PIE Example Document

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1 Projection and Definientia

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project(S, F)

Defined as

$$\exists S_1\, F,$$

where

$$S_2 := \mathsf{free\_predicates}(F),$$
$$S_1 := S_2 \setminus S.$$

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definiens(G, F, S)

Defined as

$$\mathit{project}(S, (F \land G)) \rightarrow \neg \mathit{project}(S, (F \land \neg G)).$$

2 Obtaining a Definiens with Interpolation

We specify a background knowledge base:

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kb_1

Defined as

$$\forall x\, ((\mathsf{q}x \rightarrow \mathsf{p}x) \land (\mathsf{p}x \rightarrow \mathsf{r}x) \land (\mathsf{r}x \rightarrow \mathsf{q}x)).$$

To obtain a definiens of $\exists x\, \mathsf{p}x$ in terms $\{\mathsf{q}, \mathsf{r}\}$ within kb_1 we specify an implication with the *definiens* macro:

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ex_1

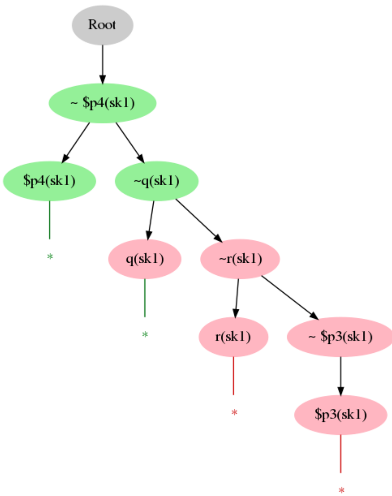
Defined as

$$\mathit{definiens}(\exists x\, \mathsf{p}x, kb_1, [\mathsf{q}, \mathsf{r}]).$$

We now invoke a utility predicate that computes and prints for a given valid implication an interpolant of its antecedent and consequent:
Input: ex_1.
Result of interpolation:

$$\exists x\, \mathsf{q}x.$$

The proof underlying interpolant extraction can be visualized, colors representing the sides with respect to interpolation. The color of the closing marks indicate the side of the connection partner:



Before we leave that example, we take a look at the expansion of ex_1:

$$\begin{array}{l} \exists p\, (\forall x\, ((\mathsf{q}x \rightarrow px) \wedge (px \rightarrow \mathsf{r}x) \wedge (\mathsf{r}x \rightarrow \mathsf{q}x)) \quad \wedge \\ \quad \exists x\, px) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rightarrow \\ \neg\exists p\, (\forall x\, ((\mathsf{q}x \rightarrow px) \wedge (px \rightarrow \mathsf{r}x) \wedge (\mathsf{r}x \rightarrow \mathsf{q}x)) \wedge \\ \qquad \neg\exists x\, px). \end{array}$$

And we invoke an external prover (*Prover9*) to validate it:
This formula is valid: ex_1.

3 Obtaining the Strongest Definiens with Elimination

The antecedent of the implication in the expansion of *definiens* specifies the strongest necessary condition of G on S within F. In case definability holds (that is, the implication is valid), this antecedent denotes the strongest definiens. In the example it has a first-order equivalent that can be computed by second-order quantifier elimination.

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ex_2

Defined as

$$\mathit{project}([\mathsf{q}, \mathsf{r}], (\mathit{kb}_1 \land \exists x\, \mathsf{p}x)).$$

Input: ex_2.
Result of elimination:

$$\forall x\, (\mathsf{r}x \rightarrow \mathsf{q}x) \land \forall x\, (\mathsf{q}x \rightarrow \mathsf{r}x) \land \exists x\, \mathsf{r}x.$$

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