

Axiomatizations of Orderings

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Formalized with the *PIE* system.

1 General Properties of Binary Relations

is_reflexive(P)

Defined as

$$\forall x Pxx.$$

is_irreflexive(P)

Defined as

$$\forall x \neg Pxx.$$

is_symmetric(P)

Defined as

$$\forall xy (Pxy \rightarrow Pyx).$$

is_asymmetric(P)

Defined as

$$\forall xy (Pxy \rightarrow \neg Pyx).$$

is_antisymmetric(*P*)

Defined as

$$\forall xy (Pxy \wedge Pyx \rightarrow x = y).$$

is_total(*P*)

Defined as

$$\forall xy (Pxy \vee Pyx).$$

is_transitive(*P*)

Defined as

$$\forall xyz (Pxy \wedge Pyz \rightarrow Pxz).$$

is_trichotomous(*P*)

Defined as

$$\begin{aligned} \forall xy ((Pxy \wedge \neg Pyx \wedge x \neq y) & \quad \vee \\ (\neg Pxy \wedge Pyx \wedge x \neq y) & \quad \vee \\ (\neg Pxy \wedge \neg Pyx \wedge x = y)). & \end{aligned}$$

is_connected(*P*)

Defined as

$$\forall xy (Pxy \vee Pyx \vee x = y).$$

2 Orderings of Binary Relations

is_total_order(*P*)

Defined as

$$is_antisymmetric(P) \wedge is_transitive(P) \wedge is_total(P).$$

is_strict_total_order(*P*)

Defined as

$$is_irreflexive(P) \wedge is_transitive(P) \wedge is_connected(P).$$

is_strict_total_order_v2(*P*)

Defined as

$$is_transitive(P) \wedge is_trichotomous(P).$$

is_partial_order(*P*)

Defined as

$$is_antisymmetric(P) \wedge is_transitive(P) \wedge is_reflexive(P).$$

is_strict_partial_order(*P*)

Defined as

$$is_irreflexive(P) \wedge is_transitive(P).$$

Some properties of orderings, for testing with theorem provers. Not all these problems are for all provers as easy as they seem – see comments in the source.

test_orderings₁

Defined as

$$is_asymmetric(p) \leftrightarrow is_antisymmetric(p) \wedge is_irreflexive(p).$$

test_orderings₂

Defined as

$$is_transitive(p) \rightarrow (is_asymmetric(p) \leftrightarrow is_irreflexive(p)).$$

test_orderings₃

Defined as

$$\begin{aligned} & is_irreflexive(\mathbf{p}) \wedge is_transitive(\mathbf{p}) \quad \rightarrow \\ & (is_connected(\mathbf{p}) \leftrightarrow is_trichotomous(\mathbf{p})). \end{aligned}$$

test_orderings₄

Defined as

$$is_strict_total_order_v_2(\mathbf{p}) \leftarrow is_strict_total_order(\mathbf{p}).$$

test_orderings₅

Defined as

$$is_strict_total_order_v_2(\mathbf{p}) \rightarrow is_strict_total_order(\mathbf{p}).$$

test_orderings₆

Defined as

$$is_strict_total_order_v_2(\mathbf{p}) \leftrightarrow is_strict_total_order(\mathbf{p}).$$

test_orderings₇

Defined as

$$is_total_order(\mathbf{p}) \rightarrow is_partial_order(\mathbf{p}).$$

test_orderings₈

Defined as

$$is_strict_total_order(\mathbf{p}) \rightarrow is_strict_partial_order(\mathbf{p}).$$

test_orderings₉

Defined as

$$is_strict_partial_order(p) \rightarrow \neg is_partial_order(p).$$

test_orderings₁₀

Defined as

$$is_strict_total_order(p) \rightarrow \neg is_total_order(p).$$

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