Logic for Computer Science - Lecture Notes

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Contents

Chapter 1

Introduction

If Arithmetic is the science that studies numbers and operations with numbers, then Logic is the science that studies *propositions* and operations with propositions.

For example, if in Arithmetic we notice that the sum of two even numbers is an even number, then in Logic we could notice that the disjunction of two true sentences is also true.

Logic sits at the intersection of philosophy, mathematics and computer science and has experienced its greatest development starting with the 1950s, because of its numerous applications in Computer Science.

In this course, we will study at an introductory level *propositional logic* and *first-order logic*.

Propositional logic is extremely simple, but the concepts that we study, the methods that we learn and the issues that we face in propositional logic generalize to other more complex logics. Moreover, propositional logic corresponds intimally to the internal organization of computers at an abstract level, in the sense that electronic circuits can be modeled as formulae in propositional logic. Propositional logic has a rich and mathematically interesting theory (examples: compactness theorem, Craig's interpolation theorem). The *satisfiability problem* for propositional logic has many applications in computer science. It is especially important both from a theoretical viewpoint (being the canonical NP-complete problem) and from a practical viewpoint as well (with applications in program verification, circuit verification, combinatorial optimization, and others).

First-order logic is an extension of propositional logic and also has numerous application in computer science, but also in mathematics. For example, all of the math that you have studied in highschool is based on a so-called first-order logic theory called **ZFC** (the **Z**ermelo–**F**raenkel set theory, together with the Axiom of **C**hoice). In Computer Science, applications of first-order logic appear in the fields of descriptive complexity, relational databases, software and hardware verification, and others. Additionally, several other logics (for example, higher-order logics) have applications in programming languages, the fundamentals of mathematics, type theory, etc.

Chapter 2

Informal Propositional Logic

Propositional logic is the logic of propositions, connected among themselves by *logical connectives* such as *or*, *and* and *not*. In this chapter, we study the basics of propositional logic.

2.1 Propositions

A *proposition* is a statement that is either true or false. Propositions are sometimes called *sentences*. Here are examples of propositions:

1. *I wear a blue shirt.*
2. *You own a laptop computer and a tablet computer, but no smartphone.*
3. $2 + 2 = 4$. (*Two plus two is four.*)
4. $1 + 1 = 1$. (*One plus one is one.*)
5. $1 + 1 \neq 1$. (*One plus one is not one.*)
6. *If* $1 + 1 = 1$, *then I'm a banana.*
7. *All natural numbers are integers.*
8. *All rational numbers are integers.*

Here are examples of things that are not propositions:

1. *Red and Black* (not a statement);

2. π (not a statement);

3. *Is it raining?* (question, not a statement);

4. *Go fish!* (imperative);

5. *x is greater than* 7 (we have here a *predicate of* x; once we set a value for x, the predicate becomes a proposition);

6. *This sentence is false.* (although a statement, it is not a proposition, since it is not either true or false: if it were true, it would need to be false and vice-versa).

Sometimes it is debatable whether something is truly a proposition. For example, we generally agree that *Snow is white* is true, but someone might argue that they have seen black snow, so the truth value of *Snow is white* is put in question. Arguing about whether something is a proposition or not is more a matter of philosophical logic than computer science logic and we will therefore not be too concerned about these sort of issues.

2.2 Atomic Propositions

Some propositions are atomic, in that they cannot be decomposed further into smaller propositions:

1. *I wear a blue shirt.*

2. *You own a laptop computer.*

3. $2 + 2 = 4$. (*Two plus two is four.*)

2.3 Conjunctions

Others however seem to be composed of smaller parts. For example, the proposition *I play games often and I study very well* is composed of two smaller propositions: *I play games often* and *I study very well*, joined together by *and*. When two propositions φ and ψ are joined by an *and*, the resulting proposition φ *and* ψ is called a *conjunction* (*the conjunction of* φ *and* ψ). The propositions φ and ψ are called the *conjuncts* of the proposition φ *and* ψ.

A conjunction is true if both of its conjuncts are true. For example, the proposition *I play games often and I study very well.* is true if both *I play games often* and *I study very well* are true. In particular, as I do not play games often, this proposition is false (when I say it).

Note that a conjunction need not use explicitly the word *and*. For example, the proposition *It is raining outside, but I have an umbrella* is also a conjunction, and it conjuncts are *It is raining outside* and *I have an umbrella*. This particular conjunction uses the adversative conjunction *but*.

Exercise 1. *Find the conjuncts of* I play at home and I study at school..

Exercise 2. *Give an example of a conjunction that is false and an example of a conjunction that is true.*

2.4 Disjunctions

Disjunctions are propositions linked together by *or*. For example, *I can install the software on my smartphone or on my tablet* is a disjunction between *I can install the software on my smartphone* and *I can install the software on my tablet.*. The two parts of the disjunction are called the *disjuncts*.

In the example above, note that the English grammar allows us to omit *I can install the software ...* in the second disjunct, as it is implicit in our understanding of the language. However, when we find the disjuncts, is helps to state them explicitly.

Exercise 3. *Find the disjuncts of* I will buy a laptop or a tablet*. Pay attention! The two disjuncts must be propositions (some words in them could be implicit and not appear in the text).*

A disjunction is true if at least one of the disjuncts is true. For example, *I am Darth Vader or I teach* is true because *I teach* is true (I do not have to worry about being Darth Vader). *I teach or I program* is also true (it happens that both disjuncts are true).

This meaning of disjunctions is called the *inclusive or*. It is standard in mathematics. Sometimes people use *or* in natural language to mean *exclusive or*. For example *Either white wins or black wins in a game of go* is an example where the *or* is exclusive. The meaning of the sentence is that *white wins* or *black wins*, but not both. Here is an example of a false proposition that uses *exclusive or*: *Either I program or I teach* (hint: false because I do both). When you see *either* in a sentence, it is a sign that you are dealing with an *exclusive or*.

In the following, by *disjunction* we will understand *inclusive or* (the standard interpretation in mathematics).

Exercise 4. *Give an example of a false disjunction and an example of a true disjunction.*

Exercise 5. *When is a disjunction φ or ψ false (depending on the truth value of φ and ψ)?*

2.5 Implications

Implications are propositions of the form *if* φ *then* ψ. The proposition φ is called the *antecedent* and the proposition ψ is called the *conclusion* (or *consequent*) of the implication.

An example of an implication is *If I get a passing grade in Logic, I will buy everyone beer.* The antecedent is *I get a passing grade in Logic.* and the conclusion is *I will buy everyone beer.* When is an implication true? Actually, it is easier to say when it is false. An implication is false if and only if the antecedent is true, but the conclusion is false. Assume that I got a passing grade in Logic. Therefore, the proposition *I get a passing grade in Logic* is true. However, I will not buy beer for everyone (just a few select friends). Therefore the proposition *I will buy everyone beer* is false. Therefore the implication *If I get a passing grade in Logic, I will buy everyone beer* as a whole is false (antecedent is true, but conclusion is false).

The meaning of implications is worth a more detailed discussion as it is somewhat controversial. This is mostly because implication as we understand it in mathematics can sometimes be subtly different from implication as we understand it in everyday life. In everyday life, when we say *If I pass Logic, I buy beer*, we understand that there is a cause-and-effect relation between passing Logic and buying beer. This subtle cause-and-effect relation is evident in a number of *if-then* statements that we use in real life: *If I have money, I will buy a car*, *If you help me, I will help you*, etc. We would never connect two unrelated sentences with an implication: the proposition *If the Earth is round, then 2+2=4* would not be very helpful, even though it is true (both the antecedent and the conclusion are true).

This implication that we use in mathematics is called *material implication* or *truth functional implication*, because the truth value of the implication as a whole depends only on the truth values of the antecedent and the conclusion, not on the antecedent and the conclusion itself. This meaning of implication sometimes does not correspond to the meaning of natural language implications, but it turns out that it is the only sensible interpretation of implications in mathematics (and computer science).

In particular, we will take both the propositions *If the Earth is flat, then* $2 + 2 = 5$ and *If the Earth is flat, then* $2 + 2 = 4$ to be true, because the antecedent is false. Implications that are true because the antecedent is false are called *vacuously true.*

Exercise 6. *What are the truth values of* If $2 + 2 = 4$, then the Earth is flat *and* If $2 + 2 = 5$, then the Earth is flat*?*

The truth value of an implication *if* φ *then* ψ, depending on the truth values of its antecedent φ and its conclusion ψ, is summarized in the truth-table below:

φ	ψ	if φ then ψ
false	*false*	*true*
false	*true*	*true*
true	*false*	*false*
true	*true*	*true*

The following example aims at convincing you that the truth table above is the only reasonable one. You must agree that every natural number is also an integer. Otherwise put, the proposition *for any number x, if x is a natural, then x is an integer* is true. In particular, you will agree that the proposition above holds for $x = -10$, $x = 10$ and $x = 1.2$. In particular, the propositions *If -10 is a natural, then -10 is an integer*, *If 10 is a natural, then 10 is an integer* and *If 1.2 is a natural, then 1.2 is an integer* must all be true. This accounts for the first, second and fourth lines of the truth table above (typically, the second line is controversial). As for the third line, false is the only reasonable truth value for an implication *if φ then ψ* where φ is true but ψ is false. Otherwise, we would have to accept propositions such as *If* $2 + 2 = 4$, *then* $2 + 2 = 5$ (antecedent $2 + 2 = 4$ true, conclusion $2 + 2 = 5$ false) as being true.

Implications are sometimes subtle to spot and identify correctly. For example, in the proposition *I will pass Logic only if I study hard* (emphasis on *only if*), the antecedent is *I will pass Logic* and the conclusion is *I study hard.* In particular, the above proposition does not have the same meaning as *If I study hard, then I will pass Logic.*

> Pay attention! In propositions of the form *I will pass Logic only if I study*, the antecedent is *I will pass Logic*, and the consequent is *I study.* This proposition does not have the same meaning as *if I study, then I will pass Logic.*

Implications can sometimes not make use of *if.* For example, take the proposition *I will pass Logic or I will drop school* (apparently a disjunction). This most likely meaning of this proposition is *If I do not pass Logic, then I will drop school.* Thankfully, both of these propositions are equivalent, in a sense that we shall study in the following lectures.

2.6 Negations

A proposition of the form *it is not the case that φ* (or simply *not ψ*) is the *negation* of φ. For example, *It is not raining* is the negation of *It is raining.*

The negation of a proposition takes the opposite truth value. For example, as I am writing this text, the proposition *It is raining* is false, and therefore the proposition *It is not raining* is true.

Exercise 7. *Give an example of a false proposition that uses both a negation and a conjunction.*

2.7 Equivalences

A proposition of the form φ *if and only if* ψ is called an *equivalence* or *double implication*. Such a proposition, as a whole, is true if φ and ψ have the same truth value (both false or both true).

For example, when I am writing this text, *It is raining if and only if it is snowing* is true. Why? Because both of the propositions *It is raining* and *It is snowing* are false.

Exercise 8. *What is the truth value of the proposition* The number 7 is odd if and only if 7 is a prime.*?*

Equivalences are, semantically speaking, conjunctions of two implications: φ *if and only if* ψ gives the same information as

$$\underbrace{\varphi \textit{ if } \psi}_{\textit{the reverse implication}} \qquad \textit{and} \qquad \underbrace{\varphi \textit{ only if } \psi}_{\textit{the direct implication}} \quad .$$

The proposition φ *if* ψ is the same as *if* ψ*, then* φ (but it has a different topic). The proposition φ *only if* ψ has the same meaning as *if* φ*, then* ψ, as we dicussed in the previous section on implications.

2.8 Logical Connectives

The words *and*, *or*, *if-then*, *not*, *only if*, *if-and-only-if* (and other similar phrases) are called *logical connectives*, as they can be used to connect smaller propositions in order to obtain larger propositions.

Pay attention! A proposition is *atomic* in propositional logic only if it cannot be decomposed into smaller propositions separated by the connectives discussed above. For example, the proposition *every natural number is an integer* is an atomic proposition (in propositional logic).

The same proposition is not necessarily atomic in other logics. For example, in first-order logic (which we study in the second half of the semester), we have additional logical connectives called *quantifiers* that can be used to construct *every natural number is an integer* from smaller propositions. Therefore, *every natural number is an integer* is not atomic in first-order logic.

2.9 Ambiguities in Natural Language

We have described above the language of propositional logic: atomic propositions connected by *and*, *or*, *not*, etc. So far, our approach has used English. However, English (or any other natural language) is not suitable for our purposes because it exhibits imprecisions in the form of ambiguitites.

Here are a few examples of ambigious propositions:

1. *John and Mary are married* (meaning 1: John and Mary are married to each other; meaning 2: John and Mary are married, but not necessarily to each other).

2. *It is not true that John is tall and Jane is short* (meaning 1: John is not tall and Jane is short; meaning 2: the conjunction *John is tall and Jane is short* is false).

In the study of the laws of logic, such ambiguities can get in the way, just like a wrong computation could impact the resistance of buildings or bridges in civil engineering.

The state of the art in Logic for over 2.000 years, from Aristotle up to the development of Symbolic Logic in the 19th century, has been to use natural language. Symbolic logic (formal logic) has changed the game by introducing languages so precise that there is no risk of misunderstandings.

Such ambiguities impede the study of propositional logic. Therefore we will design a *formal language*, the language of propositional logic, where no ambiguity can occur. The first such language that we study will be the language of *propositional logic*.

Normally, when people say *formal*, they mean it in a bad way, such as having to dress or act formally to go to dinner.

However, in computer science (and in mathematics), formal is a good thing: it means making things so precise that there is no possibility of misunderstanding.

2.10 Exercise Sheet

Exercise 9. *Establish which of the following phrases are propositions:*

1. You own a laptop computer.

2. Snow is white.

3. Snow is not white.

4. My father goes to work and I go to school.

5. It is raining outside, but I have an umbrella.
6. Either it will rain tomorrow, or it won't rain.
7. If I get a passing grade in Logic, I will celebrate.
8. $2 + 2 = 4$. (Two plus two is four.)
9. Red and Black.
10. π.
11. Is it raining?
12. Let's go fishing!
13. x is greater than 7.
14. This sentence is false.

Exercise 10. *For all propositions that you identified, establish whether they are atomic or molecular. If molecular, establish whether they are conjunctions, negations, etc.*