Virtual Classes

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Quine's virtual classes and related concepts. Virtual classes are straightforwardly expressible as macros. Formalized with the *PIE* system.

1 Quine's Virtual Classes and Virtual Relations

See [Qui70, Chap. 5 The Scope of Logic], [Qui69, Chap. I]. Quine notes there that identity (that is, equality) can be "simulated" with subtitutivity axioms for the finite vocabulary of a given formula and asks whether set theory can be handled analogousy. Virtual classes and their generalization to virtual relations provide a translation that applies to set abstraction on the right side of $\in$.

Remark: Quine cites Behmann's book [Beh27] on [Qui69, p. 19]. He notes that Behmann also introduces operations on classes and relations as mere variant notation for sentence connectives, as in the virtual theory of classes, but that there is a crucial difference as Behmanns assumes classes and relations as *values of quantifiable variables*.

1.1 Virtual Classes and Relations: Abstraction and Operations

The following macros apply to both, virtual classes and relations. Relations are like sets, except that the first argument of $\in$ and the first argument

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$Y \in_v \mathsf{setof}(X, F_X)$

Defined as

$$F_Y,$$

where

$$F_Y := F_X[X \mapsto Y].$$

$X \in_v \mathsf{complem}(Y)$

Defined as

$$\neg X \in_v Y.$$

$X \in_v \mathsf{isect}(Y, Z)$

Defined as

$$X \in_v Y \wedge X \in_v Z.$$

$X \in_v \mathsf{union}(Y, Z)$

Defined as

$$X \in_v Y \vee X \in_v Z.$$

$X \in_v \mathsf{empty}$

Defined as

$$\bot.$$

$X \in_v \mathsf{full}$

Defined as

$$\top.$$

$X \in_v \mathsf{unit}(Y)$

Defined as

$$X = Y.$$

$X \in_v \mathsf{upair}(Y, Z)$

Defined as

$$X \in_v \mathsf{union}(\mathsf{unit}(Y), \mathsf{unit}(Z)).$$

1.2 Predicates of Virtual Classes

The following macros apply to virtual *classes* only.

$X \subseteq_v Y$

Defined as

$$\forall Z \, (Z \in_v X \rightarrow Z \in_v Y),$$

where

$$Z := \text{a fresh symbol.}$$

$X \subset_v Y$

Defined as

$$X \subseteq_v Y \land \neg Y \subseteq_v X.$$

$X =_v Y$

Defined as

$$\forall Z \, (Z \in_v X \leftrightarrow Z \in_v Y),$$

where

$$Z := \text{a fresh symbol.}$$

1.3 Predicates of Virtual Relations

The following macros apply to virtual relations. The relation arity has to be supplied as first argument N.

$subseteq_v(N, X, Y)$

Defined as

$$\forall Z \, (Z \in_v X \to Z \in_v Y),$$

where

$$Z := \text{a sequence of N fresh symbols.}$$

$subset_v(N, X, Y)$

Defined as

$$subseteq_v(N, X, Y) \land \neg subseteq_v(N, Y, X).$$

$eq_v(N, X, Y)$

Defined as

$$\forall Z \, (Z \in_v X \leftrightarrow Z \in_v Y),$$

where

$$\text{mac\_make\_fresh\_arg}(\text{N}, \text{Z}).$$

1.4 Operations on Binary Virtual Relations

The macros in the following group apply to binary relations.

$[X, Y] \in_v \mathsf{product\_of\_classes}(A, B)$

Defined as

$$X \in_v A \land Y \in_v B.$$

$[X, Y] \in_v \mathsf{converse}(R)$

Defined as

$$[Y, X] \in_v R.$$

$[X, Z] \in_v \mathsf{resultant}(Q, R)$

Defined as

$$\exists Y\,([X, Y] \in_v Q \land [Y, Z] \in_v R).$$

$X \in_v \mathsf{image}(R, A)$

Defined as

$$\exists Y\,([X, Y] \in_v R \land Y \in_v A).$$

$[X, Y] \in_v \mathsf{identity}$

Defined as

$$X = Y.$$

1.5 Properties of Binary Virtual Relations

Further properties of relations can be defined as macros in terms of the previously defined operations – see [Qui69, p. 22f]. For example:

$\mathit{irreflexive}_v(R)$

Defined as

$$\mathit{subseteq}_v(2, R, \mathsf{complem}(\mathsf{identity})).$$

2 Quine's Set Abstraction in Element Position

This is discussed in [Qui70, p. 64ff].

$\mathsf{setof}_\mathsf{q}(X, F_X) \in_q Y$

Defined as

$$\exists Z\,(Z \in_q Y \land \forall X\,(X \in_q Z \leftrightarrow F_X)),$$

where

$$Z := \text{a fresh symbol}.$$

setopsq

Defined as

$$\begin{array}{ll}\forall y\, \mathsf{complem_q}(y) = \mathsf{setof_q}(\mathsf{x},\ (\mathsf{x} \in_q y)) & \land \\ \forall yz\, \mathsf{isect_q}(y,z) = \mathsf{setof_q}(\mathsf{x}, (\mathsf{x} \in_q y \land \mathsf{x} \in_q z)) & \land \\ \forall yz\, \mathsf{union_q}(y,z) = \mathsf{setof_q}(\mathsf{x}, (\mathsf{x} \in_q y \lor \mathsf{x} \in_q z)) & \land \\ \mathsf{empty_q} = \mathsf{setof_q}(\mathsf{x}, \mathsf{false}) & \land \\ \mathsf{full_q} = \mathsf{setof_q}(\mathsf{x}, \mathsf{true}) & \land \\ \forall y\, \mathsf{unit_q}(y) = \mathsf{setof_q}(\mathsf{x}, \mathsf{x} = y) & \land \\ \forall yz\, \mathsf{upair_q}(y,z) = \mathsf{union_q}(\mathsf{unit_q}(y), \mathsf{unit_q}(z)). & \end{array}$$

$Y = \mathsf{setof_q}(X, F_X)$

Defined as

$$\forall X\, (X \in_q Y \leftrightarrow F_X).$$

The expansion of *setopsq* is now:

$$\begin{array}{ll}\forall xy\, (y \in_q \mathsf{complem_q}(x) \leftrightarrow \neg y \in_q x) & \land \\ \forall xyz\, (z \in_q \mathsf{isect_q}(x,y) \leftrightarrow z \in_q x \land z \in_q y) & \land \\ \forall xyz\, (z \in_q \mathsf{union_q}(x,y) \leftrightarrow z \in_q x \lor z \in_q y) & \land \\ \forall x\, (x \in_q \mathsf{empty_q} \leftrightarrow \bot) & \land \\ \forall x\, (x \in_q \mathsf{full_q} \leftrightarrow \top) & \land \\ \forall xy\, (y \in_q \mathsf{unit_q}(x) \leftrightarrow y = x) & \land \\ \forall xy\, \mathsf{upair_q}(x,y) = \mathsf{union_q}(\mathsf{unit_q}(x), \mathsf{unit_q}(y)). & \end{array}$$

References

[Beh27] Heinrich Behmann. *Mathematik und Logik*, volume 71 of *Mathematisch-Physikalische Bibliothek*. Teubner, Leipzig, 1927.

[Qui69] Willard Van Orman Quine. *Set Theory and Its Logic*. Harvard University Press, Cambridge, MA, revised edition, 1969.

[Qui70] Willard Van Orman Quine. *The Philosophy of Logic*. Harvard University Press, Cambridge, MA, 1970.

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