$f' = \frac{(\ln x)^x}{x^{\ln x}}\left(\ln\ln x + \frac{1}{\ln x} - \frac{2\ln x}{x}\right)$$

Exercise 31.

$$\begin{aligned}
&\ln f_1 = \cos x \ln \sin x && \ln f_2 = \sin x \ln \cos x \\
&\frac{f_1'}{f_1} = -\sin x \ln \sin x + \frac{\cos^2 x}{\sin x} && \frac{f_2'}{f_2} = \cos x \ln \cos x + \frac{-\sin^2 x}{\cos x} \\
&f_1' = -\sin x \cos x (\ln \sin x)^2 + \frac{\cos^3 x \ln \sin x}{\sin x} && f_2' = \sin x \cos x (\ln \cos x)^2 - \frac{\sin^3 x \ln \cos x}{\cos x}
\end{aligned}$$

$$\implies f = \sin x \cos x (-(\ln \sin x)^2 + (\ln \cos x)^2) + \frac{\cos^3 x \ln \sin x}{\sin x} + \frac{-\sin^3 x \ln \cos x}{\cos x}$$

Exercise 32. $\ln f = \frac{1}{x} \ln x$

$\frac{f'}{f} = \frac{-1}{x^2} \ln x + \frac{1}{x^2} \qquad \implies f' = -x^{1/x-2} \ln x + x^{1/x-2}$

Exercise 33. $\ln f = 2\ln x + \frac{1}{3} \ln (3-x) - \ln (1-x) + \frac{-2}{3} \ln (3+x)$

$$\frac{f'}{f} = \frac{2}{x} + \frac{-1}{3(3-x)} - \frac{-1}{1-x} - \frac{2}{3}\frac{1}{3+x}$$

$$\begin{aligned}
f' &= 2\frac{x(3-x)^{1/3}}{(1-x)(3+x)^{2/3}} + -\frac{1}{3}\left(\frac{x^2(3-x)^{-2/3}}{(1-x)(3+x)^{2/3}}\right) + \frac{x^2(3-x)^{1/3}}{(1-x)^2(3+x)^{2/3}} - \frac{2}{3}\frac{x^2(3-x)^{1/3}}{(1-x)(3+x)^{5/3}} \\
&= \frac{x(18 - 12x + \frac{4}{3}x^2 + \frac{2}{3}x^3)}{(1-x)^2(3+x)^{5/3}(3-x)^{2/3}}
\end{aligned}$$

Exercise 34. $\ln f = \sum_{i=1}^n b_i \ln (x - a_i)$

$$\frac{f'}{f} = \sum_{i=1}^n \frac{b_i}{x - a_i} \qquad \implies \boxed{f' = \sum_{j=1}^n \frac{b_j}{x - a_j} \prod_{i=1}^n (x - a_i)^{b_i}}$$

Exercise 35.

(1) Show that $f' = rx^{r-1}$ for $f = x^r$ holds for arbitrary real r.

$$x^r = e^{r \ln x}$$

$$(e^{r \ln x})' = e^{r \ln x}\frac{r}{x} = rx^{r-1}$$

(2) For $x \leq 0$, by inspection of $x^r = e^{r \log x}$, then if $x^r > 0$, then the equality would remain valid. So then $x^r = |x^r| = |x|^r$ and so

$$\ln |f(x)| = r \ln |x|$$

$$\frac{f'(x)}{f(x)} = r\frac{1}{x} \implies f'(x) = rx^{r-1}$$

Exercise 36. Use the definition $a^x = e^{x \log a}$

(1) $\log a^x = x \log a$

Taking the exponential is a well-defined inverse function to $\log$ so taking the $\log$ of both sides of the definition, we get $\log a^x = x \log a$

(2) $(ab)^x = a^x b^x$

$$(ab)^x = e^{x \log ab} = e^{x(\log a + \log b)} = a^x b^x$$

(3) $a^x a^y = a^{x+y}$

$$a^{x+y} = e^{(x+y)\log a} = e^{x \log a} e^{y \log a} = a^x a^y$$

(4) $(a^x)^y = (a^y)^x = a^{xy}$

$$(a^x)^y = e^{xy \log a} = (a^y)^x = a^{xy}$$

(5) If $x = \log_a y$,
Using the definition

$$\log_b x = \frac{\log x}{\log b} \quad \text{if } b > 0,\ b \neq 1, \quad x > 0$$

so then

$$\log_a y = \frac{\log y}{\log a} = x \Longrightarrow \begin{array}{c} \log y = x \log a \\ e^{x \log a} = e^{\log y} = y = a^x \end{array}$$

If $y = a^x$,

$$\log y = x \log a \Longrightarrow x = \frac{\log y}{\log a} = \log_a y$$

Exercise 37. Let $f(x) = \frac{1}{2}(a^x + a^{-x})$ if $a > 0$.

$$\begin{aligned} f(x+y) &= \frac{1}{2}(a^{x+y} + a^{-(x+y)}) \\ f(x+y) + f(x-y) &= \frac{1}{2}\left(a^{x+y} + a^{-(x+y)} + a^{x-y} + a^{-(x-y)}\right) \\ f(x)f(y) &= \frac{1}{4}(a^x + a^{-x})(a^y + a^{-y}) = \frac{1}{4}\left(a^{x+y} + a^{-x-y} + a^{-(x-y)} + a^{(x-y)}\right) \end{aligned}$$

Exercise 38.

$$f(x) = e^{cx};\ f'(x) = ce^{cx};\ \boxed{f'(0) = c}$$

$$\lim_{x \to 0} \frac{e^{cx} - 1}{x} = c\left(\lim_{x \to 0} \frac{e^{cx} - 1}{cx}\right) = c\left(\lim_{cx \to 0} \frac{e^{cx} - 1}{cx}\right) = c\frac{df(cx)}{d(cx)}(0) = \frac{d}{dx}(e^{cx})(0)$$

$$\lim_{x \to 0} \frac{e^{cx} - 1}{x} = f'(0) = c$$

Exercise 39.

$$\begin{aligned} & g(x) = f(x)e^{-cx} \\ & g'(x) = f'e^{-cx} + -cg = cg - cg = 0 \\ & \boxed{f = Ke^{kx}} \end{aligned}$$

Exercise 40. Let f be a function defined everywhere on the real axis. Suppose also that f satisfies the functional equation

$$f(x+y) = f(x)f(y) \text{ for all } x \text{ and } y$$

(1)

$$f(0) = f(0)f(0) = f^2(0) \quad \begin{array}{l} \text{If } f(0) = 0\text{, then we're done} \\ \text{If } f(0) \neq 0 \text{ then } f(0) = 1 \text{ (by dividing both sides by } f(0)\text{)} \end{array}$$

(2) Take the derivative with respect to x on both sides of the functional equation.

$$\frac{df(x+y)}{d(x+y)}\frac{d(x+y)}{dx} = \frac{df(x)}{dx}f(y) \quad \Longrightarrow \frac{d(f(x+y))}{d(x+y)} = f'(x)\frac{f(x+y)}{f(x)}$$

$$\boxed{\text{Let } y = -x + y}$$

$$\frac{df(y)}{dy} = f'(x)\frac{f(y)}{f(x)} \quad \Longrightarrow f'(x)f(y) = f'(y)f(x)$$

(3) $\frac{f'(x)}{f(x)} = \frac{f'(y)}{f(y)} \quad \forall x, y$

The only way they could do that for any arbitrary x, for any arbitrary y they one could choose on *either* side, is for them to both equal a constant
$\Longrightarrow \frac{f'(y)}{f(y)} = c$

(4) Referring to Exercise 39 of the same section, $f = e^{cx}$ since $f'(0) = 1$

Exercise 41.

(1)

$$f = e^x - 1 - x \qquad f(0) = e^0 - 1 - 0 = 0 \qquad \begin{cases} e^x > 1 + x & \text{for } x > 0 \\ e^{-x} > 1 - x & \text{for } x < 0 \end{cases}$$
$$f' = e^x - 1 \gtrless 0 \text{ if } x \gtrless 0$$

(2)

$$\begin{aligned} \int_0^x e^t = e^x - 1 > x + \frac{1}{2}x^2 \quad & \implies e^x > 1 + x + \frac{1}{2}x^2 \\ -e^{-x} > -1 + x - \frac{x^2}{2} \quad & \implies e^{-x} < 1 - x + \frac{x^2}{2} \end{aligned}$$

(3)

$$\begin{aligned} \int_0^x e^t = e^x - 1 > x + \frac{1}{2}x^2 + \frac{1}{3*2}x^3 \quad & \implies e^x > 1 + x + \frac{1}{2}x^2 + \frac{1}{3*2}x^3 \\ -e^{-x} > -1 + x - \frac{x^2}{2} + \frac{x^3}{3*2} \quad & \implies e^{-x} < 1 - x + \frac{x^2}{2} - \frac{x^3}{3*2} \end{aligned}$$

(4) Suppose the nth case is true.

$$\begin{aligned} e^x &> \sum_{j=0}^n \frac{x^j}{j!} & e^{-x} &= \begin{cases} > \sum_{j=0}^{2m+1} \frac{x^j}{j!} \\ < \sum_{j=0}^{2m} \frac{x^j}{j!} \end{cases} \\ e^x &> 1 + \sum_{j=0}^n \frac{x^{j+1}}{(j+1)!} = 1 + \sum_{j=1}^{n+1} \frac{x^j}{j!} & &= \sum_{j=0}^{n+1} \frac{x^j}{j!} \\ -e^{-x} + 1 & \begin{cases} > \sum_{j=0}^{2m+1} \frac{x^{j+1}}{(j+1)!} \\ < \sum_{j=0}^{2m} \frac{x^{j+1}}{(j+1)!} \end{cases} \quad = & e^{-x} & \begin{cases} < \sum_{j=0}^{2m+2} \frac{x^j}{j!} \\ > \sum_{j=0}^{2m+1} \frac{x^j}{j!} \end{cases} \end{aligned}$$

Exercise 42. Using the result from Exercise 41,

$$\left(1 + \frac{x}{n}\right)^n = \sum_{j=0}^n \binom{n}{j} 1^{n-j} \left(\frac{x}{n}\right)^j = \sum_{j=0}^n \frac{n!}{(n-j)!j!} \frac{x^j}{n^j} = \sum_{j=0}^n \frac{n(n-1)\dots(n-j+1)}{j!} \frac{x^j}{n^j} < \sum_{j=0}^n \frac{x^j}{j!} < e^x$$

If you make this clever observation, the second inequality is easy to derive.

$$\begin{aligned} x > 0 \qquad & \frac{x}{n} > 0 \\ e^{-\frac{x}{n}} > 1 - \frac{x}{n} \quad & \implies \left(e^{-x/n}\right)^n > \left(1 - \frac{x}{n}\right)^n \\ e^{-x} > (1 - x/n)^n \quad & \implies e^x < (1 - x/n)^{-n} \end{aligned}$$

Exercise 43. $f(x, y) = x^y = e^{y \ln x}$

$$\begin{aligned} \partial_x f &= x^y y / x \\ \partial_y f &= x^y \ln x \end{aligned}$$

6.19 Exercises - The hyperbolic functions.

Exercise 7.

$$2 \sinh x \cosh x = 2 \frac{e^x - e^{-x}}{2} \frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^{2x} - e^{-x}) = \sinh 2x$$

Exercise 8.

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}) = \cosh 2x$$

Exercise 9.

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

Exercise 10.

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \left(\frac{e^x - e^{-x}}{2}\right) = e^{-x}$$

Exercise 11. Use induction.

$$(\cosh x + \sinh x)^2 = \cosh^2 x + 2\sinh x \cosh x + \sinh^2 x = \cosh 2x + \sinh 2x$$

$$\begin{aligned}
(\cosh x + \sinh x)^{n+1} &= (\cosh x + \sinh x)(\cosh nx + \sinh nx) = \\
&= \cosh nx \cosh x + \cosh nx \sinh x + \sinh nx \cosh x + \sinh x \sinh nx = \\
&= \begin{array}{l} \frac{e^{nx}+e^{-nx}}{2}\frac{e^{x}+e^{-x}}{2} + \frac{e^{nx}+e^{-nx}}{2}\frac{e^{x}-e^{-x}}{2} + \\ + \frac{e^{nx}-e^{-nx}}{2}\frac{e^{x}+e^{-x}}{2} + \frac{e^{nx}-e^{-nx}}{2}\frac{e^{x}-e^{-x}}{2} \end{array} \\
&= \cosh{(n+1)x} + \sinh{(n+1)x}
\end{aligned}$$

Exercise 12.

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x$$

6.22 Exercises - Derivatives of inverse functions, Inverses of the trigonometric functions.

Exercise 1.

$$(\cos x)' = -\sin x = -\sqrt{1-\cos^2 x}\, D\arccos x = \frac{1}{-\sqrt{1-x^2}} \quad -1 < x < 1$$

Exercise 2.

$$(\tan x)' = \sec^2 x = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1$$

$$D\arctan x = \frac{1}{1+x^2}$$

Exercise 3.

$$(\cot x)' = -\csc^2 x = -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} = -(1+\cot^2 x) \Longrightarrow arccotx = -\frac{1}{1+x^2}$$

Exercise 4.

$$(\sec y)' = \tan y \sec y = \sqrt{\sec^2 y - 1}\sec y; |\sec y| > 1\, \forall y \in \mathbb{R}$$

If we choose to restrict y such that $0 \leq y \leq \pi$, then $(\sec y)' > 0$. Then we must make $\sec y \to |\sec y|$.

$Darcsecx = \frac{1}{|x|\sqrt{x^2-1}}$

Exercise 5.

$$(\csc y)' = -\cot x \csc x = -\csc y(\sqrt{\csc^2 y - 1})$$

$$\text{Let } y \text{ such that } \frac{-\pi}{2} < y < \frac{\pi}{2}\, (\csc y) < 0$$

$$Darccscx = \frac{1}{-|x|\sqrt{x^2-1}}$$

Exercise 6.

$$(xarccotx)' = arccotx - \frac{x}{1+x^2}$$

$$\left(\frac{1}{2}\ln{(1+x^2)}\right)' = \frac{1x}{(1+x^2)}$$

$$\int arccotx = xarccotx + \frac{1}{2}\ln{(1+x^2)} + C$$

Exercise 7.

$$(xarcsecx)' = arcsecx + \frac{x}{|x|\sqrt{x^2-1}}$$

$$\left(\frac{x}{|x|}\ln|x+\sqrt{x^2-1}|\right)' = \begin{cases} \frac{1+\frac{x}{\sqrt{x^2-1}}}{|x+\sqrt{x^2-1}|} & x > 1 \\ -\frac{-1+\frac{x}{\sqrt{x^2-1}}}{|x+\sqrt{x^2-1}|} & x < -1 \end{cases} = \frac{x}{|x|\sqrt{x^2-1}}$$

$$\Longrightarrow \int arcsecxdx = xarcsecx - \frac{x}{|x|}\log|x+\sqrt{x^2-1}| + C$$

Take a note of this exercise. **When dealing with** $(\mp x^2 \pm 1)^{\frac{2j+1}{2}}; j \in \mathbb{Z}$**; try** $x \pm \sqrt{x^2 \pm 1}$ **combinations.** It'll work out.

Exercise 8.

$$\begin{gathered} (xarccscx)' = arccscx + \frac{x}{-|x|\sqrt{x^2-1}} \\ \left(\frac{x}{|x|} \ln|x + \sqrt{x^2-1}| \right)' = \begin{cases} \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{\sqrt{x^2-1}} & x > 1 \\ \frac{-1}{\sqrt{x^2-1}} & x < -1 \end{cases} \\ \implies \int arccscx = xarccscx + \frac{x}{|x|}\ln|x + \sqrt{x^2-1}| \end{gathered}$$

Exercise 9.

$$\begin{gathered} (x(\arcsin x)^2)' = (\arcsin x)^2 + \frac{2x \arcsin x}{\sqrt{1-x^2}} \\ \left(\sqrt{1-x^2}\arcsin x\right)' = \frac{-x}{\sqrt{1-x^2}}\arcsin x + 1 \\ \boxed{\int (\arcsin x)^2 = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x} \end{gathered}$$

Exercise 10.

$$\left(\frac{-\arcsin x}{x} \right)' = \frac{1}{x^2}\arcsin x - \frac{1}{x\sqrt{1-x^2}}$$

I would note how x is in the denominator of the second term. Again, reiterating,

$$\begin{gathered} (\sqrt{1-x^2})' = \frac{-x}{\sqrt{1-x^2}} \\ (y \pm \sqrt{\pm 1 \mp x^2})(y \mp \sqrt{\pm 1 \mp x^2}) = y^2 - (\pm 1 \mp x^2) \end{gathered}$$

Multiply by its "conjugate." As we see, choose y appropriately to get the desired denominator (that's achieved after differentiation). Here, pick $y = 1$.

$$\begin{gathered} (\ln(1 + \sqrt{1-x^2}))' = \frac{1}{1+\sqrt{1-x^2}}\left(\frac{-x}{\sqrt{1-x^2}} \right) = \frac{-x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}(x^2)} = \frac{-(1-\sqrt{1-x^2})}{x\sqrt{1-x^2}} \\ \implies \int \frac{\arcsin x}{x^2} = \ln\left| \frac{1-\sqrt{1-x^2}}{x} \right| - \frac{\arcsin x}{x} + C \end{gathered}$$

Exercise 11.

(1)

$$D\left(arccotx - \arctan\frac{1}{x} \right) = \frac{-1}{x^2+1} - \frac{1}{1 + \left(\frac{1}{x}\right)^2}\left(\frac{-1}{x^2} \right) = 0$$

(2) $arccotx - \arctan\frac{1}{x} = C$

Now $arccotx = \frac{\pi}{2} - \arctan x$.

$$\begin{gathered} \frac{\pi}{2} - \arctan x - \arctan\frac{1}{x} = C \implies \frac{\pi}{2} - C = \arctan x + \arctan\frac{1}{x} \\ x \to \infty \implies \frac{\pi}{2} - C = \frac{\pi}{2} + 0 \implies C = 0 \\ \text{but } x \to -\infty \\ \frac{\pi}{2} - C = -\frac{\pi}{2} + 0 \implies C = \pi \end{gathered}$$

There are problems with the choice of brances for $arccotx, \arctan\frac{1}{x}$, even though the derivatives work in all cases.

Exercise 12.

$$f' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left(\frac{1}{2}\right)$$

Exercise 13.

$$f' = \frac{-1}{\sqrt{1-\left(\frac{1-x}{\sqrt{2}}\right)^2}}\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{\sqrt{1+2x-x^2}}$$

Exercise 14. $f = \arccos \frac{1}{x}$.

$$f' = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(\frac{-1}{x^2}\right) = \frac{1}{\sqrt{x^2-1}|x|}$$

Exercise 15.

$$f(x) = \arcsin(\sin x) = \frac{1}{\sqrt{1-\sin^2 x}}\cos x = \frac{\cos x}{|\cos x|}$$

Exercise 16.

$$\frac{1}{2\ sqrtx} - \frac{1}{x+1}\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x+1)}$$

Exercise 17.

$$\frac{1}{1+x^2} + \frac{x^2}{1+x^6}$$

Exercise 18.

$$\frac{1}{\sqrt{1-\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)^2}}\left(\frac{-2x(2)}{(1+x^2)^2}\right) = \frac{\sqrt{1+x^2}}{\sqrt{1+x^2-(1-x^2)}}\frac{-4x}{(1+x^2)^2} = \frac{-4}{(1+x^2)^{3/2}\sqrt{2}}$$

Exercise 19. $f = \arctan\tan^2 x$

$$\frac{1}{1+\tan^4 x}\left(2\tan x\sec^2 x\right) = \frac{2\tan x\sec^2 x}{1+\tan^4 x}$$

Exercise 20.

$$f' = \frac{1}{1+(x+\sqrt{1+x^2})^2}(1+\frac{x}{\sqrt{1+x^2}}) = \frac{\sqrt{1+x^2}+x}{1+(x+\sqrt{1+x^2})^2}$$

Exercise 21.

$$f' = \frac{1}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}}(\cos x+\sin x) = \frac{\cos x+\sin x}{\sqrt{2\sin x\cos x}}$$

Exercise 22.

$$f' = (\arccos\sqrt{1-x^2})' = \frac{1}{-\sqrt{1-(1-x^2)}} = -\frac{1}{|x|}$$

Exercise 23.

$$f' = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\left(\frac{2}{(1-x)^2}\right) = \frac{2}{(1-x)^2+(1+x)^2} = \frac{2}{(1-2x+x^2+1+2x+x^2)} = \frac{1}{1+x^2}$$

Exercise 24. $f = (\arccos(x^2))^{-2}$

$$f' = -2(\arccos x^2)^{-3}\left(\frac{-1}{\sqrt{1-x^2}}\right)(2x) = \frac{4x(\arccos x^2)^{-3}}{\sqrt{1-x^4}}$$

Exercise 25.

$$f' = \left(\frac{1}{\arccos\frac{1}{\sqrt{x}}}\right)\left(\frac{-1}{\sqrt{1-\frac{1}{x}}}\right)\left(\frac{-1}{2x^{3/2}}\right) = \frac{1}{2\arccos\frac{1}{\sqrt{x}}\sqrt{x^3-x^2}}$$

Exercise 26. $\frac{dy}{dx} = \frac{x+y}{x-y}$.

$$\left(\arctan\frac{y}{x}\right)' = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{y'x-y}{x^2}\right) = \left(\frac{1}{2}\ln(x^2+y^2)\right)' = \frac{1}{2}\frac{1}{(x^2+y^2)}(2x+2yy')$$
$$\implies y' = \frac{x+y}{x-y}$$

Exercise 27.

$$\begin{aligned}
\ln y &= \ln(\arcsin x) - \frac{1}{2}\ln(1-x^2) \\
\frac{y'}{y} &= \frac{1}{\arcsin x}\left(\frac{1}{\sqrt{1-x^2}}\right) - \frac{1}{2}\frac{1}{1-x^2}(-2x) = \frac{1}{\arcsin x\sqrt{1-x^2}} + \frac{x}{1-x^2} \\
y &= \frac{\arcsin x}{\sqrt{1-x^2}} \\
y' &= \left(\frac{1}{1-x^2}\right) + \frac{(\arcsin x)x}{(1-x^2)^{3/2}} = \frac{\sqrt{1-x^2}+x(\arcsin x)}{(1-x^2)^{3/2}} \\
\ln y' &= \ln(\sqrt{1-x^2}+x\arcsin x) - \frac{3}{2}\ln(1-x^2) \\
\frac{y''}{y'} &= \frac{1}{\sqrt{1-x^2}+x\arcsin x}\left(\frac{-x}{\sqrt{1-x^2}} + \arcsin x + \frac{x}{\sqrt{1-x^2}}\right) - \frac{3}{2}\frac{(-2x)}{1-x^2} \\
y'' &= y'\frac{\arcsin x}{\sqrt{1-x^2}+x\arcsin x} + \frac{y'3x}{1-x^2} = \boxed{\frac{\arcsin x}{(1-x^2)^{3/2}} + \frac{(\sqrt{1-x^2}+x\arcsin x)(3x)}{(1-x^2)^{3/2}}}
\end{aligned}$$

Exercise 28.

$$f' = \frac{1}{1+x^2} - 1 + x^2 = \frac{1-1-x^2+x^4}{1+x^2} = \frac{x^4}{1+x^2} \geq 0\,\forall x$$
$$\text{since } f(0) = \arctan 0 - 0 + 0 = 0, \quad \arctan x > x - \frac{x^3}{3},\ \forall x > 0$$

Exercise 29.

$$\int\frac{dx}{\sqrt{a^2-x^2}},\ a \neq 0 \implies \boxed{\arcsin\frac{x}{a}}$$

Exercise 30.

$$\int\frac{dx}{\sqrt{2-(x+1)^2}} = \int\frac{dx}{\sqrt{2}\sqrt{1-\left(\frac{x+1}{\sqrt{2}}\right)^2}} = \arcsin\frac{x+1}{\sqrt{2}}$$

Exercise 31.

$$\int\frac{dx}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)} = \frac{1}{a}\arctan\frac{x}{a}$$

Exercise 32.

$$\frac{dx}{a(1+\left(\frac{\sqrt{ba}}{\sqrt{a}}\right)^2)} = \frac{1}{\sqrt{ba}}\arctan\frac{\sqrt{b}x}{\sqrt{a}}$$

Exercise 33.

$$\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} = \frac{4}{7}\int\frac{dx}{\left(\frac{2}{\sqrt{7}}\left(x-\frac{1}{2}\right)\right)^2+1} = \boxed{\frac{2}{\sqrt{7}}\arctan\frac{2\left(x-\frac{1}{2}\right)}{\sqrt{7}}}$$

Exercise 34.

$$\left(\frac{x^2\arctan x}{2}\right)' = x\arctan x + \frac{x^2}{2}\frac{1}{1+x^2} = x\arctan x + \frac{1}{2}\left(1 - \frac{1}{1+x^2}\right)$$
$$\left(\frac{1}{2}\left(x - \arctan x\right)\right)' = \frac{1}{2}\left(1 - \frac{1}{1+x^2}\right)$$
$$\int x\arctan x = x\arctan x + -\frac{1}{2}(x - \arctan x)$$

Exercise 35.

$$\left(\frac{x^3}{3}\arccos x\right)' = x^2\arccos x + \frac{x^3}{3}\frac{-1}{\sqrt{1-x^2}}$$
$$(x^2\sqrt{1-x^2})' = 2x\sqrt{1-x^2} + -\frac{x^3}{\sqrt{1-x^2}}$$
$$((1-x^2)^{3/2})' = \frac{3}{2}(-2x)(1-x^2)^{1/2} = -3x(1-x^2)^{1/2}$$
$$\boxed{\int x^2\arccos x = \frac{x^3}{3}\arccos x - \frac{1}{3}x^2\sqrt{1-x^2} - \frac{9}{2}(1-x^2)^{3/2}}$$

Exercise 36.

$$\left(\frac{x^2(\arctan x)^2}{2}\right)' = x(\arctan x)^2 + x^2\left(\frac{1}{1+x^2}\right)\arctan x = x(\arctan x)^2 + \left(1 - \left(\frac{1}{1+x^2}\right)\right)\arctan x$$
$$\left(\frac{(\arctan x)^2}{2}\right)' = \frac{\arctan x}{1+x^2}$$
$$(x\arctan x)' = \arctan x + \frac{x}{1+x^2}$$
$$\int x(\arctan x)^2 dx = \frac{x^2(\arctan x)^2}{2} - \left(x\arctan x - \frac{\ln(1+x^2)}{2}\right) + \frac{(\arctan x)^2}{2}$$

Exercise 37.

$$(\arctan\sqrt{x})' = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right)$$
$$(x\arctan\sqrt{x})' = \arctan\sqrt{x} + \frac{\sqrt{x}}{2(1+x)} = \arctan\sqrt{x} + \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+x)}\right)$$
$$(x\arctan\sqrt{x} + \arctan\sqrt{x} + -x^{1/2})' = \arctan\sqrt{x} + 0$$
$$\boxed{\int \arctan\sqrt{x} = x\arctan\sqrt{x} + \arctan\sqrt{x} - x^{1/2}}$$

Exercise 38. From the previous exercise,

$$\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx = (\arctan\sqrt{x})^2$$

Exercise 39. Let $x = \sin u$

$$\int \sqrt{1-x^2}dx = \int \cos^2 u du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{4}$$

Exercise 40.

$$\int \frac{xe^{\arctan x}}{(1+x^2)^{3/2}}$$

$$\left(\frac{e^{\arctan x}}{\sqrt{1+x^2}} \right)' = \frac{-xe^{\arctan x}}{(1+x^2)^{3/2}} + \frac{e^{\arctan x}}{(1+x^2)^{3/2}}$$

$$\left(\frac{xe^{\arctan x}}{\sqrt{1+x^2}} \right)' = \frac{e^{\arctan x}}{\sqrt{1+x^2}} + -\frac{x^2 e^{\arctan x}}{(1+x^2)^{3/2}} + \frac{xe^{\arctan x}}{(1+x^2)^{3/2}} = \frac{e^{\arctan x}}{(1+x^2)^{3/2}} + \frac{xe^{\arctan x}}{(1+x^2)^{3/2}}$$

$$\frac{1}{2} \left(\frac{xe^{\arctan x}}{\sqrt{1+x^2}} - \frac{e^{\arctan x}}{\sqrt{1+x^2}} \right)' = \frac{xe^{\arctan x}}{(1+x^2)^{3/2}}$$

Exercise 41. From the previous exercise,

$$\frac{1}{2} \left(\frac{xe^{\arctan x}}{\sqrt{1+x^2}} + \frac{e^{\arctan x}}{\sqrt{1+x^2}} + C \right)$$

Exercise 42.

$$\text{Since } -\left(\frac{x(1+x^2)^{-1}}{2} \right)' = \frac{x^2}{(1+x^2)^2} - \frac{1}{2(1+x^2)}$$

$$\int \frac{x^2}{(1+x^2)^2} dx = \frac{-x}{2(1+x^2)} + \frac{1}{2} \arctan x$$

Exercise 43. $\arctan e^x$.

Exercise 44.

$$\int \frac{arccote^x}{e^x} dx$$

$$(arccote^x)' = \frac{-e^x}{1+e^{3x}}$$

$$-(e^{-x} arccote^x)' = e^{-x} arccote^x + \frac{e^{-x}(-1)e^x}{1+e^{2x}} = e^{-x} arccote^x + -\left(1 - \frac{e^{2x}}{1+e^{2x}} \right)$$

$$(\ln(1+e^{2x}))' = \frac{2e^{2x}}{1+e^{2x}}$$

$$\boxed{(-e^{-x} arccote^x + x - \frac{1}{2} \ln(1+e^{2x}))'} = e^{-x} arccote^x$$

Exercise 45.

$$\int \sqrt{\frac{a+x}{a-x}} dx = \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \left(\frac{1a}{\sqrt{1-\left(\frac{x}{a}\right)^2}} + \frac{x}{\sqrt{a^2-x^2}} \right) dx = a \arcsin \frac{x}{a} + -\sqrt{a^2-x^2}$$

Exercise 46.

$$\int \sqrt{x-a}\sqrt{b-x}dx = \int \sqrt{bx - ab - x^2 + ax}dx =$$

$$= \int \sqrt{-\left(x - \left(\frac{a+b}{2}\right)\right)\left(x - \left(\frac{a+b}{2}\right)\right) + \frac{a^2+b^2}{4} - \frac{2ab}{4}} =$$

$$= \int \sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \left(\frac{a+b}{2}\right)\right)^2} = \left(\frac{a-b}{2}\right) \int \sqrt{1 - \left(\frac{x - \left(\frac{a+b}{2}\right)}{\left(\frac{a-b}{2}\right)}\right)^2} dx =$$

$$= \left(\frac{a-b}{2}\right)^2 \int \sqrt{1-u^2} =$$

$$= \boxed{\left(\frac{a-b}{2}\right)^2 \frac{\arcsin\left(\frac{2x-(a+b)}{a-b}\right)}{2} + \frac{2x-(a+b)}{2(a-b)^2}\sqrt{(a-b)^2 - (2x-(a+b))^2}}$$

Since, recall,

$$\left(\frac{\arcsin x}{2} + \frac{1}{2}x\sqrt{1-x^2}\right)' = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{2} + \frac{1}{4}\frac{x(-2x)}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Exercise 47. Wow! $\boxed{\int \frac{dx}{\sqrt{(x-a)(b-x)}}}$

$$\begin{aligned} x - a &= (b-a)\sin^2 u \\ dx &= (b-a)(2)\sin u \cos u du \\ b - x &= (a-b)\sin^2 u + b - a = \boxed{(b-a)(\cos^2 u)} \end{aligned}$$

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{(b-a)(2)\sin u \cos u du}{\sqrt{b-a}\cos u \sqrt{b-a}\sin u} = 2u = \boxed{2\arcsin\sqrt{\frac{x-a}{b-a}}}$$

6.25 Exercises - Integration by partial fractions, Integrals which can be transformed into integrals of rational functions.

Exercise 1. $\int \frac{2x+3}{(x-2)(x+5)} = \int \left(\frac{1}{x-2}\right) + \left(\frac{1}{x+5}\right) = \ln(x-2) + \ln(x+5)$

Exercise 2. $\int \frac{xdx}{(x+1)(x+2)(x+3)}$

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} = A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2)$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 5 & 4 & 3 & 1 \\ 6 & 3 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & & 0 & -1/2 \\ & 1 & 0 & 2 \\ & 0 & 1 & -3/2 \end{array}\right]$$

$$A = -1/2, \quad B = 2, \quad C = -3/2$$

$$\Longrightarrow \boxed{\frac{-1}{2}\ln(x+1) + 2\ln(x+2) + \frac{-3}{2}\ln(x+3)}$$

Exercise 3. $\int \frac{x}{(x-2)(x-1)} = \int \frac{2}{x-2} + \frac{-1}{x-1} = 2\ln x - 2 - \ln(x-1)$

Exercise 4. $\int \frac{x^4+2x-6}{x^3+x^2-2x}dx$

$$\frac{x^4+2x-6}{x^3+x^2-2x} = x-1+\frac{3(x^2-2)}{x^3+x^2-2x} \qquad \text{(do long division)}$$

$$\int x-1+\frac{3(x^2-2)}{x(x+2)(x-1)} = \frac{1}{2}x^2-x+3\int \frac{x^2-2}{x(x+2)(x-1)}$$

$$\int \frac{x^2-2}{x(x+2)(x-1)} = \int \frac{1}{x}+\frac{1/3}{x+2}+\frac{-1/3}{x-1} = \ln x+\frac{1}{3}\ln x+2-\frac{1}{3}\ln x-1$$

$$\implies \boxed{\frac{1}{2}x^2-x+3\ln x+\ln x+2-\ln x-1}$$

Exercise 5. $\int \frac{8x^3+7}{(x+1)(2x+1)^3}dx$

$$\frac{8x^3+7}{(x+1)(2x+1)^3} = \frac{A}{(2x+1)^3}+\frac{B}{(2x+1)^2}+\frac{C}{(2x+1)}+\frac{D}{(x+1)}$$

$$8x^3+7 = A(x+1)+B(2x^2+3x+1)+(4x^3+8x^2+5x+1)C+D(8x^3+12x^2+6x+1)$$

$$\implies \begin{bmatrix} 0 & 0 & 4 & 8 \\ 0 & 2 & 8 & 12 \\ 1 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 7 \end{bmatrix} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 12 \\ & 1 & 0 & 0 & -6 \\ & & 1 & 0 & 0 \\ & & 0 & 1 & 1 \end{array}\right]$$

$$A = 12, \quad B = -6, \quad C = 0, \quad D = 1$$

$$\int \frac{12}{(2x+1)^3}+\frac{-6}{(2x+1)^2}+\frac{1}{x+1} = \boxed{\frac{-6(2x+1)^{-2}}{2}+\frac{6(2x+1)^{-1}}{2}+\ln(x+1)}$$

Exercise 6. $\int \frac{4x^2+x+1}{(x-1)(x^2+x+1)}$

$$\frac{4x^2+x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \qquad \implies A(x^2+x+1)+(Bx+C)(x-1) = 4x^2+x+1$$

$$\implies \begin{bmatrix} 1 & 1 & \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad \implies A = 2, \quad B = 2, \quad C = 1$$

$$\implies \int \frac{2}{x-1}+\frac{2x+1}{x^2+x+1} = \boxed{2\ln|x-1|+\ln|x^2+x+1|}$$

Exercise 7. $\int \frac{x^4 dx}{x^4+5x^2+4}$

Doing the long division, $\frac{x^4}{x^4+5x^2+4} = 1+-\left(\frac{5x^2+4}{(x^2+1)(x^2+4)}\right)$

$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4} = \frac{5x^2+4}{(x^2+1)(x^2+4)}$$

It could be seen that $A+C=0, \quad 4A+C=0 \quad$ so $A=C=0$

$$\begin{aligned} B+D &= 5 \qquad & B &= \frac{-1}{3} \\ 4B+D &= 4 & D &= \frac{16}{3} \end{aligned}$$

$$\implies \int 1-\frac{5x^2+4}{(x^2+1)(x^2+4)} = x-\int \frac{-1/3}{x^2+1}+\frac{16/3}{x^2+4} = \boxed{x+\frac{1}{3}\arctan x+4/3\arctan x/2+C}$$

Exercise 8. $\int \frac{x+2}{x(x+1)}dx = \int \frac{1}{x+1}+\frac{2}{x(x+1)} = \ln|x+1|+2\int \frac{1}{x}-\frac{1}{x+1} = \boxed{-\ln|x+1|+2\ln x}$

Exercise 9. $\int \frac{dx}{x(x^2+1)^2} = \int \frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$

$$\frac{A(x^4+2x^2+1)+x(Bx+C)(x^2+1)+Dx^2+Ex}{x(x^2+1)^2} = \frac{A(x^4+2x^2+1)+Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex}{x(x^2+1)^2}$$

$$\implies A=1; \quad B=-1; \quad D=-1; \quad C=0; \quad E=0$$

$$\int \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} = \boxed{\ln x + -\frac{\ln|x^2+1|}{2} + \frac{(x^2+1)^{-1}}{2}}$$

Exercise 10. $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}$

Exercise 11. $\int \frac{x}{(x+1)^2}dx$

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \qquad \implies \begin{array}{l} x = A(x+1)+B \\ A=1; \quad B=-1 \end{array}$$

$$\int \left(\frac{1}{x+1} + \frac{-1}{(x+1)^2} \right) dx = \ln x+1 + \frac{1}{x+1} + C$$

Exercise 12. $\int \frac{dx}{x(x^2-1)} = \int \frac{dx}{x(x-1)(x+1)} = \int \frac{A}{x} + \frac{B}{x-1}\frac{C}{x+1}$

$$A(x^2-1)+Bx(x+1)+Cx(x-1) = Ax^2-A+Bx^2+Bx+Cx^2-Cx \qquad \implies A=-1,\ B=\frac{1}{2}=C$$

$$\int \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} = -\ln x + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1|$$

Exercise 13. $\int \frac{x^2dx}{x^2+x-6} = \int \frac{x^2dx}{(x+3)(x-2)}$

The easiest way to approach this problem is to notice that this is an improper fraction and to do long division: $\frac{x^2}{x^2+x-6} = 1 + \frac{6-x}{x^2+x-6}$

$$\frac{A}{x+3} + \frac{B}{x-2} \implies \begin{array}{c} 6 = A(x-2)+B(x+3) \\ A = \frac{-6}{5}; \quad B = \frac{6}{5} \end{array} \qquad \begin{array}{l} -x = (A+B)x - 2A + 3B \\ 2A = 3B \\ A = \frac{3B}{2} \implies \begin{array}{l} B=-2/5 \\ A-3/5 \end{array} \end{array}$$

$$\implies \int 1 + \frac{-6/5}{x+3} + \frac{6/5}{x-2} + \frac{-3/5}{x+3} + \frac{-2/5}{x-2} = \boxed{\frac{-9}{5}\ln|x+3| + \frac{4}{5}\ln|x-2| + x + C}$$

Exercise 14. $\int \frac{x+2}{(x-2)^2} = \int \frac{x-2+4}{(x-2)^2} = \ln|x-2| + \int \frac{4}{(x-2)^2} = \boxed{\ln|x-2| + -4(x-2)^{-1}}$

Exercise 15. $\int \frac{dx}{(x-2)^2(x^2-4x+5)}$

Consider the denominator with its x^2-4x+5. Usually, we would try a partial fraction form such as $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2-4x+5}$, but the algebra will get messy. Instead, it helps to be clever here.

$$\frac{1}{(x-2)^2(x^2-4x+4+1)} = \frac{1}{(x-2)^2((x-2)^2+1)} = \frac{1}{(x-2)^2} - \frac{1}{(x-2)^2+1}$$

$$\implies \int \frac{dx}{(x-2)^2(x^2-4x+5)} = \int \frac{1}{(x-2)^2} - \frac{1}{(x-2)^2+1} = \boxed{-(x-2)^{-1} - \arctan(x-2) + C}$$

Exercise 16. $\int \frac{(x-3)dx}{x^3+3x^2+2x} = \int \frac{(x-3)dx}{x(x+2)(x+1)}$

$$\int \frac{(x-3)dx}{x(x+2)(x+1)} = \int \frac{1}{(x+2)(x+1)} + -3\int \frac{1}{x(x+2)(x+1)}$$

$$\frac{1}{(x+2)(x+1)} = \frac{-1}{x+2} + \frac{1}{x+1}$$

$$\frac{1}{x(x+2)(x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}$$

Now to solve for A, B, C in the last expression, it is useful to use Gaussian elimination for this system of three linear equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 2 & & & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1/2 \end{array}\right]$$

$$\Longrightarrow \frac{1}{x(x+2)(x+1)} = \frac{1/2}{x} + \frac{1/2}{x+2} + \frac{-1}{x+1}$$

$$\Longrightarrow -\ln|x+2| + \ln|x+1| + -3/2\ln x + -3/2\ln|x+2| + 3\ln|x+1| = \boxed{-5/2\ln|x+2| + 4\ln|x+1| - 3/2\ln x}$$

Exercise 17. Use partial fraction method to integrate $\int \frac{1}{(x^2-1)^2}$. Then **build the sum**.

$$\frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)} + \frac{D}{(x+1)}$$

$$\text{Now} \begin{array}{l} (x^2-1)(x-1) = x^3 - x^2 - x + 1 \\ (x^2-1)(x+1) = x^3 + x^2 - x - 1 \end{array}$$

$$\Longrightarrow (x^2-1)(x+1) - (x^2-1)(x-1) = 2x^2 - 2$$

$$x^2 + 2x + 1$$

$$x^2 - 2x + 1$$

$$\Longrightarrow \text{(summing the above two expressions we obtain) } 2x^2 + 2$$

$$\boxed{\frac{-1}{x-1} + \frac{1/4}{x+1} + \frac{1/4}{(x-1)^2} + \frac{1/4}{(x+1)^2}}$$

$$\boxed{\int \frac{dx}{(x^2-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| + -\frac{1}{2}\left(\frac{x}{x^2-1}\right)}$$

Exercise 18. Use the method of partial fractions, where we find that

$$\int \frac{(x+1)}{x^3-1}dx = \int \frac{x+1}{(x-1)(x^2+x+1)}dx = \int \frac{-\frac{2}{3}x - \frac{1}{3}}{x^2+x+1} + \frac{\frac{2}{3}}{x-1} =$$

$$= \boxed{-\frac{1}{3}\ln|x^2+x+1| + \frac{2}{3}\ln|x-1|}$$

where we had used the following partial fraction decomposition for the given integrand

$$\frac{Ax+B}{x^2+x+1} + \frac{C}{x-1} = \frac{x+1}{x^3-1}$$

$$Ax^2 + Bx - Ax - B + Cx^2 + Cx + C = x + 1$$

$$2Ax + B - A + 2Cx + C = 1 \quad \text{(where we used the trick to take the derivative of the above equation)}$$

$$\Longrightarrow A = -C \quad B - A - A = 1$$

$$-B + C = 1; \quad A = \frac{2}{-3}, C = \frac{2}{3}\, B = -\frac{1}{2}$$

Exercise 19. $\int \frac{x^4+1}{x(x^2+1)^2}$

Again, it helps to be clever here.

$$\int \frac{x^4+1}{x(x^2+1)^2} = \int \frac{x^4+2x^2+1-2x^2}{x(x^2+1)^2} = \int \frac{(x^2+1)^2}{x(x^2+1)^2} + \frac{-2x}{(x^2+1)^2} =$$

$$= \ln x + (x^2+1)^{-1} + C$$

Exercise 20. $\int \frac{dx}{x^3(x-2)}$

Working out the algebra for the partial fractions method, we obtain

$$\frac{1}{x^3(x-2)} = \left(\frac{-1/2}{x^3}\right) + \frac{-1/4}{x^2} + \frac{-1/8}{x} + \frac{1/8}{x-2}$$

So then

$$\boxed{\int \frac{dx}{x^3(x-2)} = \frac{1}{4x^2} + \frac{1}{4x} + \frac{-1}{8}\ln x + \frac{1}{8}\ln|x-2| + C}$$

Exercise 21.

$$\begin{aligned}
\int \frac{1-x^3}{x(x^2+1)} = -\int \frac{x^3-1}{x(x^2+1)} = -\int \frac{x^2}{x^2+1} + \int \frac{1}{x(x^2+1)} = \\
= -\int \left(1 - \frac{1}{x^2+1}\right) + \int \frac{1}{x} + \frac{-x}{x^2+1} = \\
= \boxed{-x + \arctan x + \ln x - \ln|x^2+1| + C}
\end{aligned}$$

Exercise 22.

$$\begin{aligned}
\int \frac{dx}{x^4-1} = \int \left(\frac{1}{x^2+1}\right)\left(\frac{1}{x^2-1}\right) = \int \frac{1/2}{x^2-1} - \frac{1/2}{x^2+1} = \\
= \int \frac{1}{2}\left(\frac{1/2}{x-1} - \frac{1/2}{x+1}\right) - \frac{1/2}{x^2+1} = \\
= \boxed{\frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\arctan x + C}
\end{aligned}$$

Exercise 23. $\boxed{\int \frac{dx}{x^4+1}}$

I had to rely on complex numbers.

Notice that with complex numbers, *you can split up polynomial power sums*

$$\begin{aligned}
x^4+1 = (x^2+i)(x^2-i) = (x+ie(i\frac{\pi}{4}))(x-ie(i\frac{\pi}{4}))(x+e(i\frac{\pi}{4}))(x-e(i\frac{\pi}{4})) = \\
= (x+e(i\frac{3\pi}{4}))(x-e(i\frac{3\pi}{4}))(x+e(i\frac{\pi}{4}))(x-e(i\frac{\pi}{4}))
\end{aligned}$$

$$\frac{A}{(x+e(i\frac{3\pi}{4}))} + \frac{B}{(x-e(i\frac{3\pi}{4}))} + \frac{C}{(x+e(i\frac{\pi}{4}))} + \frac{D}{(x-e(i\frac{\pi}{4}))} = \frac{1}{x^4+1}$$

$$A(x^2-i)(x-e(i\frac{3\pi}{4})) + B(x^2-i)(x+e(i\frac{3\pi}{4})) + C(x^2+i)(x-e(i\frac{\pi}{4})) + D(x^2+i)(x+e(i\frac{\pi}{4})) = 1$$

$$\xrightarrow{\text{do the algebra}} \begin{aligned}
&x^3 : \; A+B+C+D = 0 \\
&x^2 : \; -Ae(-\frac{3\pi}{4}) + Be(i\frac{3\pi}{4}) - Ce(i\frac{\pi}{4}) + De(i\frac{\pi}{4}) = 0 \\
&x^1 : \; -iA - iB + iC + iD = 0 \\
&x^0 : \; -e(i\frac{\pi}{4})A + Be(i\frac{\pi}{4}) + C(-e(i\frac{3\pi}{4})) + D(e(i\frac{3\pi}{4})) = 1
\end{aligned}$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ -e(i\frac{3\pi}{4}) & e(i\frac{3\pi}{4}) & -e(i\frac{\pi}{4}) & e(i\frac{\pi}{4}) \\ -i & -i & i & i \\ -e(i\frac{\pi}{4}) & e(i\frac{\pi}{4}) & -e(i\frac{3\pi}{4}) & e(i\frac{3\pi}{4}) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To do the complex algebra for the desired Gaussian elimination procedure, I treated the complex numbers as vectors and added them and rotated them when multiplied.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ \searrow & \nwarrow & \swarrow & \nearrow & 0 \\ \downarrow & \downarrow & \uparrow & \uparrow & 0 \\ \swarrow & \nearrow & \searrow & \nwarrow & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ & 2\nwarrow & & 2\nearrow & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2\searrow & 2\nwarrow & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \\ & 2\nwarrow & & 2\nearrow & \\ 0 & 0 & 1 & 1 & 0 \\ & & 0 & 4\nwarrow & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & & & & \frac{1}{4}\nwarrow \\ & \nwarrow & & & \frac{1}{4}\uparrow \\ 0 & 0 & 1 & 0 & \frac{1}{4}\nearrow \\ & & 0 & 1 & \frac{1}{4}\swarrow \end{array}\right] =$$

$$= \left[\begin{array}{cccc|c} 1 & & & & \frac{1}{4}\nwarrow \\ & 1 & & & \frac{1}{4}\searrow \\ & & 1 & & \frac{1}{4}\nearrow \\ & & & 1 & \frac{1}{4}\swarrow \end{array}\right]$$

$$\Longrightarrow \begin{aligned} A &= \frac{1}{4}e(\frac{i3\pi}{4}) \\ B &= -\frac{1}{4}e(\frac{i3\pi}{4}) \\ C &= \frac{1}{4}e(i\frac{\pi}{4}) \\ D &= \frac{-1}{4}e(i\frac{\pi}{4}) \end{aligned} \Longrightarrow \int \frac{1/4e(\frac{i3\pi}{4})}{x+e(i\frac{3\pi}{4})} + \frac{-1/4e(i\frac{3\pi}{4})}{x-e(i\frac{3\pi}{4})} + \frac{1/4e(i\frac{\pi}{4})}{x+e(i\frac{\pi}{4})} + \frac{-1/4e(i\frac{\pi}{4})}{x-e(i\frac{\pi}{4})}$$

$$\left(\frac{1}{4}\right)\left(e(i\frac{3\pi}{4})\ln{(x+e(i\frac{3\pi}{4}))} - e(i\frac{3\pi}{4})\ln{(x-e(i\frac{3\pi}{4}))} + e(i\frac{\pi}{4})\ln{(x+e(i\frac{\pi}{4}))} - e(i\frac{\pi}{4})\ln{(x-e(i\frac{\pi}{4}))}\right)$$

After doing some complex algebra,

$$\Longrightarrow \frac{1}{4\sqrt{2}}\left(\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| - 2\arctan\left(\frac{1}{\sqrt{2}x-1}\right) - 2\arctan\left(\frac{1}{\sqrt{2}x+1}\right)\right)$$

The computation could be done to do the derivative on this, so to check our answer and reobtain the integrand.

Is there a way to solve this without complex numbers?

Exercise 24. $\int \frac{x^2dx}{(x^2+2x+2)^2}$

$$\int \frac{x^2dx}{(x^2+2x+2)^2} = \int \frac{x^2+2x+2-2x-2}{(x^2+2x+2)^2} = \left(\int \frac{1}{x^2+2x+2}\right) + (x^2+2x+2)^{-1}$$

$$\int \frac{1}{x^2+2x+2} = \int \frac{1}{(x+1)^2+1} = \arctan{(x+1)}$$

$$\boxed{\int \frac{x^2dx}{(x^2+2x+2)^2} = \arctan{(x+1)} + \frac{1}{x^2+2x+2} + C}$$

Exercise 25. $\int \frac{4x^5-1}{(x^5+x+1)^2}dx$

$$(-(x^5+x+1)^{-1})' = (x^5+x+1)^{-2}(5x^4+1) \qquad \text{(doesn't work)}$$

$$(-x(x^5+x+1)^{-1})' = (x^5+x+1)^{-2}(5x^5+x) + -(x^5+x+1)^{-1} = (x^5+x+1)^{-2}(5x^5+x-x^5-x-1)$$

$$\Longrightarrow \int \frac{4x^5-1}{(x^5+x+1)^2}dx = \boxed{-x(x^5+x+1)^{-2}}$$

Exercise 26. $\int \frac{dx}{2\sin{x}-\cos{x}+5}$ *(good example of the use of half angle substitution)*

$$\int \frac{dx}{2\sin{x}+-\cos{x}+5} = \int \frac{dx}{4SC+-C^2+S^2+5}\left(\frac{\frac{1}{C^2}}{\frac{1}{C^2}}\right) = \int \frac{\sec^2{\frac{x}{2}}dx}{4T-1+T^2+5(1+T^2)} =$$

$$= \int \frac{\sec^2{\frac{x}{2}}dx}{6T^2+4T+4} = \int \frac{\sec^2{\frac{x}{2}}dx}{6(T+\frac{1}{3})^2+\frac{10}{3}} = \int \frac{2du}{6(u+\frac{1}{3})^2+\frac{10}{3}} \qquad \text{(where } \boxed{\begin{aligned} u &= \tan{\frac{x}{2}} \\ du &= \frac{\sec^2{\frac{x}{2}}}{2}dx \end{aligned}}\text{)}$$

$$= \frac{3}{5}\int \frac{du}{\frac{9(u+\frac{1}{3})^2}{5}+1} = \boxed{\frac{1}{\sqrt{5}}\arctan\left(\frac{3(\tan{\frac{x}{2}}+\frac{1}{3})}{\sqrt{5}}\right)}$$

Exercise 27. $\int \frac{dx}{1+a\cos x}$ $(0<a<1)$ Again, using the half-angle substitution, $\boxed{\begin{aligned} u &= \tan\frac{x}{2} \\ du &= \frac{\sec^2\frac{x}{2}}{2}dx \end{aligned}}$,

$$\frac{1}{a}\int \frac{dx}{\frac{1}{a}+\cos x} = \frac{1}{a}\int\frac{dx}{\frac{1}{a}+C^2-S^2} = \frac{1}{a}\int \frac{\sec^2\frac{x}{2}dx}{\frac{1}{a}\sec^2\frac{x}{2}+1-T^2} =$$

$$= \frac{1}{a}\int\frac{\sec^2\frac{x}{2}dx}{\frac{1}{a}+1+T^2(\frac{1}{a}-1)} = \frac{1}{a}\int\frac{2du}{\frac{1}{a}+1+u^2\left(\frac{1}{a}-1\right)} = \frac{2}{1+a}\int\frac{du}{1+\left(u\sqrt{\frac{1-a}{1+a}}\right)^2} =$$

$$\frac{2}{1+a}\frac{\arctan\sqrt{\frac{1-a}{1+a}}u}{\sqrt{\frac{1-a}{1+a}}} \Longrightarrow \boxed{\frac{2}{\sqrt{1-a^2}}\arctan\left(\sqrt{\frac{1-a}{1+a}}\tan\frac{x}{2}\right)}$$

Exercise 28. $\int\frac{dx}{1+a\cos x}$ Half-angle substitution.

$$\int\frac{dx}{1+a\cos x} = \int\frac{dx}{1+a(C^2-S^2)} = \int\frac{\sec^2\frac{x}{2}dx}{\sec^2\frac{x}{2}+a(1-T^2)} =$$

$$\begin{aligned} u &= \tan\theta/2 = T \\ du &= \sec^2\theta/2\left(\frac{1}{2}\right)d\theta \end{aligned} \Longrightarrow = \int\frac{2du}{1+T^2+a(1-T^2)} = 2\int\frac{du}{(1-a)T^2+(1+a)} = \frac{2}{1-a}\int\frac{du}{u^2-\frac{a+1}{a-1}} =$$

$$= \frac{2}{1-a}\int\left(\frac{1}{u-\sqrt{\frac{a+1}{a-1}}} - \frac{1}{u+\sqrt{\frac{a+1}{a-1}}}\right)\frac{1}{2\sqrt{\frac{a+1}{a-1}}} =$$

$$= \sqrt{\frac{a-1}{a+1}}\left(\frac{1}{1-a}\right)\left(\ln\left(u-\sqrt{\frac{a+1}{a-1}}\right)-\ln\left(u+\sqrt{\frac{a+1}{a-1}}\right)\right) =$$

$$= \boxed{\frac{-1}{\sqrt{a^2-1}}\left(\ln\left(\frac{\tan\frac{x}{2}-\sqrt{\frac{a+1}{a-1}}}{\tan\frac{x}{2}+\sqrt{\frac{a+1}{a-1}}}\right)\right)}$$

Exercise 29. $\int\frac{\sin^2 x}{1+\sin^2 x}dx$

$$\int\frac{s^2}{1+s^2}dx = \int\frac{s^2+1-1}{1+s^2}dx = x+-\int\frac{dx}{1+\sin^2 x}$$

$$\int\frac{dx}{1+\sin^2 x} = \int\frac{dx}{1+\left(\frac{1-\cos 2x}{2}\right)} = \int\frac{2dx}{3-\cos 2x} = \frac{2}{3}\int\frac{dx}{1-\frac{\cos 2x}{3}} = \frac{2}{3}\int\frac{dx}{1-\left(\frac{c^2-s^2}{3}\right)} = \frac{2}{3}\int\frac{\sec^2 x dx}{\sec^2 x-\left(\frac{1-T^2}{3}\right)} =$$

$$\begin{aligned} u &= \tan x \\ du &= \sec^2 x dx \end{aligned} \Longrightarrow = \frac{2}{3}\int\frac{du}{1+u^2-\left(\frac{1-u^2}{3}\right)} = \frac{2}{3}\int\frac{du}{\frac{2}{3}+\frac{4}{3}u^2} = \int\frac{du}{1+(\sqrt{2}u)^2} =$$

$$= \frac{1}{\sqrt{2}}\arctan\sqrt{2}\tan x$$

$$\Longrightarrow \boxed{\int\frac{\sin^2 x}{1+\sin^2 x}dx = x-\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x)}$$

It seems like for here, when *dealing with squares of trig. functions*, “step up” to double angle.

Exercise 30. $\int\frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ $(ab\neq 0)$ *Take note*, we *need not* change the angle to half-angle or double-angle.

$$\frac{1}{a^2s^2+b^2c^2} = \frac{1}{a^2(1-c^2)+b^2c^2} = \frac{1}{a^2+(b^2-a^2)c^2} = \frac{1}{a^2(1+(kc)^2)} = \frac{\sec^2}{a^2(\sec^2+k^2)} = \frac{\sec^2}{a^2(1+T^2+k^2)} =$$

$$\begin{aligned} u &= \tan x \\ du &= \sec^2 x dx \end{aligned} \implies = \frac{du}{a^2(1+u^2+k^2)} = \frac{1/a^2 du}{(1+k^2)(1+\frac{u^2}{1+k^2})} = \frac{\sqrt{1+k^2}}{a^2(1+k^2)} \arctan\left(\frac{u}{\sqrt{1+k^2}}\right)$$

$$\implies \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \boxed{\frac{1}{ab}\arctan\left(\frac{a\tan x}{b}\right)}$$

Exercise 31. $\int \frac{dx}{(a\sin x + b\cos x)^2} \quad (a \neq 0)$

Note it's a good idea to *simplify*, cleverly, your constants as much as you can.

$$\int \frac{dx}{(a\sin x + b\cos x)^2} = \frac{1}{a^2}\int \frac{dx}{(\sin x + k\cos x)^2}$$

Thus, only one constant, k, is only worried about.

$$\frac{1}{(s+kc)^2} = \frac{1}{s^2+2ksc+k^2c^2} = \frac{1/c^2}{t^2+2kt+k^2} = \frac{\sec^2}{(t+k)^2} =$$

$$\begin{aligned} u &= \tan x \\ du &= \sec^2 x dx \end{aligned} \implies = \frac{du}{(u+k)^2}$$

$$\implies \boxed{\frac{1}{a^2}\int \frac{1}{(s+\frac{b}{a}c)^2} = \frac{-1}{(a^2\tan x + ab)}}$$

Again, *note*, **we need not** always step up or step down a half angle in the substitution.

Exercise 32. Note that we have a rational expression consisting of single powers of $\sin$ and $\cos$. Then use the $\tan\frac{\theta}{2}$ substitution.

$$\int \frac{\sin x}{1+\cos x + \sin x} = \int \frac{2CS}{1+2CS+C^2-S^2} = \int \frac{CS}{C(S+C)} = \int \frac{T}{(T+1)}$$

$$\text{where} \quad \begin{aligned} C &= \cos x/2 \\ S &= \sin x/2 \end{aligned} \quad \begin{aligned} u &= \tan\theta/2 \\ du &= \frac{\sec^2\theta/2 d\theta}{2} \end{aligned}$$

$$\int \frac{u}{(u+1)}\left(\frac{2du}{u^2+1}\right) = \int \frac{2udu}{(u^2+1)(u+1)}$$

$$\frac{A}{u+1} + \frac{Bu+C}{u^2+1}$$

$$Au^2 + A + Bu^2 + Cu + Bu + C = u$$

$$A = -B \quad C+B=1 \quad A+C=0 \implies C = \frac{1}{2};\ B=\frac{1}{2};\ A = -\frac{1}{2}$$

$$2\int\left(\frac{-1/2}{u+1} + \frac{\frac{1}{2}(u+1)}{u^2+1}\right)du = \int \frac{-1}{u+1} + \frac{u+1}{u^2+1}$$

$$= -\ln|u+1| + \frac{1}{2}\ln|u^2+1| + \arctan u =$$

$$= -\ln|\tan x/2 + 1| + \frac{1}{2}\ln|\sec^2 x/2| + \frac{x}{2}$$

$$\implies -(\ln|2|) + \frac{1}{2}\ln|2| + \frac{\pi}{4} = \boxed{-\frac{1}{2}\ln|2| + \frac{\pi}{4}}$$

Exercise 33. $\int \sqrt{3-x^2}dx$

$$\int (x)'\sqrt{3-x^2}dx = x\sqrt{3-x^2} - \int \frac{x(-x)}{\sqrt{3-x^2}} = x\sqrt{3-x^2} - \int \frac{-x^2+3-3}{\sqrt{3-x^2}} = x\sqrt{3-x^2} - \int \sqrt{3-x^2} + 3\int \frac{1}{\sqrt{3-x^2}}$$
$$\Longrightarrow 2\int \sqrt{3-x^2} = x\sqrt{3-x^2} + \sqrt{3}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}$$
$$\Longrightarrow \boxed{\int \sqrt{3-x^2} = \frac{x}{2}\sqrt{3-x^2} + \frac{3}{2}\arcsin\frac{x}{\sqrt{3}}}$$

Exercise 34. $\int \frac{1}{\sqrt{3-x^2}}dx = -(3-x^2)^{1/2} + C.$

$$\left(\arccos\frac{x}{\sqrt{3}}\right)' = \frac{1}{\sqrt{3}}\frac{-1}{\sqrt{1-\frac{x^2}{3}}} = -\frac{1}{\sqrt{3-x^2}}$$
$$\left(x\sqrt{3-x^2}\right)' = \sqrt{3-x^2} + \frac{-x^2}{\sqrt{3-x^2}}$$
$$\boxed{\frac{x\sqrt{3-x^2}}{2} + -\frac{3}{2}\arccos\frac{x}{\sqrt{3}}}$$

Exercise 35. $\int \frac{\sqrt{3-x^2}}{x}dx = \int \sqrt{\frac{3}{x^2}-1}dx.$

$$\frac{\sqrt{3}}{x} = \sec\theta$$
$$\sqrt{3}\cos\theta = x$$
$$dx = -\sin\theta\sqrt{3}$$
$$\int \sqrt{\sec^2\theta - 1}(-\sin\theta)\sqrt{3} = \int \tan\theta\sin\theta(-\sqrt{3}) = -\sqrt{3}\int(\sec\theta - \cos\theta) =$$
$$= -\sqrt{3}\ln|\sec\theta + \tan\theta| + \sqrt{3}\sin\theta =$$
$$= \boxed{-\sqrt{3}\ln\left|\frac{\sqrt{3}}{x} + \sqrt{\frac{3}{x^2}-1}\right| + \sqrt{3}\sqrt{1-\frac{x^2}{3}}}$$

Exercise 36. $\int \sqrt{1+\frac{1}{x}}dx$

$$\left(x\sqrt{1+\frac{1}{x}}\right)' = \sqrt{1+\frac{1}{x}} + \frac{x}{2\sqrt{1+\frac{1}{x}}}\left(\frac{-1}{x}\right) = \sqrt{1+\frac{1}{x}} + \frac{-1/2}{\sqrt{x^2+x}}$$
$$\left(\ln\left(x+\frac{1}{2}+\sqrt{x^2+x}\right)\right)' = \frac{1}{x+\frac{1}{2}+\sqrt{x^2+x}}\left(1+\frac{x+\frac{1}{2}}{\sqrt{x^2+x}}\right) = \frac{1}{\sqrt{x^2+x}}$$
$$\Longrightarrow \int\left(1+\frac{1}{x}\right)dx = x\sqrt{1+\frac{1}{x}} + \frac{1}{2}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x}\right)$$

Exercise 37.

$$(x\sqrt{x^2+6})' = \sqrt{x^2+5} + \frac{x^2}{\sqrt{x^2+5}}$$
$$(\ln(x+\sqrt{x^2+b}))' = \left(\frac{1}{x+\sqrt{x^2+b}}\right)\left(1+\frac{x}{\sqrt{x^2+b}}\right) = \frac{1}{\sqrt{x^2+b}}$$
$$\int \sqrt{x^2+5} = \frac{1}{2}\left(x\sqrt{x^2+5} + 5\ln(x+\sqrt{x^2+5})\right)$$

Exercise 38.

$$\left(\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right)' = \frac{1}{x+\frac{1}{2}+\sqrt{x^2+x+1}}\left(1+\frac{x+\frac{1}{2}}{\sqrt{x^2+x+1}}\right) = \frac{1}{\sqrt{x^2+x+1}}$$

$$\int\frac{x}{\sqrt{x^2+x+1}} = \int\frac{x+\frac{1}{2}-\frac{1}{2}}{\sqrt{x^2+x+1}} = (x^2+x+1)^{1/2} - \frac{1}{2}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$$

The trick is to note how I formed a "conjugate-able" sum from x^2+x+1's derivative.

Exercise 39.

$$\int\frac{dx}{\sqrt{x^2+x}} = \int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}} = \int\frac{2dx}{\sqrt{\left(2\left(x+\frac{1}{2}\right)\right)^2-1}}$$

$$\left(\ln\left(\sqrt{(2(x+1/2))^2-1}+2(x+1/2)\right)\right)' =$$

$$= \frac{1}{\sqrt{(2(x+1/2))^2-1}+2(x+1/2)}\left(2+\frac{2(x+1/2)2}{\sqrt{(2(x+1/2))^2-1}}\right) =$$

$$= \frac{2}{\sqrt{(2(x+1/2))^2-1}}$$

$$\boxed{\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}} = \ln\left(2\left(x+\frac{1}{2}\right)+\sqrt{(2(x+1/2))^2-1}\right)+C}$$

Exercise 40.

6.26 Miscellaneous review exercises. Exercise 1.

$$f(x) = \int_1^x \frac{\log t}{t+1}$$

$$f\left(\frac{1}{x}\right) = \int_1^{\frac{1}{x}}\frac{\log t}{t+1}dt = \int_1^x \frac{-\ln(u)}{\frac{1}{u}+1}\left(\frac{-1}{u^2}du\right) =$$

$$= \int_1^x \frac{\ln(u)}{u+u^2}du \qquad \begin{aligned} u &= \frac{1}{t} \\ du &= -\frac{1}{t^2}dt \end{aligned}$$

$$f(x)+f\left(\frac{1}{x}\right) = \int_1^x \frac{t\ln t+\ln t}{t(t+1)}dt = \int_1^x\frac{\ln t}{t}dt = \left.\frac{(\ln t)^2}{2}\right|_1^x = \boxed{\frac{(\ln x)^2}{2}}$$

$$\boxed{f(2)+f\left(\frac{1}{2}\right) = \frac{1}{2}(\ln 2)^2}$$

Exercise 2. Take the derivative of both sides, using the (first) fundamental theorem of calculus.

$$2ff' = f(x)\frac{\sin x}{2+\cos x};\Longrightarrow 2f' = \frac{\sin x}{2+\cos x}$$

At this point, it could be very easy to evaluate the integral by guessing at the solution.

$$(-\ln 2+\cos x)' = \frac{\sin x}{2+\cos x} \Longrightarrow \boxed{f = -\frac{\ln|2+\cos x|}{2}+C}$$

Otherwise, remember that for rational expressions involving single powers of $\sin$ and $\cos$, we can make a $u = \tan\theta/2$ substitution.

$$
\begin{aligned}
u &= \tan\frac{x}{2} \\
2du &= \sec^2\frac{x}{2}dx
\end{aligned}
\qquad C = \cos x/2,\ S = \sin x/2
$$

$$
\begin{aligned}
\int \frac{\sin x}{2+\cos x}dx = \int \frac{2SCdx}{2+C^2-S^2} &= \int \frac{2T}{2\sec^2 x/2 + 1 - T^2}\left(\frac{2du}{\sec^2 x/2}\right) = \\
&= 4\int \frac{udu}{(1+u^2)(3+u^2)} = 2\int \frac{T}{T^2+1} - \frac{T}{T^2+3} \\
&= 2\left(\frac{1}{2}\ln T^2+1 - \frac{1}{2}\ln T^2+3\right) = \left(\ln\left(\frac{T^2+1}{T^2+3}\right)\right) = \ln\left(\frac{2}{4+2\cos x}\right) \\
&\text{where } \tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1-\cos x}{1+\cos x}
\end{aligned}
$$

Exercise 3.

$$
\int \frac{e^x}{x}dx = e^x - \int \frac{e^x(x-1)}{x^2}xdx = e^x - \int \frac{e^x(x-1)}{x}dx \dots
$$

No way.

Exercise 4. $\int_0^{\pi/2} \ln(e^{\cos x})dx = -\cos x|_0^{3/2} = \boxed{1}$.

Exercise 5.

(1)

$$
\begin{aligned}
f &= \sqrt{4x+2}x(x+1)(x+2) \\
\ln f &= \frac{1}{2}\ln\left(\frac{4(x+2)}{x(x+1)(x+2)}\right) = \frac{1}{2}\left(\ln(4x+2) - \ln x - \ln(x+1) - \ln(x+2)\right) \\
\frac{f'}{f} &= \frac{1}{2}\left(\left(\frac{4}{4x+2}\right) - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2}\right) \qquad f'(1) = -\frac{7}{12} \\
\Longrightarrow f' &= \frac{1}{2}\sqrt{\frac{4x+2}{x(x+1)(x+2)}}\left(\left(\frac{4}{4x+2}\right) - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2}\right)
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_1^4 \pi\frac{4x+2}{x(x+2)(x+1)}dx &= 2\pi\int_1^4 \frac{(2x+1)}{x(x+2)(x+1)}dx = \\
&= 2\pi\left(\frac{1}{2}\ln x + -\frac{3}{2}\ln|x+2| + \ln|x+1|\right)\Big|_1^4 = \boxed{\pi ln\frac{25}{8}}
\end{aligned}
$$

since we can find the antiderivative through partial fractions:

$$
\begin{gathered}
\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1} = \frac{2x+1}{x(x+2)(x+1)} \\
A(x^2+3x+2) + B(x^2+x) + C(x^2+2x) = 2x+1 \\
\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1/2 \end{array}\right]
\end{gathered}
$$

Exercise 6.

(1)

$$\log x = \int_1^x \frac{1}{t}dt \quad F(x) = \int_1^x \frac{e^t}{t}dt; \quad \begin{matrix} \text{if } x > 0 \\ e^t > 1 \text{ for } t > 0 \end{matrix}$$

If $0 < x < 1$

$$\log x = \int_1^x \frac{1}{t}dt = \int_x^1 \frac{-1}{t}dt > -\int_x^1 \frac{e^t}{t} = F(x)$$

$$\boxed{\log x \leq F(x) \text{ for } x \geq 1}$$

(2)

$$F(x+a) - F(1+a) = \int_1^{x+a} \frac{e^t dt}{t} - \int_1^{a+1} \frac{e^t dt}{t} = \int_{1-a}^x \frac{e^{t+a}dt}{t+a} - \int_{1-a}^1 \frac{e^{t+a}dt}{t+a}$$
$$= e^a \int_1^x \frac{e^t}{t+a}dt$$

(3)

$$\int_1^x \frac{e^{at}}{t}dt = \int_a^{ax} \frac{e^t}{t}dt = \int_1^{ax} \frac{e^t}{t} + \int_a^1 \frac{e^t}{t} =$$
$$= \boxed{F(ax) - F(a)}$$

$$\int_1^x \frac{e^t}{t^2} = -\frac{1}{t}e^t - \int -\frac{1}{t}e^t = \boxed{-\left(\frac{e^x}{x} - e\right) + F(x)}$$

$$\int_1^x e^{1/t}dt = \int_1^{1/x} \frac{-e^u}{u^2}du = -\left(\frac{-e^u}{u} - \int -\frac{e^u}{u}\right) =$$

$$= \boxed{xe^{1/x} - e - F(1/x)} \text{ where we used the substitution } \begin{matrix} u = \frac{1}{t} \\ du = -\frac{1}{t^2}dt \end{matrix}$$

Exercise 7.

(1)

$$e^x = F(x) - F(0);\ F(x) = e^x + F(0) \Longrightarrow F(0) = 1 + F(0)$$

$0 \neq 1$. False.

(2)

$$\frac{d}{dx}\int_0^{x^2} f(t)dt = f(x^2)(2x) = -(2x)\ln 2 e^{x^2 \ln 2} \quad f(x) = -\ln 2 e^{x \ln 2}$$

$$\int_0^{x^2} -\ln 2 e^{\ln 2}dt = -\left. e^{t \ln 2}\right|_0^{x^2} = -e^{x^2 \ln 2} + 1$$

(3)

$$f(x) = 2f(x)f'(x); \Longrightarrow f(x) = \frac{x}{2} + C$$

$$\int_0^x \left(\frac{1}{2}t + c\right)dt = \left.\left(\frac{t^2}{4} + ct\right)\right|_0^x = \frac{x^2}{4} + cx$$

$$f^2(x) - 1 = \frac{x^2}{4} + Cx + C^2 - 1$$

$$\Longrightarrow C = \pm 1, f(x) = \frac{x}{2} + \pm 1$$

Exercise 8.

(1)

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)f(h) - f(x)}{h} = \frac{f(x)(hg(h))}{h} = f(x)g(h)$$
$$g(h) \to 1 \text{ as } h \to 0 \text{ so } \Longrightarrow f'(x) = f(x)$$

(2) Since for $f(x) = e^x$, we defined e^x such that $f' = f$, if

$$(e^x + g)' = e^x + g' = e^x + g$$
$$\implies e^x + g = Ce^x \implies g = (C-1)e^x \text{ but } f'(0) = 1 \text{ so } \boxed{g = e^x}$$

Exercise 9.

(1)

$$g(2x) = 2e^x g(x)$$
$$g(3x) = e^x g(2x) + e^{2x} g(x) = e^x 2e^x g + e^{2x} g = 3e^{2x} g$$

(2)

$$\text{Assume } g(nx) = ne^{(n-1)x} g$$
$$g((n+1)x) = e^x g(nx) + e^{(n+1)x} g(x) = ne^{nx} g(x) + e^{nx} g = (n+1)e^{nx} g$$

(3) From $g(x+y) = e^y g(x) + e^x g(y)$,

$$g(0) = g(0) + g(0) \implies g(0) = 0$$

$$\frac{g(x+h) - g(x)}{h} = \frac{e^h g(x) + e^x g(h) - g(x)}{h} = g(x)\left(\frac{e^h - 1}{h}\right) + \frac{e^x g(h)}{h}$$
$$f'(0) = 2 = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim h \to 0 \frac{g(h)}{h}$$

(4) $g'(x) = g(x) + 2e^x$ $\boxed{C = 2}$

Exercise 10.

$$\forall x \in \mathbb{R}, f(x+a) = bf(x); f(x+2a) = bf(x+a) = b^2 f(x)$$
$$f(x + (n+1)a) = f(x + na + a) = bf(x + na) = b^{n+1} f(x)$$
$$f(x + na) = b^n f(x)$$
$$\boxed{f(x) = b^{x/a} g(x) \text{ where } g \text{ is periodic in } a}$$

Exercise 11.

$$(\ln(fg))' = \frac{f'}{f} + \frac{g'}{g} \implies (fg)' = f'g + fg'$$
$$\left(\ln\left(\frac{f}{g}\right)\right)' = \frac{f'}{f} - \frac{g'}{g} \implies \frac{f'g - g'f}{g^2} = \left(\frac{f}{g}\right)'$$

Exercise 12. $A = \int_0^1 \frac{e^t}{t+1} dt$

(1)

$$\int_{a-1}^{a} \frac{e^{-t}}{t-a-1} dt$$
$$u = t - a \qquad \implies \int_{-1}^{0} \frac{e^{-t-a}}{t-1} dt = -\int_{1}^{0} \frac{e^{t-a}}{-t-1} dt = -e^{-a} \int_0^1 \frac{e^t}{t+1} = \boxed{-e^{-a} A}$$

(2) $\int_0^1 \frac{te^{t^2}}{t^2+1} dt = \int_0^1 \frac{\frac{1}{2} du e^u}{u+1} = \boxed{\frac{1}{2} A}$

(3) $\int_0^1 \frac{e^t}{(t+1)^2} dt = \left.\frac{-e^t}{(t+1)}\right|_0^1 - \int_0^1 \frac{-e^t}{t+1} = \boxed{\frac{-e^1}{2} + 1 + A}$

(4) $\int_0^1 e^t \ln(1+t) dt = e^t \ln(1+t) - \int \frac{e^t}{1+t} = \boxed{e \ln 2 - A}$

Exercise 13.

(1) $p(x) = c_0 + c_1x + c_2x^2; \qquad f(x) = e^x p(x) \qquad p' = c_1 + 2c_2x \qquad p'' = 2C_2$
$f' = f + e^x p'$

$$f^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} e^x (p(x))^j = f + e^x(c_1 + 2c_2x) + \left(\frac{n!}{(n-1)!}\right) + e^x(2c_2)\left(\frac{n!}{(n-2)!2!}\right)$$

$$f^{(n)}(0) = c_0 + c_1 n + n(n-1)c_2$$

(2) See generalization below.

(3)

$$p = \sum_{j=0}^{m} a_j x^j; \qquad p(0) = a_0; \qquad p^{(k)}(x) = \sum_{j=0}^{m} a_j \frac{j!}{(j-k)!} x^{j-k} = p^{(k)}(0) = a_k k!$$

$$f^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} e^x p^{(j)}(x)$$

$$f^{(n)}(0) = \sum_{j=0}^{n} \binom{n}{j} p^{(j)}(0) = \sum_{j=0}^{m} \binom{n}{j} a_j j! = \boxed{\sum_{j=0}^{m} \frac{n!}{(n-j)!} a_j}$$

So for $m = 3$, then $f^{(n)}(0) = a_0 + na_1 + n(n-1)a_2 + n(n-1)(n-2)a_3$

Exercise 14. $f(x) = x \sin ax$; $\qquad f^{(2)} = -a^2 x \sin ax + 2a \cos ax$

$f^{(2n)}(x) = (-1)^n (a^{2n} x \sin ax - 2na^{2n-1} \cos ax)$

$$f^{(2n+1)}(x) = (-1)^n (a^{2n+1} x \cos ax + a^{2n} \sin ax + 2na^{2n} \sin ax)$$
$$f^{(2n+2)}(x) = (-1)^n (-a^{2n+2} x \sin ax + a^{2n+1} \cos ax(2n+2))$$

Exercise 15.

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+m+1} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 t^{k+m} dt = \int_0^1 \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{k+m} dt = \\
&= \int_0^1 t^m \sum_{k=0}^{n} \binom{n}{k} (-t)^k dt = \int_0^1 t^m (1-t)^n dt = \\
&= -\int_1^0 (1-u)^m u^n du = \int_0^1 (1-u)^m u^n du = \\
&= \int_0^1 \sum_{j=0} 6m \binom{m}{j} (-u)^j u^n du = \boxed{\sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_0^1 t^{j+n} dt}
\end{aligned}$$

$$u1 - t$$
$$du = -dt$$

Exercise 16. $F(x) = \int_0^x f(t) dt$

(1)

$$F(x) = \int_0^x (t + |t|)^2 = \begin{cases} \int_0^x (2t)^2 dt = \frac{4}{3} x^3 & \text{if } t, x \geq 0 \\ \int_0^x 0 dt = 0 & \text{if } t, x < 0 \end{cases}$$

(2)

$$F(x) = \int_0^x f(t) dt = \begin{cases} \int_0^x (1 - t^2) dt & \text{if } |t| \leq 1 \\ \int_0^x (1 - |t|) dt & \text{if } |t| > 1 \end{cases} =$$

$$= \begin{cases} \left(t - \frac{1}{3}t^3\right)\Big|_0^x = x - \frac{x^3}{3} & & \text{if } |x| \leq 1 \\ \begin{cases} \frac{2}{3} + \int_1^x (1-t) dt = x - \frac{x^2}{2} + \frac{1}{6} & \text{if } x > 1 \\ \frac{-2}{3} + \int_{-1}^x (1+t) dt = x + \frac{x^2}{2} - \frac{1}{6} & \text{if } x < 1 \end{cases} & & \text{if } |x| \geq 1 \end{cases}$$

(3) $f(t) = e^{-|t|}$

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \int_0^x e^{-|t|}dt = \\ &= \begin{cases} \int_0^x e^{-t}dt = e^{-t}\big|_x^0 = 1 - e^{-x} & \text{if } x \geq 0 \\ \int_0^x e^t dt = e^t\big|_0^x = e^x - 1 & \text{if } x < 0 \end{cases} \end{aligned}$$

(4) $f(t) =$ max. of 1 and t^2

$$\begin{aligned} F(x) = \int_0^x f(t)dt &= \begin{cases} \int_0^x 1dt = x & \text{if } |x| \leq 1 \\ 1 + \int_1^x t^2 dt & \text{if } x > 1 \\ -\int_0^x f & \text{if } x < -1 \end{cases} = \\ &= \begin{cases} x & \text{if } |x| \leq 1 \\ 1 + \frac{1}{3}t^3\big|_1^x = \frac{x^3}{3} + \frac{2}{3} & \text{if } x > 1 \\ -\int_x^{-1} t^2 + -\int_{-1}^0 1 = \frac{1}{3}t^3\big|_{-1}^x - 1 = \frac{x^3}{3} + \frac{2}{3} & \text{if } x > 1 \end{cases} \end{aligned}$$

Exercise 17. $\int \pi f^2 = \pi \int_0^a f^2 = a^2 + a$.

$$\left(\frac{x^2 + x}{\pi}\right)' = \sqrt{\frac{2x+1}{\pi}}$$

$$\text{for } \int_0^a \frac{2x+1}{\pi} = \left(x^2 + x\right)\big|_0^a = a^2 + a$$

Exercise 18. $f(x) = e^{-2x}$.

(1) $A(t) = \int_0^t e^{-2x}dx = \left.\frac{e^{-2x}}{-2}\right|_0^t = \frac{e^{-2t}-1}{-2}$

(2) $V(t) = \pi \int_0^t e^{-4x}dx = \frac{\pi}{-4}(e^{-4t} - 1)$

(3)

$$y = e^{-2x} \Longrightarrow \frac{\ln y}{-2} = x$$

$$\begin{aligned} W(t) &= \pi \int_{e^{-2t}}^1 \left(\frac{\ln y}{-2}\right)^2 dy = \frac{\pi}{4}\left(y(\ln y)^2 - (2(y\ln y - y))\right)\big|_{e^{-2t}}^1 = \\ &= \frac{\pi}{4}\left(2 - \left(e^{-2t}4t^2 - (2(e^{-2t}(-2t) - e^{-2t}))\right)\right) = \\ &= \left(\frac{\pi}{2} - \pi t e^{-2t} - \frac{\pi}{2}e^{-2t}\right) \end{aligned}$$

where the antiderivative used was $(y(\ln y)^2)' = (\ln y)^2 + 2\ln y$

(4)

$$\frac{\frac{\pi}{4}(1 - e^{-4t})}{\left(\frac{1-e^{-2t}}{2}\right)} = \frac{\pi}{2}\frac{\left(\frac{e^{-4t}-1}{t}\right)}{\frac{e^{-2t}-1}{t}} = \boxed{\pi}$$

$$\text{where we used the limit } \lim_{x\to 0}\frac{e^{cx} - 1}{x} = c$$

Exercise 19. $\sinh c = \frac{3}{4}$

(1)

$$e^c = e^x + \sqrt{e^{2x} + 1}$$

$$\begin{aligned} \frac{e^c - e^{-c}}{2} &= \frac{e^x + \sqrt{e^{2x}+1} - \frac{1}{e^x + \sqrt{e^{2x}+1}}}{2} = \\ &= \frac{e^{2x} + 2e^x\sqrt{e^{2x}+1} + e^{2x} + 1 - 1}{2(e^x + \sqrt{e^{2x}+1})} = e^x = \frac{3}{4} \end{aligned}$$

$$\boxed{x = \ln 3 - 2\ln 2}$$

(2)

$$\begin{aligned}
\frac{e^c - e^{-c}}{2} &= \frac{e^x - \sqrt{e^{2x}-1} - \frac{1}{e^x - \sqrt{e^{2x}-1}}}{2} = \\
&= \frac{e^{2x} - 2e^x\sqrt{e^{2x}-1} + e^{2x} - 1 - 1}{2(e^x - \sqrt{e^{2x}-1})} \\
&= \frac{e^x}{+}\frac{-1}{e^x - \sqrt{e^{2x}-1}} = e^x - \frac{1}{e^c} = \frac{3}{4} \\
&\Longrightarrow \boxed{x = \ln 5 - 2\ln 2}
\end{aligned}$$

Exercise 20.

(1) True. $\ln(2^{\log 5}) = \ln 5^{\ln 2} = (\ln 2)\ln 5$.

(2) $\log_3 5 = \frac{\log_2 5}{\log_2 3}$ This is a true fact.

$$\begin{aligned}
\frac{\log_3 5}{\log_2 3} = \frac{\log_2 5}{(\log_2 3)^2} &= \log_2 5 \\
\Longrightarrow 1 = \log_2 3 \quad &\text{False}
\end{aligned}$$

(3) **Use induction**

$$\begin{aligned}
&n = 1\, 1^{-1/2} < 2\sqrt{1} \\
&n = 2\, 1 + \frac{1}{\sqrt{2}} < 2\sqrt{2} \\
&n+1 \text{ case} \quad \sum_{k=1}^{n+1} k^{-1/2} = \sum_{k=1}^{n} k^{-1/2} + \frac{1}{\sqrt{(n+1)}} < \\
&\qquad < 2\sqrt{n}\left(\frac{\sqrt{(n+1)}}{\sqrt{(n+1)}}\right) + \frac{1}{\sqrt{(n+1)}}
\end{aligned}$$

Now $(n+\frac{1}{2})^2 = n^2 + n + \frac{1}{2} > n^2 + n$, certainly. So then

$$\begin{aligned}
n + \frac{1}{2} > \sqrt{n^2+n} &\Longrightarrow n+1 > \sqrt{n^2+n} \\
&\Longrightarrow \sum_{k=1}^{n+1} k^{-1/2} < 2\sqrt{n+1}
\end{aligned}$$

(4)

$$f = (\cosh x - \sinh x - 1) = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} - 1 = e^{-x} - 1 < 0 \text{ for } x > 0$$

False.

Exercise 21. For $0 < x < \frac{\pi}{2}$,

$$\begin{aligned}
(\sin x)' = \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2} \\
(x - \sin x)' = 1 - \cos x \geq 0 \text{ for } 0 < x < \frac{\pi}{2} \\
(x - \sin x)(x = 0) = 0 \Longrightarrow \sin x < x
\end{aligned}$$

Exercise 22.

$$\frac{1}{t} < \frac{1}{t}\left(\frac{x+1}{t}\right) \qquad \text{if } 0 < x < t < x+1$$

$$\int_x^{x+1} \frac{1}{t} dt = \ln(x+1) - \ln x; \; \int_x^{x+1} \frac{x+1}{t} = \boxed{\frac{1}{x}} \text{ So } \ln\frac{x+1}{x} < \frac{1}{x}$$

Exercise 23.

$$(x-\sin x)' = 1-\cos x \geq 0 \, \forall x > 0$$

$$\text{since } (x-\sin x)(x=0) = 0; \; (x-\sin x)'(x=0)=0, \; \text{ then } x - \sin x > 0 \text{ in general for } x > 0$$

$$(\sin x - \left(x - \frac{x^3}{6} \right))' = \cos x - 1 + \frac{x^2}{2} > 0 \, \forall x > 0$$

$$\implies x - \frac{x^3}{6} < \sin x < x$$

Exercise 24. $(x^b+y^b)^{1/b} < (x^a+y^a)^{1/a}$ if $x>0, \; y>0$ and $0<a<b$

$(x^n+y^n)^{1/n} = x(1+(\frac{y}{x})^n)^{1/n}$. Without loss of generality, assume $x<y$.

Consider $(1+A^n)^{1/n}$, A constant.

$$\begin{aligned}
((1+A^n)^{1/n})' &= (\exp\left(\frac{1}{n} \ln{(1+A^n)} \right))' = (1+A^n)^{1/n} \left(\frac{-1}{n^2} \ln{(1+A^n)} + \frac{1}{n} \left(\frac{1}{1+A^n} \right) (\ln A) A^n \right) = \\
&= (1+A^n)^{1/n} \left(\frac{ -(1+A^n) \ln{(1+A^n)} + n(\ln A) A^n }{ n^2(1+A^n) } \right) = \\
&= (1+A^n)^{1/n} \left(\frac{ -\ln{(1+A^n)} - A^n \ln{(1+A^n)} + A^n \ln A^n }{ n^2(1+A^n) } \right) = \\
&= \frac{ (1+A^n)^{\frac{1-n}{n}} }{ n^2 } \left(\frac{ -\ln{(1+A^n)} + A^n \ln \left(\frac{A^n}{1+A^n} \right) }{ n^2(1+A^n) } \right) < 0 \quad \text{ since } \ln \left(\frac{A^n}{1+A^n} \right) < 0 \\
&\implies (x^b+y^b)^{1/b} < (x^a+y^a)^{1/a} \quad \text{ if } b > a
\end{aligned}$$

Exercise 25.

(1)

$$\int_0^x e^{-t} t = \left(-te^{-t} - e^{-t} \right)\Big|_0^x = -xe^{-x} - e^{-x} + 1$$

(2)

$$\begin{aligned}
\int_0^t t^2 e^{-t} dt &= -t^2 e^{-t}\Big|_0^x - \int -e^{-t}(2t) dt = -x^2 e^{-x} + 2 \int te^{-t} dt = \\
&= -x^2 e^{-x} + -2xe^{-x} - 2e^{-x} + 2
\end{aligned}$$

(3)

$$\begin{aligned}
\int_0^x t^3 e^{-t} dt &= -t^3 e^{-t}\Big|_0^x - 3\int_0^x t^2(-e^{-t}) dt = -x^3 e^{-x} + 3 \int_0^x t^2 e^{-t} dt = \\
&= -x^3 e^{-x} + 3(2)(e^{-x}) \left(e^x - 1 - x - \frac{x^2}{2!} \right)
\end{aligned}$$

(4) Assume the induction hypothesis, that

$$\int_0^x t^n e^{-t} dt = n! e^{-x} \left(e^x - \sum_{j=0}^n \frac{x^j}{j!} \right)$$

$$\begin{aligned}
\int_0^x t^{n+1} e^{-t} dt &= -t^{n+1} e^{-t}\Big|_0^x - \int (n+1)t(-e^{-t}) = -x^{n+1}e^{-x} + (n+1)n! e^{-x} \left(e^x - \sum_{j=0}^n \frac{x^j}{j!} \right) \\
&= \boxed{ (n+1)! e^{-x} \left(e^x - \sum_{j=0}^{n+1} \frac{x^j}{j!} \right) }
\end{aligned}$$

Exercise 26. Consider the hint $a_1 \sin x + b\cos x = A(a\sin x + b\cos x) + B(a\cos x - b\sin x)$. Solve for A, B in terms of

a_1, b_1, a, b. Matching up term by term the coefficients for $\sin$ and $\cos$ separately,

$$Aa^2 - abB = aa_1 \qquad -Aab + Bb^2 = -a_1 b$$
$$Ab^2 + Bab = b_1 b \qquad Aab + Ba^2 = ab_1$$
$$\boxed{A = \frac{aa_1 + bb_1}{a^2+b^2}} \qquad \boxed{B = \frac{ab_1 - a_1 b}{a^2+b^2}}$$

So if not both $a, b = 0$,

$$\int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} = \int \frac{A(a\sin x + b\cos x) + B(a\cos x - b\sin x)}{a\sin x + b\cos x} =$$
$$= \boxed{Ax + B\ln|a\sin x + b\cos x| + C}$$

Exercise 27.

(1)

$$f'(x^2) = \frac{1}{x}$$
$$\frac{df}{du} = u^{-1/2}$$
$$f(x^2) = 2x - 1$$

(2)

$$f'(\sin^2 x) = 1 - \sin^2 x \, f'(u) = 1 - u \, f = u - \frac{1}{2}u^2 + C \Longrightarrow \boxed{f(x) = x - \frac{x^2}{2} + \frac{1}{2}}$$

(3)

$$f'(\sin X) = (1 - \sin^2 x) \, f(u) = u - \frac{1}{3}u^3 + C \boxed{f(x) = x - \frac{x^3}{3} + \frac{1}{3}}$$

(4)

$$f'(\ln x) = \begin{cases} 1 & \text{for } x \leq 1 \\ x & \text{for } x > 1 \end{cases} = \begin{cases} 1 & \text{for } 0 < x \leq 1 \\ e^{\ln x} & x > 1 \end{cases}$$
$$\boxed{f(y) = \begin{cases} y & \text{for } y < 0 \\ e^y - 1 & \text{for } y > 0 \end{cases}}$$

Exercise 28.

(1)

$$Li(x) = \int_2^x \frac{dt}{\ln t} \text{ if } x \geq 2$$
$$Li(x) = \frac{x}{\ln x} - 2\frac{1}{\ln 2} - \int_2^x \frac{-1}{(\ln x)^2} dt = \frac{x}{\ln x} + \int_2^x \frac{dt}{(\ln x)^2} - \frac{2}{\ln 2}$$

(2)

$$Li(x) = \frac{x}{\ln x} - \frac{2}{\ln x} - \frac{2}{(\ln 2)^2} + \frac{x}{(\ln x)^2} - \int_a^x \frac{-2}{(\ln t)^3} dt \qquad C_2 = \frac{-2}{\ln x} + - \sum_{j=2}^{2} \frac{2(j-1)!}{(\ln 2)^j}$$
$$Li(x) = \frac{x}{\ln x} + \sum_{k=1}^{n-1} \frac{k! x}{\ln^{k+1} x} + n!\left(\frac{x}{\ln^{n+1} x} - \int_2^x \frac{-(n+1)dt}{\ln^{n+2} t} \right) \qquad C_n = -2\frac{1}{\ln 2} - \sum_{j=2}^{n} \frac{2(j-1)!}{(\ln 2)^j}$$
$$Li(x) = \frac{x}{\ln x} + \sum_{k=1}^{n} \frac{k! x}{\ln^{k+1} x} + (n+1)! \int_2^x \frac{dt}{\ln^{(n+1)+1} t} \qquad C_{n+1} = -2\frac{1}{\ln 2} - \sum_{j=2}^{n+1} \frac{2(j-1)!}{(\ln 2)^j}$$

(3)

$$Li(x) = \int_2^x \frac{dt}{\ln t} \qquad \begin{aligned} u &= \ln t \\ du &= \frac{1}{t}dt \\ e^u du &= dt\, e^u = t \end{aligned}$$

$$\boxed{Li(x) = \int_{\ln 2}^{\ln x} \frac{e^t dt}{t}}$$

(4)

$$c = 1 + \frac{1}{2}\ln 2$$

$$t = u+1 \Longrightarrow \quad \begin{aligned} \int_{c-1}^{x-1} \frac{e^{2(u+1)}du}{u} &= e^2 \int_{c-1}^{x-1} \frac{e^{2u}}{u} du = \\ &= e^2 \frac{1}{2} \int_{\frac{2(c-1)}{3}}^{\frac{2(x-1)}{3}} \frac{e^t}{\frac{t}{2}} dt = \\ &= \boxed{e^2 Li(e^{2(x-1)})} \end{aligned}$$

(5)

$$\begin{aligned} f(x) &= e^4 Li(e^{2(x-2)}) \quad -e^2 Li(e^{2(x-1)}) \\ &= \int_c^x \frac{e^{2t}}{t-2} \qquad - \int_c^x \frac{e^{2t}}{(t-1)} \end{aligned} \Longrightarrow \boxed{f'(x) = \frac{e^{2x}}{t-2} + - \frac{e^{2x}}{t-1} = e^{2x}\left(\frac{1}{t^2 - 3t + 2} \right)}$$

Exercise 29. $f(x) = \log|x|$ if $x < 0$. $\forall x < 0 \, \exists$ uniquely $\ln|x|$ since $f' = \frac{1}{x} < 0 \, \forall x < 0$.

$$-e^y = x(y) = g(y) \quad \boxed{D = \mathbb{R}}$$

Recall Theorem 3.10.

Theorem 21. *Assume f is strictly increasing and continuous on an interval $[a,b]$. Let $c = f(a), d = f(b)$ and let $-g$ be the inverse of f.*

That is $\forall y \in [c,d]$, Let $g(y)$ be that $x \in [a,b]$, such that $y = f(x)$.

Then

(1) *$-g$ is strictly increasing on $[c,d]$*
(2) *$-g$ is continuous on $[c,d]$*

Exercise 30. $f(x) = \int_0^x (1+t^3)^{-1/2} dt$ if $x \geq 0$.

(1)

$$f'(x) = \frac{1}{\sqrt{1+x^3}} > 0 \text{ for } x > 0$$

(2)

$$g'(x) = \frac{1}{f'(x)} = \sqrt{1+x^3}\, g''(x) = \frac{3x^2}{2\sqrt{1+x^3}}$$

7.4 Exercises - Introduction, The Taylor polynomials generated by a function, Calculus of Taylor polynomials. Use the following theorems for the following exercises.

Theorem 22 (Properties of Taylor polynomials, Apostol Vol. 1. Theorem 7.2.)**.** (1) *Linearity* $T_n(c_1 f + c_2 g) = c_1 T_n(f) + c_2 T_n(g)$
(2) *Differentiation* $(T_n f)' = T_{n-1}(f')$
(3) *Integration. If* $g(x) = \int_a^x f(t)dt$
$T_{n+1} g(x) = \int_a^x T_n f(t) dt$

Theorem 23 (Substitution Property, Apostol Vol. 1. Theorem 7.3.)**.** *Let $g(x) = f(cx)$, c is a constant.*

$$T_n g(x;a) = T_n f(cx;ca) \tag{12}$$

This theorem is useful for finding new Taylor polynomials without having to find the jth derivatives of the desired function.

Theorem 24. *P_n is a polynomial of degree $n \geq 1$.*
Let f, g be 2 functions with derivatives of order n at 0.

$$f(x) = P_n(x) + x^n g(x) \tag{13}$$

where $g(x) \mapsto 0$ as $x \mapsto 0$.
Then $P_n = T_n(f, x = 0)$.

Exercise 3.

$$\begin{aligned}
T_n f(x) &= \sum_{j=0}^n \frac{f^{(j)}(a)}{j!}(x-a)^j \\
a^x &= e^{x \ln a} \\
(a^x)' &= (a^x) \ln a \\
(a^x)^{(n+1)} &= (a^x(\ln a)^n)' = a^x (\ln a)^{n+1} \\
T_n(a^x) &= \sum_{j=0}^n \frac{(\ln a)^j}{j!} x^j
\end{aligned}$$

Exercise 4.

$$\begin{aligned}
&\left(\frac{1}{1+x} \right)' = \frac{-1}{(1+x)^2}; \left(\frac{1}{1+x} \right)'' = \frac{(-1)^2 2}{(1+x)^3} \\
&\left(\frac{1}{1+x} \right)^{(n+1)} = \left(\frac{(-1)^n n!}{(1+x)^{n+1}} \right)' = \frac{(-1)^{n+1}(n+1)!}{(1+x)^{n+2}} \\
&T_n \left(\frac{1}{1+x} \right) = \sum_{j=0}^n (-1)^j x^j
\end{aligned}$$

Exercise 5. $\boxed{\text{Use Theorem 7.4.}}$. Theorem 7.4 says for $f(x) = P_n(x) + x^n g(x)$, $P_n(x)$ is the Taylor polynomial.

$$\begin{aligned}
\frac{1}{1-x^2} &= \left(\sum_{j=0}^n (x^2)^j \right) + \frac{(x^2)^{n+1}}{1-x^2} = \left(\sum_{j=0}^n x^{2j} \right) + \frac{x^n x^{n+2}}{1-x^2} \\
\frac{x}{1-x^2} &= \left(\sum_{j=0}^n x^{2j+1} \right) + \frac{(x^{2n+3})}{1-x^2} \\
T_{2n+1} \left(\frac{x}{1-x^2} \right) &= \sum_{j=0}^n x^{2j+1}
\end{aligned}$$

Exercise 6.

$$(\ln(1+x))' = \frac{1}{1+x} \quad T_n \left(\frac{1}{1+x} \right) = \sum_{j=0}^n (-x)^j \quad T_n(\ln 1 + x) = \sum_{j=0}^n (-1)^j \frac{x^{j+1}}{j+1} = \sum_{j=1}^n (-1)^{j+1} \frac{x^j}{j}$$

Exercise 7.

$$\begin{aligned}
&\left(\log \sqrt{\frac{1+x}{1-x}} \right)' = \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-x}{1+x}} \frac{1}{(1-x)^2} = \frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2} \\
&\int \frac{1}{1-x^2} = \log \sqrt{\frac{1+x}{1-x}} \text{ so } \int \sum_{j=0}^n x^{2j} = \sum_{j=0}^n \frac{x^{2j+1}}{2j+1} = T_{2n+1} \left(\ln \sqrt{\frac{1+x}{1-x}} \right)
\end{aligned}$$

Exercise 8.

$$T_n \left(\frac{1}{2-x} \right) = T_n \left(\frac{1/2}{1-x/2} \right) = \frac{1}{2} T_n \left(\frac{1}{1 - \left(\frac{1}{2}\right) x} \right) = \frac{1}{2} \left(\sum_{j=0}^n (\frac{1}{2} x)^j \right) = \sum_{j=0}^n \frac{x^j}{2^{j+1}}$$

Exercise 9. We can show this in two ways.

We could write out the actual polynomial expansion.

$$(1+x)^\alpha = \sum_{j=0}^n \binom{\alpha}{j} 1^{\alpha-j}x^j = \sum_{j=0}^n \binom{\alpha}{j} x^j$$

or determine each of the coefficients of the Taylor polynomial.

$$((1+x)^\alpha)' = \alpha(1+x)^{\alpha-1};\ ((1+x)^\alpha)'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$$

$$((1+x)^\alpha)^{(n+1)} = \left(\frac{\alpha!}{(\alpha-n)!}(1+x)^{\alpha-n}\right)' = \frac{\alpha!}{(\alpha-(n+1))!}(1+x)^{\alpha-(n+1)}$$

Exercise 10. Use the substitution theorem, Apostol Vol.1. Thm. 7.3., to treat $\cos 2x$.

$$T_{2n}(\cos x) = \sum_{j=0}^n \frac{(-1)^j x^{2j}}{(2j)!};\ T_{2n}(\cos 2x) = \sum_{j=0}^n \frac{(-1)^j (2x)^{2j}}{(2j)!}$$

$$T_{2n}(\sin x^2) = T_{2n}(\frac{1}{2}(1-\cos 2x)) = \frac{1}{2}\left(1 - \sum_{j=0}^n \frac{(-1)^j(2x)^{2j}}{(2j)!}\right) = \sum_{j=1}^n \frac{(-1)^{j+1}2^{2j-1}x^{2j}}{(2j)!}$$

7.8 Exercises - Taylor's formula with remainder, Estimates for the error in Taylor's formula, Other forms of the remainder in Taylor's formula. We will use Theorem 7.7, which we learn in the preceding sections, extensively.

Theorem 25. *If for $j = 1, \dots, n+1, m \leq f^{(j)}(t) \leq < \ \forall t \in I$, I containing a,*

$$m\frac{(x-a)^{n+1}}{(n+1)!} \leq E_n(x) \leq M\frac{(x-a)^{n+1}}{(n+1)!} \qquad \textit{if } x > a \tag{14}$$

$$m\frac{(a-x)^{n+1}}{(n+1)!} \leq (-1)^{n+1}E_n(x) \leq M\frac{(a-x)^{n+1}}{(n+1)!} \qquad \textit{if } x < a \tag{15}$$

$$E_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt \tag{16}$$

Exercise 1. For $a = 0$, $|\sin^{(j)}(x)| \leq 1$ for $\forall x \in \mathbb{R}$.

$$E_n(x) \leq \frac{(x)^{2n+1}}{(2n+1)!} \text{ if } x > 0;\ (-1)^{2n+1}E_{2n}(x) \leq (+1)\frac{(-x)^{2n+1}}{(2n+1)!}$$

$$\Longrightarrow \boxed{E_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}}$$

Exercise 2.

$$\cos x = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!} + E_{2n+1}(x)\ |\cos^{(j)}(x)| \leq 1$$

$$E_{2n+1}(x) \leq \frac{x^{2n+2}}{(2n+2)!};\ (-1)^{2n+2}E_{2n+1}(x) \leq (1)\frac{(-x)^{2n+2}}{(2n+2)}$$

$$\Longrightarrow |E_{2n+1}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$$

Exercise 3.

$$\arctan x = \sum_{k=0}^{n-1} \frac{(-1)^k x^{2k+1}}{2k+1} + E_{2n}(x)$$

$$\sum_{k=0}^{n-1} \frac{(-1)^k x^{2k+1}}{2k+1}\left(\frac{(2k)!}{(2k)!}\right) = \sum_{k=0}^{n-1} \frac{(-1)^k(2k)!x^{2k+1}}{(2k+1)!} \Longrightarrow f^{(2k+1)}(0) = (-1)^k(2k)!$$

$$\frac{f^{(2n+1)}(0)x^{2n+1}}{(2n+1)!} = \frac{(-1)^n(2n)!x^{2n+1}}{(2n+1)!} = \frac{(-1)^n x^{2n+1}}{2n+1} \leq \frac{x^{2n+1}}{2n+1}$$

Note how jth derivative $(\arctan x)^{(j)}$ changes sign with each differentiation for $f^{(2j+1)}(0)$. Then we can always pick a small enough closed interval with $a = 0$ as a left or right end point to make the $f^{(2j+1)}(0)$ value the biggest for $f^{(2j+1)}(t)$.

Exercise 4.

(1)

$$x^2 = \sin x = x - \frac{x^3}{6} \Longrightarrow \frac{x^3}{6} + x^2 - x = \frac{x}{6}\left(x - \left(-3+\sqrt{15}\right)\right)\left(x-(-3-\sqrt{15})\right)$$
$$\boxed{x = \sqrt{15}-3}$$

(2)

$$E_4(r;0) = \frac{1}{4!}\int_0^r (r-t)^4 \cos t dt > 0 \sin r - r^2 = 0 + E_4(r) \leq \frac{r^5}{5!} < \frac{r}{5!} = \frac{3}{5(2)(5)(4)(3)(2)1} = \frac{\frac{3}{4}}{(5)(4)(2)5} < \frac{1}{200}$$

Exercise 5.

$$\arctan r - r^2 = r - \frac{r^3}{3} - r^2 + E_4(r;0) = 0 + E_4(r;0)$$
$$E_4(r,0) = \frac{1}{4!}\int_0^r (x-t)^4 f^{(5)}(t)dt \leq \frac{M(r^5)}{5!}\frac{r^5}{5!} = 0.065536 < \frac{7}{100}$$

$$E_4(r,0) < E_j(r,0); j > 4$$
$$\text{the 5th degree term is } \frac{f^{(5)}(0)}{5!}r^5 = \frac{24}{5!}r^5 > 0$$
$$\text{so } r^2 - \arctan r = -E_4(r,0) < 0$$

Exercise 6. Apply long division on the fraction in the integrand.

$$\int_0^1 \frac{1+x^{30}}{1+x^{60}}dx = \int_0^1 1 + \frac{x^{30}-x^{60}}{1+x^{60}}dx = 1 + \int_0^1 x^{30}\left(\frac{1-x^{30}}{1+x^{60}}\right)dx =$$
$$= 1 + c\,\frac{1}{31}x^{31}\Big|_0^1 = 1 + \frac{c}{31}$$

Exercise 7. $\int_0^{1/2} \frac{1}{1+x^4}dx$.

$$\frac{1}{1+x^4} = \sum_{j=0}^{\infty}(-x^4)^j = \sum_{j=0}^{n-1}(-x^4)^j + E_n = 1 + -x^4 + x^8 \dots$$
$$\frac{16}{17}\frac{x^{4n+1}}{(n+1)!} \leq E_n(x;0) \leq \frac{1}{(n+1)!}(x^4)^{n+1}$$
$$\Longrightarrow \int_0^{1/2} E_n = \frac{\left(\frac{1}{2}\right)^{4n+5}}{(n+1)!}\left(\frac{1}{4n+5}\right)$$
$$\int_0^{1/2}\frac{1}{1+x^4} \simeq \frac{1}{2} + \frac{-1}{5}\left(\frac{1}{2}\right)^5$$
$$0.493852 < 0.49375 < 0.493858$$

Exercise 8.

(1)

$$0 \leq x \leq \frac{1}{2}\ \sin x = x - \frac{x^3}{3!} + E_4(x)$$
$$|E_4(x)| \leq \frac{M|x|^5}{5!} = \frac{\sin\frac{1}{3}|x|^5}{5!} \leq \frac{1\left(\frac{1}{2}\right)^5}{5!}$$

(2)

$$\sin x^2 = x^2 - \frac{x^6}{6} + E_4(x^2)$$

$$\int_0^{\frac{\sqrt{2}}{2}} \sin x^2 = \left(\frac{1}{3}x^3 - \frac{x^7}{42} \right)\Big|_0^{\sqrt{2}/2} = \sqrt{2}\left(\frac{1}{12} - \frac{1}{42(16)} \right)$$

$$E_4(x^2) \leq \frac{\sqrt{2}}{64(5!)}$$

$$\int_0^{\frac{\sqrt{2}}{2}} \sin x^2 \leq \sqrt{2}\left(\frac{55}{672} + \frac{1}{64(5!)} \right) = 0.1159$$

Exercise 9.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!}\, E_6(x;0) \leq \frac{(1)x^7}{7!}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{5!} + \frac{E_6(x;0)}{x} \leq 1 - \frac{x^2}{6} + \frac{x^4}{5!} + \frac{x^6}{7!}$$

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3}\left(\frac{1}{6}\right) + \frac{1}{5(5!)} + \frac{1}{7(7!)} = 0.9461 + \frac{1}{7(7!)} = 0.9461 + 0.0000283$$

Exercise 10. $\alpha = \arctan\frac{1}{5}, \beta = 4\alpha - \frac{\pi}{4}$.

(1)

$$\tan(A+B) = \frac{(\tan A + \tan B)}{1 - \tan A \tan B}; A = B = \alpha; \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2/5}{24/25} = 5/12$$

$$\tan 4\alpha = \frac{2(\tan 2\alpha)}{1 - \tan^2(2\alpha)} = \frac{2(5/12)}{1 - (5/12)^2} = \frac{10/12}{119/144} = \frac{120}{119} A = 4\alpha, B = -\frac{\pi}{4}$$

$$\tan(4\alpha - \frac{\pi}{4}) = \tan\beta = \frac{\tan 4\alpha + \tan\left(-\frac{\pi}{4}\right)}{1 - \tan 4\alpha \tan\left(-\frac{\pi}{4}\right)} = \frac{\frac{120}{119} + -\frac{119}{119}}{1 - \frac{120}{119}(-1)} = \frac{1}{239}$$

$$\boxed{4\arctan\frac{1}{5} = \frac{\pi}{4} + \arctan\frac{1}{239}} \text{ This is incredible.}$$

(2)

$$T_{11}(\arctan x) = \sum_{k=0}^{6-1} \frac{(-1)^k x^{2k+1}}{2k+1} + E_{2(6)}(x) \qquad ; |E_{2(6)}(x)| \leq \frac{x^{2(6)+1}}{2(6)+1}$$

$$= x + \frac{-x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} \dots$$

$$\implies 3.158328957 < 16\arctan\frac{1}{5} < 3.158328958$$

(3)

$$T_3(\arctan x); x = \frac{1}{239}$$

$$-0.016736304 < -4\arctan\frac{1}{239} < -0.016736304$$

(4)

$$3.141592625$$

$$3.158328972 - 0.016736300 = 3.141592672$$

7.11 Exercises - Further remarks on the error in Taylor's formula. The o-notation; Applications to indeterminate forms.

Exercise 1.

$$2^x = \exp x \ln 2 = \boxed{1 + (x\ln 2) + \frac{x^2(\ln 2)^2}{2!}} + o(x^2)$$

Exercise 2.

$$x(\cos x) = ((x-1)+1)(\cos x) = (x-1)\cos 1 + (-\sin 1)(x-1)^2 - \frac{\cos 1(x-1)^3}{2} + \cos 1 +$$
$$+ (-\sin 1)(x-1) - \frac{\cos 1(x-1)^2}{2} + \frac{\sin 1(x-1)^3}{3!} =$$
$$= \cos 1 + (\cos 1 - \sin 1)(x-1) + \left(-\sin 1 - \frac{\cos 1}{2}\right)(x-1)^2 + \left(\frac{\sin 1 - 3\cos 1}{3!}\right)(x-1)^3 + o(x-1)^3$$

Exercise 3. Just treat the argument of $\sin x - x^2$ just like u with $u \to 0$.

$$\sin(x-x^2) = (x-x^2) - \frac{(x-x^2)^3}{3!} + \frac{(x-x^2)^5}{5!} + o(x-x^2)^5 =$$
$$= (x-x^2) - \frac{1}{6}\left(x^3 - 3x^4 + 3x^5 - x^6\right) + \frac{1}{120}\left(x^5 - 5x^6 + 10x^7 - 10x^8 + 5x^9 + x^10\right) =$$
$$= (x-x^2) - \frac{1}{6}x^3 - \frac{1}{2}x^4 + \frac{61}{120}x^5 - \frac{25}{120}x^6$$

Exercise 4.

$$\log x = \log(1 + (x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$
$$\Longrightarrow \boxed{a = 0; b = 1, c = \frac{-1}{2}}$$

Exercise 5.

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^3) \text{ as } x \to 0 \qquad \begin{aligned} 1 - \cos x &= \frac{1}{2}x^2 + o(x^3) \\ \frac{1-\cos x}{x^2} &= \frac{1}{2} + \frac{o(x^3)}{x^2} \end{aligned}$$
$$\boxed{\text{since } \frac{1-\cos x}{x^2} = \frac{1}{2} + o(x), \ \frac{1-\cos x}{x^2} \to \frac{1}{2} \text{ as } x \to 0}$$
$$\cos x = 1 - \frac{1}{2}x^2 + \frac{x^4}{4!} + o(x^5) \Longrightarrow \cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^5)$$
$$\frac{1 - \cos 2x - 2x^2}{x^4} = \frac{-\frac{2}{3}x^4 - o(x^5)}{x^4} = \frac{-2}{3} - o(x) \to -\frac{2}{3} \text{ as } x \to 0$$

Exercise 6.

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{ax - \frac{(ax)^3}{3!} + o(x^4)}{bx + o(x^2)} = \frac{a}{b}$$

Exercise 7.

$$\lim_{x\to 0} \frac{\sin 2x}{\cos 2x \sin 3x} = \lim_{x\to 0} \frac{(2x) + \frac{(2x)^3}{3!} + o(x^4)}{\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + o(x^5)\right)\left((3x) - \frac{(3x)^3}{3!} + o(x^4)\right)} = \boxed{\frac{2}{3}}$$

Exercise 8.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \boxed{-\frac{1}{6}}$$

Exercise 9.

$$\lim_{x\to 0} \frac{\ln 1 + x}{e^{2x} - 1} = \lim_{x\to 0} \frac{x - o(x)}{2x + o(x)} = \frac{1}{2}$$

Exercise 10. Don't do the trig. identity.

$$\lim_{x\to 0} \frac{1 - \cos^2 x}{x \tan x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + o(x^2)\right)^2}{x\left(x + \frac{x^3}{6!} + o(x^3)\right)} = \lim_{x\to 0} \frac{1 - (1 + x^2 + o(x^2))}{x^2 + o(x^2)} = 1$$

Exercise 11.

$$\lim_{x\to 0}\frac{x+-\frac{x^3}{6}+o(x^4)}{x-\frac{x^3}{3}}=1$$

Exercise 12.

$$\lim_{x\to 0}\frac{e^{x\ln a}-1}{e^{x\ln b}-1}=\lim_{x\to 0}\frac{x\ln a+o(x)}{x\ln b+o(x)}=\boxed{\ln a/b}$$

Exercise 13.

$$\lim_{x\to 1}\frac{(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+o(x-1)^4}{(x+2)(x+1)}=\boxed{\frac{1}{3}}$$

Exercise 14. $\boxed{1}$.

Exercise 15.

$$\lim_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3}=\lim_{x\to 0}\frac{x(2+x+\frac{x^2}{2})-2(x+x^2/2+x^3/6)}{x^3}=\lim_{x\to 0}\frac{x^3(\frac{1}{6})}{x^3}=\boxed{\frac{1}{6}}$$

Exercise 16.

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{1-\cos x}=\lim_{x\to 0}\frac{x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)-x}{x^3/2}=\boxed{-1}$$

Exercise 17.

$$\lim_{x\to\pi/2}\frac{\cos x}{x-\frac{\pi}{2}}=\frac{0+-1(x-\frac{\pi}{2})}{x-\frac{\pi}{2}}=\boxed{-1}$$

Exercise 18. $\boxed{1/6}$

Exercise 19.

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x^2}=\lim_{x\to 0}\frac{\frac{e^x+e^{-x}}{2}-\cos x}{x^2}=$$

$$=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2}-2\left(1-\frac{x^2}{2}\right)+o(x^3)}{x^2}=\boxed{2}$$

Exercise 20.

$$\lim_{x\to 0}\frac{3\tan 3x-12\tan x}{3\sin 4x-12\sin x}=\lim_{x\to 0}\frac{(4x)+\frac{(4x)^3}{3}-4\left(x+\frac{x^3}{3}\right)+o(x^4)}{(4x)-\frac{(4x)^3}{3!}-4\left(x-\frac{x^3}{3!}\right)+o(x^4)}=$$

$$=\frac{4^3-4}{\frac{-4^3+4}{2}}=\boxed{-2}$$

Exercise 21.

$$\lim_{x\to 0}\frac{a^x-a^{sinx}}{x^3}=\lim_{x\to 0}\frac{e^{x\ln a}-e^{\sin x\ln a}}{x^3}=$$

$$=\lim_{x\to 0}\frac{1+x\ln a+\frac{(x\ln a)^2}{2!}+\frac{(x\ln a)^3}{6}-\left(1+\sin x\ln a+\frac{\sin^2 x(\ln a)^2}{2}+\frac{\sin^3 x(\ln a)^3}{3!}\right)+o(x^4)}{x^3}=$$

$$=\lim_{x\to 0}\frac{\ln a(x-\left(x-\frac{x^3}{3!}\right))+\frac{(x^2\ln a)^2}{2!}-\frac{(\ln a)^2}{2}(x^2)+\frac{(lna)^3}{6}(x^3-x^3)+o(x^4)}{x^3}=\boxed{\frac{\ln a}{6}}$$

Exercise 22.

$$\lim_{x\to 0}\frac{\cos\sin x-\cos x}{x^4}=\lim_{x\to 0}\frac{1-\frac{\sin^2 x}{2!}+\frac{\sin^4 x}{4!}+o(x^5)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)}{x^4}=$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}\left(x^2-(x^2-\frac{x^4}{3})\right)+\left(\frac{x^4-x^4}{4!}\right)+o(x^5)}{x^4}=\boxed{\frac{1}{6}}$$

Exercise 23.

$$\lim x \to 1 x^{\frac{1}{1-x}} = \lim x \to 1 e^{\frac{1}{1-x}\ln x} = \exp\left(\lim x \to 1 \frac{\ln x}{1-x}\right) = \exp\left(\lim x \to 1 \frac{(x-1)+o(x-1)^2}{1-x}\right) = \boxed{e^{-1}}$$

Exercise 24.

$$\lim_{x\to 0}(x+e^{2x})^{1/x} = \exp \lim_{x\to 0} \frac{1}{x}\ln(x+e^{2x})$$

$$\lim_{x\to 0}\frac{\ln(x+e^{2x})}{x} = \lim_{x\to 0}\frac{\ln(1+x+e^{2x}-1)}{x} =$$

$$= \lim_{x\to 0}\frac{x+e^{2x}-1+o(x^2)}{x} = \lim_{x\to 0}\frac{3x+o(x^2)}{x} = 3$$

$$\implies \lim_{x\to 0}(x+e^{2x})^{1/x} = e^3$$

Exercise 25.

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x} = \lim_{x\to 0}\frac{e^{\frac{x+-\frac{x^2}{2}+o(x^2)}{x}}-e}{x} =$$

$$= \lim_{x\to 0}\frac{e^{1-\frac{x}{2}+o(x)}-e}{x} = \lim_{x\to 0}\frac{e(1+-\frac{x}{2}+o(x))-e}{x} = \frac{-e}{2}$$

Exercise 26.

$$\lim_{x\to 0}\left(\frac{(1+x)^{1/x}}{e}\right)^{1/x} = \lim_{x\to 0}\left(\exp\left(\frac{1}{x}\ln(1+x)\right)-1\right)^{1/x} = \lim_{x\to 0} e^{\frac{x-\frac{x^2}{2}+o(x^3)-x}{x^2}} = e^{-1/2}$$

Exercise 27.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arcsin x)'' = \frac{x}{(1-x^2)^{3/2}}$$

$$(\arcsin x)''' = \frac{1}{(1-x^2)^{3/2}} + \frac{3x^2}{(1-x^2)^{5/2}}$$

$$\exp\left(\lim_{x\to 0}\frac{1}{x^2}\left(\ln\left(\frac{\arcsin x}{x}\right)\right)\right) = \exp\lim_{x\to 0}\frac{1}{x^2}\ln\left(1+\left(\frac{\arcsin x}{x}-1\right)\right) =$$

$$= \exp\left(\lim_{x\to 0}\frac{1}{x^2}\ln\left(1+\frac{x+x^3/6+o(x^4)-x}{x}\right)\right) = \exp\left(\lim_{x\to 0}\frac{1}{x^2}\ln\left(1+\frac{x^2}{6}+o(x^3)\right)\right) =$$

$$= \exp\left(\lim_{x\to 0}\frac{1}{x^2}\left(\frac{x^2+o(x^3)}{6}\right)\right) = \boxed{e^{1/6}}$$

Exercise 28. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to 0}\left(\frac{e^x-1-x}{x(e^x-1)}\right) = \lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^3)}{x^2+o(x^3)} = \boxed{\frac{1}{2}}$

Exercise 29.

$$\lim_{x\to 1}\left(\frac{1}{\log x}-\frac{1}{x-1}\right) = \lim_{x\to 1}\left(\frac{(x-1)-\log x}{(x-1)\log x}\right) =$$

$$= \lim_{x\to 1}\frac{(x-1)-((x-1)-\frac{(x-1)^2}{2}+o(x-1)^3)}{(x-1)((x-1)+o(x-1)^2)} = \boxed{\frac{1}{2}}$$

Exercise 30.

$$\lim_{x\to 0}\frac{e^{ax}-e^x-x}{x^2} = \lim_{x\to 0}\frac{1+ax+\frac{(ax)^2}{2}+o(x^3)-1-x-\frac{x^2}{2}-x}{x^2}$$

if $a=2$, the limit is $\boxed{2}$

Exercise 31.

(1) Prove $\int_0^x f(t)dt = o\left(\int_0^x g(t)dt\right)$ as $x \to 0$, given $f(x) = o(g(x))$.

Consider $\lim_{x\to 0} \frac{\int_0^x f(t)dt}{\int_0^x g(t)dt}$.

Since f, g have derivatives in some interval containing 0, f, g continuous and differentiable for $|t| \leq x$.

$$\lim_{x\to 0} \frac{\int_0^x f(t)dt}{\lim_{x\to 0} g(t)dt} = \lim_{x\to 0} \frac{\left(\frac{A(x)-A(0)}{x}\right)}{\left(\frac{B(x)-B(0)}{x}\right)} = \frac{f(0)}{g(0)} = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)} = 0$$

We can do the second to last step since f, g have derivatives at 0 and thus are continuous about 0.

(2) Consider $\lim_{x\to 0} \frac{x}{e^x} = 0$. However, $\lim_{x\to 0} \frac{1}{e^x} = 1$.

Exercise 32.

(1) Use long division to find that

$$\frac{1}{1+g(x)} = 1 - g(x) + g^2(x) + \frac{-g^3(x)}{1+g(x)} = 1 - g(x) + g^2(x) + o(g^2(x))$$

(2)

Exercise 33. Given $\lim_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^3$, **use the hint.**

$$\lim_{x\to 0} g(x) = A, \quad \text{then } G(x) = A + o(1) \text{ as } x \to 0$$

Then

$$g(x) = e^3 + o(1) = \left(1 + x + \frac{f}{x}\right)^{1/x}$$
$$x\left(e^3 + o(1)\right)^x = x + x^2 + f(x) \implies \boxed{f(0) = 0}$$

$$x \exp x \ln(e^3 + o(1)) = x + x^2 + f(x)$$
$$1 \exp x \ln(e^3 + o(1)) + x \exp x \ln(e^3 + o(1))\left(\ln e^3 + o(1) + \frac{x}{e^3 + o(1)}(o'(1))\right) = 1 + 2x + f'(0)$$
$$1 + 3(0) + 0 = 1 + 0 + f'(0) \implies f'(0) = 0$$

We need to assume that in general $o(1) = x + kx^2 + o(x^2)$.

$$2 \exp x \ln(e^3 + o(1))\left(\ln(e^3 + o(1)) + \frac{x}{e^3 + o(1)}(o'(1))\right) +$$
$$+ x \exp x \ln(e^3 + o(1))\left(\ln(e^3 + o(1)) + \frac{x}{e^3 + o(1)} o'(1)\right) +$$
$$x \exp x \ln(e^3 + o(1))\left(\frac{2o'(1)}{(e^3 + o(1))} + \frac{xo''(1)}{e^3 + o(1)} + -\frac{x}{(e^3 + o(1))}(o'(1))^2\right) = 2 + f''(x)$$

$$\xrightarrow{x\to 0} 2(3) = 2 + f'(0) \implies f'(0) = 4$$

To evaluate $\lim_{x\to 0}\left(1 + \frac{f(x)}{x}\right)^{1/x}$, consider a Taylor series expansion of f.

$$\lim_{x\to 0}\left(1 + \frac{f(x)}{x}\right)^{1/x} = \lim_{x\to 0}\left(1 + \frac{0 + 0 + 4\frac{x^2}{2} + o(x^3)}{x}\right)^{1/x} = \lim_{x\to 0}\left(1 + x\left(2 + o(x)\right)\right)^{1/x} =$$
$$= \lim_{y\to 0}\lim_{x\to 0}\left(1 + x\left(2 + o(y)\right)\right)^{1/x} = \lim_{y\to 0} \exp 2 + o(y) = e^y$$

7.13 Exercises - L'Hopital's rule for the indeterminate form $0/0$. Exercise 1.

$$\lim_{x\to 2}\frac{3x^2+2x-16}{x^2-x-2}=\lim_{x\to 2}\frac{(3x+8)(x-2)}{(x-2)(x+1)}=\boxed{\frac{14}{3}}$$

Exercise 2.

$$\lim_{x\to 3}\frac{x^2-4x+3}{2x^2-13x+21}=\lim_{x\to 3}\frac{(x-3)(x-1)}{(2x-1)(x-3)}=\boxed{-2}$$

Exercise 3.

$$\lim_{x\to 0}\frac{\sinh x-\sin x}{x^3}=\frac{0}{0}=\lim_{x\to 0}\left(\frac{\cosh x-\cos x}{3x^2}\right)=\frac{0}{0}=\lim_{x\to 0}\left(\frac{\sinh x+\sin x}{6x}\right)=\lim_{x\to 0}\left(\frac{\cosh x+\cos x}{6}\right)=\boxed{\frac{1}{3}}$$

Exercise 4.

$$\lim_{x\to 0}\frac{(2-x)e^x-x-2}{x^3}=\lim_{x\to 0}\frac{-e^x+(2-x)-1}{3x^2}=\lim_{x\to 0}\frac{(2-x)(1+x+x^2/2+x^3/6+o(x^3))}{x^3}=\boxed{-\frac{1}{6}}$$

Exercise 5.

$$\lim_{x\to 0}\frac{\log(\cos ax)}{\log(\cos bx)}=\lim_{x\to 0}\frac{-\cos bx\sin axa}{-\cos ax\sin bxb}=\lim_{x\to 0}\frac{\cos axa^2}{\sin bxb^2}=\frac{a^2}{b^2}$$

Exercise 6. When it doubt, **Taylor expand**.

$$\begin{aligned}
\lim_{x\to 0^+}\frac{x-\sin x}{(x\sin x)^{3/2}}&=\lim_{x\to 0^+}\frac{1-\cos x}{\frac{3}{2}(x\sin x)^{1/2}(\sin x+x\cos x)}=\frac{2}{3}\lim_{x\to 0^+}\frac{1-(1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4))}{e^{\frac{1}{2}\ln(x\sin x)}(x-\frac{x^3}{6}+o(x^3)+x-\frac{x^3}{2}+o(x^3))}=\\
&=\frac{2}{3}\lim_{x\to 0^+}\left(\frac{\frac{x^2}{2}-\frac{x^4}{24}+o(x^4)}{\sqrt{x(x+\frac{x^3}{6}+o(x^3))}(2x-\frac{2}{3}x^3+o(x^3))}\right)=\\
&=\frac{2}{3}\lim_{x\to 0^+}\left(\frac{\frac{1}{2}-\frac{x^2}{24}+o(x^2)}{\sqrt{1+\frac{x^2}{6}+o(x^2)}(2+\frac{-2}{3}x^2+o(x^2))}\right)=\frac{2}{3}\lim_{x\to 0^+}\left(\frac{1/2}{2}\right)=\boxed{\frac{1}{6}}
\end{aligned}$$

Notice in the third step how in general we deal with powers, $(x\sin x)^{1/2}$, is to convert it into exponential form, $e^{\frac{1}{2}\ln(x\sin x)}$, but it wasn't necessary.

Exercise 7. Do L'Hopital's first.

$$\begin{aligned}
\lim_{x\to a^+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}&=\lim_{x\to a^+}\frac{\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x-a}}}{\frac{x}{\sqrt{x^2-a^2}}}=\frac{1}{2}\lim_{x\to a^+}\frac{\sqrt{x^2-a^2}}{x^{3/2}}+\frac{\sqrt{x+a}}{x}=\\
&=\frac{1}{2}\lim_{x\to a^+}\frac{\sqrt{x^2-a^2}+x^{1/2}\sqrt{x+a}}{x^{3/2}}=\frac{1}{2}\frac{\sqrt{2}a}{a^{3/2}}=\frac{\sqrt{2}}{2\sqrt{a}}
\end{aligned}$$

Exercise 8. Do L'Hopital's at the second step.

$$\begin{aligned}
\lim_{x\to 1^+}\frac{\exp(x\ln x)-x}{1-x+\ln x}&=\lim_{x\to 1^+}\frac{\exp x\ln x(\ln x+1)-1}{-1+\frac{1}{x}}=\lim_{x\to 1^+}\frac{\exp(x\ln x)(\ln x+1)^2+\frac{1}{x}\exp(x\ln x)}{-1/x^2}=\\
&=\lim_{x\to 1^+}x^2\exp(x\ln x)(\ln x+1)^2+x\exp(x\ln x)=1+1=\boxed{2}
\end{aligned}$$

Exercise 9. Keep doing L'Hopital's.

$$\lim_{x\to 0}\frac{\arcsin 2x - 2\arcsin x}{x^3} = \lim_{x\to 0}\frac{\frac{2}{\sqrt{1-(2x)^3}} - 2\frac{1}{\sqrt{1-x^2}}}{3x^2} =$$
$$= \frac{2}{3}\lim_{x\to 0}\frac{-\frac{1}{2}(1-(2x)^2)^{-3/2}(-8x) - (-\frac{1}{2})(1-x^2)^{-3/2}(-2x)}{6x} =$$
$$= \frac{1}{9}\lim_{x\to 0}\frac{4(1-(2x)^2)^{-3/2} + 4x(-\frac{3}{2})(1-(2x)^2)^{-5/2}(-8x) + -(1-x^2)^{-3/2} - x(-\frac{3}{2})(1-x^2)^{-5/2}(-2x)}{1} =$$
$$= \frac{1}{9}\frac{4-1}{1} = \boxed{\frac{1}{3}}$$

Exercise 10.

$$\lim_{x\to 0}\frac{x\cot x - 1}{x^2} = \lim_{x\to 0}\frac{x\cos x - \sin x}{x^2\sin x} = \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{2x\sin x + x^2\cos x} = \lim_{x\to 0}\frac{-\sin x}{2\sin x + x\cos x} =$$
$$= -\lim_{x\to 0}\frac{\cos x}{2\cos x + \cos x + -x\sin x} = -\lim_{x\to 0}\frac{\cos x}{3\cos x - x\sin x} = \boxed{\frac{1}{3}}$$

Exercise 11.

$$\lim_{x\to 1}\frac{\sum_{k=1}^n x^k - n}{x-1} = \lim_{x\to 1}\frac{\sum_{k=1}^n kx^{k-1}}{1} = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Exercise 12.

$$\lim_{x\to 0+}\frac{1}{x\sqrt{x}}\left(a\arctan\frac{\sqrt{x}}{a} - b\arctan\frac{\sqrt{x}}{b}\right) = \lim_{x\to 0+}\frac{\left(a\frac{1}{1+\frac{x}{a^2}}\left(\frac{1}{2a\sqrt{x}}\right) - b\frac{1}{1+\frac{x}{b^2}}\left(\frac{1}{2b\sqrt{x}}\right)\right)}{\frac{3}{2}x^{1/2}} =$$
$$= \frac{1}{3}\lim_{x\to 0+}\frac{a^2}{a^2+x}\left(\frac{1}{x}\right) - \frac{b^2}{b^2+x}\left(\frac{1}{x}\right) = \frac{1}{3}\lim_{x\to 0+}\frac{(b^2+x)a^2 - b^2(a^2+x)}{(a^2+x)(x+b^2)x} =$$
$$= \frac{1}{3}\lim_{x\to 0+}\frac{(a^2-b^2)}{(a^2-x)(b^2+x)} = \boxed{\frac{1}{3}\frac{a^2-b^2}{a^2b^2}}$$

Exercise 13.

$$\frac{(\sin 4x)(\sin 3x)}{x\sin 2x} = \frac{(2\sin 2x\cos 2x)(\sin 3x)}{x\sin 2x} = \frac{2(-2\sin 2x\sin 3x + \cos 2x 3\cos 3x)}{1} = \boxed{6} \text{ as } x\to 0 \text{ otherwise}$$
$$\frac{2\cos 2x\sin 3x}{x} \to \frac{4}{\pi} \text{ as } x\to\frac{\pi}{2}$$

We used L'Hopital's at the second to last step for $x \to 0$.

Exercise 14.

$$\lim_{x\to 0}(x^{-3}\sin 3x + ax^{-2} + b) = 0$$
$$\lim_{x\to 0}\frac{\sin 3x + ax + bx^3}{x^3} = \frac{3\cos 3x + a + 3bx^2}{3x^2} = \frac{-9\sin 3x + 6bx}{6x} = \frac{-27\cos 3x + 6b}{6} = \frac{-27+6b}{6} = 0$$

So $\boxed{b = \frac{9}{2},\, a = -3}$.

Exercise 15.

$$\lim_{x\to 0}\frac{1}{bx - \sin x}\int_0^x\frac{t^2dt}{\sqrt{a+t}} = \frac{\frac{x^2}{\sqrt{a+x}}}{b-\cos x} = \frac{2x}{\frac{1}{2\sqrt{a+x}}(1-\cos x) + \sqrt{a+x}(\sin x)} = \frac{2x\sqrt{a+x}}{\frac{1}{2}(1-\cos x) + (a+x)\sin x} =$$
$$= 2\lim_{x\to 0}\frac{\sqrt{a+x} + \frac{x}{2\sqrt{a+x}}}{\frac{\sin x}{2} + \sin x + (a+x)\cos x} = 2\frac{\sqrt{a}}{a} = 1$$
$$\Longrightarrow \boxed{a = 4}$$

Note that we had dropped the limit notation in some earlier steps and applied L'Hopital's a number of times, and we also rearranged the denominator and numerator cleverly at each step.

Exercise 16.

(1)

$$\text{angle } ABC \text{ is } \frac{x}{2}, \text{ length } BC \text{ is } \tan\frac{x}{2}$$
$$2\tan\frac{x}{2}\cos\frac{x}{2} = 2\sin\frac{x}{2} \text{ is the base length of } ABC$$
$$\tan\frac{x}{2}\sin\frac{x}{2} \text{ is the height of triangle } ABC$$
$$\implies T(x) = \tan\frac{x}{2}\sin^2\frac{x}{2} = \frac{1-\cos^2\frac{x}{2}}{\cos\frac{x}{2}}\sin\frac{x}{2} = \tan\frac{x}{2} - \frac{1}{2}\sin x$$

(2)

$$S(x) = \left(\frac{x}{2\pi}\right)(\pi(1)) - \frac{1}{2}\cos\frac{x}{2}(2\sin\frac{x}{2}) = \frac{x}{2} - \frac{\sin 2}{2}$$

(3) Use L'Hopital's theorem.

$$\frac{T(x)}{S(x)} = \frac{\tan\frac{x}{2} - \frac{1}{2}\sin x}{\frac{x-\sin x}{2}} \xrightarrow{\frac{d}{dx}} \frac{\frac{1}{2}\sec^2\frac{x}{2} - \frac{\cos x}{2}}{\frac{1-\cos x}{2}}$$
$$\xrightarrow{\frac{d}{dx}} \frac{\sec^2\frac{x}{2}\tan\frac{x}{2} + \sin x}{\sin x} \xrightarrow{\frac{d}{dx}} \frac{\sec^2\frac{x}{2}\tan^2\frac{x}{2} + \frac{1}{2}\sec^4\frac{x}{2} + \cos x}{\cos x} \to$$
$$\xrightarrow{x\to 0} \frac{3}{2}$$

Exercise 17. Use L'Hopital's rule.

$$I(t) = \frac{E}{R}(1 - e^{-\frac{Rt}{L}})$$
$$\lim_{R\to 0} I(t) = \lim_{R\to 0} \frac{E(-1)(e^{-Rt/L})\left(\frac{-t}{L}\right)}{1} = \boxed{\frac{Et}{L}}$$

Exercise 18.

$$c - k \to 0 \text{ since } c \to k$$
$$k - c = u$$
$$k - u = c$$
$$f(t) = \frac{A(\sin kt - \sin ct)}{c^2 - k^2} = \frac{A(\sin(kt) - \sin((k-u)t)}{-u(2k-u)} =$$
$$= \frac{A(\cos((k-u)t)(t)}{-(2k-u)+u} \to \frac{-At\cos kt}{2k}$$

7.17 Exercises - The symbols $+\infty$ and $-\infty$. Extension of L'Hopital's rule; Infinite limits; The behavior of $\log x$ and e^x for large x.

Exercise 15. Use L'Hopital's at the second to last step.

$$\lim_{x\to 1^{-1}} (\ln x)(\ln(1-x)) = \lim_{x\to 1^{-1}} \frac{\ln(1-x)}{\frac{1}{\ln x}} = \lim_{x\to 1^{-1}} \frac{\left(\frac{1}{1-x}\right)(-1)}{\frac{-1}{(\ln x)^2}\left(\frac{1}{x}\right)} = \lim_{x\to 1^{-1}} \frac{x}{1-x}(\ln x)^2 = \lim_{x\to 1^{-1}} \frac{2\ln x\left(\frac{1}{x}\right)}{(-1)} = \boxed{0}$$

Exercise 16. Persist in using L'Hopital's and trying all possibilities systematically.

$$\lim_{x\to 0^+} x^{x^x - 1} = \lim_{x\to 0^+} e^{(x^x-1)\ln x} = \lim_{x\to 0^+} e^{(e^{x\ln x}-1)\ln x} = \exp\lim_{x\to 0^+} \frac{\ln x}{(e^{x\ln x}-1)^{-1}} =$$
$$= \exp\lim_{x\to 0^+} \frac{1/x}{(-1)(e^{x\ln x}-1)^{-2}(e^{x\ln x}(\ln x+1))} = \exp - \lim_{x\to 0^+}\left(\frac{e^{2x\ln x} - 2e^{x\ln x} + 1}{e^{x\ln x}(x\ln x + x)}\right) =$$
$$= exp - \lim_{x\to 0^+} \frac{e^{x\ln x}(\ln x + 1) + e^{-x\ln x}(-\ln x - 1)}{(\ln x + 1 + 1)} = \exp - \lim_{x\to 0^+} \frac{e^{x\ln x} - e^{-x\ln x}}{1 + \frac{1}{\ln x + 1}} = \boxed{1}$$

Exercise 17.

$$\lim_{x\to 0^+}(x^{x^x}-1)=\lim_{x\to 0^+}\left(e^{x^x\ln x}-1\right)=\lim_{x\to 0^+}\left(e^{e^{x\ln x}\ln x}-1\right)=\left(e^{\lim_{x\to 0^+}e^{x\ln x}\ln x}-1\right)=$$
$$=e^{e^{\lim_{x\to 0^+}x\ln x}\lim_{x\to 0^+}\ln x}-1=0-1=-1$$

We used

$$\lim_{x\to 0^+}x^\alpha\log x=0\,\forall\alpha>0$$

since

$t=\frac{1}{x}$, $x^\alpha\log x=\frac{-\log t}{t^\alpha}\to 0$ as $t\to\infty$.

Exercise 18.

$$\lim_{x\to 0^-}e^{\sin x\ln(1-2^x)}=\exp\left(\lim_{x\to 0^-}\frac{\ln(1-e^{x\ln 2})}{1/\sin x}\right)=\exp\left(\lim_{x\to 0^-}\frac{\frac{1}{1-e^{x\ln 2}}\left(-\ln 2e^{x\ln 2}\right)}{\frac{-1}{\sin^2 x}\cos x}\right)=$$
$$=\exp\left(\lim_{x\to 0^-}\frac{(\sin^2 x)\ln 2e^{x\ln 2}}{(1-e^{x\ln 2})\cos x}\right)=\exp\left((\ln 2)\lim_{x\to 0^-}\frac{2\sin x\cos x}{-\ln 2e^{x\ln 2}}\right)=\boxed{1}$$

Exercise 19.

$$\lim_{x\to 0^+}e^{\frac{1}{\ln x}\ln x}=\boxed{e}$$

Exercise 20. At the end, L'Hopital's could be used to verify that indeed $\sin x\ln\sin x\to 0$ as $x\to 0$.

$$\lim_{x\to 0^+}e^{\sin x\ln\cot x}=e^{\lim_{x\to 0^+}\sin x(\ln\cos x-\ln\sin x)}=e^{\lim_{x\to 0^+}-\sin x\ln\sin x}=\boxed{1}$$

Exercise 21. Rewrite $\tan$ into $\sin$ and $\cos$ and use L'Hopital's.

$$\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}=\lim_{x\to\frac{\pi}{4}}e^{\tan 2x\ln\tan x}=\exp\lim_{x\to\frac{\pi}{4}}\frac{1}{\cos 2x}(\ln\sin x-\ln\cos x)=$$
$$=\exp\lim_{x\to\frac{\pi}{4}}\frac{\frac{1}{\sin x}\cos x-\frac{1}{\cos x}(-\sin x)}{-2\sin 2x\cos 2x}=\exp\lim_{x\to\frac{\pi}{4}}\frac{1}{-\sin^2 2x}=\boxed{e^{-1}}$$

Exercise 22.

Exercise 23. Use L'Hopital's theorem, taking derivatives of top and bottom.

$$\lim_{x\to 0^+}\exp\frac{e}{1+\ln x}\ln x=\exp e\lim_{x\to 0}\frac{\ln x}{1+\ln x}=\exp e\lim_{x\to 0}\frac{1/x}{1/x}=e^e$$

Exercise 24. Rewrite $\tan$ into $\sin$ and $\cos$ and take out $\sin$ since we could do the limit before doing L'Hopital's.

$$\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}=\lim_{x\to 1}e^{\tan\frac{\pi x}{2}\ln(2-x)}=e^{\lim_{x\to 1}\frac{\sin\pi x/2\ln(2-x)}{\cos\pi x/2}}=\exp\lim_{x\to 1}\frac{\frac{(-1)}{2-x}}{\frac{\pi}{2}\sin\pi x/2}=\boxed{\exp\frac{-2}{\pi}}$$

Exercise 26.

$$\lim_{x\to+\infty}\exp\left(x\ln\left(\frac{x+c}{x-c}\right)\right)=\exp\lim_{x\to+\infty}\frac{\ln\left(\frac{1+c/x}{1-c/x}\right)}{1/x}=\exp\lim_{x\to+\infty}\frac{\frac{1}{\left(\frac{1+c/x}{1-c/x}\right)}\left(\frac{(x-c)-(x+c)}{(x-c)^2}\right)}{\frac{-1}{x^2}}=$$
$$=\exp(2c)=4\Longrightarrow\boxed{c=\ln 2}$$

Exercise 27.

$$(1+x)^c=\exp(c\ln(1+x))=\exp(c(x-o(x)))=1+c(x-o(x))+o(x-o(x))=1+cx+o(x)$$
$$x^2\left(1+\frac{1}{x^2}\right)^{1/2}-x^2=x^2\left(1+\frac{1}{x^2}\right)^{1/2}-x^2$$

Let $x^2=\frac{1}{t}$. So $t\to 0$ as $x\to+\infty$.

$$\Longrightarrow\frac{(1+t)^{1/2}-1}{t}=\frac{1+\frac{1}{2}t-1+o(t)}{t}=\boxed{\frac{1}{2}}$$

Exercise 28.

$$(x^5+7x^4+2)^c - x = x^5\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^c - x$$

Let $\frac{1}{t}=x$ and guess that $c=\frac{1}{5}$

$$\begin{aligned}\left(\frac{1}{t^5}+\frac{7}{t^4}+2\right)^c - \frac{1}{t} &= \left(1+(7t+2t^5)\right)^{1/5}/t - \frac{1}{t} = \\ &= \frac{1+\frac{1}{5}(7t+2t^5)+o(t)-1}{t} = \boxed{\frac{7}{5}}\end{aligned}$$

Exercise 29.

$$\begin{aligned} g(x) &= xe^{x^2} \\ g'(x) &= e^{x^2}+2x^2e^{x^2} \\ g''(x) &= 2xe^{x^2}+4xe^{x^2}+4x^3e^{x^2} = 6xe^{x^2}+4x^3e^{x^2}\end{aligned}$$

$$\begin{aligned} f(x) &= \int_1^x g(t)\left(t+\frac{1}{t}\right)dt \\ f'(x) &= g(x)\left(x+\frac{1}{x}\right) - g(1)2; \\ f''(x) &= g'(x)(x+1/x)+g(x)(1-1/x^2)\end{aligned}$$

$$\begin{aligned}\frac{f''(x)}{g''(x)} &= \frac{(e^{x^2}+2x^2e^{x^2})(x+1/x)+xe^{x^2}(1-\frac{1}{x^2})}{6xe^{x^2}+4x^3e^{x^2}} = \frac{2xe^{x^2}+2x^3e^{x^2}+2xe^{x^2}}{(6xe^{x^2}+4x^3e^{x^2})} = \\ &= \frac{4x+2x^3}{6x+4x^3} = \boxed{\frac{1}{2}} \text{ as } x\to\infty\end{aligned}$$

Exercise 30.

$$\begin{aligned} g(x) &= x^ce^{2x} \\ g'(x) &= cx^{c-1}e^{2x}+2x^ce^{2x}\end{aligned} \qquad \begin{aligned} f(x) &= \int_0^x e^{2t}(3t^2+1)^{1/2}dt \\ f'(x) &= e^{2x}(3x^2+1)^{1/2}-1\end{aligned}$$

Guessing that $c=1$

$$\frac{f'(x)}{g'(x)} = \frac{e^{2x}(3x^2+1)^{1/2}-1}{2x^ce^{2x}+cx^{c-1}e^{2x}} = \frac{\sqrt{3}x(1+\frac{1}{3x^2})^{1/2}-e^{-2x}}{2x+1} = \boxed{\frac{\sqrt{3}}{2}}$$

So $\boxed{c=1}$.

Exercise 31.

Exercise 32.

(1)

$$P\left(1+\frac{r}{m}\right), P\left(1+\frac{r}{m}\right)^2, \dots P\left(1+\frac{r}{m}\right)^m$$

For each year, there are the just previously shown m compoundings, so for n years,

$$P\left(1+\frac{r}{m}\right)^{mn}$$

(2)

$$2 = e^{rt}$$

$$\frac{\ln 2}{r} = t = \boxed{11.55 years}$$

(3)

$$\begin{aligned} 2P_0 &= P_0\left(1+\frac{r}{m}\right)^{mt} \\ \ln 2 &= mt\ln(1+r/m)\end{aligned}$$

$$t = \frac{\ln 2}{m\ln(1+r/m)} = \frac{\ln 2}{4\ln(1+0.06/4)} = \boxed{11.64 years}$$

7.17 Exercises - The symbols $+\infty$ and $-\infty$. Extension of L'Hopital's rule; Infinite limits; The behavior of $\log x$ and e^x for large x. Exercise 1.

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{1000}}=\lim_{u\to\infty}e^{-u}u^500=\lim_{u\to\infty}\frac{u^{500}}{e^u}=0 \qquad \begin{aligned} u &= \frac{1}{x^2} \\ x^2 &= \frac{1}{u} \\ x^{1000} &= \frac{1}{u^500} \end{aligned}$$

Where we had used Theorem 7.11, which are two very useful limits for log and exp.

Theorem 26. *If* $a,b>0$,

$$\lim_{x\to+\infty}\frac{(\log x)^b}{x^a}=0 \tag{17}$$

$$\lim_{x\to+\infty}\frac{x^b}{e^{ax}}=0 \tag{18}$$

Proof. Trick - use the definition of the logarithm as an integral.

If $c>0, t\geq 1, t^c\geq 1\Longrightarrow t^{c-1}\geq t^{-1}$.

$$0<\ln x=\int_1^x\frac{1}{t}dt\leq\int_1^x t^{c-1}dt=\frac{1}{c}(x^c-1)<\frac{x^c}{c}$$

$$\Longrightarrow 0<\frac{(\ln x)^b}{x^a}<\frac{x^{cb-a}}{c^b}$$

$$\text{Choose } c=\frac{a}{2b},\ \frac{x^{cb-a}}{c^b}=\frac{x^{-a/2}}{c^b}\to 0 \text{ as } x\to\infty$$

$$\text{then } \frac{(\ln x)^b}{x^a}\to 0 \text{ as } x\to 0$$

For exp, Let $t=e^x$. $\ln t=x$. $\frac{x^b}{e^{ax}}=\frac{(\ln t)^b}{t^a}\to 0$ as $t\to\infty$ as $x\to\infty$. □

Exercise 2.

$$\lim_{x\to 0}\frac{\sin\frac{1}{x}}{\arctan\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{x}-o\left(\frac{1}{x}\right)}{\frac{1}{x}-o\left(\frac{1}{x}\right)}=\boxed{1}$$

Exercise 3. Use L'Hopital's.

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 3x}{\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\cos 3x}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x}{-3\sin 3x}=\boxed{-\frac{1}{3}}$$

Exercise 4. Use L'Hopital's.

$$\lim_{x\to\infty}\frac{\ln(a+be^x)}{\sqrt{a+bx^2}}=\lim_{x\to\infty}\frac{\frac{1}{a+be^x}(be^x)}{\frac{bx}{\sqrt{a+bx^2}}}=\lim_{x\to\infty}\frac{\frac{1}{ae^{-x}+b}}{\frac{1}{\sqrt{\frac{a}{x^2}+b}}}=\frac{1}{\sqrt{b}}$$

Exercise 5. Make the substitution $x=\frac{1}{t}$.

$$\lim_{x\to\infty}x^4\left(\cos\frac{1}{x}-1+\frac{1}{2x^2}\right)=\lim_{t\to\infty}\frac{1}{t^4}\left(\cos t-1+\frac{t^2}{2}\right)=\lim_{t\to\infty}\frac{t^4/4!+o(t^4)}{t^4}=\frac{1}{4!}=\frac{1}{120}$$

Exercise 6.

$$\lim_{x\to\pi}\frac{\ln|\sin x|}{\ln|\sin 2x|}=\lim_{x\to\pi}\frac{\frac{1}{\sin x}\cos x}{\frac{2}{\sin 2x}\cos 2x}=-\frac{1}{2}\lim_{x\to\pi}\frac{\sin 2x}{\sin x}=-\frac{1}{2}\lim_{x\to\pi}\frac{2\cos 2x}{\cos x}=\boxed{1}$$

Exercise 7.

$$\lim_{x\to\frac{1}{2}^-}\frac{\ln(1-2x)}{\tan\pi x}=\lim_{x\to\frac{1}{2}^-}\frac{\left(\frac{1}{1-2x}\right)(-2)}{(\sec^2\pi x)\pi}=\lim_{x\to\frac{1}{2}^-}\frac{-2(\cos^2\pi x)}{(1-2x)\pi}=$$

$$=-\frac{2}{\pi}\lim_{x\to\frac{1}{2}^-}\frac{2\cos\pi x\pi-\sin\pi x}{-2}=1$$

Exercise 8.

$$\lim_{x\to\infty}\frac{\cosh x+1}{e^x}=\lim_{x\to\infty}\frac{e^{x+1}+e^{-x-1}}{2e^x}=\frac{1}{2}\lim_{x\to\infty}e^1+\frac{1}{e^{2x+1}}=\boxed{\frac{e}{2}}$$

Exercise 9.

$$\lim_{x\to\infty}\frac{a^x}{x^b}=\lim_{x\to\infty}\frac{e^{x\ln a}}{x^b}\to\infty;\ a>1$$
$$\text{since }\lim_{x\to\infty}\frac{x^b}{e^{ax}}\text{ (in this case, }\ln a>0\text{)}$$

Exercise 10.

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x-5}{\sec x+4}=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{\tan x\sec x}=\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{1}{\cos x}\frac{\cos x}{\sin x}=\boxed{1}$$

8.5 Exercises - Introduction, Terminology and notation, A first-order differential equation for the exponential function, First-order linear differential equations.

The ordinary differential equation theorems we will use are

$$y'+P(x)y=0 \tag{19}$$
$$A(x)=\int_a^x P(t)dt$$
$$y=be^{-A(x)} \tag{20}$$

Consider $y'+P(x)y=Q(x)$; $A(x)=\int_a^x P(t)dt$.

Let $h(x)=g(x)e^{A(x)}$; g a solution.

$$h'(x)=(g'+Pg)e^A=Qe^A$$
$$\xrightarrow{\text{2nd. fund. thm. of calc.}} h(x)=h(a)+\int_a^x Q(t)e^{A(t)}dt$$
$$\text{since } h(a)=g(a)$$

$$y=g(x)=e^{-A(x)}\left(\int_a^x Q(t)e^{A(t)}dt+b\right) \tag{21}$$

We had done some of these problems previously, using an integration constant C, but following Apostol's notation for $y(a)=b$ for initial conditions is far more advantageous and superior as we seem clearly the dependence upon the initial conditions - so some of the solutions for the exercises will show corrections to the derived formula using Apostol's notation for $y(a)=b$ initial conditions.

Exercise 1. $A(x)=\int_0^x(-3)dt=-3x$

$$y=e^{3x}\left(\int_0^x e^{2t}e^{-3t}dt+0\right)=e^{3x}\left.\left(-e^{-t}\right)\right|_0^x=\boxed{-e^{2x}+e^{3x}}$$

Exercise 2. $y'-\frac{2}{x}y=x^4$. $A(x)=\int_1^x\left(\frac{-2}{t}\right)dt=-2\ln x$.

$$y=e^{-A(x)}\left(\int_a^x Q(t)e^{A(t)}dt+b\right)=e^{2\ln x\left(\int_1^x t^4e^{-2\ln t}dt+1\right)}=x^2\left(\int_1^x t^2dt+1\right)=$$
$$=-x^2+\frac{x^2}{3}(x^3-1)=\frac{x^5}{3}+\frac{2x^2}{3}$$

Exercise 3. $y'+y\tan x=\sin 2x$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, with $y=2$ when $x=0$.

$$A(x) = \int_0^x P(t)dt = \int_0^x \tan t dt = -\ln|\cos t|\Big|_0^x = -\ln\cos x$$

$$y = e^{-A(x)}\left(\int_a^x Q(t)e^{A(t)}dt + b\right) = \cos x\left(\int_a^b \sin 2te^{-\ln\cos t}dt + 2\right) =$$

$$= \cos x\left(\int_a^x 2\sin t dt + 2\right) = -2\cos^2 x + 4\cos x$$

Exercise 4. $y' + xy = x^3$. $y = 0, x = 0$.

$$A(x) = \frac{1}{2}x^2$$

$$y = e^{-\frac{1}{2}x^2}\left(\int_0^x t^3 e^{\frac{t^2}{2}}dt + 0\right) = e^{\frac{-x^2}{2}}\left(t^2 e^{\frac{t^2}{2}} - 2e^{\frac{t^2}{2}}\right)\Big|_0^x =$$

$$= x^2 - 2 + 2e^{-\frac{x^2}{2}}$$

Exercise 5. $y' + y = e^{2t}$. $y = 1, t = 0$.

$$A = x \qquad y(x) = e^{-x}\left(\int_0^x e^{2t}e^t dt + 1\right) = e^{-x}\left(\frac{e^{3t}}{3}\Big|_0^x + 1\right)$$

$$= \boxed{\frac{e^{2x}}{3} + \frac{2}{3}e^{-x}}$$

Exercise 6. $y'\sin x + y\cos x = 1; (0,\pi)$. $\Longrightarrow y' + \cot xy = \csc x$

$$A(x) = \int_a^x \cot t dt = \ln\left(\frac{\sin x}{\sin a}\right)$$

$$y(x) = e^{-\ln\left(\frac{\sin x}{\sin a}\right)}\left(\int (\csc t)e^{\ln\left(\frac{\sin t}{\sin a}\right)}dt + b\right) = \left(\frac{\sin a}{\sin x}\right)\left(\frac{x-a}{\sin a} + b\right)$$

$$\text{indeed } y = \frac{x-a}{\sin x} + \frac{b\sin a}{\sin x}$$

$$x \to 0 \text{ for } y = \frac{x-a}{\sin x} + \frac{b\sin a}{\sin x} = \frac{x + b\sin a - a}{\sin x}$$

$$\boxed{b\sin a = a} \text{ for } x \to 0$$

$$\boxed{a - b\sin a = \pi} \text{ for } x \to \pi$$

Exercise 7.

$$x(x+1)y' + y = x(x+1)^2 e^{-x^2} \Longrightarrow y' + \frac{1}{x(x+1)}y = (x+1)e^{-x^2}$$

$$A(x) = \int_a^x \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \ln\frac{x}{a} - \ln\frac{x+1}{a+1}$$

$$e^{A(x)} = \frac{(a+1)x}{a(x+1)}$$

$$y = \frac{a(x+1)}{(a+1)x}\left(\int_a^x (t+1)e^{-t^2}\left(\frac{(a+1)t}{a(t+1)}\right)dt + b\right) = \frac{(x+1)}{x}\left(\int_a^x te^{-t^2}dt + b\frac{a(x+1)}{(a+1)x}\right) =$$

$$= \frac{x+1}{2x}\left(e^{-a^2} - e^{-x^2}\right) + \frac{a(x+1)b}{(a+1)x}$$

It's easy to see that the last equation above goes to 0 as $x \to -1$.

$$y = (e^{-a^2} - e^{-x^2})(1/2)(1 + \frac{1}{x}) + \frac{ab}{a+1}(1 + \frac{1}{x}) =$$

$$\lim_{x\to 0} y = \frac{1}{x}\left(\frac{e^{-a^2} - e^{-x^2}}{2} + \frac{ab}{a+1}\right) \quad \boxed{a = 0}$$

Exercise 8. $y' + y\cot x = 2\cos x$ on $(0,\pi)$.

$$A(x) = \int_a^x \cot t dt = \ln\left(\frac{\sin x}{\sin a}\right) e^{A(x)} = \frac{\sin x}{\sin a}$$

$$y = \frac{\sin a}{\sin x}\left(\int 2\cos t \frac{\sin t}{\sin a} + b\right) = \frac{\sin a}{\sin x}\left(-\frac{\cos 2t}{2\sin a}\Big|_a^x + b\right) = \boxed{y = \sin x} \boxed{y = \frac{\cos(2a) - \cos(2x)}{2\sin x} + \frac{\sin a}{\sin x}}$$

$$= \frac{\cos(2a) + \cos 2x}{2\sin x} + \frac{\sin a}{\sin x}$$

Exercise 9. $(x-2)(x-3)y' + 2y = (x-1)(x-2)$. $y' + \frac{2}{(x-2)(x-3)}y = \frac{x-1}{x-3}$.

$$A(x) = \int_a^x \frac{2dt}{(t-2)(t-3)} = 2\int_a^x \left(\frac{1}{t-3} - \frac{1}{t-2}\right)dt = 2\left(\ln|t-3| - \ln|t-2|\right)|_a^x = 2\ln\left|\frac{x-3}{a-3}\right|\left|\frac{a-2}{x-2}\right|$$

$$\text{If } (-\infty,2),(3,+\infty), \begin{matrix} x-3 \lesseqgtr 0 \\ x-2 \lesseqgtr 0 \end{matrix} \frac{3-x}{2-x} = \frac{x-3}{x-2}$$

$$\text{If } (2,3), x-3<0, \text{ but } x-2>0$$

$$\text{If } (-\infty,2),(3,\infty)\; y = b\left(\frac{x-2}{x-3}\right)^2 \left(\left|\frac{a-3}{a-2}\right|\right)^2 + \left(\frac{x-2}{x-3}\right)^2 \left(x + \frac{1}{x-2} - a - \frac{1}{a-2}\right)$$

$$\text{If } (2,3)\; y = b\left(\frac{x-2}{x-3}\right)^2 \left(\left|\frac{a-3}{a-2}\right|\right)^2 + \left(\frac{x-2}{3-x}\right)^2 \left|\frac{a-3}{a-2}\right|^2 \int \left(\frac{t-1}{t-3}\right)\left|\frac{a-2}{a-3}\right|^2 \left(\frac{3-t}{t-2}\right)^2 =$$

$$= b\left(\frac{x-2}{3-x}\right)^2 \left|\frac{a-3}{a-2}\right|^2 + \frac{x-2}{3-x}\left(-\int \frac{(t-1)(3-t)}{(t-2)^2}\right) =$$

$$= b\left(\frac{x-2}{x-3}\right)^2 \left|\frac{a-3}{a-2}\right|^2 + \left(\frac{x-2}{x-3}\right)^2 \left(x + (x-2)^{-1} - a - (a-2)^{-1}\right)$$

Exercise 10. $s(x) = \frac{\sin x}{x}; x \neq 0\; s(0) = 1$, $T(x) = \int_0^x s(t)dt\;\; f(x) = xT(x)$

$$f' = T + x(s(x)) = T + \sin x$$
$$xf' - f = x\sin x$$
$$y' - \frac{y}{x} = \sin x$$
$$A(x) = \int_a^x \frac{-1}{t}dt = -\ln\frac{x}{a},\; e^{-A(x)} = \frac{x}{a}$$
$$y = \frac{x}{a}\left(\int_a^x \sin t\frac{a}{t} + b\right) = xT + \frac{bx}{a}$$
$$y(0) = 0 \neq 1 \text{ since } P(x) = \frac{1}{x} \text{ is not continuous at } x = 0$$

Exercise 11.

$$f(x) = 1 + \frac{1}{x}\int_1^x f(t)dt$$

$$xf(x) = x + \int_1^x f(t)dt \implies f(x) + xf' = 1 + f(x) \implies f' = \frac{1}{x}$$

$$\implies \boxed{f(x) = \ln|x| - C}$$

$$\ln|x| - C = 1 + \frac{1}{x}\int_1^x (\ln|t| - c)\,dt =$$

$$= 1 + \frac{1}{x}\left(t\ln|t| - t - ct\right)|_1^x = \ln|x| - C + \frac{1+C}{x} \implies \boxed{C = -1}\; \boxed{f(x) = \ln|x| + 1}$$

Exercise 12. Rewrite the second property we want f to have:

$$\int_1^x f(t)dt = \frac{1 - f(x)}{x} \implies f(x) = -\frac{1}{x^2} - \left(\frac{f'x - f}{x^2}\right)$$

So then

$$f' + \left(x - \frac{1}{x}\right) f = -\frac{1}{x}$$

$$A(x) = \int P(t)dt = \int \left(t - \frac{1}{t}\right) dt = \left(\frac{1}{2}x^2 - \ln x\right) - \left(\frac{1}{2}a^2 - \ln a\right)$$

$$e^{A(x)} = e^{\frac{x^2-a^2}{2}} \frac{a}{x}; e^{-A(x)} = \frac{x}{a} e^{\frac{a^2-x^2}{2}}$$

$$\boxed{f(x) = \frac{x}{a} e^{\frac{a^2-x^2}{2}} \left(\int_a^x \frac{-1}{t^2} a e^{\frac{t^2-a^2}{2}} dt + b\right)}$$

Exercise 13.

$$\begin{aligned} v &= y^k \\ v' &= ky^{k-1}y' \end{aligned} \quad v' + kPv = kQ = ky^{k-1}y' + kPy^k = kQ \text{ where } k = 1-n$$

$$\implies y'y^{-n} + Py^{1-n} = Q \implies \boxed{y' + Py = Qy^n}$$

Exercise 14. $y' - 4y = 2e^x y^{1/2}$

$$n = \frac{1}{2}\, k = 1 - \frac{1}{2} = \frac{1}{2};\; v = y^{1/2};\; v' + \frac{1}{2}(-4)v = \frac{1}{2}(2e^x) = v' - 2v = e^x$$

$$A(x) = \int_a^x P(t)dt = -2(x-a) = -2x$$

$$v = e^{2x}\left(\int e^t e^{-2t} dt + b\right) = e^{2x}\left(-e^{-x} + b\right) = \boxed{be^{2x} - e^x}$$

$$\implies y = (1+\sqrt{2})^2 e^{4x} - 2(1+\sqrt{2})e^{3x} + e^{2x}$$

$$y(x) = b^2 - 2b + 1 = 2$$

$$b = 1 + \sqrt{2}$$

Exercise 15. $y' - y = -y^2(x^2+x+1), n = 2\, k = 1 - n = -1,\; v = y^k\, v = y^{-1}$.

$$v' + kPv = kQ\, v' + (-1)(-1)v = (-1)(-(x^2+x+1)) = v' + v = x^2 + x + 1$$

$$A(x) = \int P(t)dt = \int_0^x 1dt = x$$

$$v = e^{-x}\left(\int (t^2+t+1)e^t dt + b\right) = e^{-x}\left(t^2e^t - te^t + 2e^t\right)\Big|_0^x + be^{-x} = (x^2 - x + 2) - (2e^{-x}) + be^{-x}$$

$$y = \frac{1}{x^2 - x + 2 - 2e^{-x}}$$

Exercise 16.

$$v' + -\frac{1}{x}v = 2x^2$$

$$v = e^{\ln x}\left(\int 2x^2 e^{-\ln x} dx + C\right) = x(x^2 + C) = x^3 + Cx$$

Then since $v = y^k,\ k = 1-n$,

$$y = (x^3 + Cx)^2 = x^2(x^2+C)^2;\; x \neq 0$$

$$\boxed{y = x^2(x^2-1)^2}$$

Check:

$$y' = 2x(x^2-1)^2 + x^3(4)(x^2-1)$$

$$2x^2 + x^4(4)(x^2-1) - 2x^2(x^2-1)^2 = 4x^3$$

Exercise 17. $xy' + y = y^2x^2 \log x$ on $(0, +\infty)$ with $y = \frac{1}{2}$ when $x = \frac{1}{2}, x \neq 0$.

$$y' + \frac{y}{x} = y^2 x \log x$$
$$k = 1 - n = 1 - 2 = -1, v = y^k = y^{-1}$$
$$v' + kPv = kQ; \quad v' + -\left(\frac{1}{x}\right) v = -1x \log x$$
$$v = x\left(\int \frac{-x \log x}{x} dt + C\right) = x\left(\int -\log t dt + C\right) = \boxed{y = \frac{1}{Cx + x^2 - x^2 \log x}} \quad C = -\frac{1}{2}(1 - \log\frac{1}{2}) + 2$$
$$= x(-(x \log x - x) + C) = -x^2 \log x + x^2 + Cx$$

Check:

$$y' = (C + 2x - 2x \log x - x)(-y^2)$$
$$y^2(-C + -2x + 2x \log x + x + C + x - x \log x)$$

$$y = \frac{1}{2}; x = 1\, y(x=1) = \frac{1}{2}\, b = 2$$
$$\boxed{y = \frac{1}{x(x \ln x - x + 3)}}$$

Exercise 18. $2xyy' + (1+x)y^2 = e^x$ on $(0, +\infty)$, $y = \sqrt{e}$ when $x = 1$; $y = -\sqrt{e}$ when $x = 1$.

If $x \neq 0, y \neq 0$,

$$y' + \frac{(1+x)}{2x} y = \frac{e^2}{2x} y^{-1}$$
$$k = 1 - n = 2; \quad v = y^k = y^2$$

$$v' + 2Pv = 2Q; \; v' + 2\left(\frac{1+x}{2x}\right) v = \frac{2e^x}{2x} \qquad \Longrightarrow v' + \frac{1+x}{x} v = \frac{e^x}{x} = v' + P_v = Q_v$$
$$A_v = \int_a^x P_v = \int_a^x \frac{1+t}{t} dt = \ln\left(\frac{x}{a}\right) + (x - a); \qquad e^{\int_a^x P_v dt} = e^{\ln x/a + x - a} = \frac{x}{a} e^{x-a}$$
$$v = \frac{ae^{-x+a}}{x} \left(\int_a^x \frac{e^t}{t} \frac{t}{a} e^{t-a} dt + b_v\right) = \frac{a}{x} e^{-x+a} \left(\frac{1}{2a} e^{2t-a}\Big|_a^x + b_v\right)$$
$$= \frac{a}{x} e^{-x+a} \left(\frac{e^{2x-a}}{2a} - \frac{e^a}{2a} + b_v\right)$$
$$y = \pm\sqrt{\frac{a}{x} e^{-x+a} \left(\frac{e^{2x-a}}{2a} - \frac{e^a}{2a} + b_v\right)}$$

Now $y^k = v$

$y^2 = v = e = \frac{1}{1} e^{-1+1} \left(\frac{e^{2-1}}{2(1)} - \frac{e^1}{2} + b_v\right) = b_v = e$

(1) $b_v = e \quad \Longrightarrow y = \sqrt{\frac{1}{x} e^{-x+1} \left(\frac{e^{2x-1}}{2} - \frac{e}{2} + e\right)}$

(2) $b_v = e \quad \Longrightarrow y = -\sqrt{\frac{1}{x} e^{-x+1} \left(\frac{e^{2x-1}}{2} - \frac{e}{2} + e\right)}$

(3) If we could take $a = 0$, then $\lim_{x \to 0} y = \pm\sqrt{\frac{e^x}{2x} - \frac{e^{-x+2a}}{2x} + \frac{b_v a e^{-x+a}}{x}} = \pm\sqrt{\frac{1 + x - e^{2(0)}(1-x) + 2b_v 0 e^{-x+a}}{2x}} = \pm 1$

If we consider $\lim_{x \to 0} y^2$, and let a go to 0, then

$$v = \frac{1}{x}\left(e^x - e^{-x}\right) \Longrightarrow \frac{\sinh x}{x}$$

Exercise 19. The Ricatti equation is $y' + P(x)y + Q(x)y^2 = R(x)$.

If u is a known solution, $y = u + \frac{1}{v}$ is also a solution if v satisfies a first-order ODE.

$$(u + 1/v)' = u' + \frac{-1}{v^2}v'$$
$$y' + Py + Qy^2 = R \Longrightarrow u' + \frac{-v'}{v^2} + P(u + \frac{1}{v}) + Q(u^2 + \frac{2u}{v} + \frac{1}{v^2}) = R$$
$$\Longrightarrow \boxed{v' - Pv = Q(2uv + 1)}$$

Exercise 20. $y' + y + y^2 = 2, y = 1, -2$.

(1) If $-2 \leq b < 1$,

$$y' + y + y^2 = 2 \; P = 1, Q = 1, R = 2$$
$$v' + (-P - 2Qu)v = Q \; y = u + \frac{1}{v}$$
$$u = 1$$
$$v' + (-1 - 2(1)(1))v = v' - 3v = Q = 1$$
$$v = e^{3x}\left(\int 1e^{-3t}dt + b\right) = e^{3x}\left(\left.\frac{e^{-3t}}{-3}\right|_a^x + b\right) = be^{3x} - \frac{1}{3}\left(1 - e^{3x-3a}\right)$$
$$y = 1 + \frac{3}{3be^{3x} - (1 - e^{3x-3a})}$$
$$y(0) = 1 + \frac{3}{3b - (1 - e^{-3a})} \Longrightarrow \begin{aligned} b &= 1 + \frac{1}{b} \\ b^2 - b - 1 &= 0 \\ b &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$
$$\boxed{y = 1 + \frac{3}{3be^{3x} - (1 - e^{3x})} \; b = \frac{1 - \sqrt{5}}{2}}$$

(2)

$$u = -2$$
$$v' + (-1 - 2(1)(-2))v = v' + 3v = 1$$
$$v = e^{-3x}\left(\int_a^x e^{3t}dt + b\right) = e^{-3x}\left(\frac{e^{3x} - e^{3a}}{3}\right) + be^{-3x} = \frac{1 - e^{3a-3x}}{3} + be^{-3x}$$
$$y = -2 + \frac{3}{1 - e^{3a-3x} + 3be^{-3x}}; y(0) = -2 + \frac{3}{1 - e^{3a} + 3b} \xrightarrow{a=0} y(0) = -2 + \frac{3}{3b} \Longrightarrow b = -1 \pm \sqrt{2}$$
$$\boxed{b \geq 1 \text{ or } b < -2, y = -2 + \frac{3}{1 - e^{-3x} + 3be^{-3x}} \; b = -1 \pm \sqrt{2}}$$

8.7 Exercises - Some physical problems leading to first-order linear differential equations.

Exercise 3.

(1) $y' = -\alpha y(t)$. $\quad y(T) = y_0 e^{-\alpha T} = \frac{y_0}{n}$. $\quad n = e^{\alpha T}$ so the relationship between T and n doesn't depend upon y_0. $\quad \boxed{\frac{1}{k}\ln e = T}$.

(2) $f(a) = y_0 e^{-ka}; \quad f(b) = y_0 e^{-kb}$.

$$\frac{f(a)}{f(b)} = e^{-ka+kb} \quad \Longrightarrow \ln \frac{f(a)}{f(b)} = -k(a-b); \quad \frac{1}{a-b} \ln \frac{f(a)}{f(b)} = -k$$

$$f(t) = y_0 \exp\left(\frac{\ln \frac{f(a)}{f(b)} t}{a-b} \right); \quad f(a) = y_0 \left(\frac{f(b)}{y_0} \right)^{a/b}; \quad \Longrightarrow \frac{f(a)}{(f(b))^{a/b}} = y_0^{1-\frac{a}{b}}$$

$$f(t) = \left(\frac{f(a)}{(f(b))^{a/b}} \right)^{\frac{b}{b-a}} \left(\frac{f(a)}{f(b)} \right)^{\frac{t}{a-b}} = \frac{(f(a))^{\frac{b}{b-a}}}{f(b)^{\frac{a}{b-a}}} \frac{f(a)^{-\frac{t}{b-a}}}{f(b)^{\frac{t}{a-b}}} =$$

$$= \frac{f(a)^{\frac{b-t}{b-a}}}{(f(b))^{\frac{a-t}{b-a}}} = \boxed{f(a)^{\frac{b-t}{b-a}} f(b)^{\frac{t-a}{b-a}}}$$

$$w(t) = \frac{b-t}{b-a}; \quad 1 - w(t) = \frac{b-a-(b-t)}{b-a} = \frac{t-a}{b-a}$$

Exercise 4. $F = mv' = w_0 - \frac{3}{4} v$

$$w_0 = 192$$

$$\frac{w_0}{g} = 6 = m \qquad v' = \frac{w_0}{m} - \frac{1}{8} v \Longrightarrow v' + \frac{1}{8} v = \frac{w_0}{m} = 32$$

$$v(t) = e^{-\frac{t}{8}} \left(\int_0^t \frac{w_0}{m} e^{\frac{t}{8}} dt + b \right) = e^{\frac{-t}{8}} \left(\frac{8w_0}{m} (e^{t/8} - 1) + 0 \right) = \left(256(1 - e^{-t/8}) \right)$$

$$v(10) = 256(1 - e^{-5/4}) = 256(1 - 37/128) = 182$$

$$F = mv' = w_0 - 12v \quad v' + \frac{12}{m} v = \frac{w_0}{m} = v' + 2v = 32$$

$$v(t) = e^{-2t} \left(\int_{t_0}^t e^{2x} 32 dx + b \right) = e^{-2t} \left(16(e^{2t} - e^{2t_0}) + b \right) = \left(16(1 - e^{2(t_0 - t)}) + b e^{-2t} \right)$$

$$v(10) = b e^{-2(10)} = 182 \Longrightarrow b = 182 e^{20}$$

$$v(t) = 16 + 166 e^{20-2t}$$

So then

$$v(t) = \begin{cases} 256(1 - e^{-t/8}) & \text{if } t < 10 \\ 16 + 166 e^{20-2t} & \text{if } t > 10 \end{cases}$$

Exercise 7.

(1)

$$y'(t) = (y - M)k = ky - kM \qquad y' + -ky = -kM$$

$$y = e^{kt}\left(\int -kMe^{-kt} dt + b\right) = e^{kt}(M(e^{-kt} - 1) + b) \qquad M = 60^\circ$$

$$y(0) = b = 200^\circ$$

$$y_f = e^{kT}(M(e^{-kT} - 1) + 200) = M + e^{kT}(200 - M)$$

$$\frac{1}{T} \ln \left(\frac{y_f - M}{b - M} \right) = k = \frac{1}{T}(\ln(60) - \ln(140)) = \frac{1}{T} \ln \left(\frac{3}{7} \right) = \frac{1}{30}(\ln 3 - \ln 7)$$

$$\boxed{y(t) = 60 + 140 e^{\frac{\ln 3 - \ln 7}{30} t}}$$

(2)

$$y(t) = 60 + 140 e^{-kt}; \quad k = \frac{(\ln 7 - \ln 3)}{30}$$

$$\ln \left(\frac{Y - 60}{140} \right) = -kt \Longrightarrow t_f = \frac{(\ln 140 - \ln(T - 60))}{k} \text{ for } 60 < T \leq 200$$

(3) $t_f = \frac{\ln(140) - \ln(30)}{k} = \frac{30}{\ln 7/3} \ln \frac{14}{3} = 54$ minutes

(4) $M = M(t) = M_0 - \alpha t \qquad \alpha = \frac{1}{10}$

$$y = e^{kt}\left(\int -kMe^{-kt}dt + b\right) = e^{kt}\left(\int_0^t -k(M_0 - \alpha u)e^{-ku}du + b\right) =$$
$$= -ke^{kt}\int_0^t (M_0 e^{-ku} - \alpha u e^{ku})du + be^{kt} =$$
$$= -ke\left(\left.\frac{M_0 e^{-ku}}{-k}\right|_0^t - \alpha\left.\left(\frac{ue^{-ku}}{-k} + \frac{-e^{-ku}}{k^2}\right)\right|_0^t\right) + be^{kt} =$$
$$= -ke^{kt}\left(\frac{M_0}{k}(1 - e^{-kt}) - \alpha\left(\frac{te^{-kt}}{-k} - \frac{e^{-kt}}{k^2} + \frac{1}{k^2}\right)\right) + be^{kt} =$$
$$y(t) = -M_0e^{kt} + M_0 - \alpha t - \alpha/k + \alpha e^{kt}/k + be^{kt} = (-M_0 + \alpha/k + b)e^{kt} + (M_0 - \alpha t - \alpha/k) =$$
$$\boxed{y(t) = (140 + \frac{3}{(\ln 3 - \ln 7)})e^{-kt} + (60 - \frac{t}{10} - \left(\frac{3}{\ln 3/7}\right))}$$

Exercise 8. $y'(t) = -k(y - M_0); \qquad y' + ky = kM_0.$

$$y = e^{-kt}\left(\int_{t_i}^{t_f} kM_0 e^{ku}du + b\right) = e^{-kt_f}\left(M_0(e^{kt_f} - e^{kt_i}) + b\right) = M_0(1 - e^{-k(t_f - t_i)}) + be^{-kt_f}$$
$$y(t_f) - M_0 = -M_0 e^{-k(t_f - t_i)} + be^{-kt_f}$$
$$\Longrightarrow \quad 65 - M_0 = -M_0e^{-k(5)} + 75e^{-k(5)} = (75 - M_0)e^{-5k} \Longrightarrow \ln\left(\frac{65 - M_0}{75 - M_0}\right) = -5k; \quad k = \frac{1}{5}\ln\left(\frac{75 - M_0}{65 - M_0}\right)$$
$$60 - M_0 = -M_0e^{-k(5)} + 65e^{-k(5)} = (65 - M_0)e^{-5k} \Longrightarrow \ln\left(\frac{60 - M_0}{65 - M_0}\right) = -5k$$
$$\Longrightarrow \frac{65 - M_0}{75 - M_0} = \frac{60 - M_0}{65 - M_0}$$
$$\boxed{M_0 = 55}$$

Exercise 9.

Let $y(t) =$ absolute amount of salt.
Water is leaving according to $w(t) = w_0 + (3 - 2)t = w_0 + t$.
Salt leaving $= \left(\frac{2\,gal}{min.}\right)\left(\frac{y(t)\text{ salt}}{w(t)}\right)$
So then

$$y' = \frac{-2y}{w_0 + t} \Longrightarrow \ln y = -2(\ln(w_0 + t) - \ln w_0) = -2\ln\left(\frac{w_0 + t}{w_0}\right)$$

is the equation of motion given by the problem.

$$y(t) = Ce^{\ln\left(\frac{w_0 + t}{w_0}\right)^{-2}} = C\left(\frac{w_0 + t}{w_0}\right)^{-2}$$

$$\boxed{y(t) = 50\left(\frac{100}{100 + t}\right)^2} \quad y(t = 60\,min.) = 50\left(\frac{100}{160}\right)^2 = 50\frac{25}{64} = \frac{625}{32} \simeq 19.53$$

Exercise 10.

Let y be the dissolved salt (total amount of) at t time. The (total) amount of water at any given time in the tank is $w = w_0 + t$. There is dissolved salt in mixture that is leaving the tank at any minute. There is also salt from undissolved salt in the tank that is "coming into" the dissolved salt, adding to the amount of dissolved salt in the mixture. Thus

$$y'(t) = (-2)\left(\frac{y}{w}\right) + \alpha\left(\frac{y}{w} - 3\right); \quad \alpha = \frac{-1\text{ gal}}{3\text{ min}}$$

We obtained α easily by considering only the dissolving part and how it dissolves 1 pound of salt per minute if the salt concentration, $\frac{y}{w}$ was zero, i.e. water is fresh.

$$y'(t) = \frac{-7}{3}\frac{y}{w} + 1; \qquad y' = \frac{7}{3}\frac{y}{w_0 + t} = 1$$

$$P = \frac{7/3}{w_0 + t}$$

$$\int P = \int \frac{7/3}{w_0 + t} = \frac{7}{3}\ln\left(w_0 + t\right)\Big|_0^t = \frac{7}{3}\ln\left(\frac{w_0 + t}{w_0}\right)$$

$$y = e^{-\ln\left(\frac{w_0+t}{w_0}\right)^{-7/3}}\left(\int_0^t (1)e^{\ln\left(\frac{w_0+u}{w_0}\right)^{7/3}} du + b\right) =$$

$$= \left(\frac{w_0}{w_0 + t}\right)^{7/3}\left(\int_0^t \left(\frac{w_0 + u}{w_0}\right)^{7/3} du + b\right) = \left(\frac{w_0}{w_0 + t}\right)^{7/3}\left(\frac{3w_0}{10}\left(\left(\frac{w_0 + t}{w_0}\right)^{10/3} - 1\right) + b\right) =$$

$$y = \left(\frac{100}{100 + 60}\right)^{7/3}\left(\frac{3(100)}{10}\left(\left(\frac{100 + 60}{100}\right)^{10/3} - 1\right) + 50\right) \simeq 54.78\, lbs.$$

Exercise 11. $LI'(t) + RI(t) = V(t) \qquad V(t) = E\sin\omega t.$

$$I(t) = I(0)e^{-Rt/L} + e^{-Rt/L}\int_0^t \frac{V(x)}{L}e^{Rx/L}dx$$

$$I(t) = I(0)e^{-Rt/L} + e^{-Rt/L}\frac{E}{L}\int_0^t \sin\omega x e^{Rx/L}dx$$

Using $\int e^{ax}\sin bx dx = ae^{ax}\sin bx - be^{ax}\cos bx$

$$I(t) = I(0)e^{\frac{-Rt}{L}} + \frac{Ee^{-Rt/L}}{L}\left(\frac{\frac{R}{L}e^{\frac{Rt}{L}}\sin\omega t - \omega e^{\frac{Rt}{L}}\cos\omega t}{\left(\frac{R}{L}\right)^2 + \omega^2} + \frac{\omega}{\left(\frac{R}{L}\right)^2 + \omega^2}\right)$$

$I(0) = 0 \quad$ So

$$I(t) = \frac{E}{L}\frac{\frac{R}{L}\sin\omega t - \omega\cos\omega t}{\left(\frac{R}{L}\right)^2 + \omega^2} + \frac{E\omega L}{R^2 + (\omega L)^2}e^{-Rt/L} = \frac{E(R\sin\omega t - \omega L\cos\omega t)}{R^2 + (\omega L)^2} + \frac{E\omega L}{R^2 + (\omega L)^2}e^{-Rt/L} =$$

$$\sin\alpha = \frac{\omega L}{\sqrt{R^2 + (\omega L^2)}}$$

$$\Longrightarrow I(t) = \frac{E\sin(\omega t - \alpha)}{\sqrt{R^2 + (\omega L)^2}} + \frac{E\omega L}{(R^2 + \omega^2 L^2)}e^{-Rt/L}$$

$$L = 0 \quad \sin\alpha = 0$$

Exercise 12.

$$E(t) = \begin{cases} E & \text{if } 0 < a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

$$I(t) = e^{-Rt/L}\int_0^t 0 = 0 \text{ for } t < a$$

$$I(t) = e^{-Rt/L}\int_a^t \frac{E}{L}e^{Rx/L}dx = \frac{E}{L}e^{-Rt/L}\frac{L}{R}\left(e^{Rt/L} - e^{Ra/L}\right) = \frac{E}{R}\left(1 - e^{\frac{R(a-t)}{L}}\right)$$

$$I(b) = \frac{E}{R}(1 - e^{R(a-b)/L})$$

$$\text{for } t > b, \quad I(t) = Ke^{-Rt/L}$$

$$\Longrightarrow I(t) = \frac{Ee^{-Rt/L}}{R}(e^{\frac{Rb}{L}} = e^{\frac{Ra}{l}}) \quad \text{for } I(b) = I(b)$$

Exercise 13. From Eqn. 8.22, $\frac{dx}{dt} = kx(M - x)$

$$\frac{dx}{dt} = kx(M-x) = kMx - kx^2; \quad \Longrightarrow \frac{dx}{kMx - kx^2} = dt = \frac{1/kdx}{x(M-x)} = dt$$
$$\Longrightarrow kdt = \left(\frac{1}{x} + \frac{1}{M-x}\right)\frac{1}{M}dx = \frac{\ln x + -\ln(M-x)}{M}$$
$$Mk(t-t_i) = \ln\frac{x}{M-x}; \quad e^{Mk(t-t_i)}(M-x) = x$$
$$x(t) = \frac{Me^{Mk(t-t_i)}}{1+e^{Mk(t-t_i)}} = \boxed{\frac{M}{1+e^{-Mk(t-t_i)}}}$$

Exercise 14. Note that we are given *three equally spaced times*.

$$M = x_2 + x_2 e^{-\alpha(t_2-t_0)}; \quad \frac{M-x_2}{x_2} = e^{-\alpha(t_2-t_0)}$$
$$\frac{M-x_2}{x_2}\left(\frac{x_1}{M-x_1}\right) = e^{-\alpha t_2 + \alpha t_0 + \alpha t_1 - \alpha t_0} = e^{-\alpha(t_2-t_1)}$$
$$\frac{M-x_3}{x_3}\left(\frac{x_2}{M-x_2}\right) = e^{-\alpha(t_3-t_2)} = \frac{M-x_2}{x_2}\left(\frac{x_1}{M-x_1}\right)$$
$$(M-x_3)(M-x_1)x_2^2 = x_1x_3(M-x_2)^2 = x_1x_3(M^2-2Mx_2+x_2^2) = x_2^2(M^2 - M(x_1+x_3)+x_1x_3)$$
$$(x_2^2 - x_1x_3)M^2 = M(x_2^2(x_1+x_3) + -2x_2x_1x_3) = (-x_1(x_3-x_2)+x_3(x_2-x_1))x_2$$
$$\Longrightarrow \boxed{M = x_2\frac{(x_3(x_2-x_1)-x_1(x_3-x_2))}{x_2^2 - x_1x_3}}$$

Exercise 15.

$$\frac{dx}{dt} = k(t)Mx - k(t)x^2 \qquad \frac{dx}{Mx-x^2} = k(t)dt \Longrightarrow M\int_{t_i}^t k(u)du = \ln\left(\frac{x}{M-x}\right)$$
$$\frac{x}{M-x} = e^{M\int_{t_i}^t k(u)du}$$
$$x = \frac{Me^{M\int_{t_i}^t k(u)du}}{1+e^{M\int_{t_i}^t k(u)du}} = \boxed{\frac{M}{1+e^{-M\int_{t_i}^t k(u)du}}}$$

Exercise 16.

(1) $M = 23\frac{92(23-3.9)-3.9(92-23)}{23^2-3.9(92)} = 201$

(2)
$$M = 122\left(\frac{150(122-92)-92(150-122)}{(122)^2-92(150)}\right) = 122\left(\frac{150(30)-92(28)}{(122)^2-92(150)}\right) = 216$$

(3) Reject.

8.14 Exercises - Linear equations of second order with constant coefficients, Existence of solutions of the equation $y''+by=0$, Reduction of the general equation to the special case $y''+by=0$, Uniqueness theorem for the equation $y''+by=0$, Complete solution of the equation $y''+by=0$, Complete solution of the equation $y''+ay'+by=0$.

Exercise 1. $y'' - 4y = 0 \quad y = c_1e^{2x} + c_2e^{-2x}$.

Exercise 2. $y''+4y=0 \quad y = c_1\cos(2x) + c_2\sin(2x)$.

Use Theorem 8.7.

Theorem 27. *Let $d = a^2 - 4b$ be the discrimnant of $y''+ay'+by=0$.*
Then $\forall$ solutions on $(-\infty,\infty)$ has the form

$$y = e^{-ax/2}(c_1u_1(x) + c_2u_2(x)) \tag{22}$$

where

(1) *If $a=0$, then $u_1(x)=1$ and $u_2(x)=x$*
(2) *If $d>0$, then $u_1(x) = e^{kx}$ and $u_2(x) = e^{-kx}$, where $k = \frac{\sqrt{d}}{2}$*
(3) *If $d<0$, then $u_1(x)=\cos kx$ and $u_2(x)=\sin kx$; where $k = \frac{1}{2}\sqrt{-d}$*

Exercise 3. $y'' - 4y' = 0; \quad a = -4.$

$$y = e^{2x}(c_1 e^{2x} + c_2 e^{-2x}) = \boxed{c_1 e^{4x} + c_2}$$

Exercise 4. $y'' + 4y' = 0$

$$y = e^{-2x}(c_1 e^{2x} + c_2 e^{-2x}) = \boxed{c_1 + c_2 e^{-4x}}$$

Exercise 5. $y'' - 2y' + 3y = 0 \quad d = 4 - 4(3) = -8 \quad \implies \boxed{y = e^{-x}(c_1 \sin\sqrt{2}x + c_2 \cos\sqrt{2}x)}$

Exercise 8. $y'' - 2y' + 5y = 0 \quad d = -16 \quad y = e^{x}(c_1 \cos 2x + c_2 \sin 2x)$

Exercise 9. $y'' + 2y' + y = 0 \quad d = 4 - 4(1)(1) = 0 \quad y = e^{-x}(1 + x)$

Exercise 10. $y'' - 2y' + y = 0 \quad d = 4 - 4(1)(1) = 0 \quad y = e^{x}(1 + x)$

Exercise 11. $y'' + \frac{3}{2}y' = 0 \quad y = 1,\ y' = 1;\ x = 0\ d = \frac{9}{4} > 0$

$$y = e^{\frac{-3}{4}x}(c_1 e^{\frac{3x}{4}} + c_2 e^{\frac{-3x}{4}}) = c_1 + c_2 e^{\frac{-3x}{2}}$$
$$c_2 = \frac{-2}{3} \implies y = \frac{5}{3} + \frac{-2}{3}e^{\frac{-3x}{2}}$$

Exercise 12. $y'' + 25y = 0; \quad y = -1,\ y' = 0,\ x = 3.$

$$\begin{aligned} y &= c_1 \sin 5x + c_2 \cos 5x \\ y' &= 5c_1 \cos 5x + -5c_2 \sin 5x \\ 0 &= 5c_1 \cos 15 - 5c_2 \sin 15 c_2 \sin 15 \quad = c_1 \cos 15 \end{aligned} \qquad \begin{aligned} -1 &= c_1 \sin 15 + c_2 \cos 15 \\ -1c_1(\sin 15 &+ \frac{\cos^2 15}{\sin 15}) = c_1\left(\frac{1}{\sin 15}\right) \end{aligned} \qquad \begin{aligned} c_1 &= -\sin 15 \\ c_2 &= -\cos 15 \end{aligned}$$

$$\boxed{y = -\sin 15 \sin 5x - \cos 15 \cos 5x}$$

Exercise 13. $y'' - 4y' - y = 0; \quad y = 2; \quad y' = -1$ when $x = 1$

$d = 10 - 4(1)(-1) = 20$

$$y = c_1 e^{(2+\sqrt{5})x} + c_2 e^{(2-\sqrt{5})x}$$

$$\begin{aligned} y(x=1) &= c_1 e^{2+\sqrt{5}} + c_2 e^{2-\sqrt{5}} = 2 \\ y'(x=1) &= (2+\sqrt{5})c_1 e^{2+\sqrt{5}} + (2-\sqrt{5})c_2 e^{2-\sqrt{5}} = -1 \end{aligned} \quad \implies (5+2\sqrt{5})c_1 e^{2+\sqrt{5}} + (5-2\sqrt{5})c_2 e^{2-\sqrt{5}} = 0$$

$$c_1 = \frac{2\sqrt{5}-5}{5+2\sqrt{5}} c_2 e^{-2\sqrt{5}}$$

$$\left(\frac{2\sqrt{5}-5}{5+2\sqrt{5}}\right) c_2 e^{2-\sqrt{5}} + c_2 e^{2-\sqrt{5}} = 2 = \frac{4\sqrt{5}c_2 e^{2-\sqrt{5}}}{5+2\sqrt{5}} \implies \frac{5+2\sqrt{5}}{2\sqrt{5}} e^{\sqrt{5}-2} = c_2 \quad c_1 = \frac{2\sqrt{5}-5}{2\sqrt{5}} e^{-2-\sqrt{5}}$$

$$\boxed{y = \frac{2\sqrt{5}-5}{2\sqrt{5}} e^{-2-\sqrt{5}} e^{(2+\sqrt{5})x} + \frac{5+2\sqrt{5}}{2\sqrt{5}} e^{\sqrt{5}-2} e^{(2-\sqrt{5})x}}$$

Exercise 14. $y'' + 4y' + 5y = 0$, with $y = 2$ and $y' = y''$ when $x = 0$

$16 - 4(1)(5) = -4$

$$\begin{aligned} y &= e^{-2x}(c_1 \sin 2x + c_2 \cos 2x) \quad y(x=0) = c_2 = 2 \\ y' &= -2e^{-2x}(c_1 \sin 2x + 2\cos 2x) + 2e^{-2x}(c_1 \cos 2x - 2\sin 2x) \\ y'' &= 4e^{-2x}(c_1 \sin 2x + 2\cos 2x) - 8e^{-2x}(c_1 \cos 2x - 2\sin 2x) + 4e^{-2x}(-c_1 \sin 2x - 2\cos 2x) \end{aligned}$$

$$y'(0) = -2(2) + 2(c_1) = -4 + 2c_1$$

$$y''(0) = 4(2) + (c_1) + 4(-2) = -c_1 = -4 + 2c_1 \quad c_1 = \frac{4}{3}$$

$$\boxed{y = e^{-2x}(\frac{4}{3}\sin 2x + 2\cos 2x)}$$

Exercise 15. $y'' - 4y' + 29y = 0$

$d = 16 - 4(1)(29) = -100 \quad \implies u = e^{2x}(c_1 \sin 5x + c_2 \cos 5x).$

$$\begin{gathered} v: \quad y'' + 4y' + 13y = 0 \\ d = 10 - 4(13) = -36 \quad \implies v = e^{-2x}(b_1 \sin 3x + b_2 \cos 3x) \\ v(0) = b_2 \end{gathered}$$

$$\begin{aligned} u(0) &= 1(0 + c_2) = c_2 = 0 \\ u &= e^{2x} c_1 \sin 5x \\ u' &= 2e^{2x} c_1 \sin 5x + e^{2x} c_1 5 \cos 5x \\ u'\left(\frac{\pi}{2}\right) &= 1 = 2e^{\pi} c_1(1) \quad c_1 = \frac{1}{2e^{\pi}} \\ u'(0) &= \frac{1}{2e^{\pi}} 5 \\ u'(0) &= v'(0) \implies b_1 = \frac{5}{6e^{\pi}} \end{aligned}$$

$$\begin{gathered} v = e^{-2x} b_1 \sin 3x \\ v'(0) = 3b_1 \\ \boxed{u = \frac{1}{2e^{\pi}} e^{2x} \sin 5x} \\ \boxed{v = e^{-2x} \frac{5}{6e^{\pi}} \sin 3x} \end{gathered}$$

Exercise 16.

$$\begin{aligned} &y'' - 3y' - 4y = 0 \qquad u \quad 9 - 4(1)(-4) = 25 \quad u = e^{\frac{3x}{2}}(c_1 e^{\frac{5x}{2}} + c_2 e^{-\frac{5x}{2}}) \\ &y'' + 4y' - 5y = 0 \qquad v \quad 16 - 4(1)(-5) = 36 \quad v = e^{-2x}(b_1 e^{3x} + b_2 e^{-3x}) \end{aligned}$$

$$u(0) = c_1 + c_2 = 0 v(0) \quad = b_1 + b_2 = 0 \implies \begin{aligned} u &= 2e^{\frac{3x}{2}} c_1 (\sinh \left(\frac{5x}{2}\right)) \\ v &= 2b_1 e^{-2x} (\sinh(3x)) \end{aligned}$$

$$u' = c_1 \frac{3}{2} e^{\frac{3x}{2}} (e^{\frac{5x}{2}} - e^{\frac{-5x}{2}}) + 2e^{\frac{3x}{2}} c_1 \frac{5}{2} \cosh\left(\frac{5x}{2}\right) \qquad v' = -4b_1 e^{-2x} \sinh(3x) + 6b_1 e^{-2x} \cosh(3x)$$

$$u'(0) = 5c_1 \qquad v'(0) = 6b_1$$

$$c_1 = \frac{6b_1}{5}$$

Exercise 17.

$$\begin{gathered} y'' + ky = 0 \\ d = -4(k) \\ \frac{\sqrt{d}}{2} = \frac{\sqrt{-4k}}{2} = \sqrt{-k} \end{gathered}$$

$$\begin{gathered} \text{Assume } k > 0 \\ y = c_1 \sin \sqrt{k} x + c_2 \cos \sqrt{k} x \\ y(0) = c_2 = 0 \\ y(1) = c_1 \sin \sqrt{k} 1 = 0 \implies \sqrt{k} = n\pi \end{gathered}$$

$$\begin{gathered} k < 0; \quad -k = \kappa > 0 \\ y = c_1 e^{\sqrt{\kappa} x} + c_2 e^{-\sqrt{\kappa} x}; \quad y(0) = c_1 + c_2 = 0 \quad y = c_1 \sinh \sqrt{\kappa} x \\ y = c_1 \sinh \sqrt{\kappa} 1 = 0 \, c_1 = 0 \end{gathered}$$

so if $k < 0$, there are no nontrivial solutions satisfying $f_k(0) = f_k(1) = 0$

Exercise 18. $y'' + k^2 y = 0 \qquad d = -4k^2 < 0 \qquad \frac{\sqrt{d}}{2} = \frac{2k}{2} = k > 0$

$$y = c_1 \sin kx + c_2 \cos kx \qquad y(a) = b = c_1 \sin ka + c_2 \cos ka$$
$$y' = kc_1 \cos kx - c_2 k \sin kx \qquad y'(a) = m = kc_1 \cos ka - c_2 k \sin ka$$
$$kb \cos ka = kc_1 \cos ka \sin ka + c_2 k \cos^2 ka$$
$$m \sin ka = kc_1 \cos ka \sin ka - c_2 k \sin^2 ka$$
$$kb \cos ka - m \sin ka = c_2 k \Longrightarrow c_2 = \frac{kb \cos ka - m \sin ka}{k}$$
$$c_1 \sin ka = b - c_2 \cos ka = \frac{kb}{k} - \left(\frac{kb \cos ka - m \sin ka}{k} \right) \cos ka =$$
$$= \frac{kb(1 - \cos^2 ka) + m \sin ka \cos ka}{k} = \frac{kb \sin^2 ka + m \sin ka \cos ka}{k}$$
$$c_1 = \frac{kb \sin ka + m \cos ka}{k}$$
$$y = \left(\frac{kb \sin ka + m \cos ka}{k} \right) \sin kx + \left(\frac{kb \cos ka - m \sin ka}{k} \right) \cos kx$$
$$k = 0 \Longrightarrow y = mx - ma + b$$

Exercise 19.

(1) $y = k_1 \sin x + k_2 \cos x$

$$b_2 = k_1 \sin(a_2) + k_2 \cos(a_2) = k_1 s_2 + k_2 c_2$$
$$b_1 = k_1 \sin(a_1) + k_2 \cos(a_1) = k_1 s_1 + k_2 c_1$$
$$\begin{aligned} b_2 c_1 &= k_1 s_2 c_1 + k_2 c_2 c_1 \\ -\,(b_1 c_2 &= k_1 s_1 c_2 + k_2 c_1 c_2) \\ \Longrightarrow b_2 c_1 - b_1 c_2 &= k_1 (s_2 c_1 - s_1 c_2) \end{aligned} \Longrightarrow k_1 = \frac{b_2 c_1 - b_1 c_2}{s_2 c_1 - s_1 c_2}$$
$$\begin{aligned} s_1 b_2 &= k_1 s_1 s_2 + k_2 s_1 c_2 \\ -(s_2 b_1 &= k_1 s_1 s_2 + k_2 c_1 s_2) \end{aligned} \Longrightarrow k_2 = \frac{s_1 b_2 - s_2 b_1}{s_1 c_2 - c_1 s_2}$$
$$y = \left(\frac{b_2 \cos a_1 - b_1 \cos a_2}{\sin a_2 \cos a_1 - \sin a_1 \cos a_2} \right) \sin x + \left(\frac{b_2 \sin a_1 - b_1 \sin a_2}{\sin a_1 \cos a_2 - \cos a_1 \sin a_2} \right) \cos x$$
$$y = \boxed{\frac{b_2 \cos a_1 - b_1 \cos a_2}{\sin(a_2 - a_1)} \sin x + \frac{b_2 \sin a_1 - b_1 \sin a_2}{\sin(a_1 - a_2)} \cos x} \text{ if } a_2 - a_1 \neq \pi n$$

otherwise, if $a_2 - a_1 = \pi n$; $\begin{aligned} b_2 c_1 - b_1 c_2 &= 0 \\ b_2 s_1 - b_1 s_2 &= 0 \end{aligned}$ $\quad b_2 c_1 = b_1 (-1)^n c_1;\quad$ if $c \cos(a_1) \neq 0, \quad b_2 = b_1 (-1)^n$

(2) It's true if $a_1 = a_2 = \frac{\pi}{4}$; $\qquad b_1 = b_2$.

(3) $y'' + k^2 y = 0$

$$y = A \sin kx + B \cos kx$$
$$y(a_1) = A \sin ka_1 + B \cos ka_1 = b_1 = AS_1 + BC_1$$
$$y(a_2) = A \sin ka_2 + B \cos ka_2 = b_2 = AS_2 + BC_2$$
$$\begin{bmatrix} S_1 & C_1 \\ S_2 & C_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C_2 & -C_1 \\ -S_2 & S_1 \end{bmatrix} \left(\frac{1}{S_1 C_2 - C_1 S_2} \right) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$S_1 C_2 - C_1 S_2 = \sin ka_1 \cos ka_2 - \cos ka_1 \sin ka_2 = \sin(k(a_1 - a_2))$$
$$y = \frac{b_1 \cos(ka_2) - b_2 \cos(ka_1)}{\sin(k(a_1 - a_2))} \sin ka_1 + \frac{-b_1 \sin(ka_2) + b_2 \sin(ka_1)}{\sin(k(a_1 - a_2))} \sin ka_2$$
$$\text{if } k(a_1 - a_2) \neq \pi n \quad k = 0; \quad \boxed{y = \left(\frac{b_2 - b_1}{a_2 - a_1} \right)(x - a_1) + b_1}$$

Exercise 20.

(1)

$$u_1(x) = e^x; \quad u_2(x) = e^{-x} \qquad u_2' = -e^{-x} \quad u_2'' = e^{-x} \Longrightarrow y'' - y = 0$$

(2)

$$\begin{aligned} u_1 &= e^{2x} & u_2 &= xe^{2x} \\ u_1' &= 2e^{2x} & u_2' &= e^{2x} + 2xe^{2x} \\ u_1'' &= 4e^{2x} & u_2'' &= 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x} = 4e^{2x}(1+x) \end{aligned}$$

$$u_2'' - 4u_2' + 4u_2 = 0 \Longrightarrow \boxed{y'' - 4y' + 4y = 0}$$

(3)

$$\begin{aligned} u_1(x) &= e^{-x/2}\cos x; \\ u_1' &= -\frac{1}{2}e^{-x/2}\cos x + -e^{-x/2}\sin x \\ u_1'' &= \frac{1}{4}e^{-x/2}\cos x + e^{-x/2}\sin x + -e^{-x/2}\cos x \\ &= \frac{-3}{4}e^{-x/2}\cos x + e^{-x/2}\sin x \end{aligned}$$

$$\begin{aligned} u_2(x) &= e^{-x/2}\sin x \\ u_2' &= \frac{-1}{2}e^{-x/2}\sin x + e^{-x/2}\cos x \\ u_2'' &= \frac{1}{4}e^{-x/2}\sin x + -e^{-x/2}\cos x - e^{-x/2}\sin x = \\ &= \frac{-3}{4}e^{-x/2}\sin x - e^{-x/2}\cos x \\ u_2'' &+ u_2' + \frac{5}{4}u_2 = 0 \end{aligned}$$

$$\boxed{y'' + y' + \frac{5}{4}y = 0}$$

(4) $u_1(x) = \sin(2x+1); \quad u_2(x) = \sin(2x+2)$

$$\begin{aligned} u_1' &= 2\cos(2x+1) & u_2' &= 2\cos(2x+2) \\ u_1'' &= -4\sin(2x+1) & u_2'' &= -4\sin(2x+2) \end{aligned}$$

$$\boxed{y'' + 4y = 0}$$

(5)

$$\begin{aligned} u_1 &= \cosh x \\ u_1' &= \sinh x \quad \boxed{y'' - y = 0} \\ u_1'' &= \cosh x \end{aligned}$$

Exercise 21. $w = u_1u_2' - u_2u_1'$.

(1) $w = 0 \quad \forall x \in$ open interval I,

$$\left(\frac{u_2}{u_1}\right)' = \frac{u_2'u_1 - u_1'u_2}{u_1^2} = 0 \Longrightarrow \frac{u_2}{u_1} = c$$

If $\frac{u_2}{u_1}$ is not constant, then $w(0) \neq 0$ for at least one c in I (otherwise, it'd be constant).

(2) $w' = u_1u_2'' - u_2u_1''$

Exercise 22.

(1) $w' + aw = u_1u_2'' - u_2u_1'' + a(u_1u_2' - u_2u_1') = u_1(-bu_2) + -u_2(bu_1) = 0$
$w(x) = w(0)e^{-ax}$ if $w(0) \neq 0$, then $w(x) \neq 0 \quad \forall x$.

(2) $u_1 \neq 0$ If $w(0) = 0, \quad w(x) = 0 \quad \forall x$, so $\frac{u_2}{u_1}$ constant. If $\frac{u_2}{u_1}$ constant, $w(0) = 0$ since from the previous part.

Exercise 23. Recall the properties of the Wronskian.

(1) If $W(x) = v_1(x)v_2' - v_2v_1' = v_1v_2' - v_2v_1' = 0 \quad \forall x \in I$,
then $\frac{v_2}{v_1}$ constant on I

(2) $W' = v_1v_2'' - v_2v_1''$

(3) $W' + aW = 0$ if v_1, v_2 are solutions to $y'' + ay' + by = 0$
$W(x) = W(0)e^{-ax}$
So if $W(0) \neq 0, \quad W(x) \neq 0 \quad \forall x$

Consider adding together the solutions and the solution's derivatives into some function f. By the linearity of the differential equation, we know that f is also a solution since it is a linear superposition of solutions.

$$y(x) = Av_1(x) + Bv_2(x) \qquad y'(x) = Av_1'(x) + Bv_2'(x)$$
$$f(0) = Av_1(x) + Bv_2(0) \qquad f'(0) = Av_1'(0) + Bv_2'(0)$$
$$\begin{bmatrix} v_1(0) & v_2(0) \\ v_1'(0) & v_2'(0) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} \Longrightarrow \frac{1}{W(0)} \begin{bmatrix} v_2'(0) & -v_2(0) \\ -v_1'(0) & v_1(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$
$$\text{since } W(0) \neq 0, \quad \text{this division is allowed above}$$
$$\text{so } \boxed{y(x) = \left(\frac{v_2'(0)f(0) - v_2(0)f'(0)}{W(0)} \right) v_1(x) + \left(\frac{v_1(0)f'(0) - v_1'(0)f(0)}{W(0)} \right) v_2(x)}$$

$f(0),\ f'(0)$ are initial conditions for y. $f(0),\ f'(0)$ are arbitrary.
But since $W(0) \neq 0, \quad W(0) = v_1(0)v_2'(0) - v_2(0)v_1'(0)$, we can do things like

$$f(0) = \frac{v_2'(0)f(0) - v_2(0)f'(0)}{v_1(0)v_2'(0) - v_2(0)v_1'(0)} v_1(0) + \frac{v_2(0)f'(0) - v_1'(0)f(0)}{W(0)} v_2(0)$$

8.17 Exercises - Nonhomogeneous linear equations of second order with constant coefficients, Special methods for determining a particular solution of the nonhomogeneous equation $y'' + ay' + by = R$.

Exercise 1. $y'' - y = x$ homogeneous solution $c_1e^x + c_2e^{-x}$. $y_p = -x$

$y = c_1e^x + c_2e^{-x} - x$

Exercise 2. $y'' - y' = x^2$ For the homogeneous solution

$$y'' - y' = 0 \quad d = (-1)^2 - 4(1)(0) = 1 \qquad y_h = e^{\frac{x}{2}} \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} \right) = c_2e^x + c_1$$

$$\begin{aligned} y_p &= Ax^3 + Bx^2 + Cx + D \\ y_p' &= 3Ax^2 + 2Bx + C \\ y_p'' &= 6Ax + 2B \end{aligned} \qquad \boxed{y = c_1 + c_2e^x + \frac{-1}{3}x^3 + x^2}$$

Exercise 3. $y'' + y' = x^2 + 2x$

$$e^{-\frac{x}{2}} \left(c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{x}{2}} \right) = c_1e^{-x} + c_2$$
$$P = Ax^3 + Bx^2 + Cx + D; \qquad P' = 3Ax^2 + 2Bx + C \qquad P'' = 6Ax + 2B$$
$$3Ax^2 + 2Bx + C + 6Ax + 2B = x^2 + 2x \qquad A = \frac{1}{3} \quad B = 0 \quad C = 0$$
$$\boxed{y = c_1e^{-x} + c_2 + \frac{1}{3}x^3}$$

Exercise 4. $y'' - 2y' + 3y = x^3 \qquad u = e^x(c_1 \sin \sqrt{2}x + c_2 \cos \sqrt{2}x)$

$$\begin{aligned} &3(Ax^3 + Bx^2 + Cx + D) \\ &2(3Ax^2 + 2Bx + C) \\ &(6Ax + 2B) \end{aligned} \qquad A = \frac{1}{3} \quad B = \frac{2}{3} \quad C = \frac{8}{9} \quad D = \frac{16}{27}$$
$$y = C_1e^x \sin \sqrt{2}x + C_2e^x \cos \sqrt{2}x + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{16}{27}$$

Exercise 5. $y'' - 5y' + 4y = x^2 - 2x + 1$

$y_h = e^{\frac{5x}{2}} \left(e^{\frac{3x}{2}} + e^{\frac{-3x}{2}} \right) = c_1e^{4x} + c_2e^x$

$d = \sqrt{25 - 4(4)} = 3$

$$\begin{array}{llll} 4(Ax^2 + Bx + C) & 4Ax^2 + 4Bx + 4C & & \frac{1}{2} - \frac{5}{8} + 4C = 1 \\ -5(2Ax + B) & \quad -10Ax - 5B & A = \frac{1}{4} \quad B = \frac{1}{8} & \\ 2A & \quad\quad 2A & & C = \frac{9}{32} \end{array}$$

$$\boxed{y = c_1 e^{4x} + c_2 e^{x} + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{9}{32}}$$

Exercise 6.

$$y'' + y' - 6y = 2x^3 + 5x^2 - 7x + 2$$
$$y_h = e^{\frac{-x}{2}}(e^{\frac{-5x}{2}} + e^{\frac{5x}{2}}) = e^{-3x} + e^{2x}$$
$$d = \sqrt{1 - 4(-6)} = 5$$

$$\begin{array}{lcr} y_p = Ax^3 + Bx^2 + Cx + D & & -6Ax^3 - 6Bx^2 - 6Cx - 6D \\ y_p' = 3Ax^2 + 2Bx + C & \Longrightarrow & 3Ax^2 + 2Bx + C \\ y_p'' = 6Ax + 2B & & 6Ax + 2B \end{array}$$

$$A = \frac{-1}{3} \quad B = -1 \quad C = \frac{1}{2} \quad D = \frac{-7}{12}$$

$$\boxed{y = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{3}x^3 + -x^2 + \frac{1}{2}x - \frac{7}{12}}$$

Exercise 7.

$$\begin{array}{lll} y'' - 4y = e^{2x} & v_1 = e^{2x} & v_2' = -2e^{-2x} \\ y_h = c_1 e^{2x} + c_2 e^{-2x} & v_1' = 2e^{2x} & v_2 = e^{-2x} \end{array}$$

Use Theorem 8.9.

Theorem 28. *Let v_1, v_2 be solutions to $L(y) = 0$ where $L(y) = y'' + ay' + by$*
Let $W = v_1 v_2' - v_2 v_1'$. Then $L(y) = R$ where
$y_p = t_1 v_1 + t_2 v_2$

$$t_1 = -\int v_2 \frac{R}{W(x)} dx; \qquad t_2(x) = \int v_1 \frac{R}{W} dx \tag{23}$$

$$\begin{array}{ll} t_1 = -\int e^{-2x} \frac{e^{2x}}{-4} = \frac{1}{4}x & \\ & w = e^{2x}(-2)e^{-2x} - e^{-2x} 2e^{2x} = -4 \\ t_2 = \int e^{2x} \frac{e^{2x}}{-4} = \frac{e^{4x}}{-16} & \end{array}$$

$$y_p = \frac{x}{4}e^{2x} + \frac{e^{2x}}{-16}$$
$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}e^{2x} + \frac{e^{2x}}{-16}$$

Exercise 8.

$$y'' + 4y = e^{-2x}$$
$$y_h = c_1 \sin 2x + c_2 \cos 2x \qquad w = \sin 2x - \sin 2x(2) - 2\cos 2x \cos 2x = 2$$
$$t_1 = -\int \frac{\cos 2x e^{-2x} dx}{2} = \frac{-1}{2}\int e^{-2x}\cos 2x dx \qquad (e^{-2x}\cos 2x)' = -2e^{-2x}\cos 2x + -2e^{-2x}\sin 2x$$
$$t_2 = \int \frac{\sin 2x e^{-2x}}{2} = \frac{1}{2}\int e^{-2x}\sin 2x dx \qquad (e^{-2x}\sin 2x)' = -2e^{-2x}\sin 2x + 2e^{-2x}\cos 2x$$
$$\left(\frac{e^{-2x}\sin 2x - e^{-2x}\cos 2x}{4}\right)' = e^{-2x}\cos 2x \qquad t_1 = \frac{e^{-2x}\cos 2x - e^{-2x}\sin 2x}{8}$$
$$\left(\frac{e^{-2x}\sin 2x + e^{-2x}\cos 2x}{-4}\right)' = e^{-2x}\sin 2x \qquad t_2 = \frac{e^{-2x}\sin 2x + e^{-2x}\cos 2x}{-8}$$
$$y_p = \frac{e^{-2x}\sin 2x\cos 2x - e^{-2x}\sin^2 2x}{8} + \frac{e^{-2x}\sin 2x\cos 2x + e^{-2x}\cos^2 2x}{-8} = \boxed{\frac{e^{-2x}}{8}}$$
$$\boxed{y = c_1\sin 2x + c_2\cos 2x + \frac{e^{-2x}}{8}}$$

Exercise 9. $y'' + y' - 2y = e^x \qquad d^2 = 1 - (4)(1)(-2) = 9$

$$y_h = e^{\frac{-1x}{2}}\left(e^{\frac{3x}{2}} + e^{\frac{-3x}{2}}\right) = e^x + e^{-2x}$$
$$(xe^x)' = e^x + xe^x$$
$$+(xe^x)'' = +(2e^x + xe^x) \qquad \boxed{y = c_1e^x + c_2e^{-2x} + \frac{1}{3}xe^2}$$
$$\implies 3e^x + 2xe^x$$

Exercise 10. $y'' + y' - 2y = e^{2x}$. $y_h = e^{-\frac{x}{2}}\left(c_1e^{\frac{3x}{2}} + c_2e^{-\frac{3x}{2}}\right) = c_1e^x + c_2e^{-2x}$.

$$W(x) = v_1v_2' - v_2v_1' = e^x(-2)e^{-2x} - e^{-2x}e^x = -3e^{-x}$$
$$t_1 = -\int \frac{v_2R}{W} \qquad t_2 = \int \frac{v_1R}{W}$$
$$t_1 = -\int \frac{e^{-2x}e^{2x}}{-3e^{-x}} = \frac{1}{3}e^x \qquad t_2 = \int \frac{e^xe^{2x}}{-3e^{-x}} = \frac{-1}{12}e^{4x}$$
$$y_1 = t_1v_1 + t_2v_2 = \frac{1}{3}e^xe^x + -\frac{1}{12}e^{4x}e^{-2x} = \frac{1}{4}e^{2x}$$
$$y = c_1e^x + c_2e^{-2x} + \frac{1}{4}e^{2x}$$

Exercise 11. $y'' + y' - 2y = e^x + e^{2x}$.

Consider solutions to Exercise 9,10.

$$L(y_a) = e^x;\ L(y_b) = e^{2x};L(y_a + y_b) = e^x + e^{2x}$$
$$\implies \boxed{y = c_1e^x + c_2e^{-2x} + \frac{1}{3}xe^x + \frac{1}{4}e^{2x}}$$

Exercise 12. $y'' - 2y' + y = x + 2xe^x$. $d = 4 - 4(1) = 0$. Recall the definition to be learned for this section of exercises:

Theorem 29. *Let $d = a^2 - 4b$ be the discriminant of $y'' + ay' + by = 0$. Then every solution of this equation on $(-\infty, \infty)$ has the form*

$$y = e^{-ax/2}(c_1u_1(x) + c_2u_2(x)) \tag{24}$$

(1) *If $d = 0$ then $u_1 = 1, u_2 = x$*
(2) *If $d > 0$, $u_1 = e^{kx}$; $u_2 = e^{-kx}, k = \frac{\sqrt{d}}{2}$*
(3) *If $d < 0$, $u_1 = \cos kx, u_2 = \sin kx, k = \frac{\sqrt{-d}}{2}$*

$$y_h = e^x(c_1 + c_2 x) = c_1 e^x + c_2 x e^x$$

$$t_1 = \int \frac{-xe^x(2xe^x)}{e^{2x}} = \frac{2x^3}{-3} \quad t_2 = \int \frac{e^x(2xe^x)}{e^{2x}} = x^2$$

$$W(x) = e^x(e^x + xe^x) - (xe^x)(e^x) = e^{2x}$$

$$y_p = \frac{2x^3}{-3}e^x + x^3 e^x = \boxed{\frac{x^3 e^x}{3}}$$

$$\boxed{y = \frac{x^3 e^x}{3} + c_1 e^x + c_2 x e^x}$$

Exercise 13. $y'' + 2y' + y = \frac{e^{-x}}{x^2}$

$$y_h = e^{-x}(c_1 + c_2 x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$t_1 = \int \frac{-xe^{-x}\left(\frac{e^{-x}}{x^2}\right)}{e^{-2x}} = -\ln x \quad t_2 = \int \frac{e^{-x}\left(\frac{e^{-x}}{x^2}\right)}{e^{-2x}} = \frac{-1}{x}$$

$$W = e^{-x}(e^{-x} + -xe^{-x}) - (-e^{-x})(xe^{-x}) = e^{-2x}$$

$$y_p = -\ln x e^{-x} + xe^{-x}\left(\frac{-1}{x}\right) = -\ln x e^{-x} - e^{-x}$$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + (-\ln x - 1)e^{-x}}$$

Exercise 14. $y'' + y = \cot^2 x$. $y_h = c_1 \sin x + c_2 \cos x$.

$$t_1 = \int \frac{-\cos x}{-1}\cot^2 x dx = \int \frac{\cos x \cos^2 x}{\sin^2 x} = \int \frac{\cos x(1 - \sin^2 x)}{\sin^2 x} = -\frac{1}{\sin x} + -\sin x$$

$$t_2 = \int \frac{\sin x}{-1}\cot^2 x = -\int \frac{\cos^2 x}{\sin x} = -\int \frac{1 - \sin^2 x}{\sin x} = \ln|\csc x + \cot x| + -\cos x$$

$$y_p = -1 - \sin^2 x + \cos x \ln|\csc x + \cot x| - \cos^2 x = \boxed{-2 + \cos x \ln|\csc x + \cot x|}$$

$$\boxed{y = c_1 \sin x + c_2 \cos x - 2 + \cos x \ln|\csc x + \cot x|}$$

Exercise 15. $y'' - y = \frac{2}{1+e^x}$

$y_h = c_1 e^x + c_2 e^{-x} \quad \implies W(x) = -e^x e^{-x} - e^x e^{-x} = -2$

$$t_1 = -\int \frac{e^{-x}\frac{2}{1+e^x}}{-2} = \int \frac{e^{-x}}{1+e^x} =$$

$$= \int \frac{1}{e^x} - \frac{1}{1+e^x} = -e^{-x} - \int \frac{1}{1+e^x} = -e^{-x} + \int \frac{-e^{-x}}{e^{-x}+1} = -e^{-x} + \ln(1 + e^{-x})$$

$$t_2 = \int \frac{e^x\frac{2}{1+e^x}}{-2} = -\ln|1 + e^x|$$

$$\boxed{y = -1 + e^x \ln(1 + e^{-x}) + -e^{-x}\ln(1 + e^x) + c_1 e^x + c_2 e^{-x}}$$

Exercise 16. $y'' + y' - 2y = \frac{e^x}{1+e^x}$

Discriminant: $\frac{-1\pm\sqrt{1^2-4(-2)}}{2} = -2, 1 \quad \implies y_h = c_1 e^x + c_2 e^{-2x} \qquad \implies W = e^x(-2)e^{-2x} - e^{-2x}e^x = -3e^{-x}$

$$t_1 = -\int \frac{e^{-2t}\left(\frac{e^t}{1+e^t}\right)}{-3e^{-t}} = \frac{1}{3}\int \frac{1}{1+e^t} = \frac{-1}{3}\ln\left(1+e^{-x}\right)$$

$$\begin{aligned} t_2 &= \int \frac{e^t\left(\frac{e^t}{1+e^t}\right)}{-3e^{-t}} = \frac{1}{-3}\int \frac{e^{3t}}{1+e^t} \xrightarrow{u=e^t} \frac{-1}{3}\int \frac{u^2du}{1+u} = \\ &= \frac{-1}{3}\int u + \frac{-u}{u+1} = \frac{-1}{3}\int u + -1 + \frac{1}{u+1} = \frac{-1}{3}(\frac{1}{2}u^2 - u + \ln u + 1) \\ &= \frac{-1}{6}e^{2x} + \frac{e^x}{3} + \frac{-1}{3}\ln\left(e^x+1\right) \end{aligned}$$

$$\boxed{y_1 = \frac{-1}{3}e^x\ln\left(1+e^{-x}\right) + \frac{-1}{6} + \frac{e^{-x}}{3} - \frac{e^{-2x}}{3}\ln\left(e^x+1\right) + c_1e^x + c_2e^{-2x}}$$

Exercise 17. $y' + 6y' + 9y = f(x)$; where $f(x) = 1$ for $1 \leq x \leq 2$. $f(x) = 0$ for all other x.

$$d = 36 - 4(1)(9) = 0$$

$$y_h = e^{-3x}(c_1 + c_2x) = c_1e^{-3x} + c_2xe^{-3x}$$

$$W(x) = e^{-3x}(e^{-3x} - 3xe^{-3x}) - (xe^{-3x})(-3e^{-3x}) = e^{-6x}$$

$$t_1(x) = \begin{cases} a < 1 < x & \int_1^x -te^{3t}dt = \left(\frac{-te^{3t}}{3} + \frac{e^{3t}}{9}\right)\Big|_1^x = \frac{-3xe^{3x}+e^{3x}}{9} + \frac{2}{9}e^3 \\ a < 1 < 2 < x & \int_1^2 -te^{3t}dt = \frac{-6e^6+e^6}{9} + \frac{2e^3}{9} = \frac{-5e^6}{9} + \frac{2e^3}{9} \\ 1 < a < x < 2 & \int_a^x -te^{3t}dt = \left(\frac{-te^{3t}}{3} + e^{3t}9\right)\Big|_a^x = \frac{-xe^{3x}}{3} + \frac{e^{3x}}{9} + \frac{ae^{3a}}{3} - \frac{e^{3a}}{9} \end{cases}$$

$$t_2(x) = \int \frac{e^{-3t}f(t)}{e^{-6t}} = \int e^{3t}f(t) = \int_a^x e^{3t} = \frac{1}{3}\left(e^{3x} - e^{3a}\right)$$

$$y_1 = e^{-3x}\left(\frac{-3xe^{3x}+e^{3x}}{9} + C\right) + \frac{x}{3}$$

$$\boxed{y = c_1e^{-3x} + c_2xe^{-3x} + \frac{1}{9} \text{ when } 1 \leq x \leq 2; \text{ otherwise } y = y_h}$$

Exercise 18. Start from $y'' - k^2y = R(x)$. Suppose $L(y_p) = y_p'' - k^2y_p = R(x)$.

$y_h = c_1\sinh(kx) + c_2\cosh(kx); \qquad L(y_h) = 0$

So consider $L(y_p + y_h) = R(x)$; $y_p + y_h = y_1$ is a nother particular solution.

The *key to this problem* is to *apply the integration directly on the ODE itself*, not to go the other way around by differentiating the supposed particular solution.

$$\xrightarrow{\int_0^x dt\sinh(k(x-t))} \int_0^x dt\frac{d^2y}{dt^2}(t)\sinh(k(x-t)) - k^2\int_0^x dty(t)\sinh(k(x-t)) = \int_0^x dtR(t)\sinh(k(x-t))dt$$

$$\begin{aligned} \int_0^x y''\sinh(\kappa) &= -y'(0)\sinh(kx) - \int y'\cosh(\kappa)(-k) = \\ &= -y'(0)\sinh(kx) + k(y(x) - y(0)\cosh(kx) + k\int y(t)\sinh(\kappa)) = \\ &= -y'(0)\sinh(kx) + ky(x) - ky(0)\cosh(kx) + k^2\int y(t)\sinh(\kappa) \end{aligned}$$

$$\implies y(x) - \frac{y'(0)\sinh(kx)}{k} - y(0)\cosh(kx) = \frac{1}{k}\int_0^x dtR(t)\sinh(k(x-t))dt$$

Now note that $L(y_h) = 0$, so applying $\int_0^x dt\sinh(k(x-t))$ results in 0 still.

With $y_p(x) = \frac{1}{k}\int_0^x dtR(t)\sinh{(k(x-t))}dt + \frac{y'(0)\sinh{(kx)}}{k} + y(0)\cosh{(kx)}$, we can add a homogeneous solution of $\frac{y'(0)\sinh{(kx)}}{k} + y(0)\cosh{(kx)}$ to $y_p(x)$ to obtain

$$\boxed{y_1(x) = \frac{1}{k}\int_0^x dtR(t)\sinh{(k(x-t))}dt}$$

Now for $y'' - 9y = e^{3x}$,

$$\begin{aligned}
y_1(x) &= \frac{1}{3}\int_0^x dt(e^{3t})\sinh{(3(x-t))}dt = \frac{1}{6}\int_0^x dte^{3t}\left(\frac{e^{3x-3t} - e^{-3x+3t}}{2}\right) = \frac{1}{6}\int_0^x dt(e^{3x} - e^{-3x+6t}) \\
&= \frac{1}{6}\left(e^{3x}x - e^{-3x}\left.\frac{1}{6}e^{6t}\right|_0^x\right) = \frac{1}{6}xe^{3x} - e^{-3x}\frac{e^{6x}-1}{36} = \frac{1}{6}(xe^{3x} - \frac{e^{3x}}{6} + \frac{e^{-3x}}{6}) \\
y_1' &= \frac{1}{6}(e^{3x} + 3xe^{3x} - \frac{3}{2}e^{3x} - \frac{3}{2}e^{-3x}) \\
y_1'' &= \frac{1}{6}\left(3e^{3x} + 3e^{3x} + 9xe^{3x} - \frac{9}{2}e^{3x} + \frac{9}{2}e^{-3x}\right) = \frac{1}{4}e^{3x} + \frac{3}{4}e^{-3x} + \frac{3}{2}xe^{3x} \\
y_1'' - 9y_1 &= \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}
\end{aligned}$$

Thus, we need to add homogeneous parts to our particular solution to make it work. So if

$$\boxed{y_p = \frac{xe^{3x}}{6} - \frac{e^{3x}}{y} + \frac{e^{-3x}}{6}}$$

then it could be checked easily with some computation, that this satisfies the ODE.

Exercise 19. Start from $y'' + k^2 y = R(x)$

Again, note that if $L(y_p) = y_p'' + k^2 y_p = R(x)$, $L(y_p + y_h) = R(x) + 0 = R(x)$, so $y_1 = y_p + y_h$ is also a particular solution.

$$\xrightarrow{\int_0^x dt\sin k(x-t)} \int_0^x dt\sin k(x-t)\frac{d^2y}{dt^2} + k^2\int_0^x dt\sin k(x-t)y = \int_0^x R(t)\sin k(x-t)$$

$$\begin{aligned}
\int_0^x dt\sin k(x-t)\frac{d^2y}{dt^2} &= -y'(0)\sin{(kx)} + k\int_0^x y'(t)\cos{(k(x-t))}dt = \\
&= -y'(0)\sin{(kx)} + k(y(x) - y(0)\cos{(kx)} - k\int_0^x y(t)\sin{(k(x-t))}dt) = \\
&= ky(x) - ky(0)\cos{(kx)} - y'(0)\sin{(kx)} - k^2\int_0^x y(t)\sin{(k(x-t))}dt
\end{aligned}$$

$$\Longrightarrow y(x) = \frac{1}{k}\int_0^x dtR(t)\sin k(x-t) + y(0)\cos{(kx)} + \frac{y'(0)\sin{(kx)}}{k}$$

We can add y_h with $c_1 = -y(0), \quad c_2 = \frac{-y'(0)}{k}$

$$y_1 = \frac{1}{k}\int_0^x dtR(t)\sin k(x-t)$$

Now for $y'' + 9y = \sin 3x$, then $k = 3$,

$$y_1 = \frac{1}{3}\int_0^x \sin 3t\sin 3(x-t)dt$$

$$\int_0^x \sin 3t(\sin 3x\cos 3t - \cos 3x\sin 3t)$$

$$\int_0^x sc = \int_0^x \frac{s(6t)}{2}dt = \frac{-1}{12}(c(6x) - 1)$$

$$\int_0^x s^2 = \int_0^x \frac{1-\cos{(6t)}}{2} = \frac{x}{2} - \left.\frac{\sin{(6t)}}{12}\right|_0^x = \frac{x}{2} - \frac{\sin{(6x)}}{12}$$

$$y_1 = \frac{1}{3}\left(\sin 3x\frac{-1}{12}(\cos{(6x)} - 1) - \cos 3x\left(\frac{x}{2} - \frac{\sin{(6x)}}{12}\right)\right) = \boxed{\frac{\sin 3x}{18} - \frac{x\cos 3x}{6}}$$

It could be shown with some computation that this particular solution satisfies the ODE without having to add or subtract parts of a homogeneous solution.

Exercise 20. $y'' + y = \sin x$

$y_h = c_1 \sin x + c_2 \cos x \qquad \Longrightarrow W(x) = -s^2 - c^2 = -1$

$$t_1 = -\int \frac{cs}{-1} = \frac{-\cos 2x}{4}; \quad t_2 = \int \frac{ss}{-1} = -\int \frac{1-\cos 2x}{2} = -\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)$$
$$y_p = -\frac{\sin x \cos 2x}{4} + \left(\frac{\sin 2x - 2x}{4}\right)\cos x = \frac{\sin x \cos^2 x + \sin^3 x - 2x\cos x}{4}$$
$$\boxed{y = c_1 \sin x + c_2 \cos x + \frac{\sin x \cos^2 x + \sin^3 x - 2x \cos x}{4}}$$

Exercise 21. $y'' + y = \cos x$

$y_h = c_1 \sin x + c_2 \cos x = c_1 S + c_2 C \qquad W(x) = -1$

$$t_1 = \int \frac{-CC}{-1} = \int \frac{1+\cos 2x}{2} = \frac{x + \frac{\sin 2x}{2}}{2}; \qquad t_2 = \int \frac{SC}{-1} = \frac{\cos 2x}{4}$$
$$y_p = \frac{x \sin x}{2} + \frac{\sin 2x \sin x}{4} + \frac{\cos x \cos 2x}{4}$$
$$\Longrightarrow \boxed{y = \frac{x\sin x}{2} + \frac{\sin 2x \sin x}{4} + \frac{\cos x \cos 2x}{4} + c_1 \sin x + c_2 \cos x}$$

Exercise 22. $y'' + 4y = 3x \cos x$

$y_h = c_1 \sin 2x + c_2 \cos 2x \qquad W(x) = -\sin^2 2x(2) + -\cos^2 2x(2) = -2$

$$t_1 = \int \frac{-\cos(2x)(3x\cos x)}{-2} = \frac{3}{2}\int x \cos x \cos(2x) = \frac{3}{2}\int xc(1-2s^2) = \frac{3}{2}\int xc - 3\int xcs^2 =$$
$$= \frac{3}{2}(xs + c) - 3\int x\left(\frac{c^3}{3}\right)' = \frac{3}{2}(xs+c) - (xs^3 - \int s^3) = \frac{3}{2}(xs+c) - xs^3 + \int s(1-c^2) =$$
$$= \frac{3}{2}(xs+c) - xs^3 + -c + \frac{1}{3}c^3 = \frac{3}{2}xs + \frac{c}{2} - xs^3 + \frac{1}{3}c^3$$
$$t_2 = \int \frac{\sin(2x)(3x\cos x)}{-2} = -3\int xsc^2 = \int x(c^3)' = xc^3 - \int c^3 = xc^3 - \int c(1-s^2) =$$
$$= xc^3 - s + \frac{1}{3}s^3$$
$$y_p = \left(\frac{3}{2}xs + \frac{c}{2} - xs^3 + \frac{1}{3}c^3\right)(2sc) + (xc^3 - s + \frac{s^3}{3})(1-2s^2) = \text{ (lots of algebra) } =$$
$$= \boxed{xc^2 + \frac{2}{3}s}$$
$$\Longrightarrow \boxed{y = c_1 \sin 2x + c_2 \cos 2x + x \sin x - \frac{2}{3}\cos x}$$

Remember, *persistence is key* to work through the algebra, quickly.

Exercise 23. $y'' + 4y = 3x \sin x$. From the work above, we could guess at the solution.

$(xs)' = s + xc$
$(xs)'' = 2c + -xs$

$$(xs)'' + 4(xs) = 2c - xs + 4xs = 2c + 3xs \quad (c)'' + 4c = 3c \Longrightarrow \left(\frac{-2}{3}c\right)'' + 4\left(\frac{-2}{3}c\right) = -2c$$
$$\Longrightarrow \boxed{y_h = xs - \frac{2}{3}c}$$
$$\boxed{y = x\sin x - \frac{2}{3}\cos x + c_1 \sin 2x + c_2 \cos 2x}$$

Exercise 24. $y'' - 3y' = 2e^{2x}\sin x$ Guessing and stitching together the solution seems easier to me.

$$\begin{aligned}(e^{2x}c)' &= 2e^{2x}c + -e^{2x}s \\ (e^{2x}c)'' &= 4e^{2x}c + -4e^{2x}s - e^{2x}c = 3e^{2x}c - 4e^{2x}s \\ (e^{2x}c)'' - 3(e^{2x}c)' &= 3e^{2x}c - 4e^{2x}s - 6e^{2x}c + 3e^{2x}s = \\ &= -3e^{2x}c - e^{2x}s\end{aligned}$$

$$\begin{aligned}(e^{2x}s)' &= 2e^{2x}s + e^{2x}c \\ (e^{2x}s)'' &= 4e^{2x}s + 4e^{2x}c + -se^{2x} = \\ &= 3e^{2x}s + 4e^{2x}c \\ (e^{2x}s)'' - 3(e^{2x}s)' &= 3e^{2x}s + 4e^{2x}c - 6e^{2x}s - 3e^{2x}c = \\ &= -3e^{2x}s + e^{2x}c\end{aligned}$$

$$(3e^{2x}s)'' - 3(3e^{2x}s)' + (e^{2x}c)'' - 3(e^{2x}c)' = -10e^{2x}s$$

$$\Longrightarrow \boxed{y_p = \frac{e^{2x}(3\sin x + \cos x)}{5}} \Longrightarrow \boxed{y = c_1 \sin\sqrt{3}x + c_2\cos\sqrt{3}x + \frac{e^{2x}(3\sin x + \cos x)}{-5}}$$

Exercise 25. $y'' + y = e^{2x}\cos 3x$.

$$\begin{aligned}(e^{2x}c(3x))'' &= 4e^{2x}c(3x) + -12e^{2x}s(3x) - 9e^{2x}c(3x) = -5e^{2x}c(3x) - 12e^{2x}s(3x) \\ (e^{2x}s(3x))'' &= 4e^{2x}s(3x) + 12e^{2x}c(3x) + -9e^{2x}s(3x) = -5e^{2x}s(3x) + 12e^{2x}c(3x) \\ L(e^{2x}c(3x) - 3e^{2x}s(3x)) &= -40e^{2x}\cos(3x)\end{aligned}$$

$$\boxed{y = c_1\sin x + c_2\cos x + \frac{e^{2x}\cos(3x) - 3e^{2x}\sin(3x)}{-40}}$$

8.19 Exercises - Examples of physical problems leading to linear second-order equations with constant coefficients.

In exercises 1-5, a partcile is assumed to be moving in simple harmonic motion, according to the equation $y = C\sin(kx+\alpha)$. The velocity of the particle is defined to be the derivative y'. The frequency of the motion is the reciprocal of the period. (Period $= 2\pi/k$, frequency $= k/2\pi$)

Exercise 1. Find the amplitude C if the frequency is $1/\pi$ and if the initial values of y and y' (when $x = 0$) are 2 and 4, respectively.

$$\text{frequency } = \frac{k}{2\pi} = \frac{1}{\pi} \quad \Longrightarrow k = 2$$

$$\begin{aligned} y(x=0) &= C\sin\alpha \\ y'(x=0) &= C\cos\alpha \end{aligned} \quad \Longrightarrow \frac{y(x=0)}{y'(x=0)} = \frac{1}{k}\tan\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{4} \text{ and } \boxed{C = 2\sqrt{2}}$$

Exercise 2. Find the velocity when y is zero, given that the amplitude is 7 and the frequency is 10.

$$\begin{aligned} y &= C\sin(kx+\alpha) \\ y' &= Ck\sin(kx+\alpha) \end{aligned} \quad C = 7 \quad \frac{k}{2\pi} = 10 \Longrightarrow k = 20\pi$$

$$y(x = \frac{-\alpha}{k}) = 0 \Longrightarrow y'(x = \frac{-\alpha}{k}) = \boxed{140\pi}$$

Exercise 3.

$$y = A\cos(mx+\beta)$$

$$\begin{aligned} y &= A\cos(mx+\beta) = A\cos\beta\cos(mx) - A\sin\beta\sin(mx) \\ y &= C\sin kx + \alpha = C\cos\alpha\sin kx + C\sin\alpha\cos kx \end{aligned} \quad \Longrightarrow \boxed{k = m} \text{ (since } x \text{ is arbitrary)}$$

$$\begin{aligned} -A\sin\beta &= C\cos\alpha \\ A\cos\beta &= C\sin\alpha \end{aligned} \quad \Longrightarrow \tan\alpha = \pm\cot\beta = \mp\tan\left(\frac{\pi}{2} - \beta\right) = \tan(\beta - \frac{\pi}{2})$$

$$\Longrightarrow \boxed{\alpha = \beta - \frac{\pi}{2}} \text{ and } \boxed{|C| = |A|}$$

Exercise 4. $\frac{2\pi}{T} = 4\pi$

$y = C\cos(kx+\alpha) = C\cos(kx) = \boxed{3\cos(4\pi x)}$

Exercise 5. $y = C\cos(x+\alpha) \quad y_0 = C\cos(x_0+\alpha)$

$y' = -C\sin(x+\alpha) = \pm v_0$

$v_0^2 + y_0^2 = C^2\sin^2(x+\alpha) + C^2\cos^2(x_0+\alpha) = C^2 \quad \Longrightarrow \boxed{C = \sqrt{v_0^2+y_0^2}}$

Exercise 6.

$$\begin{aligned} y &= C\cos(kx+\alpha) & y' &= -kC\sin(kx+\alpha) \\ y(0) &= C\cos(\alpha) = 1 & y'(0) &= -kC\sin(\alpha) = 2 \\ y''(0) &= -k^2C\cos(\alpha) = -12 & \frac{y''(0)}{y(0)} &= -k^2 = \frac{-12}{1} \Longrightarrow k = 2\sqrt{3} \end{aligned}$$

$$\frac{y'(0)}{y(0)} = \frac{-kC\sin(\alpha)}{C\cos(\alpha)} = \frac{2}{1} = 2 = -k\tan(\alpha) \quad \Longrightarrow \boxed{\alpha = \frac{-\pi}{6}}$$

Exercise 7. $k = \frac{2\pi}{3} \quad y = -C\sin(kx)$

$\boxed{y = -C\sin\frac{2\pi x}{3}}; \quad C > 0$

Exercise 8. Let's first solve the homogenous equation.

$$y'' + y = 0$$

$$\begin{aligned} y_h &= C_1\sin x + C_2\cos x & t_1 &= \int_0^x \frac{-\cos t(1)}{-1} = \sin x \\ W(x) &= -S^2 - C^2 = 1 & t_2 &= \int_0^x \frac{(\sin t)(1)}{-1} = (\cos x - 1) \end{aligned}$$

for $0 \le x \le 2\pi$ otherwise, for $x > 2\pi,\ t_1 = 0,\ t_2 = 0$

$$\begin{aligned} y_1 &= \sin^2 x + \cos^2 x + (1-\cos x) & y(0) &= 0 = c_2 \\ y'(x) &= C_1\cos x + \sin x & y'(0) &= c_1 = 1 \end{aligned}$$

$$y = \sin x + (1-\cos x)$$

$$\Longrightarrow \boxed{I(t) = \sin t + (1-\cos t)} \quad 0 \le t \le 2\pi$$

Exercise 9.

(1) Consider large t. Then $I(t) = F(t) + A\sin(\omega t+\alpha) \to A\sin(\omega t+\alpha)$

$$I = AS(\omega t+\alpha) = A(S(\omega t)C(\alpha) + C(\omega t)S(\alpha))$$

$$I' = \omega AC(\omega t+\alpha) = \omega A(C(\omega t)C(\alpha) - S(\omega t)S(\alpha))$$

$$I'' = -\omega^2 AS(\omega t+\alpha) = -\omega^2 A(S(\omega t)C(\alpha) + C(\omega t)S(\alpha))$$

$$I'' + RI' + I = I'' + I' + I =$$

$$= A((-\omega^2C(\alpha) + -\omega S(\alpha) + C(\alpha))S(\omega t) + (-\omega^2S(\alpha) + \omega C(\alpha) + S(\alpha))C(\omega t)) = S(\omega t)$$

$$\Longrightarrow \tan(\alpha) = \frac{-\omega}{1-\omega^2}$$

With $\tan\alpha = \frac{-\omega}{1-\omega^2}$ and the trig identities $t^2+1 = \sec^2$, $\frac{1}{C^2} = \sec^2$, and $S^2+C^2 = 1$, we can get $C = \frac{1-\omega^2}{\sqrt{1-\omega^2+\omega^4}}$, $S = \frac{-\omega}{\sqrt{1-\omega^2+\omega^4}}$

Note that the sign of S is fixed by tan.

$$A(-\omega^2C(\alpha) + -\omega S(\alpha) + C(\alpha))S(\omega t) = S(\omega t) \Longrightarrow A = \frac{1}{(1-\omega^2)C(\alpha) - \omega S(\alpha)} = \frac{D}{(1-\omega^2)^2 - \omega(-\omega)}$$

$$\Longrightarrow \boxed{A = \frac{1}{\sqrt{\omega^4-\omega^2+1}}}$$

We could immediately see that $\omega = \frac{1}{\sqrt{2}}$, $f = \frac{1}{2\pi\sqrt{2}}$ will maximize A.

(2) We could have, from the beginning, considered the problem with any R, in general.

$$A((-\omega^2 C(\alpha) + -R\omega S(\alpha) + C(\alpha))S(\omega t) + (-\omega^2 S(\alpha) + \omega R C(\alpha) + S(\alpha))C(\omega t)) = S(\omega t)$$

$$\tan\alpha = \frac{-\omega R}{1-\omega^2} \qquad \begin{aligned} S(\alpha) &= \frac{-\omega R}{\sqrt{(\omega R)^2 + (1-\omega^2)^2}} \\ C(\alpha) &= \frac{1-\omega^2}{\sqrt{(\omega R)^2 + (1-\omega^2)^2}} \end{aligned}$$

$$\implies A = \frac{1}{-\omega^2 C(\alpha) - \omega R S(\alpha) + C(\alpha)} = \frac{1}{\sqrt{(\omega R)^2 + (1-\omega^2)^2}}$$

$$(\omega R)^2 + (1-\omega^2)^2 = \omega^4 + \omega^2(-2+R^2) + 1 \xrightarrow{\frac{d}{d\omega}} 4\omega^3 + 2\omega(-2+R^2) = 2\omega(2\omega^2 + (-2+R^2)) = 0$$

$$\implies \begin{aligned} \omega &= 0 \\ 2\omega^2 &= 2 - R^2 \end{aligned} \xrightarrow{\text{to have resonances}} \boxed{R < \frac{1}{\sqrt{2}}}$$

Exercise 10. A spaceship is returning to earth. Assume that the only external force acting on it is the action of gravity, and that it falls along a straight line toward the center of the earth. The rocket fuel is consumed at a constant rate of k pounds per second and the exhaust material has a constant speed of c feet per second relative to the rocket.

Let $M(t) = M$ be the mass of the $rocket + fuel$ combination at time t. With $+y$ direction being towards earth, then the equation of motion is $F_g = +M(t)g$, where $g = 9.8m/s^2$.

$M(t)v(t) = Mv_R$ is the momentum of the rocket.
$M(t+h) = M(t) - \Delta m = M - \Delta m$ is the change in mass of the rocket due to spent fuel.
$v_e =$ velocity of the exhaust in the lab frame $= c + v_R(t)$

$$\Delta p = \Delta m(c + v_R) + (M - \Delta m)v_R(t+h) - Mv_R = M(v_R(t+h) - v_R) + -\Delta m(v_R(t+h) - v_R) + \Delta mc$$

$$\frac{\Delta p}{\Delta t} = M\left(\frac{v_R(t+h) - v_R}{\Delta t}\right) + -\Delta m\left(\frac{v_R(t+h) - v_R}{\Delta t}\right) + \left(\frac{\Delta m}{\Delta t}\right)c = M(t)g$$

$$Mv_R' + \frac{kc}{g} = M(t)g$$

$$\text{Now } M(t) = M_0 - \frac{kt}{g} \implies v_R' = g - \frac{kc/g}{M} = g - \frac{kc/g}{M_0 - \frac{kc}{g}}$$

$$v_R = gt - \frac{kc}{g}\frac{g}{-k}\ln{(M_0 - \frac{kt}{g})} = gt + c\ln{(M_0 - \frac{kt}{g})}$$

$$\boxed{y_R = \frac{gt^2}{2} + c\frac{g}{k}\left(\left(\frac{k}{g}t - M_0\right)\ln\left(M_0 - \frac{kt}{g}\right) - \frac{kt}{g}\right) + \frac{M_0 cg}{k}\ln M_0}$$

Exercise 11.

$$Mv_R' = \frac{-kc}{g}$$

$$\begin{aligned} v_R' &= \frac{-kc}{g}\left(\frac{1}{M_0 - \frac{kt}{g}}\right) \\ v_R &= c\ln{(M_0 - \frac{kt}{g})} \end{aligned} \implies y_R = c\frac{g}{k}\left(\left(\frac{kt}{g} - M_0\right)\ln\left(M_0 - \frac{kt}{g}\right) - \frac{kt}{g}\right) + \frac{M_0 cg\ln M_0}{k}$$

$$M_0 g = w \implies \boxed{y_R = c\frac{g}{k}\left(\frac{kt - w}{g}\ln\left(\frac{w - kt}{g}\right) - \frac{kt}{g}\right) + \frac{wc}{k}\ln\frac{w}{g}}$$

We could've also solved this problem with an initial velocity of v_0 and gravity. Then

$$v_R(t) = gt + c\ln{(1 - \frac{kt}{M_0 g})} + v_0$$

$$y(t) = v_0 t + \frac{1}{2}gt^2 + c\left(\left(t - \frac{M_0 g}{k}\right)\ln\left(1 - \frac{kt}{M_0 g}\right) - t\right)$$

Exercise 12.

$$\begin{aligned} Mv_R &= (M - \Delta m)(v_R(t+h)) + 0 \\ M(v_R(t+h) - v_R(t)) &= (\Delta m)v_R(t+h) \\ Mv_R' = \frac{k}{g}v_R \Longrightarrow \frac{v_R'}{v_R} &= \frac{k}{g(M_0 - \frac{kt}{g})} = \frac{k}{M_0 g(1 - \frac{kt}{M_0 g})} \\ \ln v_R = (k/w)\ln(1 - \frac{kt}{w})\left(\frac{-w}{k}\right) &= -\ln(1 - \frac{kt}{w}) \\ v_R = \frac{v_0}{1 - \frac{kt}{w}} \quad &\Longrightarrow x(t) = v_0\left(\frac{-w}{k}\right)\ln(1 - \frac{kt}{w}) = \frac{-v_0 w}{k}\ln(1 - \frac{kt}{w}) \end{aligned}$$

8.22 Exercises - Remarks concerning nonlinear differential equations, Integral curves and direction fields.

Exercise 1. $2x + 3y = C \quad \Longrightarrow y' = \frac{-2}{3}$

Exercise 2. $y = Ce^{-2x} \quad \Longrightarrow y' = -2y$

Exercise 3. $x^2 - y^2 = c \quad \Longrightarrow yy' = x \Longrightarrow y' = \frac{x}{y}; \quad y \neq 0$

Exercise 4. $xy = c \quad \Longrightarrow y' = \frac{-y}{x}; \quad x \neq 0$

Exercise 5. $y^2 = cx \quad \Longrightarrow \frac{y^2}{x} = c \Longrightarrow y' = \frac{y}{2x} \quad x \neq 0$

Exercise 6. $x^2 + y^2 + 2Cy = 1$

$$\begin{gathered} \frac{x^2}{y} + y - \frac{1}{y} = -2C \\ \frac{2xy - y'x^2}{y^2} + y' + \frac{1}{y^2}y' = 0 \\ y' = \frac{-2xy}{1 + y^2 - x^2} \end{gathered}$$

Exercise 7. $y = C(x-1)e^x$

$$\begin{gathered} \frac{y}{(x-1)e^x} = C \\ \frac{y'(x-1)e^x - (e^x + (x-1)e^x)y}{(x-1)^2 e^{2x}} = 0 \\ \boxed{y' = \frac{xy}{x-1}} \end{gathered}$$

Exercise 8. $y^4(x+2) = C(x-2)$

$$\begin{gathered} \frac{y^4(x+2)}{x-2} = C \\ \frac{(4y^3y'(x+2) + y^4)(x-2) - y^4(x+2)}{(x-2)^2} = 0 \quad \Longrightarrow 4y^3y'(x+2) + y^4 = \frac{y^4(x+2)}{x-2} \\ \boxed{y' = \frac{y}{(x-2)(x+2)}} \end{gathered}$$

Exercise 9. $y = c\cos x \quad \Longrightarrow y' = -\tan xy$

Exercise 10. $\arctan y + \arcsin x = C$

$$\frac{1}{1+y^2}y' + \frac{1}{\sqrt{1-x^2}} = 0 \quad \Longrightarrow \boxed{y' = \frac{-(1+y^2)}{\sqrt{1-x^2}}}$$

Exercise 11. All circles through the points $(1,0)$ and $(-1,0)$.

Start with the circle equation: $(x-A)^2 + (y-B)^2 = R^2$

$(1,0)$: $(1-A)^2+B^2=R^2 \implies -(1-2A+A^2+B^2=R^2)$
$(-1,0)$: $(-1-A)^2+B^2=1+2A+A^2+B^2=R^2$
$\implies 4A=0,\ A=0 \qquad 1+B^2=R^2$

$$x^2+(y-\pm\sqrt{R^2-1})^2=R^2$$
$$x^2+y^2-2By+(R^2-1)=R^2 \implies x^2+y^2-2By=1$$

B depends upon R, the radius of the circles, so we could use B as the parameter for the family of circles.

$$x^2+y^2-1=2By$$
$$\frac{x^2}{y}+y-\frac{1}{y}=2B \qquad \implies \frac{2xy-y'x^2}{y^2}+y'+\frac{1}{y^2}y'=0$$
$$\boxed{y'=\frac{2xy}{y^2-x^2+1}}$$

Exercise 12.

$$(x+A)^2+(y+B)^2=r^2$$
$$(1+A)^2+(1+B)^2=1+2A+A^2+1+2B+B^2=r^2$$
$$-\left((-1+A)^2+(-1+B)^2\right)=-\left(1-2A+A^2+1-2B+B^2=r^2\right)$$
$$\implies 4A+4B=0 \implies A=-B$$
$$(x+-B)^2+(y+B)^2=r^2$$
$$2(x-B)+2(y+B)y'=0 \qquad \implies y'=\frac{B-x}{y+B}$$
$$(y+B)y'=B-x$$
$$(1-B)^2+(1+B)^2=r^2 \implies \sqrt{2}\sqrt{(1+B^2)}=r \text{ or } \sqrt{\frac{r^2}{2}-1}=B$$
$$\implies \text{(so } B \text{ could be treated as a parameter for the family of curves)}$$

8.24 Exercises - First-order separate equations.

Exercise 1. $y'=x^3/y^2$

$\frac{1}{3}y^3=\frac{1}{4}x^4+C \quad \implies y^3=\frac{3}{4}x^4+C$

Exercise 2. $\tan x\cos y=-y'\tan y$

$\ln|\cos x|=\frac{1}{\cos y}$

Exercise 3. $(x+1)y'+y^2=0$

$\frac{1}{y}=\ln(x+1)+c$

Exercise 4. $y'=(y-1)(y-2)$

$$\left(\frac{1}{y-2}+\frac{-1}{y-1}\right)y'=1 \qquad \implies \ln(y-2)-\ln(y-1)=x$$
$$\frac{y-2}{y-1}=e^x$$

Exercise 5. $y\sqrt{1-x^2}y'=x$

$\frac{1}{2}y^2=-\sqrt{1-x^2} \qquad \implies y^2=-2\sqrt{1-x^2}$

Exercise 6. $(x-1)y'=xy$

$$\ln y=\int 1+\frac{1}{x-1}=x+\ln|x-1|$$
$$\boxed{y=e^x(x-1)+C}$$

Exercise 7. $(1-x^2)^{1/2}y' + 1 + y^2 = 0$

$\arctan y = \arccos x + C$

Exercise 8. $xy(1+x^2)y' - (1+y^2) = 0$

$$\frac{1}{2}\ln(1+y^2) = \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) + C = \ln x - \frac{1}{2\ln|1+x^2|}$$
$$y^2 = k\left(\frac{x}{\sqrt{1+x^2}}\right)^2$$

Exercise 9. $(x^2-4)y' = y$

$$\ln y = \frac{-1}{2} arctanh\Big(\frac{x}{2}\Big)$$
$$\text{since} \int \frac{1}{x^2-4}dx = \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2 - 1} =$$
$$= \frac{1}{2}\int \frac{du}{u^2-1} \text{ (where } u = \frac{x}{2} \text{)}$$
$$(\tanh(u))' = \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u} = 1 - \tanh^2 u$$
$$\Longrightarrow \boxed{y = k\exp\big(-\frac{1}{2} arctanh\Big(\frac{x}{2}\Big)\big)}$$

Exercise 10. $xyy' = 1 + x^2 + y^2 + x^2y^2$

$\frac{1}{2}\ln(1+y^2) = \ln x + \frac{1}{2}x^2 + C \qquad \Longrightarrow \boxed{y^2 = kx^2e^{x^2} - 1}$

Exercise 11. $yy' = e^{x+2y}\sin x$

$$\frac{ye^{-2y}}{-2} - \frac{e^{-2y}}{4} = \frac{e^x \sin x - e^x \cos x}{2} + C$$
$$(2y+1)e^{-2y} = -2e^x(\sin x - \cos x) + C$$

Exercise 12. $xdx + ydy = xy(xdy - ydx)$

$$y(1-x^2)dy = x(-y^2-1)dx$$
$$\frac{ydy}{1+y^2} = \frac{xdx}{x^2-1} \Longrightarrow \frac{1}{2}\ln|1+y^2| = \frac{1}{2}\ln|x^2-1| + C$$
$$1 + y^2 = (x^2-1)K \Longrightarrow \boxed{y^2 = K(x^2-1) - 1}$$

Exercise 13. $f(x) = 2 + \int_1^x f(t)dt$

$$f'(x) = f(x) \quad \Longrightarrow f(x) = Ce^x$$
$$\Longrightarrow Ce^x = 2 + Ce^x - Ce^1 \quad C = \frac{2}{e}$$
$$\boxed{f(x) = \frac{2}{e}e^x}$$

Exercise 14. $f(x)f'(x) = 5x \qquad f(0) = 1$

$$f(x)^2 = 5x^2 + C \quad \Longrightarrow f(x) = \pm\sqrt{5x^2 + C}$$
$$\boxed{f(x) = \sqrt{5x^2 + 1}}$$

Exercise 15. $f'(x) + 2xe^{f(x)} = 0 \qquad f(0) = 0$

$$e^{-y}y' = -2x \implies -e^{-y} = -x^2 + C$$
$$y = \ln{(x^2 + C)^{-1}} \qquad \implies \boxed{y = -\ln{(x^2+1)}}$$

Exercise 16. $f^2(x) + (f'(x))^2 = 1$

$$\begin{aligned} f &= -1 \\ y'^2 &= 1 - y^2 \end{aligned} \qquad y' = \pm\sqrt{1-y^2}$$
$$\pm\arcsin{(y)} = x + c \implies \boxed{f(x) = \pm\sin{(x+c)}}$$

Exercise 17.

$$\begin{aligned} &\int_a^x f(t)dt = K(x-a) \\ &f > 0 \quad \forall x \in \mathcal{R} \end{aligned} \qquad \implies \boxed{f(x) = k > 0}$$

Exercise 18.

$$\int_a^x f(t)dt = k(f(x) - f(a)) \xrightarrow{\frac{d}{dx}} f(x) = kf'(x)$$
$$\implies f(x) = Ce^{\frac{1}{k}x}; \quad C > 0$$

Exercise 19. $\int_a^x (f(t))dt = k(f(x) + f(a)) \implies f(x) = Ce^{\frac{x}{k}}$

$kCe^{\frac{x}{k}} - kCe^{\frac{a}{k}} = kCe^{\frac{x}{k}} + kCe^{a/k} \implies 2kCe^{a/k} = 0 \implies C =$
$\boxed{f = 0}$

Exercise 20. $\int_a^x f(t)dt = kf(x)f(a);$ $\qquad f(x) = kf'(x)f(a)$

$$\frac{1}{kf(a)} = \frac{f'(x)}{f(x)} \implies \ln f(x) = \left(\frac{1}{kf(a)}\right)x + C; \implies f(x) = C\exp\left(\frac{x}{kf(a)}\right)$$
$$\int_a^x f(t)dt = \left(kf(a)Ce^{\frac{t}{kf(a)}}\right)\Big|_a^x = kf(a)Ce^{\frac{x}{kf(a)}} - kf(a)Ce^{\frac{a}{kf(a)}} = kCe^{\frac{x}{kf(a)}}Ce^{\frac{a}{kf(a)}}$$
$$f(a)\left(e^{\frac{x}{kf(a)}} - e^{\frac{a}{kf(a)}}\right) = Ce^{\frac{x}{kf(a)}}e^{\frac{a}{kf(a)}}$$
$$\xrightarrow{x=a} 0 = Ce^{\frac{2a}{kf(a)}} \implies C = 0$$
$$\implies \boxed{f = 0}$$

8.26 Exercises - Homogeneous first-order equations.

Exercise 1. $f(tx, ty) = f(x, y)$ homogeneity (or homogeneity of zeroth order).

$$y' = f(x,y) = \left(\frac{x}{v}\right)' = \frac{v - xv'}{v^2} = f(x, \frac{x}{v}) = f(1, \frac{1}{v})$$
$$v - v^2 f(1, \frac{1}{v}) = xv' \implies \boxed{\ln x = \int \frac{dv}{v - v^2 f(1, \frac{1}{v})}}$$

Exercise 2. $y' = \frac{-x}{y} \qquad \implies \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C \implies \boxed{y^2 = -x^2 + C}$

Exercise 3. $y' = 1 + \frac{y}{x}$

$\frac{y}{x} = v \qquad \implies y' = v + xv' = 1 + v \implies v = \ln x$

$\boxed{y = x(\ln x + C)}$

Exercise 4. $y' = \frac{x^2 + 2y^2}{xy}$

$$y' = \frac{x^2+2y^2}{xy} = \frac{x}{y} + \frac{2y}{x}$$

$$v = \frac{y}{x} \quad \implies v + xv' = \frac{1}{v} + 2v \quad \implies \frac{v'}{\frac{1}{v}+v} = \frac{1}{x}$$

$$\frac{1}{2}\ln|1+v^2| = \ln x + C \quad \implies \boxed{y^2 = (Cx^2-1)x^2}$$

Exercise 5. $(2y^2 - x^2)y' + 3xy = 0$

$$\text{if } 2y^2 \neq x^2 \text{ , } y' = \frac{3xy}{x^2-2y^2}$$

$$\begin{aligned} y &= vx \\ y' &= v'x + v \end{aligned} \quad \implies y' = v'x + v = \frac{3vx^2}{x^2-2v^2x^2} = \frac{3v}{1-2v^2}$$

$$\implies \frac{1-2v^2}{2v(1+v^2)}v' = \frac{1}{x}$$

$$\frac{1}{2}\left(\frac{1}{v} + \frac{-3v}{1+v^2}\right)v' = \frac{1}{x} \implies \frac{1}{2}\ln v + \frac{-3}{2}\ln(1+v^2) = \ln x + C \implies \frac{v}{(1+v^2)^3} = Cx^2 = \frac{y/x}{\left(\frac{x^2+y^2}{x^2}\right)^3}$$

$$\boxed{yx^3 = C(x^2+y^2)^3}$$

However,

$$y' = \frac{3xy}{x^2-2y^2}$$

$$\begin{aligned} v &= \frac{y}{x} \\ y' &= v'x + v \end{aligned} \quad \implies v'x + v = \frac{3x^2v}{x^2-2v^2x^2} = \frac{3v}{1-2v^2}$$

$$v'x = \frac{3v}{1-2v^2} - \frac{(v-2v^3)}{1-2v^2} = \frac{2(v+v^3)}{1-2v^2} \implies \left(\frac{1}{v} + -\frac{3v}{1+v^2}\right)v' = \frac{2}{x} \implies \ln v + -\frac{3}{2}\ln|1+v^2| = 2\ln x + C$$

$$\frac{v}{(1+v^2)^{3/2}} = Cx^2 \quad \implies \frac{y^2/x^4}{(x^2+y^2)^3} = Cx^4$$

$$\implies \boxed{y^2 = C(x^2+y^2)^3}$$

Exercise 6. $xy' = y - \sqrt{x^2+y^2}$

$$\implies y' = \frac{y}{x} = \sqrt{1+\left(\frac{y}{x}\right)^2}$$

$$\begin{aligned} v &= \frac{y}{x} \\ vx &= y \end{aligned} \quad v'x + v = v - \sqrt{1+v^2} \implies \begin{aligned} v'x &= -\sqrt{1+v^2} \\ \frac{-v'}{\sqrt{1+v^2}} &= \frac{1}{x} \\ \implies \ln(v+\sqrt{1+v^2}) &= \ln x + C \text{ since} \end{aligned}$$

$$(\ln(v+\sqrt{1+v^2}))' = \frac{1}{v+\sqrt{1+v^2}}\left(1 + \frac{v}{\sqrt{1+v^2}}\right) = \frac{1}{\sqrt{1+v^2}}$$

$$v + \sqrt{1+v^2} = Cx \quad \implies 1 + v^2 = C^2x^2 - 2vCx + v^2$$

$$\implies v = \frac{Cx}{2} - \frac{1}{2Cx} \quad \implies \boxed{y = \frac{Cx^2}{2} - \frac{1}{2C}}$$

Exercise 7. $x^2y' + xy + 2y^2 = 0$

$$x^2y' = -2y^2 - xy \quad \Longrightarrow y' = \frac{-2y^2}{x^2} - \frac{y}{x} \quad v = \frac{y}{x} \Longrightarrow \begin{aligned} v'x + v &= -2v^2 - v \\ v'x &= -2(v^2+v) \end{aligned}$$

$$\frac{v'}{v(v+1)} = \frac{-2}{x} = \int v'\left(\frac{1}{v} - \frac{1}{v+1}\right) = -2\ln x + C \Longrightarrow \frac{v}{v+1} = \frac{C}{x^2}$$

$$\boxed{y = \frac{-Cx}{C - x^2}}$$

Exercise 8. $y^2 + (x^2 - xy + y^2)y' = 0$

$$\left(\frac{y}{x}\right)^2 + \left(1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2\right)y' = 0$$

$$\text{Let } v = \frac{y}{x} \quad \Longrightarrow \frac{-v^2}{1 - v + v^2} = v'x + v$$

$$v'x = \frac{-v^2}{1-v+v^2} - v = \frac{-v(1+v^2)}{1-v+v^2} \quad \Longrightarrow \frac{v^2 - v + 1}{v(1+v^2)}v' = \frac{-1}{x} = \left(\frac{1}{v} + \frac{-1}{v^2+1}\right)v' = -\frac{1}{x}$$

$$\Longrightarrow \ln v - \arctan v = -\ln x + C \Longrightarrow \ln(vx) = \arctan x + C \quad \boxed{\ln y = \arctan\frac{y}{x} + C}$$

Exercise 9. $y' = \frac{y(x^2+xy+y^2)}{x(x^2+3xy+y^2)}$

$$y' = \frac{y(x^2+xy+y^2)}{x(x^2+3xy+y^2)} = \left(\frac{y}{x}\right)\left(\frac{1 + \frac{y}{x} + \frac{y^2}{x^2}}{1 + \frac{3y}{x} + \frac{y^2}{x^2}}\right) \xrightarrow{v = \frac{y}{x}} v'x + v = v\left(\frac{1+v+v^2}{1+3v+v^2}\right) = v + \frac{-2v^2}{v^2+3v+1}$$

$$v'(1 + \frac{3}{v} + \frac{1}{v^2}) = \frac{-2}{x} \Longrightarrow v + 3\ln v + \frac{-1}{v} = -2\ln x + C$$

$$\boxed{\frac{y}{x} + 3\ln y - \frac{x}{y} = \ln x + C}$$

Exercise 10. $y' = \frac{y}{x} + \sin\frac{y}{x}$

$$\begin{aligned} \frac{y}{x} &= x \\ y' &= v + v'x \end{aligned} \quad \Longrightarrow \quad \begin{aligned} v + v'x &= v + \sin v \\ \frac{v'}{\sin v} &= \frac{1}{x} \end{aligned}$$

$$-\ln\csc v + \cot v = \ln x + C \quad \Longrightarrow \boxed{\csc v + \cot v = \frac{K}{x}}$$

Exercise 11. $x(y+4x)y' + y(x+4y) = 0$

$$y' = \frac{-y(x+4y)}{x(y+4x)} = \frac{-\frac{y}{x}(1 + \frac{4y}{x})}{\frac{y}{x} + 4} \xrightarrow{v = \frac{y}{x}} v + xv' = \frac{-v(1+4v)}{v+4}$$

$$xv' = \frac{-5v(1+v)}{v+4} \Longrightarrow \frac{-5}{x} = \frac{v+4}{v(1+v)}v' = \left(\frac{4}{v} + \frac{-3}{1+v}\right)v'$$

$$\xrightarrow{\int} 4\ln v - 3\ln(1+v) = -5\ln x + C \Longrightarrow \boxed{(yx)^4 = (x+y)^3C}$$

8.28 Miscellaneous review exercises - Some geometrical and physical problems leading to first-order equations.

Exercise 1.

$$2x + 3y = C \quad y' = -\frac{2}{3} \quad g' = \frac{3}{2} \Longrightarrow g - \frac{3}{2}x = C$$

Exercise 2.

$$xy = C \xrightarrow{d/dx} y + xy' = 0 \Longrightarrow y' = -y/x\, x \neq 0 \Longrightarrow g' = x/g \Longrightarrow \frac{1}{2}g^2 = \frac{1}{2}x^2 + C$$

Exercise 3. $x^2 + y^2 + 2Cy = 1$

$$x + yy' + Cy' = 0 \Longrightarrow y'(y+C) = -x$$
$$y' = \frac{-x}{y+C} = \frac{-x}{y + \frac{1-x^2-y^2}{2y}} = \frac{-2xy}{y^2-x^2+1}$$
$$\xrightarrow{orthogonal curves} y' = \frac{y^2-x^2+1}{2xy} = \left(\frac{1}{2x}\right) y + \frac{1}{2}\left(\frac{1}{x} - x\right) y^{-1}$$

Recognize that this is a *Ricatti equation* and we know how to solve them.

$$y' + \frac{-1}{2x} y = y^{-1}\left(\frac{-x}{2} + \frac{1}{2x}\right) \qquad n = -1$$
$$k = 1 - n = 1 - (-1) = 2$$
$$v = y^k = y^2 \qquad v' + 2\left(\frac{-1}{2x}\right) v = \frac{2}{2}\left(\frac{1}{x} - x\right)$$
$$A(x) = \int_a^x P(t)dt = \int_a^x \frac{-1}{t} = \ln\frac{a}{x} \qquad \int_a^x Qe^A = \int_a^x \left(\frac{1}{t} - t\right)\frac{a}{t} = \frac{-a}{x} + 1 - a(x-a)$$
$$\boxed{y^2 = v = -1 + \frac{x}{a} - x(x-a) + \frac{bx}{a}}$$

Exercise 4. $y^2 = Cx$.

$$\frac{y^2}{x} = C \xrightarrow{d/dx} \frac{2yy'x - y^2}{x^2} = 0$$
$$y' = \frac{y}{2x} \Longrightarrow y' = \frac{1}{\left(\frac{-y}{2x}\right)} = \frac{-2x}{y}$$
$$\Longrightarrow \boxed{y^2 + 2x^2 = C}$$

Exercise 5. $x^2 y = C$.

$$2xy + x^2 y' = 0 \quad y' = -\frac{2y}{x} \Longrightarrow y' = \frac{x}{2y}$$
$$\frac{1}{2}y^2 = \frac{x^2}{4} + C$$
$$\boxed{2y^2 - x^2 = C}$$

Exercise 6. $y = Ce^{-2x}$

$$e^{2x} y = C \Longrightarrow 2e^{2x} y + e^{2x} y' = 0$$
$$y' = -2y \xrightarrow{\text{invert}} y' = \frac{1}{2y}$$
$$\Longrightarrow \boxed{y^2 = x + C}$$

Exercise 7. $x^2 - y^2 = C$

$$2x - 2yy' = 0$$
$$y' = \frac{x}{y} \Longrightarrow y' = \frac{-y}{x}$$
$$\Longrightarrow \ln y = -\ln x + C \Longrightarrow \boxed{y = \frac{C}{x}}$$

Exercise 8. $y \sec x = C$

$$y' \sec x + y \tan x \sec x = 0$$
$$y' = -y\tan x \xrightarrow{\text{(invert)}} y' = \frac{1}{y\tan x}$$
$$\boxed{\frac{1}{2}y^2 = \ln|\sin x| + C}$$

Exercise 9. All circles through the points $(1,0)$ and $(-1,0)$ From Sec. 8.22, Ex.10, we had obtained $y' = \frac{2xy}{y^2-x^2+1}$

$\Longrightarrow y' = \frac{x^2-y^2-1}{2xy} = \frac{x^2-1}{2yx} - \frac{y}{2x} \quad \Longrightarrow y' + \frac{1}{2x}y = \frac{x^2-1}{2x}y^{-1}$

Recognize this is a *Ricatti equation*.

For $y' + Py = Qy^n$, in this case, $n=-1$, and so $k = 1-n = 1-(-1) = 2$.
Then $v = y^k$ and $v' + kPv = kQ$. In this case,
$v' + 2\left(\frac{1}{2x}\right)v = 2\left(\frac{x^2-1}{2x}\right) = v' + \frac{1}{x}v = \frac{x^2-1}{x} = x - 1/x$.

$$A(x) = \int_a^x P(t)dt = \int_a^x \frac{1}{t} = \ln\frac{x}{a}$$
$$\int (t - \frac{1}{t})\exp\left(\ln\frac{t}{a}\right)dt = \int\left(\frac{t^2}{a} - \frac{1}{a}\right)dt = \left(\frac{\frac{1}{3}t^3}{a} - \frac{t}{a}\right)\Big|_a^x$$
$$e^{-\ln\frac{x}{a}} = \frac{a}{x}$$
$$\Longrightarrow y^2 = v = \frac{\frac{1}{3}x^3 - x - \frac{1}{3}a^3 + a}{x} + \frac{ba}{x}$$

Exercise 10. All circles through the points $(1,1)$ and $(-1,-1)$.

$$(x+A)^2 + (y+B)^2 = r^2$$
$$(1+A)^2 + (1+B)^2 = 1 + 2A + A^2 + 1 + 2B + B^2 = r^2$$
$$-\left((-1+A)^2 + (-1+B)^2\right) = -\left(1 - 2A + A^2 + 1 - 2B + B^2 = r^2\right)$$
$$\Longrightarrow 4A + 4B = 0 \Longrightarrow A = -B$$
$$(x + -B)^2 + (y+B)^2 = r^2$$
$$2(x-B) + 2(y+B)y' = 0 \qquad \Longrightarrow y' = \frac{B-x}{y+B}$$
$$(y+B)y' = B - x$$

$$(1-B)^2 + (1+B)^2 = r^2 \Longrightarrow \sqrt{2}\sqrt{(1+B^2)} = r \text{ or } \sqrt{\frac{r^2}{2} - 1} = B$$
$$\Longrightarrow \text{ (so } B \text{ could be treated as a parameter for the family of curves)}$$
$$y' = \frac{-1}{\left(\frac{B-x}{y+B}\right)} = \frac{y+B}{x-B}$$
$$\frac{y'}{y+B} = \frac{1}{x-B} \Longrightarrow \boxed{y = C(x-B) - B}$$

Exercise 11. With $(0,Y) = Q$ the point that moves up wards along the positive y-axis and

$P = (x,y)$ being the point P that pursues Q,

$y' = \frac{Y-y}{X-x} = \frac{Y-y}{0-x}$ is the slope of the tangent line on a point on the trajectory of P.
The condition given, that the distance of P from the y-axis is k the distance of Q from the origin, is

$kY = x$.

$$y' = \frac{\left(\frac{1}{k}\right)x - y}{x} = f(x,y)$$
$$f(x,y) \text{ is homogeneous of zero order } \Longrightarrow y = vx \quad \text{(try this substitution)}$$
$$y' = v'x + v = \frac{1}{k} - v \Longrightarrow \frac{v'}{\frac{1}{k} - 2v} = \frac{1}{x}$$
$$\Longrightarrow -\frac{1}{2}\ln\left(\frac{1}{k} - 2v\right) = \ln x + C \Longrightarrow y = \frac{x}{2k} - \frac{1}{2C^2x}$$
$$\xrightarrow{(1,0)} \boxed{y = \frac{x}{2k} = \frac{1}{2kx}} \quad k = \frac{1}{2} \quad y = x - \frac{1}{x}$$

Exercise 12.

$$y = \frac{x}{2k} - \frac{1}{2kx}$$

Exercise 13. $y = f(x)$.

$$\begin{aligned} n\int_0^x f(t)dt = xf(x) - \int_0^x f(t)dt; (n+1)\int_0^x f(t)dt &= xy \\ ny &= xy' \\ \Longrightarrow (n_1)y = y + xy' \Longrightarrow \qquad \frac{n}{x} &= \frac{y'}{y} \\ n\ln x &= \ln y \end{aligned}$$

$$\boxed{y = Cx^n} \text{ of } y = Cx^{1/n}$$

Exercise 14.

$$\begin{gathered} n\int_0^x \pi f^2(t)dt = \int_0^x (\pi(y(x))^2 - \pi(f(t))^2)dt \\ (n+1)\int_0^x \pi f^2(t)dt = xy^2(x) = xy^2; \quad (n+1)f^2(x) = y^2 + 2xyy' \\ y' = \frac{ny}{2x} \qquad \Longrightarrow \ln y = \frac{n}{2}\ln x + C \\ \boxed{y = Cx^{n/2}} \text{ of } y = Cx^{1/2n} \end{gathered}$$

Exercise 15.

$$\pi\int_0^x f^2(t)dt = x^2 f(x) \quad \pi f^2(x) = 2xf + x^2 f' \Longrightarrow f' = \frac{\pi f^2 - 2xf}{x^2}$$

The left hand side of the last expression shown is homogeneous. Thus do the $y = vx$ substitution.

$$\begin{gathered} v'x + v = \frac{\pi v^2 x^2 - 2x^2 v}{x^2} = \pi v^2 - 2v \\ \frac{v'x}{\pi v^2 - 3v} = 1 \Longrightarrow \frac{v'}{\pi(v^2 - \frac{3v}{\pi})} = \frac{1}{x} = \frac{1}{3}\left(\frac{1}{v - \frac{3}{\pi}} - \frac{1}{v}\right)v' \\ \ln(v - \frac{3}{\pi}) - \ln v = 3\ln x + C \\ \ln\left(\frac{v - 3/\pi}{v}\right) = 3\ln x + C \\ vx - \frac{3}{\pi}xCvx^4 \\ y - \frac{3}{\pi}x = Cyx^3 \Longrightarrow \boxed{y = \frac{3x/\pi}{1 + \frac{x^3}{2}}} \end{gathered}$$

Exercise 16. $A = \int_0^a f; \quad B = \int_a^1 f$

$A - B = \int_0^a f + \int_1^a f = 2f(a) + 3a + b$

$\xrightarrow{d/dx} 2f(a) = 2f'(a) + 3 \Longrightarrow 1 = \frac{f'(a)}{f(a) - \frac{3}{2}}$

So then

$$\begin{gathered} a + C = \ln(y - \frac{3}{2}); \quad f(a) = Ce^a + \frac{3}{2} \\ f(1) = 0 = Ce^1 + \frac{3}{2}; \quad \Longrightarrow C = -\frac{3}{2e} \\ \boxed{f(x) = \frac{-3}{2}e^{x-1} + \frac{3}{2}} \end{gathered}$$

To find b,

$$
\begin{aligned}
2\left(\frac{-3}{2}e^{a-1}+\frac{3}{2}a\right)+\frac{3}{2}e^{-1}+\frac{3}{2}-\frac{3}{2} &= 2f(a)+3a+b= \\
&= 2\left(\frac{-3}{2}e^{a-1}+\frac{3}{2}\right)+3a+b \\
&\Longrightarrow \boxed{b=\frac{3}{2}e^{-1}-3}
\end{aligned}
$$

Exercise 17.

$$
\begin{aligned}
A(x) &= \int_0^x \left(f(t)-\left(\left(\frac{y(x)-1}{x}\right)t+1\right)\right)dt = x^3 \\
&= \int_0^x f+-\left(\frac{y(x)-1}{x}\right)\frac{1}{2}x^2 - x = x^3 = \\
&= \int_0^x f+-\left(\frac{y(x)-1}{2}\right)x - x = x^3 \\
&\xrightarrow{d/dx} f(x)+-\frac{1}{2}(y'x+y)-\frac{1}{2}=3x^2 \\
y' &= \frac{-2(3x^2+\frac{1}{2}-y/2)}{x} = -6x - x^{-1}+\frac{y}{x}
\end{aligned}
$$

As a leap of faith, try $y=vx$ substitution to solve $y'=-6x-x^{-1}+\frac{y}{x}$.

$$
\begin{gathered}
v'x+v=-6x-x^{-1}+v \quad v'=-6-x^{-2}; \quad v=-6x+x^{-1}+C \\
\boxed{y=-6x^2+5x+1}x
\end{gathered}
$$

Exercise 18. Assuming no friction at the orifice and energy conservation.

$$
mgh=\frac{1}{2}mv_f^2
$$

(imagine how the top layer of water is now at the bottom of the tank (final "potential energy configurations"))
$V_f=\sqrt{2gh}$ (how fast water is rushing out)

$A_0=$ cross-sectional area of the orifice.

$$
\begin{gathered}
\frac{dV}{dt}=A\frac{dh}{dt}=-c\sqrt{2gh}A_0 \\
2h^{1/2}\Big|_{h_i}^{h_f}=-c\sqrt{2g}\frac{A_0}{A}t \\
\Longrightarrow T=\frac{\sqrt{2}A}{c\sqrt{g}A_0}\left(\sqrt{h_f}-\sqrt{h_i}\right)=59.6sec
\end{gathered}
$$

Note that we included the discharge coefficient $C=0.6$.

Exercise 19. $\frac{dV}{dt}=-c\sqrt{2gh}A_0+\gamma_0=A\frac{dh}{dt}=-\kappa h^{1/2}+\gamma_0$. $\kappa=c\sqrt{2g}A_0$.

$$\frac{Adh}{\gamma_0 - \kappa h^{1/2}} = dt = (A/\gamma_0)\frac{dh}{1 - \frac{\kappa}{\gamma_0}h^{1/2}}$$

$$\left(\ln\left(1 - ah^{1/2}\right)\right)' = \frac{1}{1 - ah^{1/2}}\left(-\frac{a}{2}\frac{1}{h^{1/2}}\right) \quad \text{(where } a = \frac{\kappa}{\gamma_0}\text{)}$$

$$(h^{1/2})' = \frac{1}{2h^{1/2}}\left(\frac{1 - ah^{1/2}}{1 - ah^{1/2}}\right) = \frac{\left(\frac{1}{2} - \frac{a}{2}h^{1/2}\right)}{h^{1/2}\left(1 - ah^{1/2}\right)}$$

$$\implies \int (A/\gamma_0)\frac{dh}{1 - \frac{\kappa}{\gamma_0}h^{1/2}} = -\,(A/\gamma_0)\left(\frac{2\gamma_0^2}{\kappa^2}\ln\left(1 - \frac{\kappa}{\gamma_0}h^{1/2}\right) + \frac{2\gamma_0 h^{1/2}}{\kappa}\right)\Bigg|_{h_i}^{h_f} =$$
$$= T$$

$$\implies \exp\left(\frac{-\gamma_0}{A}t - \frac{2\gamma_0}{\kappa}\left(h_f^{1/2} - h_i^{1/2}\right)\right)\frac{\kappa^2}{2\gamma_0^2} = \frac{1 - \frac{\kappa}{\gamma_0}h_f^{1/2}}{1 - \frac{\kappa}{\gamma_0}h_i^{1/2}}$$

$$\xrightarrow{t\to\infty} h_f = \frac{\gamma_0^2}{\kappa^2} = \frac{(100in^3/s)^2}{c^2(2)(32ft/s^2)(5/3in^2)^2(12in/1ft)} = (25/24)^2$$

Exercise 20.

$$V_0 = \frac{1}{3}\pi R_0^2 H_0$$
$$V(h) = V_0 - \frac{1}{3}\pi h\left(h\frac{R_0}{H_0}\right)^2 = V_0 - \frac{1}{3}\pi\frac{R_0^2}{H_0^2}h^3 = V_0 - \alpha h^3$$
$$mg(H_0 - h) = \frac{1}{2}mv_f^2;\ \sqrt{2g(H_0 - h)} = v_f \quad \text{(energy conservation)}$$
$$cA_0 v_f = cA_0\sqrt{2gH_0\left(1 - \frac{h}{H_0}\right)} = \beta\sqrt{1 - \frac{h}{H_0}}$$
$$\frac{dV}{dt} = -\beta\sqrt{1 - h/H} = -3\alpha h^2\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{\beta}{3\alpha h^2}\sqrt{1 - \frac{h}{H}}$$
$$\int \frac{h^2/H_0^2}{\sqrt{1 - \frac{h}{H_0}}} = \frac{\beta/H_0^2}{3\alpha}T =$$
$$= H_0\int \frac{u^2du}{\sqrt{1-u}} = H_0\int\frac{(1-y)^2(-dy)}{\sqrt{y}} = -H_0\int\frac{1 - 2y + y^2}{\sqrt{y}} =$$
$$= -H_0\left(2y^{1/2} - \frac{2}{3}y^{3/2} + \frac{2}{5}y^{5/2}\right)\Bigg|_{h_i}^{h_f} =$$
$$= -H_0\left(2\left(1 - \frac{h}{H_0}\right)^{1/2} - \frac{4}{3}\left(1 - \frac{h}{H_0}\right)^{3/2} + \frac{2}{5}\left(1 - \frac{h}{H_0}\right)^{5/2}\right)\Bigg|_{h_i}^{h_f} =$$
$$= \frac{cA_0\sqrt{2gH_0}/H_0^2}{3\left(\frac{1}{3}\pi\frac{R_0^2}{H_0^2}\right)}T$$

For $h_i = 0, \quad h_f = H,$

$$H_0(2(1) - 4/3(1) + 2/5) = H_0(16/15) = cA_0\sqrt{2gH_0}T/(\pi R_0^2); T = \frac{\frac{16}{15}\sqrt{H_0}\pi R_0^2}{cA_0\sqrt{2g}} = \frac{2}{9}\frac{\pi R_0^2\sqrt{H_0}}{A_0}$$

Exercise 21. $m^2x - m + (1 - x) = 0 \quad \implies (m^2 - 1)x + 1 - m = 0, \quad \implies m = 1$

Exercise 24. Given f s.t. $2f'(x) = f\left(\frac{1}{x}\right)$ if $x > 0$, $f(1) = 2$ and $x^2y'' + axy' + by = 0$

(1)

$$2f''(x) = \frac{d}{dx} f\left(\frac{1}{x}\right) = (f')\left(\frac{-1}{x}\right)$$
$$\implies x^2 y'' = \frac{-1}{2} f' = \frac{-1}{4} f\left(\frac{1}{x}\right)$$
$$\boxed{a = 0; \quad b = \frac{1}{4}}$$

(2)

$$\begin{aligned} f(x) &= Cx^n \\ f' &= nCx^{n-1} \\ f'' &= n(n-1)Cx^{n-2} \end{aligned} \qquad \implies n(n-1)Cx^n = \frac{-1}{4} Cx^n$$
$$n^2 - n + \frac{1}{4} \implies \boxed{n = \frac{1}{2}}$$

Exercise 28. Choose the units for time to be in days first - we can convert into years later.

If no one died from accidental death, then the population will grow by e. That means, with $x = x(t)$ being the population at time t

$$\frac{dx}{dt} = x \qquad \implies x = Ce^t$$

which makes sense because if C is the original population number, then after 1 year, $x = Ce$.

With t in days, we have a decrease of $\frac{1}{100}x$ in population each day due to death. Add up the changes from the decrease due to deaths and the increase due to growth for the DE:

$$\frac{dx}{dt} = \frac{1}{365}x - \frac{1}{100}x = \frac{100 - 365}{36500}x = \frac{-265}{36500}$$
$$\implies x = Ce^{\frac{-265}{36500}t}$$

Change t units to years by multiplying the "time constant" $\frac{-265}{36500}$ by 365 days.

$$\boxed{x = 365 \exp\left(-2.65t\right)}$$

To get the total fatalities, simply integrate the deaths during each year.

$$\int_0^t y = \frac{365}{100}\frac{365}{2.65}\left(-\exp\left(-2.65t\right) + 1\right)$$

Exercise 29. For constant gravity, $\Delta K = -\Delta U$

$\implies -(0 - mgh) = \frac{1}{2}mv_f^2$

$v_f = \sqrt{2gh} = (6.37 \times 10^8\, cm)\left(\frac{1\, m}{2.54\, cm}\right)\left(\frac{1\, ft}{12\, in}\right) = 6.93\, \frac{mi}{sec} = 24940\, \frac{mi}{hr}$

The constant energy formula could also be obtained by considering

$F = \frac{GM_e m}{-r^2} = m\frac{d^r}{dt^2} = -\partial_r U$

Exercise 30. Let $y = f(x)$ be the solution to $y' = \frac{2y^2+x}{3y^2+5}$ $f(0) = 0$

(1) $y'(0) = 0$ as easily seen. Now $y'' = \frac{(4yy'+1)(3y^2+5)-(6yy')(2y^2+x)}{(3y^2+5)^2}$, so then $y''(0) = \frac{1}{5} > 0$. It is a minimum.
(2) $f'(x) \geq 2/3 \quad \forall\, x \geq 10/3$. $a = 2/3$ since f will be above this tangent line.
Suppose, in the "worst case," $f'(x) = 0$ for $0 \leq x \leq 2/3$. Then $f(x) = 0$ for $0 \leq x \leq 2/3$. Then the tangent line must be at $y = 0$ at $x = 10/3$ to remain below the graph of $f(x)$.

$\implies \frac{2}{3}x - 20/9 < f(x)$

(3) Since $f'(x) \geq \frac{2}{3}$ for each $x \geq \frac{10}{3}$, then $f \to \infty$ for $x \to \infty$ (otherwise, f would have to decrease somewhere, which would contradict the given fact about f). Rewrite the DE for y' to be

$$y' = \frac{2y^2 + x}{3y^2 + 5} \Longrightarrow (3y^2 + 5)y' = 2y^2 + x \Longrightarrow (3 + \frac{5}{y^2})y' = 2 + \frac{x}{y^2}$$

Consider

$$y' = \frac{2y^2 + x}{3y^2 + 5} = \frac{2 + \frac{x}{y^2}}{3 + \frac{5}{y^2}}$$

specifically, $\frac{x}{y^2}$. Now y must, at the very least, have some linear increase because we had already shown that $y' \geq \frac{2}{3}$. So y^2 would go to infinity faster than linear x. Thus $\lim_{x\to\infty} y' = \frac{2}{3}$. So then $(3 + \frac{5}{y^2})y' \xrightarrow{x\to\infty} (3+0)\frac{2}{3} = 2 = 2 + \frac{x}{y^2}$.

$$\boxed{0 = \frac{x}{y^2}}$$

(4) $y' = \frac{2+\frac{x}{y^2}}{3+\frac{5}{y^2}} \xrightarrow{x\to\infty} \frac{2}{3}. \Longrightarrow y = \frac{2}{3}x$ or $\frac{y}{x} = \frac{2}{3}$.

Exercise 31. Given a function f which satisfies the differential equation $xf''(x) + 3x(f'(x))^2 = 1 - e^{-x}$

(1) $c \neq 0$ for an extrenum.
$cf''(c) + 3c(f'(c))^2 = cf''(c) = 1 - e^{-c} \quad \Longrightarrow f''(c) = \frac{1-e^{-c}}{c} > 0$

(2) *Cleverly*, consider the *limit*.

$$xf''(x) + 3x(f'(x))^2 = 1 - e^{-x} \Longrightarrow f''(x) + 3(f'(x))^2 = \left(\frac{1 - e^{-x}}{x}\right) \xrightarrow{x\to 0} f''(0) + 0 = 1$$

So a critical point at $x = 0$ would be a minimum.

(3) We'll have to "cheat" a little and use the idea of power series early on here.

$f'' + 3(f')^2 = \frac{1-e^{-x}}{x}$ suggests that we consider the Taylor series of e^{-x}.

$$\frac{1 - e^{-x}}{x} = \frac{-\sum_{j=1}^{\infty} \frac{(-x)^j}{j!}}{x} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{(j+1)!}$$

This further suggests that f itself has a power series representation because its first and second order derivatives are simply a combination of infinitely many terms containing powers of x.

Then suppose $f = \sum_{j=0}^{\infty} a_j x^j$.

$$f'' + 3(f')^2 = \frac{1 - e^{-x}}{x} \Longrightarrow \begin{aligned} f' &= \sum_{j=0}^{\infty} (j+1)a_{j+1}x^j \\ f'' &= \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j \end{aligned}$$

$$\Longrightarrow \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j + 3\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (j+1)(k+1)a_{j+1}a_{k+1}x^{j+k} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{(j+1)!}$$

$$\text{If } f(0) = 0 \quad a_0 = 0$$
$$\text{If } f'(0) = 0 \quad a_1 = 0$$

Then $f = a_2x^2 + \sum_{j=3}^{\infty} a_j x^j$. Consider the x^0 terms in the DE. $(f')^2$ doesn't contribute, because f''s leading order term is x^1. So then

$$2(1)a_2 + 0 = 1 \Longrightarrow \boxed{a_2 = \frac{1}{2}} \quad \text{ i.e. } f = \frac{1}{2}x^2 + \sum_{j=3}^{\infty} a_j x^j$$

$\boxed{A = \frac{1}{2}}$ in order for $f(x) \leq Ax^2$

9.6 Exercises - Historical introduction, Definitions and field properties, The complex numbers as an extension of the real numbers, The imaginary unit i, Geometric interpretation. Modulus and argument. **Exercise 6.** Let f be a polynomial with real coefficients.

(1) Since $\overline{z_1 z_2}$

$\overline{(z_1^{n+1})} = \overline{z_1^n z_1} = \overline{z_1^n}\,\overline{z_1} = \overline{z}_1^{n+1}$

$$\overline{f(z)} = \overline{\sum a_j z^j} = \sum a_j \overline{z}^j = f(\overline{z})$$

(2) If $f(z) = 0$, then $\overline{f(z)} = f(\overline{z}) = 0$ as well.

Exercise 7. The three ordering axioms are

$$\begin{aligned} &\text{Ax. 7 If } x, y \in \mathbb{R}^+, \; x + y, xy \in \mathbb{R}^+ \\ &\text{Ax. 8 } \forall x \neq 0, \; x \in \mathbb{R}^+ \text{ or } -x \in \mathbb{R}^+ \text{ but not both} \\ &\text{Ax. 9 } 0 \notin \mathbb{R}^+ \end{aligned}$$

$x < y$ means $y - x$ positive.

Suppose i positive: $i(i) = -1$ but -1 is not positive.
Suppose $-i$ is positive. $-i(-i) = -1$ but -1 is not positive.
i is neither positive nor negative so Ax. 8 is not satisfied.

Exercise 8. Ax. 8 , Ax. 9 are satisfied.

$(a+ib)(c+id) = (ac - bd, ad + bc)$ so Ax. 7 might not be satisfied.

Exercise 9. Ax. 7, Ax. 8 , Ax. 9 are trivially satisfied (all are positive).

Exercise 10. $x > y$ $x > y$ is well defined. Ax. 8 is satisfied.

For Ax. 7, $(\frac{3}{2}, 1), (1, \frac{1}{2})$ contradicts Ax. 7 since we required the product to be positive as well if the factors are positive. We found this particular counterexample by considering factors $(a, b), (c, d)$, so the product of the two is $(ac - bd, ad + bc)$ and so we need $ac - bd - ad - bc = a(c - d) - b(c + d) < 0$

Exercise 11. See sketch.

Exercise 12.

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ w &= \frac{(az+b)(c\overline{z}+d)}{(cz+d)(c\overline{z}+d)} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{c^2|z|^2 + cd(z+\overline{z}) + d^2} \\ w + -\overline{w} &= \frac{ac|z|^2 + adz + bc\overline{z} + bd - ac|z|^2 - ad\overline{z} - bcz - bd}{|cz+d|^2} = \boxed{\frac{(ad-bc)(z-\overline{z})}{|cz+d|^2}} \end{aligned}$$

$$\text{If } ad - bc > 0$$

$$w - \overline{w} = 2Imw = \frac{ad-bc}{|cz+d|^2} 2Imz; \quad \frac{ad-bc}{|az+d|^2} > 0$$

So Imw has the same sign as Imz

9.10 Exercises - Complex exponentials, Complex-valued functions, Examples of differentiation and integration formulas.
Exercise 7.

(1)

$$
\begin{aligned}
&\text{if } m \neq n \,, \int_0^{2\pi} e^{ix(n-m)} dx = \left. \frac{e^{ix(n-m)}}{i(n-m)} \right|_0^{2\pi} = \frac{1-1}{i(n-m)} = 0 \\
&\text{if } m = n \,, \int_0^{2\pi} e^{ix(0)} dx = 2\pi
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_0^{2\pi} e^{inx} e^{-imx} dx &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx) = \\
&= \int_0^{2\pi} \cos nx \cos mx + \sin nx \sin mx + i(\sin nx \cos mx - \sin mx \cos nx) \\
\int_0^{2\pi} e^{-inx} e^{-imx} dx &= \int_0^{2\pi} (\cos nx - i \sin nx)(\cos mx - i \sin mx) = \\
&= \int_0^{2\pi} \cos nx \cos mx - \sin nx \sin mx + i(-\sin nx \cos mx - \sin mx \cos nx)
\end{aligned}
$$

Summing the two equations above

$$
\begin{aligned}
0 &= \int_0^{2\pi} 2 \cos nx \cos mx + 2 \int_0^{2\pi} i - \sin mx \cos nx \\
&\implies \int_0^{2\pi} \cos nx \cos mx = 0, \quad \int_0^{2\pi} \sin mx \cos nx = 0
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{2\pi} e^{inx} e^{-imx} dx &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx) = \\
&= \int_0^{2\pi} \cos nx \cos mx + \sin nx \sin mx + i(\sin nx \cos mx - \sin mx \cos nx) \\
\int_0^{2\pi} e^{inx} e^{imx} dx &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx + i \sin mx) = \\
&= \int_0^{2\pi} \cos nx \cos mx - \sin nx \sin mx + i(\sin nx \cos mx + \sin mx \cos nx)
\end{aligned}
$$

Subtract the two equations above

$$
\begin{aligned}
\int_0^{2\pi} \sin nx \sin mx - i \int_0^{2\pi} \sin mx \cos nx = 0 \\
\implies \int_0^{2\pi} \sin nx \sin mx = 0
\end{aligned}
$$

$$\int_0^{2\pi} e^{inx}e^{-inx} = 2\pi = \int_0^{2\pi} (\cos nx + i\sin nx)(\cos nx - i\sin nx) =$$
$$= \int_0^{2\pi} \cos^2 nx + \sin^2 nx$$
$$\int_0^{2\pi} e^{inx}e^{inx} = 0 = \int_0^{2\pi} \cos^2 nx - \sin^2 nx + i(2\cos nx \sin nx)$$
$$\Longrightarrow \int_0^{2\pi} \cos^2 nx - \sin^2 nx = 0 \quad \int_0^{2\pi} \cos nx \sin nx = 0$$

Summing the two results above, we obtain

$$2\pi = \int_0^{2\pi} 2\cos^2 nx$$
$$\Longrightarrow \int_0^{2\pi} \cos^2 nx = \pi$$
$$\text{Then also, } \int_0^{2\pi} \sin^2 nx = \pi$$

Exercise 8.

$$z = re^{i\theta} = re^{i(\theta + 2\pi m)},\ m \in \mathbb{Z}$$
$$z^{1/n} = r^{1/n}e^{i(\theta/n + 2\pi m/n)}\ m = 0, 1, \ldots n-1$$
$$\Longrightarrow z^{1/n} = Re^{i\alpha}\epsilon^m = z_1\epsilon^m$$

The roots are spaced equally by an angle $2\pi/n$

$$i = e^{i\pi/2 + i2\pi n} \Longrightarrow i^{1/3} = e^{i\pi/6},\ e^{i5\pi/6},\ e^{i3\pi/2}$$
$$i^{1/4} = e^{i\pi/8},\ e^{5i\pi/8},\ e^{9i\pi/8},\ e^{13i\pi/8}$$
$$-i = e^{-i\pi/2 + i2\pi n} \Longrightarrow (-i)^{1/4} = e^{-i\pi/8},\ e^{3i\pi/8},\ e^{7i\pi/8},\ e^{11i\pi/8}$$

Exercise 9.

$$e^{iu}e^{iv} = e^{i(u+v)} = \cos u + v + i\sin u + v =$$
$$= (\cos u + i\sin u)(\cos v + i\sin v) = \cos u\cos v - \sin u\sin v + i(\cos v\sin u + \cos u\sin v)$$
$$\Longrightarrow \sin u + v = \cos v\sin u + \cos u\sin v$$
$$\Longrightarrow \cos u + v = \cos u\cos v - \sin u\sin v$$
$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 =$$
$$= \frac{-(e^{2iz} + e^{-2iz} - 2) + (e^{2iz} + 2 + e^{-2iz})}{4} = 1$$
$$\cos iy = \frac{e^{iiy} + e^{-iiy}}{2} = \qquad \sin iy = \frac{e^{iiy} - e^{-iiy}}{2i} =$$
$$= \frac{e^{-y} + e^{y}}{2} = \cosh y \qquad = \frac{e^{-y} - e^{y}}{2i} = i\sinh y$$
$$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y} = (\cos x + i\sin x)e^{-y}$$
$$e^{-iz} = e^{-i(x+iy)} = e^{-ix}e^{y} = (\cos x - i\sin x)e^{y}$$

Thus it is clear, by mentally adding and subtracting the above results that

$$\Longrightarrow \begin{aligned} \cos z &= \cos x\cosh y - i\sin x\sinh y \\ \sin z &= i\cos x\sinh y + \sin x\cosh y \end{aligned}$$

Exercise 10.

(1) $Log(-1) = i\pi \qquad \log(i) = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$
(2) $Log(z_1z_2) = Log(|z_1||z_2|e^{i(\theta_1+\theta_2)}) = \ln|z_1||z_2| + i(\theta_1 + \theta_2 + 2n\pi) = Logz_1 + Logz_2 + i2\pi n$

(3) $Log(z_1/z_2) = Log(|z_1|/|z_2|e^{i(\theta_1 - theta_2)}) = \ln \frac{|z_1|}{|z_2|} + i(\theta_1 - \theta_2 + 2n\pi) = Logz_1 - Logz_2 + i2\pi n$
(4) $\exp(Logz) = \exp(\ln|z| + i\theta + i2\pi n) = z$

Exercise 11.

(1)
$$\begin{aligned} 1^i &= e^{iLog1} = e^{i(i2\pi n)} = e^{-2\pi n} = 1 \text{ if } n = 0 \\ i^i &= e^{iLogi} = e^{i(i\frac{\pi}{2} + i2\pi n)} = e^{-\frac{\pi}{2} - 2\pi n} = e^{-\pi/2} \text{ if } n = 0 \\ (-1)^i &= e^{iLog-1} = e^{i(i\pi + i2\pi n)} = e^{-\pi - 2\pi n} = e^{-\pi} \end{aligned}$$

(2) $z^a z^b = e^{aLogz} e^{bLogz} = e^{aLogz + bLogz} = e^{(a+b)Logz} = z^{a+b}$x

(3)
$$\begin{aligned} (z_1 z_2)^w &= e^{wLogz_1 z_2} = e^{w(Logz_1 + Logz_2 + 2\pi mi)} \\ (z_1^w z_2^w) &= e^{wLogz_1} e^{wLogz_2} = e^{w(Logz_1 + Logz_2)} \\ m &= 0 \text{ is the condition required for equality.} \end{aligned}$$

Exercise 12.

$$\text{if } L(u) = P, \; L(v) = Q,$$
$$L(u + iv) = (u + iv)'' + a(u + iv)' + b(u + iv) = u'' + au' + bu + i(v'' + av' + bv) = L(u) + iL(v) = P + iQ = R$$
$$\text{if } L(f) = R$$
$$L(u + iv) = L(u) + iL(v) = P + iQ$$
then, equating real and imaginary parts, $L(u) = P, \; L(v) = Q$

Exercise 13.

$$L(y) = -\omega^2 y + ai\omega y + by = Ae^{i\omega x} \Longrightarrow (-\omega^2 + ai\omega + b)B = A$$

We cannot let $(-\omega^2 + ai\omega b) = 0$ for a nontrivia solution. Thus $b \neq \omega^2$ or $a\omega \neq 0$.

$$B = \frac{A}{-\omega^2 + ai\omega + b}$$

Exercise 14.

$$\begin{aligned} L(\widehat{y}) = ce^{i\omega x}; \; \widehat{y} = Be^{i\omega x} &= \frac{c}{-\omega^2 + ai\omega + b} e^{i\omega x} \\ \Longrightarrow \widehat{y} &= \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} e^{i(\omega x - \alpha)} \text{ where } \tan\alpha = \frac{a\omega}{b - \omega^2} \\ \Longrightarrow \Re\widehat{y} &= \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} \cos(\omega x - \alpha) \end{aligned}$$

Exercise 15.

$$\begin{aligned} \Im(\widehat{y}) &= \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} \sin(\omega x + \alpha) \\ \Longrightarrow A &= \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}}; \quad -\tan\alpha = \frac{a\omega}{b - \omega^2} \end{aligned}$$

10.4 Exercises - Zeno's paradox, Sequences, Monotonic sequences of real numbers. **Exercise 1.** Converges to 0.

$$f(n) = \frac{n}{n+1} - \frac{n+1}{n} = \frac{n^2 - (n^2 + 2n + 1)}{n(n+1)} = \frac{-2n - 1}{n^2 + n} = \frac{\frac{-2}{n} - \frac{1}{n^2}}{1 + \frac{1}{n}} \xrightarrow{n \to \infty} 0$$

Exercise 2. Converges to -1.

$$f(n) = \frac{n^3 - (n^3 + n + n^2 + 1)}{(n+1)n} = \frac{n^2 - n - 1}{n(n+1)} = \frac{-1 - \frac{1}{n} - \frac{1}{n^2}}{1 + \frac{1}{n}} \xrightarrow{n \to \infty} -1$$

Exercise 3. Diverges since

$$|cos\frac{n\pi}{2} - L| \geq \left| \left| 1 \cos\frac{n\pi}{2} \right| - |L| \right| \geq |1 - |L||$$

Choosing $\epsilon_1 = \frac{|1 - |L||}{2}$, $|\cos\frac{n\pi}{2} - L| > \epsilon_1$ for $n = 4m$.

Exercise 4. $f(n) = \frac{1}{5} + \frac{3}{5n} - \frac{2}{5n^2} \to \lim_{n\to\infty} f(n) = \frac{1}{5}$

Exercise 5. $f(x) = \frac{x}{2^x} = \frac{x}{\exp{(x\ln 2)}} \to 0$ since $\lim_{x\to\infty} \frac{x^\alpha}{(e^x)^\beta}$.

Exercise 6. $f(n) = 1 + (-1)^n = 0$ of 1.

Thus, choosing $\epsilon_1 = \frac{|1-|L||}{2}$;

$$|f(n) - L| \geq ||f(n)| - |L|| = |1 - |L|| > \epsilon_1 \quad \text{for any } n = 2m$$

Exercise 7. $f(n) = \frac{1+(-1)^n}{n}$.

Suppose $\epsilon = \frac{3}{N}$.

So for $n > N$, $\frac{1}{N} > \frac{1}{n}$, $n \geq N = N(\epsilon) = 3/\epsilon$.

$$|f(n)| = \left|\frac{1+(-1)^n}{n}\right| \leq \frac{2}{n} < \frac{3}{n} < \frac{3}{N} = \epsilon$$

Exercise 8. $f(n) = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$

$$\begin{aligned} |f(n) - L| &= \left|\frac{(-1)^n}{n} + \frac{1+(-1)^n}{2} - L\right| \geq \left|\left|L - \frac{(-1)^n}{n}\right| - \left|\frac{1+(-1)^n}{2}\right|\right| \geq \\ &\geq \left|\left||L| - \left|\frac{(-1)^n}{n}\right|\right| - \left|\frac{1+(-1)^n}{2}\right|\right| = \left|\left||L| - \frac{1}{n}\right| = \left|\frac{1+(-1)^n}{2}\right|\right| \geq \\ &\geq \left|\left||L| - \frac{1}{n}\right| - \frac{1}{2}\right| \end{aligned}$$

Thus, consider

$$\left|\left||L| - \frac{1}{n}\right| - \frac{1}{2}\right| > \left|\left|\frac{1}{N} - |L|\right| - 1\right| = \epsilon_0 \text{ for } n > N$$

Exercise 9. $f(x) = \exp\left(\frac{1}{x}\ln 2\right)$; $\quad \lim_{x\to\infty} f(x) = 0$.

Exercise 10.

$$|f(n) - L| = |n^{(-1)^n} - L| \geq ||n^{(-1)^n}| - |L|| = ||n| - |L|| = |n| - |L| > N - |L|$$

Thus, for $n > N$, $N(\epsilon) = \epsilon + |L|$, so then $|f(n) - L| > \epsilon$.

Exercise 11. $f(n) = \frac{n^{2/3}\sin n!}{n+1}$.

$$|f(n)| = \left|\frac{n^{2/3}\sin{(n!)}}{n+1}\right| = \left|\frac{\sin{(n!)}}{n^{1/3} + n^{-2/3}}\right| \leq \left|\frac{1}{n^{1/3}}\right|$$

Thus, for $n > N$, $\quad N(\epsilon) = \frac{1}{\epsilon^3}$, $\quad |f(n)| < \epsilon$.

Exercise 12. Converges, since

$$\begin{aligned} f(n) - \frac{1}{3} &= \frac{3^{n+1} + 3(-2)^n - 3^{n+1} - (-2)^{n+1}}{3(e^{n+1} + (-2)^{n+1})} = \left(\frac{(-2)^n(3+2)}{3(3^{n+1} + (-2)^{n+1})}\right) = \\ &= \left(\frac{5}{3}\frac{(-2)^n}{3^{n+1} + (-2)^{n+1}}\right) \\ \left|f(n) - \frac{1}{3}\right| &= \frac{5}{3}\left|\frac{(-2)^n}{3^{n+1} + (-2)^{n+1}}\right| \leq \left|\frac{-(-2)^{n+1}}{3^{n+1} + (-2)^{n+1}}\right| = \\ &= \left|\frac{-1}{\left(\frac{3}{-2}\right)^{n+1} + 1}\right| = \left|\frac{1}{1 + \left(\frac{3}{-2}\right)^{n+1}}\right| < \\ &< \frac{1}{\left(\frac{3}{2}\right)^{n+1}} < \frac{1}{\left(\frac{3}{2}\right)^n} \end{aligned}$$

For $n > N$, consider $\epsilon = \left(\frac{2}{3}\right)^{-N}$, i.e. $N = \frac{-\ln\epsilon}{\ln 2/3} = N(\epsilon)$. Thus

$\boxed{L = \frac{1}{3};}$

Exercise 13.

$$f(n) = \sqrt{n+1} - \sqrt{n}$$
$$f(n) = \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \sqrt{1}\sqrt{n+1}+\sqrt{n}$$
$$|f(n)| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| \leq \frac{1}{2\sqrt{n}};$$

So then $\forall \epsilon$, we have $\epsilon = \frac{1}{2\sqrt{N}}$ and for $n > N$, $\frac{1}{2\sqrt{n}} < \frac{1}{2\sqrt{N}} = \epsilon$.

Thus $f(n)$ converges to 0.

Exercise 14.

$$f(n) = na^n = n \exp n \ln a = \frac{n}{\exp n \ln \frac{1}{a}} \to 0 \text{ since } \lim_{x\to\infty} \frac{x^a}{(e^x)^b} = 0$$

Exercise 15. $f(n) = \frac{\log_a n}{n}, a > 1$. $\lim_{n\to\infty} f(n) = 0$ since $\lim_{x\to\infty} \frac{(\log x)^a}{x^b} = 0$ for $a > 0, b > 0$

Exercise 16. $\lim_{n\to\infty} f(n) = 0$

Exercise 17. $\lim_{n\to\infty} f(n) = e^2$.

Exercise 18.

$$\left|1 + \frac{n}{n+1}\cos\frac{n\pi}{2} - L\right| \geq \left|\left|1 + \frac{n}{n+1}\cos\frac{n\pi}{2}\right| - |L|\right| = \left|\left|1 + \frac{-n}{n+1}\right| - |L|\right| =$$
$$= \left|\frac{1}{n+1} - |L|\right| > |L|$$

Choose $\epsilon_0 = \frac{|L|}{2}$. For any N, for $n > N$, $|f(n) - L| > \epsilon_0$.

Exercise 19.

$$1 + \frac{i}{2} = \frac{\sqrt{5}}{2}e^{i\alpha} \qquad \left(\frac{\sqrt{5}}{2}e^{i\alpha}\right)^{-n}, \quad \left(\frac{2}{\sqrt{5}}\right)^n e^{-ni\alpha}$$
$$1 + \left(\frac{1}{2}\right)^2 = \frac{5}{4} \qquad \lim_{n\to\infty}\left(\frac{2}{\sqrt{5}}\right)^n e^{-ni\alpha} = 0$$
$$\tan\alpha = \frac{1}{2}$$

Exercise 20. $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} e^{-\pi i n/2}$ diverges since

$$|e^{i\pi i n/2} - L| \geq ||e^{-i\pi i n/2}| - |L|| = |1 - |L|| > |L|$$

So for $\epsilon_0 = \frac{|L|}{2}$, for $n > N$, $|f(n) - |L|| > \epsilon_0$

Exercise 21. $f(n) = \frac{1}{n}e^{-i\pi n/2} \qquad |f(n)| = \frac{1}{n}$.

Suppose $N(\epsilon) = \frac{1}{\epsilon}$; then for $n > N$, $|f(n)| < \epsilon_0$

Exercise 22. $|f(n) - L| \geq ||ne^{-\pi i n/2}| - |L|| = |n - |L||$

Consider $\epsilon_0 = |1 - |L||$ for $n > N > 1$, $|f(n) - L| > \epsilon_0$.

Exercise 23. $a_n = \frac{1}{n}$.

$$|a_n| = \left|\frac{1}{n}\right| < \frac{1}{N}; \quad N(\epsilon) = \frac{1}{\epsilon}$$
$$\epsilon = 1, 0.1, 0.01, 0.001, 0.0001$$
$$N = 1, 10, 100, 1000, 10000$$

Exercise 24. $|a_n - 1| = \left|\frac{n+1-n}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}$

$N(\epsilon) = \frac{1}{\epsilon} = 1, 10, 100, 1000, 10000.$

Exercise 25. $|a_n| = \frac{1}{n}$

$N(\epsilon) = 1, 10, 100, 1000, 10000.$

Exercise 26. $|a_n| = \left|\frac{1}{n!}\right| \leq \frac{1}{\exp n \ln n} < \frac{1}{\exp n}$

For $n > N$, $\frac{1}{\exp N} = \epsilon$, so that $N = \ln 1/\epsilon$.

$$N(\epsilon) = 1, 2, 4, 6, 9$$

Exercise 27. $a_n = \frac{2n}{n^2+1}$; $|a_n| = \left|\frac{2}{n^2+1/n}\right| \leq \left|\frac{2}{n^2}\right|$.

$$N(\epsilon) = \frac{\sqrt{2}}{\sqrt{\epsilon}} = 1, 4, 14, 44, 141$$

Exercise 28. $|a_n| = \left|\frac{9}{10}\right|^n = \left(\frac{9}{10}\right)^n = e^{n \ln 9/10}$

$$\begin{aligned} N(\epsilon) = \frac{\ln \epsilon}{\ln\left(\frac{9}{10}\right)} = \frac{-\ln 1/\epsilon}{\ln{(9/10)}} = \\ = 1, 21, 43, 65, 87 \end{aligned}$$

Exercise 30. If $\forall \epsilon > 0, \ \exists N \in \mathbb{Z}^+$ such that $n > N$, $|a_n| < \epsilon$.

$$\begin{aligned} |a_n|^2 < |a_n|\epsilon < \epsilon^2 \\ |a_n^2| < \epsilon^2 \end{aligned}$$

So for $\forall \epsilon_1 > 0$, $\epsilon_1 = \epsilon^2$ and $\exists N = N(\epsilon) = N(\epsilon_1)$, so that $|a_n^2| < \epsilon_1$.

Exercise 31.

$$|a_n + b_n - (A+B)| = ||a_n - A||b_n - B|| \leq |a_n - A| + |b_n - B| < \epsilon + \epsilon = 2\epsilon$$

$$\begin{gathered} \forall \epsilon > 0, \, \exists N_A, N_B \in \mathbb{Z}^+, \, |a_n - A| < \epsilon \quad \text{if } n > N_A; |b_n - B| < \epsilon \text{ if } n > N_B \\ \text{Consider } \max{(N_A, N_B)} = N_{A+B} \\ |a_n + b_n - (A+B)| < 2\epsilon \\ \forall \epsilon_1 > 0, \, \epsilon_1 = 2\epsilon, \text{ then } \exists N_{A+B} = N_{A+B}(\epsilon_1) \in \mathbb{Z}^+ \text{ such that} \\ |(a_n + b_n) - (A+B)| < \epsilon_1 \text{ if } n > N_{A+B} \end{gathered}$$

$$\begin{gathered} |ca_n - cA| = c|a_n - A| < c\epsilon \\ \forall \epsilon_1 > 0, \, \epsilon_1 = c\epsilon; \quad \text{then } \exists N_{cA} = N(\epsilon) = N(\epsilon_1) \in \mathbb{Z}^+ \text{ such that} \\ |ca_n - cA| < \epsilon_1 \text{ for } n > N(\epsilon_1) \end{gathered}$$

Exercise 32. Given $\lim_{n\to\infty} a_n = A$,

$$\begin{gathered} \lim_{n\to\infty} (a_n - A)(a_n + A) = \lim_{n\to\infty} (a_n - A) \lim_{n\to\infty} (a_n + A) = 0(2A) = 0 \\ 2a_n b_n = (a_n + b_n)^2 - a_n^2 - b_n^2 \\ 2\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} (a_n + b_n)^2 - \lim_{n\to\infty} a_n^2 - \lim_{n\to\infty} b_n^2 = \left(\lim_{n\to\infty} (a_n + b_n)\right)^2 - A^2 - B^2 = 2AB \\ \implies \lim_{n\to\infty} a_n b_n = AB \end{gathered}$$

Exercise 33. $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$

(1)

$$\binom{-\frac{1}{2}}{1!} = \frac{-1/2}{1}$$
$$\binom{-\frac{1}{2}}{2!} = \frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{2!} = \frac{5}{8}$$
$$\binom{-\frac{1}{2}}{3!} = \frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!} = -\frac{5}{16}$$
$$\binom{-\frac{1}{2}}{4} = \frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)}{4!} = \frac{35}{128}$$
$$\binom{-\frac{1}{2}}{5} = \frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)\left(\frac{-9}{2}\right)}{5!} = \frac{-63}{256}$$

(2) $a_n = (-1)^n \binom{\frac{-1}{2}}{n}$.

$a_1 = \frac{1}{2} > 0. \quad a_2 = \frac{3}{8} > 0$

$$a_{n+1} = (-1)^{n+1}\binom{-\frac{1}{2}}{n+1} = (-1)^{n+1}\binom{\alpha}{n}\left(\frac{-\frac{1}{2}-(n+1)+1}{n+1}\right) = \frac{a_n(-1)\left(\frac{-1}{2}-n\right)}{n+1} =$$
$$= \frac{a_n(n+1/2)}{n+1} > 0$$
$$a_{n+1} = \left(\frac{n+1/2}{n+1}\right)a_n < a_n$$

Exercise 34.

(1)

$$t_n - s_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n}\left(f(1) + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) - \left(f(0) + \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\right)\right) =$$
$$= \frac{1}{n}(f(1) - f(0))$$

Since $f\left(\frac{k}{n}\right) \leq f(t) \leq f\left(\frac{k+1}{n}\right)$ for $\frac{k}{n} \leq t \leq \frac{k+1}{n}$, by f being monotonically increasing.

$$\implies s_n \leq \int_0^1 f(x)dx \leq t_n \quad \text{(from definition of integral)}$$
$$0 \leq \int_0^1 f(x)dx - s_n \leq t_n - s_n = \frac{1}{n}(f(1) - f(0))$$

(2) Use Theorem 1.9.

Theorem 30. *Every function f which is bounded on $[a,b]$ has a lower integral $\underline{I}(f)$ and an upper integral $\overline{I}(f)$ satisfying*

$$\int_a^b s(x)dx \leq \underline{I}(f) \leq \overline{I}(f) \leq \int_a^b t(x)dx$$

for all step functions s and t with $s \leq f \leq t$. The function f is integrable on $[a,b]$ iff its upper and lower integrals are equal,

$$\int_a^b f(x)dx = \underline{I}(f) = \overline{I}(f)$$

Since $f(x)$ is integrable, then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \int_0^1 f(x)dx$

(3) $\left(\frac{b-a}{n}\right) = \Delta$, $s_n = \frac{1}{\Delta}\sum_{k=0}^{n-1} f(a+k\Delta)$, $t_n = \frac{1}{\Delta}\sum_{k=1}^{n} f(a+k\Delta)$

So by increasing monotonicity of f, $s_n \leq \int_a^b f(x)dx \leq t_n$.

$$t_n - s_n = \frac{1}{\Delta}\left(f(b) + \sum_{k=1}^{n-1} f(a+k\Delta) = \sum_{k=1}^{n-1} f(a+k\Delta) - f(a)\right) = \frac{f(b)-f(a)}{\Delta}$$
$$0 \leq \int_a^b f(x)dx \leq \frac{f(b)-f(a)}{\Delta}$$

Exercise 35.

(1) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 = \int_0^1 t^2 dt = \frac{1}{3}$

(2) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{x}dx = \ln 2$

(3) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \frac{1}{1+\left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2}dx = \arctan x\big|_0^1 = \boxed{\frac{\pi}{4}}$

(4) $\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2+k^2}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{1+x^2}}dx = \ln{(x+\sqrt{1+x^2})}\big|_0^1 = \ln{(1+\sqrt{2})}$

(5) $\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n}\sin\frac{k\pi}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \sin\frac{k\pi}{n} = \int_0^1 \sin\pi x dx = \left(\frac{-\cos\pi x}{\pi}\right)\big|_0^1 = \frac{-(-1-1)}{\pi} = \boxed{\frac{2}{\pi}}$

(6) $\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n}\sin^2\frac{k\pi}{n} = \int_0^n \sin^2 x\pi = \boxed{\frac{1}{2}}$

10.9 Exercises - Infinite series, The linearity property of convergent series, Telescoping series, The geometric series.

Exercise 1. $\sum_{n=1}^\infty \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^\infty \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}$

Exercise 2. $\sum_{n=1}^\infty \frac{2}{3^{n-1}} = 2\sum_{n=0}^\infty \frac{1}{3^n} = 2\frac{1}{1-1/3} = \boxed{3}$

Exercise 3. $\sum_{n=2}^\infty \frac{1}{n^2-1} = \sum_{n=2}^\infty \frac{1/2}{n-1} - \frac{1/2}{n+1} = \sum_{n=2}^\infty \left(\frac{1/2}{n-1} - \frac{1/2}{n}\right) + -\left(\frac{1/2}{n+1} - \frac{1/2}{n}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Exercise 4. $\sum_{n=1}^\infty \frac{2^n+3^n}{6^n} = \sum_{n=1}^\infty \left(\frac{1}{3}\right)^n + \sum_{n=1}^\infty \left(\frac{1}{2}\right)^n = \frac{1/3}{1-1/3} + \frac{1/2}{1-1/2} = \frac{1}{2} + 1 = \frac{3}{2}$

Exercise 5. $\sum_{n=1}^\infty \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \sum_{n=1}^\infty \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1$

Exercise 6.

$$\sum_{n=1}^\infty \frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=1}^\infty \frac{3/2}{(n+2)(n+3)} + \frac{-1/2}{(n+1)(n+2)} = \sum_{n=1}^\infty \frac{1}{(n+2)(n+3)} + \frac{1}{2}\left(\frac{-1}{6}\right) =$$
$$= \sum_{n=1}^\infty \frac{1}{1+2} - \frac{1}{n+3} - \frac{1}{12} = \frac{1}{3} - \frac{1}{12} = \boxed{\frac{1}{4}}$$

Exercise 7. $\sum_{n=1}^\infty \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^\infty \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1$

Exercise 8. $\sum_{n=1}^\infty \frac{2^n+n^2+n}{2^{n+1}n(n+1)} = 1 = \sum_{n=1}^\infty \frac{1}{2n(n+1)} + \frac{1}{2^{n+1}} = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n} - \frac{1}{n+1} + \frac{1/4}{1-1/2} = \frac{1}{2} + \frac{1}{2} = 1$

Exercise 9. $\sum_{n=1}^\infty \frac{(-1)^{n-1}(2n+1)}{n(n+1)} = \sum_{n=1}^\infty (-1)^{n-1}\left(\frac{1}{n} + \frac{1}{n+1}\right)$.

$$(-1)^{n-1}\left(\frac{1}{n} + \frac{1}{n+1}\right) + (-1)^n\left(\frac{1}{n+1} + \frac{1}{n+2}\right) = (-1)^n\left(\frac{-1}{n} + \frac{1}{n+2}\right)$$
$$\sum_{j=1}^\infty \left(\frac{-1}{2j-1} + \frac{1}{2j+1}\right) = \frac{1}{2}\sum_{j=1}^\infty \left(\frac{-1}{(j-1/2)} + \frac{1}{(j+1/2)}\right) = \frac{1}{2}\frac{-1}{1/2} = -1$$

Exercise 10.

$$\sum_{n=2}^\infty \frac{\log{((1+\frac{1}{n})^n(1+n))}}{(\log n^n)(\log{(n+1)^{n+1}})} = \sum_{n=2}^\infty \frac{\log\left(\left(\frac{n+1}{n}\right)^n(1+n)\right)}{\log{(n+1)^{n+1}}\log n^n} =$$
$$= \sum_{n=2}^\infty \frac{log(n+1)^{n+1} - \log n^n}{\log{(n+1)^{n+1}}\log n^n} = \sum_{n=2}^\infty \frac{1}{\log n^n} - \frac{1}{\log{(n+1)^{n+1}}} =$$
$$= \frac{1}{2\log 2} = \log_2\sqrt{e}$$

since if $\frac{1}{2\log 2} = y$, then $y = \log_2\sqrt{2}$.

Exercise 11. $\sum_{n=1}^\infty nx^n = x\frac{d}{dx}\sum_{n=1}^\infty x^n = x\left(\frac{x}{1-x}\right)' = \frac{x}{(1-x)^2}$.

Exercise 12.

$$\sum_{n=1}^{\infty} n^2 x^n = x\frac{d}{dx}\sum_{n=1}^{\infty} nx^n = x\left(\frac{x}{(1-x)^2}\right)' =$$
$$= x\frac{(1-x)^2+2(1-x)x}{(1-x)^4} = \frac{x(1-x^2)}{(1-x)^4} = \frac{x(1+x)}{(1-x)^3}$$

Exercise 13.

$$\sum_{n=1}^{\infty} n^3 x^n = x\frac{d}{dx}\sum_{n=1}^{\infty} n^2 x^n = x\left(\frac{x+x^2}{(1-x)^3}\right)' =$$
$$= x\frac{(1+2x)(1-x)^3+3(1-x)^2(x)(x+1)}{(1-x)^6} = x\frac{(1+2x)(1-x)+3x(x+1)}{(1-x)^4} = \frac{x(x^2+4x+1)}{(1-x)^4}$$

Exercise 14. $\sum_{n=1}^{\infty} n^4 x^4 = x\frac{d}{dx}\sum_{n=1}^{\infty} n^3 x^3 = x\left(\frac{x^3+4x^2+x}{(1-x)^4}\right)'$.

$$\ln\left(\frac{x^3+4x^2+x}{(1-x)^4}\right) = \ln(x^3+4x^2+x) - 4\ln(1-x)$$
$$(\ln f)' = \frac{1}{f}f' = \frac{3x^2+8x+1}{x^3+4x^2+x} + 4\frac{1}{1-x};$$
$$f' = \frac{(3x^2+8x+1)(1-x)}{(1-x)^5} + \frac{4(x^3+4x^2+x)}{(1-x)^5} =$$
$$= \frac{3x^2+8x+1-3x^3-8x^2-x+4x^3+16x^2+4x}{(1-x)^5} = \frac{x^3+11x^2+11x+1}{(1-x)^5}$$
$$\Longrightarrow \frac{x^4+11x^3-11x^2+x}{(1-x)^5}$$

Exercise 15. $\sum_{n=1}^{\infty} \frac{x^n}{n} = x\sum_{n=1}^{\infty}\int_0^x t^{n-1}dt = \int_0^x dt\sum_{n=1}^{\infty} t^{n-1} = \int_0^x \frac{1}{1-t} = -\ln(1-x)$.

Exercise 16.

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = \sum_{j=1}^{\infty}\int_0^x t^{2j-2}dt = \int_0^x dt\sum_{j=1}^{\infty}(t^2)^{j-1} =$$
$$= \int_0^x \frac{dt}{1-t^2} = \int_0^x dt\left(\frac{1/2}{1-t} + \frac{1/2}{1+t}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Exercise 17. $\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^{n+1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}$

Exercise 18.

$$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2!}x^n = \sum_{n=0}^{\infty}\left(\frac{x^{n+2}}{2}\right)'' = \frac{d^2}{dx^2}\frac{x^2}{2}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\frac{x}{1-x} + \frac{x^2}{2(1-x)^2}\right)$$
$$= \frac{1}{1-x} + \frac{x}{(1-x)^2} + \frac{x}{(1-x)^2} + \frac{x^3}{(1-x)^3} = \frac{1-2x+x^2+2x-2x^2+x^2}{(1-x)^3} = \frac{1}{(1-x)^3}$$

Exercise 19.

$$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)}{3!}x^n = \sum_{n=0}^{\infty}\frac{d^3}{dx^3}\frac{x^{n+3}}{3!} = \frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{d^2}{dx^2}\frac{x^{(n+1)+2}}{2} =$$
$$= \frac{1}{3}\frac{d}{dx}\sum_{n=1}^{\infty}\frac{d^2}{dx^2}\frac{x^{n+2}}{2} = \frac{1}{3}\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{d^2}{dx^2}\frac{x^{n+2}}{2} - \frac{d^2}{dx^2}\left(\frac{x^2}{2}\right)\right) =$$
$$= \frac{1}{3}\frac{d}{dx}\left(\frac{1}{(1-x)^3} - 1\right) = \frac{1}{3}\frac{-3(-1)}{(1-x)^4} = \frac{1}{(1-x)^4}$$

Exercise 20. $\sum_{n=1}^{\infty} n^k x^n = \frac{P_k(x)}{(1-x)^{k+1}}$

$$\sum_{n=1}^{\infty} n^{k+1}x^n = x\frac{d}{dx}\sum_{n=1}^{\infty} n^k x^n = x\frac{d}{dx}\left(\frac{P_n(x)}{(1-x)^{k+1}}\right) = x\left(\frac{P_k'(x)(1-x)^{k+1} + (k+1)(1-x)^k P_k(x)}{(1-x)^{2k+2}}\right) =$$
$$= x\left(\frac{P_k'(x)(1-x) + (k+1)P_k(x)}{(1-x)^{k+2}}\right) = \frac{(k+1)xP_k(x) + x(1-x)P_k'(x)}{(1-x)^{k+2}}$$

$((k+1)P_k(x) + (1-x)P_k'(x))x$ has x as its lowest degree term from $xP_x'(x)$ and $(k+1)x^{k+1} + -kx^{k+1} = x^{k+1}$ highest degree term is obtained from $(k+1)P_k(x) + -xP_k'(x)$.

Exercise 21. $\sum_{n=0}^{\infty}\binom{n+k}{k}x^n = \frac{1}{(1-x)^{k+1}} = \frac{d^k}{dx^k}\sum_{n=0}^{\infty}\frac{x^{n+k}}{k!}$.

$$\sum_{n=0}^{\infty}\binom{n+k+1}{k+1}x^n = \sum_{n=0}^{\infty}\frac{d^{k+1}}{dx^{k+1}}\frac{x^{n+k+1}}{(k+1)!} = \frac{1}{(k+1)}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{d^k}{dx^k}\frac{x^{(n+1)+k}}{k!} =$$
$$= \frac{1}{k+1}\frac{d}{dx}\sum_{k=1}^{\infty}\frac{d^k}{dx^k}\frac{x^{n+k}}{k!} = \left(\frac{1}{k+1}\right)\frac{d}{dx}\sum_{n=0}^{\infty}\frac{d^k}{dx^k}\frac{x^{n+k}}{k!} - \frac{d^k}{dx^k}\frac{x^k}{k!} =$$
$$= \left(\frac{1}{k+1}\right)\frac{d}{dx}\left(\frac{1}{(1-x)^{k+1}} - 1\right) = \frac{1}{(1-x)^{k+2}}$$

Exercise 22.

(1) $\sum_{n=2}^{\infty}\frac{n-1}{n!} = \sum_{n=2}^{\infty}\frac{1}{(n-1)!} - \sum_{n=2}^{\infty}\frac{1}{n!} = \sum_{n=1}^{\infty}\frac{1}{n!} - \sum_{n=2}^{\infty}\frac{1}{n!} = \boxed{1}$.
(2) $\sum_{n=2}^{\infty}\frac{n}{n!} + \sum_{n=2}^{\infty}\frac{1}{n!} = \sum_{n=2}^{\infty}\frac{1}{(n-1)!} + \sum_{n=0}^{\infty}\frac{1}{n!} - 1 - 1 = \sum_{n=1}^{\infty}\frac{1}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!} - 2 = \sum_{n=0}^{\infty}\frac{2}{n!} - 3 = \boxed{2e-3}$.
(3)
$$\sum_{n=2}^{\infty}\frac{(n-1)(n+1)}{n!} = \sum_{n=2}^{\infty}\frac{n^2}{n!} + \sum_{n=2}^{\infty}\frac{-1}{n!} = \sum_{n=2}^{\infty}\frac{n}{(n-1)!} + \sum_{n=2}^{\infty}\frac{-1}{n!} =$$
$$= \sum_{n=1}^{\infty}\frac{n+1}{n!} + \sum_{n=2}^{\infty}-\frac{1}{n!} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!} + 1 = \sum_{n=0}^{\infty}\frac{1}{n!} + 1 = \boxed{e+1}$$

Exercise 23.

(1) $x\frac{d}{dx}\left(x\frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^n}{n!}\right) = x\frac{d}{dx}\left(x\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}\right) = x\frac{d}{dx}\sum_{n=1}^{\infty}\frac{nx^n}{n!} = \sum_{n=1}^{\infty}\frac{n^2x^n}{n!} = x\frac{d}{dx}\left(x\frac{d}{dx}e^x\right) = x^2e^x + xe^x$
(2) $x\frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{n^2x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{n^3x^n}{n!} = x\frac{d}{dx}\left((x^2+x)e^x\right) = x\left((2x+1)e^x + (x^2+x)e^x\right) = (x^3+3x^2+x)e^x$

x = 1 $\boxed{k=5}$

Exercise 24.

(1) $\sum_{n=2}^{\infty}(-1)^n = \sum_{n=2}^{\infty}(-1)^n(n-(n-1))$. Identical.
(2) $\sum_{n=2}^{\infty}(1-1) = \sum_{n=2}^{\infty}(-1)^n$. Not identical.
(3) Not identical. $\sum_{n=2}^{\infty}(-1)^n$ vs. $(\sum_{n=2}^{\infty}(-1+1)) + 1$.
(4) Identical. $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^n\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n(2-1) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

Exercise 25.

(1)
$$1 + x^2 + x^4 + \dots + x^{2n} + \dots = \frac{1}{1-x^2} \quad \text{if } |x| < 1$$
$$\implies 1 + 0 + x^2 + 0 + x^4 + \dots = \frac{1}{1-x^2} \quad \text{if } |x| < 1$$
(2) Thm. 10.2. $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha\sum_{n=1}^{\infty}a_n + \beta\sum_{n=1}^{\infty}b_n$. So then
$$\sum_{j=0}^{\infty}x^j - \sum_{j=0}^{\infty}\frac{x^j + (-x)^j}{2} = \sum_{j=0}^{\infty}\frac{x^j - (-x)^j}{2} = \sum_{j=0}^{\infty}x^{2j+1} = \frac{1}{1-x} - \frac{1}{1-x^2} = \frac{x}{1-x^2}$$
(3)
$$\sum_{j=0}^{\infty}(x^2)^j + \sum_{j=0}^{\infty}x^j = \sum_{j=0}^{\infty}(x^j - x^{2j}) = \frac{x}{1-x^2}$$

10.14 Exercises - Tests for convergence, Comparison tests for series of nonnegative terms, The integral test. We'll be using the integral test.

Theorem 31 (Integral Test)**.**
Let f be a positive decreasing function, defined for all real $x \geq 1$.
For $\forall n \geq 1$, let $s_n = \sum_{k=1}^n f(k)$ and $t_n = \sum_1^n f(x)dx$.
Then both sequences $\{s_n\}$ and $\{t_n\}$ converge or both diverge.

Exercise 1.

$$\frac{3}{4j-3} + \frac{-1}{4j-1} = \frac{3(4j-1)+(-1)(4j-3)}{(4j-3)(4j-1)} = \frac{8j}{(4j-3)(4j-1)}$$

$$\sum_{j=1}^n \frac{j}{(4j-3)(4j-1)} = \sum_{j=1}^n \left(\frac{3/8}{4j-3} + \frac{-1/8}{4j-1} \right)$$

$$\int_1^n \left(\frac{3/8}{4x-3} + \frac{-1/8}{4x-1} \right) dx = \left((3/8)\frac{\ln{(4x-3)}}{4} + (-1/8)\frac{\ln{(4x-1)}}{4} \right)\Big|_1^n =$$

$$= \frac{1}{32} \ln \frac{(4x-3)^3}{4x-1}\Big|_1^n = \frac{1}{32}\ln\left(\frac{3(4n-3)^3}{4n-1} \right)$$

$$\lim_{n\to\infty} \frac{1}{32} \ln \left(\frac{3(4n-3)^3}{4n-1} \right) = \lim_{n\to\infty} \int_1^n \frac{xdx}{(4x-3)(4x-1)} dx = \infty, \quad \text{so} \sum_{j=1}^n \frac{j}{(4j-3)(4j-1)} \text{ diverges as well}$$

Exercise 2.

$$\sum_{j=1}^\infty \frac{\sqrt{2j-1}\log{(4j+1)}}{j(j+1)} = \sum_{j=1}^\infty a_j$$

$$a_j \leq \frac{\sqrt{4j+1}\log{(4j+1)}}{j(j+1/4)} = \frac{4\log{(4j+1)}}{j(4j+1)^{1/2}}\left(\frac{(4j+1)}{(4j+1)} \right) = 4\frac{\log{(4j+1)(4+1/j)}}{(4j+1)^{3/2}} \leq$$

$$\leq \frac{16\log{(4j+1)}}{(4j+1)^{3/2}} = b_j$$

Now use the integral test on $\sum b_j$ to determine the convergence of $\sum b_j$.

$$\int_1^n \frac{\log{(ax+1)}}{(ax+1)^{3/2}} dx = \int \left(\frac{(-2)}{a(ax+1)^{1/2}} \right)' \log{(ax+1)}dx =$$

$$= \frac{-2}{a(ax+1)^{1/2}}\log{(ax+1)} - \int \frac{-2}{a(ax+1)^{1/2}}\left(\frac{a}{ax+1} \right) =$$

$$= \frac{-2}{a(ax+1)^{1/2}}\log{(ax+1)} + \frac{-4}{a(ax+1)^{1/2}}$$

$$\lim_{n\to\infty} \int_1^n \frac{log(ax+1)}{(ax+1)^{3/2}}dx = \lim_{n\to\infty} \left(\frac{-2\log{(an+1)}}{a(an+1)^{1/2}} + \frac{2\log{(a+1)}}{a(a+1)^{1/2}} + \frac{-4}{a}\left(\left(\frac{1}{an+1} \right)^{1/2} - \frac{1}{(a+1)^{1/2}} \right) \right) =$$

$$= \frac{2\log{(a+1)}}{a(a+1)^{1/2}} + \frac{4}{a(a+1)^{1/2}}$$

Then by integral test, $\sum b_j$ converges. Since $\sum b_j$ converges, then $\sum a_j$ converges by comparison test.

Exercise 3. $\sum_{j=1}^\infty \frac{j+1}{2^j}$.

$$\int_1^n \frac{x+1}{e^{x\ln 2}}dx = \int_1^n (xe^{-x\ln 2} + e^{-x\ln 2})dx = \left(\frac{xe^{-x\ln 2}}{-\ln 2} + \frac{e^{-x\ln 2}}{-\ln 2} + \frac{-e^{-x\ln 2}}{(\ln 2)^2} \right)\Big|_1^n$$

$$= \frac{ne^{-n\ln 2}}{-\ln 2} + \frac{e^{-\ln 2}}{\ln 2} + - \left(\frac{1}{(\ln 2)^2} + \frac{1}{\ln 2} \right)e^{-n\ln 2} + \left(\frac{1}{(\ln 2)^2} + \frac{1}{\ln 2} \right)e^{-\ln 2}$$

$$\lim_{n\to\infty} \int_1^n \frac{x+1}{e^{x\ln 2}}dx = \boxed{\left(\frac{2}{\ln 2} + \frac{1}{(\ln 2)^2} \right)\left(\frac{1}{2} \right)}$$

By integral test, $\sum_{j=1}^{\infty} \frac{j+1}{2^j}$ converges.

Exercise 4. $\sum_{j=1}^{\infty} \frac{j^2}{2^j}$.

$$\int_1^n \frac{x^2}{2^x} dx = \int_1^n \frac{x^2}{e^{x \ln 2}} = \left(\frac{x^2 e^{-x \ln 2}}{-\ln 2} + \frac{2xe^{-x \ln 2}}{-(-\ln 2)^2} + \frac{2e^{-x\ln 2}}{(-\ln 2)^3} \right)\Big|_1^n$$

$$\lim_{n\to\infty} \int_1^n \frac{x^2}{2^x} dx = e^{-\ln 2} \left(\frac{1}{\ln 2} + \frac{2}{(\ln 2)^2} + \frac{2}{(\ln 2)^3} \right) = \frac{1}{2} \left(\frac{1}{\ln 2} + \frac{2}{(\ln 2)^2} + \frac{2}{(\ln 2)^3} \right)$$

By integral test, $\sum_{j=1}^{\infty} \frac{j^2}{2^j}$ converges.

Exercise 5.

$$\sum_{j=1}^{\infty} \frac{|\sin jx|}{j^2} = \sum_{j=1}^{\infty} a_j \leq \sum_{j=1}^{\infty} \frac{1}{j^2}$$

$\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges since

$$\lim_{n\to\infty} \int_1^n \frac{1}{x^s} dx = \lim_{n\to\infty} \frac{x^{-s+1}}{-s+1}\Big|_1^n = \lim_{n\to\infty} \left(\frac{1}{1-s} \right) \left(\frac{1}{n^{s-1}} - 1 \right) = \frac{1}{s-1} \text{ if } s > 1$$

$\sum_{j=1}^{\infty} \frac{|sinjx|}{j^2}$ converges by comparison test and integral test.

Exercise 6.

$$\sum_{j=1}^{\infty} \frac{2+(-1)^j}{2^j} = \sum_{j=1}^{\infty} \left(\frac{1}{2^{2j-1}} + \frac{3}{2^{2j}} \right) = \frac{2(1/4)}{1-1/4} + \frac{3(1/4)}{1-1/4} = \boxed{\frac{4}{3}}$$

Exercise 7. $\sum_{j=1}^{\infty} \frac{j!}{(j+2)!}$.

$$a_j = \frac{j!}{(j+2)!} = a_j = \frac{1}{(j+1)(j+2)} \leq \frac{1}{j^2} = b_j$$

Since $\sum b_j$ converges, $\sum a_j$ converges, by comparison test.

Exercise 8. $\sum_{j=2}^{\infty} \frac{\log j}{j\sqrt{j+1}} = \sum_{j=2}^{\infty} a_j \leq \sum_{j=2}^{\infty} \frac{\log j}{j^{3/2}}$

$$\int_2^n \frac{\log x}{x^{3/2}} dx = \int_2^n (-2x^{-1/2})' \log x = (-2x^{-1/2} \log x)\Big|_2^n - \int_2^n -\frac{2x^{-1/2}}{x} dx =$$
$$= (-2) \left(\frac{\log n}{n^{1/2}} - \frac{\log 2}{2^{1/2}} \right) + -4x^{-1/2}\Big|_2^n$$
$$\lim_{n\to\infty} \int_2^n \frac{\log x}{x^{3/2}} dx = 2^{1/2} \log 2 + \frac{4}{\sqrt{2}}$$

So $\sum a_j$ converges by comparison test.

Exercise 9. $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j(j+1)}} = \sum_{j=1}^{\infty} a_j$. Let $b_j = \frac{1}{j}$.

$$\lim_{j\to\infty} \frac{a_j}{b_j} = \lim_{j\to\infty} \frac{j}{\sqrt{j(j+1)}} = \lim_{j\to\infty} \frac{1}{\sqrt{1+1/j}} = 1$$

By limit comparison test, since $\sum b_j$ diverges, $\sum a_j$ diverges.

Exercise 10. $\sum_{j=1}^{\infty} \frac{1+\sqrt{j}}{(j+1)^3-1} = \sum_{j=1}^{\infty} a_j$

$b_j = \frac{1}{j^{5/2}}$

$$\lim_{j\to\infty} \frac{a_j}{b_j} = \lim_{j\to\infty} \left(\frac{1+\sqrt{j}}{(j+1)^3-1} \right) j^{5/2} = \lim_{j\to\infty} \frac{j^3 + j^{5/2}}{(j+1)^3-1} = \lim_{j\to\infty} \frac{1+1/j^{1/2}}{(1+1/j)^3 - \frac{1}{j^3}}$$

By limit comparison test, since $\sum b_j$ converges, $\sum a_j$ converges.

Exercise 11. $\sum_{j=2}^{\infty} \frac{1}{(\log j)^s} = \sum a_j$

$\boxed{\text{If } s \leq 0, \sum a_j \text{ diverges since } \lim_{j\to\infty} a_j \neq 0}$

If $0 < s \leq 1$, $\quad \frac{1}{(\log j)^s} > \frac{1}{j^s}$, and since $\sum_{j=2}^{\infty} \frac{1}{j^s}$ diverges for $0 < s < 1$, so does $\sum \frac{1}{(\log j)^s}$

$$\int (\log x)^{-s} = \int \left(\frac{1}{x}(\log x)^{-s} \right) x = \int \left(\frac{(\log x)^{-s+1}}{(1-s)} \right)' x = \frac{(\log x)^{-s+1}}{(1-s)} x - \int \frac{(\log x)^{1-s}}{1-s}$$

Thus, if $s > 1$ has any decimal part, or is an integer, its integral will diverge, so that by integral test, the series diverges.

Exercise 12. $\sum_{j=1}^{\infty} \frac{|a_j|}{10^j}$; $\quad |a_j| < 10$.

$$\sum_{j=1}^{\infty} \frac{|a_j|}{10^j} < \sum_{j=1}^{\infty} \frac{10}{10^j} = \sum_{j=0}^{\infty} \frac{1}{10^j} = \frac{1}{1 - 1/10} = \boxed{\frac{10}{9}}$$

Exercise 13. $\sum_{j=1}^{\infty} \frac{1}{1000j+1} < \sum_{j=1}^{\infty} \frac{1}{1000j} = \frac{1}{1000} \sum_{j=1}^{\infty} \frac{1}{j}$

The series diverges since $\sum \frac{1}{j}$ diverges.

Exercise 14.

$$\sum_{j=1}^{\infty} \frac{j \cos^2 (j\pi/3)}{2^j} \leq \sum_{j=1}^{\infty} \frac{j}{2^j} = \sum_{j=1}^{\infty} \frac{j}{e^{j \ln 2}}$$

$$\int_1^{\infty} \frac{x}{e^{kx}} = \int_1^{\infty} x e^{-kx} = \left(\frac{xe^{-kx}}{-k} - \frac{e^{-kx}}{(-k)^2} \right) \Bigg|_1^{\infty} = \frac{e^{-k}}{k} + \frac{e^{-k}}{(-k)^2}$$

Exercise 15. $\sum_{j=3}^{\infty} \frac{1}{j \log j (\log (\log j))^s}$

$$\int \frac{1}{x \log x (\log (\log x))^s} = \int \left(\frac{(\ln (\ln x))^{-s+1}}{-s+1} \right)' = \begin{cases} \frac{(\ln (\ln x))^{-s+1}}{-s+1} & \text{if } s > 1,\ s < 1,\ s \neq 0 \\ \ln (\ln (\ln x)) & \text{if } s = 1 \\ \ln (\ln x) & \text{if } s = 0 \end{cases}$$

Converges, by integral test, for $s > 1$

Exercise 16. Converges by integral test since

$$\int_1^{\infty} x e^{-x^2} = \left(\frac{e^{-x^2}}{-2} \right) \Bigg|_1^{\infty} = 0 + \boxed{\frac{e^{-1}}{2}}$$

Exercise 17. I drew a picture to help me see what's going on.

$$\frac{\sqrt{x}}{1+x^2} \leq \frac{\sqrt{x}}{1}$$

$$\int_0^{1/n} \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_0^{1/n} = \frac{2}{3} \left(\frac{1}{n} \right)^{3/2}$$

$$\sum_{j=2}^{\infty} \frac{2}{3} \left(\frac{1}{j} \right)^{3/2} = \frac{2}{3} \sum_{j=2}^{\infty} \frac{1}{j^{3/2}}$$

So $\sum_1 n \int_0^{1/n} \frac{\sqrt{x}}{1+x^2} dx$ converges by comparison test.

Exercise 18.

$$(e^{-\sqrt{x}})' = e^{-\sqrt{x}}\frac{-1}{2\sqrt{x}}$$

$$(\sqrt{x}e^{-\sqrt{x}})' = \frac{-e^{-\sqrt{x}}}{2} + \frac{1}{2\sqrt{x}}e^{-\sqrt{x}}(\sqrt{x}e^{-\sqrt{x}} + e^{-\sqrt{x}})' \quad = \frac{e^{-\sqrt{x}}}{2}$$

$$\int_n^{n+1} e^{-\sqrt{x}}dx = 2\,(\sqrt{x}+1)e^{-\sqrt{x}}\Big|_n^{n+1} = 2\left(\frac{\sqrt{x}}{e^{\sqrt{x}}} + \frac{1}{e^{\sqrt{x}}}\right)\Big|_n^{n+1} =$$

$$= 2\left(\frac{\sqrt{n+1}}{e^{\sqrt{n+1}}} + \frac{1}{e^{\sqrt{n+1}}} - \frac{\sqrt{n}}{e^{\sqrt{n}}} - \frac{1}{e^{\sqrt{n}}}\right)$$

$$2\sum_{j=1}^{\infty}\left(\frac{\sqrt{j+1}}{e^{\sqrt{j+1}}} - \frac{\sqrt{j}}{e^{\sqrt{j}}} + \frac{1}{e^{\sqrt{j+1}}} - \frac{1}{e^{\sqrt{j}}}\right) = \boxed{-\frac{1}{e}}$$

Note the use of telescoping sum in the last step. The series converges.

Exercise 19. $\int_1^n f(x)dx = \int_1^n \log x dx = \left.(x\ln x - x)\right|_1^n = n\ln n - n + 1$.

$$\sum_{k=1}^{n-1} lnk \leq \int_1^n \ln x \leq \sum_{k=2}^{n}\ln(k) \Longrightarrow \sum_{k=1}^{n-1} lnk \leq n\ln n - n + 1 \leq \sum_{k=2}^{n}\ln(k)$$

$$\exp\left(\sum_{k=1}^{n-1}\ln k\right) = (n-1)! \leq n^n e^{-n+1}$$

$$\exp\left(\sum_{k=2}^{n}\ln k\right) = (n)! \geq n^n e^{-n+1}$$

$$\Longrightarrow \frac{e^{1/n}}{e} < \frac{(n!)^{1/n}}{n} < \frac{e^{1/n}n^{1/n}}{e}$$

10.16 Exercises - The root test and the ratio test for series of nonnegative terms. **Exercise 1.** $\sum_{j=1}^{\infty}\frac{(j!)^2}{(2j)!}$

$$\frac{((j+1)!)^2}{(2j+2)}\left(\frac{(2j)!}{(j!)^2}\right) = \frac{(j+1)^2}{(2j+2)(2j+1)} = \frac{j^2+2j+1}{4j^2+6j+2} \xrightarrow{j\to\infty} \frac{1}{4}$$

Converges by ratio test.

Exercise 2. $\sum_{j=1}^{\infty}\frac{(j!)^2}{2^{j^2}}$.

$$\frac{((j+1)!)^2}{2^{(j+1)^2}}\frac{2^{j^2}}{(j!)^2} = \frac{(j+1)^2 2^{j^2}}{2^{j^2+2j+1}} = \frac{j^2+2j+1}{2e^{j\ln 2}} \xrightarrow{j\to\infty} 0$$

Converges by ratio test.

Exercise 3. $\sum_{j=1}^{\infty}\frac{2^j j!}{j^j}$

$$\frac{2^{j+1}(j+1)!}{(j+1)^{j+1}}\frac{j^j}{2^j j!} = \frac{2(j+1)}{(j+1)}\left(\frac{1}{1+1/j}\right)^j \xrightarrow{j\to\infty} \frac{2}{e} < 1$$

Converges by ratio test.

Exercise 4. $\sum_{j=1}^{\infty}\frac{3^j j!}{j^j}$

$$\frac{3^{j+1}(j+1)!}{(j+1)^{j+1}}\left(\frac{j^j}{3^j j!}\right) = 3\left(\frac{1}{(1+1/j)^j}\right) \xrightarrow{j\to\infty} \frac{3}{e} > 1$$

Diverges by ratio test.

Exercise 5. $\sum_{j=1}^{\infty}\frac{j!}{3^j}$.

$$\frac{(j+1)!}{3^{j+1}}\frac{3^j}{j!} = \frac{j+1}{3}$$

Diverges by ratio test.

Exercise 6. $\sum_{j=1}^{\infty}\frac{j!}{2^{2j}}$

$$\frac{(j+1)!}{2^{2(j+1)}}\frac{2^{2j}}{j!} = \frac{(j+1)}{4}$$

Diverges.

Exercise 7. $\sum_{j=2}^{\infty} \frac{1}{(\log j)^{1/j}}$

Draw a picture to see what's going on.

$$\sum_{j=2}^{\infty} \frac{1}{(\log j)^{1/j}} = \sum_{j=2}^{\infty} \exp\left(\frac{-1}{j} \ln \log j\right)$$

$$0 < \frac{\ln \log j}{j} \quad \text{ for } j > 3$$

$$\frac{\ln(\log j)}{j} < \frac{\ln j}{j} \quad \implies \lim_{j\to\infty} \frac{\ln(\log j)}{j} < \lim_{j\to\infty} \frac{\ln j}{j} = 0$$

$$\implies \lim_{j\to\infty} \frac{-1}{j} \ln(\log j) = 0 \quad \text{ so then}$$

$$\lim_{j\to\infty} \exp\left(\frac{-1}{j} \ln(\log j)\right) = 1$$

$\sum_{j=2}^{\infty} \frac{1}{(\log j)^{1/j}}$ diverges because the a_j term doesn't go to zero.

Exercise 8. $\sum_{j=1}^{\infty} (j^{1/j} - 1)^j$

$$((j^{1/j} - 1)^j)^{1/j} = (e^{\frac{1}{j}\ln j} - 1) \xrightarrow{j\to\infty} 0$$

Converges by root test.

Exercise 9. $\sum_{j=1}^{\infty} e^{-j^2}$

$$(e^{-j^2})^{1/j} = e^{-j} \xrightarrow{j\to\infty} 0$$

Converges by root test.

Exercise 10. I systematically tried ratio test and then root test. Both were inconclusive.

Consider comparison with $\sum \frac{1}{j}$.

$$\left(\frac{e^{j^2} - j}{je^{j^2}}\right) j = \frac{e^{j^2} - j}{e^{j^2}} = 1 - \frac{j}{e^{j^2}} \xrightarrow{j\to\infty} 1$$

By limit comparison test, since $\sum \frac{1}{j}$ diverges, so does $\sum \left(\frac{1}{j} - \frac{1}{e^{j^2}}\right)$.

Exercise 11. $\sum_{j=1}^{\infty} \frac{(1000)^j}{j!} = e^{1000}$

Exercise 12. $\sum_{j=1}^{\infty} \frac{j^{j+1/j}}{(j+1/j)^j}$.

$$a_j^{1/j} = \frac{j^{1/j^2}}{1 + \frac{1}{j^2}} \qquad \lim_{j\to\infty} a_j^{1/j} = \lim_{j\to\infty} \frac{\exp\left(\frac{1}{j^2}\ln j\right)}{1 + \frac{1}{j^2}} = 1$$

Note that root test is inconclusive.

$$a_j = \frac{\exp\left(\frac{1}{j}\ln j\right)}{\left(1 + \frac{1}{j}\right)^j} \geq \frac{\exp\left(\frac{1}{j}\ln j\right)}{\left(1 + \frac{1}{j^2}\right)^{j^2}}$$

$$\lim_{j\to\infty} a_j \geq \lim_{j\to\infty} \frac{\exp\left(\frac{1}{j}\ln j\right)}{\left(1 + \frac{1}{j^2}\right)^{j^2}} = \frac{1}{e} > 0$$

Diverges since $\lim_{j\to\infty} a_j > 0$.

Exercise 13. $\sum_{j=1}^{\infty} \frac{j^3(\sqrt{2} + (-1)^j)^j}{3^j}$.

$$\left(\frac{j^3(\sqrt{2}+(-1)^j)^j}{3^j}\right)^{1/j} = \frac{j^{3/j}(\sqrt{2}+(-1)^j)}{3} = \frac{e^{\frac{3}{j}\ln j}(\sqrt{2}+(-1)^j)}{3} \xrightarrow{j\to\infty} \frac{(\sqrt{2}+(-1)^j)}{3} < 1$$

Converges by root test.

Exercise 14. $\sum_{j=1}^{\infty} r^j |\sin jx|$.

If $0 < r < 1$.

$$\sum_{j=1}^{\infty} r^j |\sin jx| < \sum_{j=1}^{\infty} r^j$$

$$\text{so by comparison test, } \sum_{j=1}^{\infty} r^j |\sin jx| \text{ converges for } 0 < r < 1$$

If $r \geq 1$,

$$\lim_{j\to\infty} r^j |\sin jx| \neq 0 \text{ so } \sum_{j=1}^{\infty} r^j |\sin jx| \text{ diverges, unless } jx = \pi j$$

Exercise 15.

(1) $c_j = b_j - \frac{b_{j+1}a_{j+1}}{a_j} > 0 \quad \forall j \geq N$. Then there must be a positive number r that's in between c_j and 0.

$$a_j b_j - a_{j+1} b_{j+1} \geq r a_j$$

$$r \sum_{j=N}^{n} a_j \leq \sum_{j=N}^{n} (a_j b_j - a_{j+1} b_{j+1}) = a_N b_N - a_{n+1} b_{n+1} \leq a_N b_N$$

$$\implies \sum_{j=N}^{n} a_j \leq \frac{a_N b_N}{r}$$

(2) $c_n < 0$

$$b_j - \frac{b_{j+1}a_{j+1}}{a_j} < 0 \qquad\qquad \sum \frac{1}{b_j} \text{ diverges, so}$$

$$a_j b_j < b_{j+1} a_{j+1} \implies \frac{b_j}{b_{j+1}} < \frac{a_{j+1}}{a_j} \qquad \lim_{j\to\infty} \frac{b_j}{b_{j+1}} \geq 1 \text{ by ratio test}$$

$$\xrightarrow{j\to\infty} 1 \leq \frac{b_j}{b_{j+1}} < \frac{a_{j+1}}{a_j}$$

So by ratio test, $\sum a_j$ diverges.

Exercise 16. $b_{n+1} = n$; $b_n = n - 1$.

$c_n = n - 1 - \frac{n a_{n+1}}{a_n} \geq r \implies \frac{a_{n+1}}{a_n} \leq 1 - \frac{1}{n} - \frac{r}{n}$.

Using Exercise 15, $\sum a_n$ converges.

$\sum \frac{1}{b_n}$ diverges since $\sum \frac{1}{b_n}$ is a harmonic series of $s = 1$.

$$n - 1 - \frac{n a_{n+1}}{a_n} \leq 0 \implies 1 - \frac{1}{n} \leq \frac{a_{n+1}}{a_n}$$

Exercise 17. For some $N \geq 1,\ s > 1,\ M > 0$, and given that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + \frac{f(n)}{n^s} = 1 - \left(\frac{A - \frac{f(n)}{n^{s-1}}}{n}\right)$$

Consider $A - \frac{f(n)}{n^{s-1}}$.

Since $|f(n)| < M$, $f(n)$ is finite, so consider s larger than 1 and n going to infinity so that $\frac{f(n)}{n^{s-1}} \to 0$.

Using Exercise 16, for $\sum a_j$ to converge, $A - \frac{f(n)}{n^{s-1}} = 1 + r$ where $r > 0$, for all $n \geq N$, where N is some positive number.
Let $r = \frac{M}{N^{s-1}}$ so that

$$A = 1 + r + \frac{f(n)}{n^{s-1}} > 1$$

If $A > 1$, then $\sum a_n$ converges.

If $A = 1$, then consider using Exercise 15 and $b_n = n \log n$.

$$\begin{aligned} c_n &= b_n - b_{n+1}\frac{a_{n+1}}{a_n} = (n-1)\log(n-1) - n\log n \left(\frac{a_{n+1}}{a_n}\right) \\ &= (n-1)\log(n-1) - n\log n\left(1 - \frac{1}{n} + \frac{f(n)}{n^s}\right) = (n-1)\log\left(\frac{(n-1)}{n}\right) - n\log n\left(\frac{f(n)}{n^s}\right) = \\ &= -(n-1)\log\left(\frac{n}{(n-1)}\right) - n\log n\left(\frac{f(n)}{n^s}\right) \end{aligned}$$

$$\text{since } \frac{\log n}{n^{s-1}} \xrightarrow{n\to\infty} 0,$$

$$-(n-1)\log\left(\frac{n}{(n-1)}\right) - n\log n\left(\frac{f(n)}{n^s}\right) < 0 \quad \text{for } n \text{ large enough}$$

$$\text{since } c_n < 0 \text{ for } n \geq N \text{ for some } N > 0, \text{ then by Exercise 15, } \sum a_n \text{ is divergent.}$$

Given that $A < 1$, then for $A - \frac{f(n)}{n^{s-1}}$, choose $N > 0$ so that $\frac{M}{n^{s-1}} < \epsilon < 1$ and that $A - \frac{f(n)}{n^{s-1}} \leq A + \frac{M}{n^{s-1}} = A + \epsilon \leq 1$. We can always choose ϵ small enough because there's always a real number in between A and 1 (Axiom of Archimedes).

$$A - \frac{f(n)}{n^{s-1}} \leq 1 \quad \implies -\left(A - \frac{f(n)}{n^{s-1}}\right) \geq -1$$

$$\implies \text{using Exercise 16, } \frac{a_{n+1}}{a_n} = 1 - \frac{A + \frac{f(n)}{n^{s-1}}}{n} \geq 1 - \frac{1}{n} \text{ for all } n \geq N$$

Exercise 18.

$$\left(\frac{1\cdot 3\cdot 5\dots(2n+1)}{2\cdot 4\cdot 6\dots(2n+2)}\cdot\frac{2\cdot 4\cdot 6\dots(2n)}{1\cdot 3\cdot 5\dots(2n-1)}\right)^k = \left(\frac{2n+1}{2n+2}\right)^k \xrightarrow{n\to\infty} 1$$

Ratio test fails.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \left(\frac{2n+1}{2n+2}\right)^k = \left(1 + \frac{-1}{2n+2}\right)^k = \left(1 + \frac{-1/2}{n+1}\right)^k = \\ &= \sum_{j=0}^k \binom{k}{j}\left(\frac{-1/2}{n+1}\right)^j = 1 + k\left(\frac{-1/2}{n+1}\right) + \sum_{j=2}^k \binom{k}{j}\left(\frac{-1/2}{n+1}\right)^j \end{aligned}$$

$$\text{Note that for } k < \infty, \sum_{j=2}^k \binom{k}{j}\left(\frac{-1/2}{n+1}\right)^j < \infty$$

$$\text{Let } \left|\sum_{j=2}^k \binom{k}{j}\left(\frac{-1/2}{n+1}\right)^j\right| \leq M$$

$$k/2 = A > 1 \text{ or } k > 2 \text{ means } \sum a_j \text{ converges} \quad k/2 = A \leq 1 \text{ or } k \leq 2 \text{ means } \sum a_j \text{ diverges}$$

10.20 Exercises - Alternating series, Conditional and absolute convergence, The convergence tests of Drichlet and Abel. We will be using Leibniz's test alot, initially.

Theorem 32 (Leibniz's Rule)**.** *If a_j is a monotonically decreasing sequence with limit 0,*
$\sum_{j=1}^\infty (-1)^{j-1}a_j$ converges.

If $S = \sum_{j=1}^\infty a_j$, $\quad s_n = \sum_{j=1}^n (-1)^{j-1}a_j$,

$$0 < (-1)^j(S - s_j) < a_{j+1}$$

Exercise 1. $\sum_{j=1}^\infty \frac{(-1)^{j+1}}{\sqrt{j}}$. $\lim_{j\to\infty} \frac{1}{\sqrt{j}} = 0$ Converges conditionally.

Exercise 2. $\sum_{j=1}^\infty (-1)^j \frac{\sqrt{j}}{j+100}$ $\qquad \lim_{j\to\infty} \frac{\sqrt{j}}{j+100} = 0$. Converges by Leibniz's test.

$\frac{1}{\sqrt{j} + \frac{100}{\sqrt{j}}} \geq \frac{1}{101\sqrt{j}}$, so by comparison test, the series diverges absolutely. So the alternating series converges conditionally by comparison test.

Exercise 3. $\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^s}$ If $s > 1$, then the series absolutely converges. $\lim_{j\to\infty} \frac{1}{j^s} = 0$ if $s > 0$. Converges conditionally for $0 < s < 1$. Otherwise, if $s < 0$ the series diverges absolutely.

Exercise 4. $\sum_{j=1}^{\infty}(-1)^j \left(\frac{1\cdot 3\cdot 5 \dots (2j-1)}{2\cdot 4\cdot 6 \dots (2j)} \right)^3$.

$$\frac{a_{j+1}}{a_j} = \frac{1\cdot 3\cdot 5\cdot (2j+1)}{2\cdot 4\cdot 6 \dots (2j+2)} \frac{2\cdot 4\cdot 6 \dots (2j)}{1\cdot 3\cdot 5 \dots (2j-1)} = \frac{2j+1}{2j+2}$$

Absolutely converges.

Exercise 5. $\sum_{j=1}^{\infty} \frac{(-1)^{j(j-1)/2}}{2^j}$ converges since $\lim_{j\to\infty} \frac{(-1)^{j(j-1)/2}}{2^j} = 0$; $\quad \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1/2}{1-1/2} = \boxed{1}$. Absolutely converges.

Exercise 6. $\sum_{j=1}^{\infty}(-1)^j \left(\frac{2j+100}{3j+1} \right)^j$.

$$\exp\left(j \ln\left(\frac{2j+100}{3j+1} \right) \right) = \exp\left(j\ln\left(\frac{2}{3} + \frac{298}{9j+3} \right) \right) \leq \exp\left(j \left(\ln\left(\frac{2}{3} \right) + \frac{1}{\frac{2}{3}} \left(\frac{298}{9j+3} \right) \right) \right) =$$

$$= \exp\left(j\ln\frac{2}{3} + \frac{146}{3+1/j} \right)$$

$$0 \leq \lim_{j\to\infty} \exp\left(j\ln\left(\frac{2j+100}{3j+1} \right) \right) \leq \lim_{j\to\infty}$$

$$exp\left(j\ln\frac{2}{3} + \frac{146}{3+1/j} \right) = 0$$

$$\Longrightarrow \lim_{j\to\infty} \exp\left(j\ln\left(\frac{2j+100}{3j+1} \right) \right) = 0$$

So the alternating series converges.

$$\left(\frac{2j+100}{3j+1} \right) < \frac{2j+100}{3j} < \frac{2.5j}{3j} = \frac{5}{6} \quad (\text{ for } j \geq 200)$$

$$\Longrightarrow \left(\frac{2j+100}{3j+1} \right)^j < \left(\frac{5}{6} \right)^j \text{ for } j \geq 200$$

So the series absolutely converges by comparison with a geometric series.

Exercise 7. $\sum_{j=2}^{\infty} \frac{(-1)^j}{\sqrt{j}+(-1)^j}$.

$$\lim_{j\to\infty} \frac{1}{\sqrt{j}+(-1)^j} \text{ doesn't exist since ???}$$

To show divergence, we usually think of either *taking the general term and finding the limit* (and if it goes to a nonzero constant, then it diverges), or we use ratio, root, comparison test on the general term. Since this is an alternating series, I've observed that the general term *is a sum of two adjacent terms, one even and one odd.*

$$\frac{(-1)^j}{\sqrt{j}+(-1)^j}$$

$$\frac{(-1)^{2j}}{\sqrt{2j}+(-1)^{2j}} + \frac{(-1)^{2j+1}}{\sqrt{2j+1}+(-1)^{2j+1}} = \frac{1}{\sqrt{2j}+1} + \frac{-1}{\sqrt{2j+1}+1} = \frac{\sqrt{2j+1}-1-(\sqrt{2j}+1)}{(\sqrt{2j}+1)(\sqrt{2j+1}-1)} =$$

$$= \frac{\sqrt{2j+1}-\sqrt{2j}-2}{(\sqrt{2j}+1)(\sqrt{2j+1}-1)} = \frac{\sqrt{2j}\sqrt{1+\frac{1}{2j}}-\sqrt{2j}-2}{(\sqrt{2j}+1)(\sqrt{2j}\sqrt{1+\frac{1}{2j}}-1)} =$$

$$\xrightarrow{\text{for } j \text{ large}} \simeq \frac{\sqrt{2j}\left(1+\frac{1}{4j}\right)-\sqrt{2j}-2}{(\sqrt{2j}+1)(\sqrt{2j}\left(1+\frac{1}{4j}\right)-1)} = \frac{-2}{j}\left(\frac{1-\frac{1}{4\sqrt{2j}}}{2-\frac{1}{2j}+\frac{1}{2\sqrt{2}j^{3/2}}} \right)$$

Every term, since we considered any j, will contain -2. So we factor it out. Then

$$\frac{1}{j}\left(\frac{1-\frac{1}{4\sqrt{2j}}}{2-\frac{1}{2j}+\frac{1}{2\sqrt{2j}j^{3/2}}} \right) > \frac{1}{j}\left(\frac{1-\frac{1}{4\sqrt{2j}}}{4-\frac{1}{\sqrt{2j}}} \right) = \frac{1}{4j}$$

By comparison test to $\frac{1}{j}$ the series diverges.

Exercise 8. Using the theorem

Theorem 33.
Assume $\sum |a_j|$ converges

Then $\sum a_j$ converges and $|\sum a_j| \leq \sum |a_j|$.

So using the contrapositive,

If $\sum a_j$ diverges,
$\sum |a_j|$ diverges.

$$\frac{1}{j^{1/j}} = \frac{1}{e^{\frac{1}{j}\ln j}} \qquad \lim_{j\to\infty} \frac{1}{j^{1/j}} = \frac{1}{\exp\left(\lim_{j\to\infty} \frac{1}{j}\ln j\right)} = 1$$

Diverges absolutely.

Exercise 9. $\sum_{j=1}^{\infty}(-1)^j \frac{j^2}{1+j^2}$ Diverges absolutely.

$$\begin{aligned} \frac{(2j)^2}{1+(2j)^2} - \frac{(2j-1)^2}{1+(2j-1)^2} &= \frac{4j^2}{1+4j^2}\left(\frac{4j^2-4j+2}{4j^2-4j+2}\right) - \frac{(4j^2-4j+1)}{(4j^2-4j+2)}\frac{(1+4j^2)}{1+4j^2} \\ &= \frac{4j-1}{2(1+4j^2)(2j^2-2j+1)} \end{aligned}$$

$$\frac{4j-1}{2(1+4j^2)(2j^2-2j+1)}(j^3) = \frac{4-1/j}{2(4+1/j^2)(2-2/j+1/j^2)} = \frac{1}{4}$$

By limit comparison test, with $\sum \frac{1}{j^3}$, $\sum_{j=1}^{\infty} \frac{j^2}{1+j^2}$ converges.

Exercise 10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(e^n+e^{-n})}$

$$\lim_{n\to\infty} \frac{1}{\log(e^n+e^{-n})} = \lim_{n\to\infty} \frac{1}{n} = 0$$

The series converges.

$$\lim_{n\to\infty} \frac{n}{\log(e^n+e^{-n})} = \lim_{n\to\infty} \frac{n}{\log e^n + \log(1+e^{-2n})} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+\log(1+e^{-2n})}{n}\right)} = 1$$

Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\log(e^n+e^{-n})}$ diverges.

Exercise 11. $\sum_{j=1}^{\infty} \frac{(-1)^j}{j\log^2(j+1)}$

$\lim_{j\to\infty} \frac{1}{j\log^2(j+1)} = 0$ so by Leibniz's test, the alternating series.

$$\frac{1}{n\log^2(n+1)} < \frac{1}{n\log^2(n)}$$
$$\int \frac{1}{n\log^2 n} = \int \left(\frac{-1}{\log n}\right)' = \frac{-1}{\log n} \xrightarrow{n\to\infty} \frac{1}{\log 2}$$

Converges by comparison test to $\frac{1}{n\log^2 n}$, which converges by integral test. So the series absolutely converges.

Exercise 12. $\sum_{j=1}^{\infty} \frac{(-1)^j}{\log(1+1/j)}$ diverges absolutely.

$$\frac{(-1)}{\log\left(1+\frac{1}{2j-1}\right)}+\frac{1}{\log\left(1+\frac{1}{2j}\right)}=\frac{-1}{\log\left(\frac{2j}{2j-1}\right)}+\frac{1}{\log\left(\frac{2j+1}{2j}\right)}=\frac{-\log\left(\frac{2j+1}{2j}\right)+\log\left(\frac{2j}{2j-1}\right)}{\log\left(\frac{2j}{2j-1}\right)\log\left(\frac{2j+1}{2j}\right)}=$$

$$=\frac{\log\left(\frac{(2j)^2}{4j^2-1}\right)}{\log\left(\frac{2j}{2j-1}\right)\left(\log\left(1+\frac{1}{2j}\right)\right)}=\frac{\log\left(1+\frac{1}{4j^2-1}\right)}{\log\left(1+\frac{1}{2j-1}\right)\log\left(1+\frac{1}{2j}\right)}=$$

$$=\frac{\frac{1}{4j^2-1}+o\left(\frac{1}{4j^2-1}\right)}{\left(\frac{1}{2j-1}+o\left(\frac{1}{2j-1}\right)\right)\left(\frac{1}{2j}+o\left(\frac{1}{2j}\right)\right)}\approx$$

$$\approx\frac{4j^2-2j}{4j^2-1}=\frac{1-\frac{1}{2j}}{1-\frac{1}{4j^2}}\xrightarrow{j\to\infty}1$$

So the alternating series diverges.

Exercise 13. $\sum_{j=1}^{\infty}\frac{(-1)^j j^{37}}{(j+1)!}$ Use the ratio test.

$$\frac{a_{j+1}}{a_j}=\frac{(j+1)^37}{(j+2)!}\frac{(j+1)!}{j^37}=\left(\frac{1}{j+2}\right)\left(1+\frac{1}{j}\right)^{37}\to 0$$

Converges for $\sum|a_j|$. Then $\sum a_j$ converges. The series absolutely converges.

Exercise 14. $\sum_{n=1}^{\infty}(-1)^n\int_n^{n+1}\frac{e^{-x}}{x}dx$

$$\int_n^{n+1}\frac{e^{-x}}{x}dx\leq\int_n^{n+1}\frac{1}{e^{2x}}dx=\left.\frac{e^{-2x}}{-2}\right|_n^{n+1}=\left(\frac{-1}{2}\right)\left(\frac{1}{e^{2(n+1)}}-\frac{1}{e^{2n}}\right)=$$

$$=\frac{e^2-1}{2e^{2n+2}}<1$$

Converges absolutely.

Exercise 15. $\sum_{j=1}^{n}\sin(\log j)$

$\lim_{j\to\infty}\sin(\log j)$ doesn't exist. So the series is divergent.

Exercise 16. $\sum_{j=1}^{\infty}\log\left(j\sin\frac{1}{j}\right)$ Note that

$$\log\left(j\sin\frac{1}{j}\right)=\log\left(\frac{\sin 1/j}{1/j}\right)$$
$$\lim_{j\to\infty}\log\left(\frac{\sin 1/j}{1/j}\right)=\log\left(\lim_{j\to\infty}\frac{\sin 1/j}{1/j}\right)=\log 1=0$$

$$\sin\frac{1}{j}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{j}\right)^{2k+1}}{(2k+1)!}(-1)^k$$

$$\log\left(j\sum_{k=0}^{\infty}\frac{\left(\frac{1}{j}\right)^{2k+1}(-1)^k}{(2k+1)!}\right)=\log\left(1+\frac{-1}{6j^2}+\sum_{k=2}^{\infty}\frac{\left(\frac{1}{j}\right)^{2k}(-1)^k}{(2k+1)!}\right)\geq$$

$$\geq\log\left(1+-\frac{1}{6j^2}\right)\geq\frac{-1}{6j^2}$$

The series absolutely converges.

Exercise 17. $\sum_{j=1}^{\infty}(-1)^j\left(1-j\sin\frac{1}{j}\right)$

$$\left(1 - x\sin\frac{1}{x}\right)' = -\sin\frac{1}{x} - x\cos\frac{1}{x}\left(\frac{-1}{x^2}\right) =$$
$$= -\sin\frac{1}{x} + \frac{1}{x}\cos\frac{1}{x} = \frac{-x\sin\frac{1}{x} + \cos\frac{1}{x}}{x}$$

$$\sin\frac{1}{x} = \sum_{j=0}^\infty \frac{\left(\frac{1}{x}\right)^{2j+1}(-1)^j}{(2j+1)!}$$

$$\frac{-x\sin\frac{1}{x} + \cos\frac{1}{x}}{x} =$$
$$= \frac{-1 + \frac{\left(\frac{1}{x}\right)^3(+1)}{3!} + \sum_{j=2}^\infty \frac{\left(\frac{1}{x}\right)^{2j+1}(-1)^j}{(2j+1)!} + 1 - \left(\frac{1}{x}\right)^2/2 + \sum_{j=2}^\infty \frac{\left(\frac{1}{x}\right)^{2j}(-1)^j}{(2j)!}}{x} < 0 \text{ for } x \text{ large enough}$$

$\sum_{j=1}^\infty (-1)^j\left(1 - j\sin\frac{1}{j}\right)$ converges since $a_j = 1 - j\sin\frac{1}{j}$ is monotonically decreasing sequence with limit 0.

$$1 - j\sin\frac{1}{j} = 1 - j\sum_{k=0}^\infty \frac{\left(\frac{1}{j}\right)^{2k+1}(-1)^k}{(2k+1)!} = 1 - j\left(\frac{1}{j} + \sum_{k=1}^\infty \frac{\left(\frac{1}{j}\right)^{2k+1}(-1)^k}{(2k+1)!}\right) =$$
$$= 1 - \left(1 + \sum_{k=1}^\infty \frac{\left(\frac{1}{j}\right)^{2k}(-1)^k}{(2k+1)!}\right) = \sum_{k=1}^\infty \frac{\left(\frac{1}{j}\right)^{2k}(-1)^{k+1}}{(2k+1)!} \leq \frac{1}{6j^2}$$

The series converges absolutely since the term itself is a series that is dominated by $\frac{1}{6j^2}$, so that by comparison test, the series must converge.

Exercise 18. $\sum_{j=1}^\infty (-1)^j\left(1 - \cos\frac{1}{j}\right)$.

$$(\cos\frac{1}{x})' = \left(-\sin\frac{1}{x}\left(\frac{-1}{x^2}\right)\right) = \frac{-1}{x^2}\sin\frac{1}{x} < 0$$

$\sum_{j=1}^\infty (-1)^j(1 - \cos\frac{1}{j})$ converges since $a_j = (1 - \cos\frac{1}{j})$ is monotonically decreasing to 0

$$(1 - \cos\frac{1}{j}) = 1 - \sum_{k=0}^\infty \frac{(1/j)^{2k}(-1)^k}{(2k)!} = \sum_{k=1}^\infty \frac{(1/j)^{2k}(-1)^{k+1}}{(2k)!} \leq \frac{1}{2j^2}$$

So the series converges absolutely, by comparison test with $\sum\frac{1}{j^2}$ which converges.

Exercise 19. $\sum_{j=1}^\infty (-1)^j \arctan\frac{1}{2j+1}$.

$$(\arctan\left(\frac{1}{2j+1}\right))' = \frac{1}{1 + \left(\frac{1}{2j+1}\right)^2}\left(\frac{-1}{(2j+1)^2}\right)(2) = \frac{-2}{(2j+1)^2 + 1} < 0$$

$\sum_{j=1}^\infty (-1)^j \arctan\frac{1}{2j+1}$ converges, since $a_j = \arctan\frac{1}{2j+1}$ is monotonically decreasing to 0

$$\frac{1}{1+x^2} = (\arctan x)' = \sum_{j=0}^\infty (-x^2)^j = \sum_{j=0}^\infty (-1)^j x^{2j}$$
$$\Longrightarrow \arctan x = \sum_{j=0}^\infty (-1)^j \frac{x^{2j+1}}{2j+1}$$

$$\arctan\frac{1}{2j+1} = \sum_{k=0}^\infty \frac{(-1)^k\left(\frac{1}{2j+1}\right)^{2k+1}}{(2k+1)} = \frac{1}{2j+1} + (-1)\left(\frac{\frac{1}{(2j+1)^3}}{3}\right) + \sum_{k=2}^\infty \frac{(-1)^k\left(\frac{1}{2j+1}\right)^{2k+1}}{2k+1} >$$
$$> \frac{1}{2j+1} + (-1)\frac{1}{3(2j+1)^3} = \frac{3(4j^2+4j+1) + (-1)}{3(2j+1)^3} = \frac{12j^2+12j+2}{3(2j+1)^3} > \frac{2j}{(2j+1)^2} > \frac{2}{9}\frac{1}{j} \quad \text{for } j > 2$$

So by comparison test to $\sum\frac{1}{j}$, $\sum\arctan\frac{1}{2j+1}$ diverges absolutely. The series is conditionally convergent.

Exercise 20. $\sum_{j=1}^\infty (-1)^j\left(\frac{\pi}{2} - \arctan\log j\right)$

$$\left(\frac{\pi}{2} - \arctan \log n\right)' = \left(\frac{-1}{1+(\log n)^2}\right)\left(\frac{1}{n}\right)$$

$$\frac{\pi}{2} - \arctan \log n \geq \frac{\pi}{2} - \arctan (n-1) \to \frac{\pi}{2} - \arctan n \quad \text{just change indices}$$

$$\int_0^n \left(\frac{\pi}{2} - \arctan x\right) dx = \frac{\pi}{2}x - \left(x \arctan x - \frac{1}{2}\ln(1+x^2)\right)\Big|_0^n$$

$$= \frac{\pi}{2}n - \left(n \arctan n - \frac{1}{2}\left(\ln\left(1+n^2\right)\right)\right) = n\left(\frac{\pi}{2} - \arctan n\right) + \frac{1}{2}\ln(1+n^2)$$

$$\lim_{n\to\infty}\left(n\left(\frac{\pi}{2} - \arctan n\right) + \frac{1}{2}\ln(1+n^2)\right) \to \infty$$

Then by the integral test, $\sum \frac{\pi}{2} - \arctan \log n$ diverges absolutely. So the alternating series is conditionally convergent.

Exercise 21. $\sum_{j=1}^{\infty} \log\left(1 + \frac{1}{|\sin j|}\right)$

$\lim_{j\to\infty} \log\left(1 + \frac{1}{|\sin j|}\right)$ doesn't exist and $\log\left(1+\frac{1}{|\sin j|}\right) > 0 \quad \forall j$ so the series diverges.

Exercise 22. $\sum_{j=2}^{\infty} \sin\left(j\pi + \frac{1}{\log j}\right)$

$$\sin\left(2j\pi + \frac{1}{\log 2j}\right) + \sin\left((2j+1)\pi + \frac{1}{\log(2j+1)}\right) =$$

$$= \sin(2j\pi)\cos\frac{1}{\log 2j} + \sin\left(\frac{1}{\log 2j}\right)\cos(2\pi j) + \sin(2j+1)\pi\cos\frac{1}{\log(2j+1)} + \cos(2j+1)\pi\sin\left(\frac{1}{\log 2j+1}\right) =$$

$$= \sin\frac{1}{\log 2j} - \sin\frac{1}{\log 2j+1} =$$

$$= \sum_{k=0}^{\infty}\frac{\left(\frac{1}{\log 2j}\right)^{2k+1}(-1)^k}{(2k+1)!} - \sum_{k=0}^{\infty}\frac{\left(\frac{1}{\log(2j+1)}\right)^{2k+1}(-1)^k}{(2k+1)!}$$

$$\sin\left(\frac{1}{\log 2j}\right) - \sin\left(\frac{1}{\log 2j+1}\right) =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\left(\frac{1}{\log 2j}\right)^{2k+1} - \left(\frac{1}{2j+1}\right)^{2k+1}\right)$$

$$0 < \log 2j < \log 2j+1 \text{ so } \left(\frac{1}{\log 2j}\right)^{2k+1} - \left(\frac{1}{\log 2j+1}\right)^{2k+1} > 0$$

and since for $j > 1$, $\left(\frac{1}{\log 2j}\right)$ and $\left(\frac{1}{\log 2j+1}\right)$ are < 1 and so we are adding smaller and smaller amounts

$$< \frac{1}{\log 2j} - \frac{\log 2j+1}{=}$$

$$= \frac{\log(2j+1) - \log 2j}{\log 2j \log 2j+1} \leq \frac{\log\left(1+\frac{1}{2j}\right)}{(\log(2j))^2} \leq \frac{\frac{1}{2j}}{(\log(2j))^2}$$

$$\int_1^n \frac{1}{2j(\log 2j)^2} = -\frac{1}{\log 2j}\Big|_1^n \xrightarrow{n\to\infty} \frac{1}{\log 2}$$

So the series converges by using integral test, showing that $\sum \frac{1}{2j(\log 2j)^2}$ converges, so by comparison test, the series converges.

Exercise 33. $\sum_{n=1} n^n z^n$

$$\left|\sum_{n=1} n^n z^n\right| = \left|\sum_{j=1}^{\infty}(jz)^j\right| = \sum \left(e^{\ln j} z\right)^j$$

$$\sum_{j=1}^{n}(e^{\ln j} z)^j =$$

$$= \frac{e^{\ln j} z - (e^{\ln j} z)^n}{1 - e^{\ln j} z} \longrightarrow \infty$$

So $z = 0$

Exercise 34. $\sum_{j=1}^{\infty} \frac{(-1)^j z^{3j}}{j!} = \sum_{j=1}^{\infty} \frac{(-z^3)^j}{j!} = e^{-z^3}$. $\boxed{\mathbb{C}}$.

Exercise 35. $\sum_{j=0}^{\infty} \frac{z^j}{3^j} = \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j z^j$

$\sum z^j$ be convergent or $\sum_{j=1}^{n} z^j$ bounded.

$\boxed{|z| < 3}$ and $|z| = 3$ if $z \neq 3$

Exercise 36. $\sum_{j=1}^{\infty} \frac{z^j}{j^j}$ $\boxed{\{z\} = \mathbb{C}}$ since

$$\left|\left(\frac{z}{j}\right)\right| < 1 \text{ for } j \geq N > |z|$$

Exercise 37. $\sum_{j=1}^{\infty} \frac{(-1)^j}{z+j}$

By Leibniz's Rule, since $\frac{1}{z+j} \xrightarrow{j\to\infty} 0$, then the series converges. However, z cannot be equal to any negative integer since one term in the series will then blow up.

Exercise 38. $\sum_{j=1}^{\infty} \frac{z^j}{\sqrt{j}} \log\left(\frac{2j+1}{j}\right)$.

$$\frac{z^j}{\sqrt{j}} \log\left(2 + \frac{1}{j}\right) = z^j \frac{\log\left(2 + \frac{1}{j}\right)}{\sqrt{j}}$$

Since $\lim_{j\to\infty} \frac{\log\left(2+\frac{1}{j}\right)}{\sqrt{j}} = 0$ so that $\frac{\log\left(2+\frac{1}{j}\right)}{\sqrt{j}}$ is a monotonically convergent sequence.

Then by Dirchlet's test, $\sum_{j=1}^{n} z^j$ must be bounded. $|z| > 1$, and $|z| = 1$ if $z \neq 1$.

Exercise 39. $\sum_{j=1}^{\infty} \left(1 + \frac{1}{5j+1}\right)^{j^2} |z|^{17j} = \sum_{j=1}^{\infty} \left(1 + \frac{1}{5j+1}\right)^{j^2} (|z|^{17})^j$

$$\left(\left(1 + \frac{1}{5j+1}\right)^j |z|^{17}\right)^j$$

$$\lim_{j\to\infty} \left(1 + \frac{1}{5j+1}\right)^j \leq \lim_{j\to\infty} \left(1 + \frac{1/5}{j}\right)^j = e^{1/5}$$

$$e^{1/5}|z|^{17} < 1 \Longrightarrow \boxed{|z| < e^{-1/85}}$$

Exercise 40. $\sum_{j=0}^{\infty} \frac{(z-1)^j}{(j+2)!}$

$$\left|\sum_{j=0}^{\infty} \frac{(z-1)^j}{(j+2)!}\right| \leq \sum_{j=0}^{\infty} \frac{|(z-1)|^j}{(j+2)!} \leq \sum_{j=0}^{\infty} \frac{|(z-1)|^{j+2}}{(j+2)!} = \boxed{e^{|z-1|} - 1 - \frac{|z-1|}{1!}}$$

The series converges $\forall\, z$.

Exercise 41.

$$\sum_{j=1}^{\infty} \frac{(-1)^j (z-1)^j}{j} = \sum_{j=1}^{\infty} \frac{(1-z)^j}{j} = \log\left(1 - (1-z)\right) = \boxed{\log z}$$

So the series converges $\forall\, z$ except for $z = 0$.

Exercise 42. $\sum_{j=1}^{\infty} \frac{(2z+3)^j}{j\log{(j+1)}}$

$$\lim_{j\to\infty} \frac{1}{j\log{(j+1)}} = 0 \text{ so } \frac{1}{j\log{(j+1)}} \text{ is a monotonically convergent sequence}$$

$$\sum (2z+3)^j \qquad \begin{aligned} |2z+3| &< 1 \\ |z+\frac{3}{2}| &< \frac{1}{2} \end{aligned} \qquad \text{then } \sum (2z+3)^j \text{ converges}$$

$$\left(\frac{1}{x\log{(x+1)}}\right)' = \frac{-1}{(x\log{(x+1)})^2}\left(\log{(x+1)} + \frac{x}{x+1}\right) < 0; \quad \text{for } x > 0$$

By Dirichlet's test, $\sum_{j=1}^{\infty} \frac{(2z+3)^j}{j\log{(j+1)}}$ converges for $\boxed{|z+\frac{3}{2}| \leq \frac{1}{2}; \quad z \neq -1}$.

Exercise 43. $\sum_{j=1}^{\infty} \frac{(-1)^j}{(2j-1)}\left(\frac{1-z}{1+z}\right)^j = \sum_{j=1}^{\infty} \frac{(-1)^j/2}{j-\frac{1}{2}}\left(\frac{1-z}{1+z}\right)^j$

$\Longrightarrow \sum_{j=1}^{\infty} \frac{\left(\frac{z-1}{z+1}\right)^j}{2j-1} = \lim_{j\to\infty} \frac{1}{2j-1} = 0$ and $\frac{1}{2j-1}$ is monotonically decreasing.
So $\frac{1}{2j-1}$ is a monotonically decreasing convergent sequence of real terms.

For $\left|\frac{z-1}{z+1}\right| < 1$,

$$\left|\frac{z-1}{z+1}\right| < 1 \Longrightarrow |z-1| < |z+1|$$

$$\begin{aligned} (u-1)^2 + v^2 &< (u+1)^2 + v^2 \\ u^2 - 2u + 1 &< u^2 + 2u + 1 \qquad \boxed{\Re(z) > 0; \quad z \neq 0} \\ -u &< u \end{aligned}$$

Exercise 44.

$$\sum_{j=1}^{\infty}\left(\frac{z}{2z+1}\right)^j = \sum_{j=1}^{\infty}\left(\frac{1}{2} + \frac{-1/2}{2z+1}\right)^j = \sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j\left(1 - \frac{1}{2z+1}\right)^j$$

$$\left(\frac{1}{2}\right)^j \text{ is a monotonically decreasing, convergent sequence}$$

$$\text{Now we want } \left|1 - \frac{1}{2z+1}\right| = \left|\frac{2z+1-1}{2z+}\right| = \left|\frac{2z}{2z+1}\right| < 1$$

$$\begin{aligned} |2z| &< |2z+1| \\ |z| &< |z+1/2| \\ u^2 + v^2 &< (u+1/2)^2 + v^2 = u^2 + u + 1/4 + v^2 \\ \frac{-1}{4} &< u \Longrightarrow \boxed{\Re(z) > -\frac{1}{4}} \end{aligned}$$

$$\text{if } \Re(z) = \frac{-1}{4}$$

$$\frac{2\left(-\frac{1}{4} + iv\right)}{2\left(\frac{-1}{4} + iv\right) + 1} = \frac{\frac{-1}{2} + 2iv}{\frac{1}{2} + 2iv}$$

$$\Longrightarrow \boxed{z \neq \frac{-1}{4}}$$

Exercise 45. $\sum_{j=1}^{\infty} \frac{j}{j+1}\left(\frac{z}{2z+1}\right)^j$.

$$\left(\frac{x}{x+1}\right)' = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

$$\frac{j}{j+1} \text{ is a monotonically increasing and convergent sequence}$$

$$\frac{1}{2} + \frac{-1}{2(2z+1)} = \left|\frac{z}{2z+1}\right| < 1$$

$$\left|\frac{2z+1}{z}\right| > 1 \Longrightarrow \boxed{|2 + \frac{1}{z}| > 1}$$

Exercise 46. $\sum_{j=1}^{\infty} \frac{1}{(1+|z|^2)^j} = \sum_{j=1}^{\infty}\left(\frac{1}{1+|z|^2}\right)^j$

$$\begin{aligned} \frac{1}{1+|z|^2} &< 1 \\ 1 &< 1+|z|^2 \qquad \forall z \text{ except } z=0 \\ 0 &< |z|^2 \end{aligned}$$

Exercise 47. $\sum_{j=1}^{\infty}(-1)^j \frac{2^j \sin^{2j} x}{j}$

Use Dirichlet's Test. $\frac{1}{j}$ is a monotonically decreasing sequence converging to zero. Consider $(-2)^j \sin^{2j} x$. The condition for convergence is

$$|(-2\sin^2 x)| < 1 \quad \text{for } j \geq N \text{ for some } N$$
$$\implies x \in (\frac{-\pi}{4} + n\pi, \frac{\pi}{4} + n\pi), \quad n \in \mathbb{Z}$$

Exercise 49. $\sum a_j$ converges.

$\sum a_j \frac{1}{a_j}$ diverges.
Then since $\sum a_j$ is a convergent series (by Abel's test), $\frac{1}{a_j}$ is a divergent sequence.
Then $\sum \frac{1}{a_j}$ is divergent (since $\lim_{j\to\infty} \frac{1}{a_j}$ doesn't exist).

Exercise 50. $\sum |a_j|$ converges.

$\sum |a_j|$ converges, then $\sum a_j$ converges.
$|a_j|^2 = a_j^2$. $|a_j|$ converges, then

$$\lim_{j\to\infty} |a_j| = 0 \qquad \lim_{j\to\infty} \frac{|a_j|^2}{a_j^2} = 1$$
$$\lim_{j\to\infty} |a_j|^2 = 0$$

By limit comparison test, $\sum a_j^2$ converges.

Counterexample: $\sum \left(\frac{1}{j}\right)^2$ converges, but $\sum \frac{1}{j}$ diverges.

Exercise 51. Given $\sum a_j, \quad a_j \geq 0$. $\sum a_j$ converges.

$$\lim_{j\to\infty} a_j = 0$$
$$\sum \sqrt{a_j}\frac{1}{(j)^p} \qquad \lim_{j\to\infty} \sqrt{a_j} = \left(\lim_{j\to\infty} a_j\right)^{1/2} = 0$$
$$\int \sum_{j=0}^{n-1} x^j = \sum_{j=0}^{n-1} \frac{x^{j+1}}{j+1} = \sum_{j=1}^{n} \frac{x^j}{j} = \int \frac{1-x^n}{1-x}$$

A counterexample would be $\frac{\sqrt{a_j}}{j^p} = \sqrt{\frac{a_j}{j}}$.

Exercise 52.

(1) $\sum a_j$ converges absolutely, then if $\sum |a_j|$ converges, $\sum a_j^2$ converges.

$$\frac{a_j^2}{1+a_j^2} = 1 + \frac{-1}{1+a_j^2}$$
$$\frac{a_j^2}{1+a_j^2} \leq a_j^2 \text{ since } \sum a_j^2 \text{ converges }, \sum \frac{a_j^2}{1+a_j^2} \text{ converges}$$

(2) $\sum a_j$ converges absolutely, $\lim_{j\to\infty} |a_j| = 0$
$\sum \left|\frac{a_j}{1+a_j}\right| = \sum |a_j| \left(\frac{1}{|1+a_j|}\right)$.
By Abel's test, since

$$\lim_{j\to\infty} \frac{1}{|1+a_j|} = \frac{1}{|1+\lim_{j\to\infty} a_j|} = 1 \quad \text{shows that } \frac{1}{|1+a_j|} \geq 0 \text{ is a monotonically convergent sequence}$$

By Abel's test, $\sum \frac{a_j}{1+a_j}$ is convergent.

10.22 Miscellaneous review exercises - Rearrangements of series. **Exercise 1.**

(1)
$$a_j = \sqrt{j+1} - \sqrt{j} = \sqrt{j}\sqrt{1+\frac{1}{j}} - \sqrt{j} = \sqrt{j}\left(1 + \frac{1}{2}\left(\frac{1}{j}\right) + o\left(\frac{1}{j}\right) + -1\right) =$$
$$= \sqrt{j}\left(\frac{1}{2}\left(\frac{1}{j}\right) + o\left(\frac{1}{j}\right)\right)$$
$$\lim_{j\to\infty} a_j = 0$$

(2)
$$a_j = (j+1)^c - j^c = j^c\left(\left(1+\frac{1}{j}\right)^c - 1\right) = (j^c)\left(1 + c\left(\frac{1}{j}\right) + o\left(\frac{1}{j}\right) - 1\right) =$$
$$= j^c\left(c\left(\frac{1}{j}\right) + o\left(\frac{1}{j}\right)\right) = \left(cj^{c-1} + j^c o\left(\frac{1}{j}\right)\right)$$
$$\boxed{\begin{array}{ll} \text{if } c > 1, & a_j \text{ diverges} \\ \text{if } c = 1, & \lim_{j\to\infty} a_j = 1 \\ \text{if } c < 1, & \lim_{j\to\infty} a_j = 0 \end{array}}$$

Exercise 2.

(1)
$$(1+x^n)^{\frac{1}{n}} = \exp\left(\frac{1}{n}\ln(1+x^n)\right) = \exp\left(\frac{1}{n}\sum_{j=1}^{\infty}\frac{(x^n)^j(-1)^{j-1}}{j}\right) \xrightarrow{n\to\infty} \boxed{1}$$

(2) $\lim_{n\to\infty}(a^n+b^n)^{1/n} = \lim_{n\to\infty} a\left(1+\left(\frac{b}{n}\right)^n\right)^{1/n} = a$ if $a > b$.

Exercise 3. $a_{n+1} = \frac{a_n + a_{n-1}}{2} = \frac{a_{n-1}+a_{n-2}}{2^2} + \frac{a_{n-2}+a_{n-3}}{2^2}$

10.24 Exercises - Improper integrals.

Exercise 1. $\int_0^\infty \frac{x}{\sqrt{x^4+1}}dx$

$$\lim_{x\to\infty}\left(\frac{x}{\sqrt{x^4+1}}\right)\left(\frac{1}{1/x}\right) = \lim_{x\to\infty}\frac{x^2}{x^2\sqrt{1+1/x^4}} = 1$$

Since $\int_1^\infty \frac{1}{x}$ diverges, so does $\int_0^\infty \frac{x}{\sqrt{x^4+1}}dx$

Exercise 2.

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \int_0^\infty e^{-x^2}dx + \int_0^{-\infty} e^{-x^2}dx = \int_0^\infty e^{-x^2}dx + -\int_\infty^0 e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$$
$$\int_0^\infty e^{-x^2}dx \leq \int_0^\infty e^{-x}dx = -e^{-x^2}\Big|_0^\infty = \boxed{1}$$

Converges by theorem.

Exercise 3. $\int_0^\infty \frac{1}{\sqrt{x^3+1}}dx$

Exercise 4. $\int_0^\infty \frac{1}{\sqrt{e^x}}dx$

Exercise 5. $\int_{0^+}^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$

Exercise 6. $\int_{0^+}^1 \frac{\log x}{\sqrt{x}}dx$

Exercise 7. $\int_{0^+}^{1-} \frac{\log x}{1-x}dx$

Exercise 8. $\int_{-\infty}^\infty \frac{x}{\cosh x}dx$

Exercise 9. $\int_{0^+}^{1-} \frac{dx}{\sqrt{x}\log x}$

Exercise 10. $\int^{\infty} \frac{dx}{x(\log x)^s}$

11.7 Exercises - Pointwise convergence of sequences of functions, Uniform convergence of sequences of functions, Uniform convergence of sequences of functions, Uniform convergence and continuity, Uniform convergence and integration, A sufficient condition for uniform convergence, Power series. Circle of convergence. **Exercise 1.** $\sum_{j=0}^{\infty} \frac{z^j}{2^j} = \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j$

Using the *comparison test*,
$\left|\frac{z}{2}\right|^j \leq t^j; \qquad \left|\frac{z}{2}\right| < 1 \Longrightarrow |z| < 2$

Suppose $|z| = 2$, $z \neq 2$ $\sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j = \sum_{j=0}^{\infty} (e^{2i\theta})^j$

Now $\sum_{j=0}^{n} (e^{2i\theta})^j \leq \frac{1}{\sin\theta} + 1$, since

$$\sum_{j=1}^{n} (e^{2i\theta})^j = \frac{e^{2i\theta} - e^{2i\theta(n+1)}}{1 - e^{2i\theta}} = \frac{e^{in\theta} - e^{-i\theta n} e^{i\theta n + i\theta}}{-e^{-i\theta} + e^{i\theta}} = \frac{\sin n\theta e^{i\theta n + i\theta}}{\sin\theta} < \frac{1}{\sin\theta}$$

So $\sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j$ converges for $|z| = 24$, $z \neq 2$

Exercise 2. $\sum_{j=0}^{\infty} \frac{z^j}{(j+1)2^j}$

Use *ratio test* .

$$\frac{a_{j+1}}{a_j} = \frac{z^{j+1}}{(j+2)2^{j+1}} \frac{(j+1)2^j}{z^j} = \frac{z}{2} \frac{(j+1)}{(j+2)} = \frac{z}{2} \frac{(1+1/j)}{(1+2/j)} \xrightarrow{j \to \infty} \frac{z}{2}$$

If $|z| < 2$, $\sum_{j=0}^{\infty} a_j$ converges, if $|z| > 2$, $\sum a_j$ diverges.

If $|z| = 2$, $\sum \frac{z^j}{(j+1)2^j} = \sum (e^{2i\theta})^j \left(\frac{1}{j+1}\right)$

Now $\frac{1}{j+1}$ is a monotonically decreasing sequence of real terms.

$\sum (e^{2i\theta})^j$ is a bounded series.

By Dirichlet's test, $\sum a_j$ converges if $|z| = 2$, $\quad z \neq 2$

Exercise 3. $\sum_{j=0}^{\infty} \frac{(z+3)^j}{(j+1)2^j}$

Use ratio test:

$$\frac{a_{j+1}}{a_j} = \left(\frac{(z+3)^{j+1}}{(j+2)2^{j+1}}\right) \frac{(j+1)2^j}{(z+3)^j} = \frac{(z+3)}{2} \left(\frac{j+1}{j+2}\right) = \frac{(z+3)}{2} \left(\frac{1+1/j}{1+2/j}\right) \xrightarrow{j \to \infty} \frac{z+3}{2}$$

Using Theorem 11.7,

Theorem 34 (Existence of a circle of convergence)**.**

$$\textit{Assume} \sum a_j z^j \quad \begin{matrix} \textit{converges for at least } z_1 \neq 0 \\ \textit{diverges for at least one } z_2 \neq 0 \end{matrix}$$

$$\exists\, r > 0, \quad \textit{such that} \sum a_j z^j \quad \begin{matrix} \textit{absolutely converges for } |z| < r \\ \textit{diverges for } |z| > r \end{matrix}$$

We can plug in real numbers to satisfy the condition $\frac{|z+3|}{2} < 1$ for convergence.
$\sum a_j$ converges for $|z+3| < 2$; $\quad$ diverges for $|z+3| > 2$.

Consider $|z+3| = 2$; $\quad z \neq 1$ $\sum a_j = \sum (e^{2i\theta})^j \left(\frac{1}{j+1}\right)$. Since $\frac{1}{j+1}$ is a monotonically decreasing sequence of real numbers and $\sum (e^{2i\theta})^j$ is a bounded series, by Dirichlet's test,
$\sum a_j$ converges for $|z+3| = 2$; $\quad z \neq -1$.

Exercise 4. $\sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} z^{2j}}{2j} = -\sum_{j=1}^{\infty} \frac{(-1)^{j-1}(2z)^{2j}}{(2j)}$. Look at what the terms look like.

Consider using Leibniz's Rule, Theorem 10.14.

Theorem 35 (Leibniz's rule)**.** *If a_j is a monotonic decreasing sequence and* $\lim_{j\to\infty} a_j = 0$*, then* $\sum_{j=1}^{\infty} (-1)^{j-1} a_j$ *converges.*

$$\frac{(-1)^j 2^{2j} z^{2j}}{2j} = \frac{(-1)^j (2z)^{2j}}{2j}$$

$$\text{Consider } \left|\frac{(2z)^{2j}}{2j}\right| = \frac{|2z|^{2j}}{2j} = \frac{(2|z|)^{2j}}{2j}$$

$$\text{Consider } 2|z| = M^{1/2} < \infty$$

$$\implies \frac{(2|z|)^{2j}}{2j} = \frac{M^j}{2j} = \frac{e^{j \ln M}}{2j}$$

$$\text{converges for } 0 < 2|z| = M \leq 1 \implies |z| \leq \frac{1}{2}$$

(we use Theorem 11.6 at this point, because real numbers are included in complex numbers).

Theorem 36. *Assume* $\sum a_j z^j$ *converges for some* $z = z_1 \neq 0$.

(1) $\sum a_j z^j$ *converges absolutely* $\forall z$ *with* $|z| < |z_1|$.
(2) $\sum a_j z^j$ *converges uniformly on every circular disk with center at* 0 *and* $R < |z_1|$

We had *first used Leibniz's test* to find az_1 on the real line.

$$\frac{2^{4j} z^{4j}(4j(1-\frac{1}{4z^2}) - 2)}{4j(4j-2)} = \frac{2^{4j} z^{4j}((1-\frac{1}{4z^2}) - \frac{2}{4j})}{(4j-2)} \xrightarrow{j\to\infty} \frac{2^{4j} z^{4j}(1-\frac{1}{4z^2})}{(4j-2)}$$

If $|z| > \frac{1}{2}$, the series diverges (by a_jth general term test).

Exercise 5. $\sum_{j=1}^{\infty}(1-(-2)^j)z^j$.

Try limit comparison test .
The first step is to test *absolute convergence* first; it's easier.

$$\frac{\left|(1-(-2)^j)z^j\right|}{|(-2z)^j|} = \left|\left(\frac{1}{-2}\right)^j - 1\right| \xrightarrow{j\to\infty} 1$$

According to limit comparison test, for $\sum(1-(-2)^j)z^j$ to converge, $\sum(-2z)^j$ must converge.
So if $|z| < \frac{1}{2}$, then $\sum_{j=1}^{\infty}(1-(-2)^j)z^j$ absolutely converges.
If $|z| = \frac{1}{2}$, $\quad z \neq \frac{-1}{2}$,

$$\sum(1-(-2)^j)z^j = \sum z^j - \sum(-e^{2i\theta})^j =$$
$$= 0 - \sum(e^{2i\theta + \pi i})^j < \frac{1}{\sin(\theta + \pi)}$$

If $z = \frac{-1}{2}$,

$$\sum\left(-\frac{1}{2}\right)^j - \sum 1^j \to \infty$$

Exercise 6. $\sum_{j=1}^{\infty} \frac{j! z^j}{j^j}$

A **very big hint** is to use Exercise 19 on pp. 399, in the section for Exercises 10.14.

Since $\sum_{j=1}^{n-1} f(j) \le \int_1^n f(x)dx$

$$\begin{aligned}
\sum_{j=1}^{n-1} \ln j \le \int_1^n \ln x = n\ln n - n + 1 \quad &\le \sum_{j=2}^{n} \ln j \\
(n-1)! \le n^n e^{-n} n \qquad\qquad &\le n!
\end{aligned}$$

$$\frac{n!}{n^n} \ge ne^{-n}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} \ge \lim_{n\to\infty} ne^{-n} = 0$$

$$\lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} \frac{n^2}{e^n} = 0$$

$$\implies \boxed{\lim_{n\to\infty} \frac{n!}{n^n} = 0}$$

So then since $\frac{n!}{n^n}$ is a monotonically decreasing convergent sequence of real terms; if $\sum z$ is a bounded series, then by Dirichlet's test, $\sum \frac{n!z^n}{n^n}$ is convergent.

$|z| < 1$; $\quad |z| = 1$ if $z \neq 1$

Try the ratio test, because it's clear from the results of the ratio test where convergence and divergence begins and ends.

$$\frac{(n+1)!|z|^{n+1}}{(n+1)^{n+1}} \frac{n^n}{n!|z|^n} = \left(\frac{n}{n+1}\right)^n |z| = $$
$$= \left(\frac{1}{1+1/n}\right)^n |z| \xrightarrow{n\to\infty} \frac{1}{e}|z|$$

Converges for $\boxed{|z| < e}$.

Now try plugging in a complexified e:

$$\frac{e^j}{\left(\frac{j^j}{j!}\right)} = \frac{j!e^j e^{i2\theta j}}{(j)^j} \ge \frac{j^j e^{-j} j e^j e^{i2\theta j}}{j^j} e^{i2\theta j} \to \infty$$

So the series diverges for $|z| = e$.

Exercise 7. $\sum_{j=0}^{\infty} \frac{(-1)^j (z+1)^j}{j^2+1}$

By the ratio test,

$$\frac{|z+1|^{j+1}}{j^2+2j+2} \frac{j^2+1}{|z+1|^j} = |z+1| \frac{1+1/j^2}{1+2/j+2/j^2} \xrightarrow{j\to\infty} |z+1|$$

The series absolutely converges for $|z+1| < 1$.

For $z = 0$, $\sum \frac{(-1)^j}{j^2+1}$ converges since $\frac{1}{j^2+1} \xrightarrow{j\to\infty} 0$

For $z = -2$, $\sum \frac{(-1)^j(-1)^j}{j^2+1} = \sum \frac{1}{j^2+1}$ and $\frac{1}{j^2+1} < \frac{1}{j^2}$, but $\sum \frac{1}{j^2}$ is convergent (by integral test). So the series converges for $z = -2$.

By Dirichlet's test, the series converges as well, if we treat $a_j = (-1)^j (z+1)^j$ and $b_j = \frac{1}{j^2+1}$ to be the monotonically decreasing sequence.

$\implies \boxed{|z+1| \le 1}$ for convergence x

Exercise 8. $\sum_{j=0}^{\infty} a^{j^2} z^j$, $0 < a < 1$

Use the *root test* .

$$(a^{j^2} z^j)^{1/j} = a^j z \xrightarrow{j\to\infty} 0$$

So the series converges $\forall\, z \in \mathbb{C}$

Exercise 9. $\sum_{j=1}^{\infty} \frac{(j!)^2}{(2j)!} z^j$

Use the *ratio test*

$$\frac{a_{j+1}}{a_j} = \frac{((j+1)!)^2 z^{j+1}}{(2(j+1))!}\frac{(2j)!}{(j!)^2 z^j} = \frac{(j+1)^2 z}{(2j+2)(2j+1)} \xrightarrow{j\to\infty} \frac{1(z)}{4}$$

$$\text{for } |z|<4, \sum a_j \text{ absolutely converges}$$

$$\text{for } |z|>4, \sum a_j \text{ diverges}$$

Let's test the boundary for convergence.

$$\frac{(j!)^2}{(2j)!}4^j = \frac{(j!)^2}{(2j)!}e^{j\ln 4} = \frac{(j!)^2}{(2j-2)!}\frac{e^{j\ln 4}}{(2j)(2j-1)} \geq \frac{(j!)^2}{(2j-2)!}\frac{e^{j\ln 4}}{(2j)^2} \geq$$

$$\geq \frac{(j!)^2}{(2j-2)!}\frac{j^{2j}}{(j!)^2} = \frac{j^{2j}}{(2j-2)!} \to \boxed{\infty}$$

where we had used

$$(n-1)! \leq n^n e^{-n} n \leq n! \qquad \Longrightarrow ne^{-n} \leq \frac{n!}{n^n}$$

$$\frac{n}{e^n} \leq \frac{n!}{n^n} \qquad \Longrightarrow \frac{n^n}{n!} \leq \frac{e^n}{n}$$

Exercise 10. $\sum_{j=1}^{\infty} \frac{3^{\sqrt{j}} z^j}{j} = \sum_{j=1}^{\infty} \frac{e^{\sqrt{j}\ln 3} z^j}{j}$

$$\frac{e^{\sqrt{j+1}\ln 3} z^{j+1}}{j+1}\frac{j}{e^{\sqrt{j}\ln 3} z^j} = \left(\frac{1}{j+1}\right) e^{(\sqrt{j+1}-\sqrt{j})\ln 3} z$$

$$\sqrt{j+1}-\sqrt{j} = \sqrt{1+\frac{1}{j}} - 1 \simeq 1 + \frac{1}{2}\left(\frac{1}{j}\right) - 1 = \frac{1}{2j} \qquad \text{(for large } j\text{)}$$

$$\text{(for large } j\text{)} \qquad \left(\frac{j}{j+1}\right) e^{\frac{1}{2j}} z \to 0$$

$$\Longrightarrow \text{Converges} \quad \forall z \in \mathbb{C}$$

Exercise 11. $\sum_{j=1}^{\infty} \left(\frac{1\cdot 3\cdot 5\ldots(2j-1)}{2\cdot 4\cdot 6\ldots(2j)}\right)^3 z^j$

$$\frac{a_{j+1}}{a_j} = \left(\frac{1\cdot 3\cdot 5\ldots(2j+1)}{2\cdot 4\cdot 6\ldots(2j+2)}\right)^3 z^{j+1}\left(\frac{2\cdot 4\cdot 6\ldots(2j)}{1\cdot 3\cdot 5\ldots(2j-1)}\right)^3 \frac{1}{z^j} = \left(\frac{2j+1}{2j+2}\right)^3 z = \left(1+\frac{-1/2}{j+2}\right)^3 z$$

If $|z|<1$, it converges by ratio test, if $|z|=1$, then it converges by Gauss test

$$\frac{a_{j+1}}{a_j} = \left(1+\frac{-1/2}{j+2}\right)^3 z = \sum_{k=0}^{3}\binom{3}{k}\left(\frac{-1/2}{j+2}\right)^k |z| =$$

$$= |z|\left(1+3\left(\frac{-1/2}{j+2}\right)+3\left(\frac{-1/2}{j+2}\right)^2+\left(\frac{-1/2}{j+2}\right)^3\right) \xrightarrow{j\to\infty} |z|$$

$$\boxed{\text{diverges for } |z|>1 \text{ (by ratio test)}}$$

Exercise 12. $\sum_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{j^2} z^j$

$$\left(\left(1+\frac{1}{j}\right)^{j^2} z^j\right)^{1/j} = \left(1+\frac{1}{j}\right)^j z \xrightarrow{j\to\infty} e^1 z$$

$$|z| < \frac{1}{e}, \quad \sum(1+\frac{1}{j})^{j^2} z^j \text{ converges by root test}$$

$$|z| > \frac{1}{e}, \quad \sum(1+\frac{1}{j})^{j^2} z^j \text{ diverges by root test}$$

$$\boxed{\frac{1}{e} = r}$$

Exercise 13. $\sum_{j=0}^{\infty}(\sin aj)z^j \qquad a>0$

$|\sin{(aj)}z^j| \leq |z|^j$
By comparison test, $\sum_{j=0}^{\infty}(\sin aj)z^j$ converges, since $\sum_{j=0}^{\infty}|z|^j$ converges absolutely, for $|z|<1$.

Note that if $a=\pi$, the series is zero.
$\sum_{j=0}^{\infty}(\sin aj) \to \infty$ for $a=2\pi$ so $\boxed{\text{r=1}indeed.}$

Exercise 14. $\sum_{j=0}^{\infty}(\sinh aj)z^j = \sum_{j=0}^{\infty}\left(\frac{e^{aj}-e^{-aj}}{2}\right)z^j = \frac{1}{2}\left(\sum_{j=0}^{\infty}(e^a z)^j - \sum_{j=0}^{\infty}\left(\frac{z}{e}a\right)^j\right); \qquad a>0$

If $|z|<\frac{1}{e^a}$, then $\sum \sinh ajz^j$ converges. So then the radius of convergence is $r=\frac{1}{e^a}$

Exercise 15. $\sum_{j=1}^{\infty}\frac{z^j}{a^j+b^j}$. Assume $a<b$

$$\frac{z^j}{b^j(1+\left(\frac{a}{b}\right)^j)}$$

$$\text{(ratio test)}\ \frac{z^{j+1}}{b^{j+1}(1+\left(\frac{a}{b}\right)^{j+1})}\left(\frac{b^j\left(1+\left(\frac{a}{b}\right)^j\right)}{z^j}\right) = \frac{z}{b}\left(\frac{1+\left(\frac{a}{b}\right)^j}{1+\left(\frac{a}{b}\right)^{j+1}}\right) \xrightarrow{j\to\infty} \left(\frac{z}{b}\right)$$

So then $|z| \lessgtr b$ converges (diverges) by ratio test.
If $a=b$,

$$\sum_{j=1}^{\infty}\frac{z^j}{2a^j} = \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{z}{a}\right)^j$$

By comparison test with x^j, if $|z| \lessgtr |a|$, the series converges (diverges).

Exercise 16. $\sum_{j=1}^{\infty}\left(\frac{a^j}{j}+\frac{b^j}{j^2}\right)z^j$ Use ratio test on each of the sums, separately.

$$\frac{(a|z|)^{j+1}}{j+1}\frac{j}{(a|z|)^j} = \frac{a|z|}{1+\frac{1}{j}} \xrightarrow{j\to\infty} a|z|$$
$$\implies |z|<\frac{1}{a} \text{ then the series converges}$$
$$\frac{(b|z|)^{j+1}}{(j+1)^2}\frac{j^2}{(b|z|)^j} = b|z|\left(\frac{1}{1+\frac{1}{j}}\right)^2 \xrightarrow{j\to\infty} b|z|$$
$$\implies |z|<\frac{1}{b}$$

$\boxed{\text{So if } a \gtrless b, \text{ then } r=a;(b)}$

Exercise 17. $\int_0^1 f_n(x) = \int_0^1 nxe^{-nx^2} = \left.\frac{e^{-nx^2}}{-2}\right|_0^1 = \frac{e^{-n}-1}{-2} \xrightarrow{n\to\infty} \frac{1}{2}$

However,
$\lim_{n\to\infty} nxe^{-nx^2}=0$
This example shows that the operations of integration and limit cannot always be interchanged. We need uniform convergence. **Exercise 18.** $f_n(a)=\frac{\sin nx}{n} \qquad \lim_{n\to\infty}\frac{\sin nx}{n}=0$

$f(x)=\lim_{n\to\infty}f_n(x)$
$f_n'(x)=\frac{n\cos nx}{n} \qquad \implies \lim_{n\to\infty}f_n'(0)=1$
This example shows that differentiation and limit cannot always be interchanged.

Exercise 19. $\sum_{j=1}^{\infty}\frac{\sin jx}{j^2}=f(x)$

$$\frac{\sin jx}{j^2} \leq \frac{1}{j^2} \qquad \forall\, x \in \mathbb{R}$$

$$\text{by comparison test, } \sum_{j=1}^{\infty} \frac{|\sin jx|}{j^2} \text{ converges, so } \sum_{j=1}^{\infty} \frac{\sin jx}{j^2} \text{ converges}$$

$$\left| \frac{\sin jx}{j^2} \right| \leq \frac{1}{N^2} \qquad \forall\, x \in \mathbb{R}; \qquad \forall j \geq N$$

$\sum N \frac{1}{N^2}$ converges, so $\sum \frac{\sin jx}{j^2}$ uniformly converges.

Then by Thm., since $\frac{\sin jx}{j^2}$ are continuous, $\sum \frac{\sin jx}{j^2}$ is continuous.

Since $\sum \frac{\sin jx}{j^2}$ uniformly converges.

$$\int_0^{\pi} \sum \frac{\sin jx}{j^2} = \sum \int_0^{\pi} \frac{\sin jx}{j^2} = \sum \left. \frac{\cos jx}{-j^3} \right|_0^{\pi} = \sum \frac{(-1)^j - 1}{-j^3} =$$

$$= \boxed{2 \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3}}$$

Exercise 20. It is known that $\sum_{j=1}^{\infty} \frac{\cos jx}{j^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$ if $0 \leq x \leq 2\pi$

(1) $x = 2\pi$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{(2\pi)^2}{4} - \frac{2\pi^2}{2} + \frac{\pi^2}{6} = \frac{\pi^2}{6}$$

(2) As shown in Ex. 19, $\sum \frac{\cos jx}{j^2}$ is uniformly convergent on $\mathbb{R}$

$$\sum \int \frac{\cos jx}{j^2} = \sum \left. \left(\frac{\sin jx}{j^3} \right) \right|_0^{\pi/2} = \sum \frac{(-1)^{j+1}}{(2j-1)^3}$$

$$\int \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} = \left. \left(\frac{1}{3} \frac{x^3}{4} - \frac{\pi x^2}{4} + \frac{\pi^2}{6} x \right) \right|_0^{\pi/2} =$$

$$= (\pi)^3 \left(\frac{1}{12(8)} - \frac{1}{16} + \frac{1}{12} \right) = \boxed{(\pi)^3 \frac{1}{32}}$$

11.13 Exercises - Properties of functions represented by real power series, The Taylor's series generated by a function, A sufficient condition for convergence of a Taylor's series, Power-series expansions for the exponential and trigonometric functions, Bernstein's Theorem. Sufficient Condition for convergence.

Theorem 37 (Bernstein's Theorem)**.** *Assume* $\forall x \in [0, r],\ f(x),\ f^{(j)}(x) \geq 0 \quad \forall j \in \mathbb{N}$.
Then if $0 \leq x < r$

$$\sum^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \textit{converges} \tag{25}$$

Proof. If $x = 0$, we're done. Assume $0 < x < r$.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + E_n(x)$$

$$E_n(x) = \frac{x^{n+1}}{n!} \int_0^1 u^n f^{(n+1)}(x - xu) du$$

$$F_n(x) = \frac{E_n(x)}{x^{n+1}} = \frac{1}{n!} \int_0^1 u^n f^{(n+1)}(x - xu) du$$

Since $f^{(n+1)} > 0, \quad f^{(n+1)}(x(1-u)) \leq f^{(n+1)}(r(1-u))$

$$\Longrightarrow F_n(x) \leq F_n(r) \Longrightarrow \frac{E_n(x)}{x^{n+1}} \leq \frac{E_n(r)}{r^{n+1}}$$

For $f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j + E_n(x) \Longrightarrow E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} E_n(r)$

$$f(r) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} r^j + E_n(r) \geq E_n(r) \text{ since } f^{(j)}(0) \geq 0 \quad \forall j$$

So then $0 \leq E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} f(r)$
$n \to \infty$ and $f(t)$ will be some non-infinite value, so $E_n(x) \xrightarrow{n\to\infty} 0$. □

Exercise 1. $\sum_{j=0}^{\infty} (-1)^j x^{2j}$

Consider absolute convergence. $\lim_{j\to\infty} (x^2)^j = 0$ If $x^2 < 1$

If $|x| = 1$, then consider

$$x^{2(2j)} - x^{2(2j+1)} = x^{4j}(1-x)$$
$$\sum_{j=0}^{\infty} (1-x)x^{4j} = (1-x)\sum_{j=0}^{\infty} (x^4)^j$$
$$\boxed{\text{Indeed } \sum_{j=0}^{\infty} (-1)^j x^{2j} \text{ converges for } |x| \leq 1}$$

Exercise 2. $\sum_{j=0}^{\infty} \frac{x^j}{3^{j+1}} = \frac{1}{3}\sum_{j=0}^{\infty} \left(\frac{x}{3}\right)^j$ The series converges for $\boxed{|x| < 3}$

Exercise 3. $\sum j = 0^{\infty} jx^j$

$$\int_0^x \sum_{j=0}^{\infty} jt^{j-1} = \sum_{j=0}^{\infty} x^j$$

So the series converges for $|x| < 1$. Note that we had used the integrability of power series.

Exercise 4. $\sum_{j=0}^{\infty} (-1)^j jx^j$.

$$jx^j je^{j\ln x}$$
$$\lim_{j\to\infty} je^{j\ln x} = \begin{cases} \infty & \text{if } x > 1 \\ 0 & \text{if } 0 < x < 1 \end{cases}$$

If $x = 1$,

$$(2j)x^{2j} - (2j+1)x^{2j+1} = x^{2j}(2j + -(2j+1)x) = \boxed{-1} \sum(-1) = \infty$$

So $\sum(-1)^j jx^j$ converges only for $0 \leq x < 1, \quad |x| < 1$.

Exercise 5. $\sum_{j=0}^{\infty} (-2)^j \frac{j+2}{j+1} x^j = \sum_{j=0}^{\infty} (-1)^j \left(\frac{j+2}{j+1}\right)(2x)^j$

$$\lim_{j\to\infty} \left(\frac{j+2}{j+1}\right)(2x)^j = \lim_{j\to\infty} (2x)^j \quad \text{ if } |2x| < 1$$

So when $|x| < \frac{1}{2}$, the series converges.

$$\left(\frac{2j+2}{2j+1}\right) - \left(\frac{2j+3}{2j+2}\right) = \frac{(2j+2)(2j+2) - (2j+3)(2j+1)}{(2j+1)(2j+2)}$$
$$\frac{4j^2 + 8j + 4 - (4j^2 + 5j + 3)}{4j^2 + 6j + 2} = \frac{3j+1}{4j^2 + 6j + 2} = \frac{(3j+1)/2}{2j^2 + 3j + 1} = \frac{(3j+1)/2}{(2j+1)(j+1)}$$
$$\frac{3j+1}{4j^2 + 6j + 2} < \frac{3j+1}{4j^2 + \frac{4}{3}} < \frac{3(j+1/3)}{4(j^2 + 1/3)} < \frac{3}{4}\left(\frac{1}{j}\right) + \frac{1}{12j^2}$$

Thus, it diverges, by comparison test with $\frac{1}{j}$ for $x = \frac{1}{2}$.

Theorem 38. *Let f be represented by $f(x) = \sum_{j=0}^{\infty} a_j (x-a)^j$ in the $(a-r, a+r)$ interval of convergence*

(1) $\sum_{j=1}^{\infty} ja_j(x-a)^{j-1}$ *also has radius of convergence r.*

(2) $f'(x)$ *exists* $\forall x \in (a-r, a+r)$ *and*

$$f'(x) = \sum_{j=1}^{\infty} j a_j (x-a)^{j-1} \tag{26}$$

Exercise 6. $\sum_{j=1}^{\infty} \frac{(2x)^j}{j} = \sum_{j=1}^{\infty} \frac{e^{j \ln 2x}}{j} \implies \boxed{|x| < \frac{1}{2}}$ it'll converge, since by comparison test, $\frac{e^{j\ln 2x}}{j} < \frac{1}{j^2}$ if $0 < x < \frac{1}{2}$.

Exercise 7. $\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \left(\frac{x}{2}\right)^{2j}$.

$$\sum_{j=0}^{\infty} \frac{x^{2j}}{2(j+1/2)2^{2j}} < \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j+1} j} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{x}{2}\right)^{2j} \left(\frac{1}{j}\right) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{e^{2j \ln \frac{x}{2}}}{j}$$

$$\text{since by comparison test,} \quad \frac{e^{2j \ln \frac{x}{2}}}{j} < \frac{1}{j^2} \quad \text{if } 0 < x < 2$$

$$\text{If } x = \pm 2 \text{ ,} \quad \begin{aligned} &\frac{1}{2(2j)+1} - \frac{1}{2(2j+1)+1} = \frac{1}{4j+1} - \frac{1}{4j+3} = \frac{2}{(4j+1)(4j+3)} \leq \\ &\leq \frac{1}{8}\frac{1}{j^2} \text{ (converges by comparison test to } \sum \frac{1}{j^2} \text{)} \end{aligned}$$

For $\boxed{|x| < 2}$, $\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \left(\frac{x}{2}\right)^{2j}$ converges.

Exercise 8. $\sum_{j=0}^{\infty} \frac{(-1)^j x^{3j}}{j!} = \sum_{j=0}^{\infty} \frac{(-x^3)^j}{j!} = e^{-x^3}$, which converges $\forall\, x \in \mathbb{R}$

Exercise 9. $\sum_{j=0}^{\infty} \frac{x^j}{(j+3)!} = \frac{1}{x^3} \sum_{j=0}^{\infty} \frac{x^{j+3}}{(j+3)!} = \frac{1}{x^3} \sum_{j=+3}^{\infty} \frac{x^j}{j!} = \frac{1}{x^3}(e^x - x^2/2 - x - 1)$ Thus, it converges for $\forall\, x \in \mathbb{R}$.

Exercise 10. $\sum_{j=0}^{\infty} \frac{(x-1)^j}{(j+2)!} = \frac{1}{(x-1)^2} \sum_{j=0}^{\infty} \frac{(x-1)^{j+2}}{(j+2)!} = \frac{1}{(x-1)^2} \left(\sum_{j=2}^{\infty} \frac{(x-1)^j}{j!} \right) = \frac{1}{(x-1)^2} \left(e^{x-1} - (x-1) - 1 \right) = \boxed{\frac{e^{x-1} - x}{(x-1)^2}}$

Exercise 11. $a^x = e^{x \log a} = \sum_{j=0}^{\infty} \frac{(\log ax)^j}{j!}$

$\frac{(\log ax)^{j+1}}{(j+1)!} \frac{j!}{(\log ax)^j} = \frac{(\log ax)}{j+1} \xrightarrow{j\to\infty} 0$. By ratio test, $\sum_{j=0}^{\infty} \frac{(\log ax)^j}{j!}$ converges for all x.

Exercise 12.

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \left(\sum_{j=0}^{\infty} \frac{x^j}{j!} - \sum_{j=0}^{\infty} \frac{(-x)^j}{j!} \right) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!}$$

$$\frac{x^{2j+3}}{(2j+3)!} \frac{(2j+1)!}{x^{2j+1}} = \frac{x^2}{(2j+3)(2j+2)} \xrightarrow{j\to\infty} 0$$

Exercise 13.

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1 - \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!}(-1)^j}{2} = \sum_{j=1}^{\infty} \frac{2^{2j-1} x^{2j} (-1)^{j+1}}{(2j)!}$$

$$\frac{2^{2j+1} x^{2j+2}}{(2j+2)!} \frac{(2j)!}{2^{2j-1} x^{2j}} = \frac{4x^2}{(2j+2)(2j+1)} \xrightarrow{j\to\infty} 0$$

So the series converges $\forall\, x$

Exercise 14. $\frac{1}{1-\frac{x}{2}} = \sum_{j=0}^{\infty} \left(\frac{x}{2}\right)^j \qquad \frac{1}{2-x} = \sum_{j=0}^{\infty} \frac{x^j}{2^{j+1}}$

$$\frac{x^{j+1}}{2^{j+2}} \frac{2^{j+1}}{x^j} = \frac{x}{2} < 1 \quad \implies \text{(the series converges for } |x| < 2\text{, by ratio test)}$$

If $x = 2$, the series would diverge

$$\text{If } x = -2 \ \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-2)^j}{2^j} = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j = 0 \text{ but } \frac{1}{2-(-2)} = \frac{1}{4}\text{x}$$

Exercise 15. $e^{-x^2} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!}$

$\frac{x^{2j+2}}{(j+1)!}\frac{j!}{x^{2j}} = \frac{x^2}{j+1} \xrightarrow{j\to\infty} 0$

Exercise 16. $\sin 3x = \sin 2x \cos x + \sin x \cos 2x = 3\sin x - 4\sin^3 x$.

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4} = \frac{3}{4}\left(\sum_{j=0}^{\infty} \frac{x^{2j+1}(-1)^j}{(2j+1)!} - \sum_{j=0}^{\infty} \frac{(3x)^{2j+1}(-1)^j}{(2j+1)!}\right) =$$

$$= \frac{3}{4}\left(\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}(1-3^{2j+1})}{(2j+1)!}\right) = \frac{3}{4}\left(\sum_{j=0}^{\infty} \frac{(-1)^{j+1}(3^{2j+1}-1)x^{2j+1}}{(2j+1)!}\right)$$

Exercise 17. $\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}(\log(1+x) - \log(1-x)) = \frac{1}{2}\left(\sum_{j=1} \frac{(+x)^j}{j}(-1)^{j-1} - \sum_{j=1} \frac{(x^j)(-1)^j}{j}\right)$

$$\ln(1+x) = \sum_{j=0}^{\infty} \frac{(-x)^{j+1}}{j+1}(-1) = \sum_{j=1}^{\infty} \frac{(-x)^j}{j}(-1) \qquad -\ln(1-x) = \sum_{j=0}^{\infty} \frac{x^{j+1}}{j+1} = \sum_{j=1}^{\infty} \frac{x^j}{j} \qquad \Longrightarrow -\sum_{j=1}^{\infty} \frac{((-1)^j+1)x^j}{j}$$

$$\Longrightarrow \sum^{\infty} \boxed{\frac{x^{2j+1}}{2j+1}}$$

$$\frac{x^{2j+3}}{2j+3}\frac{2j+1}{x^{2j+1}} \xrightarrow{j\to\infty} x^2$$

$$|x^2| < 1, \qquad \text{converges, with radius of convergence of } 1$$

Exercise 18. $\frac{3x}{1+x-2x^2} = \frac{1}{1-x} - \frac{1}{1+2x} = \sum_{j=0}^{\infty} \frac{x^j}{j} - \sum_{j=0}^{\infty} \frac{(-2x)^j}{j} = \sum_{j=0}^{\infty} \frac{x^j}{j}(1-(-2)^j)$

$$\frac{x^{j+1}|(1-(-2)^{j+1})|}{j+1}\frac{j}{|x^j(1-(-2)^j)|} = x\left|\frac{\left(\frac{1}{(-2)^j}+2\right)}{\left(\frac{1}{(-2)^j}-1\right)}\right| \xrightarrow{j\to\infty} 2x < 1$$

$$|x| < \frac{1}{2}$$

$$\text{For } x = \frac{1}{2}$$

$$\frac{x^{2j}}{2j}(1-(-2)^{2j}) + \frac{x^{2j+1}}{2j+1}(1-(-2)^{2j+1}) = \frac{x^{2j}((2j+1)(1-2^{2j}) + x(1+2^{2j+1})(2j))}{(2j)(2j+1)}$$

$$\frac{\left(\frac{1}{2}\right)^{2j}(2j+1-(2j+1)2^{2j}+2jx+2^{2j+2}jx)}{(2j)(2j+1)} \xrightarrow{j\to\infty} \frac{-(2j+1)+4j}{(2j)(2j+1)} = \frac{2j-1}{(2j)(2j+1)} \to 0$$

So $\frac{3x}{1+x-2x^2}$ converges for $|x| \leq \frac{1}{2}$

Exercise 19. $\frac{12-5x}{6-5x-x^2} = \sum_{j=0}^{\infty}\left(1+\frac{(-1)^j}{6^j}\right)x^j \quad (|x|<1)$.

$$\frac{12-5x}{6-5x-x^2} = \frac{5x-12}{(x+6)(x-1)} = \frac{1}{1-x} + \frac{6}{6+x} = \sum_{j=0}^{\infty} x^j + \sum_{j=0}^{\infty}\left(\frac{-x}{6}\right)^j = \sum_{j=0}^{\infty} x^j(1+\left(\frac{-1}{6}\right)^j)$$

$|x| < 1$ since for $x = 1$, $\lim_{j\to\infty}(1+\left(\frac{-1}{6}\right)^j) = 1$.

Exercise 20. $\frac{1}{x^2+x+1} = \frac{2}{\sqrt{3}}\sum_{j=0}^{\infty} \sin\frac{2\pi(j+1)}{3}x^j \quad (|x|<1)$

$$x^3 - 1 = (x-1)(x^2+x+1) \qquad \frac{1}{1-x^3} = \sum_{j=0}^{\infty}(x^3)^j$$

$$\frac{1}{x^2+x+1} = \frac{x-1}{x^3-1} = \frac{1-x}{1-x^3} \qquad \frac{1-x}{1-x^3} = \sum_{j=0}^{\infty}\left((x^3)^j - x^{3j+1}\right)$$

By induction, it could be observed that $x^{3j}, -x^{3j+1}, 0$ appear in sequences of 3 terms.

$\frac{2}{\sqrt{3}} \sin \frac{2\pi(j+1)}{3}$ accounts for this.

$\Longrightarrow \frac{2}{\sqrt{3}} \sum_{j=0}^{\infty} \sin \frac{2\pi(j+1)}{3} x^j \qquad |x| < 1$

$\sin \frac{2\pi(j+1)}{3} x^j < x^j$ (so by comparison test to $\sum x^j$, the radius of convergence is 1)

Exercise 21.

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j; \qquad \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{j=1}^{\infty} j x^{j-1} = \sum_{j=0}^{\infty} (j+1) x^j$$

$$\frac{1}{(1-x)(1-x^2)} = \frac{1/4}{1-x} + \frac{1/2}{(1-x)^2} + \frac{1/4}{1+x} = \frac{1}{4}\left(\sum_{j=0}^{\infty} (x^j + (-x)^j) + 2\sum_{j=1}^{\infty} j x^{j-1}\right)$$

$$\frac{x}{(1-x)(1-x^2)} = \frac{1}{4}\left(\sum_{j=0}^{\infty} x^{j+1}(1+(-1)^j) + 2\sum_{j=1}^{\infty} j x^j\right) = \frac{1}{4}\sum_{j=1}^{\infty} x^j (1 + (-1)^{j-1} + 2j)$$

Exercise 22.

$$\sin\left(2x + \frac{\pi}{4}\right) = \sum_{j=0}^{\infty} \frac{(2x)^{2j+1}}{(2j+1)!}(-1)^j; \qquad \cos 2x = \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!}(-1)^j$$

$$\sin 2x = \sum_{j=0}^{\infty} \frac{(2x)^{2j+1}}{(2j+1)!}(-1)^j$$

$$\sum_{j=0}^{\infty} a_j x^j = \frac{\sqrt{2}}{2}(\sin 2x + \cos 2x)$$

$$\cos 2x = \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!}(-1)^j$$

$$\text{For } j = 98, \quad a_j = \frac{2^{98}(-1)^{49}}{98!}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{-2^{98}}{98!}\left(\frac{\sqrt{2}}{2}\right)}$$

Exercise 23.

$$\begin{aligned}
f(x) &= (2+x^2)^{5/2} \\
f'(x) &= \frac{5}{2}(2+x^2)^{3/2}(2x) = 5x(2+x^2)^{3/2} \\
f''(x) &= 5(2+x^2)^{3/2} + \frac{15}{2}x(2+x^2)^{1/2}(2x) = 5(2+x^2)^{3/2} + 15x^2(2+x^2)^{1/2} \\
f'''(x) &= \frac{15}{2}(2+x^2)^{1/2}(2x) + 30x(2+x^2)^{1/2} + \frac{15x^2}{2}(2+x^2)^{-1/2}(2x) \\
f''''(x) &= 15(2+x^2)^{1/2} + \frac{15}{2}(2+x^2)^{-1/2}(2x)x + 30(2+x^2)^{1/2} + \frac{30x}{2}(2+x^2)^{1/2}(2x) + \\
&+ 45x^2(2+x^2)^{-1/2} + 15x^3\left(-\frac{1}{2}\right)(2+x^2)^{-3/2}(2x)
\end{aligned}$$

$$\boxed{2^{5/2} + 0x + \frac{5(2^{3/2})x^2}{2!} + \frac{0x^3}{3!} + \frac{45}{4!}\sqrt{2}x^4}$$

Exercise 24. $f(x) = e^{-1/x^2}$ if $x \neq 0$ and let $f(0) = 0$

(1)

$$f(x) = \sum_{j=0}^{\infty} \frac{\left(\frac{-1}{x^2}\right)^j}{j!} = 1 + \frac{-1}{x^2} + \sum_{j=2}^{\infty} \frac{\left(\frac{-1}{x^2}\right)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{-2j}}{j!}$$

$$f^{(k)} = \sum_{j=0}^{\infty} (-1)^j \frac{(-2j)(-2j-1)\dots(-2j-(k-1))}{j!} x^{-2j} = \sum_{j=0}^{\infty} (-1)^j \frac{(-2j)!}{(-2j-k)!j!} x^{-2j}$$

Use *ratio test* :

$$\frac{(-2(j+1))!}{(-2j-2-k)!(j+1)!}\frac{j!(-2j-k)!}{(-2j)!}\frac{x^{-2(j+1)}}{x^{-2j}} = \frac{(-2j-k)(-2j-k-1)}{(j+1)(-2j)(-2j-1)}x^{-2} \xrightarrow{j\to\infty} 0$$

Thus, by ratio test, every order of derivative exists for every x on the real line since the series representing the derivative converges for every x.

(2) $f(x) = \sum_{j=0}^{\infty} \frac{-x^{-2j}}{j!}$. There are no nonzero terms of positive power, i.e. no x^j; $\quad j \geq 1$.

$\implies f^{(j)}(0) = 0 \quad \forall\, j \geq 1$

11.16 Exercises - Power series and differential equations, binomial series.

Exercise 1. For $(1-x^2)y'' - 2xy' + 6y = 0$,

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1} \qquad f(0) = 1 \implies a_0 = 1 \qquad f'(0) = 0 \implies a_1 = 0$$

$$y'' = \sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} = \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j$$

$$2(1)a_2 + 3(2)a_3 x + -2(1)a_1 x + 6a_0 + 6a_1 x + \sum_{j=2}^{\infty}((j+2)(j+1)a_{j+2} - j(j-1)a_j - 2ja_j + 6a_j)x^j =$$

$$= 2a_2 + 6 + 6a_3 x + \sum_{j=2}^{\infty}((j+2)(j+1)a_{j+2} + a_j(j+3)(j-2))x^j$$

$$\implies \begin{aligned} a_2 &= -3 \\ a_3 &= 0 \end{aligned} \qquad \boxed{a_{j+2} = \frac{(j+3)(j-2)}{(j+2)(j+1)}a_j}$$

For $j = 2$, $\quad a_4 = 0$, so then $a_{j+2} = 0$ for $j = 2, 4, \dots$. Likewise, since $a_3 = 0$, then $a_{j+2} = 0$ for $j = 3, 5, \dots$.
$\implies f(x) = 1 - 3x^2$

Exercise 2. Using the work from above, then for $(1-x^2)y'' - 2xy' + 12y = 0$

$$f(0) = 0 \implies a_0 = 0 \qquad f'(0) = 2 \implies a_1 = 2$$

$$2(1)a_2 + 3(2)a_3 x + -2(1)a_1 x + 12a_0 + 12a_1 x + \sum_{j=2}^{\infty}((j+2)(j+1)a_{j+2} - j(j-1)a_j - 2ja_j + 12a_j)x^j =$$

$$= 2a_2 + 6a_3 x + -4x + 0 + 24x + \sum_{j=2}^{\infty}((j+2)(j+1)a_{j+2} - a_j(j+4)(j-3))x^j$$

$$\implies \begin{aligned} a_2 &= 0 \\ a_3 &= -10/3 \end{aligned} \qquad \boxed{a_{j+2} = \frac{(j+4)(j-3)}{(j+2)(j+1)}a_j}$$

For $j = 3$, $\quad a_5 = 0$, so then $a_{j+2} = 0$ for $j = 3, 5, \dots$. Likewise, since $a_2 = 0$, then $a_{j+2} = 0$ for $j = 2, 4, \dots$.
$\implies f(x) = -10/3x^3 + 2$

Exercise 3. $f(x) = \sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!}$; $\quad \frac{d^4 y}{dx^4} = y$

$$
\begin{aligned}
(x^4)' &= 4x^3 \\
(x^4)'' &= 12x^2 \\
(x^4)''' &= 24x \\
(x^4)'''' &= 24
\end{aligned}
\qquad
\begin{aligned}
\left(\frac{x^{4j}}{(4j)!}\right)' &= \frac{x^{4j-1}}{(4j-1)!} \\
\left(\frac{x^{4j}}{(4j)!}\right)'' &= \frac{x^{4j-2}}{(4j-2)!} \\
\left(\frac{x^{4j}}{(4j)!}\right)''' &= \frac{x^{4j-3}}{(4j-3)!} \\
\left(\frac{x^{4j}}{(4j)!}\right)'''' &= \frac{x^{4j-4}}{(4j-4)!}
\end{aligned}
\qquad
\sum_{j=1}^{\infty} \frac{x^{4j-4}}{(4j-4)!} = y'''' = \boxed{\sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!}} = f(x)
$$

To test convergence, use the ratio test

$$
\frac{x^{4j+4}}{(4j+4)!}\frac{(4j)!}{x^{4j}} = \frac{x^4}{(4j+4)(4j+3)(4j+2)(4j+1)} \xrightarrow{j\to\infty} 0 \quad \forall\, x \in \mathbb{R}
$$

So the series converges on $\mathbb{R}$.

Exercise 4. $f(x) = \sum_{j=0}^{\infty} \frac{x^j}{(j!)^2}$ $\qquad xy'' + y' - y = 0$

$$
\begin{aligned}
y' &= \sum_{j=1}^{\infty} \frac{jx^{j-1}}{(j!)^2} = \sum_{j=0}^{\infty} \frac{(j+1)x^j}{((j+1)!)^2} \\
y'' &= \sum_{j=1}^{\infty} \frac{(j+1)jx^{j-1}}{((j+1)!)^2}
\end{aligned}
$$

$$
\sum_{j=1}^{\infty} \left(\frac{(j+1)j}{((j+1)!)^2} + \frac{(j+1)}{((j+1)!)^2} - \frac{1}{(j!)^2} \right) x^j + \frac{1}{1!} - 1 = \sum_{j=1}^{\infty} \left(\frac{1}{(j!)^2} - \frac{1}{(j!)^2} \right) = 0
$$

Exercise 5. $f(x) = 1 + \sum_{j=1}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-2)}{(3j)!} x^{3j}; \quad y'' = x^a y + b \quad$ (Find a and b)

$$
\begin{aligned}
f' &= \sum_{j=1}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-2)}{(3j-1)!} x^{3j-1} \\
f'' &= \sum_{j=1}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-2)}{(3j-2)!} x^{3j-2} = \sum_{j=1}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-5)}{(3j-3)!} x^{3j-2} = \\
&= x + \sum_{j=2}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-5)}{(3j-3)!} x^{3j-2} = x + \sum_{j=1}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-2)}{(3j)!} x^{3j+1} \\
x^a f &= -x^a + \sum_{j=1}^{\infty} \frac{1\cdot 4\cdot 7 \ldots (3j-2)}{(3j)!} x^{3j+a}
\end{aligned}
$$

So then $\boxed{a = 1; \quad b = 0}$.

$$
\frac{1\cdot 4\cdot 7 \ldots (3j+1)}{(3j+3)!} x^{3j+3} \left(\frac{(3j)!}{1\cdot 4\cdot 7 \ldots (3j-2)} \right) \frac{1}{x^{3j}} = \frac{1(3j+1)}{(3j-2)(3j+3)(3j+2)(3j+1)} x^3 \xrightarrow{j\to\infty} 0
$$

So the series converges for all x.

Exercise 6. $f(x) = \sum_{j=0}^{\infty} \frac{x^{2j}}{j!}; \quad y' = 2xy.$

$$
f' = \sum_{j=1}^{\infty} \frac{2jx^{2j-1}}{j!} = 2\sum_{j=1}^{\infty} \frac{x^{2j-1}}{(j-1)!} = 2\sum_{j=0}^{\infty} \frac{x^{2j+1}}{j!} = 2xf
$$

$$
\frac{x^{2j+2}}{(j+1)!}\frac{j!}{x^{2j}} = \frac{x^2}{j+1} \xrightarrow{j\to\infty} 0 \quad \forall x
$$

By ratio test, f converges $\forall\, x \in \mathbb{R}$.

Exercise 7. $f(x) = \sum_{j=2}^{\infty} \frac{x^j}{j!} \quad y' = x + y$

$$f' = \sum_{j=2}^{\infty} \frac{x^{j-1}}{(j-1)!} = \sum_{j=1}^{\infty} \frac{x^j}{j!} = x + y$$

$$\frac{x^{j+1}}{(j+1)!}\frac{j!}{x^j} = \frac{x}{j} \xrightarrow{j\to\infty} 0$$

So the series converges $\forall\, x \in \mathbb{R}$ by ratio test.

Exercise 8.

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (kx)^{2j}}{(2j)!}$$

$$f' = \sum_{j=1}^{\infty} \frac{(-1)^j (kx)^{2j-1} k}{(2j-1)!} = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} (kx)^{2j+1} k}{(2j+1)!} \qquad f'' - k^2 f = 0$$

$$f'' = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} (kx)^{2j}}{(2j)!} k^2$$

$$\frac{(kx)^{2j+1}}{(2j+1)!}\frac{(2j)!}{(kx)^{2j}} = \frac{kx}{2k+1} \xrightarrow{j\to\infty} 0 \quad \text{by ratio test, } f \text{ converges } \forall x \in \mathbb{R}.$$

Exercise 9.

$$f'' = \sum_{j=1}^{\infty} \frac{(3x)^{2j-1}}{(2j-1)!} 9 = \sum_{j=0}^{\infty} \frac{9(3x)^{2j+1}}{(2j+1)!}$$

$$9(f - x) = 9(x + \sum_{j=0}^{\infty} \frac{(3x)^{2j+1}}{(2j+1)!} - x)$$

$$\frac{(3x)^{2j+3}}{(2j+3)!}\frac{(2j+1)!}{(3x)^{2j+1}} = \frac{9x^2}{(2j+3)(2j+2)} \xrightarrow{j\to\infty} 0$$

(by ratio test, f converges $\forall x \in \mathbb{R}$)

Exercise 10. $J_0(x) = \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j}}{(j!)^2 2^{2j}}$ $\qquad J_1(x) = \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j+1}}{j!(j+1)!2^{2j+1}}$.

(1)

$$\frac{x^{2j+2}}{((j+1)!)^2 2^{2j+2}}\frac{(j!)^2 2^{2j}}{x^{2j}} = \frac{x^2}{(j+1)^2 4} \xrightarrow{j\to\infty} 0 \qquad \text{by ratio test, } f \text{ converges } \forall x \in \mathbb{R}$$

$$\frac{x^{2j+3}}{(j+2)!2^{2j+3}}\frac{j!2^{2j+1}}{x^{2j+1}} = \frac{x^2}{(j+2)(j+1)4} \xrightarrow{j\to\infty} 0 \qquad \text{by ratio test, } f \text{ converges } \forall x \in \mathbb{R}$$

(2)

$$J_0'(x) = \sum_{j=1}^{\infty}(-1)^j \frac{x^{2j-1}}{(j-1)!(j!)2^{2j-1}} = \sum_{j=0}^{\infty}(-1)^{j+1}\frac{x^{2j+1}}{j!(j+1)!2^{2j+1}} = -J_1(x)$$

(3)

$$j_1(x) = xJ_1(x) = \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j+2}}{j!(j+1)!2^{2j+1}}$$

$$j_0(x) = xJ_0(x) = \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j+1}}{(j!)^2 2^{2j}}$$

$$j_1' = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(j!)^2 2^{2j}}$$

$$\Longrightarrow j_0 = j_1'$$

Exercise 11. $x^2 y'' + xy' + (x^2 - n^2)y = 0$.

$n = 0 \Longrightarrow x^2y'' + xy' + (x^2)y = 0$

$$J_0 = \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j}}{(j!)^2 2^{2j}} = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{x^{2j-2}}{((j-1)!)^2 2^{2j-2}};$$
$$J_0' = \sum_{j=1}^{\infty}(-1)^j \frac{x^{2j-1}}{j!(j-1)!2^{2j-1}};$$
$$J_0'' = \sum_{j=1}^{\infty}(-1)^j \frac{x^{2j-2}}{j!(j-1)!}\frac{(2j-1)}{2^{2j-1}}$$
$$\sum_{j=1}^{\infty}(-1)^j\left(\frac{(2j-1)}{j!(j-1)!2^{2j-1}} + \frac{1}{j!(j-1)!2^{2j-1}} + \frac{-2j}{((j-1)!)j!2^{2j-1}}\right) = 0$$

$n = 1 \Longrightarrow x^2y'' + xy' + (x^2-1)y = 0$

$$J_1(x) = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j+1}}{j!(j+1)!2^{2j+1}} = \frac{x}{2} + \sum_{j=1}^{\infty}\frac{(-1)^j x^{2j+1}}{j!(j+1)!2^{2j+1}} = \frac{x}{2} + \sum_{j=1}^{\infty}\frac{(-1)^{j-1}x^{2j-1}}{(j-1)!(j)!2^{2j-1}}$$
$$J_1' = \frac{1}{2} + \sum_{j=1}^{\infty}\frac{(-1)^j(2j+1)x^{2j}}{j!(j+1)!2^{2j+1}}$$
$$J_1'' = \sum_{j=1}^{\infty}\frac{(-1)^j(2j+1)(2j)x^{2j-1}}{(j!)(j+1)!2^{2j+1}}$$
$$x^2J_1'' + xJ_1' + (x^2-1)J_1 =$$
$$= \sum_{j=1}^{\infty}x^{2j+1}\left(\left(\frac{(-1)^j(2j+1)}{(j!)(j+1)!2^{2j+1}}\right)((2j)+(1)) + \frac{(-1)^j(-1)}{(j-1)!j!}\left(\frac{j+1}{j+1}\right)\left(\frac{j}{j}\right)\left(\frac{1}{2^{2j-1}}\right)\left(\frac{2^2}{2^2}\right) - \frac{(-1)^j}{(j+1)!(j!)2^{2j+1}}\right) +$$
$$+\frac{x}{2} - \frac{x}{2} =$$
$$= \sum_{j=1}^{\infty}\left(\frac{(-1)^j x^{2j-1}}{(j!)(j+1)!2^{2j+1}}\right)((2j+1)(2j+1) + (-1)(2j)(2j+2) - 1) = 0$$

Exercise 12. $y' = x^2 + y^2$; $\quad y = 1$ when $x = 0$.

$$y = a_0 + a_1 x + a_2 x^2 + \sum_{j=3}^{\infty} a_j x^j$$

$$y'(0) = 0 + 1^2 = 1 \qquad \begin{aligned} y^2 &= a_0^2 + a_1^2 x^2 + a_2^2 x^4 + \left(\sum_{j=3}^{\infty} a_j x^j \right)^2 + \\ &+ 2a_0 a_1 x + 2a_0 a_2 x^2 + 2a_0 \sum_{j=3}^{\infty} a_j x^j + \\ &+ 2a_1 a_2 x^3 + 2a_1 \sum_{j=3}^{\infty} a_j x^{j+1} + 2a_2 \sum_{j=3}^{\infty} a_j x^{j+2} \end{aligned}$$

$$y' = a_1 + 2a_2 x + \sum_{j=3}^{\infty} j a_j x^{j-1}$$

$\boxed{a_1 = 1}$ since $y'(0) = 1$

Consider the first few terms of $x^2 + y^2$

$$a_0^2 + 2a_0 a_1 x + a_1^2 x^2 + 2a_0 a_2 x^2 + x^2 = a_1 + 2a_2 x + 3a_3 x^2 \quad \Longrightarrow \quad \begin{aligned} a_1 = 1 = a_0^2 &\Longrightarrow \boxed{a_0 = 1} \\ 2a_2 = 2a_0 a_1 &\Longrightarrow \boxed{a_2 = 1} \\ 3a_3 = a_1^2 + 2a_0 a_2 + 1 = 4 &\Longrightarrow \boxed{a_3 = \frac{4}{3}} \end{aligned}$$

Exercise 13. $y' = 1 + xy^2$ with $y = 0$ when $x = 0$ $\quad \Longrightarrow a_0 = 0$

$$y = a_1 x + a_2 x^2 + a_3 x^3 + \sum_{j=4}^{\infty} a_j x^j$$

$$y = \sum_{j=1}^{\infty} a_j x^j$$

$$y = \sum_{j=1}^{\infty} j a_j x^{j-1} = \sum_{j=0}^{\infty} (j+1) a_{j+1} x 6j$$

$$\begin{aligned} y^2 &= a_1^2 x^2 + a_2^2 x^4 + a_3 x^6 + \left(\sum_{j=4}^{\infty} a_j x^j \right)^2 + \\ &+ 2a_1 a_2 x^3 + 2a_1 a_3 x^4 + 2a_1 \sum_{j=4}^{\infty} a_j x^{j+1} + \\ &+ 2a_2 a_3 x^5 + 2a_2 \sum_{j=4}^{\infty} a_j x^{j+2} + 2a_3 \sum_{j=4}^{\infty} a_j x^{j+3} \end{aligned}$$

$\boxed{a_1 = 1}$

$x: \quad 2a_2 = 0 \quad \Longrightarrow a_2 = 0 \qquad x^2: \quad 3a_3 = 0 \quad \Longrightarrow a_3 = 0$

$x^3: \quad 4a_4 = 1^2 \quad \Longrightarrow \boxed{a_4 = \frac{1}{4}}$

$x^4: \quad 5a_5 = 0 \quad \Longrightarrow a_5 = 0 \qquad x^5: \quad 6a_6 = 0 \quad \Longrightarrow a_6 = 0$

$x^6: \quad 7a_7 = 2a_1 a_4 + 2a_2 a_3 \quad \Longrightarrow \boxed{a_7 = \frac{1}{14}}$

$x^7: \quad 8a_8 = 0 + 2a_2 a_4 = 0 \quad \Longrightarrow a_8 = 0 \qquad x^8: \quad 9a_9 = 0 \quad \Longrightarrow a_9 = 0$

$x^9: \quad 10a_{10} = \left(\frac{1}{4}\right)^2 + 2(1)\frac{1}{14} \quad \Longrightarrow \boxed{a_{10} = \frac{23}{1120}}$

Exercise 14. $y' = x + y^2 \qquad y = 0$ when $x = 0 \quad \Longrightarrow a_0 = 0$

$y'(0) = 0 + 0 = 0 \quad \Longrightarrow a_1 = 0$

$$y = \sum_{j=2}^{\infty} a_j x^j \qquad y' = \sum_{j=2}^{\infty} j a_j x^{j-1} = \sum_{j=1}^{\infty} (j+1) a_{j+1} x^j$$

$$y^2 = \left(a_2 x^2 + a_3 x^3 + a_4 x^4 + \sum_{j=5}^{\infty} a_j x^j \right)^2 =$$

$$= a_2 x^4 + a_3^2 x^6 + a_4^2 x^8 + \left(\sum_{j=5}^{\infty} a_j x^j \right)^2 +$$

$$+ 2a_2 a_3 x^5 + 2a_2 a_4 x^6 + 2a_2 x^2 \sum_{j=5}^{\infty} a_j x^j + 2a_3 a_4 x^y + 2a_3 x^3 \sum_{j=5}^{\infty} a_j x^j + 2a_4 x^4 \sum_{j=5}^{\infty} a_j x^j$$

$$y' = x + y^2$$

$x : 2a + 2 = 1 + 0 \Longrightarrow \boxed{a_2 = \frac{1}{2}}$

$x^2 : 3a_3 = 0 \Longrightarrow a_3 = 0$

$x^3 : 4a_4 = 0 \Longrightarrow a_4 = 0$

$x^4 : 5a_5 = a_2^2 \Longrightarrow \boxed{a_5 = \frac{1}{20}}$

$x^5 : 6a_6 = 0 \Longrightarrow a_6 = 0 x^6 : 7a_7 = 0 \Longrightarrow a_7 = 0$

$x^7 : 8a_8 = 2(\frac{1}{2})(\frac{1}{20}) \Longrightarrow \boxed{a_8 = \frac{1}{160}}$

$x^8 : 9a_9 = 0 \Longrightarrow a_9 = 0$

$x^9 : 10a_{10} = 0 \Longrightarrow a_{10} = 0$

$x^{10} : 11a_{11} = 2(\frac{1}{2})(\frac{1}{160}) + \left(\frac{1}{20}\right)^2 \Longrightarrow \boxed{a_{11} = \frac{7}{8800}}$

Exercise 15. $y' = \alpha y$

$\Longrightarrow \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j = \alpha \sum_{j=0}^{\infty} a_j x^j$

$a_{j+1} = \frac{\alpha a_j}{(j+1)}$

$\boxed{a_j = \frac{\alpha^j}{j!} a_0}$ x (by induction)

Exercise 16. $y'' = xy$

$$y'' = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j =$$

$$= 2a_2 + \sum_{j=0}^{\infty} (j+3)(j+2) a_{j+3} x^{j+1} = \sum_{j=0}^{\infty} a_j x^{j+1}$$

$$\Longrightarrow a_2 = 0 \text{ and } a_{j+3} = \frac{a_j}{(j+3)(j+2)}$$

$$j = 0 \qquad a + 3 = \frac{a_0}{3 \cdot 2} \qquad j = 1 \qquad a_4 = \frac{a_1}{4 \cdot 3}$$

$$j = 3 \qquad a_6 = \frac{a_3}{6 \cdot 5} \qquad j = 4 \qquad a_7 = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

$$\boxed{a_{3j} = \frac{a_0}{(3j)!} \prod_{k=0}^{j-1} (3k+1)}; \qquad \boxed{a_{3j+1} = \frac{a_1}{(3j+1)!} \prod_{k=0}^{j-1} (3k+2)}$$

Exercise 17. $y'' + xy' + y = 0$

$$
\begin{aligned}
y &= \sum_{j=0}^{\infty} a_j x^j \\
y' &= \sum_{j=1}^{\infty} j a_j x^{j-1} \\
y'' &= \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} \\
&= \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j
\end{aligned}
$$

$$y'' + xy' + y = \sum_{j=1}^{\infty} x^j((j+2)(j+1)a_{j+2} + ja_j + a_j) = 0 \Longrightarrow a_{j+2} = \frac{-a_j}{(j+2)}$$

$$
\begin{aligned}
a_2 &= \frac{-a_0}{2} \qquad & a_3 &= \frac{-a_1}{3} \\
a_4 &= \frac{-a_2}{4} \qquad & a_5 &= \frac{-a_3}{5} = \frac{a_1}{15}
\end{aligned}
$$

$$\boxed{a_{2j} = \frac{(-1)^j a_0}{(2j)!!} \qquad a_{2j+1} = \frac{a_1}{(2j+1)!!}(-1)^j} \text{ (could be shown by induction)}$$

Exercise 18. Recall that

$$
\begin{aligned}
y &= \sum_{j=0}^{\infty} a_j x^j \\
y' &= \sum_{j=1}^{\infty} j a_j x^{j-1} \\
&= \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j
\end{aligned}
$$

Knowing this, we could *cleverly observe* that $e^{-2x} = \sum_{j=0}^{\infty}(2a_j + (j+1)a_{j+1})x^j$ is actually a *differential equation*!!!

$$\Longrightarrow e^{-2x} = y' + 2y$$

Solving this ODE using $y(x) = e^{-A(x)}\left(\int_a^x Q(t)e^{A(t)}dt + y(a)\right)$ where $A(x) = \int_a^x P(t)dt$,

$$\boxed{y = e^{-2x}(x+1)}$$

We had obtained the necessary initial conditions to solve this ODE from the information given, that $a_0 = 1$, so that $y(0) = 1$.
By doing some simple computation and comparison of powers with e^{-2x}, then $a_1 = 2$, $a_2 = -2$, $a_3 = 4/3$

Exercise 19. $\cos x = \sum_{j=0}^{\infty} a_j(j+2)x^j$ for $f(x) = \sum_{j=0}^{\infty} a_j x^j$.

Using $\cos x = \sum_{j=0}^{\infty} \frac{(x)^{2j}}{(2j)!}(-1)^j$ representation, we can immediately conclude that for odd terms, $a_{2j+1} = 0$ and by matching powers of x,

$$
\begin{aligned}
a_{2j}(2j+2) &= (-1)^j \frac{1}{(2j)!} \\
a_5 &= 0 \\
a_6(6+2) &= \frac{(-1)^3}{6!} \Longrightarrow \boxed{a_6 = \frac{-7}{8!}}
\end{aligned}
$$

Now notice that for $\cos x = \sum_{j=0}^{\infty} a_j(j+2)x^j = \sum_{j=1}^{\infty} ja_jx^j + 2\sum_{j=0}^{\infty} a_jx^j$ is actually a differential equation, $\cos x = xy' + 2y$. We can solve this first-order ODE using

$y(x) = e^{-A(x)}\left(\int_a^x Q(t)e^{A(t)}dt + y(a)\right)$ where $A(x) = \int_a^x P(t)dt$. Then solving $y' + \frac{2y}{x} = \frac{\cos x}{x}$,

$$y = \frac{1}{x^2}(x\sin x + \cos x - (a\sin a + \cos a) + b)$$

Plugging 0 as a good guess back into the ODE, $\cos 0 = 1 = y(0)(2) \implies y(0) = \frac{1}{2}$ With this initial condition, we get

$$\boxed{f(x) = \frac{\sin x}{x} + \frac{\cos x - 1}{x^2}} \text{ if } x \neq 0$$

So $f(0) = \frac{1}{2}$ and $f(\pi) = \frac{-2}{\pi^2}$

Exercise 20.

(1)

$$\begin{aligned}(1-x)^{-1/2} &= \sum_{j=0}^{\infty} \binom{-1/2}{j} (-x)^j = \\ &= 1 + \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}x^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}x^3 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)}{4!}x^4 + \\ &+ \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)\left(\frac{-9}{2}\right)}{5!}x^5 + \dots = \\ &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3 + \frac{35}{128}x^5 + \frac{63}{256}x^5 + \dots\end{aligned}$$

(2) To make the notation clear, $(1-x)^{-1/2} = \sum_{j=0}^{\infty} \binom{-1/2}{j}(-x)^j = \sum_{j=0}^{\infty} b_j x^j = \sum_{j=0}^{\infty} a_j$

Now

$$\frac{\binom{\alpha}{j+1}}{\binom{\alpha}{j}} = \frac{\alpha(\alpha-1)\dots(\alpha-(j+1)+1)}{(j+1)!}\frac{j!}{\alpha(\alpha-1)\dots(\alpha-j+1)} = \frac{(\alpha - j)}{(j+1)}$$

So for $\alpha = \frac{-1}{2}$,

$$\frac{a_{j+1}}{a_j} = -\left(\frac{1/2 + j}{j+1}\right)(-x) < x$$

Using this, we further find that

$$b_{j+1} < b_j \frac{1}{50}$$

$$b_{j+2} < b_{j+1}\frac{1}{50} < b_j \left(\frac{1}{50}\right)^2$$

For $x = \frac{1}{50}$. So by induction, $b_{n+j} < b_n \left(\frac{1}{50}\right)^j$

$$r_n = \sum_{j=1}^{\infty} a_{n+j} < \sum_{j=1}^{\infty} a_n \left(\frac{1}{50}\right)^j = a_n \frac{1/50}{1 - 1/50} = \frac{a_n}{49}$$

$$\boxed{r_n < \frac{a_n}{49}}$$

(3) Note that $(1-x)^{-1/2} = \left(1 - \frac{1}{50}\right)^{-1/2} = \left(\frac{49}{50}\right)^{-1/2} = \frac{5\sqrt{2}}{7}$

$$\frac{7}{5}\left(1 - \frac{1}{50}\right)^{-1/2} = 1 + \frac{1}{100} + \frac{3}{2}\left(\frac{1}{2(50)}\right)^2 + \frac{5}{2}\left(\frac{1}{2(50)}\right)^3 + \frac{35}{8}\left(\frac{1}{2(50)}\right)^4 + \frac{63}{8}\left(\frac{1}{2(50)}\right)^5$$

$$\boxed{\sqrt{2} \simeq 1.4142135624}$$

Exercise 21.

(1) $\frac{1732}{1000}\left(1 - \frac{176}{3000000}\right)^{-1/2} = \frac{1732}{1000}\left(\frac{3000000}{2999824}\right)^{1/2}$
Obviously, $(3000000)^{1/2} = 1000\sqrt{3}$ so that we have $1732\,(3/2999824)^{1/2}$.
With long multiplication, we could show easily that $1732 * 1732 = 2999824$ (it's harder to divide). So then

$$\boxed{\frac{1732}{1000}\left(1 - \frac{176}{3000000}\right)^{-1/2} = \sqrt{3}}$$

(2)

Exercise 22. $\arcsin x = \int \frac{1}{\sqrt{1-x^2}}$

$(1-x^2)^{-1/2} = \sum_{j=0}^{\infty} \binom{\alpha}{j}(-x^2)^j = 1 + \sum_{j=1}^{\infty} \binom{\alpha}{j}(-x^2)^j$

$$\Longrightarrow \arcsin x = x + \sum_{j=1}^{\infty} \binom{-1/2}{j} \frac{(-1)^j}{(2j+1)} x^{2j+1}$$

$$\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\dots\left(\frac{-1}{2}-j+1\right)}{j(j-1)\dots(2)(1)} = (-1)^j \frac{(1)(3)\dots(1+2j-2)}{(2j)!!} = (-1)^j \frac{(2j-1)!!}{(2j)!!}$$

$$\Longrightarrow \boxed{\arcsin x = x + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{(2j)!!} \frac{x^{2j+1}}{2j+1}}$$

13.21 Exercises - The conic sections, Eccentricity of conic sections, Polar equations for conic sections.

Exercise 1. F is in the positive half-plane determined by N.

$$\begin{aligned} \|X-F\| &= ed(X,L) \\ \|X-F\| &= e|(X-F)\cdot N + d| \end{aligned}$$

Exercise 2.

(1)

$$\begin{aligned} \|X-F\| &= ed(X,L) \\ \|X-F\| &= e|(X-F)\cdot N + d| \\ F = 0 \Longrightarrow \|X\| &= e|(X\cdot N)+d|; \quad r = e|r\cos\theta + d| \\ \Longrightarrow r = e(r\cos\theta + d) &\Longrightarrow r = \frac{ed}{1-e\cos\theta} \end{aligned}$$

(2) The right branch for the hyperbola is given by $r = \frac{ed}{1-e\cos\theta}$ because $X\cdot N > 0$. The left branch for $e > 1$,

$$\begin{aligned} \|X-F\| = ed(X,L) = e|(X-F)\cdot N + d| = \\ = e|X\cdot N + d| = -e(d + r\cos\theta) = r \\ r = \frac{-ed}{(1+e\cos\theta)} \end{aligned}$$

Exercise 3. For points below the horizontal directrix,

$$\|X-F\| = ed(X,L)$$

$$F = 0 \Longrightarrow \|X\| = ed(X,L) = e(|(X-F)\cdot N - d|) = e|X\cdot N - d| = e|r\sin\theta - d|$$

$$\text{Now Thm. 13.18 says } r = \frac{ed}{e\cos\theta + 1} \quad \text{if } 0 < e \leq 1$$

$$\Longrightarrow r = e(d - r\sin\theta) \Longrightarrow r = \frac{ed}{1+e\sin\theta}$$

For the "right" or upper-half branch of a hyperbola.

$$\|X-F\| = e(|(X-F)\cdot N - d|) = e(r\sin\theta - d) = r$$

$$r = \frac{-ed}{1-e\sin\theta} = \frac{ed}{e\sin\theta - 1}$$

Exercise 4. $\|X-F\| = ed(X,L)$; $\|X\| = e|(X-F)\cdot N - d| = e|r\cos\theta - d| = e(d - r\cos\theta) \quad \Longrightarrow r = \frac{ed}{1+e\cos\theta}$

$\boxed{e = 1, d = 2}$.

Exercise 5. $r = \frac{3}{1+\frac{1}{2}\cos\theta} = \frac{6\left(\frac{1}{2}\right)}{1+\frac{1}{2}\cos\theta}$. $e = \frac{1}{2}$; $d = 6$

Exercise 6. $r = \frac{6}{3+\cos\theta} = \frac{2}{1+\frac{1}{3}\cos\theta}$. $e = \frac{1}{3}$; $d = 6$.

Exercise 7. $r = \frac{1}{\frac{-1}{2}+\cos\theta}$.

$$ed(X,L) = \|X-F\| = e|(X-F)\cdot N - d| = e|r\cos\theta - d| = er\cos\theta - ed = r$$
$$\frac{-ed}{1-e\cos\theta} = r = \frac{ed}{e\cos\theta - 1}$$

So for $r = \frac{2}{2\cos\theta - 1}$, $\quad e = 2, d = 1$.

Exercise 8. $r = \frac{4}{1+2\cos\theta} \quad e = 2,\ d = 2$.

Exercise 9. $r = \frac{4}{1+\cos\theta} \quad e = 1\, d = 4$.

Exercise 10. $3x + 4y = 25 \Longrightarrow \frac{3}{5}x + \frac{4}{5}y = 5$. $N = \left(\frac{3}{5}, \frac{4}{5}\right)$.

$L = \{x = P + tA\}, \quad N\cdot X = N\cdot P$.

To find the distance from the focus, at the origin, to the directrix,

$$dN = P + tA; \quad dN\cdot N = d = N\cdot P$$

So for this problem, $d = 5$.

$$r = \|X-F\| = ed(X,L) = e|(X-F)\cdot N - d| = e|X\cdot N - d| = \left|\frac{3}{5}r\cos\theta + \frac{4}{5}r\sin\theta - 5\right|$$
$$r = \frac{1}{2}\left(5 - \frac{3}{5}r\cos\theta - \frac{4}{5}r\sin\theta\right)$$
$$r = \frac{5/2}{1 + \frac{3}{10}\cos\theta + \frac{4}{10}\sin\theta}$$

Exercise 11. $e = 1, \quad 4x + 3y = 25 \quad \frac{4}{5}x + \frac{3}{5}y = 5; \quad N = \left(\frac{4}{5}, \frac{3}{5}\right)$.

$d = 5$.

$$\|X-F\| = ed(X,L) = e|(X-F)\cdot N - d| = e\left|\frac{4}{5}r\cos\theta + \frac{3}{5}r\sin\theta - 5\right|$$
$$r = 5 - r\left(\frac{4}{5}\cos\theta + \frac{3}{5}\sin\theta\right)$$
$$r = \frac{5}{1 + \frac{4}{5}\cos\theta + \frac{3}{5}\sin\theta}$$

Exercise 12. $e = 2$, hyperbola, so there's 2 branches.

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \frac{1}{\sqrt{2}} \quad L = \{x = P + tA\} \quad X\cdot N = N\cdot P$$
$$dN = P + tA; \quad dN\cdot N = d = N\cdot P = \frac{1}{\sqrt{2}}$$

Note that the sign of d here tells you what side the focus, at the origin, lies on.

$$\|X-F\| = ed(X,L) = \|X\| = e|(X-F)\cdot N - d| = e(d - \frac{1}{\sqrt{2}}r\cos\theta - \frac{1}{\sqrt{2}}r\sin\theta)$$
$$r = \frac{2/\sqrt{2}}{1 + \frac{2}{\sqrt{2}}\cos\theta + \frac{2}{\sqrt{2}}\sin\theta}$$

But for the right side branch,

$$\|X-F\| = ed(X,L) = \|X\| = e|(X-F)\cdot N - d| = -e(d - \frac{1}{\sqrt{2}}r\cos\theta - \frac{1}{\sqrt{2}}r\sin\theta)$$
$$r = \frac{-2/\sqrt{2}}{1 - \frac{2}{\sqrt{2}}\cos\theta - \frac{2}{\sqrt{2}}\sin\theta}$$

Exercise 13. $e = 1$ parabola.

(1)

$$\|X-F\| = \|X\| = ed(X,L) = 1|(X-F)\cdot N - d| = d - X\cdot N = d - r\cos\frac{\pi}{3}$$

$$d = r\left(\frac{3}{2}\right) = \frac{3}{2}\times 10^8 mi$$

$$\boxed{r = \frac{\frac{3}{2}\times 10^8 mi}{1+\cos\theta}} \qquad \theta = 0, \quad r = \frac{3}{4}\times 10^6 mi$$

(2) Focus is in the positive half-plane determined by N.

$$\|X-F\| = \|X\| = ed(X,L) = |(X-F)\cdot N + d| = r\cos\theta + d$$

$$d = r(1-\cos\theta) = 10^8 mi(1-\cos\frac{\theta}{3}) = \frac{1}{2}\times 10^8 mi$$

$$\boxed{r = \frac{d}{1-\cos\theta} = \frac{\frac{1}{2}\times 10^8 mi}{1-\cos\theta}} \quad \boxed{r(\theta=\pi) = \frac{1}{4}\times 10^8 mi}$$

13.24 Exercises - Conic sections symmetric about the origin, Cartesian equations for the conic sections.
Quick Review.

Consider symmetry about the origin.

$$\|X-F\| = ed(X,L) = e|(X-F)\cdot N - d| = e|X\cdot N - F\cdot N - d| = |eX\cdot N - e(F\cdot N + d)|$$

$$\|X-F\|^2 = \|X\|^2 - 2X\cdot F + \|F\|^2 = e^2(X\cdot N)^2 - 2aeX\cdot N + a^2$$

$$X\to -X; \qquad X\cdot F = aeX\cdot N$$

$$X = (F - aeN) = 0 \Longrightarrow F = aeN; \quad F\cdot N = ae;\ a = \frac{ed}{1-e^2} \quad F = \frac{e^2 d}{1-e^2}N$$

$$\Longrightarrow \|X\|^2 + (ae)^2 = e^2(X\cdot N)^2 + a^2$$

if $X = \pm aN$; $\quad \|X\|^2 + (ae)^2 = e^2(X\cdot N)^2 + a^2$ is satisfied

if $X = \pm bN'$; $\quad b^2 + (ae)^2 = e^2(0) + a^2 \quad b^2 = a^2(1-e^2)$

Exercise 1. $b^2 = a^2(1-e^2) \quad \frac{x^2}{100} + \frac{y^2}{36} = 1$

$\sqrt{1-\frac{b^2}{a^2}} = e \Longrightarrow \boxed{e = \frac{4}{5}}$.

$|F| = |aeN| = 10\left(\frac{4}{5}\right) = 8$. $f = (\pm 8, 0)$. $(0,0)$ center. Vertices $(\pm 0, 6)$.

Exercise 2. $\frac{y^2}{100} + \frac{x^2}{36} = 1$. $\frac{4}{5} = e$; $\quad f = (0, \pm 8)$.

$(0,0)$ center; vertices $(\pm 6, 0), (0, \pm 10)$.

Exercise 3. $\frac{(x-2)^2}{16} + \frac{(y-3)^2}{9} = 1$. Center $(2,-3)$. $|F| = ae = 4\frac{\sqrt{7}}{4} = \sqrt{7}$.

$\sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4} = e$; $\quad (2+\sqrt{7}, -3)$, $(2-\sqrt{7},-3)$ foci.
Vertices $(6,-3), (-2,-3), (2,6), (2,-12)$.

Exercise 4. $\frac{x^2}{\left(\frac{25}{9}\right)} + y^2 = 1$. Center $x = (0,0)$. $|F| = ae = \left(\frac{5}{3}\right)\left(\frac{4}{5}\right) = \frac{4}{3}$.

$e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$. Foci: $(\pm\frac{4}{3}, 0)$. Vertices $(\pm\frac{5}{3}, 0), (0, \pm 1)$.

Exercise 5. $\frac{y^2}{(1/4)} + \frac{x^2}{(1/3)} = 1$

$|F| = ae = \frac{1}{\sqrt{3}}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{3}}$.

$\sqrt{1-\frac{1/4}{1/3}} = \frac{1}{2} = e$. Foci: $\left(\frac{\pm 1}{2\sqrt{3}}, 0\right)$. Center $(0,0)$.

Vertices $(\pm 1/\sqrt{3}, 0), (0, \pm 1/2)$.

Exercise 6. Center $(-1,-2)$.

$\sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{16}{25}} = \frac{3}{5} = e; \quad |F| = ae = 5\frac{3}{5} = 3.$

Foci: $(-1,-1)$, $(-1,-5)$.

Vertices: $(-1,3),(-1,-7),(3,-2),(-5,-2)$.

Exercise 7. $F = ae = \frac{3}{4}$. $a = 1$, $e = \frac{3}{4}$. $b^2 = a^2(1-e^2); \quad b^2 = 1\left(\frac{1}{4}\right)$.

$\boxed{x^2 + 4y^2 = 1}$.

Exercise 8. $2a = 4$. $a^2 = 4$. $2b = 3$. $b^2 = 9/4$. $\quad \implies \frac{(x+3)^2}{4} + \frac{(y-4)^2}{4} = 1$

Exercise 9. $\frac{(x+3)^2}{9/4} + \frac{(y-4)^2}{4} = 1$.

Exercise 10. $2a = 6$, $a = 3$. $\frac{(x+4)^2}{9} + \frac{(y-2)^2}{1} = 1$.

Exercise 11. $2a = 10$, $a = 5$. $|F| = ae = 5e = 4\, e = 4/5$. $b^2 = a^2(1-e^2) = 25\left(1-\frac{16}{25}\right) = 9$.

Exercise 12. $\frac{(x-2)^2}{a^2} + \frac{(y-1)^2}{b^2} = 1;$

$a = 4$ from $(6,1)$. $b = 2$ from $(2,3)$. $\implies \frac{(x-2)^2}{4^2} + \frac{(y-1)^2}{4} = 1$

Exercise 13. $b^2 = a^2(1-e^2)$.

$\frac{x^2}{100} - \frac{y^2}{64} = 1; \quad b^2 = 100(1-e^2) = -64. \quad 1 + \frac{64}{100} = e^2.$
Center $(0,0)$. $\quad e = \frac{2\sqrt{41}}{10} = \frac{\sqrt{41}}{5}$.

Vertices; $(\pm 10, 0)$. $F = ae = 2\sqrt{41}$. Foci: $(\pm 2\sqrt{41}, 0)$.

$\frac{x^2}{100} = \frac{y^2}{64} + 1 \xrightarrow{x,y\to\infty}; \quad y = \frac{\pm 4}{5}x$

Exercise 14. $\frac{y^2}{100} - \frac{x^2}{64} = 1$; Center $(0,0)$, $a^2 = 100; \quad b^2 = -64$.

$b^2 = a^2(1-e^2). \quad e = \frac{\sqrt{41}}{5}$. Vertices $(0,\pm 10)$. $F = ae = (0, \pm 2\sqrt{41})$.
$\frac{x^2}{64} + 1 = \frac{y^2}{100} \xrightarrow{x,y\to\infty} \frac{\pm 5}{4}x = y$.

Exercise 15. $\frac{(x+3)^2}{4} - (y-3)^2 = 1$.

Center $(-3,3)$. $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{-1}{4}} = \frac{\sqrt{5}}{2}$.

Foci: $ae = 2\frac{\sqrt{5}}{2} = \sqrt{5}$. $(-3+\sqrt{5},3),(-3-\sqrt{5},3)$.

Vertices: $(-3,4),(-3,2); \quad (1,3),(-7,3)$.

$\frac{(x+3)^2}{4} = 1 + (y-3)^2 \xrightarrow{x,y\to\infty} \boxed{\frac{\pm(x+3)}{2} = y - 3}$

Exercise 16. $\frac{x^2}{144/9} - \frac{y^2}{144/16} = 1 = \frac{x^2}{16} - \frac{y^2}{9}$.

$e = \sqrt{1-\frac{-9}{16}} = \frac{5}{4}$. Center $(0,0)$. $|F| = ae = 5$. Foci: $(5,0),(-5,0)$. Vertices $(\pm 4, 0)$.

Exercise 17. $20 = 5y^2 - 4x^2$. Center $(0,0)$. $|F| = ae = 2\left(\frac{3}{2}\right) = 3$. Foci: $(0,\pm 3)$. $1 = \frac{y^2}{4} - \frac{x^2}{5}$. $e = \sqrt{1-\frac{-5}{4}} = \frac{3}{2}$.

Vertices: $(0,\pm 2)$

Exercise 18. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} = 1$.

Center $(1,-2)$. $e = \sqrt{1-\frac{-9}{4}} = \frac{\sqrt{13}}{2}; \quad |F| = 2\frac{\sqrt{13}}{2} = \sqrt{13}$. Foci: $(1+\sqrt{13},-2),(1-\sqrt{13},-2)$.

Vertices: $(5,-2),(-3,-2)$.

Exercise 19. $F = ae = 2(2) = 4$.

$\frac{x^2}{4} + \frac{y^2}{-12} = 1. \quad \frac{y^2}{12} + 1 = \frac{x^2}{4} \xrightarrow{x,y\to\infty} y = \pm\sqrt{3}x$.

$b^2 = a^2(1-e^2) = 4(1-4) = -12.$

Exercise 20. $F = ae = \sqrt{2} = (1)e. \quad b^2 = a^2(1-e^2) = 1(1-2) = -1. \implies y^2 - x^2 = 1.$

Exercise 21. $\frac{x^2}{4} - \frac{y^2}{16} = 1$

Exercise 22. $(y-4)^2 - \frac{(x+1)^2}{-3} = 1$ where

$F = ae = |-2| = ae.$ $b^2 = a^2(1-e^2) = 1(1-4) = -3$

Exercise 23. $\pm\frac{(x-2)^2}{a^2} \mp \frac{(y+3)^2}{b^2} = 1$

$$\begin{array}{l} (3,-1) \implies \pm\dfrac{1}{a^2} \mp \dfrac{4}{b^2} = 1 \\ (-1,0) \implies \pm\dfrac{9}{a^2} \mp \dfrac{9}{b^2} = 1 \end{array} \implies \frac{(y+3)^2}{27/8} - \frac{(x-2)^2}{(27/5)} = 1$$

Exercise 24. $\frac{x^2-1}{3} = y^2.$ $\frac{2x}{3} = 2yy'. \quad yy' = \frac{x}{3}.$

$3x - 2y = C.$ $m = \frac{3}{2} \quad \implies y_0\frac{9}{2} = x_0.$ $\frac{81}{4}y_0^2 - 1 = 3y_0^2 \implies y_0 = \frac{\pm 2}{\sqrt{69}}.$

The asymptotes of $y^2 = \frac{x^2-1}{3}$ are $y = \pm\frac{x}{\sqrt{3}}$.

$$3\left(\frac{\pm 9}{\sqrt{69}}\right) - 2\left(\pm\frac{2}{\sqrt{69}}\right) = \frac{\pm 23}{\sqrt{69}} = C$$

$$\boxed{3x \pm \sqrt{\frac{23}{3}} = 2y}$$

Exercise 25. $\frac{\pm x^2}{a^2} + \mp\frac{y^2}{b^2} = 1 \quad \frac{\pm x^2}{a^2} \mp \frac{y^2}{4a^2} = 1.$

$(3,-5) \to \pm 9 \mp \frac{25}{4} = a^2; \quad a^2 = \frac{11}{4}.$

$\implies \boxed{\dfrac{x^2}{11/4} - \dfrac{y^2}{11} = 1}.$

Quick Review of Parabolas.

F on positive half plane to N.

$\|X - F\| = e|(X-F)\cdot N + d|$

Let $N = \vec{e}_x$; $d = 2c$; $F = (c,0)$; $e = 1$.

$$(x-c)^2 + y^2 = e^2((x-c)+2c)^2 = (x-c)^2 + 4c(x-c) + 4c^2$$
$$y^2 = 4cx$$

Thus, for ellipses, the vertex is equidistant to the focus and directrix (confirming the other definition).

Let $N = \vec{e}_y$, $d = 2c$; $\quad F = (0,c)$, $e = 1$.

$$x^2 + (y-c)^2 = ((y-c)+2c)^2 = (y-c)^2 + 4c(y-c) + 4c^2$$
$$x^2 = 4cy$$

Exercise 26. $4c = -8 \quad (0,0)$ vertex. $y = 0$ symmetry axis. $x = 5$ directrix.

Exercise 27. $4c = 3$. Vertex: $(0,0)$. Symmetry axis: $y = 0$. Directrix: $x = -3/4$.

Exercise 28. $(y-1)^2 = 12(x - \frac{1}{2})$. $4c = 12$, $c = 3$. Symmetry axis: $y = 1$. Directrix: $\left(\frac{-5}{2}, 1\right)$.

Exercise 29. $x^2/6 = y$. $4c = \frac{1}{6} \quad c = \frac{1}{24}$. Vertex: $(0,0)$. Directrix: $y = -\frac{1}{24}$. Symmetry axis: $x = 0$.

Exercise 30. $x^2 + 8y = 0. \quad 4c = \frac{-1}{8}; \quad c = \frac{-1}{32}.$ $y = \frac{1}{32}$ directrix; $x = 0$ axis.

Exercise 31. $(x+2)^2 = 4(y + \frac{9}{4})$. $4c = 4; \quad c = 1$. Center $(-2, -9/4)$. Directrix: $y = -13/4$. Axis: $x = -2$.

Exercise 32. $y = -x^2$.

Exercise 33. $x^2 = 8y$.

Exercise 34. $(y-3) = -8(x+4)^2$.

Exercise 35. $c = \frac{5}{4} \quad 5(x - \frac{7}{4}) = (y+1)^2$

Exercise 36. $y = ax^2 + bx + c$

$$(0,1) \to c = 1 \quad (1,0) \to 0 = a + b + 1(2,0) \to 0 = 4a + 2b + 1 \qquad a = \frac{1}{2} \qquad \Longrightarrow y = \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

Exercise 37. $4c(x-1) = (y-3)^2$. $4c(-2) = (-4)^2 = 16$. $c = -2$. $\quad -8(x-1) = (y-3)^2$.

Exercise 38. $\|X - F\| = ed(X, L) = |(X - F) \cdot N - d|$

$L = \{(x,y) | 2x + y = 10; \frac{2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} = \frac{10}{\sqrt{5}}\}$.
$d = N = x_L \quad dN \cdot N = d = x_L \cdot N = \frac{10}{\sqrt{5}}$.

$$F = 0 \Longrightarrow \|X\|^2 = |X \cdot N - d|^2 = \left(\frac{-2}{\sqrt{5}}x + -\frac{y}{\sqrt{5}} + \frac{10}{\sqrt{5}} \right)^2 = x^2 + y^2$$
$$5x^2 + 5y^2 = (-2x - y + 10)^2 = 4x^2 + y^2 + 100 + 4xy - 40x - 20y$$
$$\Longrightarrow x^2 + 4y^2 - 4xy + 40x + 20y - 100 = 0$$

13.25 Miscellaneous exercises on conic sections.

Exercise 1.

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad y^2 = b^2 - \left(\frac{bx}{a} \right)^2 = b^2 \left(1 - \left(\frac{x}{a} \right)^2 \right)$$
$$y = 2 \int_{-a}^{a} b\sqrt{1 - \left(\frac{x}{a} \right)^2} dx = 2 \int_{-1}^{1} ab\sqrt{1 - x^2} dx = (ab) \text{ area of a circle of radius 1}$$

Exercise 2.

(1) Without loss of generality, let the major axis be $2a$ in the x-axis. $y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$

$$V = \int_{-a}^{a} \pi b^2 \left(1 - \frac{x^2}{a^2} \right) dx = \pi b^2 a \int_{-1}^{1} (1 - x^2) dx = \frac{4}{3}\pi (1)^3 b^2 a$$

(2) If rotated about the minor axis, suppose, without loss of generality, $2a$ is the minor axis (just note that $\frac{x^2}{a^2} + \frac{b^2}{a^2} = 1$ have x, y, a, b as dummy labels).
$\Longrightarrow V = \frac{4}{3}\pi(1)^3 b^2 a$, where $2a$ is the minor axis, $2b$ is the major axis.

Exercise 3. $\frac{x^2}{(3/A)} + \frac{y^2}{(3/B)} = 1 \quad By^2 = 3 - Ax^2 \quad \Longrightarrow y^2 = \frac{3}{B} - \frac{Ax^2}{B}; \quad y = \sqrt{\frac{3}{B} - \frac{Ax^2}{B}}$. So the area inside this ellipse is

$$2\sqrt{\frac{1}{B}} \int_{-\sqrt{3/A}}^{\sqrt{3/A}} \sqrt{3 - Ax^2} dx = 2\sqrt{\frac{3}{B}} \int_{-\sqrt{3/A}}^{\sqrt{3/A}} \sqrt{1 - \frac{x^2}{\left(\frac{3}{A}\right)}}$$

For the other ellipse equation, $\frac{x^2}{3/(A+B)} + \frac{y^2}{3/(A-B)} = 1. \quad y^2 = \left(\frac{3}{A-B} \right) \left(1 - \frac{x^2}{(3/(A+B))} \right); \quad y = \sqrt{\frac{3}{A-B}} \sqrt{1 - \frac{x^2}{(3/(A+B))}}$.
Thus, the area inside this ellipse is

$$2\sqrt{\frac{3}{A-B}} \int_{-\sqrt{\frac{3}{A+B}}}^{\sqrt{\frac{3}{A+B}}} \sqrt{1 - \left(\frac{x}{\sqrt{\frac{3}{A+B}}} \right)^2}$$

Equating the two areas after making an appropriate scale change,

$$2\sqrt{\frac{3}{B}}\sqrt{\frac{3}{A}} \int_{-1}^{1} \sqrt{1 - x^2} dx = 2\sqrt{\frac{3}{A-B}}\sqrt{\frac{3}{A+B}} \int_{-1}^{1} \sqrt{1 - x^2} dx$$

Thus $A^2 - B^2 = AB \implies A^2 - BA - B^2$. Simply try treating B as a number and solve the quadratic equation in terms of A.

$$A = \frac{B \pm \sqrt{B^2 - 4(1)(-B^2)}}{2(1)} = \frac{B \pm B\sqrt{5}}{2} = \frac{B(1+\sqrt{5})}{2}$$

Exercise 4. $y = -\frac{4h}{b^2}x^2$.

$$\int_{-b/2}^{b/2} \left(\frac{4h}{-b^2}x^2 + h \right) = \left. \frac{4h}{-3b^2}x^3 \right|_{-b/2}^{b/2} + h\left(\frac{b}{2} + \frac{b}{2}\right) = \frac{2hb}{3}$$

Exercise 5. $y^2 = 8x$. $\int_0^2 \pi 8tdt = 4\pi(2)^2 = 16pi$

Exercise 6. $y^2 = 2(x-1)$. $y^2 = 4(x-2)$.

(1)

$$\begin{aligned} A &= 2\int_1^2 \sqrt{2(x-1)} + 2\int_2^3 \sqrt{2(x-1)} - 2\sqrt{x-2} = \\ &= 2\sqrt{2}\,\frac{2}{3}(x-1)^{3/2}\Big|_1^2 + 2\sqrt{2}\frac{2}{3}\,(x-1)^{3/2}\Big|_2^3 - 4\frac{2}{3}\,(x-2)^{3/2}\Big|_2^3 \\ &= 2\sqrt{2}\frac{2}{3} + \sqrt{2}\frac{4}{3}(2)^{3/2} - 2\sqrt{2}\frac{2}{3} - 4\frac{2}{3} = 8/3 \end{aligned}$$

(2)

$$\int_1^2 2(x-1) = 2\left(\frac{1}{2}x^2 - x\right)\Big|_1^2 = 2\left(\frac{1}{2}(4-1) - (2-1)\right) = 1$$

$$\int_2^3 \left(2(x-1) - 4(x-2)\right)dx = \int_2^3 (-2x+6)dx = -x^2\big|_2^3 + 6x\big|_2^3 = (-9+4+6(3-2)) = 1$$

$$\implies V = \pi\int_1^2 2(x-1) + \pi\int_2^3 (2(x-1) - 4(x-2)) = 2\pi$$

(3) $\frac{y^2}{2} + 1 = x, \quad \frac{y^2}{4} + 2 = x$

$$\begin{aligned} 2\pi\int_0^2 \left(\left(\frac{y^2}{4} + 2\right)^2 - \left(\frac{y^2}{2}+1\right)^2 \right) &= 2\pi\int_0^2 \left(\frac{-3y^4}{16} + 3\right)dy = 2\pi\left(\frac{-3}{80}y^5 + 3y\right)\Big|_0^2 = \\ &= 2\pi\left(\frac{-3(32)}{80} + 6\right) = 2\pi\left(\frac{-96+480}{80}\right) = 2\pi\left(\frac{384}{80}\right) = \pi\frac{48}{5} \end{aligned}$$

Exercise 7. By Apostol's definition of conic sections, we are basically given the conic section definition with $e = \frac{1}{2}$. So just plug in the pt. $(0,4)$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \xrightarrow{(0,4)} b = 4 \qquad b^2 = a^2(1-e^2) = 16 = a^2\left(1 - \left(\frac{1}{2}\right)^2\right)$$

$$\frac{x^2}{64/3} + \frac{y^2}{16} = 1$$

Exercise 8. $F = 0 \quad \|X - F\| = \|X\| = ed(X,F) = |X \cdot N + d| = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}$ because for the directrix

$$y + x = -1$$

$$N = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \frac{1}{\sqrt{3}}y + \frac{x}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$X_L = P + tA$$

$$X_L \cdot NN \cdot P = -1/\sqrt{2}^{dN} = X_L \quad X_L \cdot N = d = -1/\sqrt{2}$$

So by squaring both sides of the vector equation,

$$\begin{aligned}x^2+y^2 &= \frac{x^2}{2}+xy+\frac{y^2}{2}+\frac{1}{2}+x+y\\ \frac{x^2}{2}+\frac{y^2}{2}-xy-x-y &= \frac{1}{2}\\ x^2+y^2-2xy-2x-2y &= 1\end{aligned}$$

Exercise 9. Center $(1/2, 2)$ because we equate the asymptotes to see where they intersect: $y = 2x+1 = -2x+3$.

$$\begin{gathered}\frac{(y-2)^2}{a^2}-\frac{(x-1/2)^2}{a^2/4}=1 \xrightarrow{(0,0)} \frac{4}{a^2}-\frac{1}{a^2}=\frac{3}{a^2}=1\\ \frac{(y-2)^2}{3}-\frac{(x-1/2)^2}{3/4}=1\end{gathered}$$

Exercise 10. $px^2+(p+2)y^2=p^2+2p$. $\quad \frac{x^2}{p+2}+\frac{y^2}{p}=1$.

(1) Since $p+2>p$, the foci must lie on the x axis. $a^2=p+2$; $\quad b^2=a^2(1-e^2)=p=(p+2)(1-e^2)$. $e=\sqrt{\frac{2}{p+2}}$
$F=ae=\sqrt{2}$. $\quad (\pm\sqrt{2},0)$.

(2) $F=ae=\sqrt{2}=a(\sqrt{3}) \implies a=\sqrt{\frac{2}{3}}$; $\quad b^2=\frac{2}{3}(1-3)=\frac{-4}{3}$.

$$\frac{x^2}{2/3}-\frac{y^2}{4/3}=1$$

Exercise 11. $e=1$ for an ellipse.

$$\begin{aligned}&\|X-F\|=|X\cdot N-a|=a-X\cdot N\\ &\|-X-F\|=\|X+F\|=|-X\cdot N-a|=a+X\cdot N\\ &\|X-F\|+\|X+F\|=2a\end{aligned}$$

Exercise 12.

$$\begin{gathered}\|X-F\|=e|(X-F)\cdot N-d|=e(d-(X-F)\cdot N)\\ \|X+F\|=ed(X,L)=e|(X-F)\cdot N+d|=e(-d-(X-F)\cdot N)\\ \|X-F\|-\|X+F\|=2ed\\ X\to -X \quad \text{so for the other branch, } \|X+F\|-\|X-F\|=2ed\end{gathered}$$

Exercise 13.

(1) $\frac{(tx)^2}{a^2}+\frac{(by)^2}{b^2}=1 \quad \left(\frac{b}{t}\right)^2=\frac{a^2(1-e^2)}{t^2}=\left(\frac{a}{t}\right)^2(1-e^2)$

(2) $b_1^2=a_1^2(1-e^2) \quad b_2^2=a_2^2(1-e^2)$.

$$1-\frac{b_1^2}{a_1^2}=1-\frac{b_2^2}{a_2^2};\quad \frac{b_1^2}{a_1^2}=\frac{b_2^2}{a_2^2}$$

$$\frac{x_1^2}{a_1^2}+\frac{y^2}{b_1^2}=1=\frac{\left(\left(\frac{b_2}{b_1}\right)x\right)^2}{a_2^2}+\frac{\left(\frac{b_2}{b_1}y\right)^2}{(b_2y)^2}$$

(3) $\pm\frac{(tx)^2}{a^2}\mp\frac{(ty)^2}{b^2}=1 \quad \left(\frac{b}{t}\right)^2=\frac{-a^2(e^2-1)}{t^2}=-\left(\frac{a}{t}\right)^2(e^2-1)$.

$$\begin{gathered}b_1^2=a_1^2(e^2-1) \quad b_2^2 = a_2^2(e^2-1)\\ \frac{b_1^2}{a_1^2}+1=e^2 \quad \frac{b_2^2}{a_2^2}+1=e^2 \quad \frac{b_1^2}{a_1^2}=\frac{b_2^2}{a_2^2}\\ \pm\left(\frac{x}{a_1}\right)^2\mp\left(\frac{y}{b_1}\right)^2=\pm\left(\frac{\left(\frac{b_2}{b_1}x\right)}{a_2}\right)^2\mp\left(\frac{\frac{b_2}{b_1}y}{b_2}\right)^2=1\end{gathered}$$

Exercise 14. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \Longrightarrow \frac{x}{a^2} + \frac{y}{b^2}y' = 0$

$\Longrightarrow y' = \frac{-b^2 x}{ya^2} = \frac{-a^2(1-e^2)x}{a^2 y} = \frac{(e^2-1)x}{y}$

Exercise 15.

(1) $y = ax^2 + bx + c \quad ty = a(tx)^2 + btx + c \to y = atx^2 + bx + c/t = y = Ax^2 + b + C$
(2) $y = tx^2$, $t \neq 0$

Exercise 16. $x - y + 4 = 0$ $y = 4\sqrt{x} \quad (y^2 = 16x)$; $y' = 2x^{-1/2}$.

$y'(x=4) = 1 \quad (x,y) = (4,8)$.

Exercise 17.

(1) If we treat the two given parabolas, $y^2 = 4p(x-a)$ and $x^2 = 4qy$, as two vector objects free from any specific coordinate system then we observe that we can disregard the sign of q and p and simply state that they are both positive. What matters is that we observe that p and q are the distance of the foci to the vertex for each of the respective parabolas.

Second, observe that a is not given. By diagram, if p, q are given, a must be moved along the x-axis to fit the tangency condition. Thus, in terms of doing the algebra, just eliminate p and q from the relations.

If (h,k) is the point of contact,

$$\begin{aligned} x^2 &= 4qy \qquad & y^2 &= 4p(x-a) \quad y = 2\sqrt{p}\sqrt{x-a} \\ \frac{x}{2q} &= y' & y' &= \sqrt{p}\frac{1}{\sqrt{x-a}} \\ y'(h) = \frac{h}{2q} & & y'(h) &= \sqrt{p}\frac{1}{\sqrt{h-a}} \end{aligned}$$

$$\text{(Tangent condition)} \left(\frac{h}{2q}\right)^2 = \frac{p}{h-a} \quad \Longrightarrow (h^2)(h-a) = (2q)^2 p$$

$$\text{(one point of contact condition)} \quad \text{with } q = \frac{h^2}{4k},\ p = \frac{k^2}{4(h-a)}$$

$$\Longrightarrow h^2(h-a) = \left(\frac{h^2}{2k}\right)^2 \frac{k^2}{4(h-a)} \Longrightarrow (h-a)^2 = \frac{h^2}{16}$$

$$\Longrightarrow h = \frac{2a \pm a/2}{15/8} = \boxed{4a/3}$$

(2)

$$\frac{h}{2q} = \frac{\sqrt{p}}{\sqrt{h-a}}$$

$$\frac{2a}{3q} = \frac{\sqrt{p}}{\sqrt{a/3}} = \frac{\sqrt{3p}}{\sqrt{a}}; \quad 2a\sqrt{a} = 3\sqrt{3pq}$$

$$\Longrightarrow 4a^3 = 27pq^2$$

Exercise 18. First hint: Vector methods triumph over algebraic manipulations of Cartesian coordinates. Think of the locus in terms of vector objects that are coordinate-free and the conic section will emerge. I mean, try evaluating $\|P - A\|^2 = (x-2)^2 + (x-3)^2 = (x+y)^2$

$A = (2,3), \quad N = \frac{1}{\sqrt{2}}(1,1), \quad X = (x,y)$.

$\|X - A\| = x + y = \sqrt{2}(X \cdot N) = \sqrt{2}(X \cdot N - (F \cdot N - d))$

where $F \cdot N = d = A \cdot N = \frac{5}{\sqrt{2}}$

$d =$ distance from focus to the directrix .

$y = x + 1$ (axis of the hyperbola)

$d = \frac{5}{\sqrt{2}} = \sqrt{(2-x)^2 + (3-y)^2} = \sqrt{2}(2-x) \qquad x = -\frac{1}{2},\ y = \frac{1}{2}$

$\left(\frac{-1}{2}, \frac{1}{2}\right)$ must also be the center. $y - \frac{1}{2} = -\left(x + \frac{1}{2}\right)$ is the directrix.

$\left(y - \frac{1}{2}\right) = \alpha\left(x + \frac{1}{2}\right)$ is the general form of the asymptote.

Consider asymptotes in general. $\|X-F\| = ed(X,L)$.

$$\frac{\|X-F\|}{d(X,L)} = e = \frac{\|X-F\|}{|X\cdot N - (F\cdot N - d)|} = \frac{\|X-F\|}{(X-F)\cdot N + d|}$$

For $\|X-F\| \to \infty$, $\quad \|X-F\| > d$. To keep ratio of e, $X-F$ must be ultimately directed by N by a ratio of e.

$$\Longrightarrow e = \frac{\|X-F\|}{\|X-F\|\cos\phi} = \frac{1}{\cos\phi}$$

e.g. Consider $N = \vec{e}_x$. $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Longrightarrow y = \frac{b}{a}x = \sqrt{e^2-1}x$.

From the vector equation,

$$(X-F)\cdot N = (x-c,y)\cdot N = \sqrt{(x-c)^2+y^2}\cos\phi = x-c$$

$$\frac{\sqrt{(x-c)^2+y^2}}{x-c} = \frac{1}{\cos\phi} = e; \quad \frac{(x-c)^2+y^2}{(x-c)^2} = e^2;$$

$$\frac{y^2}{(x-c)^2} = e^2 - 1 \Longrightarrow y = \sqrt{e^2-1}x$$

For our problem, consider the conic section approaching the asymptote. Then the conic section will look more like those linear asymptotes.

$$\sqrt{(x-2)^2+(y-3)^2} = x+y$$

$$\xrightarrow{y-\frac{1}{2}=\alpha\left(x+\frac{1}{2}\right)} \sqrt{\left(\left(x+\frac{1}{2}\right)-\frac{5}{2}\right)^2 + \left(\alpha\left(x+\frac{1}{2}\right)-\frac{5}{2}\right)^2} = x + \alpha\left(x+\frac{1}{2}\right)+\frac{1}{2}$$

$$\Longrightarrow \sqrt{(1+\alpha^2)\left(x+\frac{1}{2}\right)^2 - 5(1+\alpha)\left(x+\frac{1}{2}\right)+\frac{25}{2}} \xrightarrow{x\to\infty} \frac{\sqrt{1+\alpha^2}}{(1+\alpha)}$$

$$\Longrightarrow \alpha = 0$$

The asymptotes are $y = \frac{1}{2}$ and $x = \frac{-1}{2}$.

In the second part, each quadrant must be checked. So far, I only have that quadrant II is filled: points in quadrant III and quadrant IV cannot satisfy the given condition. To see this, consider quadrant II.

$$\|x-A\| = -x+y = \sqrt{2}(x,y)\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

For quadrant II, $N = \left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. By diagram, $(X-F)\cdot N > 0$ and $X\cdot N > 0$.

$$A\cdot N = \frac{1}{\sqrt{2}} \quad -A\cdot N = \frac{-1}{\sqrt{2}} \quad d = \frac{1}{\sqrt{2}}$$

$$|(X-F)\cdot N + d| = (X-F)\cdot N + d$$

The equation for the axis of the conic section is $y = -(x-5)$.

By taking the asymptotic limit like above, we can show that $\alpha = 0$ again. We only sketch the part of the hyperbola in quadrant II.

By similar procedure, I found that quadrant III, IV cannot satisfy the condition.

Exercise 19.

$$\|X-F\| = d(X,L) = |(X-F)\cdot N + d|$$

$$x^2+y^2 = (X\cdot N + d_1)^2 = y^2 + 2yd_1 + d_1^2$$

$$\xrightarrow{F=0} x^2 = 2yd_1 + d_1^2 \qquad y_1' = \frac{x}{d_1}$$

$$\|X-F\| = |(X-F)\cdot N - d_2| = d_2 - (X-F)\cdot N$$

$$\xrightarrow{F=0} \quad \|X\| = d_2 - y$$

$$x^2+y^2 = d_2^2 - 2d_2y + y^2$$

$$\Longrightarrow x^2 = d_2 - 2d_2y \qquad y_2' = \frac{-x}{d_2}$$

$$\text{Point of intersection } x_0^2 = 2y_0d_1 + d_1^2 = d_2^2 - 2d_2y_0$$
$$2(d_1+d_2)y_0 = d_2^2 - d_1^2 \Longrightarrow y_0 = \frac{d_2 - d_1}{2}$$
$$x_0^2 = d_2(d_2 - 2y_0) = d_2 d_1$$
$$\Longrightarrow \begin{aligned} y_1' &= \frac{\pm\sqrt{d_2d_1}}{d_1} = \pm\sqrt{\frac{d_2}{d_1}} \\ y_2' &= \frac{\mp\sqrt{d_2d_1}}{d_2} = \mp\frac{1}{\sqrt{\frac{d_2}{d_1}}} \end{aligned}$$

Exercise 20.

(1) Use $X \to -X$ symmetry.

$$\|X-F\| = ed(X,L) = e|(X-F)\cdot N + d| = e|X\cdot N - F\cdot N + d| = |eX\cdot N - a|$$
$$\|X\|^2 - 2X\cdot F + \|F\|^2 = e^2(X\cdot N)^2 - 2ea(X\cdot N) + a^2$$
$$X \to -X \Longrightarrow \|X\|^2 + 2X\cdot F + \|F\|^2 = e^2(X\cdot N)^2 + 2ea(X\cdot N) + a^2$$
$$\Longrightarrow \|X^2\| + \|F\|^2 = e^2(X\cdot N)^2 + a^2$$
$$x^2 + y^2 + c^2 = e^2x^2 + a^2 \qquad |F| = c = ae$$
$$\left(\frac{a^2-c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2 \Longrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

(2) $\frac{x}{a^2} + \frac{yy'}{a^2-c^2} = 0 \Longrightarrow y' = \frac{-(a^2-c^2)x}{ya^2}$

$$xy(y')^2 = \left(\frac{a^2-c^2}{a^2}\right)^2\frac{x^3}{y}$$
$$(x^2-y^2-c^2)y' = \left(x^2 + \frac{a^2-c^2}{a^2}x^2 - a^2\right)\left(\frac{-(a^2-c^2)x}{ya^2}\right) =$$
$$= (a^4 + -(a^2-c^2)x^2 - a^2x^2)\frac{(a^2-c^2)}{a^4}\frac{x}{y}$$
$$-xy = \frac{-xy^2}{y} = -\frac{x(a^2-c^2-\left(\frac{a^2-c^2}{a^2}\right)x^2)}{y} =$$
$$= (a^2-c^2)(-a^4 + a^2x^2)x/(a^4y)$$
$$\Longrightarrow xyy'^2 + (x^2-y^2-c^2)y' - xy = 0$$

(3) For y', consider $-\frac{1}{y'}$ at every (x,y).

$$xy\left(\frac{-1}{y'}\right)^2 + (x^2-y^2-c^2)\frac{-1}{y'} - xy = 0 = \frac{-xy}{y'} + (x^2-y^2-c^2)\frac{1}{y'} + xy$$
$$\xrightarrow{\text{if } y'\neq 0} -xy + (x^2-y^2-c^2)y' + xy(y')^2 = 0$$

Thus $S \to S$ since the defining differential equation is invariant under the transformation of the slope.

Exercise 21. For a circle centered at C, then $\|X - C\| = r_0^2$ for all points X on that circle.

For the condition of being tangent to a given line, $L = P + tA$, then $(X_C - C)\cdot A = 0$ and the point lies on the circle so $\|X_C - C\| = r_0^2$.

Call the point that all the circles pass through F. Then $\|C - F\| = \|C - X_C\|$. $\|C - X_C\|$ is by definition $d(X, L)$, the distance from the circle center to the line. $\|C - F\| = \|C - X_0\|$ is by definition a parabola.

Exercise 22. Consider a circle that's part of the mentioned family that has its center directly below the given circle with radius r_0, and center Q.

It's given that the center is equidistant from the point of tangency and the line. This hints at a parabola because the parabola's vertex is equidistant from the focus and the directrix. Thus, we need to show that $d(X, L)$ is equal to the distance from the circle center C to the bottom point of Q.

Let N be a unit normal vector pointing from the line towards the focus, placing the focus in the positive half-plane.
Let C be the center of an arbitrary circle in the family and r_1 its radius.
Let X_1 be the point of tangency between circle Q and circle C.
We want $\|(Q + r_0 N) - C\| = \|X_2 - C\|$.

The tangency condition between circle Q and C means that

$$(X_1 - C) = -\alpha(X_1 - Q);\ \alpha > 0 \qquad \alpha = \frac{r_1}{r_0}$$

$$Q - r_0 N - C = Q - X_1 - r_0 N - C + X_1$$

$$\xrightarrow{\text{take the magnitude}} \|Q - X_1\|^2 + \|X_1 - C\|^2 + r_0^2 + 2(Q - X_1)(X_1 - C) + 2(X_1 - Q)r_0 N + 2(C - X)r_0 N$$

$$r_0^2 + r_1^2 + r_0^2 + 2\alpha r_0^2 + 2r_0(1 + \alpha)(x_1 - Q) \cdot N$$

$$2r_0^2 + r_1^2 + 2r_1 r_0 + 2(r_1 + r_0)(X_1 - Q) \cdot N$$

I had thought the key is to use *the law of cosines* to evaluate $(X_1 - Q) \cdot N = \frac{1}{\alpha}(C - X_1) \cdot N$.

Length $l = d(X, L) = d(C, L)$.

But that just gets us back to the same place.

I had found the solution by a clever construction. But to come to that conclusion it required me to be "unstuck" - if something doesn't work, move onto the next - don't try to make something work and go in circles. And persistence is key because there can be many **false eurekas**.

Again, consider a particular circle with its center C_2 right below the given Q circle that just makes C_2 tangent with the given line L_2. The directrix is not going to be L_2 but L_1, a line translated below L_2, line of tangency, by r_0, so that $\|Q - C_2\| = r_2 + r_0 = d(C_2, L_1)$. It is a clever artificial construction.

Let's show this for any circle C of radius r_1 in the family.

Tangent to the circle Q condition: $X_1 - C = \alpha(Q - X_1)$.

So then $\|Q - C\| = r_1 + r_0$

Tangent to the line $L_2 = B_2 + tA_2$: $(X_2 - C) \cdot A_2 = 0$

$\|X_2 - C\| = r_1$

Consider L_1, a line translated by r_0 from L_2 away from Q.

If $L_2 = B_2 + tA_2, \quad L_1 = B_2 - r_0 N + tA_2$.

Since $X_2 - C = r_1(-N)$ then $X_2 - r_0 N - C = (r_1 + r_0)(-N)$ will point from C to L_1, because $(X_2 - r_0 N) = ((B_2 + tA_2) - r_0 N) \in L_1$.

$\implies \|Q - C\| = r_1 + r_0 = \|X_2 - r_0 N - C\| = d(C, L_1)$.

Exercise 23. Without loss of generality, use $y^2 = 4cx$.

The latus rectum intersect the parabola at $(c, +2c)$, $(c, -2c)$.

Thus $4c =$ length of latus rectum $= 2d = 2($ distance from focus to directrix $)$.

$$y = 2\sqrt{cx} \quad y' = \sqrt{c}/\sqrt{x} \quad y'(c) = \pm 1$$

Tangent lines: $y = \pm(x + c)$.

$\xrightarrow{\text{intersection}} +(x + c) = -(x + c) \quad x = -c$ (at the directrix)

Exercise 24. Center of circle is given to be 0.

Collinear with center and center not between them: $P = \alpha Q; \quad \alpha > 0$

$$\|P\| \, \|Q\| = r_0^2 = \alpha \, \|Q\|^2$$

For the line defined in Cartesian coordinates as $x+2y-5=0$, the vector form of this line is given by

$$X_L = B + tA \quad X_L \cdot N = (x,y)\cdot\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = N\cdot B + 0 = \sqrt{5}$$

$$A = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \text{ is a vector that's perpendicular to } N \;;$$

$$B = (1,2) \text{ since we can simply plug it in to satisfy the equation}$$

$$X_L = (1,2) + t\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \quad t\in\mathbb{R}$$

$$Q = B + tA \Longrightarrow \|Q\|^2 = B^2 + 2tB\cdot A + t^2A^2 = 5 + t(0) + t^2 = 5 + t^2$$

$$(5+t^2)(\alpha) = r_0^2 = 4 \quad \alpha = \frac{4}{5+t^2}$$

$$P = \alpha Q = \frac{4}{5+t^2}\left((1,2) + t\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\right)$$

14.4 Exercises - Vector-valued functions of a real variable, Algebraic operations. Components; Limits, derivatives, and integrals. **Exercise 1.** $F' = (1, 2t, 3t^2, 4t^3)$.

Exercise 2. $F' = (-\sin t, 2\sin t\cos t, 2\cos 2t, \sec^2 t)$

Exercise 3. $F' = \left(\frac{1}{\sqrt{1-t^2}}, \frac{-1}{\sqrt{1-t^2}}\right)$

Exercise 4. $F' = (2e^t, 3e^t)$.

Exercise 5. $F' = \left(\sinh t, 2\cosh 2t, -3e^{-3t}\right)$

Exercise 6. $\left(\frac{t}{1+t^2}, \frac{1}{1+t^2}, \frac{-2t}{(1+t^2)^2}\right)$

Exercise 7. $F' = \left(\frac{2}{1+t^2} + \frac{-4t^2}{(1+t^2)^2}, \frac{4t}{(1+t^2)^2}, 0\right)$.

$$F'\cdot F = \frac{4t(1+t^2)}{(1+t^2)^3} + \frac{-4t^2(2t)}{(1+t^2)^3} + \frac{4t-4t^3}{(1+t^2)^3} = 0$$

Exercise 8. $\left(\frac{1}{2}, \frac{2}{3}, e^1 - 1\right)$

Exercise 9.

$$(-\cos t, \sin t, -\ln|\cos t|)|_0^{\pi/4} = \left(\frac{-\sqrt{3}}{2} + 1, \frac{\sqrt{2}}{2}, -\ln\frac{\sqrt{2}}{2}\right)$$

Exercise 10. $(\ln(1+e^t), t - \ln(1+e^t))|_0^1 = \left(\ln\left(\frac{1+e}{2}\right), 1 - \ln\left(\frac{1+e}{2}\right)\right)$

Exercise 11. $\left(te^t - e^t, t^2e^t - 2te^t + 2e^t, -te^{-t} - e^{-t}\right)\Big|_0^1 = (1, e-2, -2e^{-1}+1)$

Exercise 12. $(2,-4,1) = A$.

$\int_0^1 (te^{2t}, t\cosh 2t, 2te^{-2t})dt = \left(\frac{1}{2}te^{2t} + \frac{-1}{4}e^{2t}, \frac{t\sinh 2t}{2} - \frac{\cosh 2t}{4}, 2\left(\frac{-1}{2}te^{-2t} - \frac{e^{-2t}}{4}\right)\right)\Big|_0^1 =$
$= \left(\frac{1}{4}e^2 + \frac{1}{4}, \frac{\sinh 2}{2} - \frac{\cosh 2}{4} + \frac{1}{4}, 2\left(\frac{-3}{4}e^{-2} + \frac{1}{4}\right)\right)$.
$A\cdot B = \frac{1}{2}e^2 + -2\sinh 2 + \cosh 2 + \frac{-3}{2}e^{-2}$.

Exercise 13. $F'(t) = B = 1 = \|F'(t)\|\,|B|\cos\theta(t)$.

Given $\theta(t) = \theta_0$ constant, $\|F'(t)\|$ must be a constant.

$\|F'(t)\|^2 = F'(t)\cdot F'(t) = g \qquad g' = 2F''(t)\cdot F'(t) = 0$ since $\|F'\|^2$ constant .
$\Longrightarrow F''(t)\cdot F'(t) = 0$

Exercise 14.

$$F' = 2e^{2t}A + -2e^{-2t}B$$
$$F'' = 4e^{2t}A + 4e^{-2t}B = 4(F)$$

Exercise 15. $G' = F' \times F' + F \times F'' = F \times F''$

Exercise 16.

$$G = F \cdot (F' \times F'')$$
$$G' = F' \cdot (F' \times F'') + F \cdot (F'' \times F'' + F' \times F''') = F \cdot (F' \times F''')$$

Exercise 17. If $\lim_{t \to p} F(t) = A, \quad \forall j$th component,

$$\forall \sqrt{\frac{\epsilon}{n}} > 0, \exists \delta_j > 0 \quad \text{such that } |F_j(t) - A_j| < \sqrt{\frac{\epsilon}{n}} \quad \text{if } |t-p| < \delta_j$$
$$\text{Consider } \min_{j=1,\dots n} \delta_j = \delta_0$$
$$\sum_{j=1}^n |F_j(t) - A_j|^2 < \sum_{j=1}^n \left(\sqrt{\frac{\epsilon}{n}} \right)^2 = \epsilon \quad \text{whenever } |t-p| < \delta_0$$
$$\implies \lim_{t \to p} \|F(t) - A\| = 0$$

If $\lim_{t \to p} \|F(t) - A\| = 0, \quad \forall \epsilon > 0, \exists \delta > 0$ such that $\sqrt{\sum_{j=1}^n (F_j(t) - A_j)^2} < \epsilon \quad$ if $|t-p| < \delta$.

$\implies \sum_{j=1}^n (F_j(t) - A_j)^2 < \epsilon$
$\epsilon > \sum_{j=1}^n (F_j(t) - A_j)^2 > (F_k(t) - A_k)^2 > 0$

$\implies \epsilon > |F_k(t) - A_k|$ if $|t-p| < \delta$.

Exercise 18. If F is differentiable on I, then

$$F' = \sum_{j=1}^n f_j' \vec{e}_j \qquad f_j' = \lim_{h \to 0} \frac{1}{h}(f_j(t+h) - f_j(t))$$
$$F' = \sum_{j=1}^n \lim_{h \to 0} \frac{1}{h}(f_j(t+h) - f_j(t)) = \lim_{h \to 0} \frac{1}{h} \sum_{j=1}^n (f_j(t+h) - f_j(t))e_j = \lim h \to 0 \frac{1}{h}(F(t+h) - F(t))$$

If $F'(t) = \lim_{hto0} \frac{1}{h}(F(t+h) - F(t)) = \lim_{h \to 0} \frac{1}{h} \sum_{j=1}^n (f_j(t+h) - f_j(t))e_j =$
$= \sum_{j=1}^n \lim_{h \to 0} \frac{1}{h}(f_j(t+h) - f_j(t))e_j = \sum_{j=1}^n f_j'(t)e_j$

So F' is differentiable.

Exercise 19. $F'(t) = 0, \forall j = 1 \dots n, f_j'(t) = 0$. By one-dimensional zero-derivative theorem, $f_j(t) = c_j$ constant. Thus $F(t) = \sum_{j=1}^n c_j \vec{e}_j = C$ on an open interval I.

Exercise 20. $\frac{1}{6}t^3 A + \frac{1}{2}t^2 B + Ct + D$

Exercise 21. $Y'(x) + p(x)Y(x) = Q(x)$. Then $\forall j = 1, \dots, n$

$$y_j'(x) + p(x)y_j(x) = Q_j(x)$$

Since p, Q are continuous on I, and given this initial value condition $y_k(a) = b_k$,

$$y_j(x) = e^{-\int_a^x p(t)dt} \left(b_j + \int_a^x Q_j(t) e^{\int_a^t p(u)du} dt \right)$$
$$\implies \sum_{j=1}^n j_j(x) = Y(x) = e^{-\int_a^x p} \left(B + \int_a^x Q e^{\int_a^t p} dt \right)$$

Exercise 22.

$$\begin{aligned}
tF' = F + tA &\Longrightarrow F' + tF'' = F' + A \\
&\Longrightarrow tF'' = A \\
F''(t) &= A/t \\
&\Longrightarrow F'(t) = A\ln t + B \\
&\Longrightarrow F(t) = A(t\ln t - t) + Bt + C \\
F(1) &= A(-1) + B + C = 2A
\end{aligned}$$

$$\begin{aligned}
tF' = F + tA &\Longrightarrow At\ln t + Bt = A(t\ln t - t) + Bt + C + tA \\
C &= 0, \quad B = 3A \\
F(t) &= A(t\ln t - t) + 3At \\
F(3) &= A(3\ln 3 - 3) + 9A = \boxed{3A\ln 3 + 6A}
\end{aligned}$$

Exercise 23.

$$\begin{aligned}
F'(x) &= e^x A + xe^x A + -\frac{1}{x^2}\int_1^x F(t)dt + \frac{1}{x}F(x) = \\
&= e^x A + xe^x A + e^x A - \frac{F(x)}{x} + \frac{F(x)}{x} = 2e^x A + xe^x A = (2+x)e^x A
\end{aligned}$$

$$F'(x) = (2+x)e^x A; \qquad F(x) = 2e^x A + A(xe^x - e^x) + C = Axe^x + e^x A + C$$

$$\begin{aligned}
\int_1^x (Ate^t + e^t A + C)dt &= \left(A(te^t - e^t) + e^t A + Ct\right)\Big|_1^x = \\
&= A(xe^x - e^x) + e^x A + C(x-1) - eA = Axe^x + C(x-1) - eA \\
xe^x A + Ae^x + \frac{C(x-1)}{x} - \frac{eA}{x} &\Longrightarrow C = eA
\end{aligned}$$

Exercise 24. $F'(t) = \alpha(t)F(t)$

$$\begin{aligned}
&\Longrightarrow f_k'(t) = \alpha(t)f_k(t); \quad \ln\left(\frac{f_k(t)}{f_k(a)}\right) = \int_a^t \alpha(x)dx \\
&f_k = f_k(a)e^{\int_a^t \alpha} \\
&F(t) = \sum_{j=1}^n f_j(a)e6\int_a^t \alpha e_j = e^{\int_a^t \alpha}\sum_{j=1}^n f_j(a)e_j = u(t)A
\end{aligned}$$

14.19 Exercises - Velocity and acceleration in polar coordinates, Plane motion with radial acceleration, Cylindrical coordinates.

Exercise 1.

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta = \vec{e}_r + r\vec{e}_\theta \quad (\theta = t) \\
\mathbf{a} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\vec{e}_\theta = -r\vec{e}_r + 2\vec{e}_\theta \\
\mathbf{v} &= \vec{e}_r + t\vec{e}_\theta = \cos t\vec{e}_x + \sin t\vec{e}_y + -t\sin t\vec{e}_x + t\cos t\vec{e}_y = \\
&= (\cos t - t\sin t)\vec{e}_x + (\sin t + t\cos t)\vec{e}_y \\
\mathbf{a} &= (-t\cos t - 2\sin t)\vec{e}_x + (-t\sin t + 2\cos t)\vec{e}_y
\end{aligned}$$

Exercise 2.

$$\begin{aligned}
\mathbf{v} &= \vec{e}_r + r\vec{e}_\theta + \vec{e}_z = (\cos t - t\sin t)\vec{e}_x + (\sin t + t\cos t)\vec{e}_y + \vec{e}_z \\
\mathbf{a} &= (-t\cos t - 2\sin t)\vec{e}_x + (t\sin t + 2\cos t)\vec{e}_y
\end{aligned}$$

Exercise 3. (a).

$$\begin{aligned}
r = \sin t; \theta = t; z = \log\sec t; \theta \leq t < \frac{\pi}{2} \\
(r\cos\theta)^2 + (r\sin\theta - \frac{1}{2})^2 = r^2 - r\sin\theta + \frac{1}{4} = \frac{1}{4}
\end{aligned}$$

(b).

$$\mathbf{v} = \frac{dt}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \log\sec t\vec{e}_z = \cos t\vec{e}_r + r\vec{e}_\theta + \frac{\tan\theta\sec\theta}{\sec\theta}\vec{e}_z$$
$$v_z = \tan\theta;\ v^2 = \cos^2 t + r^2 + \tan^2\theta = \sec^2\theta$$
$$\cos\phi = \frac{\tan\theta}{\sec\theta} = \sin\theta = r = \sin t$$
$$\boxed{\phi = \arccos(\sin\theta)}$$

Exercise 6.

$$A = \int R^2(\theta)d\theta = \int_0^{2\pi}\frac{1}{2}e^{2c\theta}d\theta = \left.\frac{e^{2c\theta}}{4c}\right|_0^{2\pi} = \frac{e^{4\pi c}-1}{4c}$$

Exercise 7.

$$\int_0^\pi \frac{1}{2}\sin^4\theta d\theta = \frac{1}{2}\int_0^\pi \sin^2\theta(1-\cos^2\theta)d\theta = \frac{1}{2}\int_0^\pi\left(\left(\frac{1-\cos 2\theta}{2}\right) - \left(\frac{1-\cos 4\theta}{2(4)}\right)\right)d\theta \quad = 3\pi/16$$

Exercise 15. Place target at the center (without loss of generality). The strategy is to break up $\mathbf{v}$ into the polar coordinate unit vectors.

$$\mathbf{r} = r\vec{e}_r$$
$$\mathbf{v} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$
$$\frac{dr}{dt} = v_r = v\cos(\pi-\alpha) = -v\cos\alpha$$
$$r\frac{d\theta}{dt} = v\sin\alpha$$
$$\frac{v\sin\alpha}{-v\cos\alpha} = -\tan\alpha = \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = r\frac{d\theta}{dr};\ \frac{1}{r}\frac{dr}{d\theta} = -\tan\alpha$$
$$r = e^{-\tan\alpha\theta}$$

Exercise 17.

A first order differential equation of the form $y' = f(x,y)$ is homogeneous if $f(tx,ty) = f(x,y)$. Then

$$f(r\cos\theta, r\sin\theta) = f(\cos\theta,\sin\theta) = f(\theta)$$

We find that

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$
$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = f(\theta)$$

Exercise 18.

$$\mathbf{v} = \omega\mathbf{k}\times\mathbf{r}$$
$$\mathbf{v} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$
$$\mathbf{v}\cdot\vec{e}_r = 0,\ \text{so}\ \frac{dt}{dt} = 0;\ \omega\mathbf{k}\times\mathbf{r} = r\frac{d\theta}{dt}\vec{e}_\theta = \omega r\vec{e}_\theta = r\frac{d\theta}{dt}$$
$$|\omega\mathbf{k}\times\mathbf{r}|^2 = \omega^2 r^2 = r^2\left(\frac{d\theta}{dt}\right)^2$$
$$\boxed{\omega = \left|\frac{d\theta}{dt}\right|,\ \omega > 0}$$

Exercise 19. (a)

$$\mathbf{v} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta = r\frac{d\theta}{dt}\vec{e}_\theta;\ \frac{dr}{dt} = 0;\ \vec{e}_\theta = \vec{e}_z \times \vec{e}_r$$

$$\mathbf{v} = r\frac{d\theta}{dt}\vec{e}_z \times \vec{e}_r = \frac{d\theta}{dt}\vec{e}_z \times r\vec{e}_r = \omega \times \mathbf{r}$$

(b).

$$a = v' = \omega' \times r + \omega \times r' = \alpha \times r + \omega \times r'$$

$$\omega \times r' = \omega \times (\omega \times r) = (\omega \cdot r)\omega - \omega^2 r$$

(c).

$$\text{Now } \omega \cdot r = 0 \Longrightarrow a = -\omega^2 r$$

Exercise 20.

The distance $|\mathbf{r}_p(t) - \mathbf{r}_q(t)|$ is independent of t, so $\frac{d}{dt}|\mathbf{r}_p(t) - \mathbf{r}_q(t)| = 0$, which implies

$$\frac{d}{dt}(r_p^2 - 2r_p \cdot r_q + r_q^2) = 2r_p \cdot \frac{dr_p}{dt} - 2\frac{dr_p}{dt} \cdot r_q - 2r_p \cdot \frac{dr_q}{dt} + 2\frac{dr_q}{dt} \cdot r_q = 0$$

$$\frac{dr_p}{dt} \cdot (r_q - r_p) = -\frac{dr_q}{dt} \cdot (r_p - r_q)$$

Suppose

$$v_p = \omega_p \times r_p$$
$$v_q = \omega_q \times r_q$$

Then

$$v_p \cdot (r_q - r_p) = -v_q \cdot (r_p - r_q)$$
$$\omega_p \times r_p \cdot r_q = -\omega_q \times r_q \cdot r_p = \quad = \omega_q \times r_p \cdot r_q$$
$$((\omega_p - \omega_q) \times r_p) \cdot r_q = 0$$

Thus, $\omega_p = \omega_q$.