

Logic for Computer Science - Lecture Notes

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Chapter 1

Motivation and introduction

First-order logic, what we will be studying next, is an extension of propositional logic, extension that brings more expressivity. The additional expressivity is necessary in order to model certain statements that cannot be expressed in propositional logic.

In propositional logic, we cannot express naturally the following statement: *All men are mortal*.

To model a statement in propositional logic, we identify the atomic propositions. Then we associate to each atomic proposition a propositional variable. The atomic propositions are the propositions that cannot be split into one or more smaller propositions, linked among them by the logical connectives of propositional logic: \neg , \wedge , \vee , \rightarrow and \leftrightarrow .

We notice that the statement *All men are mortal* cannot be decomposed into smaller statements linked among them by the logical connectives of propositional logic, as is described above. Therefore, in propositional logic, the statement is atomic. So we associate to the entire statement a propositional variable $\mathbf{p} \in A$.

Let us now model the statement *Socrates is a man*. Obviously, to this second statement we must associate another propositional variable $\mathbf{q} \in A$. Let us assume that \mathbf{p} and \mathbf{q} are true. Formally, we work in a truth assignment $\tau : A \rightarrow B$ where $\tau(\mathbf{p}) = 1$ and $\tau(\mathbf{q}) = 1$. Can we draw the conclusion that *Socrates is mortal* in the truth assignment τ ?

No, because to the statement *Socrates is mortal* we should associate a third propositional variable $\mathbf{r} \in A$. We cannot draw any conclusion on $\tau(\mathbf{r})$ from $\tau(\mathbf{p}) = 1$ and $\tau(\mathbf{q}) = 1$. So, from the semantics of propositional logic, we cannot draw the conclusion that \mathbf{r} is true in any truth assignment that makes

both **p** and **q** true. This is despite the fact that, in any world where *All men are mortal* and *Socrates is a man*, we can draw the conclusion that *Socrates is mortal* without failure. This difference between reality and our modelling indicates that our modelling is not sufficient for our purposes.

First-order logic includes, in addition to propositional logic, the notion of *quantifier* and the notion of *predicate*. The universal quantifier is denoted by \forall and the existential quantifier is denoted by \exists .

A predicate is a statement whose truth value depends on zero or more parameters. For example, for the statements above, we will be using two predicates: **Man** and **Mortal**. The predicate **Man** is the predicate that denotes the quality of being a man: **Man**(**x**) is true iff **x** is a man. The predicate **Mortal** is true when its argument is mortal. As the predicates above have only one argument/parameter, they are called *unary* predicates. Predicates generalize propositional variables by the fact that they can take arguments. Actually, propositional variable can be regarded as predicates with no arguments.

In this way, the statement *All men are mortal* will be modelled by the formula

$$(\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x))),$$

which is read as follows: *for any x, if Man of x, then Mortal of x*. The statement *Socrates is a man* shall be modelled by the formula **Man**(**s**), where **s** is a *constant* that denotes Socrates, just like 0 denotes the natural number zero. For example, **Man**(**s**) is true (as **s** stands for a particular man – Socrates), but **Man**(**l**) is false if **l** is a constant standing for the dog *Lassie*.

The statement *Socrates is mortal* shall be represented by **Mortal**(**s**) (recall that the constant **s** stands for Socrates). The statement **Mortal**(**s**) is true, as Socrates is mortal; likewise, the statement **Mortal**(**l**) is also true.

We shall see that in first-order logic, the formula **Mortal**(**s**) is a logical consequence of the formulae $(\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x)))$ and respectively **Man**(**s**). Therefore, first-order logic is sufficiently expressive to explain theoretically the argument by which we deduce that *Socrates is mortal* from the facts that *All men are mortal* and *Socrates is a man*.

Chapter 2

Structures and signatures

You have certainly met already several first-order logic formulae, without necessarily knowing that you are dealing with first-order logic. Consider the following formula:

$$\varphi = \left(\forall x. (\forall y. (x < y \rightarrow \exists z. (x < z \wedge z < y))) \right).$$

The formula makes use of a binary predicate, $<$, that is defined as follows: $<(x, y)$ is true if x is strictly smaller than y . In order to simplify our writing, we use the infix notation $(x < y)$ instead of the prefixed notation $(<(x, y))$ for many binary predicates (including for $<$).

Is the formula φ above true? The formula states that between any two values of the variables x, y there is a third value, of the variable z . The formula is true if the domain of the variables x, y, z is \mathbb{R} , but it is false if the domain is \mathbb{N} (between any two real numbers there exists a third, but between two consecutive naturals there is no other natural number).

Generally, first-order formulas refer to a particular *mathematical structure*.

Definition 1 (Mathematical structure). A mathematical structure is a tuple $S = (D, Pred, Fun)$ where:

- D is a non-empty set called the domain of the structure;
- each $P \in Pred$ is a predicate (of a certain arity) over the set D ;
- each $f \in Fun$ is a function (of a certain arity) over the set D .

Here are a few examples of mathematical structures:

1. $(\mathbb{N}, \{<, =\}, \{+, 0, 1\})$;

The domain of the structure is the set of naturals. The structure contains two predicates: $<$ and $=$, both of arity 2. The predicate $<$ is the *smaller than* predicate on naturals, and the predicate $=$ is the *equality* predicate over natural numbers.

The structure also contains three functions. The binary function $+$: $\mathbb{N}^2 \rightarrow \mathbb{N}$ is the addition function for naturals, and the functions 0 : $\mathbb{N}^0 \rightarrow \mathbb{N}$ and respectively 1 : $\mathbb{N}^0 \rightarrow \mathbb{N}$ are the arity 0 functions (also called constant functions or simply constants) 0 and respectively 1.

2. $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$;

This structure contains two binary predicates, $<$ and $=$, as well as four functions over \mathbb{R} : the binary function $+$, the unary function $-$ (unary minus) and the constants $0, 1 \in \mathbb{R}$.

3. $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$;

This structure is similar to that above, but the domain is the set of integers.

4. $(B, \emptyset, \{\cdot, +, -\})$;

This structure is a boolean algebra, where the domain is the set truth values and the functions are those that we studied in the first half of the semester. Such structures, without any predicates, are called *algebraic structures*.

5. $(\mathbb{R}, \{<\}, \emptyset)$.

This structure contains only a predicate of arity 2 (the *less than* relation over \mathbb{R}) and no function. Structures without functions are called relational structures. Relational structures with a finite domain are called relational data bases and you will study them in your second year.

Whenever we want to evaluate the truth value of a first-order formula we need a mathematical structure. Recall our previous formula:

$$\varphi = \left(\forall x. (\forall y. (x < y \rightarrow \exists z. (x < z \wedge z < y))) \right).$$

This formula is true in the structure $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ (between any two distinct real numbers there is another real number), but it is false in the structure $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$ (because it is not true that between any two distinct integers there is a third integer – for example there is no such integer between two consecutive integers).

It is possible for two different structure to have a set of predicates and a set of functions with the same names. For example, the structures above, $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ and respectively $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$. Even if the

predicate $<\in \mathbb{R}^2$ is different from the predicate $<\in \mathbb{Z}^2$, they both have the same name: $<$.

Generally, in Mathematics and in Computer Science, we do not make any difference between a predicate and its name or between a function and its name. However, in Logic, the difference is extremely important. In particular, if we refer to the name of a function, we shall use the phrase “functional symbol” (i.e., symbol standing for a function). When we refer to the name of a predicate, we shall use the phrase “predicate symbol” (or “relational symbol”). Why is the difference between a predicate and a predicate symbol important? Because we shall need to associate to the same predicate symbol several predicates, similarly to how we can associate several values to a program variable in an imperative language.

When we are interested only in the function and predicate names (not the function or predicates themselves), we work with signatures:

Definition 2 (Signature). A signature Σ is a tuple $\Sigma = (\mathcal{P}, \mathcal{F})$, where \mathcal{P} is a set of predicate symbols and \mathcal{F} is a set of functional symbols. Each predicate or functional symbol s has an associate natural number called its arity denoted by $ar(s)$.

To a signature we can associate many structures:

Definition 3 (Σ -structure). If $\Sigma = (\mathcal{P}, \mathcal{F})$ is a signature, a Σ -structure is any structure $S = (D, Pred, Fun)$ so that for each predicate symbol $P \in \mathcal{P}$, exists a predicate $P^S \in Pred$ of corresponding arity, and for every functional symbol $f \in \mathcal{F}$, there is a function $f^S \in Fun$ of corresponding arity.

Example 4. Let $\Sigma = (\{\mathbf{P}, \mathbf{Q}\}, \{\mathbf{f}, \mathbf{i}, \mathbf{a}, \mathbf{b}\})$, where \mathbf{P} and \mathbf{Q} are predicate symbols of arity $ar(\mathbf{P}) = ar(\mathbf{Q}) = 2$ and $\mathbf{f}, \mathbf{i}, \mathbf{a}, \mathbf{b}$ are function symbols having the following arities: $ar(\mathbf{f}) = 2$, $ar(\mathbf{i}) = 1$ and $ar(\mathbf{a}) = ar(\mathbf{b}) = 0$.

We have that $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ and $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$ are Σ -structures.

Remark. As you can observe in Example 4, for predicate symbols (e.g., \mathbf{P}, \mathbf{Q}) we use a different color than the color used for functional symbols (e.g., $\mathbf{f}, \mathbf{i}, \mathbf{a}, \mathbf{b}$). For predicates and functions we use the normal font for mathematical formulas.

To remember!

Structure = domain + predicates + functions

Signature = predicate symbols + functional symbols

To a signature Σ we can associate many structures, which are called Σ -structures.

Notation. The set of predicate symbols of arity n is denoted by $\mathcal{P}_n = \{P \mid \text{ar}(P) = n\}$, and the set of functional symbols of arity n is $\mathcal{F}_n = \{f \mid \text{ar}(f) = n\}$. For the particular case when $n = 0$, \mathcal{F}_0 is the set of constant symbols (that is, functional symbols with arity 0).

2.1 Exercise sheet

Exercise 5. Identify the predicates and the functions in the text below. What is their domain?

John is a student. Any student learns Logic. Anyone learning Logic passes the exam. Any student is a person. There is a person who did not pass the exam. Therefore: not all persons are students.

Exercise 6. Identify the predicates and the functions in the text below. What is their domain?

The sum of two even numbers is even.

Exercise 7. Identify the predicates and the functions in the text below. What is their domain?

In chess, the queen can make a move from one square to another iff the bishop or the rook can make the same move.

Exercise 8. Identify the predicates and the functions in the text below. What is their domain?

The sum of two numbers greater than zero is greater than zero.

Exercise 9. Identify the predicates and the functions in the text below. What is their domain?

The number 7 is prime.

Exercise 10. Identify the predicates and the functions in the text below. What is their domain?

Any even number greater than 2 is the sum of two primes.

Exercise 11. *Identify the predicates and the functions in the text below. What is their domain?*

If the Earth is flat, then $2 + 2 = 5$.

Exercise 12. *Identify the predicates and the functions in the text below. What is their domain?*

For any $\epsilon \in (0, \infty)$, there exists $\delta_\epsilon \in (0, \infty)$ such that for any $x \in \mathbb{R}$ with $d(x_0, x) < \delta_\epsilon$, we have $d(f(x_0), f(x)) < \epsilon$.

Chapter 3

The Syntax of First-Order Logic

In this chapter we present the syntax of first-order logic formula. The language of first-order logic is parameterised by a signature. A difference to propositional logic is that there are several first-order logic languages, one first-order language for each signature Σ . In propositional logic, there was just one language, \mathbb{PL} .

Next, we shall fix a signature Σ that contains the predicate symbols in \mathcal{P} and the functional symbols in \mathcal{F} .

3.1 The Alphabet of First-Order Logic

Just as propositional logic formulae, the formulae in first-order logic are strings of characters over a certain alphabet. Unlike propositional logic, the alphabet is now richer. The alphabet of first-order logic consists of the follows “characters”:

1. the logical connectives already known: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \perp$, as well as two new *quantifiers*: \forall, \exists ;
2. variables: we will assume that a countably infinite set of variables $\mathcal{X} = \{x, y, z, x', y', x_1, z'', \dots\}$ is also part of the alphabet (not to be confused with propositional variables in propositional logic – they are two fundamentally different notions);
3. auxilliary symbols: $(,), \cdot, \circ, (,),$ and $;$;
4. non-logical symbols, that are specific to each signature $\Sigma = (\mathcal{P}, \mathcal{F})$: the functional symbols in \mathcal{F} and the predicate symbols in \mathcal{P} .

3.2 Terms

Definition 13. The set of terms, \mathcal{T} , is the smallest set having the following properties:

1. $\mathcal{F}_0 \subseteq \mathcal{T}$ (any constant symbol is a term);
2. $X \subseteq \mathcal{T}$ (any variable is a term);
3. if $f \in \mathcal{F}_n$ (with $n > 0$) and $t_1, \dots, t_n \in \mathcal{T}$, then $f(t_1, \dots, t_n) \in \mathcal{T}$ (a functional symbol of arity n applied to n terms is a term).

Remark. The elements of the set \mathcal{T} are often called Σ -terms, because the definition of \mathcal{T} depends on Σ .

Terms are essentially built by applying functional symbols to variables and constant symbols.

Example 14. Recall $\Sigma = (\{\mathbf{P}, \mathbf{Q}\}, \{\mathbf{f}, \mathbf{i}, \mathbf{a}, \mathbf{b}\})$ from Example 4, where $ar(\mathbf{P}) = ar(\mathbf{Q}) = 2$, $ar(\mathbf{f}) = 2$, $ar(\mathbf{i}) = 1$, $ar(\mathbf{a}) = ar(\mathbf{b}) = 0$. Here are several example of terms: \mathbf{a} , \mathbf{b} , \mathbf{x} , \mathbf{y} , \mathbf{x}_1 , \mathbf{y}' , $\mathbf{i}(\mathbf{a})$, $\mathbf{i}(\mathbf{x})$, $\mathbf{i}(\mathbf{i}(\mathbf{a}))$, $\mathbf{i}(\mathbf{i}(\mathbf{x}))$, $\mathbf{f}(\mathbf{a}, \mathbf{b})$, $\mathbf{i}(\mathbf{f}(\mathbf{a}, \mathbf{b}))$, $\mathbf{f}(\mathbf{f}(\mathbf{x}, \mathbf{a}), \mathbf{f}(\mathbf{y}, \mathbf{y}))$.

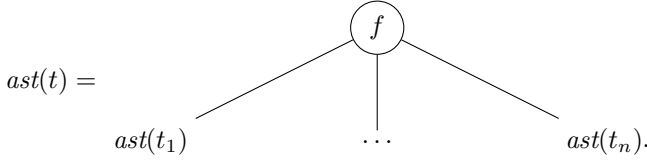
Exercise 15. Identify the Σ -terms in the following list :

1. $\mathbf{i}(\mathbf{i}(\mathbf{x}))$;
2. \mathbf{i} ;
3. $\mathbf{f}(\mathbf{x}, \mathbf{x})$;
4. $\mathbf{P}(\mathbf{a}, \mathbf{b})$;
5. $\mathbf{i}(\mathbf{a}, \mathbf{a})$;
6. $\mathbf{f}(\mathbf{i}(\mathbf{x}), \mathbf{i}(\mathbf{x}))$;
7. $\mathbf{f}(\mathbf{i}(\mathbf{x}, \mathbf{x}))$;
8. $\mathbf{a}(\mathbf{i}(\mathbf{x}))$.

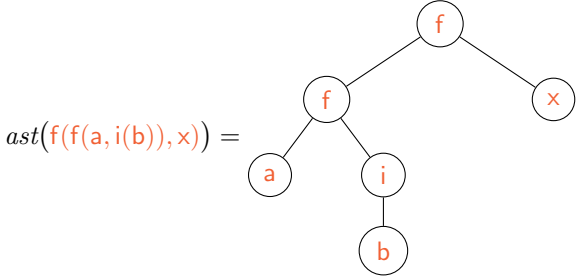
We denote terms by the letters t, s, t_1, t_2, s_1, t' , etc. Even if terms are usually written as a string of characters, they have an associated abstract syntax tree defined as follows:

1. if $t = c$ and $c \in \mathcal{F}_0$, then $ast(t) = \bigcirc c$;
2. if $t = x$ and $x \in \mathcal{X}$, then $ast(t) = \bigcirc x$;

3. if $t = f(t_1, \dots, t_n)$ and $f \in \mathcal{F}_n$ ($n > 0$), $t_1, \dots, t_n \in \mathcal{T}$, then



Remark. Even if formally terms are defined as strings of characters over the alphabet described above, these must be understood as being trees. In any software program that handles terms, these are stored as a rooted tree. Here is the tree associated to the term $f(f(a, i(b)), x)$:



Exercise 16. Compute the abstract syntax trees for all terms in Example 14.

3.3 Atomic formulae

Definition 17 (Atomic formula). An atomic formula is any string of characters of the form $P(t_1, \dots, t_n)$, unde $P \in \mathcal{P}_n$, where $P \in \mathcal{P}_n$ is a predicate symbol of arity n , and $t_1, \dots, t_n \in \mathcal{T}$ are terms.

Example 18. Continuing the previous example, we work over the signature

$$\Sigma = (\{P, Q\}, \{f, i, a, b\}),$$

where $ar(P) = ar(Q) = 2$, $ar(f) = 2$, $ar(i) = 1$, $ar(a) = ar(b) = 0$.

Here are a few examples of atomic formulae: $P(a, b)$, $P(x, y)$, $Q(i(i(x)), f(x, x))$, $Q(a, b)$, $P(f(f(a, i(x)), b), i(x))$.

Exercise 19. Explain why $P(a)$, $P, i(i(x))$ are not atomic formulae over the signature in Example 18.

3.4 First-Order Formulae

Definition 20 (First-Order Formula). The set of first-order formulae, written FOL , is the smallest set with the following properties:

1. (base case) any atomic formula is a formula (that is $P(t_1, \dots, t_n) \in \text{FOL}$ for any predicate symbol $P \in \mathcal{P}_n$ and any terms t_1, \dots, t_n ; if $n = 0$, we write P instead of $P()$);
2. (inductive cases) for any formulae $\varphi, \varphi_1, \varphi_2 \in \text{FOL}$, for any variable $x \in \mathcal{X}$, we have:
 - (a) $\neg\varphi_1 \in \text{FOL}$;
 - (b) $(\varphi_1 \wedge \varphi_2) \in \text{FOL}$;
 - (c) $(\varphi_1 \vee \varphi_2) \in \text{FOL}$;
 - (d) $(\varphi_1 \rightarrow \varphi_2) \in \text{FOL}$;
 - (e) $(\varphi_1 \leftrightarrow \varphi_2) \in \text{FOL}$;
 - (f) $(\forall x.\varphi) \in \text{FOL}$;
 - (g) $(\exists x.\varphi) \in \text{FOL}$.

Remark. In Definition 20, we find the logical connectives $\neg, \wedge, \vee, \rightarrow$ and \leftrightarrow from propositional logic. The predicate symbols of arity 0 play the role of propositional variables (for now, at the syntactic level). The constructions $(\forall x.\varphi)$ and $(\exists x.\varphi)$ are new.

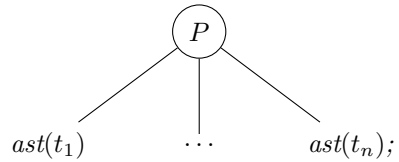
Example 21. Recall $\Sigma = (\{\mathbf{P}, \mathbf{Q}\}, \{\mathbf{f}, \mathbf{i}, \mathbf{a}, \mathbf{b}\})$ from Example 14, where $\text{ar}(\mathbf{P}) = \text{ar}(\mathbf{Q}) = 2$, $\text{ar}(\mathbf{f}) = 2$, $\text{ar}(\mathbf{i}) = 1$ și $\text{ar}(\mathbf{a}) = \text{ar}(\mathbf{b}) = 0$.

Here are several example of first-order formulae:

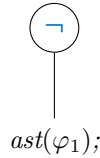
1. $\mathbf{P}(\mathbf{a}, \mathbf{b})$;
2. $\mathbf{Q}(\mathbf{a}, \mathbf{b})$;
3. $\mathbf{P}(\mathbf{a}, \mathbf{x})$;
4. $\neg\mathbf{P}(\mathbf{a}, \mathbf{b})$;
5. $(\mathbf{P}(\mathbf{a}, \mathbf{b}) \wedge \neg\mathbf{Q}(\mathbf{a}, \mathbf{b}))$;
6. $(\mathbf{P}(\mathbf{a}, \mathbf{b}) \vee \neg\mathbf{Q}(\mathbf{x}, \mathbf{y}))$;
7. $(\mathbf{P}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{P}(\mathbf{a}, \mathbf{b}))$;
8. $((\mathbf{P}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{P}(\mathbf{a}, \mathbf{b})) \leftrightarrow (\mathbf{P}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{P}(\mathbf{a}, \mathbf{b})))$;
9. $(\forall \mathbf{x}.\mathbf{P}(\mathbf{a}, \mathbf{x}))$;
10. $(\exists \mathbf{x}.\neg\mathbf{Q}(\mathbf{x}, \mathbf{y}))$.

Definition 22 (The Abstract Syntax Tree of formulae in FOL). *Formulae have an associated abstract syntax tree defined as follows:*

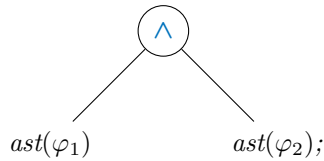
1. if $\varphi = P(t_1, \dots, t_n)$, then $ast(\varphi) =$



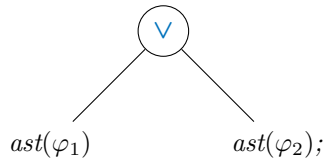
2. if $\varphi = \neg \varphi_1$, then $ast(\varphi) =$



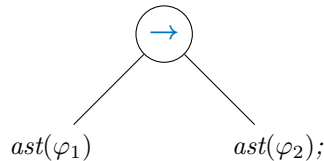
3. if $\varphi = (\varphi_1 \wedge \varphi_2)$, then $ast(\varphi) =$



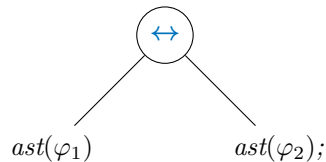
4. if $\varphi = (\varphi_1 \vee \varphi_2)$, then $ast(\varphi) =$



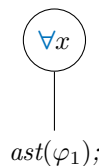
5. if $\varphi = (\varphi_1 \rightarrow \varphi_2)$, then $ast(\varphi) =$



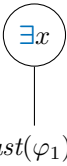
6. if $\varphi = (\varphi_1 \leftrightarrow \varphi_2)$, then $ast(\varphi) =$



7. if $\varphi = (\forall x. \varphi_1)$, then $ast(\varphi) =$



8. if $\varphi = (\exists x.\varphi_1)$, then $ast(\varphi) =$



Exercise 23. Compute the abstract syntax tree of the formulae shown in Example 21.

3.5 Model English sentences as formulae in FOL

Next, we will explain the signature used to model in first-order logic the statements *All men are mortal*, *Socrates is a man* and respectively *Socrates is mortal*.

First, we identify the predicates in the text. We have two unary predicates *is a man* and respectively *is mortal*. We choose the predicate symbol **H** for the first predicate and the predicate symbol **M** for the second predicate. We also have one constant in the text: Socrates. We choose the functional symbol **s** (of arity 0) for this constant. Therefore, to model the statements above, we shall work in the signature

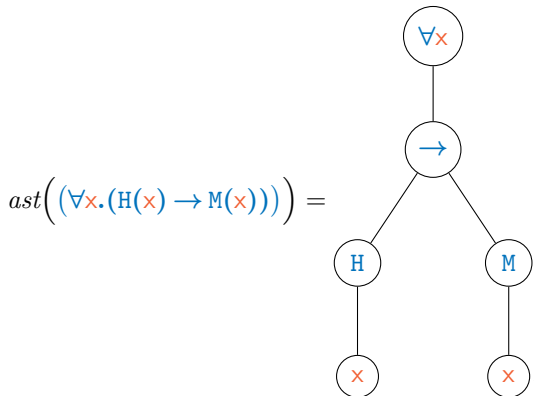
$$\Sigma = (\{\mathbf{H}, \mathbf{M}\}, \{\mathbf{s}\}),$$

where **H** and **M** are predicate symbols of arity $ar(\mathbf{H}) = ar(\mathbf{M}) = 1$, and **s** is a functional symbol of arity $ar(\mathbf{s}) = 0$.

The statement *All men are mortal* will be modelled by the first-order formula

$$(\forall x.(\mathbf{H}(x) \rightarrow \mathbf{M}(x))),$$

whose abstract syntax tree is:



The statement *Scorates is a man* shall be modelled by the atomic formula $\mathbf{H}(\mathbf{s})$, and the statement *Socrates is mortal* by the atomic formula $\mathbf{M}(\mathbf{s})$.

For the signature $\Sigma = (\{\mathbf{H}, \mathbf{M}\}, \{\mathbf{s}\})$ fixed above, there exist several possible Σ -structures. An example of a Σ -structure would be $S = (D, \{\mathbf{H}^S, \mathbf{M}^S\}, \{\mathbf{s}^S\})$ defined as follows:

1. D is the set of all beings on Earth;
2. $\mathbf{H}^S(x)$ is true for any being x that is a man;
3. $\mathbf{M}^S(x)$ is true of any being x (all of the elements in the domain are mortal);
4. \mathbf{s}^S is Socrates (Socrates, being a being, belongs to the set D).

Anticipating a little bit (we shall discuss the semantics of first-order formulae in the next lecture), all tree formulae discussed in this section, $(\forall x.(\mathbf{H}(x) \rightarrow \mathbf{M}(x)))$, $\mathbf{H}(\mathbf{s})$ and respectively $\mathbf{M}(\mathbf{s})$, are true in the structure S defined above.

In fact, the quality of the argument *All men are mortal; Socrates is a man; so: Socrates is mortal* is given by the fact that the formula $\mathbf{M}(\mathbf{s})$ is necessarily true in *any* structure in which the formulae $(\forall x.(\mathbf{H}(x) \rightarrow \mathbf{M}(x)))$ and respectively $\mathbf{H}(\mathbf{s})$ are true, not just in the structure S above.

3.6 Model arithmetic sentences as formulae in FOL

Consider the signature $\Sigma = (\{<, =\}, \{+, -, 0, 1\})$, where $<$ and $=$ are predicate symbols of arity 2, $+$ is a functional symbol of arity 2, $-$ is a functional symbol of arity 1, and 0 and 1 are constant symbols.

Here are a few first-order formulae in the first-order language associated to the signature Σ :

1. $(\forall x.(\forall y.(<(x, y) \rightarrow \exists z.(<(x, z) \wedge <(z, y))))$;
2. $(\forall x.(\forall y.(\exists z.(=(+(x, y), z))))$;
3. $(\forall x.(<(0, x) \vee =(0, x)))$;
4. $(\forall x.(\exists y.(=(x, -(y))))$;
5. $=(+(x, y), z)$.

Many times, in the case of binary predicate symbols and binary functional symbols, we use the infix notation (e.g., $x < y$ instead of $<(x, y)$). In this case, we could write the formulae above as follows:

1. $(\forall x. (\forall y. (x < y \rightarrow \exists z. (x < z \wedge z < y))))$;
2. $(\forall x. (\forall y. (\exists z. (x + y = z))))$;
3. $(\forall x. (0 < x \vee 0 = x))$;
4. $(\forall x. (\exists y. (x = -(y))))$;
5. $x + y = z$.

Two of the possible Σ -structures are $S_1 = (\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ and $S_2 = (\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$, where the predicates and functions are those known from mathematics (with the remark that $-$ is the unary minus function).

Anticipating the next lecture, on the semantics of first-order formulae, the first formula is false in S_2 and true in S_1 . The second and the fourth formula are true both in S_1 and in S_2 . The third formulae is false both in S_1 and in S_2 . The truth value of the fifth formula depends not only of the structure where we evaluate the truth value of the formula, but also on the values of the variables x, y, z . Because the variables x, y, z are not protected by a quantifier in formula number 5, they are called *free variables*. Formula number 5 is *satisfiable* both in the structure S_1 as well as in the structure S_2 , because in both cases there are values for the variables x, y, z that make the formula true (e.g. the values 1, 2, 3 for x, y , and respectively z).

3.7 Exercises

Exercise 24. *Identify a signature for the following statements and model the statements as formulae in first-order logic over that signature.*

John is a student. Any student learns Logic. Anyone learning Logic passes the exam. Any student is a person. There is a person who did not pass the exam. Therefore: not all persons are students.

Exercise 25. *Consider $S = (\mathbb{R}, \{Nat, Int, Prime, Even, >\}, \{+, 0, 1, 2\})$, a structure where Nat , Int , $Prime$, $Even$ are unary predicates with the following meaning: $Nat(u) = "u \text{ is a natural number}"$, $Int(u) = "u \text{ is an integer number}"$, $Prime(u) = "u \text{ is a prime}"$ and $Even(u) = "u \text{ is an even number}"$. The binary predicate $>$ is the "greater than" relation over real numbers. The*

function $+$ is the addition function for real numbers. The constants $0, 1, 2$ are what you would expect.

1. Find a signature Σ so that the structure S is a Σ structure;
2. Model the following statements as first-order formulae in the signature associated to the structure S above:
 - (a) Any natural number is also an integer.
 - (b) The sum of any two natural numbers is a natural number.
 - (c) No matter how we would choose a natural number, there is prime number that is greater than the number we chose.
 - (d) If any natural number is a prime number, then zero is a prime number.
 - (e) No matter how we choose a prime number, there is a prime number greater than it.
 - (f) The sum of two even numbers is an even number.
 - (g) Any prime number greater than 2 is odd.
 - (h) Any prime number can be written as the sum of four prime numbers.
 - (i) The sum of two even numbers is an odd number.
 - (j) Any even number is the sum of two primes.

Exercise 26. Give examples of 5 terms over the signature in Exercise 25 (1.) and compute their abstract syntax tree.

Exercise 27. Give examples of 5 formulae over the signature in Exercise 25 (1.) and compute their abstract syntax tree.

Exercise 28. Compute the abstract syntax tree of the following formulae:

1. $(P(x) \vee (P(y) \wedge \neg P(z)))$;
2. $((\neg \neg P(x) \vee P(y)) \rightarrow (P(x) \wedge \neg P(z)))$;
3. $(\forall x. (\forall y. ((\neg \neg P(x) \vee P(y)) \rightarrow (P(x) \wedge \neg P(z))))$;
4. $(\forall x. (\forall y. ((\neg \neg P(x) \vee P(y)) \rightarrow (\exists x. (P(x) \wedge \neg P(x)))))$;
5. $(\forall x'. \neg (\forall x. (P(x) \wedge (\exists y. ((Q(x, y) \vee \neg Q(z, z)) \rightarrow (\exists z'. P(z'))))))$.

Chapter 4

The Semantics of First-Order Formulae

The syntax of first-order logic tells us what is, from a syntactical point of view, a formula in first-order logic. On the other hand, the *semantics* of the logic refers to the *meaning* of formulae. The semantics of a formula (or its meaning) is a truth value. In general, just as for propositional logic, the truth value of a formula depends not only on the formula itself, but also on some extra information. In the case of first-order logic, the extra information is *mathematical structure* in which the formula is evaluated.

We recall that a signature $\Sigma = (\mathcal{P}, \mathcal{F})$ is a pair consisting of predicate symbols \mathcal{P} and functional symbols \mathcal{F} , each symbol having an arity.

In this chapter, we work over the signature $\Sigma = (\{\mathbf{P}\}, \{\mathbf{f}, \mathbf{i}, \mathbf{e}\})$, where \mathbf{P} is a predicate symbol of arity 2, and \mathbf{f} , \mathbf{i} and \mathbf{e} are function symbols of arity 2, 1 and respectively 0. Therefore, $\mathcal{P}_2 = \{\mathbf{P}\}$, $\mathcal{P}_1 = \emptyset$, $\mathcal{P}_0 = \emptyset$, $\mathcal{F}_2 = \{\mathbf{f}\}$, $\mathcal{F}_1 = \{\mathbf{i}\}$, and $\mathcal{F}_0 = \{\mathbf{e}\}$.

We recall that, if $\Sigma = (\mathcal{P}, \mathcal{F})$ is a signature, a Σ -structure is a tuple $S = (D, \text{Pred}, \text{Fun})$, where:

- D is a non-empty set called the *domain* of the structure;
- for each predicate symbol $P \in \mathcal{P}$ there is a predicate $P^S \in \text{Pred}$ having the same arity as P ;
- for each function symbol $f \in \mathcal{F}$ there is a predicate $f^S \in \text{Pred}$ is a function having the same arity as f .

Example 29. Here are some examples of Σ -structures:

1. $S_1 = (\mathbb{Z}, \{=\}, \{+, -, 0\})$;

2. $S_2 = (\mathbb{R}^*, \{=\}, \{\times, \cdot^{-1}, 1\})$;
3. $S_3 = (\mathbb{N}, \{=\}, \{+, s, 0\})$;
4. $S_4 = (\mathbb{N}, \{<\}, \{+, s, 0\})$.
5. $S_5 = (\mathbb{Z}, \{<\}, \{+, -, 0\})$.

The structure S_1 has the domain \mathbb{Z} (the set of integers), the predicate associated to the predicate symbol \mathbf{P} is $=$ (the equality predicate for integers), the function $+$ is the addition function for integers and it is associated to the function symbol \mathbf{f} , $-$ is the unary minus function associated to the function symbol \mathbf{i} , and the constant symbol \mathbf{e} has 0 as the associated constant.

The structure S_2 has the domain \mathbb{R}^* (the set of positive reals), the predicate associated to the predicate symbol \mathbf{P} is $=$ (the equality predicate over positive reals), the function \times is the multiplication function over positive reals associated to the function symbol \mathbf{f} , \cdot^{-1} is the unary function associated to the function symbol \mathbf{i} that computes the inverse of a positive real number (e.g. $5^{-1} = \frac{1}{5}$, and $\frac{1}{10}^{-1} = 10$), and the constant 1 is associated to the constant symbol \mathbf{e} .

The structure S_3 has the domain \mathbb{N} (the set of naturals), the predicate associated to the predicate symbol \mathbf{P} is $=$ (the equality predicate for natural numbers), the function $+$ is the addition function for naturals associated to the function symbol \mathbf{f} , s is the successor function (which associates to a natural number the next natural number – e.g., $s(7) = 8$) associated to the function symbol \mathbf{i} , and the constant 0 is associated to the constant symbol \mathbf{e} .

The structure S_4 has the domain \mathbb{N} (the set of naturals), the predicate associated to the symbol \mathbf{P} is $<$ (the smaller than relation over naturals), the function $+$ is the addition function for naturals associated to the function symbol \mathbf{f} , s is the successor function associated to the function symbol \mathbf{i} , and the constant 0 is associated to the constant symbol \mathbf{e} .

The structure S_5 is similar to S_1 , but the predicate symbol \mathbf{P} is associated with the less than relation instead of equality.

Using the notation above, we have that $\mathbf{P}^{S_4} = <$, $\mathbf{f}^{S_2} = \times$, and $\mathbf{e}^{S_1} = 0$.

4.1 Assignments

As in the propositional logic, in order to obtain the truth value of a formula in a structure, we have to start by fixing some concrete values to the syntactic symbols over which the formula is built. In the case of first order logic, we start with the variables.

Definition 30 (Assignment). *Let Σ be a signature and let S be Σ -structure with the domain D .*

An S -assignment is any function

$$\alpha : \mathcal{X} \rightarrow D.$$

Example 31. *Consider the S_1 -assignment $\alpha_1 : \mathcal{X} \rightarrow \mathbb{Z}$ defined as follows:*

1. $\alpha_1(\mathbf{x}_1) = 5$;
2. $\alpha_1(\mathbf{x}_2) = 5$;
3. $\alpha_1(\mathbf{x}_3) = 6$;
4. $\alpha_1(x) = 0$ for all $x \in \mathcal{X} \setminus \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

Example 32. *Consider the S_1 -assignment $\alpha_2 : \mathcal{X} \rightarrow \mathbb{Z}$ defined as follows:*

1. $\alpha_2(\mathbf{x}_1) = 6$;
2. $\alpha_2(\mathbf{x}_2) = 5$;
3. $\alpha_2(\mathbf{x}_3) = 6$;
4. $\alpha_2(x) = 0$ for all $x \in \mathcal{X} \setminus \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

In the presence of an assignment α we can compute the value of a term in the assignment α . For this, we use an extension of α , denoted $\bar{\alpha}$,

$$\bar{\alpha} : \mathcal{T} \rightarrow D,$$

which is defined below.

Definition 33 (The Value of a Term in an Assignment). *Given an S -assignment α and a term $t \in \mathcal{T}$ over the signature Σ , the value of the term t in the assignment α is an element of the domain D , denoted by $\bar{\alpha}(t)$, and computed recursively as follows:*

1. $\bar{\alpha}(c) = c^S$ if $c \in \mathcal{F}_0$ is a constant symbol;
2. $\bar{\alpha}(x) = \alpha(x)$ if $x \in \mathcal{X}$ is a variable;
3. $\bar{\alpha}(f(t_1, \dots, t_n)) = f^S(\bar{\alpha}(t_1), \dots, \bar{\alpha}(t_n))$ if $f \in \mathcal{F}_n$ is a function symbol of arity n , and t_1, \dots, t_n are terms.

Example 34. Continuing Example 31, where α_1 is an S_1 -assignment, we have that

$$\begin{aligned}
 \overline{\alpha_1}(\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e})) &= \overline{\alpha_1}(\mathbf{i}(\mathbf{x}_1)) + \overline{\alpha_1}(\mathbf{e}) \\
 &= -(\overline{\alpha_1}(\mathbf{x}_1)) + \mathbf{e}^{S_1} \\
 &= -(\alpha_1(\mathbf{x}_1)) + 0 \\
 &= -5 + 0 \\
 &= -5.
 \end{aligned}$$

Therefore, the value of the term $\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e})$ in the assignment α_1 is -5 .

Definition 35 (Updating an Assignment). Given an assignment α , a variable $x \in \mathcal{X}$ and a value $u \in D$, we denote by $\alpha[x \mapsto u]$ a new assignment, obtained from α by updating the value of the variable x to u , formally defined as follows:

$$\alpha[x \mapsto u] : \mathcal{X} \rightarrow D, \text{ s.t.}$$

1. $(\alpha[x \mapsto u])(x) = u$;
2. $(\alpha[x \mapsto u])(y) = \alpha(y)$, for any $y \in \mathcal{X} \setminus \{x\}$.

Example 36. For example, the assignment $\alpha_1[\mathbf{x}_1 \mapsto 6]$ is exactly the same as the assignment α_2 defined earlier. The value of the term $\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e})$ in the assignment $\alpha_1[\mathbf{x}_1 \mapsto 6]$ is $\overline{\alpha_1[\mathbf{x}_1 \mapsto 6]}(\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e})) = -6$.

Exercise 37. Compute the values below:

1. $\overline{\alpha_1[\mathbf{x}_1 \mapsto 10]}(\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e}))$;
2. $\overline{\alpha_1[\mathbf{x}_2 \mapsto 10]}(\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e}))$;
3. $\overline{\alpha_1[\mathbf{x}_2 \mapsto 10][\mathbf{x}_1 \mapsto 10]}(\mathbf{f}(\mathbf{i}(\mathbf{x}_1), \mathbf{e}))$.

4.2 The Truth Value of a First-Order Logic Formula

Let Σ be a signature, let S be a Σ -structure and let α be an S -assignment. Having fixed the elements above, we may compute the truth value of a first-order formula constructed over the signature Σ .

By writing $S, \alpha \models \varphi$ we denote the fact that the formula φ is true in the structure S with the assignment α . By writing $S, \alpha \not\models \varphi$ we denote the fact

that the formula φ is not true (is false) in the structure S with the assignment α .

The notation $S, \alpha \models \varphi$ can also be read as S satisfies φ with the assignment α , and $S, \alpha \not\models \varphi$ can also be read as S does not satisfy φ with the assignment α .

Definition 38. The fact that a structure S satisfies a formula φ with an assignment α (or equivalently, that φ is true in the structure S with the assignment α) is defined inductively as follows:

1. $S, \alpha \models P(t_1, \dots, t_n)$ iff $P^S(\bar{\alpha}(t_1), \dots, \bar{\alpha}(t_n))$;
2. $S, \alpha \models \neg \varphi$ iff $S, \alpha \not\models \varphi$;
3. $S, \alpha \models (\varphi_1 \wedge \varphi_2)$ iff $S, \alpha \models \varphi_1$ and $S, \alpha \models \varphi_2$;
4. $S, \alpha \models (\varphi_1 \vee \varphi_2)$ iff $S, \alpha \models \varphi_1$ or $S, \alpha \models \varphi_2$;
5. $S, \alpha \models (\varphi_1 \rightarrow \varphi_2)$ iff $S, \alpha \not\models \varphi_1$ or $S, \alpha \models \varphi_2$;
6. $S, \alpha \models (\varphi_1 \leftrightarrow \varphi_2)$ iff (1) both $S, \alpha \models \varphi_1$ and $S, \alpha \models \varphi_2$, or (2) $S, \alpha \not\models \varphi_1$ and $S, \alpha \not\models \varphi_2$;
7. $S, \alpha \models (\exists x. \varphi)$ iff there is $u \in D$ such that $S, \alpha[x \mapsto u] \models \varphi$;
8. $S, \alpha \models (\forall x. \varphi)$ iff for all $u \in D$, we have that $S, \alpha[x \mapsto u] \models \varphi$.

Example 39. We shall work over the signature $\Sigma = (\{\mathbf{P}\}, \{\mathbf{f}, \mathbf{i}, \mathbf{e}\})$, the Σ -structure $S_1 = (\mathbb{Z}, \{=\}, \{+, -, 0\})$ and the S_1 -assignments α_1, α_2 .

We have that

$$\begin{aligned}
 S_1, \alpha_1 \models \mathbf{P}(\mathbf{x}_1, \mathbf{x}_1) &\text{ iff } & P^{S_1}(\bar{\alpha}_1(\mathbf{x}_1), \bar{\alpha}_1(\mathbf{x}_1)) \\
 &\text{ iff } & \bar{\alpha}_1(\mathbf{x}_1) = \bar{\alpha}_1(\mathbf{x}_1) \\
 &\text{ iff } & \alpha_1(\mathbf{x}_1) = \alpha_1(\mathbf{x}_1) \\
 &\text{ iff } & 5 = 5.
 \end{aligned}$$

As $5 = 5$, we have that $S_1, \alpha_1 \models \mathbf{P}(\mathbf{x}_1, \mathbf{x}_1)$, meaning that the formula $\mathbf{P}(\mathbf{x}_1, \mathbf{x}_1)$ is true in the structure S_1 with the assignment α_1 . Equivalently, we say that S_1 satisfies $\mathbf{P}(\mathbf{x}_1, \mathbf{x}_1)$ with the assignment α_1 .

Example 40. Continuing the previous example, we have that

$$\begin{aligned}
 S_1, \alpha_1 \models \mathbf{P}(\mathbf{x}_1, \mathbf{x}_3) &\text{ iff } & P^{S_1}(\bar{\alpha}_1(\mathbf{x}_1), \bar{\alpha}_1(\mathbf{x}_3)) \\
 &\text{ iff } & \bar{\alpha}_1(\mathbf{x}_1) = \bar{\alpha}_1(\mathbf{x}_3) \\
 &\text{ iff } & \alpha_1(\mathbf{x}_1) = \alpha_1(\mathbf{x}_3) \\
 &\text{ iff } & 5 = 6.
 \end{aligned}$$

As $5 \neq 6$, we have that $S_1, \alpha_1 \not\models P(x_1, x_3)$, meaning that $P(x_1, x_3)$ is false in the structure S_1 with the assignment α_1 . Equivalently, we say that S_1 does not satisfy $P(x_1, x_3)$ with the assignment α_1 .

Example 41. Continuing the previous example, we have that

$$\begin{aligned}
 S_1, \alpha_1 \models \neg P(x_1, x_3) & \text{ iff} & S_1, \alpha_1 \not\models P(x_1, x_3) \\
 & \text{ iff} & \text{not } P^{S_1}(\overline{\alpha_1}(x_1), \overline{\alpha_1}(x_3)) \\
 & \text{ iff} & \text{not } \overline{\alpha_1}(x_1) = \overline{\alpha_1}(x_3) \\
 & \text{ iff} & \overline{\alpha_1}(x_1) \neq \overline{\alpha_1}(x_3) \\
 & \text{ iff} & \alpha_1(x_1) \neq \alpha_1(x_3) \\
 & \text{ iff} & 5 \neq 6.
 \end{aligned}$$

As $5 \neq 6$, we have that $S_1, \alpha_1 \models \neg P(x_1, x_3)$, meaning that the formula $\neg P(x_1, x_3)$ is true in the structure S_1 with the assignment α_1 . Equivalently, we say that S_1 satisfies $\neg P(x_1, x_3)$ with the assignment α_1 .

Example 42. Continuing the previous example, we have that

$$\begin{aligned}
 S_1, \alpha_1 \models P(x_1, x_1) \wedge \neg P(x_1, x_3) & \text{ iff} \\
 S_1, \alpha_1 \models P(x_1, x_1) \text{ și } S_1, \alpha_1 \models \neg P(x_1, x_3) & \text{ iff} \\
 \dots \text{ and } \dots & \text{ iff} \\
 5 = 5 \text{ and } 5 \neq 6. &
 \end{aligned}$$

As $5 = 5$ și $5 \neq 6$, we have that $S_1, \alpha_1 \models P(x_1, x_1) \wedge \neg P(x_1, x_3)$.

Example 43. Continuing the previous example, we have that

$$S_1, \alpha_1 \models P(x_1, x_3) \vee P(x_1, x_1) \text{ iff } S_1, \alpha_1 \models P(x_1, x_3) \text{ or } S_1, \alpha_1 \models P(x_1, x_1).$$

We have already seen that $S_1, \alpha_1 \models P(x_1, x_3)$, so $S_1, \alpha_1 \models P(x_1, x_3) \vee P(x_1, x_1)$ (even if $S_1, \alpha_1 \not\models P(x_1, x_1)$).

Example 44. Continuing the previous example, we have that

$$\begin{aligned}
 S_1, \alpha_1 \models \exists x_1. P(x_1, x_3) & \text{ iff} \\
 \text{exists } u \in D \text{ a.î. } S_1, \alpha_1[x_1 \mapsto u] \models P(x_1, x_3) & \text{ iff} \\
 \text{exists } u \in D \text{ a.î. } P^{S_1}(\overline{\alpha_1[x_1 \mapsto u]}(x_1), \overline{\alpha_1[x_1 \mapsto u]}(x_3)) & \text{ iff} \\
 \text{exists } u \in D \text{ a.î. } \overline{\alpha_1[x_1 \mapsto u]}(x_1) = \overline{\alpha_1[x_1 \mapsto u]}(x_3) & \text{ iff} \\
 \text{exists } u \in D \text{ a.î. } \alpha_1[x_1 \mapsto u](x_1) = \alpha_1[x_1 \mapsto u](x_3) & \text{ iff} \\
 \text{exists } u \in D \text{ a.î. } u = \alpha_1(x_3) & \text{ iff} \\
 \text{exists } u \in D \text{ a.î. } u = 6. &
 \end{aligned}$$

Since there exists u (namely $u = 6$) so that $u = 6$, we have that $S_1, \alpha_1 \models \exists x_1. P(x_1, x_3)$.

Example 45. Continuing the previous example, we have that

$$\begin{array}{ll}
 S_1, \alpha_1 \models \forall x_1. \exists x_3. P(x_1, x_3) & \text{iff} \\
 \text{for all } u \in D, \text{ we have } S_1, \alpha_1[x_1 \mapsto u] \models \exists x_3. P(x_1, x_3) & \text{iff} \\
 \text{for all } u \in D, \text{ there is } v \in D \text{ a.i. } S_1, \alpha_1[x_1 \mapsto u][x_3 \mapsto v] \models P(x_1, x_3) & \text{iff} \\
 \dots & \text{iff} \\
 \text{for all } u \in D, \text{ there is } v \in D \text{ a.i. } u = v. &
 \end{array}$$

As for any integer u , there exists an integer v so that $u = v$, we have that $S_1, \alpha_1 \models \forall x_1. \exists x_3. P(x_1, x_3)$.

Exercise 46. Show that $S_1, \alpha_1 \models \forall x_1. \exists x_3. P(x_1, i(x_3))$.

4.3 Satisfiability in a Fixed Structure

Definition 47 (Satisfiability in a Fixed Structure). A formula φ is satisfiable in a structure S if there exists an S -assignment α such that

$$S, \alpha \models \varphi.$$

Example 48. The formula $P(x_1, x_3)$ is satisfiable in the structure S_1 because there is an assignment, for instance α_2 , such that $S_1, \alpha_2 \models P(x_1, x_3)$.

Exercise 49. Show that $\neg P(x_1, x_1)$ is not satisfiable in S_1 (because, for each assignment α , we have $S_1, \alpha \not\models \neg P(x_1, x_1)$).

4.4 Validity in a Fixed Structure

Definition 50 (Validity in a Fixed Structure). A formula φ is valid in a structure S if, for any S -assignment α , we have that

$$S, \alpha \models \varphi.$$

Example 51. The formula $P(x_1, x_3)$ is not valid in the structure S_1 , because there exists an assignment, namely α_1 , having the property that $S_1, \alpha_1 \not\models P(x_1, x_3)$.

Exercise 52. Show that $P(x_1, x_1)$ is valid in the structure S_1 (because, for any assignment α , $S_1, \alpha \models P(x_1, x_1)$).

4.5 Satisfiability

Definition 53 (Satisfiability). A formula φ is satisfiable if there exists a structure S and an S -assignment α such that

$$S, \alpha \models \varphi.$$

Example 54. The formula $\neg P(x_1, x_1)$ is satisfiable, because there exists a structure (namely S_5) and an S_5 -assignment (namely α_1) so that $S_5, \alpha_1 \models \neg P(x_1, x_1)$ (because $5 \not\prec 5$).

Remark. Because S_5 and S_1 have the same domain, the assignment α_1 is both an S_1 -assignment as well as an S_5 -assignment.

Remark. A formula could be unsatisfiable in a fixed structure (for example $\neg P(x_1, x_1)$ is not satisfiable in the structure S_1) but it could still be satisfiable (for example $\neg P(x_1, x_1)$, as we have seen above).

4.6 Validity

Definition 55 (Validity). A formula φ is valid if, for any structure S and for any S -assignment α , we have that

$$S, \alpha \models \varphi.$$

Example 56. The formula $P(x_1, x_1)$ is not valid, because $S_5, \alpha_1 \not\models P(x_1, x_1)$.

The formula $P(x_1, x_1) \rightarrow P(x_1, x_1)$ is valid.

Remark. A formula could be valid in a fixed structure (for example $P(x_1, x_1)$ is valid in the structure S_1) and still not be valid (for example, $P(x_1, x_1)$ is not valid because $S_5, \alpha_1 \not\models P(x_1, x_1)$).

4.7 Semantical Consequence

Definition 57. A formula φ is a semantical (or logical) consequence of the formulae $\varphi_1, \dots, \varphi_n$ in a fixed structure S , denoted by $\varphi_1, \dots, \varphi_n \models_S \varphi$, if, for any S -assignment α for which $S, \alpha \models \varphi_1, S, \alpha \models \varphi_2, \dots, S, \alpha \models \varphi_n$, we also have that $S, \alpha \models \varphi$.

Example 58. We have $P(x, y) \models_{S_1} P(y, x)$, because, for any S_1 -assignment α with the property that $S_1, \alpha \models P(x, y)$ (meaning that $\alpha(x) = \alpha(y)$), we also have $S_1, \alpha \models P(y, x)$ (meaning $\alpha(y) = \alpha(x)$).

We have that $P(x, y) \not\models_{S_5} P(y, x)$, because, for the assignment $\alpha(x) = 5, \alpha(y) = 6$, we have $S_5, \alpha \models P(x, y)$ (that is, $5 < 6$), but $S_5, \alpha \not\models P(y, x)$ ($6 \not< 5$).

Definition 59. A formula φ is a *semantical (or logical) consequence* of the formulae $\varphi_1, \dots, \varphi_n$, denoted by $\varphi_1, \dots, \varphi_n \models \varphi$, if

$$\varphi_1, \dots, \varphi_n \models_S \varphi$$

for any structure S .

Example 60. We have $P(x, y) \not\models P(y, x)$, because there exists a structure (namely S_5) so that $P(x, y) \not\models_{S_5} P(y, x)$.

Exercise 61. Show that

$$\forall x. \forall y. \forall z. (P(x, y) \wedge P(y, z) \rightarrow P(x, z)), P(x_1, x_2), P(x_2, x_3) \models P(x_1, x_3).$$

Of course, in the previous notations (as in the propositional logic), the list $\varphi_1, \varphi_2, \dots, \varphi_n$ denotes the set having as elements the enumerated formulae.

4.8 Consistent set of formulae

Definition 62. A set of formulae Γ is *consistent* in a structure S if, by definition,

$$\text{there exists } \alpha \text{ s.t. } S, \alpha \models \varphi \text{ for any } \varphi \in \Gamma.$$

Example 63. We have that $\{P(x, y), P(y, x)\}$ is consistent in S_1 , but not in S_4 .

Definition 64. A set of formulae Γ is *consistent* if, by definition, there exists a structure in which Γ is consistent.

Example 65. We have that $\{P(x, y), P(y, x)\}$ is consistent (because, for example, it is consistent in S_1).

Example 66. We have that $\{P(x, y), \neg P(x, y)\}$ is not consistent.

4.9 Exercises

We recall here the structures show in Example 29:

1. $S_1 = (\mathbb{Z}, \{=\}, \{+, -, 0\})$;
2. $S_2 = (\mathbb{R}^*, \{=\}, \{\times, \cdot^{-1}, 1\})$;
3. $S_3 = (\mathbb{N}, \{=\}, \{+, s, 0\})$;
4. $S_4 = (\mathbb{N}, \{<\}, \{+, s, 0\})$.

$$5. S_5 = (\mathbb{Z}, \{<\}, \{+, -, 0\}).$$

These structures will be used in the exercises below.

Exercise 67. Establish whether the following sentences hold:

1. $S_1, \alpha_1 \models P(x_2, x_3);$
2. $S_1, \alpha_1 \models \neg P(x_2, x_3);$
3. $S_1, \alpha_1 \models \neg P(x_2, x_3) \wedge P(x_1, x_1);$
4. $S_1, \alpha_1 \models \exists x_3. P(x_2, x_3);$
5. $S_1, \alpha_1 \models \forall x_2. \exists x_3. P(x_2, x_3);$
6. $S_1, \alpha_1 \models \exists x_3. \forall x_2. P(x_2, x_3);$
7. $S_1, \alpha_2 \models \forall x_2. \exists x_3. P(x_2, i(x_3));$

Exercise 68. Find for each item below a S_2 -assignment α_3 such that:

1. $S_2, \alpha_3 \models P(x_1, x_2);$
2. $S_2, \alpha_3 \models P(f(x_1, x_2), x_3);$
3. $S_2, \alpha_3 \models P(f(x_1, x_2), i(x_3));$
4. $S_2, \alpha_3 \models P(x, e);$
5. $S_2, \alpha_3 \models \exists y. P(x, i(y));$
6. $S_2, \alpha_3 \models \forall y. \exists x. P(x, i(y)).$

Exercise 69. Show that the formulae below are valid in S_2 :

1. $\forall x. \exists y. P(x, i(y));$
2. $\forall x. P(f(x, e), x);$
3. $\forall x. P(x, i(i(x))).$

Exercise 70. Show that $\forall x. \exists y. P(x, i(y))$ is not valid in S_3 .

Exercise 71. Find a formula that is satisfiable in S_1 but not satisfiable in S_3 .

Exercise 72. Show that $\forall x. \exists y. P(x, y)$ is not valid.

Exercise 73. Show that $(\forall x. P(x, x)) \rightarrow \exists x_2. P(x_1, x_2)$ is valid.

Exercise 74. Show that $\forall x. \exists y. P(y, x)$ is not valid.

Exercise 75. Show that $\forall x. \neg P(x, x)$ is satisfiable.

Exercise 76. Show that $\forall x. \neg P(x, x) \wedge \exists x. P(x, x)$ is not satisfiable.

Chapter 5

Natural Deduction

In this chapter we present the natural deduction for first-order logic. We define the notion of *substitution*, we recall several notions that we discussed for natural deduction in propositional logic and we present an extended deductive system together with their soundness and completeness properties.

Remark. *The rules of the deductive system of the natural deduction includes the rules presented for propositional logic. The latter are explained again and exemplified on first-order formulas. The natural deduction of FOL also includes rules for quantifiers. These rules are completely new.*

5.1 Substitutions

Definition 77. *A substitution is a function $\sigma : \mathcal{X} \rightarrow \mathcal{T}$, with the property that $\sigma(x) \neq x$ for a finite number of variables $x \in \mathcal{X}$.*

Definition 78. *If $\sigma : \mathcal{X} \rightarrow \mathcal{T}$ is a substitution, the set $\text{dom}(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$ is the domain of the substitution σ .*

Remark. *By definition, the domain of a substitution is a finite set.*

Definition 79. *If $\sigma : \mathcal{X} \rightarrow \mathcal{T}$ is a substitution, then the unique extension of the substitution σ to the set of terms is the function $\sigma^\# : \mathcal{T} \rightarrow \mathcal{T}$, recursively defined as follows:*

1. $\sigma^\#(x) = \sigma(x)$, for any $x \in \mathcal{X}$;
2. $\sigma^\#(c) = c$, for any constant symbol $c \in \mathcal{F}_0$;
3. $\sigma^\#(f(t_1, \dots, t_n)) = f(\sigma^\#(t_1), \dots, \sigma^\#(t_n))$, for any functional symbol $f \in \mathcal{F}_n$ of arity $n \in \mathbb{N}^*$ and any terms $t_1, \dots, t_n \in \mathcal{T}$.

The substitutions are noted by $\sigma, \tau, \sigma_0, \tau_1, \sigma',$ etc.

Remark. If $t \in \mathcal{T}$ is a term, then $\sigma^\#(t) \in \mathcal{T}$ is the term obtained from t by applying to the substitution σ or the term obtained by applying the substitution σ on the term t .

Practically, in order to obtain $\sigma^\#(t)$ from t , all occurrences of a variable x in t are replaced simultaneously with the corresponding term $\sigma(x)$.

Example 80. Let consider the substitution $\sigma_1 : \mathcal{X} \rightarrow \mathcal{T}$ defined as follows:

1. $\sigma_1(\mathbf{x}_1) = \mathbf{x}_2$;
2. $\sigma_1(\mathbf{x}_2) = \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)$;
3. $\sigma_1(x) = x$ for all $x \in \mathcal{X} \setminus \{\mathbf{x}_1, \mathbf{x}_2\}$.

Let consider the term $t = \mathbf{f}(\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2), \mathbf{f}(\mathbf{x}_3, \mathbf{e}))$. We have that:

$$\begin{aligned}
 \sigma_1^\#(t) &= \sigma_1^\#(\mathbf{f}(\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2), \mathbf{f}(\mathbf{x}_3, \mathbf{e}))) \\
 &= \mathbf{f}(\sigma_1^\#(\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2)), \sigma_1^\#(\mathbf{f}(\mathbf{x}_3, \mathbf{e}))) \\
 &= \mathbf{f}(\mathbf{f}(\sigma_1^\#(\mathbf{x}_1), \sigma_1^\#(\mathbf{x}_2)), \mathbf{f}(\sigma_1^\#(\mathbf{x}_3), \sigma_1^\#(\mathbf{e}))) \\
 &= \mathbf{f}(\mathbf{f}(\sigma_1(\mathbf{x}_1), \sigma_1(\mathbf{x}_2)), \mathbf{f}(\sigma_1(\mathbf{x}_3), \mathbf{e})) \\
 &= \mathbf{f}(\mathbf{f}(\mathbf{x}_2, \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)), \mathbf{f}(\mathbf{x}_3, \mathbf{e})).
 \end{aligned}$$

Note that by applying a substitution over a term, we replace (in the same time) all occurrences of the variables from the domain of the substitution with the corresponding terms.

Notation. If $\text{dom}(\sigma) = \{x_1, \dots, x_n\}$, then the substitution σ can be also written as:

$$\sigma = \{x_1 \mapsto \sigma(x_1), \dots, x_n \mapsto \sigma(x_n)\}.$$

Attention, it is not a set, but only some notation for substitutions.

Example 81. For the substitution from the previous example, we have

$$\sigma_1 = \{\mathbf{x}_1 \mapsto \mathbf{x}_2, \mathbf{x}_2 \mapsto \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)\}.$$

Definition 82. If $\sigma : \mathcal{X} \rightarrow \mathcal{T}$ is a substitution and $V \subseteq \mathcal{X}$ is a subset of variables, then the restriction of the substitution σ to the set V is another substitution $\sigma|_V : \mathcal{X} \rightarrow \mathcal{T}$, defined as follows:

1. $\sigma|_V(x) = \sigma(x)$ for any $x \in V$;

2. $\sigma|_V(x) = x$ for any $x \in \mathcal{X} \setminus V$.

Example 83. For the substitution σ_1 in Example 80, we have $\sigma_1|_{\{\mathbf{x}_1\}} = \{\mathbf{x}_1 \mapsto \mathbf{x}_2\}$ și $\sigma_1|_{\{\mathbf{x}_2\}} = \{\mathbf{x}_2 \mapsto \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)\}$.

In other words, by restricting a substitution to a set of variables, we remove some the other variables from the domain of the substitution.

Definition 84. for any substitution $\sigma : \mathcal{X} \rightarrow \mathcal{T}$, the extention of σ to the set of formulae is the function $\sigma^b : \text{FOL} \rightarrow \text{FOL}$, defined as:

1. $\sigma^b(P(t_1, \dots, t_n)) = P(\sigma^b(t_1), \dots, \sigma^b(t_n))$;
2. $\sigma^b(\neg\varphi) = \neg\sigma^b(\varphi)$;
3. $\sigma^b((\varphi_1 \wedge \varphi_2)) = (\sigma^b(\varphi_1) \wedge \sigma^b(\varphi_2))$;
4. $\sigma^b((\varphi_1 \vee \varphi_2)) = (\sigma^b(\varphi_1) \vee \sigma^b(\varphi_2))$;
5. $\sigma^b((\varphi_1 \rightarrow \varphi_2)) = (\sigma^b(\varphi_1) \rightarrow \sigma^b(\varphi_2))$;
6. $\sigma^b((\varphi_1 \leftrightarrow \varphi_2)) = (\sigma^b(\varphi_1) \leftrightarrow \sigma^b(\varphi_2))$;
7. $\sigma^b((\forall x.\varphi)) = (\forall x.(\rho^b(\varphi)))$, where $\rho = \sigma|_{\text{dom}(\sigma) \setminus \{x\}}$;
8. $\sigma^b((\exists x.\varphi)) = (\exists x.(\rho^b(\varphi)))$, where $\rho = \sigma|_{\text{dom}(\sigma) \setminus \{x\}}$;

In other words, to obtain the formula $\sigma^b(\varphi)$ from φ , every free occurrence of the variable x from the formula φ is replaced by the term $\sigma(x)$.

Example 85. Using the substitution σ_1 in Example 80, we have that:

$$\begin{aligned}
 & \sigma_1^b\left(\left(\forall \mathbf{x}_2. \mathbf{P}(\mathbf{x}_1, \mathbf{x}_2)\right) \wedge \mathbf{P}(\mathbf{x}_2, \mathbf{x}_2)\right) = \\
 & \sigma_1^b\left(\left(\forall \mathbf{x}_2. \mathbf{P}(\mathbf{x}_1, \mathbf{x}_2)\right)\right) \wedge \sigma_1^b\left(\mathbf{P}(\mathbf{x}_2, \mathbf{x}_2)\right) = \\
 & \left(\forall \mathbf{x}_2. \sigma_1|_{\{\mathbf{x}_1\}}^b\left(\mathbf{P}(\mathbf{x}_1, \mathbf{x}_2)\right)\right) \wedge \mathbf{P}\left(\sigma_1|_{\{\mathbf{x}_2\}}^b(\mathbf{x}_2), \sigma_1|_{\{\mathbf{x}_2\}}^b(\mathbf{x}_2)\right) = \\
 & \left(\forall \mathbf{x}_2. \mathbf{P}\left(\sigma_1|_{\{\mathbf{x}_1\}}^b(\mathbf{x}_1), \sigma_1|_{\{\mathbf{x}_1\}}^b(\mathbf{x}_2)\right)\right) \wedge \mathbf{P}\left(\sigma_1(\mathbf{x}_2), \sigma_1(\mathbf{x}_2)\right) = \\
 & \left(\forall \mathbf{x}_2. \mathbf{P}\left(\sigma_1|_{\{\mathbf{x}_1\}}(\mathbf{x}_1), \sigma_1|_{\{\mathbf{x}_1\}}(\mathbf{x}_2)\right)\right) \wedge \mathbf{P}(\mathbf{f}(\mathbf{x}_3, \mathbf{x}_4), \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)) = \\
 & \left(\forall \mathbf{x}_2. \mathbf{P}\left(\sigma_1(\mathbf{x}_1), \mathbf{x}_2\right)\right) \wedge \mathbf{P}(\mathbf{f}(\mathbf{x}_3, \mathbf{x}_4), \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)) = \\
 & \left(\forall \mathbf{x}_2. \mathbf{P}(\mathbf{x}_2, \mathbf{x}_2)\right) \wedge \mathbf{P}(\mathbf{f}(\mathbf{x}_3, \mathbf{x}_4), \mathbf{f}(\mathbf{x}_3, \mathbf{x}_4)).
 \end{aligned}$$

Remark. Attention: the bound occurrences of variables are NOT replaced when applying the substitution! In Example 85, the occurrence of the variable \mathbf{x}_2 in $(\forall \mathbf{x}_2. \mathbf{P}(\mathbf{x}_1, \mathbf{x}_2))$ is bound.

Notation. According to Notation 5.1, for the substitutions with a finite domain, we also use the notation $\{x_1 \mapsto \sigma(x_1), \dots, x_n \mapsto \sigma(x_n)\}$. We will also use substitutions without associating a name to them, since they are simple having the form $\{x \mapsto t\}$. In order to express the fact that we apply this substitution to a formula, according to our notation, we should write $\{x \mapsto t\}(\varphi)$. However, in the literature other notations are preferred that we will also use. One variant is to write $\varphi[t/x]$. Another one is $\varphi[x \mapsto t]$. In this document we will use the latter notation, namely $\varphi[x \mapsto t]$.

5.2 Sequences

Definition 86 (Sequence). A sequence is a pair formed by a set of formulae $\{\varphi_1, \dots, \varphi_n\} \subseteq \text{FOL}$ and a formula $\varphi \in \text{FOL}$, denoted by:

$$\{\varphi_1, \dots, \varphi_n\} \vdash \varphi.$$

sometimes we read the notation $\{\varphi_1, \dots, \varphi_n\} \vdash \varphi$ as φ is a syntactic consequence from $\{\varphi_1, \dots, \varphi_n\}$. Usually, we will note with $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ the set of hypothesis and we will write the sequence as $\Gamma \vdash \varphi$.

Remark. We recall that the usual notation in the literature allows us to write $\varphi_1, \dots, \varphi_n \vdash \varphi$ (without curly brackets) instead of $\{\varphi_1, \dots, \varphi_n\} \vdash \varphi$. However, we have to remember that on the left side of the symbol \vdash is all the time a set. The notation without brackets allows us to write $\varphi_1, \dots, \varphi_n, \psi \vdash \varphi$ instead of $\{\varphi_1, \dots, \varphi_n\} \cup \{\psi\} \vdash \varphi$.

Example 87. In many examples from this material, we work with the signature $\Sigma = (\{\mathbf{P}, \mathbf{Q}\}, \{\mathbf{a}, \mathbf{b}, \mathbf{f}, \mathbf{g}\})$, where the predicate symbols \mathbf{P} and \mathbf{Q} have arity 1, the functional symbols \mathbf{f} and \mathbf{g} have arity 1, and the symbols \mathbf{a} and \mathbf{b} are constants (of arity 0).

Example 88. Let consider the signature Σ from Example 87. The following are some examples of sequences:

1. $\{\mathbf{P}(\mathbf{a}), \mathbf{Q}(\mathbf{a})\} \vdash (\mathbf{P}(\mathbf{a}) \wedge \mathbf{Q}(\mathbf{a}));$
2. $\{\forall x. \mathbf{Q}(x), \mathbf{P}(\mathbf{a})\} \vdash (\mathbf{P}(\mathbf{a}) \wedge \mathbf{Q}(\mathbf{a}));$
3. $\{\exists x. \mathbf{Q}(x)\} \vdash \mathbf{Q}(\mathbf{a}).$

Later we will see that the first two sequences from above are valid, and the last one is not valid.

It is often convenient to write sequence without curly braces as in the next example:

Example 89. The sequences in Example 88 can be written without curly braces as follows:

1. $P(a), Q(a) \vdash (P(a) \wedge Q(a));$
2. $\forall x. Q(x), P(a) \vdash (P(a) \wedge Q(a));$
3. $\exists x. Q(x) \vdash Q(a).$

5.3 Inference rules

Definition 90. An inference rule is a tuple formed from:

1. a set of sequences S_1, \dots, S_n , called hypothesis of the rule;
2. a sequence S called conclusion of the rule;
3. a condition for the applicability of the rule;
4. a name.

An inference rule is noted as follows:

$$\text{NAME} \frac{S_1 \quad \dots \quad S_n}{S} \text{ condition.}$$

Remark. The inference rules that have $n = 0$ hypothesis, are called axioms. Also, the applicability conditions may be absent.

Example 91. The following are some inference rules from propositional logic:

$$\begin{array}{ccc} \wedge i \frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}, & \wedge e_1 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_1}, & \wedge e_2 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_2}. \end{array}$$

As in the case of propositional logic, all three inference rules from above are correct. None of them has a applicability condition. The following is an example of inference rule with $n = 0$ hypothesis, but with one condition.

$$\text{HYPOTHESIS} \frac{}{\Gamma \vdash \varphi} \varphi \in \Gamma.$$

Below we have an example of incorrect inference rule (in a way that we will clarify later, but that can be already perceived).

$$\text{REGULĂ INCORECTĂ} \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}.$$

Remark. The hypothesis of the inference rule, as the conclusion, are in fact patterns for sequences and not the sequences themselves. These patterns can be instantiated, meaning that an inference rule (presented above) has several instances obtained by replacing mathematical variables $\varphi, \varphi', \Gamma$ with concrete formulae. For instance, here is a new instance for the rule $\wedge i$ from above:

$$\wedge i \frac{\frac{\{P(a), Q(a)\} \vdash P(a) \quad \{P(a), Q(a)\} \vdash Q(a)}{\{P(a), Q(a)\} \vdash (P(a) \wedge Q(a))}; \quad \frac{\{P(a), Q(a), Q(b)\} \vdash (P(a) \wedge Q(a)) \quad \{P(a), Q(a), Q(b)\} \vdash P(a)}{\{P(a), Q(a), Q(b)\} \vdash ((P(a) \wedge Q(a)) \wedge P(a))}.$$

We first replaced the mathematical variable Γ with the set of formulae $\{P(a), Q(a)\}$, the mathematical variable φ with the formula $P(a)$ and the mathematical variable φ' with the formula $Q(a)$. Exercise: establish with what was replaced each mathematical variable from the second instance.

Here is an example of rule that is not an instance of a rule $\wedge i$ (exercise: explain why not):

$$? \frac{\{P(a), Q(a)\} \vdash P(a) \quad \{P(a), Q(a)\} \vdash Q(a)}{\{P(a), Q(a)\} \vdash (P(a) \wedge Q(a))};$$

5.4 Deductive system

Definition 92. A deductive system is a set of inference rules.

Example 93. Let consider the deductive system D_1 , formed from the following four inference rules:

$$\begin{array}{l} \text{HYPOTHESIS} \frac{}{\Gamma \vdash \varphi, \varphi \in \Gamma} \quad \wedge i \frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}, \quad \wedge e_1 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_1}, \\ \wedge e_2 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_2}. \end{array}$$

5.5 Formal proof

Definition 94 (Formal proof). A formal proof in a deductive system is a list of sequences

1. S_1
2. S_2
- ...
- $n. S_n,$

with the property that each sequence S_i is justified by a inference rule of the deductive system applied on the previous sequences (S_1, \dots, S_{i-1}) , meaning that S_i is the conclusion of an instance of an inference rule from the deductive rule, rule that uses as hypothesis sequences chosen from S_1, \dots, S_{i-1} . In addition, if the inference rule has a condition, this condition has to be true. Note also that any prefix of a formal proof is also a proof.

Example 95. Here is an example of formal proof in the deductive system D_1 from above:

1. $\{P(a), Q(a)\} \vdash P(a);$ (HYPOTHESIS)
2. $\{P(a), Q(a)\} \vdash Q(a);$ (HYPOTHESIS)
3. $\{P(a), Q(a)\} \vdash (P(a) \wedge Q(a));$ ($\wedge i, 1, 2$)
4. $\{P(a), Q(a)\} \vdash (Q(a) \wedge (P(a) \wedge Q(a))).$ ($\wedge i, 2, 3$)

As in the case of propositional logic, each line has the name of the applied inference rule and the lines where the needed hypothesis are found (in the same order used to present the deductive system).

Remark. The definition of the formal proof in the first order logic is the same as in the case of propositional logic. However, we will see later that in order to apply the new inference rules, associated to the quantifiers, we will use new annotations for the lines of the formal proof.

Definition 96 (Valid sequence). A sequence $\Gamma \vdash \varphi$ is valid in a deductive system D if there is a formal proof S_1, \dots, S_n in D such that $S_n = \Gamma \vdash \varphi$.

Example 97. The sequence $\{P(a), Q(a)\} \vdash (P(a) \wedge Q(a))$ is valid in the deductive system D_1 from above because is the last sequence from the following formal proof:

1. $\{P(a), Q(a)\} \vdash P(a);$ (HYPOTHESIS)
2. $\{P(a), Q(a)\} \vdash Q(a);$ (HYPOTHESIS)
3. $\{P(a), Q(a)\} \vdash (P(a) \wedge Q(a));$ ($\wedge i, 1, 2$)

Remark. Attention! Do not mix the notions of valid sequence in a deductive system and the notion of valid formula.

5.6 Natural deduction

Natural deduction is a deductive system for the first order logic. In other words, the deductive system for first order logic includes all the rules of natural deduction from propositional logic. In addition, for first order logic we have new rules for the introduction and elimination of quantifiers. In this section we will present each inference rule from natural deduction of first order logic.

5.6.1 Rules for conjunctions

We already saw the inference rules for the introduction and elimination for the "and" connector:

$$\wedge_i \frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}, \quad \wedge_{e1} \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_1}, \quad \wedge_{e2} \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_2}.$$

This deductive system is called *natural* because the inference rules mimic the reasoning of humans, based in essence on an intuitive semantics for the notion of truth:

1. The rule for the introduction of the connector \wedge indicates that we can prove a conjunction $(\varphi_1 \wedge \varphi_2)$ from the set of hypothesis Γ if we already know that each part of the conjunction, φ_1 and respectively φ_2 , are consequences of the hypotheses from Γ .

In other words, in order to prove a conjunction from a set of hypotheses, is enough to establish individually that each part of the conjunction is a consequence of the hypothesis.

2. For the \wedge connector, we have two rules for elimination. First elimination rule for the \wedge connector says that if we already established that some conjunction $(\varphi_1 \wedge \varphi_2)$ is the consequence of a set Γ of hypotheses, then the left side of the conjunction, φ_1 , is a consequence of the set Γ .

The second rule is symmetric with respect to the first and says that we can conclude that the right side of the conjunction is the consequence of a set of formulae if the conjunction is the consequence of this set of formulae.

Here is an example of formal proof that uses the inference rules for the connector \wedge :

1. $\{(P(a) \wedge Q(a)), \forall x.P(x)\} \vdash (P(a) \wedge Q(a));$ (HYPOTHESIS)
2. $\{(P(a) \wedge Q(a)), \forall x.P(x)\} \vdash \forall x.P(x);$ (HYPOTHESIS)

3. $\{(P(a) \wedge Q(a)), \forall x.P(x)\} \vdash P(a);$ ($\wedge e_1$, 1)
4. $\{(P(a) \wedge Q(a)), \forall x.P(x)\} \vdash (P(a) \wedge \forall x.P(x)).$ ($\wedge i$, 3, 2)

5.6.2 Rules for implication

The rule for the elimination of the implication, also called *modus ponens* in latin, is one of the most important rules of inference that we apply.

$$\rightarrow e \frac{\Gamma \vdash (\varphi_1 \rightarrow \varphi_2) \quad \Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_2}$$

The rule shows that, supposing that we proved $\varphi \rightarrow \varphi'$ (from Γ) and in addition we proved that φ (also from Γ), then we can prove φ' (from Γ).

Here is an example of formal proof that uses the rule for the elimination of the implication:

1. $\{(P(a) \rightarrow \forall x.P(x)), (P(a) \wedge Q(a))\} \vdash (P(a) \wedge Q(a));$ (HYPOTHESIS)
2. $\{(P(a) \rightarrow \forall x.P(x)), (P(a) \wedge Q(a))\} \vdash P(a);$ ($\wedge e_1$, 1)
3. $\{(P(a) \rightarrow \forall x.P(x)), (P(a) \wedge Q(a))\} \vdash (P(a) \rightarrow \forall x.P(x));$ (HYPOTHESIS)
4. $\{(P(a) \rightarrow \forall x.P(x)), (P(a) \wedge Q(a))\} \vdash \forall x.P(x).$ ($\rightarrow e$, 3, 1)

This proof shows that the sequence $\{(P(a) \rightarrow \forall x.P(x)), (P(a) \wedge Q(a))\} \vdash \forall x.P(x)$ is valid, meaning that the formula $\forall x.P(x)$ is a consequence of the set of formulae $\{(P(a) \rightarrow \forall x.P(x)), (P(a) \wedge Q(a))\}$. Observe the order in which the lines 3 and 1 appear in the explanation for line 4: they follow the same order, fixed by the inference rule.

Exercise 98. *Prove that the following sequences are valid:*

1. $\{((P(a) \wedge Q(a)) \rightarrow \forall x.P(x)), P(a), Q(a)\} \vdash \forall x.P(x);$
2. $\{(P(a) \rightarrow \forall x.P(x)), P(a), Q(a)\} \vdash (Q(a) \wedge \forall x.P(x)).$

The rule for introducing the implication is subtle. In order to prove that an implication $(\varphi_1 \rightarrow \varphi_2)$ follows from Γ , we suppose φ_1 (besides Γ) and prove φ_2 . In other words, in the hypothesis for the rule, we add the formula φ_1 to the formulae from Γ . The rule may be written in two equivalent ways, that differ only by the fact that the first rule uses the convention referring to the curly brackets around premises from the notations of sequences, while in the second one the brackets appear explicitly:

$$\rightarrow i \frac{\Gamma, \varphi_1 \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \rightarrow \varphi_2)}, \quad \rightarrow i \frac{\Gamma \cup \{\varphi_1\} \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \rightarrow \varphi_2)}.$$

It is important to observe and understand for rule of introduction of implication is that the set of premises changes. In the conclusion we have that the formula $(\varphi_1 \rightarrow \varphi_2)$ follows from Γ , while in the hypothesis we have to prove that φ_2 follows from the premises $\Gamma \cup \{\varphi_1\}$. In other words, intuitively, in order to prove an implication $(\varphi_1 \rightarrow \varphi_2)$, we suppose the antecedent φ_1 and prove φ_2 .

Example 99. *Let's prove that the sequence $\{\} \vdash (P(a) \rightarrow P(a))$ is valid:*

1. $\{P(a)\} \vdash P(a);$ (HYPOTHESIS)
2. $\{\} \vdash (P(a) \rightarrow P(a)).$ ($\rightarrow i$, 1)

Example 100. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b))\} \vdash (P(a) \rightarrow P(b))$ is valid:

1. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b)), P(a)\} \vdash (P(a) \rightarrow Q(a));$ (HYPOTHESIS)
2. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b)), P(a)\} \vdash P(a);$ (HYPOTHESIS)
3. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b)), P(a)\} \vdash Q(a);$ ($\rightarrow e$, 1, 2)
4. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b)), P(a)\} \vdash (Q(a) \rightarrow P(b));$ (HYPOTHESIS)
5. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b)), P(a)\} \vdash P(b);$ ($\rightarrow e$, 4, 3)
6. $\{(P(a) \rightarrow Q(a)), (Q(a) \rightarrow P(b))\} \vdash (P(a) \rightarrow P(b)).$ ($\rightarrow i$, 5)

Exercise 101. *Prove that the following sequences are valid:*

1. $\{((P(a) \wedge Q(a)) \rightarrow P(b)), P(a), Q(a)\} \vdash P(b);$
2. $\{((P(a) \wedge Q(a)) \rightarrow P(b))\} \vdash (P(a) \rightarrow (Q(a) \rightarrow P(b)));$
3. $\{(P(a) \rightarrow (Q(a) \rightarrow P(b)))\} \vdash ((P(a) \wedge Q(a)) \rightarrow P(b)).$

5.6.3 Rules for disjunction

The connector \vee has two introduction rules:

$$\vee i_1 \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2)}, \quad \vee i_2 \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2)}.$$

The first rule shows that if we know φ_1 (from Γ), then we also know $(\varphi_1 \vee \varphi_2)$ (from Γ), no matter what φ_2 is. The second rule of elimination is symmetric, for the right side of the disjunction.

Example 102. *Let's prove that $\{(P(a) \wedge Q(a))\} \vdash (P(a) \vee Q(a))$ is valid:*

1. $\{(P(a) \wedge Q(a))\} \vdash (P(a) \wedge Q(a));$ (HYPOTHESIS)
2. $\{(P(a) \wedge Q(a))\} \vdash P(a);$ ($\wedge e_1, 1$)
3. $\{(P(a) \wedge Q(a))\} \vdash (P(a) \vee Q(a)).$ ($\vee i_1, 2$)

Another formal proof of the sequence is the following:

1. $\{(P(a) \wedge Q(a))\} \vdash (P(a) \wedge Q(a));$ (HYPOTHESIS)
2. $\{(P(a) \wedge Q(a))\} \vdash Q(a);$ ($\wedge e_2, 1$)
3. $\{(P(a) \wedge Q(a))\} \vdash (P(a) \vee Q(a)).$ ($\vee i_2, 2$)

The proof for the elimination of disjunction is a little more complicated, being another rule in which the set of premises of the sequences changes from hypotheses to conclusion:

$$\vee e \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \quad \Gamma, \varphi_1 \vdash \varphi' \quad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi'}$$

The first hypothesis of the rule, $\Gamma \vdash (\varphi_1 \vee \varphi_2)$, is easy to understand: in order to “remove” a disjunction, we need a disjunction between the hypotheses (disjunction that we want to “eliminate”). The last two hypothesis of the elimination rule of disjunction has to be understood intuitively as follows. From the first hypothesis we know $(\varphi_1 \vee \varphi_2)$ (from Γ); in other words, at least one of the formulae φ_1 and respectively φ_2 follows from Γ . The hypotheses 2 and 3 indicates that, no matter which of the formulae φ_1 or φ_2 holds, in any case φ' holds. That is, if we suppose φ_1 (besides Γ), φ' holds, and if we suppose φ_2 (besides Γ), φ' still holds. And therefore the conclusion indicates that φ' holds no matter which one of the formulae φ_1 and respectively φ_2 would hold.

Example 103. *Let us prove that the sequence $\{(P(a) \vee Q(a))\} \vdash (Q(a) \vee P(a))$ is valid:*

1. $\{(P(a) \vee Q(a)), P(a)\} \vdash P(a);$ (HYPOTHESIS)
2. $\{(P(a) \vee Q(a)), P(a)\} \vdash (Q(a) \vee P(a));$ ($\vee i_2, 1$)
3. $\{(P(a) \vee Q(a)), Q(a)\} \vdash Q(a);$ (HYPOTHESIS)
4. $\{(P(a) \vee Q(a)), Q(a)\} \vdash (Q(a) \vee P(a));$ ($\vee i_1, 1$)

5. $\{(P(a) \vee Q(a))\} \vdash (P(a) \vee Q(a));$ (HYPOTHESIS)
6. $\{(P(a) \vee Q(a))\} \vdash (Q(a) \vee P(a)).$ ($\vee e$, 5, 2, 4)

Note the way in which the set of premises changes from a sequence to the other in the formal proof, following the inference rules.

Exercise 104. Find a formal proof for the sequence

$$\{(P(a) \vee Q(a)), (P(a) \rightarrow P(b)), (Q(a) \rightarrow P(b))\} \vdash P(b).$$

5.6.4 Rules for negation

The rules for the introduction and elimination of the negation are presented together with a rule for the elimination of \perp :

$$\neg i \frac{\Gamma, \varphi \vdash \perp}{\Gamma \vdash \neg \varphi} \qquad \neg e \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \neg \varphi}{\Gamma \vdash \perp} \qquad \perp e \frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi}$$

Let us recall that \perp is a logical connector of arity 0. In other words, the connector \perp does not change in the formula. The semantics of the formula \perp is such that it is false in any structure and any assignment. In other words, \perp is a contradiction.

The first rule, for the introduction of negation, is easy to explain intuitively: how can we prove that a formula of the form $\neg \varphi$ follows from Γ ? We suppose, in addition besides Γ , that we have φ and prove that from Γ and φ follows a contradiction ($\Gamma, \varphi \vdash \perp$). In this way, we prove that $\neg \varphi$ follows from Γ .

The second rule, for the elimination of negation, indicates that if a formula φ , as well as its negation, $\neg \varphi$, follow from the same set of premises Γ , then, from Γ also follow a contradiction, \perp . A set Γ from which follow a contradiction is called *inconsistent*.

The third rule indicates that, if Γ is an inconsistent set of formulae, then any formula φ follow from Γ .

There is no rule for the introduction of \perp (or, the rule for the elimination of negation can be considered also as being the rule for the introduction of \perp).

Example 105. Let us prove that the sequence $\{P(a)\} \vdash \neg \neg P(a)$ is valid:

1. $\{P(a), \neg P(a)\} \vdash P(a);$ (HYPOTHESIS)
2. $\{P(a), \neg P(a)\} \vdash \neg P(a);$ (HYPOTHESIS)
3. $\{P(a), \neg P(a)\} \vdash \perp;$ ($\neg e$, 1, 2)

$$4. \{P(a)\} \vdash \neg\neg P(a). \quad (\neg i, 3)$$

Example 106. Let us prove that the sequence $\{P(a), \neg P(a)\} \vdash P(b)$ is valid:

$$1. \{P(a), \neg P(a)\} \vdash P(a); \quad (\text{HYPOTHESIS})$$

$$2. \{P(a), \neg P(a)\} \vdash \neg P(a); \quad (\text{HYPOTHESIS})$$

$$3. \{P(a), \neg P(a)\} \vdash \perp; \quad (\neg e, 1, 2)$$

$$4. \{P(a), \neg P(a)\} \vdash P(b). \quad (\perp e, 3)$$

Elimination of double negation

In the case of propositional logic, we also saw the following rule for the elimination of the double negation:

$$\neg\neg e \quad \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}$$

Example 107. Let us prove that the sequence $\{(\neg P(a) \rightarrow Q(a)), \neg Q(a)\} \vdash P(a)$ is valid:

$$1. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a), \neg P(a)\} \vdash \neg P(a); \quad (\text{HYPOTHESIS})$$

$$2. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a), \neg P(a)\} \vdash (\neg P(a) \rightarrow Q(a)); \quad (\text{HYPOTHESIS})$$

$$3. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a), \neg P(a)\} \vdash Q(a); \quad (\rightarrow e, 2, 1)$$

$$4. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a), \neg P(a)\} \vdash \neg Q(a); \quad (\text{HYPOTHESIS})$$

$$5. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a), \neg P(a)\} \vdash \perp; \quad (\neg i, 4, 3)$$

$$6. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a)\} \vdash \neg\neg P(a); \quad (\neg i, 5)$$

$$7. \{(\neg P(a) \rightarrow Q(a)), \neg Q(a)\} \vdash P(a). \quad (\neg\neg e, 6)$$

Example 108. Let us prove that the sequence $\{\} \vdash (P(a) \vee \neg P(a))$ is valid:

$$1. \{\neg(P(a) \vee \neg P(a)), P(a)\} \vdash \neg(P(a) \vee \neg P(a)); \quad (\text{HYPOTHESIS})$$

$$2. \{\neg(P(a) \vee \neg P(a)), P(a)\} \vdash P(a); \quad (\text{HYPOTHESIS})$$

$$3. \{\neg(P(a) \vee \neg P(a)), P(a)\} \vdash (P(a) \vee \neg P(a)); \quad (\vee i_1, 2)$$

$$4. \{\neg(P(a) \vee \neg P(a)), P(a)\} \vdash \perp; \quad (\neg e, 1, 3)$$

$$5. \{\neg(P(a) \vee \neg P(a))\} \vdash \neg P(a); \quad (\neg i, 4)$$

6. $\{\neg(P(a) \vee \neg P(a))\} \vdash (P(a) \vee \neg P(a));$ ($\vee i_2$, 5)
7. $\{\neg(P(a) \vee \neg P(a))\} \vdash \neg(P(a) \vee \neg P(a));$ (HYPOTHESIS)
8. $\{\neg(P(a) \vee \neg P(a))\} \vdash \perp;$ ($\neg e$, 7, 6)
9. $\{\} \vdash \neg\neg(P(a) \vee \neg P(a));$ ($\neg i$, 8)
10. $\{\} \vdash (P(a) \vee \neg P(a)).$ ($\neg\neg e$, 9)

5.6.5 Elimination of universal quantifier

The rule for the elimination of the universal quantifier is:

$$\forall e \frac{\Gamma \vdash (\forall x.\varphi)}{\Gamma \vdash \varphi[x \mapsto t]} \text{ vars}(t) \cap \text{bound}(\varphi) = \emptyset$$

The elimination rule for the universal quantifier is quite simple: if we know that $(\forall x.\varphi)$ is a syntactic consequence from Γ , then we can instantiate the bound variable x with any term t such that t has no bound variables in φ .

Exercise 109. *Question: does the previous rule make sense if x does not appear in φ ? For instance, from $(\forall x.\varphi)$ can we deduce $\Gamma \vdash P(a)[x \mapsto b]$?*

Example 110. *Let us go back to an example previously discussed in which we have the two affirmations: All men are mortal and Socrate is a man. Can we conclude that Socrate is mortal? In order to answer to the question, we have to prove the sequence:*

$$\{\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x)), \text{Man}(s)\} \vdash \text{Mortal}(s),$$

where **Man** and **Mortal** are predicates of arity 1 and **s** is a constant (functional symbol of arity 0) associated to the name Socrates. Here is the formal proof for the sequence:

1. $\{\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x)), \text{Man}(s)\} \vdash \forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x))$ (HYP)
2. $\{\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x)), \text{Man}(s)\} \vdash (\text{Man}(s) \rightarrow \text{Mortal}(s))$ ($\forall e$, 1, s)
3. $\{\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x)), \text{Man}(s)\} \vdash \text{Man}(s)$ (HYP)
4. $\{\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x)), \text{Man}(s)\} \vdash \text{Mortal}(s)$ ($\rightarrow e$, 2, 3)

Note that at step 2 of the proof, we used the rule $\forall e$ which instantiates in the formula $\forall x.(\text{Man}(x) \rightarrow \text{Mortal}(x))$ the bound variable x with s : $(\text{Man}(s) \rightarrow \text{Mortal}(s))$. In natural language, this is similar with deducing by reasoning that If Socrates is a man, then he is mortal from All men are mortal.

5.6.6 Introduction of existential quantifier

There is a duality of rules for the introduction and the elimination of quantifiers in the sense that the rule for introducing the existential quantifier from below can be seen as a dual rule for the elimination of universal quantifier:

$$\exists i \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash (\exists x.\varphi)} \text{ vars}(t) \cap \text{bound}(\varphi) = \emptyset$$

The rule indicates that we can deduce $(\exists x.\varphi)$ when $\varphi[x \mapsto t]$ is a semantical consequence from Γ . Informally, if there is a concrete x — namely t — such that $\varphi[x \mapsto t]$ is true (such that t has no bound variables in φ), we conclude that $(\exists x.\varphi)$ is true.

Example 111. *Let us prove that the sequence $\{P(a)\} \vdash \exists x.P(x)$ is valid:*

1. $\{P(a)\} \vdash P(a)$ (HYPOTHESIS)
2. $\{P(a)\} \vdash \exists x.P(x)$ ($\exists i, 1$)

Note that in this case φ is $P(x)$ and $\varphi[x \mapsto a]$ is $P(x)[x \mapsto a]$.

Example 112. *Let us prove that the sequence $\{\forall x.(P(x) \rightarrow Q(x)), P(a)\} \vdash \exists x.Q(x)$ is valid:*

1. $\{\forall x.(P(x) \rightarrow Q(x)), P(a)\} \vdash \forall x.(P(x) \rightarrow Q(x))$ (HYPOTHESIS)
2. $\{\forall x.(P(x) \rightarrow Q(x)), P(a)\} \vdash P(a)$ (HYPOTHESIS)
3. $\{\forall x.(P(x) \rightarrow Q(x)), P(a)\} \vdash (P(a) \rightarrow Q(a))$ ($\forall e, 1, a$)
4. $\{\forall x.(P(x) \rightarrow Q(x)), P(a)\} \vdash Q(a)$ ($\rightarrow e, 3, 2$)
5. $\{\forall x.(P(x) \rightarrow Q(x)), P(a)\} \vdash \exists x.Q(x)$ ($\exists i, 4$)

5.6.7 Introduction of universal quantifier.

The rule for the introduction of the universal quantifier is:

$$\forall i \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash (\forall x.\varphi)} x_0 \notin \text{vars}(\Gamma, \varphi)$$

The rule from above says that we can conclude $\Gamma \vdash \forall x.\varphi$ if we first prove that $\varphi[x \mapsto x_0]$ is a syntactical consequence from Γ , where x_0 is a *new* variable: it does not appear in other formulae and we make no assumption over it.

Example 113. Let us prove that the sequence $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash \forall x.Q(x)$ is valid:

1. $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash \forall x.(P(x) \rightarrow Q(x))$ (HYPOTHESIS)
2. $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash \forall x.P(x)$ (HYPOTHESIS)
3. $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash (P(x_0) \rightarrow Q(x_0))$ ($\forall e, 1, x_0$)
4. $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash P(x_0)$ ($\forall e, 2, x_0$)
5. $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash Q(x_0)$ ($\rightarrow e, 3, 4$)
6. $\{\forall x.(P(x) \rightarrow Q(x)), \forall x.P(x)\} \vdash \forall x.Q(x)$ ($\forall i, 5$)

Note that for the sequences 3, 4 and 5, we use the variable x_0 about which we make no assumption. Therefore, intuitively, $Q(x_0)$ holds for any x_0 .

Exercise 114. Prove that the following sequences are valid:

1. $\{\forall x.(P(x) \wedge Q(x))\} \vdash \forall x.P(x)$;
2. $\{\forall x.Q(x), P(a)\} \vdash P(a) \wedge Q(a)$;
3. $\{\forall x.P(x), \forall x.Q(x)\} \vdash \forall x.(P(x) \wedge Q(x))$.

5.6.8 Elimination of existential quantifier

The rule for the elimination of the existential quantifier is the following:

$$\exists e \frac{\Gamma \vdash (\exists x.\varphi) \quad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi} \quad x_0 \notin \text{vars}(\Gamma, \varphi, \psi)$$

The first hypothesis of the rule is $\Gamma \vdash (\exists x.\varphi)$, which, intuitively, ensures us that there is at least one term (they can be several) that can replace x such that φ is a syntactical consequence from Γ . However, we don't know which are these terms (in the case they are several). We only know that there is at least one and we call it x_0 . In order to prove the conclusion, that ψ is a syntactical consequence from Γ , we have to analyze several cases for x_0 . This is summarized by the second hypothesis of the rule, where we have to prove that ψ is a syntactical consequence from $\Gamma \cup \{\varphi[x \mapsto x_0]\}$.

Example 115. Let us prove that the sequence $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x)\} \vdash \exists x.Q(x)$ is valid:

1. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x)\} \vdash \exists x.P(x)$ (HYPOTHESIS)

2. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x), P(x_0)\} \vdash P(x_0)$ (HYPOTHESIS)
3. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x), P(x_0)\} \vdash \forall x.(P(x) \rightarrow Q(x))$ (HYPOTHESIS)
4. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x), P(x_0)\} \vdash (P(x_0) \rightarrow Q(x_0))$ ($\forall e, 3, x_0$)
5. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x), P(x_0)\} \vdash Q(x_0)$ ($\rightarrow e, 4, 2$)
6. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x), P(x_0)\} \vdash \exists x.Q(x)$ ($\exists i, 5$)
7. $\{\forall x.(P(x) \rightarrow Q(x)), \exists x.P(x)\} \vdash \exists x.Q(x)$ ($\exists e, 1, 6$)

Note that in order to prove the 7th sequence, we used sequences 1 and 6. The former one was proved by steps 2, 3, 4 and 5, where we also used as a hypothesis the formula $P(x_0) (= P(x)[x \mapsto x_0])$.

5.6.9 Other rules

Another useful rule, that does not necessary corresponds to some operator, is the extension rule, that was also presented in the case of natural deduction for propositional logic:

$$\text{EXTENSION} \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi}$$

This rule indicates the fact that, if φ is a consequence from a set of formulas Γ , then φ is also a consequence of $\Gamma \cup \{\varphi'\}$ (for any φ'). In other words, we can extend the set of premises of a valid sequence and we get another valid sequence.

Example 116. Here is a proof of $\{P(a), \neg Q(a), P(f(a)), (P(b) \wedge Q(b))\} \vdash \neg\neg P(a)$:

1. $\{P(a), \neg P(a)\} \vdash P(a)$; (HYPOTHESIS)
2. $\{P(a), \neg P(a)\} \vdash \neg P(a)$; (HYPOTHESIS)
3. $\{P(a), \neg P(a)\} \vdash \perp$; ($\neg e, 1, 2$)
4. $\{P(a)\} \vdash \neg\neg P(a)$; ($\neg i, 3$)
5. $\{P(a), \neg Q(a)\} \vdash \neg\neg P(a)$; (EXTINDERE, 4)
6. $\{P(a), \neg Q(a), P(f(a))\} \vdash \neg\neg P(a)$; (EXTINDERE, 5)
7. $\{P(a), \neg Q(a), P(f(a)), (P(b) \wedge Q(b))\} \vdash \neg\neg P(a)$. (EXTINDERE, 6)

5.7 Natural deduction system

The natural deduction for first order logic is the deductive system formed by all rules presented in the previous section. Here is the sum up of all rules:

$$\begin{array}{c}
\wedge i \frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}, \quad \wedge e_1 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_1}, \\
\wedge e_2 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_2}, \quad \rightarrow e \frac{\Gamma \vdash (\varphi_1 \rightarrow \varphi_2) \quad \Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_2}, \\
\rightarrow i \frac{\Gamma, \varphi_1 \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \rightarrow \varphi_2)}, \quad \vee i_1 \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2)}, \quad \vee i_2 \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2)}, \\
\vee e \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \quad \Gamma, \varphi_1 \vdash \varphi' \quad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi'}, \\
\neg e \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \neg \varphi}{\Gamma \vdash \perp}, \quad \neg i \frac{\Gamma, \varphi \vdash \perp}{\Gamma \vdash \neg \varphi}, \quad \perp e \frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi}, \\
\text{HYPOTHESIS} \frac{}{\Gamma \vdash \varphi} \varphi \in \Gamma, \quad \text{EXTINDERE} \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi}, \quad \neg \neg e \frac{\Gamma \vdash \neg \neg \varphi}{\Gamma \vdash \varphi}, \\
\forall e \frac{\Gamma \vdash (\forall x. \varphi)}{\Gamma \vdash \varphi[x \mapsto t]} \text{vars}(t) \cap \text{bound}(\varphi) = \emptyset \\
\exists i \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash (\exists x. \varphi)} \text{vars}(t) \cap \text{bound}(\varphi) = \emptyset \\
\forall i \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash (\forall x. \varphi)} x_0 \notin \text{vars}(\Gamma, \varphi) \\
\exists e \frac{\Gamma \vdash (\exists x. \varphi) \quad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi} x_0 \notin \text{vars}(\Gamma, \varphi, \psi)
\end{array}$$

In proofs we are allowed to use derived rules (including the ones presented for propositional logic).

5.8 Soundness and Completeness of Natural Deduction for First Order Logic

Theorem 117 (Soundness for natural deduction). *For any set Γ of formulae and any formula φ , if the sequence $\Gamma \vdash \varphi$ is valid, then $\Gamma \models \varphi$.*

Exercise 118. *Prove Theorem 117.*

Theorem 119 (Completeness for natural deduction). *For any set Γ of formulae and any formula φ , if $\Gamma \models \varphi$ then the sequence $\Gamma \vdash \varphi$ is valid.*

The proof of the completeness theorem exceeds the level of this course.

Remark. *Note that, using the two soundness and completeness theorems, the relation \vdash coincides with \models , even if they have different meanings.*

5.9 Exercises

Exercise 120. *Prove that the next sequences are valid:*

1. $\{((P(a) \wedge Q(a)) \wedge \forall x.P(x))\} \vdash (Q(a) \wedge \forall x.P(x));$
2. $\{((P(a) \wedge Q(a)) \wedge \forall x.P(x)), \forall x.Q(x)\} \vdash (\forall x.Q(x) \wedge Q(a));$
3. $\{((P(a) \wedge Q(a)) \wedge \forall x.P(x))\} \vdash (\forall x.P(x) \wedge (Q(a) \wedge P(a)));$
4. $\{((P(a) \wedge Q(a)) \rightarrow \forall x.P(x)), P(a), Q(a)\} \vdash \forall x.P(x);$
5. $\{(P(a) \rightarrow \forall x.P(x)), P(a), Q(a)\} \vdash (Q(a) \wedge \forall x.P(x));$
6. $\{(P(a) \rightarrow P(b)), (Q(a) \rightarrow P(b))\} \vdash ((P(a) \vee Q(a)) \rightarrow P(b));$
7. $\{\neg(P(a) \wedge Q(a))\} \vdash (\neg P(a) \vee \neg Q(a));$
8. $\{\neg(\neg P(a) \vee \neg Q(a))\} \vdash (P(a) \wedge Q(a));$
9. $\{\neg(\neg P(a) \wedge \neg Q(a))\} \vdash (P(a) \vee Q(a));$

Exercise 121. *Which of the following sequences are valid?*

1. $\{\forall x.(P(x) \wedge Q(x))\} \vdash \forall x.P(x);$
2. $\{\forall x.Q(x), P(a)\} \vdash (P(a) \wedge Q(a));$
3. $\{\forall x.P(x), \forall x.Q(x)\} \vdash \forall x.(P(x) \wedge Q(x));$
4. $\{\exists x.\exists y.P(x, y)\} \vdash \exists y.\exists x.P(x, y);$

5. $\{\exists x. \forall y. P(x, y)\} \vdash \forall y. \exists x. P(x, y)$; What about $\{\forall y. \exists x. P(x, y)\} \vdash \exists x. \forall y. P(x, y)$?
6. $\{\neg(\exists x. P(x))\} \vdash \forall x. \neg P(x)$;
7. $\{\forall x. \neg P(x)\} \vdash \neg(\exists x. P(x))$;