Logic for Computer Science - Lecture Notes

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Chapter 1

Motivation and introduction

First-order logic, what we will be studying next, is an extension of propositional logic, extension that brings more expressivity. The additional expressivity is necessary in order to model certain statements that cannot be expressed in propositional logic.

In propositional logic, we cannot express naturally the following statement: *All men are mortal.*

To model a statement in propositional logic, we identify the atomic propositions. Then we associate to each atomic proposition a propositional variable. The atomic propositions are the propositions that cannot be split into one or more smaller propositions, linked among them by the logical connectives of propositional logic: $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

We notice that the statement *All men are mortal* cannot be decomposed into smaller statements linked among them by the logical connectives of propositional logic, as is described above. Therefore, in propositional logic, the statement is atomic. So we associate to the entire statement a propositional variable $\texttt{p} \in A$.

Let us now model the statement *Socrates is a man.* Obviously, to this second statement we must associate another propositional variable $\texttt{q} \in A$. Let us assume that $\texttt{p}$ and $\texttt{q}$ are true. Formally, we work in a truth assignment $\tau : A \to B$ where $\tau(\texttt{p}) = 1$ and $\tau(\texttt{q}) = 1$. Can we draw the conclusion that *Socrates is mortal* in the truth assignment τ?

No, because to the statement *Socrates is mortal* we should associate a third propositional variable $\texttt{r} \in A$. We cannot draw any conclusion on $\tau(\texttt{r})$ from $\tau(\texttt{p}) = 1$ and $\tau(\texttt{q}) = 1$. So, from the semantics of propositional logic, we cannot draw the conclusion that $\texttt{r}$ is true in any truth assignment that makes both $\mathtt{p}$ and $\mathtt{q}$ true. This is despite the fact that, in any world where *All men are mortal* and *Socrates is a man*, we can draw the conclusion that *Socrates* is mortal without failure. This difference between reality and our modelling indicates that our modelling is not sufficient for our purposes.

First-order logic includes, in addition to propositional logic, the notion of *quantifier* and the notion of *predicate*. The universal quantifier is denoted by $\forall$ and the existential quantifier is denoted by $\exists$.

A predicate is a statement whose truth value depends on zero or more parameters. For example, for the statements above, we will be using two predicates: $\mathtt{Man}$ and $\mathtt{Mortal}$. The predicate $\mathtt{Man}$ is the predicate that denotes the quality of being a man: $\mathtt{Man}(\mathsf{x})$ is true iff x is a man. The predicate $\mathtt{Mortal}$ is true when its argument is mortal. As the predicates above have only one argument/parameter, they are called *unary* predicates. Predicates generalize propositional variables by the fact that they can take arguments. Actually, propositional variable can be regarded as predicates with no arguments.

In this way, the statement *All men are mortal* will be modelled by the formula

$$\big(\forall\mathsf{x}.(\mathtt{Man}(\mathsf{x}) \rightarrow \mathtt{Mortal}(\mathsf{x}))\big),$$

which is read as follows: *for any* x*, if* $\mathtt{Man}$ *of* x*, then* $\mathtt{Mortal}$ *of* x. The statement *Socrate is a men* shall be modelled by the formula $\mathtt{Man}(s)$, where s is a *constant* that denotes Socrates, just like 0 denotes the natural number *zero*. For example, $\mathtt{Man}(\mathsf{s})$ is true (as s stands for a particular man – Socrates), but $\mathtt{Man}(\mathsf{l})$ is false if l is a constant standing for the dog *Lassie*.

The statement *Socrates is mortal* shall be represented by $\mathtt{Mortal}(\mathsf{s})$ (recall that the constant s stands for Socrates). The statement $\mathtt{Mortal}(\mathsf{s})$ is true, as Socrates is mortal; likewise, the statement $\mathtt{Mortal}(\mathsf{l})$ is also true.

We shall see that in first-order logic, the formula $\mathtt{Mortal}(\mathsf{s})$ is a logical consequence of the formulae $\big(\forall\mathsf{x}.(\mathtt{Man}(\mathsf{x}) \rightarrow \mathtt{Mortal}(\mathsf{x}))\big)$ and respectively $\mathtt{Man}(\mathsf{s})$. Therefore, first-order logic is sufficiently expressive to explain theoretically the argument by which we deduce that *Socrates is mortal* from the facts that *All men are mortal* and *Socrates is a man*.

Chapter 2

Structures and signatures

You have certainly met already several first-order logic formulae, without necessarily knowing that you are dealing with first-order logic. Consider the following formula:

$$\varphi = \Big(\forall \mathsf{x}.(\forall \mathsf{y}.(\mathsf{x} < \mathsf{y} \rightarrow \exists \mathsf{z}.(\mathsf{x} < \mathsf{z} \wedge \mathsf{z} < \mathsf{y})))\Big).$$

The formula makes use of a binary predicate, $<$, that is defined as follows: $<(x, y)$ is true if x is strictly smaller than y. In order to simplify our writing, we use the infixed notation $(x < y)$ instead of the prefixed notation $(<(x, y))$ for many binary predicates (including for $<$).

Is the formula φ above true? The formula states that between any two values of the variables x, y there is a third value, of the variable z. The formula is true if the domain of the variables $\mathsf{x}, \mathsf{y}, \mathsf{z}$ is $\mathbb{R}$, but it is false if the domain is $\mathbb{N}$ (between any two real numbers there exists a third, but between two consecutive naturals there is no other natural number).

Generally, first-order formulas refer to a particular *mathematical structure*.

Definition 1 (Mathematical structure)**.** *A* mathematical structure *is a tuple* $S = (D, Pred, Fun)$ *where:*

- D *is a non-empty set called the* domain *of the structure;*
- *each* $P \in Pred$ *is a predicate (of a certain arity) over the set* D*;*
- *each* $f \in Pred$ *is a function (of a certain arity) over the set* D*.*

Here are a few examples of mathematical structures:

1. $(\mathbb{N}, \{<, =\}, \{+, 0, 1\})$;

The domain of the structure is the set of naturals. The structure contains two predicates: $<$ and $=$, both of arity 2. The predicate $<$ is the *smaller than* predicate on naturals, and the predicate $=$ is the *equality* predicate over natural numbers.

The structure also contains three functions. The binary function $+ : \mathbb{N}^2 \to \mathbb{N}$ is the addition function for naturals, and the functions $0 : \mathbb{N}^0 \to \mathbb{N}$ and respectively $1 : \mathbb{N}^0 \to \mathbb{N}$ are the arity 0 functions (also called constant functions or simply constants) 0 and respectively 1.

2. $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$;

   This structure contains two binary predicates, $<$ and $=$, as well as four functions over $\mathbb{R}$: the binary function $+$, the unary function $-$ (unary minus) and the constants $0, 1 \in \mathbb{R}$.

3. $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$;

   This structure is similar to that above, but the domain is the set of integers.

4. $(B, \emptyset, \{\cdot, +, \text{-}\})$;

   This structure is a boolean algebra, where the domain is the set truth values and the functions are those that we studied in the first half of the semester. Such structures, without any predicates, are called *algebraic structures*.

5. $(\mathbb{R}, \{<\}, \emptyset)$.

   This structure contains only a predicate of arity 2 (the *less than* relation over $\mathbb{R}$) and no function. Structures without functions are called relational structures. Relational structures with a finite domain are called relational data bases and you will study them in your second year.

Whenever we want to evaluate the truth value of a first-order formula we need a mathematical structure. Recall our previous formula:

$$\varphi = \Big(\forall x.\big(\forall y.(x < y \rightarrow \exists z.(x < z \wedge z < y))\big)\Big).$$

This formula is true in the structure $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ (between any two distinct real numbers there is another real number), but it is false in the structure $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$ (because it is not true that between any two distinct integers there is a third integer – for example there is no such integer between two consecutive integers).

It is possible for two different structure to have a set of predicates and a set of functions with the same names. For example, the structures above, $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ and respectively $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$. Even if the predicate $<\in \mathbb{R}^2$ is different from the predicate $<\in \mathbb{Z}^2$, they both have the same name: $<$.

Generally, in Mathematics and in Computer Science, we do not make any difference between a predicate and its name or between a function and its name. However, in Logic, the difference is extremely important. In particular, if we refer to the name of a function, we shall use the phrase "functional symbol" (i.e., symbol standing for a function). When we refer to the name of a predicate, we shall use the phrase "predicate symbol" (or "relational symbol"). Why is the difference between a predicate and a predicate symbol important? Because we shall need to associate to the same predicate symbol several predicates, similarly to how we can associate several values to a program variable in an imperative language.

When we are interested only in the function and predicate names (not the function or predicates themselves), we work with signatures:

Definition 2 (Signature)**.** *A* signature Σ *is a tuple* $\Sigma = (\mathcal{P}, \mathcal{F})$*, where* $\mathcal{P}$ *is a set of predicate symbols and* $\mathcal{F}$ *is a set of functional symbols. Each predicate or functional symbol* s *has an associate natural number called its* arity *denoted by* $ar(s)$*.*

To a signature we can associate many structures:

Definition 3 (Σ-structure)**.** *If* $\Sigma = (\mathcal{P}, \mathcal{F})$ *is a signature, a* Σ-structure *is any structure* $S = (D, Pred, Fun)$ *so that for each predicate symbol* $P \in \mathcal{P}$*, exists a predicate* $P^S \in Pred$ *of corresponding arity, and for every functional symbol* $f \in \mathcal{F}$*, there is a function* $f^S \in Fun$ *of corresponding arity.*

Example 4. *Let* $\Sigma = (\{\mathsf{P}, \mathsf{Q}\}, \{\mathsf{f}, \mathsf{i}, \mathsf{a}, \mathsf{b}\})$*, where* P *and* Q *are predicate symbols of arity* $ar(\mathsf{P}) = ar(\mathsf{Q}) = 2$ *and* $\mathsf{f}, \mathsf{i}, \mathsf{a}, \mathsf{b}$ *are function symbols having the following arrities:* $ar(\mathsf{f}) = 2$*,* $ar(\mathsf{i}) = 1$ *and* $ar(\mathsf{a}) = ar(\mathsf{b}) = 0$*.*

We have that $(\mathbb{R}, \{<, =\}, \{+, -, 0, 1\})$ *and* $(\mathbb{Z}, \{<, =\}, \{+, -, 0, 1\})$ *are* Σ*-structures.*

Remark. *As you can observe in Example 4, for predicate symbols (e.g.,* P*,* Q*) we use a different color than the color used for functional symbols (e.g.,* f*,* i*,* a*,* b*). For oredicates and and functions we use the normal font for mathematical formulas.*

To remember!
Structure = domain + predicates + functions Signature = predicate symbols + functional symbols To a signature Σ we can associate many structures, which are called Σ-structures.

Notation. *The set of predicate symbols of arity* n *is denoted by* $\mathcal{P}_n = \{P \mid ar(P) = n\}$, *and the set of functional symbols of arity* n *is* $\mathcal{F}_n = \{f \mid ar(f) = n\}$. *For the particular case when* $n = 0$, $\mathcal{F}_0$ *is the set of constant symbols (that is, functional symbols with arity 0).*

2.1 Exercise sheet

Exercise 5. *Identify the predicates and the functions in the text below. What is their domain?*

John is a student. Any student learns Logic. Anyone learning Logic passes the exam. Any student is a person. There is a person who did not pass the exam. Therefore: not all persons are students.

Exercise 6. *Identify the predicates and the functions in the text below. What is their domain?*

The sum of two even numbers is even.

Exercise 7. *Identify the predicates and the functions in the text below. What is their domain?*

In chess, the quuen can make a move from one square to another iff the bishop or the rook can make the same move.

Exercise 8. *Identify the predicates and the functions in the text below. What is their domain?*

The sum of two numbers greater than zero is greater than zero.

Exercise 9. *Identify the predicates and the functions in the text below. What is their domain?*

The number 7 is prime.

Exercise 10. *Identify the predicates and the functions in the text below. What is their domain?*

Any even number greater than 2 is the sum of two primes.

Exercise 11. *Identify the predicates and the functions in the text below. What is their domain?*

If the Earth is flat, then $2 + 2 = 5$.

Exercise 12. *Identify the predicates and the functions in the text below. What is their domain?*

For any $\epsilon \in (0, \infty)$, there exists $\delta_\epsilon \in (0, \infty)$ such that for any $x \in \mathbb{R}$ with $d(x_0, x) < \delta_\epsilon$, we have $d(f(x_0), f(x)) < \epsilon$.