Categorical Semantics for Intuitionistic Belief

Cosimo Perini Brogi
DiMa - Università degli Studi di Genova

Seminario di Logica e Filosofia della Scienza
Università degli Studi di Firenze

24 Maggio 2019

Outline

Knowledge and Verification

We adopt the view that an ***an intuitionistic epistemic state (belief or knowledge) is the result of verification*** *where a verification is evidence considered sufficiently conclusive for practical purposes.*

[Artemov and Protopopescu 2014]

Knowledge and Verification

We adopt the view that an ***an intuitionistic epistemic state (belief or knowledge) is the result of verification*** *where a verification is evidence considered sufficiently conclusive for practical purposes.*

[Artemov and Protopopescu 2014]

Brouwer-Heyting-Kolgomorov Interpretation

According to the BHK semantics, a proposition A is true if there is a proof of it, and false if one can show that assuming A leads to a contradiction. More precisely:

- there is no proof of $\bot$;
- a proof p of $A \wedge B$ consists of a pair $\langle a, b \rangle$ where a is a proof of A and b is a proof of B;
- a proof p of $A \vee B$ is a pair $\langle n, q \rangle$ where $n = 0$ and q proves A, or $n = 1$ and q proves B;
- a proof p of $A \rightarrow B$ is a rule which transforms any proof q of A into a proof $p(q)$ of B.

Extending BHK

If we add an epistemic modal operator $\Box$ to the language of intuitionistic propositional logic, we must extend the BHK interpretation to any formula $\Box A$.
The epistemic clause adopted is

- a proof p of $\Box A$ is a conclusive evidence of *verification* that A has a proof.

Extending BHK

1. Every proof is a verification;
2. That something is a proof is itself capable of proof.

Considering 1. and this extended semantics we have that

$$\text{Intuitionistic Truth} \Rightarrow \text{Intuitionistic Knowledge/Belief}$$

since, in general,

$$\text{it is proved that } A \text{ is verified}$$

is a weaker statement than

$$\text{it is proved that } A$$

Extending BHK

1. Every proof is a verification;
2. That something is a proof is itself capable of proof.

Considering 1. and this extended semantics we have that

$$\text{Intuitionistic Truth} \Rightarrow \text{Intuitionistic Knowledge/Belief}$$

since, in general,

$$\text{it is proved that } A \text{ is verified}$$

is a weaker statement than

$$\text{it is proved that } A$$

Extending BHK

1. Every proof is a verification;
2. That something is a proof is itself capable of proof.

Considering 1. and this extended semantics we have that

$$\text{Intuitionistic Truth} \Rightarrow \text{Intuitionistic Knowledge/Belief}$$

since, in general,

$$\text{it is proved that } A \text{ is verified}$$

is a weaker statement than

$$\text{it is proved that } A$$

In other terms

We can read any formula $\Box A$ as asserting that A has a proof which is not necessarily specified in the process of verification, or more generally that it is verified that A holds in some not specified constructive sense.
This allows to apply intuitionistic epistemic reasoning in various contexts which are not necessarily in the standard domain of BHK; for instance:

- Testimony of authority;
- Zero-knowledge protocols;
- Highly probable truth.

In other terms

We can read any formula $\Box A$ as asserting that A has a proof which is not necessarily specified in the process of verification, or more generally that it is verified that A holds in some not specified constructive sense.
This allows to apply intuitionistic epistemic reasoning in various contexts which are not necessarily in the standard domain of BHK; for instance:

- Testimony of authority;
- Zero-knowledge protocols;
- Highly probable truth.

$\mathbb{IEL}^-$

Axioms

1. Axioms of propositional intuitionistic logic;
2. $\Box(A \to B) \to \Box A \to \Box B$; (K-scheme)
3. $A \to \Box A$. (co-reflection)

Rules

$$\frac{A \to B \qquad A}{B}\ SR$$

Some Meta-results

Lemma

For $\mathbb{IEL}^-$ the following hold:

(i) *The necessitation rule is derivable.*

(ii) *The deduction theorem holds.*

(iii) *Uniform substitution holds.*

(iv) $\mathbb{IEL}^-$ *is a normal modal system.*

(v) $\mathbb{IEL}^- \vdash \Box A \to \Box\Box A$ *and* $\mathbb{IEL}^- \vdash \neg\Box A \to \Box\neg\Box A$.

Relational Semantics

An $\mathbb{IEL}^-$-model is given by

- A model $\langle W, \leq, v\rangle$ for intuitionistic propositional logic;
- A binary relation $E \subseteq W \times W$ such that:
  - $E(x) \subseteq x\!\uparrow$ for any $x \in W$;
  - if $x \leq y$, then $E(y) \subseteq E(x)$
- v is an evaluation map which extends to a forcing relation $\vDash$ whose epistemic clause is
  - $x \vDash \Box A$ iff $y \vDash A$ for any $y \in E(x)$.

Soundness and Completeness

Lemma (Monotonicity)

For each model and a formula A, if $x \vDash A$ and $x \leq y$, then $y \vDash A$.

Lemma (Soundness)

If $\mathbb{IEL}^- \vdash A$ then A holds in any $\mathbb{IEL}^-$-model.

Theorem (Completeness)

If A holds in any $\mathbb{IEL}^-$-model, then $\mathbb{IEL}^- \vdash A$.

Proof.

By constructing a canonical model in the standard way. □

Soundness and Completeness

Lemma (Monotonicity)

For each model and a formula A, if $x \vDash A$ and $x \leq y$, then $y \vDash A$.

Lemma (Soundness)

If $\mathbb{IEL}^- \vdash A$ then A holds in any $\mathbb{IEL}^-$-model.

Theorem (Completeness)

If A holds in any $\mathbb{IEL}^-$-model, then $\mathbb{IEL}^- \vdash A$.

Proof.

By constructing a canonical model in the standard way. □

Factivity of Knowledge

Lemma

$\mathbb{IEL}^- \nvdash \Box A \to A$

Proof.

Consider the following model:

$$\begin{tikzcd} \bullet \arrow[r, "\leq"'] \arrow[r, bend left, dotted, "E"] & \bullet\, p \arrow[loop above, dotted, "E"] \end{tikzcd}$$

$\square$

Natural Deduction for $\mathbb{IEL}^-$

Consider the calculus obtained by extending the propositional fragment of NJ with the following rules:

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots \\ \Box A_1\end{matrix} \quad \cdots \quad \begin{matrix}\Gamma_n \\ \vdots \\ \Box A_n\end{matrix} \quad \begin{matrix}[A_1, \cdots, A_n] \\ \vdots \\ B\end{matrix}}{\Box B}\ \Box\textit{-elim} \qquad \text{and} \qquad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{\Box A}\ \Box\textit{-intro}$$

where all $A_1, \cdots, A_n$ are discharged in $\Box - elim$.

We call this calculus IEL^-.

Equivalence of $\mathbb{IEL}^-$ and IEL^-

Lemma

$\mathbb{IEL}^- \equiv \mathsf{IEL}^-$

Proof.

In both directions one proceeds by induction on the derivation.

$\Rightarrow$: The K-scheme is derivable as follows

$$\dfrac{\Box(A \to B) \qquad \Box A \qquad \dfrac{[A \to B] \qquad [A]}{B}}{\Box B}\ ;$$

The co-reflection scheme is just $\dfrac{A}{\Box A}$.

$\Leftarrow$: By applying the deduction theorem for $\mathbb{IEL}^-$ to the inductive hypothesis in the $\Box - elim$ case, and SR with co-reflection and with K as necessary; for the $\Box - intro$ case, SR with co-reflection scheme gives the result.

Equivalence of $\mathbb{IEL}^-$ and IEL^-

Lemma

$\mathbb{IEL}^- \equiv \mathsf{IEL}^-$

Proof.

In both directions one proceeds by induction on the derivation.

$\Rightarrow$: The K-scheme is derivable as follows

$$\dfrac{\Box(A \to B) \qquad \Box A \qquad \dfrac{[A \to B] \qquad [A]}{B}}{\Box B};$$

The co-reflection scheme is just $\dfrac{A}{\Box A}$.

$\Leftarrow$: By applying the deduction theorem for $\mathbb{IEL}^-$ to the inductive hypothesis in the $\Box - elim$ case, and SR with co-reflection and with K as necessary; for the $\Box - intro$ case, SR with co-reflection scheme gives the result.

Equivalence of $\mathbb{IEL}^-$ and IEL^-

Lemma

$\mathbb{IEL}^- \equiv \mathsf{IEL}^-$

Proof.

In both directions one proceeds by induction on the derivation.

- $\Rightarrow$: The K-scheme is derivable as follows

$$\dfrac{\Box(A \to B) \qquad \Box A \qquad \dfrac{[A \to B] \qquad [A]}{B}}{\Box B}\;;$$

  The co-reflection scheme is just $\dfrac{A}{\Box A}$.

- $\Leftarrow$: By applying the deduction theorem for $\mathbb{IEL}^-$ to the inductive hypothesis in the $\Box - elim$ case, and SR with co-reflection and with K as necessary; for the $\Box - intro$ case, SR with co-reflection scheme gives the result.

Equivalence of $\mathbb{IEL}^-$ and IEL^-

Lemma

$\mathbb{IEL}^- \equiv \mathsf{IEL}^-$

Proof.

In both directions one proceeds by induction on the derivation.

- ⇒: The K-scheme is derivable as follows

$$\dfrac{\Box(A \to B) \qquad \Box A \qquad \dfrac{[A \to B] \qquad [A]}{B}}{\Box B} \; ;$$

The co-reflection scheme is just $\dfrac{A}{\Box A}$.

- ⇐: By applying the deduction theorem for $\mathbb{IEL}^-$ to the inductive hypothesis in the $\Box - elim$ case, and SR with co-reflection and with K as necessary; for the $\Box - intro$ case, SR with co-reflection scheme gives the result.

□

Proofs-as-Terms

IEL^- can be easily turned into a typed system with a modal operator by extending the syntax for the simply typed λ-calculus as follows:

$$T ::= 1 \mid 0 \mid p \mid A \to B \mid A \times B \mid A + B \mid \Box A$$

$$\begin{aligned} t ::= \quad & \star \mid x \mid \lambda x : A.t : B \mid t_1\, t_2 \mid (t_1, t_2) \mid \pi_1(t) \mid \pi_2(t) \mid \\ & \mathtt{inl}(a : A) \mid \mathtt{inr}(b : B) \mid \mathtt{case}(t)\mathtt{of}\,\mathtt{inl}(x) \Rightarrow s_1 | \mathtt{inr}(y) \Rightarrow s_2 \mid \\ & \mathtt{box}(t) : \Box A \mid \mathtt{unbox}(t : B)[\vec{s} : \vec{\Box A} | \vec{x} : \vec{A}] : \Box B \ . \end{aligned}$$

In this way, we can handle IEL^--deductions algebraically.
For instance, we can define a λ-theory for these deductions by imposing specific equations between terms which we can read as (potentially bi-directional) "proof-rewritings".

Rewritings

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \Box A\end{array} \qquad [A]}{\Box A}\ \Box{-}elim \quad \rightsquigarrow \quad \begin{array}{c}\Gamma\\ \vdots\\ \Box A\end{array}$$

Rewritings

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \Box A\end{array} \qquad [A]}{\Box A}\ \Box\!-\!elim \quad \rightsquigarrow \quad \begin{array}{c}\Gamma \\ \vdots \\ \Box A\end{array}$$

Rewritings

$$
\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{\Box A}\ \Box\text{-}intro \qquad \begin{array}{c}[A]\\ \vdots\\ B\end{array}}{\Box B}\ \Box\text{-}elim
\quad \rightsquigarrow \quad
\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\\ \vdots\\ B\end{array}}{\Box B}\ \Box\text{-}intro
$$

Rewritings

$$\dfrac{\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{\Box A}\ \Box\text{-}intro \qquad \begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{\Box B}\ \Box\text{-}elim \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ A \\ \vdots \\ B\end{matrix}}{\Box B}\ \Box\text{-}intro$$

Rewritings

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}[\vec{A}]^1\\ \vdots\\ B\end{matrix}}{\Box B}\ \Box{-}elim{:}1 \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{C}\end{matrix} \qquad \begin{matrix}\Theta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}[B,\vec{C},\vec{D}]^2\\ \vdots\\ E\end{matrix}}{\Box E}\ \Box{-}elim{:}2$$

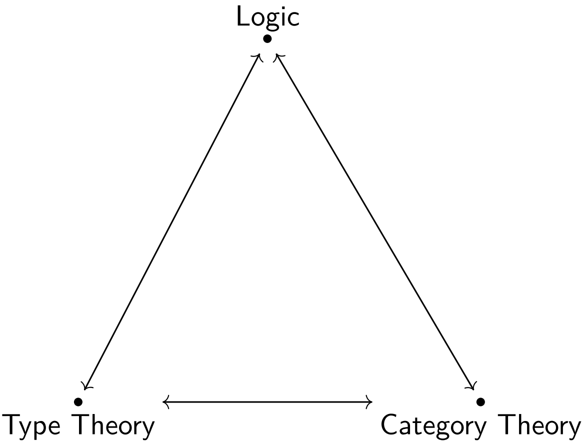
$$\rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{C}\end{matrix} \qquad \begin{matrix}\Theta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}\begin{matrix}[\vec{A}]^1\\ \vdots\\ B\end{matrix} \quad [\vec{C} \quad \vec{D}]^1\\ \vdots\\ E\end{matrix}}{\Box E}\ \Box{-}elim{:}1$$

Rewritings

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}[\vec{A}]^1\\ \vdots\\ B\end{matrix}}{\Box B}\,\Box{-}elim{:}1 \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{C}\end{matrix} \qquad \begin{matrix}\Theta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}[B,\vec{C},\vec{D}]^2\\ \vdots\\ E\end{matrix}}{\Box E}\,\Box{-}elim{:}2$$

$$\rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ \Box\vec{A}\end{matrix} \qquad \begin{matrix}\Delta\\ \vdots\\ \Box\vec{C}\end{matrix} \qquad \begin{matrix}\Theta\\ \vdots\\ \Box\vec{D}\end{matrix} \qquad \begin{matrix}\begin{matrix}[\vec{A}]^1\\ \vdots\\ B\end{matrix} \quad [\vec{C} \quad \vec{D}]^1\\ \vdots\\ E\end{matrix}}{\Box E}\,\Box{-}elim{:}1$$

Curry-Howard-Lambek Correspondence



Curry-Howard-Lambek Correspondence

Logic	Type Theory	Category Theory
proposition	type	object
proof	term	arrow
theorem	inhabitant	element-arrow
conjunction	product type	product
true	unit type	terminal object
implication	function type	exponential
disjunction	sum type	coproduct
false	empty type	initial object

CCCs

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\mathsf{NJ}} B$ between the corresponding formulae;
- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\mathsf{NJ}} A \wedge B \qquad \text{iff} \qquad C \vdash_{\mathsf{NJ}} A \,\&\, C \vdash_{\mathsf{NJ}} B;$$

- Implication is modelled by exponentials, by using the adjunction

$$A \wedge B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} B \to C;$$

- Top is modelled by terminal object 1.

CCCs

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\mathsf{NJ}} B$ between the corresponding formulae;
- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\mathsf{NJ}} A \wedge B \qquad \text{iff} \qquad C \vdash_{\mathsf{NJ}} A \,\&\, C \vdash_{\mathsf{NJ}} B;$$

- Implication is modelled by exponentials, by using the adjunction

$$A \wedge B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} B \to C;$$

- Top is modelled by terminal object 1.

CCCs

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\mathsf{NJ}} B$ between the corresponding formulae;
- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\mathsf{NJ}} A \wedge B \qquad \text{iff} \qquad C \vdash_{\mathsf{NJ}} A \,\&\, C \vdash_{\mathsf{NJ}} B;$$

- Implication is modelled by exponentials, by using the adjunction

$$A \wedge B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} B \to C;$$

- Top is modelled by terminal object 1.

CCCs

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\mathsf{NJ}} B$ between the corresponding formulae;
- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\mathsf{NJ}} A \wedge B \qquad \text{iff} \qquad C \vdash_{\mathsf{NJ}} A \,\&\, C \vdash_{\mathsf{NJ}} B;$$

- Implication is modelled by exponentials, by using the adjunction

$$A \wedge B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} B \to C;$$

- Top is modelled by terminal object 1.

CCCs

Cartesian closed categories (CCCs) provide the semantics *and* proof theory of disjunction-free propositional intuitionistic logic:

- Objects correspond to formulae, and an arrow from A to B corresponds to a deduction $A \vdash_{\mathsf{NJ}} B$ between the corresponding formulae;
- Conjunction is modelled by products, by using the adjunction

$$C \vdash_{\mathsf{NJ}} A \wedge B \qquad \text{iff} \qquad C \vdash_{\mathsf{NJ}} A \,\&\, C \vdash_{\mathsf{NJ}} B;$$

- Implication is modelled by exponentials, by using the adjunction

$$A \wedge B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} B \to C;$$

- Top is modelled by terminal object 1.

Bi-CCCs

If a CCC has finite coproducts it is called bi-cartesian closed. Bi-CCCs provide the semantics *and* proof theory of propositional intuitionistic logic:

- Disjunction is modelled by coproducts, by using the adjunction

$$A \vee B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} C \,\&\, B \vdash_{\mathsf{NJ}} C;$$

- Bottom is modelled by initial object 0.

It follows that Bi-CCCs give categorical models for simple type theory with sum types (objects are types and arrows are typed terms).

Bi-CCCs

If a CCC has finite coproducts it is called bi-cartesian closed. Bi-CCCs provide the semantics *and* proof theory of propositional intuitionistic logic:

- Disjunction is modelled by coproducts, by using the adjunction

$$A \vee B \vdash_{\mathsf{NJ}} C \qquad \text{iff} \qquad A \vdash_{\mathsf{NJ}} C \,\&\, B \vdash_{\mathsf{NJ}} C;$$

- Bottom is modelled by initial object 0.

It follows that Bi-CCCs give categorical models for simple type theory with sum types (objects are types and arrows are typed terms).

Monoidal Endofunctors

Given a CCC $\mathscr{C}$, an endofunctor $F : \mathscr{C} \to \mathscr{C}$ is *monoidal* when

- there exists a natural transformation

$$m_{A,B} : FA \times FB \to F(A \times B);$$

- there exists a morphism

$$m_1 : 1 \to F1,$$

preserving the monoidal structure of $\mathscr{C}$.
These are called structure morphisms of F.

Pointed Endofunctors

Given *any* category $\mathscr{C}$, an endofunctor $F : \mathscr{C} \to \mathscr{C}$ is *pointed* iff there exists a natural transformation

$$\pi : Id_{\mathscr{C}} \Rightarrow F$$
$$\pi_A : A \to FA$$

$$\begin{tikzcd} A \arrow[r, "\pi_A"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "\pi_B"'] & FB \end{tikzcd}$$

(the pointer of F).

IEL$^-$-categories

An IEL$^-$-category is a bi-CCC $\mathscr{C}$, equipped with a pointed monoidal endofunctor $\Box : \mathscr{C} \to \mathscr{C}$.
We use m for the structure morphism of $\Box$, and k for the pointer of $\Box$.

Soundness

Theorem

Let $\mathscr{C}$ be an IEL^-*-category. Then there is a canonical interpretation* $[\![]\!]$ *of* IEL^- *in* $\mathscr{C}$ *such that*

- *A formula A is mapped to an object $[\![A]\!]$ of $\mathscr{C}$;*
- *A deduction $t: A_1, \cdots, A_n \vdash B$ is mapped to an arrow $[\![t]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!]$;*
- *For any two deductions t and s which are equal modulo rewritings, we have $[\![t]\!] = [\![s]\!]$.*

Soundness

Proof.

By structural induction on $t:\vec{A}\vdash B$.
The intuitionistic cases are given by the previous remarks concerning bi-CCCs.

The deduction $\dfrac{f:\Gamma\vdash A}{\Box A}\,\Box{-}intro$ is mapped to $k_{[\![A]\!]}\circ[\![f]\!]$.
The deduction

$$\frac{f_1:\Gamma_1\vdash\Box A_1 \qquad \cdots \qquad f_n:\Gamma_n\vdash\Box A_n \qquad g:[A_1,\cdots,A_n]\vdash B}{\Box B}\,\Box{-}elim$$

is mapped to $(\Box[\![g]\!])\circ m_{[\![A_1]\!],\ldots,[\![A_n]\!]}\circ[\![f_1]\!]\times\cdots\times[\![f_n]\!]$, where $m_{A_1,\ldots,A_n}$ is defined inductively as $m_{A_1,\ldots,A_{n-1},A_n}{:=}m_{A_1\times\cdots\times A_{n-1},A_n}\circ(m_{A_1,\ldots,A_m})\times id_{\Box A_m}$.

The desired equalities hold by functoriality of $\Box$ and naturality of m and k. $\Box$

Soundness

Proof.

By structural induction on $t : \vec{A} \vdash B$.
The intuitionistic cases are given by the previous remarks concerning bi-CCCs.

The deduction $\dfrac{f : \Gamma \vdash A}{\Box A}\,{\scriptstyle \Box-intro}$ is mapped to $k_{[\![A]\!]} \circ [\![f]\!]$.

The deduction

$$\dfrac{f_1 : \Gamma_1 \vdash \Box A_1 \qquad \cdots \qquad f_n : \Gamma_n \vdash \Box A_n \qquad g : [A_1, \cdots, A_n] \vdash B}{\Box B}\,{\scriptstyle \Box-elim}$$

is mapped to $(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\ldots,[\![A_n]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!]$, where $m_{A_1,\ldots,A_n}$ is defined inductively as $m_{A_1,\ldots,A_{n-1},A_n} := m_{A_1 \times \cdots \times A_{n-1},A_n} \circ (m_{A_1,\ldots,A_m}) \times id_{\Box A_m}$.

The desired equalities hold by functoriality of $\Box$ and naturality of m and k. $\Box$

Soundness

Proof.

By structural induction on $t : \vec{A} \vdash B$.
The intuitionistic cases are given by the previous remarks concerning bi-CCCs.

The deduction $\dfrac{f : \Gamma \vdash A}{\Box A}\ \Box\text{-}intro$ is mapped to $k_{[\![A]\!]} \circ [\![f]\!]$.

The deduction

$$\dfrac{f_1 : \Gamma_1 \vdash \Box A_1 \qquad \cdots \qquad f_n : \Gamma_n \vdash \Box A_n \qquad g : [A_1, \cdots, A_n] \vdash B}{\Box B}\ \Box\text{-}elim$$

is mapped to $(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!]$, where $m_{A_1,\dots,A_n}$ is defined inductively as $m_{A_1,\dots,A_{n-1},A_n} := m_{A_1 \times \cdots \times A_{n-1}, A_n} \circ (m_{A_1,\dots,A_m}) \times id_{\Box A_m}$.

The desired equalities hold by functoriality of $\Box$ and naturality of m and k. $\Box$

Soundness

Proof.

By structural induction on $t : \vec{A} \vdash B$.
The intuitionistic cases are given by the previous remarks concerning bi-CCCs.

The deduction $\dfrac{f : \Gamma \vdash A}{\Box A}\ \Box{-}intro$ is mapped to $k_{[\![A]\!]} \circ [\![f]\!]$.
The deduction

$$\dfrac{f_1 : \Gamma_1 \vdash \Box A_1 \qquad \cdots \qquad f_n : \Gamma_n \vdash \Box A_n \qquad g : [A_1, \cdots, A_n] \vdash B}{\Box B}\ \Box{-}elim$$

is mapped to $(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\dots,[\![A_n]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!]$, where $m_{A_1,\dots,A_n}$ is defined inductively as $m_{A_1,\dots,A_{n-1},A_n} := m_{A_1 \times \cdots \times A_{n-1},A_n} \circ (m_{A_1,\dots,A_m}) \times id_{\Box A_m}$.

The desired equalities hold by functoriality of $\Box$ and naturality of m and k. $\square$

Completeness

Theorem

If the interpretation of two IEL^-*-deductions is equal in all* IEL^-*-categories, then the two deductions are equal modulo rewritings in* IEL^-.

Proof.

By constructing a term model. Consider the following category $\mathscr{M}$:

- its objects are formulae;
- an arrow $f : A \to B$ is an IEL^--deduction $A \vdash B$;
- identities are given by assuming a formula;
- composition is given by transitivity of deductions.

Then $\mathscr{M}$ has a bi-cartesian closed structure given by the properties of conjunction, implication, and disjunction in NJ.

Completeness

Theorem

If the interpretation of two IEL^-*-deductions is equal in all* IEL^-*-categories, then the two deductions are equal modulo rewritings in* IEL^-*.*

Proof.

By constructing a term model. Consider the following category $\mathscr{M}$:

- its objects are formulae;
- an arrow $f : A \to B$ is an IEL^--deduction $A \vdash B$;
- identities are given by assuming a formula;
- composition is given by transitivity of deductions.

Then $\mathscr{M}$ has a bi-cartesian closed structure given by the properties of conjunction, implication, and disjunction in NJ.

Completeness

Proof.

The modal operator $\Box$ induces a functor by mapping A to $\Box A$, and

$$
\begin{array}{c} A_1, \cdots, A_n \\ \vdots \\ B \end{array}
\quad \longmapsto \quad
\dfrac{\Box(A_1 \wedge \cdots \wedge A_n) \qquad \begin{array}{c} [A_1 \wedge \cdots \wedge A_n] \\ \vdots \\ B \end{array}}{\Box B}\ \Box\text{-}elim
$$

which preserves identities by the first rewriting rule, and preserves composition by the third rewriting rule.

The structure morphism is given by

$$
\dfrac{\dfrac{\Box A \wedge \Box B}{\Box A} \qquad \dfrac{\Box A \wedge \Box B}{\Box B} \qquad \dfrac{[A] \qquad [B]}{A \wedge B}}{\Box(A \wedge B)}
$$

whose properties follow from the third rewriting rule.

Completeness

Proof.

Finally, the pointer is given by $\dfrac{A}{\Box A}$
whose naturality follows from the second rewriting rule.
Thus, $\mathscr{M}$ is an IEL^--category. Moreover it proves the statement of the theorem:

- Assume an equality between IEL^--deductions holds in all IEL^--categories. Then it holds in $\mathscr{M}$. Therefore the two deductions are equal modulo rewritings.

$\square$

Rewritings Return

Recall the second rewriting on IEL$^-$-deductions:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{\Box A}\,\Box\text{–}intro \qquad \begin{array}{c}[A]\\ \vdots\\ B\end{array}}{\Box B}\,\Box\text{–}elim \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\\ \vdots\\ B\end{array}}{\Box B}\,\Box\text{–}intro$$

From a proof-theoretic perspective it lacks generality!

Rewritings Return

The following one seems to behave better

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1\\ \vdots\\ A_1\end{matrix}}{\Box A_1}\,\Box\text{-}intro \qquad \cdots \qquad \dfrac{\begin{matrix}\Gamma_n\\ \vdots\\ A_n\end{matrix}}{\Box A_n}\,\Box\text{-}intro \qquad \begin{matrix}[A_1,\cdots,A_n]\\ \vdots\\ B\end{matrix}}{\Box B}\,\Box\text{-}elim$$

$\rightsquigarrow$

$$\dfrac{\begin{matrix}\Gamma_1 & \cdots & \Gamma_n\\ \vdots & & \vdots\\ A_1 & \cdots & A_n\\ & \vdots & \\ & B & \end{matrix}}{\Box B}\,\Box\text{-}intro$$

But now the algebraic structure of deductions has changed!

Rewritings Return

The following one seems to behave better

$$
\dfrac{\dfrac{\begin{matrix}\Gamma_1\\ \vdots\\ A_1\end{matrix}}{\Box A_1}\,\Box\text{-}intro \quad \cdots \quad \dfrac{\begin{matrix}\Gamma_n\\ \vdots\\ A_n\end{matrix}}{\Box A_n}\,\Box\text{-}intro \qquad \begin{matrix}[A_1,\cdots,A_n]\\ \vdots\\ B\end{matrix}}{\Box B}\,\Box\text{-}elim
$$

$$
\rightsquigarrow \qquad \dfrac{\begin{matrix}\Gamma_1 & \ldots & \Gamma_n\\ \vdots & & \vdots\\ A_1 & \cdots & A_n\\ & \vdots & \\ & B & \end{matrix}}{\Box B}\,\Box\text{-}intro
$$

But now the algebraic structure of deductions has changed!

Monoidal Natural Transformations

Given a CCC $\mathscr{C}$, and monoidal endofunctors $F, G : \mathscr{C} \to \mathscr{C}$, a natural transformation $\alpha : F \Rightarrow G$ is *monoidal* when the following commute

$$\begin{tikzcd} FA \times FB \arrow[r, "m^F_{A,B}"] \arrow[d, "\alpha_A \times \alpha_B"'] & F(A \times B) \arrow[d, "\alpha_{A \times B}"] \\ GA \times GB \arrow[r, "m^G_{A,B}"'] & G(A \times B) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, "m^F_1"] \arrow[d, equal] & F1 \arrow[d, "\alpha_1"] \\ 1 \arrow[r, "m^G_1"'] & G(1) \end{tikzcd}$$

Adequacy

Theorem

IEL$^-$ *is sound and complete for the class of bi-cartesian closed categories with monoidal pointed endofunctor whose pointer is monoidal.*

Proof.

We proceed as before by constructing the canonical interpretation and the term model.

To prove soundness, we note that the general rewriting follows from the monoidal condition on the pointer k:

$$\begin{tikzcd}
A \times B \arrow[r, equal] \arrow[d, "k_A \times k_B"'] & A \times B \arrow[d, "k_{A \times B}"] \\
\Box A \times \Box B \arrow[r, "m_{A,B}"'] & \Box(A \times B)
\end{tikzcd}$$

To prove completeness, we note that the monoidal condition on k is just a special case of that rewriting up to $\wedge$-detours.

Adequacy

Theorem

IEL$^-$ *is sound and complete for the class of bi-cartesian closed categories with monoidal pointed endofunctor whose pointer is monoidal.*

Proof.

We proceed as before by constructing the canonical interpretation and the term model.

To prove soundness, we note that the general rewriting follows from the monoidal condition on the pointer k:

$$\begin{tikzcd} A \times B \arrow[r, equal] \arrow[d, "k_A \times k_B"'] & A \times B \arrow[d, "k_{A \times B}"] \\ \Box A \times \Box B \arrow[r, "m_{A,B}"'] & \Box(A \times B) \end{tikzcd}$$

To prove completeness, we note that the monoidal condition on k is just a special case of that rewriting up to $\wedge$-detours. □

Adequacy

Theorem

IEL^- *is sound and complete for the class of bi-cartesian closed categories with monoidal pointed endofunctor whose pointer is monoidal.*

Proof.

We proceed as before by constructing the canonical interpretation and the term model.

To prove soundness, we note that the general rewriting follows from the monoidal condition on the pointer k:

$$\begin{tikzcd} A \times B \arrow[r, equal] \arrow[d, "k_A \times k_B"'] & A \times B \arrow[d, "k_{A \times B}"] \\ \Box A \times \Box B \arrow[r, "m_{A,B}"'] & \Box(A \times B) \end{tikzcd}$$

To prove completeness, we note that the monoidal condition on k is just a special case of that rewriting up to $\wedge$-detours. $\Box$

Adequacy

Theorem

IEL$^-$ *is sound and complete for the class of bi-cartesian closed categories with monoidal pointed endofunctor whose pointer is monoidal.*

Proof.

We proceed as before by constructing the canonical interpretation and the term model.

To prove soundness, we note that the general rewriting follows from the monoidal condition on the pointer k:

$$\begin{tikzcd} A \times B \arrow[r, equal] \arrow[d, "k_A \times k_B"'] & A \times B \arrow[d, "k_{A\times B}"] \\ \Box A \times \Box B \arrow[r, "m_{A,B}"'] & \Box(A \times B) \end{tikzcd}$$

To prove completeness, we note that the monoidal condition on k is just a special case of that rewriting up to $\wedge$-detours. $\square$

Intuitionistic Epistemic Natural Deduction

One could define an equivalent natural deduction calculus for $\mathbb{IEL}^-$ characterised by a single rule:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\\ \Box A_1\end{array} \quad \cdots \quad \begin{array}{c}\Gamma_n\\ \vdots\\ \Box A_n\end{array} \quad \begin{array}{c}[A_1, \cdots, A_n]^1, \Delta\\ \vdots\\ B\end{array}}{\Box B}\ \Box\text{-}intro{:}1$$

and proceed as before with producing an equational theory for rewritings which can be modelled in a suitable categorical context.

Intuitionistic Epistemic Natural Deduction

Even if this calculus clearly lacks of elegance, a certain symmetry is restored when considering a natural deduction system for $\mathbb{IEL} := \mathbb{IEL}^- + \Box A \to \neg\neg A$:

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots \\ \Box A_1\end{matrix} \qquad \cdots \qquad \begin{matrix}\Gamma_n \\ \vdots \\ \Box A_n\end{matrix} \qquad \begin{matrix}[A_1, \cdots, A_n]^1, \Delta \\ \vdots \\ B\end{matrix}}{\Box B}\ \Box\text{−}intro{:}1 \qquad\qquad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ \Box A\end{matrix}}{\neg\neg A}\ \Box\text{−}elim$$

However, there are some ...

(Open) Problems

- It is not clear which kind of rewritings can be safely imposed on proof-terms;
- As a consequence, the categorical model based on monoidal pointed endofunctors seems inadequate for the "single rule" calculus for intuitionistic belief;
- A fortiori, it is unclear which is the structure of the full calculus for intuitionistic knowledge;

But there is also ...

The Big One

Normalization is still a desideratum!

- Translating IEL^{-}-proof-terms into simply typed λ-calculus seems (to me) a promising strategy (and I'm working on it);
- At the same time, categorical intuition may be useful for extending Tait's method by defining appropriate reducibility candidates;
- Suggestions?

Thank you for listening!