Axiomatizations of Orderings

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Formalized with the *PIE* system.

1 General Properties of Binary Relations

$is\_reflexive(P)$

Defined as

$$\forall x\, Pxx.$$

$is\_irreflexive(P)$

Defined as

$$\forall x\, \neg Pxx.$$

$is\_symmetric(P)$

Defined as

$$\forall xy\, (Pxy \to Pyx).$$

$is\_asymmetric(P)$

Defined as

$$\forall xy\, (Pxy \to \neg Pyx).$$

$is\_antisymmetric(P)$

Defined as

$$\forall xy\,(Pxy \wedge Pyx \rightarrow x = y).$$

$is\_total(P)$

Defined as

$$\forall xy\,(Pxy \vee Pyx).$$

$is\_transitive(P)$

Defined as

$$\forall xyz\,(Pxy \wedge Pyz \rightarrow Pxz).$$

$is\_trichotomous(P)$

Defined as

$$\begin{aligned}\forall xy\,((Pxy \wedge \neg Pyx \wedge x \neq y) &\vee \\ (\neg Pxy \wedge Pyx \wedge x \neq y) &\vee \\ (\neg Pxy \wedge \neg Pyx \wedge x = y)).\end{aligned}$$

$is\_connected(P)$

Defined as

$$\forall xy\,(Pxy \vee Pyx \vee x = y).$$

2 Orderings of Binary Relations

$is\_total\_order(P)$

Defined as

$$is\_antisymmetric(P) \wedge is\_transitive(P) \wedge is\_total(P).$$

$is\_strict\_total\_order(P)$

Defined as

$$is\_irreflexive(P) \wedge is\_transitive(P) \wedge is\_connected(P).$$

$is\_strict\_total\_order\_v_2(P)$

Defined as

$$is\_transitive(P) \wedge is\_trichotomous(P).$$

$is\_partial\_order(P)$

Defined as

$$is\_antisymmetric(P) \wedge is\_transitive(P) \wedge is\_reflexive(P).$$

$is\_strict\_partial\_order(P)$

Defined as

$$is\_irreflexive(P) \wedge is\_transitive(P).$$

Some properties of orderings, for testing with theorem provers. Not all these problems are for all provers as easy as they seem – see comments in the source.

$test\_orderings_1$

Defined as

$$is\_asymmetric(\mathsf{p}) \leftrightarrow is\_antisymmetric(\mathsf{p}) \wedge is\_irreflexive(\mathsf{p}).$$

$test\_orderings_2$

Defined as

$$is\_transitive(\mathsf{p}) \rightarrow (is\_asymmetric(\mathsf{p}) \leftrightarrow is\_irreflexive(\mathsf{p})).$$

$test\_orderings_3$

Defined as

$$is\_irreflexive(\mathsf{p}) \wedge is\_transitive(\mathsf{p}) \rightarrow (is\_connected(\mathsf{p}) \leftrightarrow is\_trichotomous(\mathsf{p})).$$

$test\_orderings_4$

Defined as

$$is\_strict\_total\_order\_v_2(\mathsf{p}) \leftarrow is\_strict\_total\_order(\mathsf{p}).$$

$test\_orderings_5$

Defined as

$$is\_strict\_total\_order\_v_2(\mathsf{p}) \rightarrow is\_strict\_total\_order(\mathsf{p}).$$

$test\_orderings_6$

Defined as

$$is\_strict\_total\_order\_v_2(\mathsf{p}) \leftrightarrow is\_strict\_total\_order(\mathsf{p}).$$

$test\_orderings_7$

Defined as

$$is\_total\_order(\mathsf{p}) \rightarrow is\_partial\_order(\mathsf{p}).$$

$test\_orderings_8$

Defined as

$$is\_strict\_total\_order(\mathsf{p}) \rightarrow is\_strict\_partial\_order(\mathsf{p}).$$

$test\_orderings_{9}$

Defined as

$$is\_strict\_partial\_order(\mathsf{p}) \rightarrow \neg is\_partial\_order(\mathsf{p}).$$

$test\_orderings_{10}$

Defined as

$$is\_strict\_total\_order(\mathsf{p}) \rightarrow \neg is\_total\_order(\mathsf{p}).$$

Index