# An Introduction to Normal Modal Logics
## End Session

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Logic Group Working Seminar

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# Past Seminars

1. **Normal Modal Logics**
   Axiomatic Calculi and Canonical Model Theorems

   ⇝ S. Popkorn, *First Steps in Modal Logic*, CUP 1994

2. **Gödel-Löb Logic**
   Modal Completeness and Solovay Theorem

   ⇝ G. Boolos, *The Logic of Provability*, CUP 1995

3. **Structural Proof Theory**
   Cut Elimination and Some Corollaries

   ⇝ S. Negri, J. von Plato, *Structural Proof Theory*, CUP 2001

   ⇝ A. Troelstra, H. Swichtenberg, *Basic Proof Theory* 2nd ed., CUP 2000

4. **Labelled Sequent Calculi**
   Cut Elimination "for" Kripke Frames with First Order Properties

   ⇝ S. Negri, J. von Plato, *Proof Analysis*, CUP 2011

   ⇝ S. Negri, R. Dyckhoff, *Geometrization of First Order Logic*, The Bulletin of Symbolic Logic, 21(2) 2015

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# Today Seminar

## Labels, Trees, Hypersequents

Tree Hypersequents and Labelled Calculi

- $\rightsquigarrow$ S. Negri, J. von Plato, *Proof Analysis*, CUP 2011
- $\rightsquigarrow$ F. Poggiolesi, *Gentzen Calculi for Modal Propositional Logic*, Springer 2011
- $\rightsquigarrow$ R. Goré, R. Ramanayake, *Labelled tree sequents, Tree hypersequents and Nested (Deep) Sequents*, in Advances in Modal Logic (AiML 2012), College Publications 2012

# Outline

# Löb Rule

**Remark**

For any $\mathcal{M} \in ITF$ $x \vDash_{\mathcal{M}} \Box A$ iff for any $y$ s.t. $Rxy$, $y \vDash_{\mathcal{M}} \Box A \to A$.

*Proof.*

If $x \vDash_{\mathcal{M}} \Box A$, then, for any $y$ such that $Rxy$, $y \vDash_{\mathcal{M}} A$, therefore $y \vDash_{\mathcal{M}} \Box A \to A$. Conversely, if for any $y$ such that $Rxy$ $y \vDash_{\mathcal{M}} \Box A \to A$, then assuming $x \nvDash_{\mathcal{M}} \Box A$, there exists a $y$ such that $Rxy$ and $y \nvDash_{\mathcal{M}} A$, so that $y \nvDash_{\mathcal{M}} \Box A$, therefore there is a $z$ such that $Ryz$ and $z \nvDash_{\mathcal{M}} A$, $\ldots$, contra $\mathcal{M} \in ITF$. $\Box$

# Löb Rule

## Remark

For any $\mathcal{M} \in ITF$ $x \vDash_{\mathcal{M}} \Box A$ iff for any $y$ s.t. $Rxy$, $y \vDash_{\mathcal{M}} \Box A \to A$.

## Proof.

If $x \vDash_{\mathcal{M}} \Box A$, then, for any $y$ such that $Rxy$, $y \vDash_{\mathcal{M}} A$, therefore $y \vDash_{\mathcal{M}} \Box A \to A$. Conversely, if for any $y$ such that $Rxy$ $y \vDash_{\mathcal{M}} \Box A \to A$, then assuming $x \nvDash_{\mathcal{M}} \Box A$, there exists a $y$ such that $Rxy$ and $y \nvDash_{\mathcal{M}} A$, so that $y \nvDash_{\mathcal{M}} \Box A$, therefore there is a $z$ such that $Ryz$ and $z \nvDash_{\mathcal{M}} A$, $\ldots$, contra $\mathcal{M} \in ITF$.

$\Box$

# G3GL

Let G3GL denote the following modification/extension of G3K:

**Initial sequents:** $+ \quad x : \Box A, \Gamma \Rightarrow \Delta, x : \Box A$

**Rules:** $+ \quad \dfrac{}{Rxx, \Gamma \Rightarrow \Delta}$ *Irref* $\quad \& \quad \dfrac{Rxy, Ryz, Rxz, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \Gamma \Rightarrow \Delta}$ *Trans*

$$\frac{Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ \Box_L{-}K$$

$$\frac{x : \Box A, Rxy, \Gamma \Rightarrow \Delta, y : \Box A \qquad y : A, x : \Box A, Rxy, \Gamma \Rightarrow \Delta}{x : \Box A, Rxy, \Gamma \Rightarrow \Delta}\ \Box_L{-}A$$

$\Box_L - A$ and $\Box_L - K$ substitute the standard rules for $\Box$.
Moreover, in $\Box_L - K$, we require $y$ to be a fresh variable

# Structural Properties

## Lemma

For G3GL the following hold:

(i) Weakening and substitution rules are hp admissible;

(ii) All rules are invertible;

(iii) Let the range of $x$ in a derivation $\delta$ be the finite set of worlds $y$ such that $Rxy$ or the relational terms $Rxx_1, x_1Rx_2, \cdots, x_nRy$ appear in the antecedents of sequents in $\delta$; let also ranges be ordered by set inclusion. Then contraction rules are range-preserving admissible.

## Proof.

To (i): By induction on the height of the derivation.

To (ii): The relevant case is $\Gamma \overset{0}{\Rightarrow} \Delta, x : \Box A$. If $x : \Box A$ is not principal, then $Rxy, y : \Box A; \Gamma \overset{0}{\Rightarrow} \Delta, y : A$ or we have *Irref*. But if $x : \Box A$ is principal, then $\Gamma \equiv x : \Box A, \Gamma'$ and we have $Rxy, y : \Box A, x : \Box A, \Gamma' \overset{m+1}{\Rightarrow} \Delta, y : A$ by $\Box_L{-}A$ with $Rxy, y : \Box A, x : \Box A, \Gamma' \overset{0}{\Rightarrow} \Delta, y : A, y : \Box A$ and $y : A, Rxy, y : \Box A, x : \Box A, \Gamma' \overset{m}{\Rightarrow} \Delta, y : A$.

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# Structural Properties

## Proof.

To (iii): By simultaneous induction for left and right contraction, with primary induction on the contraction formula, and secondary induction on the height of the derivation.

Relevant case: Let $x : \Box A$ to be contracted on the right. If it is not principal, use SIH and then the rule. Otherwise, the premise must be $Rxy, y : \Box A, \Gamma \Rightarrow \Delta, x : \Box A, y : A$, so we can invert $\Box_L - K$ obtaining $Rxy, y : \Box A, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A, y : A$ and by IH we have the result.

$\Box$

# Cut Elimination for G3GL

## Cut Elimination Theorem

The cut rule is admissible in G3GL.

## Proof.

By induction on the weight of the cut, given by the cut formula complexity, the range of the cut label in the derivation and the cut height.
First note that if a loop occurs in the the conclusion of a *Cut*, then we can obtain that conclusion by *Irref* and *Trans*. Otherwise, there cannot be any loop in the premisses, and by the restrictions on variables we cannot introduce a loop by $\Box_L - K$.
Then if $y \in \text{rg}(x)$, then $\text{rg}(y) \subset \text{rg}(x)$.
Now, if the cut formula is not principal in the premisses of *Cut*, we can lift the cut. Otherwise, if $x : \Box A$ is principal in both premisses, then it must be introduced by $\Box_L$ rules. Hence we can

1. eliminate the cut of $x : \Box A$ in the conclusion of $\Box_L - K$ and in the first premiss of $\Box_L - A$ by TIH, obtaining the sequent $Rxz, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : \Box A$;
2. eliminate the cut of $x : \Box A$ in the conclusion of $\Box_L - K$ and in the second premiss of $\Box_L - A$ by TIH, obtaining the sequent $Rxz, z : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$;
3. substitute the eigenvariable in the premiss of $\Box_L - K$ to match that one of 1. and cut $z : \Box A$, obtaining by SIH $Rxz, Rxz, \Gamma, \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta, z : A$ without cut;
4. eliminate the cut of $z : A$ between 3. and 2. by IH, obtaining by contraction $Rxz, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$, as desired.

$\Box$

# Cut Elimination for G3GL

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4. eliminate the cut of $z : A$ between 3. and 2. by IH, obtaining by contraction $Rxz, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$, as desired.

$\Box$

# Tree Hypersequents

Recall that a sequent is a pair $X \Rightarrow Y$ where $X$ and $Y$ are finite *multisets* of formulas.

A tree hypersequent (THS) is defined as follows:

- Every sequent is a THS
- if $\mathcal{S}$ is a sequent and $G_1, \cdots, G_n$ are THS, then $\mathcal{S}/G_1; \cdots; G_n$ is a THS.

The intended interpretation I of a THS is

$$(X \Rightarrow Y)^{\mathsf{I}} := \bigwedge X \to \bigvee Y$$

$$(\mathcal{S}/G_1; \cdots; G_n)^{\mathsf{I}} := \mathcal{S}^{\mathsf{I}} \vee \Box G_1^{\mathsf{I}} \vee \cdots \vee \Box G_n^{\mathsf{I}}.$$

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# Tree Hypersequents

Two sequents $\mathcal{S}_1$, $\mathcal{S}_2$ have *equivalent position* in $G$ and $H$ (and we write $G\{\mathcal{S}_1\} \sim H\{\mathcal{S}_2\}$) if

- $G \equiv \mathcal{S}_1$ and $H \equiv \mathcal{S}_2$, or
- $G \equiv \mathcal{S}_1/\underline{X}_1$ and $H \equiv \mathcal{S}_2/\underline{X}_2$, or
- $G \equiv \mathcal{T}_1/(H_1\{\mathcal{S}_1\}; \underline{X}_1) \sim \mathcal{T}_2/(H_2\{\mathcal{S}_2\}; \underline{X}_2) \equiv H$ where $H_1\{\mathcal{S}_1\} \sim H_2\{\mathcal{S}_2\}$, $\mathcal{T}_1, \mathcal{T}_2$ are sequents and $\underline{X}_1, \underline{X}_2$ are sequences of THS.

# Cut Rule

Let $X \Rightarrow Y \otimes U \Rightarrow V := X, U \Rightarrow Y, V$.
For $H\{\mathcal{S}\} \sim H'\{\mathcal{S}'\}$ define $H\{\mathcal{S}\} \star H'\{\mathcal{S}'\}$ as follows:

- $\mathcal{S} \star \mathcal{S}' := \mathcal{S} \otimes \mathcal{S}'$
- $(\mathcal{S}/\underline{X}) \star (\mathcal{S}'/\underline{Y}) := \mathcal{S} \otimes \mathcal{S}'/\underline{X}; \underline{Y}$
- $(\mathcal{T}/H\{\mathcal{S}\}; \underline{X}) \star (\mathcal{T}'/H'\{\mathcal{S}'\}; \underline{Y}) := \mathcal{T} \otimes \mathcal{T}'/H\{\mathcal{S}\} \star H'\{\mathcal{S}'\}; \underline{X}; \underline{Y}$ where $\mathcal{T}, \mathcal{T}'$ are sequents.

## The Rule

For $G\{X \Rightarrow Y, A\}$ and $G'\{A, U \Rightarrow V\}$
s.t. $G\{X \Rightarrow Y, A\} \sim G'\{A, U \Rightarrow V\}$ define

$$\frac{G\{X \Rightarrow Y, A\} \qquad G'\{A, U \Rightarrow V\}}{G\{X \Rightarrow Y\} \star G'\{U \Rightarrow V\}}\, Cut$$

Note that $Cut$ merges trees, so that its conclusion is a THS indeed

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# Labelled Tree Sequents

Let $\mathcal{R}$ denote a set of relational terms.
We say that $\mathcal{R}$ is a tree when the frame it defines is a tree (possibly the empty one).
A labelled tree sequent (LTS) is a labelled sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ where

- $\mathcal{R}$ is a tree, and
- if $\mathcal{R} = \varnothing$, then all formulas in the sequent have the same label;
- otherwise, each label occurring in $\Gamma$ or $\Delta$ occurs in $\mathcal{R}$ also.

# Cut Rule

Note first that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are tree, $\mathcal{R}_1 \cup \mathcal{R}_2$ might not be so.

## The Rule

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \qquad \mathcal{R}, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \, Cut$$

# Translation

## $\mathbb{TL}$

For a label $x$ define $\mathbb{TL}_x$:THS→LTS as follows:

- $\mathbb{TL}_x(X \Rightarrow Y) := x : X \Rightarrow x : Y$
- $\mathbb{TL}_x(X \Rightarrow Y/G_1; \cdots ; G_n) :=$ $(\bigotimes_{j=1}^{n} \mathbb{TL}_{x_j}(G_j)) \otimes (Rxx_1, Rxx_2, \cdots , Rxx_n, x : X \Rightarrow x : Y)$,

where $(\mathcal{R}_1, X \Rightarrow Y) \otimes (\mathcal{R}_2, U \Rightarrow V) := \mathcal{R}_1 \cup \mathcal{R}_2, X, U \Rightarrow Y, V$.

## $\mathbb{LT}$

For any LTS $\mathcal{R}, \Gamma \Rightarrow \Delta$, define $\mathbb{LT}$ :LTS→THS as follows:

- if $\mathcal{R} = \varnothing$, then $\mathbb{LT}(x : \Gamma \Rightarrow x : \Delta) := \Gamma \Rightarrow \Delta$
- otherwise, if $x$ is the root of $\mathcal{R} \equiv \{Rxy_1, \cdots , Rxy_n\}$, let $v_i := \{z : Ry_iz\}$; then $\mathbb{LT}(\mathcal{R}, \Gamma \Rightarrow \Delta) :=$

$$\Gamma_x \Rightarrow \Delta_x/\mathbb{LT}(\mathcal{R}_{v_1}, \Gamma_{v_1} \Rightarrow \Delta_{v_1}); \cdots ; \mathbb{LT}(\mathcal{R}_{v_n}, \Gamma_{v_n} \Rightarrow \Delta_{v_n}).$$

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  $\upsilon_i := \{z : Ry_i z\}$; then $\mathbb{LT}(\mathcal{R}, \Gamma \Rightarrow \Delta) :=$

$$\Gamma_x \Rightarrow \Delta_x / \mathbb{LT}(\mathcal{R}_{\upsilon_1}, \Gamma_{\upsilon_1} \Rightarrow \Delta_{\upsilon_1}); \cdots ; \mathbb{LT}(\mathcal{R}_{\upsilon_n}, \Gamma_{\upsilon_n} \Rightarrow \Delta_{\upsilon_n}).$$

# Substitution

Let $Lbl$ be set of labels, and $Var(\mathcal{S}) \subset Lbl$ the finite set of labels occurring in the labelled sequent $\mathcal{S}$. A renaming of $\mathcal{S}$ consists of a one-to-one function $f_{\mathcal{S}} : Var(\mathcal{S}) \rightarrowtail Lbl$.
For $\mathcal{S}'$ and a renaming $f_{\mathcal{S}}$, let $\mathcal{S}'_{f_{\mathcal{S}}}$ the labelled sequent obtained by substituting in $\mathcal{S}'$ $x$ with $f_{\mathcal{S}}(x)$ for any $x \in Dom(f_{\mathcal{S}}) \cap Var(\mathcal{S}')$.

Note that if $\mathcal{S}'$ is a LTS, $\mathcal{S}'_{f_{\mathcal{S}}}$ needs not to be a LTS, but it is so for $\mathcal{S}$ LTS.

**Lemma**

Let $G$ be a THS and $\mathcal{S}$ a LTS. Then $\mathbb{LT}(\mathbb{TL}_x(G)) = G$, and $\mathbb{TL}_x(\mathbb{LT}(\mathcal{S})) = \mathcal{S}_{f_{\mathcal{S}}}$ for some $f_{\mathcal{S}}$.

**Proof.**

By definition of LTS, we cannot loose variables applying $\mathbb{LT}$ to $\mathcal{S}$; however a renaming might be necessary, to obtain the desired equality, since $\mathbb{TL}$ does assign variables.
The equality $\mathbb{LT}(\mathbb{TL}_x(G)) = G$ holds by definitions of the translations.

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The equality $\mathbb{LT}(\mathbb{TL}_x(G)) = G$ holds by definitions of the translations.

# Substitution Lemma

Let C be a LTS calculus, and $\mathcal{S}$ a LTS. Let $f_{\mathcal{S}}$ be a renaming of $\mathcal{S}$. If $\mathcal{S}$ is derivable in C with height $n$, then so is $\mathcal{S}_{f_{\mathcal{S}}}$.

*Proof.*

Let $\delta$ be the derivation of $\mathcal{S}$ with height $n$. We proceed by induction on $n$:

- if $n = 0$, then $\mathcal{S}$ is initial, so that $\mathcal{S}_{f_{\mathcal{S}}}$ is initial too.
- if $n > 0$, and $\rho$ is the last rule of $\delta$ (from $\mathcal{S}_1, \cdots, \mathcal{S}_n$ to $\mathcal{S}$) then $(\mathcal{S}_i)_{f_{\mathcal{S}}}$ might not be a LTS, since $\bigcup_i Var(\mathcal{S}_i)$ may contain labels that do not occur in $\mathcal{S}$. Hence define $g : \bigcup_i Var(\mathcal{S}_i) \setminus Dom(f_{\mathcal{S}}) \rightarrowtail Lbl \setminus Im(f_{\mathcal{S}})$. Then the composition $f_{\mathcal{S}} \circ g$ is a renaming of $\mathcal{S}_i$'s. By applying the IH to the $\mathcal{S}_i$'s and then applying $\rho$, we have a derivation of $((\mathcal{S})_g)_{f_{\mathcal{S}}} = \mathcal{S}_{f_{\mathcal{S}}}$.

$\square$

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Let $\delta$ be the derivation of $\mathcal{S}$ with height $n$. We proceed by induction on $n$:

- if $n = 0$, then $\mathcal{S}$ is initial, so that $\mathcal{S}_{f_{\mathcal{S}}}$ is initial too.
- if $n > 0$, and $\rho$ is the last rule of $\delta$ (from $\mathcal{S}_1, \cdots, \mathcal{S}_n$ to $\mathcal{S}$) then $(\mathcal{S}_i)_{f_{\mathcal{S}}}$ might not be a LTS, since $\bigcup_i Var(\mathcal{S}_i)$ may contain labels that do not occur in $\mathcal{S}$. Hence define $g : \bigcup_i Var(\mathcal{S}_i) \setminus Dom(f_{\mathcal{S}}) \rightarrowtail Lbl \setminus Im(f_{\mathcal{S}})$. Then the composition $f_{\mathcal{S}} \circ g$ is a renaming of $\mathcal{S}_i$'s. By applying the IH to the $\mathcal{S}_i$'s and then applying $\rho$, we have a derivation of $((\mathcal{S})_g)_{f_{\mathcal{S}}} = \mathcal{S}_{f_{\mathcal{S}}}$.

□

# Induced Rules

$$\frac{\mathcal{R}, \overbrace{Rxy, y : \Box A}^{\text{principal}}, \Gamma \Rightarrow \Delta, \overbrace{y : A}^{\text{principal}}}{\mathcal{R}, \Gamma \Rightarrow \Delta, \underbrace{x : \Box A}_{\text{principal}}} \quad \overset{\longmapsto}{\mathtt{LT}} \quad \frac{G\{X \Rightarrow Y / \Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}}$$

$$\frac{G\{\overbrace{X \Rightarrow Y}^{x} / \overbrace{\Box A \Rightarrow A}^{y}\}}{G\{\underbrace{X \Rightarrow Y, \Box A}_{x} / \underbrace{\emptyset}_{y}\}} \quad \overset{\longmapsto}{\mathtt{TL}} \quad \frac{\mathcal{R}, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}$$

# Induced Calculus

## Translation Lemma

Let C be a THS calculus. Then the following hold:

(i) $\mathsf{C} \overset{n}{\vdash} G$ iff $\mathbb{TL}\mathsf{C} \overset{n}{\vdash} \mathbb{TL}G$

(ii) $\mathbb{TL}\mathsf{C} \overset{n}{\vdash} \mathcal{S}$ iff $\mathsf{C} \overset{n}{\vdash} \mathbb{LT}\mathcal{S}$.

## Proof.

To (i): Let $\delta$ be a derivation of $G$ in C with height $n$. Then by substituting each THS $H$ in $\delta$ with $\mathbb{TL}H$ and any rule $\rho$ in $\delta$ with $\mathbb{TL}(\rho)$ we obtain a derivation of $\mathbb{TL}G$ whose height is $n$ by construction.

To (ii): We can reason analogously to (i).

$\square$

## Corollary

For any THS calculus C and any formula $A$, $\mathsf{C} \vdash \Rightarrow A$ iff $\mathbb{TL}\mathsf{C} \vdash \Rightarrow x : A$.

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# THSGL

Let THSGL denote the following THS calculus for $\mathbb{GL}$:

**Initial** $THS$**:** $G\{p, X \Rightarrow Y, p\}$ $\quad G\{\Box A, X \Rightarrow Y, \Box A\}$

**Propositional rules:**

$$\frac{G\{X \Rightarrow Y, A\}}{G\{\neg A, X \Rightarrow Y\}}\neg A \qquad \frac{G\{A, X \Rightarrow Y\}}{G\{X \Rightarrow Y, \neg A\}}\neg K$$

$$\frac{G\{A, B, X \Rightarrow Y\}}{G\{A \wedge B, X \Rightarrow Y\}}\wedge A \qquad \frac{G\{X \Rightarrow Y, A\} \qquad G\{X \Rightarrow Y, B\}}{G\{X \Rightarrow Y, A \wedge B\}}\wedge K$$

**Modal rules:**

$$\frac{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V, \Box A/\underline{X})\} \qquad G\{\Box A, X \Rightarrow Y/(A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}\Box A$$

$$\frac{G\{X \Rightarrow Y/\Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}}\Box K$$

**Special logical rule:**

$$\frac{G\{\Box A, X \Rightarrow Y/(\Box A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}4$$

# Structural Properties

## Lemma

For THSGL the following hold:

(i) Weakening rules are hp admissible;

(ii) All rules are invertible;

(iii) Contraction rules are admissible.

## Proof.

To (i): By induction on the height of the derivation.

To (ii): By induction on the height of the derivation.

To (iii): By simultaneous induction for left and right contraction, with primary induction on the contraction formula, and secondary induction on the height of the derivation.

$\square$

# Cut Elimination

## Cut Elimination Theorem

The cut rule is admissible in THSGL.

## Proof.

By induction on the cut formula, with secondary induction on the position of the cut, and ternary induction on the height of the cut.

The position of a sequent $\mathcal{S}$ in a THS $G$ in a derivation $\delta$ is given by the difference between the length of the longest THS occurring in $\delta$ and the number of sequents that precede $\mathcal{S}$.

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$\square$

# $\mathbb{TL}$THSGL

By applying the map $\mathbb{TL}$ to THSGL we obtain the following LTS calculus:

**Initial** $LTS$**:** $\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p \qquad \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta, x : \Box A$

**Propositional rules:**

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A}{\mathcal{R}, x : \neg A, \Gamma \Rightarrow \Delta} \neg A \qquad \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \neg A} \neg K$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta} \wedge A \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \qquad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B} \wedge K$$

**Modal rules:**

$$\frac{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta, y : \Box A \qquad \mathcal{R}, Rxy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta} \mathbb{TL}\Box A$$

$$\frac{\mathcal{R}, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} \mathbb{TL}\Box K$$

**Special logical rule:**

$$\frac{\mathcal{R}, Rxy, x : \Box A, y : \Box A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta} \mathbb{TL}4$$

Note three main differences from G3GL:

- $\mathcal{R}$ is a tree;
- $\mathbb{TL}4$ does not appear in G3GL;
- *Irref* and *Trans* do not appear in $\mathbb{TL}$THSGL.

# Structural Properties

**Lemma**

Both weakening and contraction rules are admissible in $\mathbb{TL}$THSGL.

**Proof.**

It follows by the definition $\mathbb{TL}$ and the translation lemma since the corresponding results hold for THSGL. □

**Lemma**

Let $\delta$ be a derivation in G3GL $+ \mathbb{TL}4$ and assume that $\delta$ does not contain any occurrence of *Irref*. Then *Trans* can be eliminated in $\delta$.

**Proof.**

Assume *Trans* occurs $s+1$-times in $\delta$, and let $\rho$ be the topmost occurrence of *Trans*:

$$\frac{Rxy, Ryz, Rxz, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \Gamma' \Rightarrow \Delta'}\ \rho$$

Let $w(\rho)$ denote the number of rules above $\rho$ that make $Rxz$ principal. We proceed by induction on $w(\rho)$.

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Let $w(\rho)$ denote the number of rules above $\rho$ that make $Rxz$ principal. We proceed by induction on $w(\rho)$.

# Structural Properties

## Proof.

- if $w(\rho) = 0$, then $Rxz$ is not principal in any rule above $\rho$, then it can be eliminated and we obtain a derivation with $s$ occurrences of *Trans*;
- if $w(\rho) = k + 1$, then since the derivation above $\rho$ does not contain *Irref* nor *Trans*, $Rxz$ can be principal just by $\mathbb{TL}4$ or $\mathbb{TL}\Box A$:
  - if $Rxz$ is principal by $\mathbb{TL}\Box A$, we have

$$\dfrac{\begin{array}{c} Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta', z : \Box B \\ Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, z : B, \Gamma' \Rightarrow \Delta' \end{array}}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta'} \ \mathbb{TL}\Box A$$

$$\vdots$$

$$\dfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta} \ \rho$$

Then we can weaken the premises with $y : \Box B$, then apply $\mathbb{TL}\Box A$ with $Ryz$ principal formula and $\mathbb{TL}4$ with $Rxy$ principal. This way $w(\rho) = k$, and we apply IH;

# Structural Properties

## Proof.

- if $Rxz$ is principal by $\mathbb{TL}4$, we have

$$\cfrac{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, z : \Box B, \Gamma' \Rightarrow \Delta'}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}4$$

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$$\cfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \rho$$

Then we can weaken with $y : \Box B$, then apply $\mathbb{TL}4$ with $Ryz$ principal formula, and again $\mathbb{TL}4$ with $Rxy$ principal, so that $w(\rho) = k$ and we can use IH.

The result now follows by induction on $s$.

□

# Cut Elimination

## Adequacy Theorem

For any formula $A$, $\mathbb{TL}$THSGL $\vdash \Rightarrow x : A$ iff G3GL $\vdash \Rightarrow x : A$.

## Cut Elimination Theorem

The cut rule is admissible for THSGL.

## Proof.

It follows by the corresponding result for G3GL along with the translation lemma. $\square$

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# Cut Elimination

## Adequacy Theorem

For any formula $A$, $\mathbb{TL}$THSGL $\vdash \Rightarrow x : A$ iff G3GL $\vdash \Rightarrow x : A$.

## Proof.

To $\Rightarrow$: It suffices to prove that $\mathbb{TL}4$ is admissible in G3GL. Assume then that $\mathcal{R}, Rxy, y : \Box A, x : \Box A, \Gamma \Rightarrow \Delta$ is derived. Consider now

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : \Box A \Rightarrow z : \Box A}{Rxz, x : \Box A, z : \Box A \Rightarrow z : A, z : \Box A} \quad \dfrac{z : A \Rightarrow z : A}{Rxz, z : A, x : \Box A, z : \Box A \Rightarrow z : A}}{Rxz, x : \Box A, z : \Box A \Rightarrow z : A}\ \mathbb{TL}\Box A}{Rxy, Ryz, Rxz, x : \Box A, z : \Box A \Rightarrow z : A}\ LW}{Rxy, Ryz, x : \Box A, z : \Box A \Rightarrow z : A}\ (Trans)}{Rxy, x : \Box A \Rightarrow y : \Box A}\ \mathbb{TL}\Box K$$

By applying contractions and *Cut* we obtain $Rxy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta$ which is turned into a contraction- and cut-free derivation by contraction admissibility and cut elimination for G3GL.

# Cut Elimination

## Adequacy Theorem

For any formula $A$, $\mathbb{TL}$THSGL $\vdash \Rightarrow x : A$ iff G3GL $\vdash \Rightarrow x : A$.

## Proof.

To $\Leftarrow$: Note that in any derivation in G3GL, a label can disappear only by $\Box_L - K$ rule when it occurs in a relational term $Rxy$ with $x \not\equiv y$. Now if in the derivation $\delta$ of $\Rightarrow x : A$ *Irref* occurs, we could not have empty antecedent in the final sequent, therefore $\delta$ does not contain *Irref*. By the previous lemma, $\Rightarrow x : A$ is derivable in G3GL $+ \mathbb{TL}4 -$*Trans*$-$*Irref*. Moreover such a derivation involves only LTS, for, by inspection of the rules in G3GL $+ \mathbb{TL}4 -$*Trans*$-$*Irref*, we see that each rule, read upwards, preserves LTS: since $\Rightarrow x : A$ is LTS, we have the result.

$\Box$

# Questions

By means of Negri's and Poggiolesi's calculi we have uncontroversial syntactic cut elimination for provability logic.

(Do you recall its long history, until the mechanized proof in $\mathrm{HOL}$ for multiset version of the original sequent calculus for $\mathbb{GL}$?)

As Blamey and Humberstone state:

> *[T]he move from truth-functional to modal logic is not one best made simply by adding a new primitive connective with new rules governing it, but rather by extending one's conception of the objects to be manipulated by such rules.*

By applying Goré and Ramanayake's translation lemma to THSGL we see that the construction of tree-labelled and -hypersequent calculi has a "common" structure.

## (Kind of) Psychoanalytic Question

What mathematical structure is behind these calculi?
Could we extend this construction to different non-classical operators?

# Questions

We have strongly well-behaved calculi for Gödel-Löb provability logic, and it seems easy to construct intuitionistic versions of them. Are there results concerning interpretability logics, and Solovay logic?

## Questions about Interpretability

Is it possible to extend these calculi to capture $\mathbb{IL}$ and other provability & interpretability logics?
Should we use *standard* Veltman semantics or *generalized* Veltman semantics?

- $\Rightarrow$ Mixing labels for worlds and neighbourhoods?
- $\Rightarrow$ How to "index" THS?

# Questions

Recall that Solovay theorem can be proved (as far as *everybody* knows) *only* by encoding a Kripke countermodel in our arithmetical language.

## Main Question

Is it possible to develop a syntactic proof of Solovay theorem, for instance by encoding THS, or LTS?

Related to previous questions: Could we prove *syntactically* Berarducci arithmetical completeness theorem for $\mathbb{ILM}$?

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