PARAMETRIC EQUATIONS OF THE OLOID

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Introduction

The Oloid is a developable or torse or ruled surface. It is based upon joining the points between two identical circles, perpendicular to one another in space and separated such that each has its centre lying on the circumference of the other. Points on one circle are then joined to points on the other with a straight line - hence a "ruled surface".

The parametrization of the unit circle in the plane and centred at the origin is typically given as $(\sin t, \cos t)$ over $\{t \in \mathbb{R} | -\pi \leq t \leq \pi\}$. This is a good basis for the parametrization of one of the circles, but not for the other. As will be demonstrated, for the remarkable properties of the Oloid to be realised, the mapping of a point on one circle to its corresponding straight line end point on the other circle requires the arc of the end point to have a different parametrization to that of the start. Implementations that assume a naive connection of points typically exhibit concave surfaces and are not Oloids. I call such surfaces Fauxloids.

A derivation is given for the equations provided in [1]. A further alternative parametrization with a rational goal is also shown and when rendered with a low polygon count approximates the ideal more accurately.

Inversion with Respect to a Circle

Point A on the unit circle centred at the origin O lies on the tangent line that crosses the x-axis at point P as shown in figure 0.1. The point P' is the *inverse* of P with respect to the circle. Similarly, P is the inverse of P'. If the x coordinate of P' is $\cos t$, then the x coordinate of P is $\frac{1}{cost}$. This bidirectional relationship is exploited in the following sections.

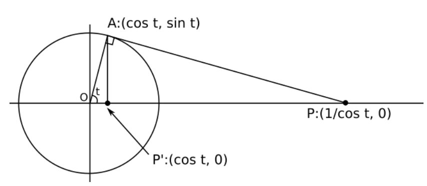


Figure 0.1. Inversion with respect to a circle

Parametrization using Transcendental Functions

Points U and V lie on unit circles centred at $C_U = (0, -\frac{1}{2})$ and $C_V = (0, \frac{1}{2})$ respectively as shown in figure 0.2. U and V correspond to points A and B in [1].

$$U(t) = \left[\sin t, -\frac{1}{2} - \cos t, 0\right] \tag{0.1}$$

The triangle $T'UV$ is formed by the joining line UV and the intersection point T' of the tangent lines through U and V. Point T' is also the inverse of point $T_U = (0, -\frac{1}{2} - \cos t)$.

By inversion with respect to the circle, the distance $C_U T'$ is

$$|C_U T'| = \frac{1}{\cos t}$$

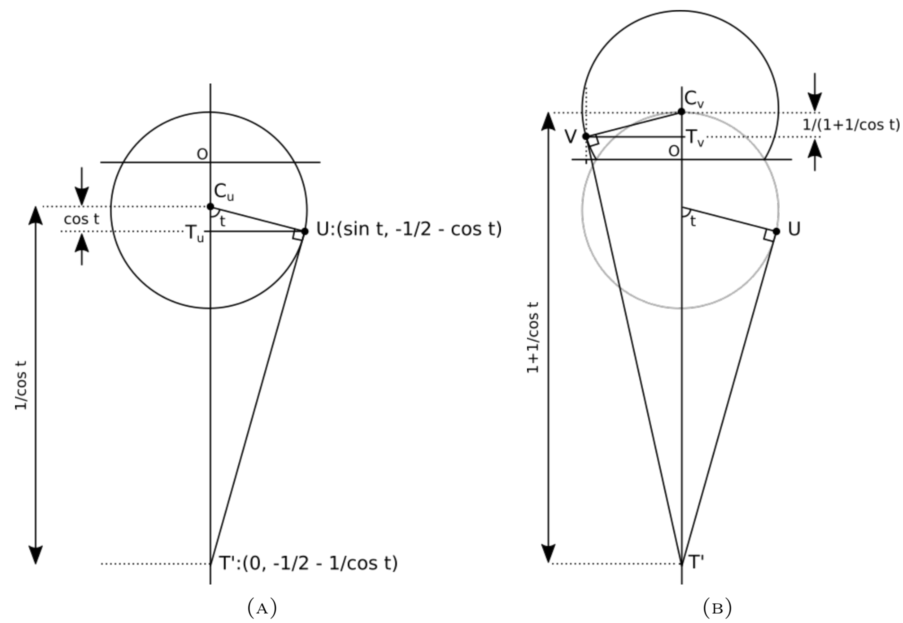


FIGURE 0.2. Distances $|C_U T'|$and $|T' C_V|$

For simplicity, the circle centred at C_V is rotated from yz into the xy plane. Since the circle centres are unit distance apart, the length $T'C_V$ is therefore

$$|T'C_V| = 1 + \frac{1}{\cos t}$$

T_V is the inverse of T' with respect to the upper circle, making the displacement along the y axis

$$|C_V T_V| = \frac{1}{1 + \frac{1}{\cos t}} = \frac{\cos t}{1 + \cos t}$$

By Pythagoras, the squared distance of V to the y axis is

$$|VT_V|^2 = 1 - \frac{\cos^2 t}{(1+\cos t)^2} = \frac{1+2\cos t}{(1+\cos t)^2}$$

Rotating back into the yz plane it follows that V has the parametrization

$$V(t) = \left[0, \frac{1}{2} - \frac{\cos t}{1+\cos t}, \pm\frac{\sqrt{1+2\cos t}}{1+\cos t}\right] \tag{0.2}$$

Alternative Quasi-Rational Parametrization

The approach taken for an alternative parametrization is identical to the one already given. This time as shown in figure 0.3, the ubiquitous rational parametrization of the circle is used as a starting point

$$U(t) = \left[\frac{2t}{1+t^2}, -\frac{1}{2} - \frac{1-t^2}{1+t^2}, 0\right] \tag{0.3}$$

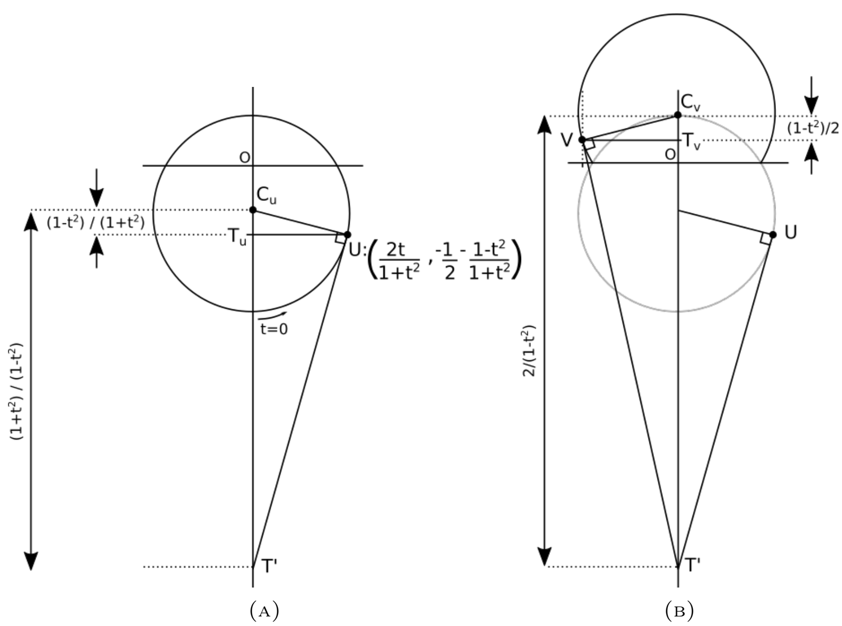


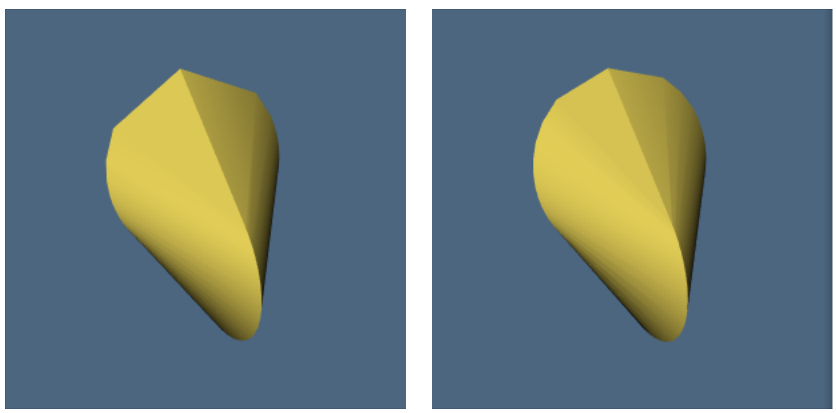
Figure 0.3. Alternative rational parametrization

By applying the same methodology of inversion with respect to a circle, it follows that

$$V(t) = \left[0, \frac{t^2}{2}, \pm\frac{1}{2}\sqrt{(3-t^2)(1+t^2)}\right] \tag{0.4}$$

Comparison of the Parametrizations

When rendered in a 3D graphics program, both parametrizations give pleasing results. The first method uses $\left\{t \in \mathbb{R} \mid -\frac{2}{3}\pi \leq t \leq \frac{2}{3}\pi\right\}$while the second uses $\left\{t \in \mathbb{R} \mid -\sqrt{3} \leq t \leq \sqrt{3}\right\}$. There is no escape from irrational constants in the second method despite its rational intentions.



(A) Original

(B) Quasi-rational

FIGURE 0.4. Comparison of Oloids using 28 sub-divisions per circular arc

Figures 0.4a and 0.4b show the Oloid rendered with a low polygon count. The quasi-rational parametrization is appreciably closer to the ideal. Figure 0.5 shows how both parametrizations project the circular arc traced by V on the plane and their different trajectories. The vertical axis tracks the parameter t. The steeper slope of the original parametrization indicates a less uniform distribution of sample points than the quasi-rational solution. This is reflected in the relative surface quality at low polygon counts.

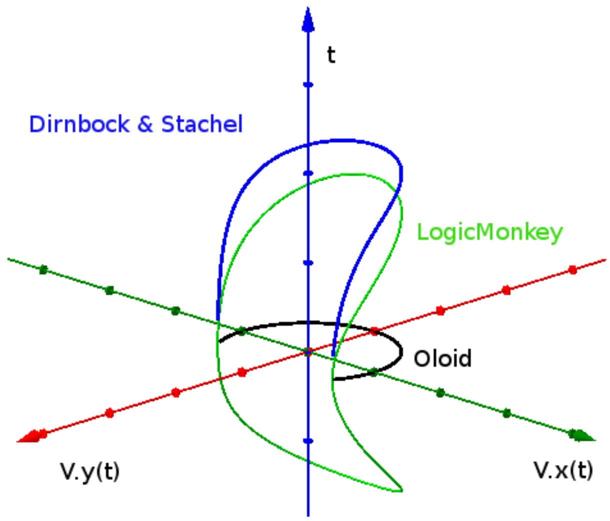


FIGURE 0.5. Functions over full range of parameter t

The Oloid property $|UV| = \sqrt{3}$ is readily proven using equations 0.1 and 0.2 as in [1]. Similarly, using equations 0.3 and 0.4

$$|UV|^2 = \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{t^2}{2} - \left(-\frac{1}{2} - \frac{1-t^2}{1+t^2}\right)\right)^2 + \left(\frac{\sqrt{(3-t^2)(1+t^2)}}{2}\right)^2 = 3$$

C++ Source Code using VTK

`https://github.com/logicmonkey/surfaces/blob/master/Oloid/Oloid.cxx`

`https://github.com/logicmonkey/surfaces/blob/master/rOloid/rOloid.cxx`

References

[1] Dirnbock H., Stachel H. The Development of the Oloid. Journal for Geometry and Graphics Volume 1 (1997), No. 2, 105–118