Circumscription

Revision: February 13, 2018; Rendered: May 25, 2021

Definition of predicate circumscription. Formalized with the *PIE* system.

1 Definition of Predicate Circumscription

$xcirc(S, F)$

Defined as

$$F \land \neg xraise(S, F).$$

A version of parallel predicate circumscription [Lif94]. The F parameter is the circumscribed formula. The S parameter specifies the roles of the predicates in the circumscription. It is a list of specifiers of the following form, where p is a predicate and the second component indicates positive, negative, or both polarities:

p is to be minimized	p-`n`
p is to be maximized	p-`p`
p is varying	p-`pn`
p is fixed	p is not mentioned in S

In some cases it might be necessary to specify explicitly also the arity of the respective predicates, e.g. $p/1$-`n`.

The S argument is considered complementary to the scope argument to *circ* in [Wer12] (like in forgetting instead of projection).

$xraise(S, F)$

Defined as

$$\exists Q\, (F_1 \land T_1 \land \neg T_2),$$

where

$$\begin{aligned}
F_1 &:= F[S \mapsto Q],\\
S_2 &:= S \text{ with arities from } F,\\
T_1 &:= \text{transfer clauses } S_2 \to Q,\\
T_2 &:= \text{transfer clauses } Q \to S_2.
\end{aligned}$$

The second-order subformula on which *xcirc* is based, Similar to *raise* [Wer12], however the scope argument S is considered complementary (like in forgetting instead of projection).

2 Some Examples

These are examples from [Lif94].

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], \mathsf{pa})$.
Result of elimination:

$$\mathsf{pa} \land \forall x\, (\mathsf{p}x \to x = \mathsf{a}).$$

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], \neg\mathsf{pa})$.
Result of elimination:

$$\forall x\, \neg\mathsf{p}x.$$

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], (\mathsf{pa} \land \mathsf{pb}))$.
Result of elimination:

$$\mathsf{pa} \land \mathsf{pb} \land \forall x\, (\mathsf{p}x \to x = \mathsf{a} \lor x = \mathsf{b}).$$

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], (\mathsf{pa} \lor \mathsf{pb}))$.
Result of elimination:

$$(\mathsf{pa} \lor \mathsf{pb}) \land \forall x\, (\mathsf{p}x \to (\mathsf{pa} \to x = \mathsf{a}) \land (\mathsf{pb} \to x = \mathsf{b})).$$

This formula is valid: $\mathit{last\_result} \leftrightarrow \forall x\, (\mathsf{p}x \leftrightarrow x = \mathsf{a}) \lor \forall x\, (\mathsf{p}x \leftrightarrow x = \mathsf{b})$.

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], (\neg\mathsf{pa} \lor \mathsf{pb}))$.
Result of elimination:

$$\forall x\, \neg\mathsf{p}x.$$

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], (\mathsf{pa} \lor (\mathsf{pb} \land \mathsf{pc})))$.
Result of elimination:

$$\begin{aligned}
&(\mathsf{pa} \lor (\mathsf{pb} \land \mathsf{pc})) \land\\
&\forall x\, (\mathsf{p}x \to (\mathsf{pa} \to x = \mathsf{a}) \land (\mathsf{pb} \land \mathsf{pc} \to x = \mathsf{b} \lor x = \mathsf{c})).
\end{aligned}$$

This formula is valid: $\mathit{last\_result} \leftrightarrow \forall x\, (\mathsf{p}x \leftrightarrow x = \mathsf{a}) \lor (\forall x\, (\mathsf{p}x \leftrightarrow x = \mathsf{b} \lor x = \mathsf{c}) \land \mathsf{a} \neq \mathsf{b} \land \mathsf{a} \neq \mathsf{c})$.

Input: $\mathit{xcirc}([\mathsf{p\text{-}n}], \forall x\, \mathsf{p}x)$.
Result of elimination:

$$\forall x\, \mathsf{p}x.$$

Input: $xcirc([\mathsf{p\text{-}n}], \forall x\, (\mathsf{q}x \rightarrow \mathsf{p}x))$.
Result of elimination:

$$\begin{array}{l}(\forall x\, (\mathsf{q}x \rightarrow \mathsf{p}x) \rightarrow \forall x\, (\mathsf{p}x \rightarrow \mathsf{q}x)) \land \\ \forall x\, (\mathsf{q}x \rightarrow \mathsf{p}x).\end{array}$$

Input: $xcirc([\mathsf{p\text{-}n}], \exists x\, \mathsf{p}x)$.
Result of elimination:

$$\forall x\, (\mathsf{p}x \rightarrow \forall y\, (\mathsf{p}y \rightarrow x = y)) \land \exists x\, \mathsf{p}x.$$

This formula is valid: $last\_result \leftrightarrow \exists x\, \forall y\, (\mathsf{p}y \leftrightarrow x = y)$.

Input: $xcirc([\mathsf{p\text{-}n}], \forall x\, \mathsf{p}xx)$.
Result of elimination:

$$\begin{array}{l}(\forall xy\, (\mathsf{p}xy \rightarrow x = y) \lor \exists x\, \neg \mathsf{p}xx) \land \\ \forall x\, \mathsf{p}xx.\end{array}$$

Input: $xcirc([\mathsf{p\text{-}n}, \mathsf{q\text{-}pn}], \forall x\, (\mathsf{q}x \rightarrow \mathsf{p}x))$.
Result of elimination:

$$\forall x\, (\mathsf{q}x \rightarrow \mathsf{p}x) \land \forall x\, \neg \mathsf{p}x.$$

This formula is valid: $last\_result \leftrightarrow \forall x\, \neg \mathsf{p}x \land \forall x\, \neg \mathsf{q}x$.

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block_axioms

Defined as

$$\begin{array}{ll}\forall x\, (\mathsf{block}(x) \land \neg \mathsf{ab}(x) \rightarrow \mathsf{ontable}(x)) & \land \\ \neg \mathsf{ontable}(\mathsf{b}_1) & \land \\ \mathsf{block}(\mathsf{b}_1) & \land \\ \mathsf{block}(\mathsf{b}_2) & \land \\ \mathsf{b}_1 \neq \mathsf{b}_2. & \end{array}$$

Input: $xcirc([\mathsf{ab\text{-}n}, \mathsf{ontable\text{-}pn}], block\_axioms)$.
Result of elimination:

$$\begin{array}{ll}\mathsf{b}_1 \neq \mathsf{b}_2 & \land \\ \neg \mathsf{ontable}(\mathsf{b}_1) & \land \\ \mathsf{block}(\mathsf{b}_1) & \land \\ \mathsf{block}(\mathsf{b}_2) & \land \\ (\mathsf{ab}(\mathsf{b}_1) \rightarrow \forall x\, (\mathsf{ab}(x) \rightarrow x = \mathsf{b}_1 \land \mathsf{block}(x))) & \land \\ \forall x\, (\mathsf{block}(x) \rightarrow \mathsf{ab}(x) \lor \mathsf{ontable}(x)). & \end{array}$$

This formula is valid: $last\_result \leftrightarrow block\_axioms \land \forall x\, (\mathsf{ab}(x) \leftrightarrow x = \mathsf{b}_1)$.

2.1 Auxiliary Macros

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last_result

Defined as

$$X,$$

where

$$\text{last\_ppl\_result(X).}$$

References

[Lif94] Vladimir Lifschitz. Circumscription. In Dov M. Gabbay, C. J. Hogger, and J. A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 3, pages 298–352. Oxford University Press, 1994.

[Wer12] Christoph Wernhard. Projection and scope-determined circumscription. *Journal of Symbolic Computation*, 47:1089–1108, 2012.

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