M392C: Orderability/3-Manifold Groups

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Abstract

Class taught by Cameron Gordon, notes taken by Reese Lance. The notes are not live texed. They are post-mortem-texed, that is, taken by hand during class and typed later. Some of my own thoughts are interjected, but rarely. I initially thought to try to separate my thoughts from the professor's but it becomes too difficult. As such I will also try to expand on examples which are mentioned in passing in class, spell out proofs which are glossed over, and add insight where I think it is helpful. This helps to justify the existence of this set of notes, as opposed to live-texed notes (which I believe there are for this class, thanks to Jackson van Dyke), which are probably slightly better for a faithful representation of what is being taught in the classroom. Especially because some of my own content is interspersed throughout these notes, any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu) or if you can find any other way to communicate with me, that is also fine. At the moment I'm trying to get the notes written, and worrying about making the format not look like trash later. I'm also not going to track theorem and lemma numbers, though Cameron does, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers, which is what would have been the plan. Thanks to Arun Debray whose formatting choices inspired my own.

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CHAPTER 1

Overview and Preliminary Group Theory

Lecture 1, Jan 21. The book we will be following is "Ordered Groups and Topology" by Clay and Rolfsen, a pdf of a draft of which can be found here. Orderability was originally developed as a group theoretic concept, and only (somewhat) recently has topology entered the picture. In fact, some of the bare bones definitions can be made entirely in Set:

Definition: Given a set X, a strict, total order (STO) on X is a binary relation $<$, satisfying

$$i)\text{ Transitivity}: \forall\, x, y, z \in X, x < y, y < z \Rightarrow x < z$$
$$ii)\text{ Totality: } \forall\, x, y \in X, \text{exactly one of the following holds:}$$
$$x < y,\;\; y < x,\;\; \text{or } x = y$$

For convenience, we may also define the STO $>$ as induced by an STO:

$$x > y \iff x \neq y \text{ and } x \not< y$$

If we upgrade to Grp, we can use the multiplication structure to talk about invariance:
Definition: Given a group, G, and an STO, $<$, on the underlying set, $<$ is called a left order (LO) on G if

$$g < h \Rightarrow fg < fh\, \forall\, f \in G$$

and we call G left orderable if there exists a left ordering of G. Similarly, we can define right-orderability in the obvious way. One might ask how different the concepts of left and right orders are. It is clear that for abelian groups, they are one and the same, that is: all left orders are also right orders. In general this may not hold. If it does,
Definition: Given an LO, $<$, on G, it is called a bi order (BO) if it is also an RO, and we say G is bi orderable if it admits a bi order.
To return to the above question, given an LO $<$, we can induce an R, $\prec$ O in the following way:

$$g \prec h \iff h^{-1} < g^{-1}$$

Lemma: *The above binary relation is an RO on G.*
Proof: Transitivity and totality of $\prec$ follow immediately from $<$ being an STO. To show it is a right order, consider elements g, h, f of G such that

$$g \prec h$$

Then

$$h^{-1} < g^{-1}$$
$$\Rightarrow f^{-1}h^{-1} < f^{-1}g^{-1}$$
$$\Rightarrow (f^{-1}h^{-1})^{-1} \succ (f^{-1}g^{-1})^{-1}$$
$$\Rightarrow gf \prec hf$$

□

So every left order induces a right order, i.e. all left orderable groups are also right orderable. Note that this does not imply that all left orders are bi orders. That would require that all left orders are **themselves** right orders. All we have shown is that a left order **induces** a right order. Indeed, it is not the case: There are groups which are left orderable but not bi orderable. We obviously can restrict the LO to an LO on a subgroup $H < G$ (denoting subgroup here), just as an STO can be restricted to a subset $A \subset X$.

Example: $(\mathbb{R}, +)$ is bi ordered, with the usual ordering. Its subgroups $\mathbb{Z}, \mathbb{Q}$, etc. can inherit this ordering.

Here is a series of elementary lemmas:

Lemma: *Given $<$ an LO on G,*

$$i)\ g > 1, h > 1 \Rightarrow gh > 1$$
$$ii)\ g > 1 \Rightarrow g^{-1} < 1$$
$$iii)\ < \ is\ BO \iff g < h \Rightarrow f^{-1}gf < f^{-1}hf\ \forall f \in G$$
$$(invariant\ under\ conjugation)$$

Proof:

$$i)\ h > 1, \Rightarrow gh > g > 1$$
$$ii)\ g > 1 \Rightarrow g^{-1}g > g^{-1} \Rightarrow 1 > g^{-1}$$

iii) Assume $<$ is also an RO. Then we can clearly apply left invariance on f^{-1} and right invariance on f to obtain the result. If $<$ is conjugation invariant, then

$$g < h \Rightarrow f^{-1}gf < f^{-1}hf$$
$$gf < hf$$

Using left invariance in the final line.

□

Lemma: *Given $<$ a BO on G,*

$$i)\ g < h \Rightarrow g^{-1} > h^{-1}$$
$$ii)\ g_1 < h_1, g_2 < h_2 \Rightarrow g_1g_2 < h_1h_2$$

Proof:

$$i)\ g < h \Rightarrow g^{-1}g < g^{-1}h$$
$$1 < g^{-1}h \Rightarrow h^{-1} < g^{-1}hh^{-1}$$
$$\Rightarrow h^{-1} < g^{-1}$$
$$ii)\ g_1 < h_1 \Rightarrow g_1g_2 < h_1g_2 < h_1h_2$$

The final inequality follows because $g_2 < h_2$, applying left invariance of h_1.

□

Statement $i)$ above need not hold for an LO on a group, although we showed that it always holds in the case that $h = 1$. So we've only seen one example of an orderable group, the real numbers. We'd like to see some more, but explicit orderings can be very difficult to write down, as we'll see later. For now, we can give an important class of non examples via the following:

Lemma: *If G is left orderable, it is torsion-free.*

Proof: Given $g \in G/\{1\}$, wlog assume $g > 1$. By left invariance,

$$g^n > g^{n-1} > \cdots > 1$$

For any n, this series does not terminate, since the inequalities are strict.

□

In particular, this rules out all finite groups from being left orderable. Another class of not-necessarily-examples is quotient groups. Because subgroups inherit orders, we might guess that quotient groups also do. By simply quotienting something orderable to get something finite, we see that this is not the case. For example, the quotient

$$\mathbb{Z} \to \mathbb{Z}_n$$

does not transport an order.

Remark: If $\{G_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of groups, the direct product of G_λ is the set

$$\prod_{\lambda\in\Lambda} G_\lambda$$

with multiplication defined componentwise, i.e. the elements of the product are λ[1]-tuples, where multiplication is performed in the individual components where we know how to multiply.

Definition: A well order (WO) is an STO on a set X such that for every nonempty $A \subset X$, A has a least element, that is $\exists\, x_0 \in A$ such that $x_0 < x\ \forall\, x \in A$. We might want to know which sets can be well ordered. It might seem like all sets can do this, and indeed that statement is equivalent to the axiom of choice [2]

[1] To be pedantic, these should be labelled as Card(Λ)-tuples

[2] I only understand this on an intuitive level, since I've never had any formal contact with the axiom of choice.

Theorem: *The family G_λ is* $\begin{pmatrix} LO \\ BO \end{pmatrix}$[3] *iff the direct product is* $\begin{pmatrix} LO \\ BO \end{pmatrix}$

Proof: The $\Leftarrow$ implication is easy because we realize each G_λ as a subgroup of the direct product, so that it inherits any ordering. In the other direction, assume we have an $\begin{pmatrix} LO \\ BO \end{pmatrix}$ ordering $<_\lambda$ on each group G_λ. To construct an $\begin{pmatrix} LO \\ BO \end{pmatrix}$ ordering on the direct product, take two elements, $(g_\lambda), (h_\lambda)$. Assuming AoC, we can well order the indexing set, Λ. Choose a well ordering, $\prec$. Assuming $(g_\lambda) \neq (h_\lambda)$, the set of indices where $g_i \neq h_i$ is a nonempty subset of Λ, and thus has a least element, λ_0. We define

$$(g_\lambda) < (h_\lambda) \iff g_i <_{\lambda_0} h_i$$

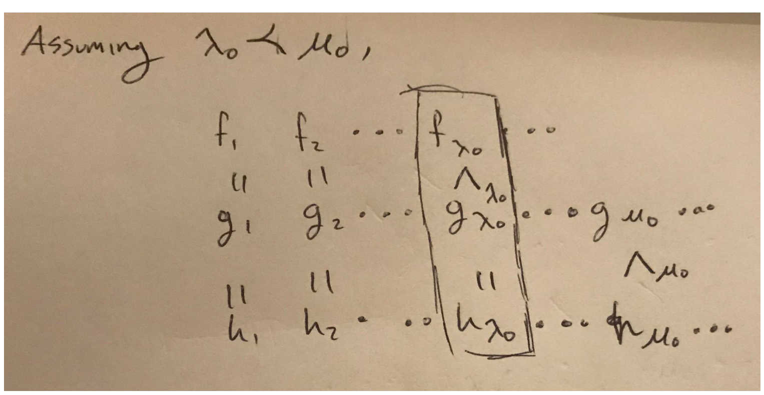
Intuitively, this is the same as comparing numbers. Start at their first digit, compare. If they're the same, go to the next digit, and continue comparing until they're not equal, and compare there. To show this is a transitive relation, suppose $f < g, g < h$. Then for each relation, we have a least index where the elements being compared differ λ_0, μ_0. If it happens that $\lambda_0 = \mu_0$, then we have $f_{\lambda_0} <_{\lambda_0} g_{\lambda_0}$ and $g_{\lambda_0} <_{\lambda_0} h_{\lambda_0}$, and $f_i = g_i = h_i$ for $i \prec \lambda_0$. But $<_{\lambda_0}$ was an STO so by transitivity, $f_{\lambda_0} <_{\lambda_0} h_{\lambda_0} \Rightarrow f < h$. If $\lambda_0 \prec \mu_0$, again $f_i = g_i = h_i$ for $i \prec \lambda_0$. We know g and h agree on all indices $j \prec \mu_0$, in particular at λ_0, i.e.

$$g_{\lambda_0} = h_{\lambda_0}$$

Combining this with the fact that $f < g$, and they disagree at λ_0, we have

$$\begin{gathered} f_{\lambda_0} <_{\lambda_0} g_{\lambda_0} = h_{\lambda_0} \\ \Rightarrow f < h \end{gathered}$$

The case where $\mu_0 \prec \lambda_0$ follows in the same manner. During class, I remember being completely confused by this, and getting lost in symbols, and had to revisit after class to understand it, so I drew a little picture that may help put things in place[4]:



[3]Meaning each individual G_λ is _____orderable

[4]After having made the "diagram" I realized how simple the idea was, and probably nobody had any problem understanding this proof, but I went to the trouble of drawing it so I may as well use it.

So this is a transitive binary relation. It is totally ordered because the $<_\lambda$ are. It is left(right) invariant because multiplciation doesn't change the index λ_0. To belabor the point, if we have $g < h$, then $g_i = h_i$ for $i \prec \lambda_0$ and $g_{\lambda_0} <_{\lambda_0} h_{\lambda_0}$. Thus, for $i \prec \lambda_0$,

$$(fg)_i = f_i g_i = f_i h_i \text{ because } g_i = h_i$$
$$(fg)_{\lambda_0} = f_{\lambda_0} g_{\lambda_0} <_{\lambda_0} f_{\lambda_0} h_{\lambda_0}$$
$$\Rightarrow fg < fh$$

which shows $<$ is an LO on the direct product.

$\square$

We now know that we can build orderable groups out of known orderable groups by simply taking their direct product.

Definition: Given a family of indexed groups $\{G_\lambda\}_{\lambda \in \Lambda}$, the direct sum, $\sum_{\lambda \in \Lambda} G_\lambda$ is the subgroup of the direct product $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of only those elements such that all but finitely many coordinates are 1.

Corollary: *G_λ is $\begin{pmatrix} LO \\ BO \end{pmatrix}$ iff the direct sum is $\begin{pmatrix} LO \\ BO \end{pmatrix}$.*

Proof: The direct sum is a subgroup of the direct product, so it inherits the order.

$\square$

Corollary: *Free abelian groups are BO.*

Proof: The free abelian group on a set Λ is just a direct sum over $\mathbb{Z}$:

$$Free(\Lambda)_{Ab} = \sum_{\lambda \in \Lambda} \mathbb{Z}$$

$\square$

Definition: Let $<$ be an LO on G. The positive cone, $P_<$ is the set of positive elements:

$$P_\lambda = \{g \in G |\, g > 1\}$$

Lemma:

$$i)\ P_\lambda \textit{ is a sub semi} - \textit{group}.$$
$$ii)\ G = P_\lambda \sqcup P_\lambda^{-1} \sqcup 1$$
$$iii)\ < \textit{is a BO} \iff f^{-1} P_\lambda f \subset P_\lambda\ \forall f \in G$$

Proof: We have already proven all the facts needed for this lemma. The only interseting part is that P_λ is a sub *semi*-group rather than a subgroup. Recall the distinguishing features of a semi group is it is a group that need not have an identity element, and elements need not have multiplicative inverses. Indeed that is the case here, there are neither. The identity element does not satisfy $1 < 1$, and if $g \in P_\lambda$, i.e. $g > 1$, then $g^{-1} < 1$, i.e. $g^{-1} \notin P_\lambda$.

$\square$

Definition: $P \subset G$ is a positive cone if it satisfies properties $i)$ and $ii)$ above.

Lemma: *If $P \subset G$ is a positive cone, then $g < h \iff g^{-1}h \in P$ defines an LO on G, with*

$P_< = P$.
Combining the previous two lemmas, we have a correspondence

$$\{LO's\ on\ G\} \Longleftrightarrow \{positive\ cones\ in\ G\}$$
$$\{BO's\ on\ G\} \Longleftrightarrow \{conjugation\ invariant\ positive\ cones\ on\ G\}$$

This correspondence is the essence of where topology comes into the discussion. Left and right orderability are algebraic concepts. Positive cones are nice subsets of G, i.e. elements of the power set, which can be topologized. So the space of left orders corresponds to a subspace of a topological space. Let's revisit the topology on a power set[5]. Given a set, X, its power set can be viewed as $P(X) = \mathrm{Hom}(X, \{0,1\})$. We can always view functions $X \to Y$ as a nice[6] subset of $X \times Y$, so if we give $\{0,1\}$ the discrete topology $P(X) = \mathrm{Hom}(X, \{0,1\}) \subset \{0,1\}^{\mathrm{Card}(X)}$ which is given the product topology[7]. So the space of left orders on a group is a topological space, and in some cases it may look odd. It could be a cantor set, or other weird things. The cardinalities of these spaces is interesting. Some groups admit only finitely many LO's. In such a case, the order is always 2^n, for some n. If a group does admit infinitely many LO's, it is always uncountably many. Following Tychonoff's theorem, the space of orders is compact, and we will see a proof of this later.
Theorem: Free(2) *is LO*.
Proof: An explicit ordering was constructed, but I didn't pay close enough attention. It was gross. Look at Jackson's notes for a discussion of this.
Corollary: Any free group on a countable number of generators is also orderable.
Proof: Any free group on a countable number of generators is isomorphic to a subgroup of Free(2). This comes from covering space theory[8]. We'll end with a theorem that might give insight to how this will arise in the future:
Theorem: *Given an SES*,

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

i) If H,Q are LO $\Rightarrow$ G is LO
ii) If Q is LO and H has a conjugation invariant BO, then G is BO.

[5]Because I forgot it already.
[6]Include conditions here so that you ensure you actually get functions which are well defined. This is not a note to myself, I'm not going to come back and do this.
[7]A nice way to think of the product topology is that it is the coarsest topology which makes projections continuous, projections being the main structure you get when you have a product of spaces. This is a nice way to think of a lot of 'derived' topologies, like quotients, products, etc. What topology makes the important thing continuous? That's the natural topology to give the 'derived' object.
[8]I know the general idea of this statement, but I'd like to come back and flesh this out. I've forgotten too much alg top already.

CHAPTER 2

Extensions, Torsion, and Orderability

Lecture 2, Jan 23. So last time, we stated the theorem related to extensions of G. We will now prove it:
Theorem: *Given an SES,*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

i) If H,Q are LO $\Rightarrow$ G is LO
ii) If Q is LO and H has a conjugation invariant BO, then G is BO.
Proof: i) To define a left order on G, we will specify the positive elements. Let $H = \text{Ker } \varphi < G$, and let P_H, P_Q be positive cones on H and Q respectively. Define

$$P = \varphi^{-1}(P_Q) \sqcup P_H$$

To show it is actually a positive cone, we need to show two things: That it is closed under multiplication, and $G = P \sqcup P^{-1} \sqcup 1$. For i), take $g, h \in P$, then there are 3 cases corresponding to where g and h live:

1. If $g, h \in P_H$, then $gh \in P_H \subset P$ becuase P_H is a positive cone.
2. If, $g, h \in \varphi^{-1}(P_Q)$, then $\varphi(g), \varphi(h) \in P_Q \Rightarrow \varphi(gh) = \varphi(g)\varphi(h) \in P_Q \Rightarrow gh \in \varphi^{-1}(P_Q)$.
3. WLOG, if $g \in \varphi^{-1}(P_Q), h \in P_H)$, then $\varphi(gh) = \varphi(g)\varphi(h) = \varphi(g)e = \varphi(g) \in P_Q \Rightarrow gh \in \varphi^{-1}(P_Q)$.

ii) By exactness,

$$G = H/\{1\} \sqcup \varphi^{-1}(Q/\{1\}) \sqcup \{1\}$$

But

$$\varphi^{-1}(Q/\{1\}) = \varphi^{-1}(P_Q) \sqcup \varphi^{-1}(P_Q^{-1}) \text{ and } H/\{1\} = P_H \sqcup P_H^{-1}$$

So substituting into the first equation gives the result.

$\square$

Example: Let X^2 be the Klein Bottle, and

$$K = \langle a, b \mid b^{-1}ab = a^{-1} \rangle$$

be its fundamental group. Then there is an SES

$$1 \longrightarrow \langle a \rangle = \mathbb{Z} \longrightarrow K \longrightarrow \mathbb{Z} \longrightarrow 1$$

So K is LO. However, K is not BO. If it was, then

$$\begin{aligned} a > 1 \iff & b^{-1}ab > 1 \text{ by LO} \\ & \Rightarrow a^{-1} > 1 \end{aligned}$$

which is a contradiction. Recall $\mathbb{Z}$ has only two orders, so there are 4 choices of LO on K.
Theorem: *These are the only left orders on K.*
Proof: It suffices to show that each determines a unique LO[1]. We first note that all elements of K can be written as $a^m b^n$. Take an LO on K, $<$. If $a > 1$, $b > 1$, then $a^k < b \; \forall \; k \in \mathbb{Z}$.
Claim: $a^m b^n > 1 \iff n > 0$ or $n = 0$ and $m > 0$. This defines a positive cone. The $n = 0$ case is trivial. If $n > 0$, then $a^{-k}b > 1 \; \forall \; k$, by the above claim. Then $b^n > b$ so $a^{-k}b^n > a^{-k}b > 1$. If $n < 0$, then $a^m b^n = b^n a^{\pm m} = (a^{\mp m} b^{-n})^{-1}$. By the first statement, this is > 1, so its inverse is less than 1, which shows the result.

In general, if $<$ is an LO on G, and $\alpha : G \to G$ is an automorphism, $(<, \alpha)$ induces an LO on G by $g <_\alpha h \iff \alpha(g) < \alpha(h)$. There exist automorphisms of K, α_1, α_2, such that

$$\alpha_1(a) = a, \quad \alpha_1(b) = b^{-1}$$

and

$$\alpha_2(a^{-1}) = a, \quad \alpha_2(b) = b^{-1}$$

So $<_{(i)}$ is the unique LO on K determined by i). The LO $<_{(ii)}$ is determined by $(<_{(i)}, \alpha_1)$. $<_{(iii)}$ is determined by $(<_{(i)}, \alpha_2)$, and $<_{(iv)}$ is determined by $(<_{(i)}, \alpha_1\alpha_2)$.

□

As we mentioned before, if a group has finitely many left orders, the number of left orders is 2^n for some n. K is an example of that fact, which has 2^2 left orders, each factor of two coming from a choice on $\mathbb{Z}$. As an exercise, you can show that $\forall \; n > 0$, there is a group with 2^n left orders[2].
Corollary: *For any LO on K, $h \in \langle a \rangle$, $g \in K/\langle a \rangle$, $g > 1 \Rightarrow g > h$.*
Suppose M is a closed, orientable 3-fold. We may ask "Is $\pi_1(M)$ LO?" The only strong statement we have is that it must be torsion free if we hope for the answer to be yes. So already we have examples of spaces whose fundamental group is not LO. If $M = L(p, q)$ is a Lens space, we know $\pi_1(M) = \mathbb{Z}_p$, so we have an infinite class of non examples. If $M = M_1 \# M_2$, then we know $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, e.g. $X = M \# L(p, q) \Rightarrow \pi_1(X)$ is not LO. However, $T^3 = S^1 \times S^1 \times S^1$ is LO, since the property "being LO" commutes with direct product. $M = \#_n(S^1 \times S^2) \cong F_n$ is also LO.

[1] I don't understand this remark.
[2] Hint: examine the exact sequence we just used and iterate.

Until now, we might be led to believe that being torsion free implies that a group is LO. This is not the case:

Example: Take X^2 the Klein Bottle, $K = \pi_1(X^2)$ and consider $X = X^2 \times I$. This is a non orientable 3-fold. Take the 2-fold cover

$$p : T^2 \to X^2$$

so that

$$p_*(\pi_1(T^2)) = \langle a, b^2 \rangle \cong \mathbb{Z} \times \mathbb{Z} < K$$

and let N be the mapping cylinder of p, i.e.

$$N = (T^2 \times I) \sqcup X^2 \Big/ \Big\{ (x,0) \sim p(x) \; \forall \, x \in T^2 \Big\}$$

and N_1, N_2 two copies of N. Then of course, $\pi_1(N_i) = K \equiv \langle a_i, b_i \mid \dots \rangle$, since there is a strong deformation retraction $N \to X^2$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Then

$$M = N_1 \sqcup_\varphi N_2$$

is a closed, orientable 3-fold, and $\pi_1(M) = K_1 *_{\mathbb{Z}\times\mathbb{Z}} K_2$, which is torsion free.

Theorem: *$H_1(M)$ is finite, so $\pi_1(M)$ is not LO.*

Proof: φ is determined by, up to isotopy, $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$, and we denote $H_1(\partial N_i) = \langle a_i, 2b_i \rangle$. Then φ_* is a matrix, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, so that

$$\varphi_*(a_1) = pa_2 + 2qb_2$$
$$\varphi_*(2b_1) = ra_2 + 2sb_2$$

Denote

$$H_1(N_i) = \langle a_i, b_i \mid a_i^2 \rangle$$

Then $H_1(M)$ is presented by

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{pmatrix}$$

so that $H_1(M)$ is finite $\iff$ $\det(A) \neq 0$ $\iff$ $q \neq 0$ $\iff$ $\varphi_*(a_1) \neq \pm a_2$. Suppose $\pi_1(M)$ were LO, then we would induce an LO on the subsets ∂N_i. There are only 4 LO's on $\pi_1(N_i)$.

Corollary: *For any LO on $\pi_1(N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$ summand of $\pi_1(\partial N) = \langle a, b^2 \rangle$ such that $h \in \langle a \rangle$ and $g \in \pi_1(\partial N)/\{1\}, g > 1$ then $g > h$.*

$$\Rightarrow \varphi_*(\pm a_2)$$

which shows the result.

CHAPTER 3

An Equivalent Condition to LO

Lecture 3, Jan 28. **Theorem:** A, B *torsion free groups* $\Rightarrow A *_C B$ *is torsion free.*

$\square$

Lemma: *Let* $G = Homeo_+(\mathbb{R}) := \{$*Orientation preserving homeomorphisms of* $\mathbb{R}\}$*. Then* G *is LO.*

The proof of this lemma follows from a more general fact:

Definition: If $<$ is an STO on a set X, $\underline{\mathcal{B}(X,<)}$ is the set of set of order preserving bijections $\varphi : X \to X$ (i.e. $x < y \Rightarrow \varphi(x) < \varphi(y)$). This is a group under composition, and in fact is a subgroup of the group of the automorphism group.

Theorem: $\mathcal{B}(X,<)$ *is* LO.

Proof: Let $\prec$ be a well ordering on X[1], and let

$$f \neq g \in \mathcal{B}(X,<)$$

Then define $[f \neq g] := \{x \in X \mid f(x) \neq g(x)\}$ as the set of points on which f and g disagree. This set is nonempty because $f \neq g$, so let x_0 be its $\prec$-least element. Then define $f < g \iff f(x_0) < g(x_0)$. This is an LO on $\mathcal{B}(X,<)$.

$\square$

From this, the above lemma follows: If we let $<$ be the standard ordering on $\mathbb{R}$, then the set of order preserving bijections is equal to the set of orientation preserving homeomorphisms, so the above theorem applies.

Remark: Let $\prec_x$ be a well ordering on $\mathbb{R}$ such that x is the least element on $\mathbb{R}$[2]. Then $<_x$ is the induced LO on $Homeo_+(\mathbb{R})$, and is distinct from all other orders induced by a different $x' \in \mathbb{R}$, so there are uncountably many LO's on $\mathcal{B}(\mathbb{R},<)$.

Given $x, y \in \mathbb{R}, \exists g \in Homeo_+(\mathbb{R})$ s.t.

$$g(x) > x, g(y) < y$$

Then $g >_x 1$ and $g <_y 1$ iff $<_x \neq <_y$.

Corollary: *G is LO iff G acts faithfully on an STO'd set* $(X,<)$*.*

[1] It's still not clear to me how much of a problem it is that we are making a choice of WO in these constructions. It's clear that it isn't a problem for these existence proofs, but there is no uniqueness property satsified here. I don't know how much of a concern that is. The mere property of being left orderable is already interesting enough, so maybe it doesn't matter that these aren't natural constructions.

[2] It's not clear to me that this can be done for any $x \in \mathbb{R}$.

Proof: One direction is the previous proof, the other direction is seen from the fact that G acts faithfully on itself by left multiplication.
Corollary: *Any subgroup of $Homeo_+(\mathbb{R})$ is LO.*
In fact, this characterizes countable LO groups:
Theorem: *If G is countable, G is LO iff G is isomorphic to a subgroup of $Homeo_+(\mathbb{R})$.*
Proof: For the forward direction, we will show something more general:
Theorem: *If $(G, <)$ is countable, there exists an LO on $Homeo_+(\mathbb{R})$, and an order preserving injection homomorphism*

$$(G, <) \to (Homeo_+(\mathbb{R}), <)$$

Sketch of Proof: If $G \neq \{1\}$, if it is infinite, let $g_1, g_2, \ldots$ be some indexing of the elements of G. Define an embedding

$$e : G \to \mathbb{R}$$

via

$$e(g_1) = 0, \quad \text{and}$$

$$g_{n+1} \begin{pmatrix} > \\ < \end{pmatrix} g_i \;\forall\, 1 \leq i \leq n \Rightarrow e(g_{n+1}) = \begin{pmatrix} \max\{e(g_i)|\; 1 \leq i \leq n+1\} \\ \min\{e(g_i)|\; 1 \leq i \leq n-1\} \end{pmatrix}$$

If the first condition doesn't hold, then let $g_l = \max\{g_i, 1 \leq i \leq n, g_i < g_{n+1}\}$, and similarly for g_r defined via the min. Then set

$$e(g_{n+1}) = \frac{e(g_l) + e(g_r)}{2}$$

So if g_{n+1} is greater or less than everything before it, just stick it on the end, if not, stick it in the middle. We note that e is order preserving by construction. Also if the first condition does hold, then the first $n+1$ elements are integers. In fact, $e(g_{n+1})$ is an integer iff the first condition holds. We also know

$$\begin{gathered} g > 1 \Rightarrow g^2 > g > 1 \text{ and } g^{-1} < g \\ g < 1 \Rightarrow g^2 < g < 1 \text{ and } g^{-1} > g \\ \Rightarrow \mathbb{Z} \subset e(G) \equiv \Gamma \end{gathered}$$

Then $G \curvearrowright \Gamma$ via $g(e(a)) = e(ga)$, which is order preserving.
We want to extend this to an action on $\mathbb{R}$. This is a continuous action, so it extends to an action when we throw in limit points, i.e. $G \curvearrowright \overline{\Gamma}$. But $\mathbb{R}/\overline{\Gamma}$ is a countable disjoint union of open intervals, $\sqcup_i (a_i, b_i)_i$. So G is defined on the set of $\{a_i, b_i\}$, and we may extend affinely to the intervals (a_i, b_i).

$\square$

To define an LO on $Homeo_+(\mathbb{R})$ that restricts to the LO on Γ from G, let $\gamma \in \Gamma$. Then

$$g \begin{pmatrix} > 1 \\ < 1 \end{pmatrix} \iff g(\alpha) \begin{pmatrix} > \gamma \\ < \gamma \end{pmatrix}$$

Then let $\prec$ be a WO on $\mathbb{R}$ such that γ is the $\prec$-least element of $\mathbb{R}$, and let $\ll$ be the LO on $Homeo_+(\mathbb{R})$ induced by $\prec$. Then

$$g \begin{pmatrix} > 1 \\ < 1 \end{pmatrix} \iff g \begin{pmatrix} \gg 1 \\ \ll 1 \end{pmatrix}$$

For prime 3-manifold groups, we can eliminate the faithful condition, so you need only act on $\mathbb{R}$. This will have important implications for things like foliations in the future.

Now we will talk about an application of orderability, in fact one which really got people interested in orderability in the first place.
Definitions: If R is a unital ring, $a \in R$ is a
i) unit if $\exists b \in R$ such that $ab = ba = 1$.
ii) 0-divisor if $a \neq 0$ and $\exists b \neq 0 \in R$ such that $ab = 0$ or $ba = 0$.
iii) non-trivial idempotent if $a \notin \{0, 1\}$ and $a^2 = a$.
Definition: If G is a group and R a ring, the group ring is defined as

$$RG := \left\{ \sum_g r_g g \mid g \in G, r_g \in R, r_g = 0 \text{ for all but finitely many } g \right\}$$

Then one can check that RG is a ring. Furthermore, if $g \in G$, r is a unit in R, then rg is a unit in RG.
Definition: A unit in RG is non-trivial if it is not of the above form.
Group rings always have trivial units if R has units, but the question of when they have non trivial units can be interesting.
Theorem: *Suppose G has torsion and K is a field of characteristic 0. Then*
i) KG has 0-divisors
ii) KG has non-trivial units
iii) KG has non-trivial idempotents.
Proof: i) Let g have order $n \geq 2$. Then

$$\sigma := 1 + g + g^2 + \cdots + g^{n-1} \in KG$$
$$\Rightarrow g\sigma = \sigma \Rightarrow (1-g)\sigma = 0 \Rightarrow \sigma \text{ is a 0-divisor.}$$

ii)

$$\sigma^2 = 1\sigma + g\sigma + g^2\sigma + \cdots + g^{n-1}\sigma = n\sigma$$
$$\Rightarrow (1-\sigma)\left(1 - \frac{1}{n-1}\sigma\right) = 1 \Rightarrow (1-\sigma) \text{ is a non-trivial unit}$$

We note that if $n = 2$, $(1-\sigma)$ is a trivial unit, in which case we may also compute

$$(1-2\sigma)\left(1 - \frac{2}{3}\sigma\right) = 1$$

iii)

$$\left(\frac{1}{n}\sigma\right)^2 = \frac{1}{n^2}\sigma^2 = \frac{1}{n}\sigma$$

which gives nontrivial idempotents.

$\square$

The upshot of this theorem is that when G has torsion, we have a bunch of "bad stuff" in the group ring. However, if G is torsion-free,

Theorem: (Kaplansky Conjecture) *If G is a torsion-free group and K is a field, then KG does not have any non-trivial units, 0-divisors, or non-trivial idempotents.*

It was later proven that all conditions are equivalent, yet it is still not known if any of them are true. Since orderable groups are torsion free, they obey the Kaplansky Conjecture, which historically spawned interest in orderable groups.

CHAPTER 4

More Applications of Orderability and $\mathbb{Z} \times \mathbb{Z}$

Lecture 4, Jan 30. We will now directly prove that left orderable groups satisfy the Kaplansky Conjecture, without using the fact that LO groups are torsion-free:
Proof: i) Note:

$$\left(\sum_{i=1}^{m} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j h_j\right) = \sum_{i}^{n}\sum_{j}^{m} \alpha_i \beta_j (g_i h_j)$$

and assume, WLOG, $g_i < g_{i+1}, h_i < h_{i+1}$. Let $g_k h_l$ be the minimal element of

$$S := \{g_i h_j\} \subset G$$

Then $h_1 < h_j \; \forall \; j > 1$, so $gh_1 < gh_j \Rightarrow l = 1$. Also, $gh_1 = g'h_1 \Rightarrow g = g'$, so $g_k h_1$ is unique. We similarly define a unique maximal element, $g_r h_n$. These two elements are distinct, so that there are at least two terms on the RHS of the equation. If we supposed that the equation above is equal to 1, this shows that there are no non-trivial units, and if we suppose it were equal to 0, this shows that there are no non-trivial 0-divisors. It is a general phenomenon that ii) implies iii), so we are done.

□

Another related application is the
Conjecture: (Isomorphism Conjecture) *If G is torsion-free, then*

$$\mathbb{Z}G \cong \mathbb{Z}H \Rightarrow G \cong H$$

Note that there is G finite such that the conjecture is false[1]
Corollary: *G LO $\Rightarrow$ G satisfies the Isomorphism Conjecture.*
Proof: Define $U_{\mathbb{Z}G} := \{\textit{units in } \mathbb{Z}G\} \cong \mathbb{Z}_2 \times G$. It is easily seen that this has a group structure. Sppose $\mathbb{Z}G \cong \mathbb{Z}H$. If $\mathbb{Z}G$ has no 0-divisors, then $\mathbb{Z}H$ has no 0-divisors, so H is torsion free , by a previous theorem[2]. Then $H < U_{\mathbb{Z}H} \cong U_{\mathbb{Z}G} \cong \mathbb{Z}_2 \times G$, so H can be realized as a subgroup of $Z_2 \times G$, which is LO, so H is LO, becuase subgroups inherit orderings. But LO groups do not have 0 divisors, as they are torsion free, so

$$U_{\mathbb{Z}H} \cong \mathbb{Z}_2 \times H$$
$$\Rightarrow \mathbb{Z}_2 \times H \cong \mathbb{Z}_2 \times G \Rightarrow G \cong H$$

[1] Hertweck 2001, "A counterexample to the isomorphism problem for integral group rings".
[2] We proved the contrapositive of this statement earlier.

There are details to check about cancelling the factors of $\mathbb{Z}_2$. It certainly doesn't hold in generality, but it does in this trivial case[3].

$\square$

We will now discuss Bi-Orders on $\mathbb{Z} \times \mathbb{Z}$. Take a line in $\mathbb{R}^2$ through the origin with irrational slope α. This gives an LO via the positive cone which declares every point in the integer lattice above the line is positive, i.e.

$$P = \{(m, n) \mid n > m\alpha\}$$

For each irrational slope, we have a unique order, thus $\mathbb{Z} \times \mathbb{Z}$ has uncountably many LO's. If α is rational, then we can define an LO on the integer lattice as long as we decide consistently what to do with points that lie on the line. Points which lie on the line are a subset of $\mathbb{Z}$, so we may pick a $\mathbb{Z}$ LO on there, P_0. Then

$$P = P_0 \sqcup \{(m, n) \mid n > m\alpha\}$$

is an LO. It can be shown that these are all of the LO's on $\mathbb{Z} \times \mathbb{Z}$. Each α yields 2 LO's if it is irrational and 4 LO's if it is rational, and this generalizes in the obvious way to the n-dimensional integer lattice.

We will now make the complete argument for topologizing the space of LO's of a group. We recall some point set topology results and definitions:
Lemma: *X_λ Hausdorff $\Rightarrow \prod X_\lambda$ Hausdorff.*
Definition: X is totally disconnected if the only non-empty connected subspaces are the singletons.
Lemma: *X_λ totally disconnected $\Rightarrow \prod X_\lambda$ totally disconnected.*
This implies that Pow(X) is compact, Hausdorff, and totally disconnected for any space X. For $x \in X$, we may define the projection $\pi_x : Pow(X) \to \{0, 1\} \subset \{0, 1\}^X$ via

$$\pi_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

and define the sets in Pow(X).

$$U_x = \pi_x^{-1}(1), \qquad V_x = \pi_x^{-1}(0)$$

It is readily seen that $U_x = V_x^c$. These U_x and V_x then form a basis for Pow(X), if we allow them to vary over x. This coincides with the product topology.

[3]Just for the sake of synthesis, if the Kaplansky Conjecture were true, the theorem would be "Torsion free groups satisfy IC", and its proof would not reference orderability or topology.

CHAPTER 5

The topology on LO(G) and checking for Orderability

Lecture 5, Feb 4. We know Pow(X) is compact, Hausdorff, and totally disconnected. Given $B \subset X$, define

$$I = \{A \subset X \mid B \not\subset A\}$$
$$II = \{A \subset X \mid A \cap B \neq \emptyset\}$$

These sets are both open.
Theorem: *$LO(G)$ is closed in Pow(G), and thus compact.*
Proof: WTS Pow(G) is open, i.e. Pow(G)/LO(G) is open. So we take $A \subset G$ such that A is not a positive cone. Then either

1. A is not closed under multiplication
2. $\exists g \mid g^{-1} \in A$
3. $e \in A$
4. $A \cup A^{-1} \neq G/\{e\}$

If we show all of these conditions are open conditions, we are done. This can be seen by reformulating each statement in terms of the open sets which we already know:

1. $\iff A \in U_g \cap U_h \cap V_{gh}$
2. $\iff A \in U_g \cap U_{g^{-1}}$
3. $\iff A \in U_e$
4. $\iff A \in \cup_{g \neq e}(V_g \cap V_{g^{-1}})$

All these sets are open, so A has a neighborhood of subsets which are also not positive cones, hence $LO(G)$ is closed in Pow(G).

□

A similar result holds for bi-orders.
An aside on the Cantor set:
Definition: A Cantor space is a topological space which is totally disconnected, perfect, compact, and uncountable. Equivalently, a Cantor space is a topological space which is

homeomorphic to the Cantor set, i.e. the middle thirds removed set.
If $x \in [0,1]$, we can consider its ternary expansion

$$x = 0.x_1 x_2 \cdots = \sum_i \frac{x_i}{3^i}$$

for $x_i \in \{0,2\}$. So we can define the Cantor set as

$$C = \left\{ x \in [0,1] \mid x = 0.x_1 x_2 \ldots \mid x_i \in \{0,2\} \right\}$$

Then if we give $\{0, 2^{\mathbb{N}}\}$ the product topology,

$$0.x_1 x_2 \cdots \mapsto (x_1, x_2, \ldots)$$

is a homeomorphism, so $C \cong \{0,2\}^{\mathbb{N}}$. In fact, $\{0,2\}^{\mathbb{N}}$ is the canonical example of a Cantor space. So G countable $\Rightarrow$ LO(G) is homeomorphic to a subspace of C. Furthermore, since totally disconnectedness and compactness are inherited by LO(G), if LO(G) has no isolated points, it is a Cantor space.
Example[1]**:** $n > 1 \Rightarrow$ LO($\mathbb{Z}^n$) $\cong C$.
Example[2]**:** LO(F_n) $\cong C$.
Definition: If $X \subset G$ is a subgroup, define

$$S(X) := \{\textit{semigroup generated by } X\}$$

Theorem: *G is LO iff* $\forall$ *finite* $F \subset G/\{e\}, \exists\, \epsilon : F \to \{\pm 1\}$ *s.t.*

$$\boxed{1 \notin S\left(\left\{ f^{\epsilon(f)} \mid f \in F \right\}\right)}$$

Proof: We note the RHS of the boxed equation as $S(F,\epsilon)$.
$\Rightarrow$: Define $\epsilon(f) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ if $f \begin{pmatrix} > 1 \\ < 1 \end{pmatrix}$. Then

$$S(F,\epsilon) > 1, \Rightarrow 1 \notin S$$

$\Leftarrow$: Define $Q(F,\epsilon) = \{Q \subset G/\{1\} \mid S(F,\epsilon) \subset Q, S(F,\epsilon)^{-1} \cap Q = \emptyset\}$
We note $Q(F,\epsilon) \neq \emptyset$ implies it satisfies the boxed equation. We also define

$$Q(F) = \bigcup_{\epsilon} Q(F,\epsilon)$$

which is a finite union. We wish to show $Q(F)$ is closed, i.e. $S(G)/Q(F,\epsilon)$ is open. Take A in the complement. Then either $1 \in A$, $S(F,\epsilon) \not\subset A$, or $S(F,\epsilon)^{-1} \cap A \neq \emptyset$. Similar to the flavor of the lemma we proved a while ago, formulating all of these conditions in terms of the basis sets show that these are all open conditions.

[1](Sikora 2004?) "Topology on the Spaces of Orderings of Groups"
[2](McCleary 1985) "Free lattice-ordered groups represented as *o*-2 transitive *l*-permutation group"

$\square$

Note that if $F \subset F'$, then

$$S(F, \epsilon'|_{F'}) \subset S(F', \epsilon')$$
$$Q(F') \subset Q(F)$$

So given a finite number of finite subsets, $F_1, \ldots, F_n$, of $G/\{1\}$, then

$$\bigcap_{i=1}^{n} Q(F_i) \supset Q(F_1 \cup \cdots \cup F_n) \neq \emptyset$$

because the boxed equation holds. Thus we say that $\{Q(F)\}$ has the finite intersection property, and each one is closed.

In theory, this theorem turns the orderability question into a combinatorial/computational problem. Since we are checking finite subgroups, there exists algorithms such that if G is not left orderable, there exists a finite subgroup which does not satisfy the boxed equation, and an algorithm will find it eventually[3],[4]. Since $S(G)$ is compact,

$$\bigcap_{F \subset G/\{1\}} Q(F) \neq \emptyset$$

Then let $P \in \bigcap Q(F)$. In the next lecture, we will prove that this indeed forms a positive cone.

[3] I'm not sure I understand this point. Even though the subgroups themselves are finite, isn't it possible that there are infinitely many finite subgroups? So that checking all such subgroups would take infinitely long?

[4] Here is a good link to read more about the computational aspect of this problem.

CHAPTER 6

Group properties and Burns-Hale Theorem

Lecture 6, Feb 6. We claim P is a positive cone for G.
Proof: i) $e \notin P$, bc $e \notin Q(F)$.
ii)$g, h \in P \Rightarrow gh \in P$: Take $g, h \in P$. Then if $F = \{g, h\}$, there exists $\epsilon(g), \epsilon(h) \in \{\pm 1\}$ such that

$$S\left(g^{\epsilon(g)}, h^{\epsilon(h)}\right) \subset P \quad \text{and} \quad S\left(g^{\epsilon(g)}, h^{\epsilon(h)}\right)^{-1} \cap P = \varnothing$$
$$\Rightarrow \epsilon(g) = \epsilon(h) = 1$$
$$\Rightarrow gh \in S\left(g^{\epsilon(g)}, h^{\epsilon(h)}\right) \subset P$$

iii) $P \cap P^{-1} = \varnothing$: Let $g \in P$, $F = \{g\} \Rightarrow S(g) \subset P$, so that $S(g)^{-1} \cap P = \varnothing \Rightarrow g^{-1} \notin P$.
iv) $P \sqcup P = G/\{1\}$: Take $g \in G$, and let $F = \{g\}$. Then $\exists \epsilon = \pm 1$ such that $S(g^{\epsilon}) \subset P$ and $S(g^{-\epsilon}) \cap P = \varnothing$, which implies $g^{\epsilon} \in P$.

□

Remark: A similar result holds for BO.
New topic:
Definition: If P is a property of a group[1], G locally has the property P, or <u>G is locally P</u>, if every finitely generated subgroup has the property P.
Fun fact: A group G is locally locally P iff it is locally P.
Definition: A property P is called a local property if (G is locally $P \Rightarrow G$ is P).
Remark: LO and BO are local properties.
Theorem: *G is locally LO/BO iff G is LO/BO.*
Pf: $\Leftarrow$: Obvious
$\Rightarrow$: F finite $\subset G/\{1\}$. Then $\langle F \rangle \subset G$ is LO.

$$\Rightarrow \exists \epsilon \mid \text{the boxed equation holds.}$$

This property applies for all F, which implies G Is LO.
Corollary: *An Abelian group is BO iff it is torsion-free.*
Proof: We proved the forward direction before. $\Leftarrow$: G is BO iff G is locally BO. Then H f.g. subgroup of G is torsion-free $\Rightarrow H \cong \mathbb{Z}^n$, which is LO. Because BO is a local property, we are done.

[1]Whatever that means. Some examples might be 'finite', 'abelian', 'torsion-free', etc.

□

Corollary: *An arbitrary free group is LO.*
Proof: F free group, H finitely generated in $F \Rightarrow H \cong F_n$, a free group of rank n, which we showed earlier was LO.

□

Theorem: *G_λ is LO $\iff *_\lambda G_\lambda$ is LO.*
Proof: $\Leftarrow$: G_λ is a subgroup of $*_\lambda G_\lambda$.
$\Rightarrow$: $\exists$ homeomorphism $\varphi : *_\lambda G_\lambda \to \Pi_\lambda G_\lambda$ via

$$g_\lambda \mapsto (1, \ldots, 1, g_\lambda, 1, \ldots, 1)$$

We then get an exact sequence:

$$1 \longrightarrow H \longrightarrow *_\lambda G_\lambda \longrightarrow \Pi_\lambda G_\lambda \longrightarrow 1$$

where $H = Ker\varphi$. Then by the Kurosh subgroup theorem, $H = (*_\mu H_\mu)*F$, where H_μ is a subgroup of a conjugate of G_{λ_μ}, which is a subgroup of the free product, and F is a free group. But $H = Ker\varphi$, and $\varphi|_{G_\lambda}$ is injective for all λ, so

$$\Rightarrow H \cap g^{-1} G_\lambda g = \{1\} \ \forall \lambda, \forall g$$
$$\Rightarrow H = F$$

So H and F are both LO, so

$$1 \longrightarrow LO \longrightarrow *_\lambda G_\lambda \longrightarrow LO \longrightarrow 1$$

implies the central group is also LO, by a theorem from way back when.

□

Definition: *P is a property of a group G, then G is residually P if $\forall g \in G/\{1\}, \exists$ epimorphism[2] $\varphi : G \to H$ such that G has property P, and $\varphi(g) \neq 1$.*
We note again that G is residually residually P iff G is residually P, and we call a property residual iff G is residually P implies G is P.
Example: Finiteness is not residual, e.g. $\mathbb{Z}$.
Lemma: *If a property P is closed under taking subgroups and direct products, then P is a residual property.*
Proof: Suppose G is residually P. Then for all $g \in G/\{1\}$, $\exists$ epimorphism $\varphi_g : G \to H_g$ such that H_g is P, and $\varphi_g(g) \neq 1$. This induces an injection

$$\varphi : G \to \Pi_{g \in G/\{1\}} H_g$$

H_g is P, $\Rightarrow \Pi_g H_g$ is $P \Rightarrow G$ inherits P since it is a subgroup.

□

[2]Given a category C, an epimorphism is a morphism $f : A \to B$ such that for any other morphisms $g, h : B \to Z$, $gf = gh \Rightarrow g = h$. In a concrete category, i.e. one which comes equipped with a faithful functor to **Set**, surjections are epimorphisms, though I'm not sure if it's always an equivalence. Certainly there are some cases where it is. We are generally working in such a category, so it is appropriate to think of this as a surjection in most cases.

Corollary: *LO and BO are residual properties.*
There are some open questions related to residual properties, for example the geometrization conjecture[3]. Another question is: If G is a residually finite group such that

$$FQ(G) = FQ(F_2)$$

where FQ is the set of finite quotients, does that imply $G \cong F_2$? This is a currently open question. Note that $FQ(F_2)$ is a very concrete object, it is the set of finite groups with two generators.
Another question due to Grothendeick: If G_1 and G_2 are finitely presented and residually finite, $FQ(G_1) = FQ(G_2) \Rightarrow G_1 \cong G_2$? This question is much less concrete.
We now move on to a very powerful theorem in orderability: A strengthening of the fact that LO is a local property. We also adopt the convention that $\{1\}$ is *not* orderable, otherwise we would have to write "except for the trivial group" everywhere.
Theorem: (Burns-Hale) *G is LO $\iff$ every non-trivial finitely generated subgroup has an LO quotient.*
LO being local requires every finitely generated subgroup to be LO, but this theorem tells us it's even better than that: every finitely generated subgroup simply needs to have a LO quotient.
Proof: $\Leftarrow$: Define $F = \{g_1, \ldots, g_n\} \subset G/\{1\}, n \geq 1$. Then we will induct on n to show that you can find an ϵ satisfying the boxed equation.
$n = 1$: $\langle g_1 \rangle$ has an LO quotient. Since the quotient is LO, it must be infinite, so $|\langle g_1 \rangle|$ itself must be infinite, so the order of g_1 is infinite, so $1 \notin S(g_1)$.
If $n > 1$, then there is a surjection

$$\varphi : \langle g_1, \ldots, g_n \rangle \to L$$

for some L LO. We may assume the first k generators satisfy

$$\begin{gathered} g_k \mapsto 1 \\ \Rightarrow g_{k+1} \not\mapsto 1 \\ g_{k+2} \not\mapsto 1 \\ \vdots \\ g_n \not\mapsto 1 \end{gathered}$$

for $k < n$. By the inductive hypothesis, there exists $\epsilon_1, \ldots, \epsilon)n \in \{\pm 1\}$ such that $1 \notin S\left(\{g_i^{\epsilon_i} | \, 1 \leq i \leq k\}\right)$. Then let $<$ be an LO on L. Define $\epsilon_i \in \{\pm 1\}$ so that

$$\varphi(g_i^{\epsilon_i}) > 1$$

Then $1 \notin S\left(\{g_i^{\epsilon_i} | \, 1 \leq i \leq k\}\right)$.

□

Definition: G is <u>indicable</u> if $G = \{1\}$ or there exists an epimorphism $G \to \mathbb{Z}$.
Corollary: G locally indicable $\Rightarrow$ G is LO.

[3] I know nothing about this, that's just something Cameron said.

Remark: G has an LO quotient $\not\Rightarrow$ G is LO, e.g. $\mathbb{Z}*\mathbb{Z}_2$. Kill the right factor, and you have $\mathbb{Z}$, which is LO, but the group itself has torsion, so is not LO.
All hope is not lost though:
Theorem: *If G is a group such that every f.g. subgroup of infinite index is indicable, then*

$$G \text{ is LO} \iff G \text{ has an LO quotient}$$

The previous example doesn't satisfy the hypothesis (as it must not, since it isn't LO but has an LO quotient) because the subgroup corresponding to the generator for $\mathbb{Z}_2$ is f.g. and has infinite index, yet is not indicable.

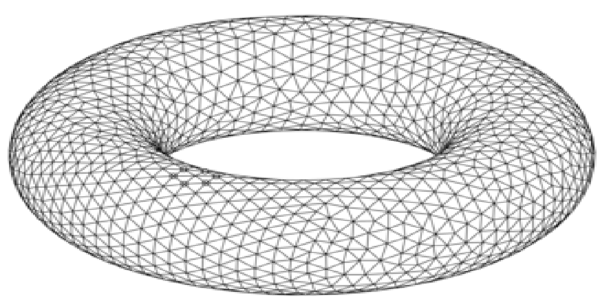
CHAPTER 7

Introduction to Surfaces

Lecture 7, Feb 11. We will now prove the previously stated theorem:
Proof: The forward direction is obvious. For the reverse implication, we want to show G is LO, assuming it has an LO quotient. This occurs iff every finitely generated, non-trivial subgroup has an LO quotient. Let H be such a subgroup. Then if it has infinite index, it is indicable by assumption, so H has $\mathbb{Z}$ as a quotient. Then by Burns-Hale, G is LO.
If $[G : H]$ is finite, then we have an epimorphism $\varphi : G \to L$ for some L LO, and infinite, and would like to restrict that to a quotient $\varphi|_H : H \to \varphi(H) < L$. As long as the restriction is not trivial, the proof is done. Because $[G : H]$ is finite, $[Q : \varphi(H)]$ is finite, thus $\varphi(H)$ is nontrivial (if it were trivial, it would have infinite index), so $\varphi_| H : H \to \varphi(H)$ is a quotient to an LO subgroup, $\varphi(H)$, so we are done.

□

We will now talk about how all these ideas apply to surfaces, or 2-manifolds. Eventually we will talk about 3-manifolds.
Definition: Given a manifold, M, a <u>triangulation</u> of M is a homeomorphism $M \to K$ for K a locally finite simplicial complex. It is now known that every n-manifold has a triangulation for $n \leq 3$.



A triangulation of the Torus

I think of triangulations as "coverings by n-simplices". This might be reminiscent of the computations we learn early on in homology, e.g. breaking the gluing diagram of the torus up into two triangles along the diagonal. This is actually not a triangulation because the edges of the triangle, for example, are not embedded. They are two to one at

the endpoints, so you have a cell complex, but not a simplicial complex. We use the words surface and 2-manifold interchangably. There is a classification of surfaces, split into the compact and non-compact case, but we don't discuss the non-compact case:
Theorem: *If M is a closed surface, then either $M \cong S^2$, $\#_g T^2$, or $\#_g \mathbb{P}^2$.*
So we may identify a closed surface by two pieces of data: what its genus is, and if it is orientable or non orientable. Since we have such a nice classification, we are probably in a decent position to answer the question: Which surface groups are LO? By Burns-Hale, we need only consider finitely generated subgroups of surface groups, and their quotients.
Lemma: *If M is a closed n-manifold, N is a connected N-manifold, and $f : M \to N$ is injective, then f is a homeomorphism.*
Lemma: *Let S be a noncompact surface. Then $H_2(S) = 0$.*
Proof: Pick a triangulation of S, $\varphi : S \to K$. Then by unioning the facets of the simplicial complex, we can get a sequence of compact surfaces $S_1, S_2, \ldots S$ such that

$$S = \bigcup_{i=1}^{n} S_i$$

By the above lemma, each S_i has non empty boundary, so that $S_i \cong$ a 1-simplex, i.e. $H_2(S_i) = 0$. Every two cycle is contained in some S_i, so $H_2(S) = 0$.

□

Lemma: *Let S be a surface, δ a circle component of S s.t. $\pi_1(\delta) \to \pi_1(S)$ is not injective. Then $S \cong D^2$.*
Proof: For S compact, you can check this is true by exhausting cases via the classification. If non-compact[1], then define $S^* := S \cup D^2$, glued along δ. Then we have the commutative diagram

$$\begin{array}{ccc} \pi_1(\delta) & \longrightarrow & \pi_1(S) \\ \downarrow & & \downarrow \\ H_1(\delta) & \longrightarrow & H_1(S) \end{array}$$

The left arrow is an isomorphism, but the top is not, so $H_1(\delta) \to H_1(S)$ is not an iso. But the Mayer-Vietoris sequence gives

$$\ldots \longrightarrow H_2(S^*) \longrightarrow H_1(\delta) \longrightarrow H_1(S) \longrightarrow \ldots$$

so that $H_2(S^*) \cong \ker H_1(\delta) \to H_1(S) \Rightarrow H_2(S^*) \neq 0$, a contradiction.

□

Theorem: (Scott core theorem, Compact core theorem) *Let S be a surface and $\pi_1(S)$ finitely generated. Then there exists a compact, connected $S_0 \subset S$ such that $i_* : \pi_1(S_0) \to \pi_1(S)$ is an isomporphism. S_0 is called a compact core of S.*
I wasn't present for this lecture, and to my understanding this proof requires surgery theory which I don't know, so I'm skipping over it.

[1] Having trouble with a picture in my head. Surely not all surfaces have such a boundary component, so you've got to assume that S is one which does have such a boundary component. The thing I'm thinking of in my head is an infinite pair of pants, but I'd like to see some other examples. I know there's a classification of non compact surfaces, but I haven't seen it yet.

CHAPTER 8

Lecture 8

Lecture 8, Feb 13. From the Burns-Hale theorem, we have

$$G \text{ locally indicable} \Rightarrow G \text{ LO or } G = \{1\}$$

Theorem: *S be a surface $S \ncong \mathbb{R}P^2$. Then $\pi_1(S)$ is LO.*
Proof: Let $H < \pi_1(S)$, finitely generated and non-trivial. There is a connected covering space $\tilde{S} \to S$ such that $\pi_1(\tilde{S}) = H$. By the compact core theorem,

$$H \cong \pi_1(S_0) \text{ for some } S_0 \text{ a compact surface}, S_0 \subset \tilde{S}$$
$$\Rightarrow \pi_1(S_0) \cong H \neq \{1\}$$

$S_0 \cong \mathbb{R}P^2$ would imply $\tilde{S} \cong \mathbb{R}P^2 \Rightarrow S \cong \mathbb{R}P^2$, a contradiction. So

$$S_0 \ncong \mathbb{R}P^2$$

But S_0 is not simply connected by assumption, so $H_1(S_0)$ is infinite, using the classification of compact surfaces. Then $\exists$ an epimorphism $H_1(S_0) \to \mathbb{Z}$, and the abelianization map, so we can compose to get an empimorphism φ:

$$\begin{tikzcd}
H_1(S_0) \arrow[r, two heads] & \mathbb{Z} \\
H = \pi_1(S_0) \arrow[u, two heads] \arrow[ur, two heads, "\varphi"'] &
\end{tikzcd}$$

So $\pi_1(S_0) = H$ is indicable, so $\pi_1(S)$ is locally indicable, and thus LO.

$\square$

This gives us the definitive answer to our overall question about surface groups
Corollary: I*f S is a surface, S is LO $\iff$ $S \ncong \mathbb{R}P^2$ and $\pi_1(S) \neq \{1\}$.*

$\square$

Remark: i) $S = K \Rightarrow \pi_1(S)$ is LO, but not BO, for K the Klein bottle[1].
ii) Locally indicable implies LO, but the converse does not hold.
iii) It can be shown if S is a non compact surface, then $\pi_1(S)$ is free with countable rank.

[1] This fails because locally indicable does not imply BO, as it did for LO, and there is no analogous statement.

iv) It can also be shown that $\pi_1(S) = \{1\} \iff S \cong S^2$ or $S \cong D^2/X$ for X some closed subset of S^1.
The question of surfaces is settled, so we will now move on to 3-folds.
We will consider 3-folds which are connected, orientable, possibly with boundary, and possibly non compact.
Definition: If M_1 and M_2 are oriented 3-folds, and

$$B_i \subset \text{int } M_i \text{ such that } B_i \cong B_3 \text{, the 3-ball}$$

Then we may define the <u>connect sum</u> of M_1 and M_2 by

$$M \equiv M_1 \# M_2 := (M_1/\text{int } B_1) \cup_h (M_2/\text{int } B_2)$$

where $h : \partial B_1 \to \partial B_2$ is an orientation reversing homeomorphism. It turns out $M_1 \# M_2$ is well defined up to order preserving homeomorphism.
Basically we take two 3-folds and delete a ball from each of them, then glue them together at the ball. The homeomorphism is orientation reversing because the second manifold is glued on as a mirror image. This definition can be made for any fixed dimension.
We can represent # as an operation on 3-manifolds. This operation is commutative, associative, and has unit S^3.
Definition: If M is a 3-manifold, M is called <u>prime</u> if

$$M = M_1 \# M_2 \Rightarrow M \cong M_1, M_2 \cong S^3 \text{ or the permutation}$$

Basically, a prime manifold is one which is not a non-trivial connect sum.
Theorem: (Kneser, Milnor: Prime decomposition of 3-manifolds) *Let M be a compact, oriented, 3-manifold. Then*

$$M \cong \#_{i=1}^n M_i$$

where the homeomorphism is orientation preserving, and M_i are prime, $M_i \ncong S^3$, and M_i are unique up to orientation preserving homeo, i.e. unique as oriented topological manifolds, modulo obvious re-ordering. The $n = 0$ case corresponds to S^3.
If M is compact,

$$\begin{gathered} M \cong \#_{i=1}^n M_i \\ \Rightarrow \pi_1(M) \cong *_{i=1}^n \pi_1(M_i) \\ \Rightarrow \pi_1(M) \; LO \iff \pi_1(M_i) \; LO \end{gathered}$$

The main idea here is that when asking if a compact 3-manifold is LO, it is acceptable to assume M is prime, because once we know all the prime 3-manifolds, we know all the compact 3-manifolds. However, there do exist non compact 3-manifolds that do not have a prime decomposition.
Definition: M is <u>irreducible</u> if $\forall S^2 \subset M$ bounds a 3-ball, $B^3 \subset M$.
Lemma: *M is irreducible iff M is prime and $M \ncong S^1 \times S^2$.*
This says that, apart from $S^1 \times S^2$, being irreducible is equivalent to being prime.
Theorem: (Perelman) *Let M be a closed 3-manifold with universal cover $\widetilde{M}$. Then,*

i) $\pi_1(M)$ is finite $\Rightarrow \widetilde{M} \cong S^3$, and $\pi_1(M) \curvearrowright S^3$ freely with quotient M as a subgroup of $SO(4)$.
ii) $\pi_1(M)$ infinite and M irreducible $\Rightarrow \widetilde{M} \cong \mathbb{R}^3$. As we know,

$$\begin{aligned} \pi_1(M) \text{ infinite} &\Rightarrow \widetilde{M} \text{ is non-compact} \\ M \text{ irreducible } &\Rightarrow \pi_2(M) = 0 \text{ we will see this implication latter} \\ &\Rightarrow \widetilde{M} \text{ contractible.} \end{aligned}$$

But there exist contractible, non-compact 3-manifolds without boundary which are not homeomorphic to $\mathbb{R}^3$, so restricting to closed is necessary.
It turns out that 3-manifolds with finite fundamental group are all Seifert fiber spaces, which we will encounter later.
Example: $\mathbb{Z}_p \curvearrowright \mathbb{C}_2$ by

$$(z, w) \mapsto \left(e^{2\pi i/p}z, e^{2q\pi i/p}w\right)$$

Restricting to $S^3 \subset \mathbb{C}^2$, the action is free[2]. Then the quotient is called a lens space:

$$L(p, q) := S^3/\mathbb{Z}_p$$

and

$$\pi_1(L(p, q)) = \mathbb{Z}_p$$

Theorem: (Reidemeister) $L(p, q) \cong^{o.p.} L(p', q') \iff$

$$q \equiv q' mod p \quad \text{and} \quad qq' \equiv 1 mod p$$

One direction of the proof is obvious.
Theorem: (Perelman) *M, M' closed 3-manifolds, M prime and not a lens space, then*

$$\pi_1(M) \cong \pi_1(M') \Rightarrow M' \cong M$$

so prime 3-manifolds are "almost" determined by their fundamental groups.
Remark: Restriction to prime 3-manifolds is necessary:
If M is an oriented 3-manifold such that $M \cong^{o.p.} -M$, *e.g.* $M = L(3, 1)$ or Poincare homology sphere, then $\pi_1(M\#M) \cong \pi_1(M\#(-M))$. But by Kneser-Milnor, $M\#M \ncong M\#(-M))$.

[2] Is it just because we're throwing out 0? I don't think anything else is fixed by this action. This construction is a little weird and hard to visualize.