M392C: Complex Geometry

Reese Lance

Fall 2020

Abstract

Class taught by Berndt Siebert at the University of Texas at Austin, notes taken by Reese Lance. The notes are not live texed. They are post-mortem-texed, that is, taken by hand during class and typed later. Some of my own thoughts are interjected, but rarely. Often I will spell out proofs which are glossed over, mostly for my own benefit. This helps to justify the existence of this set of notes, as opposed to live-texed notes (which are often available for classes at this university), which are probably better for a faithful representation of what is being taught in the classroom. Any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu). At the moment I'm trying to get the notes written, and worrying about the format later, possibly never. I'm also not going to track theorem and lemma numbers, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers next to things, which is what would have been the plan. Thanks to Arun Debray whose formatting choices inspired my own. Also the dates got messed up at some point so just ignore them. I don't know why anyone would care anyway.

Table of Contents

CHAPTER 1

Overview of Complex Analysis

Lecture 1, Aug 27. When working with several complex variables, the use of i as an index becomes problematic, since we want to assign $\sqrt{-1} = i$. So for instance, the standard coordinates

$$\mathbb{C}^n = \{z_1, \ldots, z_n \mid z_i = x_i + iy_i, x_i, y_i \in \mathbb{R}\}$$

might appear confusing. However, it is always clear from context which i is meant: There is never an instance where you we will be indexing over the complex numbers. One important concept we will discuss is the:

Theorem: (Cauchy Integral Formula, version 1) *If $f : U \to \mathbb{C}$ is a holomorphic function, and $\gamma : I \to U$ is a simple, closed curve in U, then for any $z \in \Omega_{int}$ defined by the Jordan Curve theorem applied to γ satisfies:*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

WLOG, we may treat γ as the unit disk, in which case CIF is written as

$$f(z) = \frac{1}{2\pi i} \int_{\|\zeta\|=1} \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

Here, the curve we are interested in is γ, but we require f to be defined on a larger, open set U, but this is an unnecessary requirement: for CIF to hold, we need only require that f be continuous and holomorphic on Ω_{int}. We can see version 1 implies version 2 by taking increasingly large disks inside of Ω_{int}: Apply version 1 of CIF on a unit disk of radius $1 - \epsilon \subset \Omega_{int}$. Then we have the CIF formula for all points $z \in B(0, 1 - \epsilon)$. By perturbing the smaller disk slightly, a continuity argument says that the new integral,

$$\frac{1}{2\pi i} \int_{\|\zeta\|=1-\epsilon} \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

converges. But of course, the value of $f(z)$ doesn't change, so neither does the integral. Then taking the limit $\epsilon \to 0$ shows the result. This is the more natural way to state the CIF, since it doesn't require the extra ad-hoc neighborhood U.

In a first encounter with complex analysis, we typically write down definitions formally and manipulate them calculus style:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and } \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and similarly for the differential forms

$$dz = dx + idy, \quad \text{and } d\overline{z} = dx - idy$$

If we consider a differentiable function $f = g + ih : \mathbb{C} \to \mathbb{C}$, we can write down its Jacobian

$$D_p f = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}(p)$$

If we wish to change coordinates to $z = x + iy$, we might want to write

$$D_p f = \begin{pmatrix} \partial_z f & \partial_{\overline{z}} f \\ \partial_z \overline{f} & \partial_{\overline{z}} \overline{f} \end{pmatrix}$$

But as we know, $\overline{(\partial_z f)} = \partial_{\overline{z}} \overline{f}$, so this Jacobian has inherent redundancy: the two entries which are diagonal from each other are related by the complex conjugate. In other words, this matrix has the form

$$D_p f = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$$

So in order to compute the real jacobian, if you're working in complex coordinates, you must know $\partial_z f$ and $\partial_{\overline{z}} f$, it is not enough to just know one. If, in addition, f is holomorphic, then the Cauchy-Riemann equations imply $\partial_{\overline{z}} f = 0$, so all the information about the total derivative of f is contained in $\partial z f$. So for $f \in \mathcal{O}(\mathbb{C})$[1],

$$D_p f = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}$$

Now we move on to several variables, with the notion of holomorphicity:
Definition: If $U \subset \mathbb{C}^n$, then $f : U \to \mathbb{C}^n$ is holomorphic if the function

$$w \mapsto f(z_1, \ldots, z_{i-1}, w, z_{i+1}, \ldots, z_n) \text{ is holomorphic } \forall i \in \{1, \ldots, n\}$$

i.e. if f is holomorphic in each of its variables. Then, if $f \in \mathcal{O}(U)$, we can write down a version of CIF by applying the 1-d version repeatedly to each entry:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\|z_i - w_i\| = \epsilon_i} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n$$

here, the natural domain is
Definition: A poly disc, $D_\epsilon(w)$, is a product of disks,

$$\epsilon = (\epsilon_1, \ldots, \epsilon_n), \quad D_\epsilon(w) := \{z \mid \|z_i - w_i\| < \epsilon_i\}$$

[1] this denotes the set of holomorphic functions on $\mathbb{C}$, which is referred to by Siebert as a "structure sheaf", which is a phrase that scares me

For $n > 1$, this is distinct from the unit disk, even when $\epsilon = (1, \ldots, 1)$: in $n = 2$, $(i - \epsilon, i - \epsilon) \in D_{(1,1)}(0)$, but $(i - \epsilon, i - \epsilon)$ has distance ~ 2 from the origin, and so is not in the unit disk. As such, their boundaries are also not the same:

$$\partial \Delta^n = S^{2n-1} \ncong \{z \in \mathbb{C}^n \mid \|z\| = 1\}$$

The RHS is sometimes referred to as the Shilov boundary.
As in one variable, the CIF gives us the power series expansions:

$$f(z) = \sum a_I z^I$$

where I is the multi-index $I = (i_1, \ldots, i_n)$, and a_I are complex coefficients that can be obtained using the CIF.
Another interesting feature unique to $n > 1$ complex analysis is the concept of domains of holomorphy. A domain of holomorphy, Ω is characterized by the property that there is a holomorphic function on Ω which cannot be extended to a larger set.
In the case of Δ^2, this means that

$$f \in \mathcal{O}(\Delta^2 \backslash \{0\}) \Rightarrow \quad f \text{ extends over } 0$$

i.e. the map given by restriction

$$\mathcal{O}(\Delta^2) \to \mathcal{O}(\Delta^2 \backslash \{0\}) \quad \text{is a bijection}$$

This fact is a consequence of
Theorem (Hartogs): *Let f be a holomorphic function on a set $G \setminus K$ where G is an open subset of $\mathbb{C}^n$, for $(n \geq 2)$ and K is a compact subset of G. If the complement $G \setminus K$ is connected, then f can be extended to a unique holomorphic function on G.*
Switching gears again, suppose you have a holomorphic function f on a poly disc, and you want to study the zeros of this function.
Definition: A Weierstrass polynomial is a function on $\mathbb{C}^n = \{w, z\}$ for $w \in \mathbb{C}$ and $z \in \mathbb{C}^{n-1}$, of the form

$$w^d + a_1(z) w^{d-1} + \cdots + a_d(z)$$

where a_i are holomorphic functions on $U \subset \mathbb{C}^{n-1}$. If you fix the z coordinate, f is a polynomial in w of degree d, with d roots. We will talk more about why this is a desirable form in the next lecture.

CHAPTER 2

Hartogs Theorem and Domains of Holomorphy

Lecture 2, Sept 1. In this lecture, we will go through proofs for Hartogs' and Weierstrass Theorems.
Theorem (Hartog): *Let $\epsilon, \epsilon' \in \mathbb{R}^n_{\geq 0}$, and $\epsilon'_i > \epsilon_i$, $n \geq 2$. Then the restriction map*

$$\varphi : \mathcal{O}(D_\epsilon) \to \mathcal{O}(D_\epsilon \setminus \overline{D_{\epsilon'}})$$

is bijective.
Proof: To prove injectivity, we will use a result from 1 variable complex analysis:
Theorem (Identity Theorem): *If $f, g, : U \to \mathbb{C}$ are holomorphic functions on an open subset $U \subset \mathbb{C}$ such that there exists a non-empty open subset $V \subset U$ such that $f|_V = g|_V$, then $f = g$.*

Knowing this, if we assume that $\varphi(f) = \varphi(g)$, then if we restrict to a single variable, say the first coordinate, and fix all the others, then $\varphi(f)(z) = \varphi(g)(z)$ for all $z \in D_\epsilon \setminus \overline{D_{\epsilon'}}$. An open set minus a closed set is open, and $\epsilon'_i > \epsilon_i$ implies $D_\epsilon \setminus \overline{D_{\epsilon'}}$ is non empty, so we satisfy the conditions to apply 1-dimensional Identity Theorem, setting $U = D_\epsilon$ and $V = D_\epsilon \setminus \overline{D_{\epsilon'}}$, so that $f = g$, and φ is injective.
To show surjectivity, define a projection

$$\pi : D_\epsilon \setminus \overline{D_{\epsilon'}} \to D_{\hat{\epsilon}} \setminus \overline{D_{\hat{\epsilon}'}}$$

where a hat denotes an omission of the first entry, so this is a projection from $\mathbb{C}^n \to \mathbb{C}^{n-1}$. Then we have

$$\pi^{-1}(\hat{z}) = \begin{cases} \Delta_{\epsilon_1} \setminus \overline{\Delta_{\epsilon'_1}} \times \{\hat{z}\} & z \in \overline{D_{\hat{\epsilon}'}} \\ \Delta_{\epsilon 1} \times \{\hat{z}\} & \text{else} \end{cases}$$

Then take $f \in \mathcal{O}(D_\epsilon \setminus \overline{D_{\epsilon'}})$, and we want to find a function on D_ϵ which restricts to f on $D_\epsilon \setminus \overline{D_{\epsilon'}}$, and we will use CIF to define

$$g := \frac{1}{2\pi i} \int_{\|\zeta\| = \frac{\epsilon_1 + \epsilon'_1}{2}} \frac{f(\zeta, \hat{z})}{\zeta - z_1} d\zeta \in \mathcal{O}\left(D_{\left(\frac{\epsilon_1 + \epsilon'_1}{2}, \hat{\epsilon}\right)} \right)$$

So we are applying CIF in one dimension, specifically the first variable, so that g is holomorphic in the first variable, and f is holomorphic in the final $n-1$ variables, so g is holomorphic in each of its variables, and thus is holomorphic on this restricted polydisc.

Then, if we restrict to the "annulus" polydisc $D_{\left(\frac{\epsilon_1+\epsilon_1'}{2},\hat{\epsilon}\right)} \setminus \overline{D_{\epsilon'}}$, the Cauchy Integral Theorem tells us that $g = f$, as required.

$\square$

Question: What is the topology of the domain $D_\epsilon \setminus \overline{D_{\epsilon'}}$?
If we look in the 2D case, we can define a map

$$\mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}$$
$$(z_1, z_2) \mapsto (\|z_1\|, \|z_2\|)$$

Then the fiber over each point in $\mathbb{R}^2_{\geq 0}$ is $S^1 \times S^1$.
To answer the question, we know that the poly discs have the topology of a ball. So if you remove a closed ball of smaller radius, you are left with a space that retracts onto the sphere, so $D_\epsilon \setminus \overline{D_{\epsilon'}}$ is topologically a sphere, S^{2n-1}. As such, it is connected and simply-connected.
Now we talk about domains of convergence of power series, $\sum_{k\in\mathbb{N}^n} a_k z^k$. Since the domain of convergence in $n = 1$ is a disk, we might guess that in n dimensions, the domain of convergence will be a polydisc. To prove this, we will use a lemma:
Lemma: *If $w \in \mathbb{C}^{n*}$, and*

$$\forall\, k,\ |a_k w^k| \leq C$$

for some C, i.e. there is a uniform bound on the norm of the terms, then $\sum_k a_k z^k$ converges absolutely in D_w.
Proof: Same as dimension 1 proof.

$\square$

So it turns out that the polydisc guess is not exactly right, but is close. Instead,
Definition: Given the map $\kappa : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ sending $(z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|)$, a Reinhardt domain is the pre-image of a domain $U \subset \mathbb{R}^n_{\geq 0}$ under κ. In this case, a domain is just a connected, open subset.

The fiber of a point $x \in \mathbb{R}^n_{\geq 0}$ with all non-zero entries is an n-torus. If not all the entries are non-zero, then the number of coordinate axes which x intersects determines the dimension of the resulting torus. These are just the fibers over single points, the Reinhardt domains are the pre-images over any connected, open set, so they can look very crazy.
Then we may define
Definition: The Logarithm map is the map

$$\mathrm{Log} : \mathbb{C}^{n*} \to \mathbb{R}^n$$
$$(z_1, \ldots, z_n) \mapsto (\log(|z_1|), \ldots, \log(|z_n|)$$

Definition: If $U \subset \mathbb{C}^n$ is a Reinhardt domain, then U is logarithmically convex if $\mathrm{Log}(U \cap C^{n*})$ is a convex domain.
Proposition: *If U is the domain of convergence for some power series $\sum_k a_k z^k$, then U is a logarithmically convex Reinhardt domain. In fact, this is an equivalence, but we'll only prove the*

stated direction.

Proof: Assume that $\sum a_k z^k$ converges for $|z| = \delta$ and $|z| = \sigma$. Then there exists C such that

$$|a_k|\delta^k, \ \ |a_k|\sigma^k < C$$

Then let $\chi \in \mathbb{R}^n_{\geq 0}$ such that

$$\log\chi = \alpha\log\rho + \beta\log\sigma, \quad \alpha + \beta = 1$$

then, if $k = (k_1, \dots, k_n)$,

$$\begin{aligned}
\log\chi^k &= \log\chi_1^{k_1} \dots \chi_n^{k_n} \\
&= \sum_{i=1}^{n} k_i \mathrm{Log}\chi_i \\
&= \alpha \sum k_i \log\rho_i + \beta \sum \log\sigma_i \\
&= \alpha \log \rho^k + \beta\log\sigma^k \\
&\leq \alpha\log\frac{C}{|a_k|} \\
|a_k\chi^k| &< C
\end{aligned}$$

The final line implies, by our earlier lemma, that $\sum_k a_k z^k$ converges on D_χ.

□

Now we will discuss the Identity theorem in $n \geq 2$.

Theorem: *If f, g are holomorphic functions on $U \subset \mathbb{C}^n$, and $\exists V \subset U$ open such that $f|_V = g|_V$, then $f = g$.*

Recall the proof in $n = 1$: We wish to show Ω, the set of points on which f and g disagree, is closed, open, and nonempty in $\mathbb{C}$. To see it is closed, we note $f - g$ is a continuous function, and singletons are closed in T_1 spaces, the pre-image of a closed set is closed, so $(f-g)^{-1}(0)$ is a closed set, and $(f-g)^{-1}(0) = \Omega$. To show it is open, if we take some $c \in \Omega$, so that $f(c) = g(c)$, then because f and g are holomorphic, we can expand them into a Taylor series, each of which have non-zero radii of convergence (in this case, poly radius, since they will converge on a polydisc), and f and g agree on the polydisc, so Ω is open. Ω is non-empty by definition.

□

Proposition: *For $f : U \subset \mathbb{C}^n \to \mathbb{C}$ holomorphic and non-constant, f is open.*

Proof: If f is non-constant, for any $z \in U$, there is some line through z on which f is non-constant, i.e. there is another point on the line, w, such that $f(z) \neq f(w)$. Applying the 1-dimensional open mapping theorem, we have the result.

□

We will do some preparatory work to talk about Weierstrass preparation theorem next lecture:

Lemma: *Let $\epsilon > \epsilon' > 0$, and $f \in \mathcal{O}(\Delta_\epsilon)$, and $\lambda_1, \ldots, \lambda_d \in \Delta_{\epsilon'}$ are the zeros of f counted with multiplicities (so if λ_i has multiplicity r, then it appears r times in the list $\{\lambda_1, \ldots, \lambda_d\}$). Then for $k \geq 0$,*

$$\sum_i^n \lambda_i^k = \frac{1}{2\pi i} \int_{|z|=\epsilon'} z^k \frac{f'(z)}{f(z)} dz$$

The LHS is what is known as a Newton polynomial. Note that the case of $k = 0$, we recover the typical method of counting zeros.
Proof: By the residue theorem, the RHS is

$$\sum_{w \in \Delta_{\epsilon'}(0)} \text{Res}\left(z^k \frac{f'(z)}{f(z)}, w\right)$$

We will compute this for fixed λ. In this case, we can rewrite

$$f(z) = (z-\lambda)^r h(z)$$

where $h(z) \neq 0$. Then the residue is

$$\begin{aligned}
\text{Res}\left(z^k \frac{f'(z)}{f(z)}, \lambda\right) &= \text{Res}\left(z^k \frac{r(z-\lambda)^{r-1}h(z) + (z-\lambda)^r h'(z)}{(z-\lambda)^r h(z)}, \lambda\right) \\
&= \text{Res}\left(z^k \left(\frac{1}{z-\lambda} + \frac{h'(z)}{h(z)}\right), \lambda\right) \\
&= \lambda^k r + 0
\end{aligned}$$

Then summing over all λ gives us the result.

$\square$

CHAPTER 3

Weierstrass Preparation Theorem

Lecture 3, Sept 3. Consider the ring of symmetric polynomial functions. This has obvious basis of the elementary symmetric polynomials: if there are d variables, there is one elementary symmetric polynomial of degree n for each $n \leq d$, made by adding products of each variable. For example, the $n = 3$ elementary symmetric polynomial is

$$\sum_{1 \leq j < k < l \leq d} \lambda_j \lambda_k \lambda_l$$

It is a theorem that you can also pick a basis as these Newton polynomials, or "power sums", i.e. there are transformations called "Newton identities" which express these Newton polynomials in terms of the elementary symmetric polynomials. One can look them up, it becomes complicated with higher degrees. For now, all we need to know is that it can be done.

Theorem (Weierstrass Preparation Theorem): *Adopt the coordinates (z, w) for $z \in \mathbb{C}$ and $w \in \mathbb{C}^{n-1}$. Then if $U \subset \mathbb{C}^{n-1}$, $f \in \mathcal{O}(U \times \Delta_\epsilon)$ such that $f|_{U \times \partial \Delta_\epsilon}$ has no zeros of f, then $\forall\ \epsilon' < \epsilon$ sufficiently close, $\exists$ a Weierstrass polynomial g, such that $f|_{U \times \Delta_{\epsilon'}} = gh$, for some $h \in \mathcal{O}(U' \times \Delta_{\epsilon'})$, for $U' \subset U$, and $h(0) \neq 0$.*

Proof: We may assume that $f|_{U' \times \Delta_{\epsilon'}}$ has no zeros. Then for $w \in U$, let $\lambda_1(w), \ldots, \lambda_{d(w)}(w)$ be the set of zeros of $f|_{\{w\} \times \Delta_{\epsilon'}}$. By the previous lemma, the function

$$d(w) = \sum_{i=1}^{d(w)} \lambda_i^0 = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{\partial_z f}{f} dz$$

is locally constant wrt w, and if we take U connected, it is independent of w. So we define g pointwise:

$$g(z, w) = \prod_{i=1}^{d} \Big(z - \lambda_i(w)\Big) = z^d + \alpha_1(w) z^{d-1} + \cdots + \alpha_d(w)$$
$$\text{where } \alpha_i = (-1)^i \delta_i(\lambda_1, \ldots, \lambda_d)$$

where δ_i is the elementary symmetric polynomial. Then the fact from last lecture: The elementary symmetric functions can be expressed in terms of the Newton polynomials. But as the lemma showed, the Newton polynomials, for every k, are just integrals, and as such are holomorphic. So the α_i are holomorphic, and so is g. Then defining $h = \frac{f}{g}$ gives the result.

$\square$

The reason this theorem is important in complex analysis is because it gives us tools to study the 0 loci of holomorphic functions, so-called analytic sets. It is saying that to study the zeros of f in this region, it is sufficient to study g, a Weierstrass polynomial, which is a factor of f.
Example: Apply WPT to the polynomial $f(z_1, z_2) = z_1^3 z_2 + z_1 z_2 + z_1^2 z_2^2 + z_2^2 + z_1 z_2^3$

We recall that given a real vector space, V, we can always get a complex vector space by complexifying V to $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Note that there is a canonical isomorphism $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V$ decomposing $V_{\mathbb{C}}$ into "real" and "complex" parts. Then the complex structure on $V \oplus V$ is the map $(v, w) \to (-w, v)$. If we think of (v, w) as $v + iw$, this choice of complex structure is natural. This is an important point because, given an abstract complex vector space, there is no notion of real or imaginary parts. The complex vector spaces which are obtained as tensor products of real vector spaces have this natural decomposition.
Definition: Given a complex vector space, V, a real structure on V is a conjugate linear involution $\kappa : V \to V$.

If an abstract complex vector has a real structure, then the fixed locus of κ, V^κ, defines an isomorphism

$$V^\kappa \otimes_{\mathbb{R}} \mathbb{C} \to V$$

which gives a decomposition of V into real and imaginary parts. Any complex vector space can be given a real structure, so that one may decompose vectors into real and imaginary parts, but only those which arise as a tensor product have this structure a priori. If handed an abstract complex vector space, one must impose this structure by picking a basis of V. But what would happen if you wanted to complexify a complex vector space? This happens already in complex manifold theory, where we study the complexified tangent bundle. But the tangent bundle already has a complex structure, that's the definition of a complex manifold! If you complexify a complex vector space, V, you get an isomorphism $V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$, where $\overline{V}$ is the complex vector space with the same underlying real vector space as V, but with the opposite complex structure, $-I$.
Finally, we will talk about analytic sets
Definition: An analytic set $Z \subset U$ is a set such that $\forall\ z \in U$, $\exists V \subset U$ open such that $\exists f_1, \ldots, f_\gamma \in \mathcal{O}(V), Z \cap V = \{z \in V \mid f_1(z) = \cdots = f_\gamma(z) = 0\}$.

So an analytic set locally looks like the zero locus of finitely many holomorphic functions.
Definition: The germ of an analytic set at $0 \in \mathbb{C}^n$ is an equivalence class of pairs (U, Z), where U is an open neighborhood of $0 \in \mathbb{C}^n$ and Z is an analytic subset of U. The equivalence relation imposed is $(U, Z) \sim (U', Z')$ iff $\exists V \subset (U \cap U')$ a neighborhood of 0 such that $Z \cap V = Z' \cap V$.

So the usual notion of a germ, that is setting things to be equivalent if they agree locally, but now locally means "on an analytic subset". If α_i are elements of a ring, we denote $I(\alpha_i)$ to be the ideal generated by those elements. The germ of an analytic set Z is denoted $(Z, 0)$, so for example, the germ of a zero locus of an ideal I is written $(Z(I), 0)$.
Lemma: *The set of germs is in bijection with the set of finitely generated ideals in the ring of convergent power series of $\mathbb{C}^n$.*
Proof: Given a germ of an analytic set, let $f_1, \ldots, f_n$ be those functions which realize the definition of an analytic set, say at a point z. Then $(f_1, \ldots, f_n)$ generated a finitely generated ideal in the ring of convergent power series, since f_i are holomorphic. Conversely, given a finitely generated ideal let $f_1, \ldots, f_n$ be a finite generating set. Then $Z(f_1, \ldots, f_n) := \{z \in U \mid f_i(z) = 0\}$ is analytic, and so we can take its germ. To do this, we made a choice of generators f_i, but the germ is well defined under this choice. This comes from the general fact that if I is finitely generated, then $Z(I)$ is well defined: If g_i is another generating set, then

$$\forall\, \alpha, f_\alpha = \sum_i \lambda_i g_i, \quad \text{for } \lambda_i \in \mathcal{O}_{\mathbb{C}^n,0}$$
$$\Rightarrow Z(f_i) \subset Z(g_i)$$

and vice versa[1].

□

This lemma is very reminiscent of Hilbert's Nullstellensatz.

Definition: If I is an ideal, the <u>radical</u> of I is the set $\sqrt{I} := \{z \in R \mid \exists d > 0 \text{ s.t. } z^d \in I\}$, and if I is an ideal such that $I = \sqrt{I}$, then we call I a <u>radical ideal</u>.

Lemma: $Z(I) = Z(\sqrt{I})$.
Proof:

$$z_0 \in Z(I) \Rightarrow f(z_0) = 0 \forall\, f \in I$$
$$\forall\, g \in \sqrt{I}, g^k = f \text{ for some } f \in I$$
$$\Rightarrow g^k(z_0) \equiv \underbrace{g(z_0) \ldots g(z_0)}_{\text{n times}} = f(z_0) = 0 \Rightarrow g(z_0) = 0$$

Conversely, $I \subset \sqrt{I} \Rightarrow Z(I) \subset Z(\sqrt{I})$.

□

Properties of $\mathcal{O}_{\mathbb{C}^n,0}$: i) It is a UFD, ii) It is Noetherian, i.e. all ideals are finitely generated.

[1]Siebert did not prove the statement that $(Z(I), 0)$ was well defined, and I doubt my own proof here. I used neither the fact that we took germs of $Z(I)$ instead of just $Z(I)$, nor the fact that it was finitely generated. If my proof is correct, then the bijection statement could have been made without any mention of germs, which is probably wrong.

In the future, we will study the basic correspondence we have just outlined:

$$\begin{aligned} \left\{ I \subset \mathcal{O}_{\mathbb{C}^n,0} \mid I = \sqrt{I} \right\} &\longleftrightarrow \{ (Z,0) \subset (\mathbb{C}^n,0) \text{ analytic } \} \\ I &\longmapsto Z(I) \\ \{ f \in \mathcal{O}_{\mathbb{C}^n,0} \mid Z \subset Z(f) \} =: I(Z) &\longleftarrow\!\shortmid (Z,0) \end{aligned}$$

CHAPTER 4

Differential Topology and Local Rings

Lecture 4, Sept 8. If we have a holomorphic function $f : U \subset \mathbb{C}^m \to \mathbb{C}^n$, and adopt the coordinates (z_i) on the domain and (w_i) on the codomain, with $z_i = x_i + iy_i$ and $w_i = u_i + iv_i$, and $f_i = g_i + ih_i$, the Jacobian of f is written as

$$J_{\mathbb{R}}(f)(z) = \begin{pmatrix} \partial_{x_j} g_i & \dots & \partial_{y_j} g_i \\ \vdots & & \vdots \\ \partial_{x_j} h_i & \dots & \partial_{y_j} h_i \end{pmatrix} \in M(2n \times 2m, \mathbb{R})$$

so this is a real matrix, and suppose we want to study its complexification. We recall that $\cdot \otimes_{\mathbb{R}} \mathbb{C}$ is a functor, so that we may apply it to maps as well, and it acts on maps by tensoring with the identity on the second component, so

$$J_{\mathbb{C}}(f)(z) = J_{\mathbb{R}}(f)(z) \otimes_{\mathbb{R}} Id_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}$$

Note[1]: $\det J_{\mathbb{R}}(f)(z) = \det J_{\mathbb{C}}(f)(z)$. We can write down this Jacobian for any real differentiable map. But what if f also happens to be holomorphic? As we discussed in the first lecture[2], we have

$$J_{\mathbb{C}}(f)(z) = \begin{pmatrix} \partial_{z_j} f_i & \dots & \partial_{\overline{z_j}} \\ \vdots & & \vdots \\ \partial_{z_j} \overline{f_j} & \dots & \partial_{\overline{z_j}} \overline{f_j} \end{pmatrix} = \begin{pmatrix} \partial_{z_j} f_i & 0 \\ 0 & \overline{\partial_{z_j} f_i} \end{pmatrix}$$

So the Jacobian is really determined by one, smaller matrix in the upper left, which we will denote $J(f)(z)$. Then of course, $\det J_{\mathbb{C}}(f)(z) = |\det J(f)(z)|^2 \geq 0$. As a result, f is orientation preserving iff it is non-singular, and it cannot reverse orientation. We will see later that this is exactly the reason that complex manifolds are always orientable.
Definition: A map $f : U \to V$, for $U, V \subset \mathbb{C}^n$ open is <u>biholomorphic</u> if it is holomorphic and bijective, and its inverse is holomorphic.

This is the complex analogue of a diffeomorphism. We recall the

[1] I actually don't know why this is. I know there's a formula to tensor two matrices together, but there has to be a better explanation than that. Maybe thinking about det as the map on the top ext power

[2] Maybe it was second I don't remember

Theorem (Inverse Function Theorem:) *If f is a map $f : U \to V$ for $U, V \subset \mathbb{C}^n$, and z is a regular value (or equivalently*[3]*, det $J(f)(z) \neq 0$), then there exist open sets $U', V' \subset \mathbb{C}^n$ such that $f|_{U'}$ is a biholomorphism onto V'.*

Proof: We note that z is a regular value of f iff det $J_{\mathbb{R}}(f)(z) \neq 0$, so the real inverse function theorem tells us that there exists a smooth inverse to f, $g : V' \to U'$. To see that it is also holomorphic, we show it satisfies the CR equations[4]:

$$\begin{gathered}
0 = \partial_{\overline{z_j}}(f^{-1} \circ f) \\
= \sum_{k=1}^{n} (\partial_{w_k} f^{-1} \cdot \partial_{\overline{z_j}} f_k + \partial_{\overline{w_k}} f^{-1} \cdot \partial_{\overline{z_j}} \overline{f_k}) \\
= \sum_{k=1}^{n} \overline{\partial_{z_j} f_k} \cdot \partial_{\overline{w_k}} f^{-1} \\
\Rightarrow J(f) = (\partial_{z_j} f_k)_{ij} \text{ invertible } \Rightarrow \forall\, k \ \ \partial_{w_k} f^{-1} = 0
\end{gathered}$$

so that f^{-1} is holomorphic.

□

Lemma: *If $f : U \to V$ is holomorphic and bijective, then f is a biholomorphism.*

We note that in the real case (replacing holomorphic and biholomorphic with smooth and diffeomorphic, respectively), this is not true. For example, take $y = x^3$.

Theorem (Implicit Function Theorem)[5]**:** *Let $U \subset \mathbb{C}^m \times \mathbb{C}^n$ have coordinates (z, w), and $f : U \to \mathbb{C}^n$ be holomorphic. If $J(f) = (\partial_{w_j} f_i))_{ij}$ is invertible at $(z_0, w_0) \in U$, and $f(z_0, w_0) = 0$, then $\exists\, U_1 \subset \mathbb{C}^m, U_2 \subset \mathbb{C}^n$ open, $U_1 \times U_2 \subset U$, $(z_0, w_0) \in U_1 \times U_2$, $g : U_1 \times U_2$ such that*

$$\{(z, w) \in U_1 \times U_2 \mid f(z, w) = 0\} = \Gamma_g = \{(z, g(z)) \mid z \in U_1\}$$

Proof: Again we have the existence of such a function g from the real version of the implicit function theorem, and apply the chain rule to show that g is in fact holomorphic.

□

We will now work our way up to the analytic Nullstellensatz. We recall that, essentially by definition, $\mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \ldots, z_n\}$.

Definition: A ring is called local if it has a unique maximal ideal.

Lemma: *The ring described above is local.*

[3] It took me about 30 minutes to figure out, but they're equivalent because f is a map from $\mathbb{C}^n \to \mathbb{C}^n$, so the differential is a map between vector spaces of the same dimension, thus surjectivity implies isomorphism. Felt quite silly upon realizing this.

[4] The first equality holds becuase the identity map is holomorphic

[5] Bernd notes that in Germany, during oral exams for undergraduates, students would reliably fail at correctly formally stating the implicit function theorem. Indeed, this exact scenario happened to me in an undergraduate class last year. One part of one of our problems on an exam was to simply state the implicit function theorem, and i only received about half credit on that part.

Proof: Claim the unique maximal ideal is

$$\mathcal{M} := \left\{ f = \sum a_I z^I \mid a_0 = 0 \right\} = \{ f \in \mathcal{O}_{\mathbb{C}^n,0} \mid f(0) = 0 \}$$

This clearly defines an ideal. To show it is maximal, there is a well defined evaluation homomorphism:

$$\begin{aligned} ev_0 : \mathcal{O}_{\mathbb{C}^n,0} &\to \mathbb{C} \\ f &\mapsto f(0) = a_0 \end{aligned}$$

We see that the quotient $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{M} \cong \mathbb{C}$, and we recall that an ideal is maximal iff the quotient ring is a field, which $\mathbb{C}$ is.

□

We note that this property of locality is not unique to holomorphic functions: Smooth and continuous functions also share this property, for example.

Lemma: *$\mathcal{O}_{\mathbb{C}^n,0}$ is an integral domain*

Proof: There is a tedious, algebraic method to prove this lemma, but the geometric way is cleaner, via the identity theorem. WLOG, we may assume $f, g \in \mathcal{O}(D_\epsilon)$. Then if $f \not\equiv 0$, $\exists\, z \in D_\epsilon$ such that $f(z) \neq 0 \Rightarrow \exists D_{\epsilon'}(z)$ such that $f|_{D_{\epsilon'}(z)} \neq 0$. But then $g|_{D_{\epsilon'}(z)} \equiv 0$, so the identity theorem shows $g = 0$.

□

Lemma: *$\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

Idea of Proof: The Wierstrass Preparation Theorem reduces this proof to a known statement about polynomials.

□

Definition: A hypersurface is the set of zeros $Z(f) \subset U \subset \mathbb{C}^n$ for $f \in \mathcal{O}(U \setminus \{0\})$. The germ of a hypersurface is then the germ of the zero locus.

Corollary: *Any germ of hypersurfaces decomposes uniquely into germs of irreducible hypersurfaces.*

Proof: If we decompose f into irreducibles, $f = g_1 \dots g_r$, then $Z(f) = \cup Z(g_i)$. Then we must show the $Z(g_i)$ are irreducible. To do so, we must use a lemma about hypersurface containment:

Lemma: *If $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ and g irreducible, then $Z(g) \subset Z(f) \Rightarrow g \mid f$.*

With this lemma, the result can be found.

□

CHAPTER 5

Lec 5

Lecture 5, Sept 10. I missed lecture 5

CHAPTER 6

Noether Normalization

Lecture 6, Sept 15. **Theorem (Local Finite Mapping/Noether Normalizing):** *Let $X \subset (\mathbb{C}^n, 0)$ be a germ of analytic sets. Then there exists $d \leq n$, and a linear coordinate system $(z_1, \ldots, z_d, z_{d+1}, \ldots, z_n)$ such that the projection*

$$\pi : \mathbb{C}^n \to \mathbb{C}^d, \ \ (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_d)$$

induces a proper, open surjection $X \to (\mathbb{C}^d, 0)$ with finite fibers, i.e. is a branched cover. We refer to d as the dimension of X.

Non-example: Take $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ such that $(x, y, z) \mapsto (x, y)$, and take $X = Z(y - zx)$. Then X is irreducible, but π does not have finite fibers: $\pi^{-1}(0, 0) = \{0, 0\} \times \mathbb{C}$. This is an example of a blow-up. The theorem does not apply because $\mathcal{O}_{\mathbb{C}^2,0} \to \mathcal{O}_{X,0}$ is not an integral ring extension, which is a necessary step for the algebraic proof of the above theorem.

A priori, we may believe d depends on the choice of projection. There are 3 ways to show that this notion of dimension is well defined: a) We can show that outside of a nowhere dense, analytic subset, this is an unbranched covering. Then working locally, if we assume there is another covering π' of dimension d', then we have a diffeomorphism from a subset of $\mathbb{C}^d$ to $\mathbb{C}^{d'}$, and we are done. For the second method, we define

Definition: The Krull dimension of a commutative ring R is the supremum of the lengths of all chains of prime ideals:

$$\dim R = \sup\{k \mid f_0 \subset f_1 \subset \cdots \subset f_k \in \mathcal{O}_{X,0}\}$$

where f_i are prime ideals, and the containments are proper

So 0-dimensional rings correspond to maximal ideals. If you take a hypersurface inside X, that has codimension 1, then you continue until you reach an irreducible component. So if X is irreducible, $\mathcal{O}_{X,0}$ is an integral domain. The final way of showing this is related to stratifications.

Definition: If X is an analytic subset, then the singular locus

$$X_{\text{sing}} = \{x \in X \mid X \text{ is not a submfld at } x\} \subset X$$

is the locus where X is not a submanifold.

X_{sing} can have any codimension. For example, take two copies of $\mathbb{C}^2$ and have them meet at the origin of $\mathbb{C}^4$ transversely, this has codim 2. Of course, the containment $X_{\text{sing}} \subset X$ must be a proper containment. Then we may take the series of successive singularities: the next step would be $(X_{\text{sing}})_{\text{sing}}$. Each set is also analytic, thought that must be proved, and the dimension decreases with each step, so this series will always terminate. Then we define the dimension of X to be complex dimension dim $X \setminus X_{\text{sing}}$.

So now we have an idea of what the zero loci of holomorphic functions in severeal comlex variables looks like. They are closed, they admit these stratifications, locally they are coverings of domains in $\mathbb{C}^d$, and they have dimension d.

Onto complex manifolds.

CHAPTER 7

Complex Projective Space

Lecture 7, Sept 17. We will start with projective space. Bernd remarks that $\mathbb{C}P^n$ is arguably the most important complex manifold, even moreso for compact complex manifolds. As a set,

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\}) \Big/ \mathbb{C}^*$$

where $\mathbb{C}^*$ acts diagonally. This induces the quotient topology. We note there is a surjection $S^{2n+1} \to \mathbb{C}P^n$, the antipodal map, which implies the latter is compact. The standard charts on $\mathbb{C}P^n$ is

$$U_i := \{[z_0, \dots, z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$$

Clearly these U_i cover $\mathbb{C}P^n$, and re biholomorphic to $\mathbb{C}^n$ via

$$\varphi_i : U_i \to \mathbb{C}^n \text{ parameterized by } w_0, \dots \hat{w}_i, \dots, w_n$$
$$[z_0, \dots, z_n] \mapsto \frac{1}{z_i}(z_0, \dots, \hat{z}_i, \dots, z_n)$$

It is easily seen that the transition functions are biholomorphic.
Similarly, if V is a finite dimensional complex vector space, then can define $\mathbb{C}P(V)$ via charts, but we then need to worry about change of basis. If we have a change of basis $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, i.e. a linear isomorphism, we get a functor from complex vector spaces to comlex manifolds, $\mathbb{P}$, which sends

$$T \mapsto \mathbb{P}(T) : \mathbb{C}P^n \to \mathbb{C}P^n$$

is a biholomorphism. Note we induce a map on the quotient simply by linearity. Another way to view this, we have a group homomorphism

$$F : GL(n+1, \mathbb{C}) \to \{\text{biholomorphisms of } \mathbb{C}P^n\}$$

What is Ker F? Clearly, all the scalar diagonal matrices, i.e. those of the form cA for $c \in \mathbb{C}^*$ and $A \in GL(n, \mathbb{C}^{n+1})$, because multiplication by a scalar prerves the $\mathbb{C}P^n$ class, by definition. It turns out this is the entire kernel, and the quotient by this normal subgroup Ker F is called the projective linear group, $PGL(n+1, \mathbb{C}^{n+1})$.

There is an easy way to write down submanifolds of projective space via the homogeneous coordinates.

Definition: If $f \in \mathbb{C}[z_0, \ldots, z_n]$, then it is called homogeneous of degree d if $\exists d$ s.t. $\forall\, \lambda \in \mathbb{C}^*$,

$$f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n)$$

If f is as above, we may now define the zero locus $Z(f) := \{[z_0, \ldots, z_n] \in \mathbb{C}P^n \mid f(z_0, \ldots, z_n) = 0\}$. Note this is well defined because f is homogeneous: if

$$f(z_0, \ldots, z_n) = 0$$
$$\Rightarrow f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n) = 0$$

Also note that f cannot be a function on $\mathbb{C}P^n$, by the maximum principal: a non-constant holomorphic function does not have local maxima. But because $\mathbb{C}P^n$ is compact, every f would achieve a maximum, and thus must be constant. f would be holomorphic, since it is a polynomial in z_i, thus it must not be a function[1]. Locally, we can easily compute its zero locus via charts: on U_i, $\varphi_i(Z(f) \cap U_i) = Z(f_i)$, where $f_i(w_0, \ldots, \hat{w}_i, \ldots, w_n) = f(w_0, \ldots, 1, \ldots, w_n)$ with the 1 in the ith index. This f_i is the "dehomogeneized" form of f.

Definition: A complex hypersurface in $\mathbb{C}P^n$ is a set of the form $Z(f)$ for $f \in \mathbb{C}[z_0, \ldots, z_n] \setminus \mathbb{C}$. As usual, given an ideal of homogeneous elements, $S \subset \mathbb{C}[z_0, \ldots, z_n]$, we have a corresponding zero locus,

$$Z(S) := \bigcap_{f \in S} Z(f)$$

As before, these are analytic subsets, and the implicit function theorem shows that these are also submanifolds. We note that because $\mathbb{C}[z_0, \ldots, z_n]$ is Noetherian, all ideals are finitely generated. Then because the zero locus of S is the same as the zero locus of the ideal it generates then pick a finite generating set, WLOG, we can treat S as being finite.

Theorem (Chow): *$X \subset \mathbb{C}P^n$ is analytic $\Rightarrow X = Z(S)$ for some S.*

i.e. every $\mathbb{C}$-analytic set in $\mathbb{C}P^n$ is algebraic, i.e. defined globally by the zero set of homogeneous polynomials.

Question: Is every complex manifold algebraic?

In algebraic geometry, we look at the field $K(X)$, the field of rational functions on X, with transcendence degree[2] $\dim X$.

In complex analysis, there is a similar idea:

Definition: A meromorphic function on a complex manifold X is an equivalence class of pairs (U, f) with $U \subset X$ open and f a holomorphic function on U, $X \setminus U$ analytic and nowhere dense, such that $\forall\, x \in X, \exists\, V \ni x \subset X$ open, $g, h \in \mathcal{O}(V)$ such that h has no zeros on $V \setminus U$ and

$$f|_{V \cap U} = \frac{g}{h}\Big|_{V \setminus U}$$

[1] I think this means that it would not be a well defined function, despite the fact that its zero locus is well defined. I don't know how else it could fail to be a function.

[2] don't really know what this is.

where $(U,f) \sim (V,g) \iff f|_{U\cap V} = g|_{U\cap V}$. Here we denote $\mathcal{K}(X)$ as the field of meromorphic functions on X.

Rather long winded definition, but the idea is that locally these should just look like quotients of holomorphic functions. Recall some of the behavior of single variable complex functions near singularities: Near an essential singularity, the function maps a neighborhood to the entire plane minus one or two points. Near a pole, you experience a blow up, and the Riemann extension theorem says that if you don't blow up[3], then the singularity is removable.
Example: Take $f = \frac{z}{w}$ on $\mathbb{C}^2_{z,w}$. Clearly there is a pole along the $w = 0$ line. If you approach the horizontal axis ($w = 0$) from a vertical direction, you will get infinity. However, if you approach from an angle suffiiciently close to vertical, say approaching the origin following the square root function, you get 0. So singularities are more complciated in multiple dimensions, because you cannot resolve this issue even by compactifying.

We note that multiplication on $K(X)$ is given by

$$(U,f)\cdot(V,g) = (U\cap V, f\cdot g)$$

and inversion given by

$$(U,f)^{-1} = (U\setminus Z(f), f^{-1})$$

and one can check that this is compatible with the equivalence classes.
Definition: The algebraic dimension of X a compact complex manifold is the transcendence degree of $\mathcal{K}(X)$.

So given X, one can look at the field of rational functions, which is a subfield of $\mathcal{K}(X)$. It is not clear that these are the same, but another version of Chow's theorem says that any rational function is algebraic, so indeed they are the same field, i.e. that any meromorphic function can be written as the quotient of two homogeneous polynomials of the same degree. So
Theorem: (Remmert-Siegel) *If X is a compact, complex manifold, then the algebraic dimension is the transcendence degree of $\mathcal{K}(X)$, which is less than or equal to $\dim X = n$.*
Proof sketch: Cover the manifold with polydiscs. Then we take $f_1,\dots,f_{n+1} \in \mathcal{K}(X)$. Then we want to build an $F \in \mathbb{C}[x_1,\dots,x_{n+1}]$ such that $F(f_1,\dots,f_{n+1}) = 0$ and has zeros of high order at the centers of the polydiscs. So say the degree of F is m, and the order of vanishing is, say, m'. Then we recall the Schwarz' lemma: If you have a holomorphic function on the disk which vanishes at the origin to order k, then you can bound the growth of the function polynomially, with coefficients the norm of f. So for each polydisc, there is a smaller polydisc on which we can estimate the growth of F. You then argue that you can choose m and m' such that the Schwarz lemma must give a contradiction unless $F(f_1,\dots,f_{n+1}) = 0$, as desired.

In the next lecture, we will go through this proof in more detail.

[3] ie the function is locally bounded near all of its zeros

CHAPTER 8

Proof of Remmert-Siegel

Lecture 8, Sept . We will now go through a proof of the previous theorem. So first, we pick f_i as $n+1$ meromorphic functions on X. As we discussed last time, these functions locally look like quotients of holomorphic functions, i.e. $\forall\ x \in X, \exists\ U_x \subset X$ open such that $x \in U$ and $f_i|_{U_x} = \frac{g_{i,x}}{h_{i,x}}$, for some $g_{i,x} \in \mathcal{O}(U_x)$ and $h_{i,x} \in \mathcal{O}(U_x) \setminus \{0\}$. We also know if $g_{i,x}, h_{i,x} \in \mathcal{O}_{X,x}$ we may assume g and h are relatively prime, since $\mathcal{O}_{X,x}$ is a UFD. So at a point, the germs of these functions can be taken to be relatively prime.

Lemma: *If $g_{i,x}$ and $h_{i,x}$ are relatively prime at $\mathcal{O}(X,x)$, by possibly shrinking U_x, it holds that they are relatively prime on the whole neighborhood U_x, i.e. $\forall y \in U_x$, they are relatively prime in $\mathcal{O}_{X,y}$.*

Now the U_x give an open covering of X. By shrinking U_x to a chart, we choose refinements of $\{U_x\}$ such that

$$\begin{array}{ccccccccc}
\{x\} & \hookrightarrow & W_x & \hookrightarrow & V_x & \hookrightarrow & \overline{V_x} & \hookrightarrow & U_x \\
 & & \downarrow \wr & & \downarrow \wr & & & & \\
 & & D_{\frac{1}{2}}(0) & \hookrightarrow & D_1(0) & & \longrightarrow & & \mathbb{C}^n
\end{array}$$

Then because X is compact, the open cover $\{W_x\}$ has a finite subcover, $\{W_{x_i}\}$, with centers $x_1, \ldots, x_N$, and we forget the double indexing, so define $\{W_k\} = \{W_{x_k}\}$, and the same for the V_x, and g and h. Now we will construct the F. We want it to satisfy

$$\partial^I F\left(\frac{g_{1,k}}{h_{1,k}}, \ldots, \frac{g_{n+1,k}}{h_{n+1,k}}\right)(x_k) = 0 \quad \forall |I| < m'$$

so all the holomorphic derivatives, that is derivatives wrt z_i, not $\overline{z_i}$, as those vanish anyway, vanish. On the intersection $U_k \cap U_l$,

$$\frac{g_{i,k}}{h_{i,k}} = \frac{g_{i,l}}{h_{i,l}}$$

$$\Rightarrow h_{i,k} = h_{i,l} \cdot \varphi_{i,kl} \text{ where } \varphi_{i,kl} := \frac{g_{i,k}}{g_{i,l}}$$

We then claim that $\varphi_{i,kl}$ has no zeros or poles[1]. To see this, take some $y \in U_k \cap U_l$. Then by unique factorization, $\varphi_{i,kl} = \frac{\alpha}{\beta}$ for α, β coprime in $\mathcal{O}_{X,y}$. So $\varphi_{i,kl}$ is a unit in $\mathcal{O}(U_k \cap U_l)$. We then define

$$\varphi_{kl} = \prod_{i=1}^{n+1} \varphi_{i,kl}$$

which is holomorphic and bounded[2] on $\overline{V}_k \cap \overline{V}_l$. Also define

$$C := \max\{\|\varphi_{kl}\|\}_{k,l=1,\dots,N}$$

where $\|\cdot\|$ is the sup norm on $\overline{V}_k \cap \overline{V}_l$. Becuase this is closed, we have a max instead of sup. We note

$$\varphi_{kl} \cdot \varphi_{lk} = \left(\prod_{i=1}^{n+1} \frac{g_{i,k}}{g_{i,l}}\right) \cdot \left(\prod_{i=1}^{n+1} \frac{g_{i,l}}{g_{i,k}}\right) = 1$$
$$\Rightarrow C \geq 1$$

Generally, we are going to use this φ_{kl} to clear denominators. Specifically, if we have $F\left(\frac{g_{1,k}}{h_{1,k}}, \dots, \frac{g_{n+1,k}}{h_{n+1,k}}\right)$ of degree m, then define $G_k \in \mathcal{O}(U_k)$ by

$$F = \frac{G_k}{\left(\prod_{i=1}^{n+1} h_{i,k}\right)^m}$$

We have that

$$G_k = \varphi_{lk}^m G_l$$

Now, assume we have

$$\binom{m+n+1}{m} > N\binom{m'-1+n}{m'-1}$$

and call this the $(*)$ condition. Claim $(*)$ implies there exists F of degree m satisfying

$$\partial^I F\left(\frac{g_{ik}}{h_{ik}}\right)(x_k) = 0$$

Indeed this is just a dimension count. There will be a nonzero kernel, so we can choose such an F. Now we may apply Schwarz' lemma, which says

$$f \in \mathcal{O}(\overline{D_\epsilon(0)}), \partial_z^k f(0) = 0 k = 0, \dots, m'-1$$
$$\Rightarrow \forall\, z \in \overline{D_\epsilon(0)},\ f(z) \leq \|f\| \cdot \left(\frac{|z|}{\epsilon}\right)^{m'}$$

[1] Can you not just see this from the equality $h_{i,k} = h_{i,l} \cdot \varphi_{i,kl}$? If φ had any zeros/poles, either $h_{i,k}$ has to have them as well, or $h_{i,l}$ has to cancel them somehow, which also shouldn't be possible.

[2] I think this is because we showed there are no poles/zeros.

This holds for higher dimensions. So we have that

$$\partial^I F(\dots)(x_k) = 0 \Rightarrow \partial^I G_k(x_k) = 0 \forall\, k, \forall\, |I| < m'$$
$$\Rightarrow \forall\, x \in \overline{W}_k,\ \ |G_k(x_k)| \le \left(\frac{1}{1}\right)^{m'} \cdot C',\ \ C' := \|G_k\|\,|_{W_k}$$

Now fix k, and $x \in \overline{V}_k$, which may not be in W_k, and write $C' = |G_k(x)|$. Now, X is covered by W_l, so $x \in W_l$ for some l. But on W_l, we have the estimate

$$C' = |G_k(x)| = |G_l(x)| \cdot |\varphi_{lk}^m(x)| \le \frac{C'}{2^{m'}} \cdot C^m$$
$$\Rightarrow \left|1 - \frac{C^m}{2^{m'}}\right| C' \le 0$$

this inequality will lead to a contradiction unless $G_k \equiv 0$ if we can show the first term is strictly larger than 0, forcing $C' = 0$. To do that, need to find m, m' such that

$$\frac{C^m}{2^{m'}} < 1$$

So because $C \ge 1$, we may write it as $C = 2^\lambda$ for some $\lambda \in \mathbb{R}_{\ge 0}$. Then

$$\frac{C^m}{2^{m'}} = 2^{\lambda m - m'} < 1 \iff m' > \lambda m$$

Recall we also wanted m and m' to satisfy

$$\binom{m+n+1}{m} > N \binom{m'-1+n}{m'-1}$$

Writing out the binomial expansion, one would see

$$\exists\, a, b \mid \binom{m+n+1}{m} > a \cdot m^{n+1} \ \text{ for } m >> 0$$
$$N \cdot \binom{m'-1+n}{m'-1} < b \cdot (m')^n \text{ for } m' >> 0$$

Now take $m' = 2\lambda m$, so we satisfy $m' > \lambda m$. Then we can show

$$(*) \Leftarrow b \cdot (2\lambda)^n m^n = \frac{b \cdot (2\lambda)^n}{m} \cdot m^{n+1} < a \cdot m^{n+1}$$

The inequality follows because we can choose m so large that the coefficient in front of m^{n+1} because arbitrarily small. Working backwards from this conclusion, we can construct a proof of the theorem.

□

We may consider complex tori $\pi : \mathbb{C}^n \setminus \mathbb{Z}^{2n}$, endowed with the quotient topology. In fact, π is a local homeomorphism, making the complex torus a complex manifold. More generally, if V is a complex vector space, and Γ is a free abelian discrete group of maximal rank $2n$. Then under a basis $b : V \to \mathbb{C}^n$,

$$\Gamma \mapsto \mathbb{Z}^n + \mathbb{Z}\tau_1 + \cdots + \mathbb{Z}\tau_n$$

Then as a $\mathbb{C}^\infty$ manifold, $X \cong (S^1)^n$, but $C^n \setminus \Gamma \cong \mathbb{C}^n \setminus \Gamma' \iff \exists\, A \in GL(n, \mathbb{C})\ \ A(\Gamma) = \Gamma'$.

CHAPTER 9

Complex Tori and Holomorphic Group Actions

Lecture 9, . Weierstrass $\mathfrak{p}$-functions: We recall the Louiville theorem, which states that any bounded holomorphic function on $\mathbb{C}$ is a constant. We want to consider periodic functions, but by Louiville, that would give us a function on the torus, which is bounded, and thus constant. However, meromorphic functions don't have this restriction, so we examine some such functions. One such function is the Weierstrass $\mathfrak{p}$-function, given by, for $\Gamma = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$

$$\mathfrak{p}(z) := \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \quad z \in \mathbb{C}, \notin \Gamma$$

So we create a function with second order poles at each ω, with an essential singularity at the origin. It is not obvious that this converges, but it does. It is also Γ-periodic: $\forall\, \omega \in \Gamma, \mathfrak{p}(z+\omega) = \mathfrak{p}(z)$. It is easy to see its derivative is Γ periodic:

$$\mathfrak{p}'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}$$

But since the original function is just the integral of a periodic function, it is also periodic. So these are two meromorphic functions on a 1-dimensional manifold. By Remmert-Siegel, then, these are algebraically dependent. In fact,

$$\mathfrak{p}'^2 = 4\mathfrak{p}^3 - \mathfrak{g}_2\mathfrak{p} - \mathfrak{g}_3$$

where

$$\mathfrak{g}_2 = \mathfrak{g}_2(\tau) = 60 \cdot \sum_{(m,n) \neq (0,0)} \frac{1}{(m+n\tau)^4} \quad \in \mathcal{O}(\mathbb{H})$$
$$\mathfrak{g}_3 = \mathfrak{g}_3(\tau) = 140 \cdot \sum_{(m,n) \neq (0,0)} \frac{1}{(m+n\tau)^6} \quad \in \mathcal{O}(\mathbb{H})$$

These are called the elliptic modular forms of weight 4 and 6. Now, we may define the map $\mathbb{C} \setminus \Gamma \to \mathbb{C}^2 \to \mathbb{C}P^2$ via

$$z \mapsto (\mathfrak{p}(z), \mathfrak{p}'(z)) \mapsto [\mathfrak{p}(z), \mathfrak{p}'(z), 1]$$

which extends to a map from $\mathbb{C} \to \mathbb{C}P^2$. We can also quotient $\mathbb{C}$ by Γ, call the result E, then we induce a map $E \to \mathbb{C}P^2$. Then we have
Theorem: *$E = \mathbb{C} \setminus \mathbb{Z} + \mathbb{Z}\tau, \tau \in \mathbb{H}$ is biholomorphic to the plane cubic curve $C \subset \mathbb{C}P^2$ with*

$$C \cap \mathbb{C}^2_{x,y} = z(y^2 - 4x^3 - \mathfrak{g}_2(\tau)x - \mathfrak{g}_3(\tau)$$

This is all summarized in the diagram

$$\begin{tikzcd}
\mathbb{C} \setminus \Gamma \arrow[rr] \arrow[d] & & \mathbb{C}^2 \arrow[d] \\
\mathbb{C} \arrow[rr] \arrow[d] \arrow[urr, dashed, "\psi"] & & \mathbb{C}P^2 \arrow[d] \\
E \arrow[r] \arrow[urr, dashed, "\varphi"] & \{[x,y,z] \mid yz^2 = 4x^2 - g_2xz^2 - g_3z^3\} &
\end{tikzcd}$$

Then we may define
Definition: An elliptic curve is a smooth plane cubic curve, along with the data of a point on the curve, e, which corresponds to the unit in the torus.

"Definition": A complex lie group is a lie group where the word differentiable is replaced with the word complex/holomorphic.
Examples: a) Linear groups: $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{Sp}(n,\mathbb{C})$. Note $U(n,\mathbb{C})$ is not a complex Lie group, because the definition requires complex conjugation, which is not a holomorphic operation.

These are non-compact and non-abelian if $n > 1$.
b) $X = \mathbb{C}^n \setminus \Gamma$ is an abelian, compact, complex Lie group.
c) Finite and discrete groups is a 0-dimensional complex lie group.
Proposition: *If G is a compact, complex Lie group, then G is abelian.*
Proof: For $g \in G$, consider

$$\Phi_g : G \to G, \quad h \mapsto g^{-1} \cdot h \cdot g$$

Then G is abelian iff $\forall g \in G, \Phi_g = Id_G$. Denote $\mathfrak{g}$ as the Lie algebra of G. Then apply the maximum principle to the components of

$$\begin{gathered} Ad : G \to \mathrm{Aut}(\mathfrak{g}) \cong \mathrm{GL}(n,\mathbb{C}) \\ g \mapsto (\Phi_g)_A|_e \end{gathered}$$

This is a holomorphic map, so it must be constant. $Ad_e = Id$, so $Ad = Id_G$. Because G is compact, the exponential map is surjective. It also commutes with Φ_g, so that

$$\Phi_g(\exp(\xi)) = \exp((\Phi_g)_*\xi) = \exp\xi$$

so that Φ is the identity.

$\square$

If X is a topological space with a continuous group action by G, then $X \to X \setminus G$ makes $X \setminus G$ into a topological space. Two important notions we need are
Definition: An action is free if every stabilizer group trivial.
Definition: If G is a topological group, the action is proper if the map

$$\begin{gathered} G \times X \to X \times X \\ (g, x) \mapsto (x, g \cdot x) \end{gathered}$$

is proper.

If G is compact, of course all actions are proper, but if G is not, then the notion of being proper is interesting.
Theorem: *If G is a complex Lie group which acts freely and properly on a complex manifold, X, then X/G is naturally a complex manifold, and the quotient map is holomorphic. In fact, $\pi : X \to X/G$ is a principal G-bundle.*

Examples of the previous theorem: a) The action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ which results in $\mathbb{C}P^n$.
b) The action of a full rank lattice Γ on $\mathbb{C}^n$ that results in an elliptic curve.

CHAPTER 10

Holomorphic group actions II

Lecture 10, Sept 29. **Theorem:** *Let $G \times X \to X$ be a proper and free topological group action. Then*
i) X/G is Hausdorff.
ii) $q : X \to X/G$ is a principal G-bundle.

Proofs of this fact are more available in the smooth and holomorphic categories. We have $\forall\ x \in X,\ q^{-1}(x) = \{G \cdot x\} \cong G$. The idea is then to find a "slice" of the group action, Z, which is a submanifold of X which hits each fiber in only one point, and has complimentary dimension to the fiber, i.e. $\dim q^{-1}(x) + \dim Z = \dim X$. So locally we are just looking for a transverse submanifold to $\{G \cdot x\}$, which is done by splitting your chart into $\mathbb{R}^{n-k}$ and $\mathbb{R}^k$, but showing this fully is non-trivial.

Example:
i) $\mathbb{C}P^n = \left(\mathbb{C}^{n+1}/\{0\}\right)/\mathbb{C}^*$
ii) the complex tori $\mathbb{C}^n \to \mathbb{C}^n/\Gamma$

Non-Example: For $\alpha \in \mathbb{R} \setminus Q$, let $\mathbb{Z} \curvearrowright S^1 = \mathbb{R}/\mathbb{Z}$ by $x \mapsto x + n\alpha$. In this action, all orbits are dense.
Proposition: *If G is discrete, X locally compact, then the action $G \curvearrowright X$ is proper and free iff G acts properly discontinuously, i.e. $\forall\ x \in X, \exists U \ni x$ such that $\forall g \in G \setminus \{e\}, g(U) \cap U = \emptyset$*

Example: Linear $\mathbb{C}^*$ actions: $\mathbb{C}^* \curvearrowright \mathbb{C}^n$ by

$$\lambda(z_1, \ldots, z_n) = (\lambda^{a_1} z_1, \ldots, \lambda^{a_n} z_n)$$

for integers a_i. When is $\mathbb{C}^n/\mathbb{C}^*$ a complex manifold?
Clearly if we leave the origin in, the result is not Hausdorff, because for any U containing the origin, and for any $\lambda \in \mathbb{C}^*$, $0 \in \lambda \cdot U \cap U$, so the action cannot be proper and free, and the result is non Hausdorff. If we examine the $n = 2$ case, the case $a_2 < 0$, then $\mathbb{C}^2/\mathbb{C}^* \cong \mathbb{C}P^1 \sqcup \{*\}$, which is non-Hausdorff, and $(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^* \cong \mathbb{C}P^1$. If we have $a_1 < 0, a_2 = 0$, then we collapse one of the axes, namely every horizontal line in $\mathbb{C}^2$ collapses down to a point in the vertical line at the origin, yielding $\mathbb{C}^2/\mathbb{C}^* \cong \mathbb{C} \sqcup \mathbb{C}$, so you would have to remove the whole axis to get something Hausdorff, namely $\mathbb{C}$. If one is positive and one negative, then you get $\mathbb{C}P^1 \sqcup \{*\}$, as before. These are under

the umbrella of geometric invariant theory (GIT), where we look at a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, and let G act on it by acting on $\mathbb{C}^n$. Typically in GIT we take G to be a complex reductive group, which is a group which is the complexification of a compact, real Lie group, for example GL or SL. These act on $V = \mathbb{C}^n$, and the quotient V/G is typically not a nice space. To fix this, you restrict to the stable locus, $\forall \lambda : \mathbb{C}^* \to G$ a homomorphism, $V^S := \{v \in V \mid \lambda(v) \equiv O_v \text{ is closed } \Rightarrow G_v \text{ is finite}\}$, and then V^S/G is typically a nice space.

We recall the uniformization theorem in 1D:
Theorem (Uniformization): *If Σ is a simply connected Riemann surface, it is conformally equivalent to either the open unit disk, the complex plane, or $\mathbb{C}P^1$.*

If we take Σ to be a Riemann surface with universal cover D, the open unit disk, then there is a deck transformation group, Γ which acts on D, such that

$$\Sigma = \mathbb{H}/\Gamma \cong D/\Gamma$$

But as we know,

$$\mathrm{Aut}(D) = SU(1,n) \cong SL(2,\mathbb{R})/\{\pm 1\}$$

with Γ as a discrete subgroup, and the group itself acts as mobius transformations. Such an example is called a ball quotient.

If we have a closed Riemann surface, C, and $E = \mathbb{C}/\Gamma$ is an elliptic curve, and $G \in (E, +)$ is a finite group acting on $\mathbb{C}$, then $G \times (C \times E) \to C \times E$ by

$$(g, (x, y)) \mapsto (g \cdot x, y + g)$$

is a free and proper action.
Definition: A hyper-elliptic curve is a Riemann surface which arises as a 2-to-1 cover of $\mathbb{C}P^1$.

Of course, for any double branched cover, there is a $\mathbb{Z}_2$ action swapping the branches, and every elliptic curve has a $\mathbb{Z}_2$ action sending $z \mapsto -z$.

If we define $X := (C \times E)/G$, then $\pi : X \to C/G$ is holomorphic, with $\pi^{-1}([z]) \cong E/G_z$, and π is an example of what we call an elliptic fibration.

CHAPTER 11

Calculus and Holomorphic Vector Bundles

Lecture 11, Oct 1. Some more examples of quotient manifolds:
Example (Hopf manifolds) The elliptic curve arose as a quotient of $\mathbb{C}^*/\mathbb{Z}$. We can generalize this to several variables in the following way: Given $\lambda \in \mathbb{C}^*$, $|\lambda| > 1$, there is an action $\mathbb{Z} \times \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}$ by

$$k \cdot (z_1, \ldots, z_n) \mapsto (\lambda^k z_1, \ldots, \lambda^k z_n)$$

This action is proper and free. It looks like this is just a product of n elliptic curves, but this is an action of $\mathbb{Z}$, not $\mathbb{Z}^n$. So we define $X := (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$. Rather than a product of elliptic curves, there is a diffeomorphism $X \cong S^1 \times S^{2n-1}$. This construction is useful because, as a real manifold, it is not clear why $S^1 \times S^{2n-1}$ should have a complex structure, yet this diffeomorphism makes it clear by our previous theorem.

Example (Iwasawa manifolds) The real Heisenberg group is given by matrices of the form

$$H := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \; z_i \in \mathbb{R} \right\} \subset \mathrm{GL}(3, \mathbb{R})$$

and the complex Heisenberg group is the same, but with complex coefficients, $\cong \mathbb{C}^3$. By restricting our coefficients to the Gaussian integers, $\mathbb{Z} + \mathbb{Z}i$, we get a discrete subgroup $\Gamma = H \cap \mathrm{GL}(3, \mathbb{Z} + \mathbb{Z}i)$, with an action on H by multiplication. Of course, any such action is free, but it is also proper, so that $X := H/\Gamma$ is a complex manifold, called an Iwasawa manifold. We may define a projection $\pi : X \to E \times E$, where $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$. In that case, π is a holomorphic fiber bundle with fibers E, but $X \ncong E \times E \times E$.

Example (Grassmannians): We define

$$G(k, n) := \{W \subset \mathbb{C}^n \text{ linear subspace with } \dim W = k\}$$

Note $G(1, n) \cong G(n-1, n) \cong \mathbb{C}P^{n-1}$. So the first interesting[1] Grassmannian is $G(2, 4)$, and it is the only one with $n = 4$. In general, how do we get charts? $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $G(k, n)$, so it suffices to check on $W := \mathbb{C}^k \times \{0\}$. An element of $G(k, n)$ near

[1] As in, not projective space

W is viewed as the graph of a map from a complimentary (not necessarily orthogonal) subspace of dimension $n-k$, so that we have charts given by

$$M((n-k)\times k, \mathbb{C}) \to G(k,n)$$
$$A \mapsto \operatorname{im}\left(\begin{pmatrix} E \\ A \end{pmatrix} : \mathbb{C}^k \to \mathbb{C}^n\right)$$

it can be checked that the transition functions are holomorphic.

We have the Plucker embedding: If $L \in G(k,n)$ is spanned by linearly independent vectors $v_1, \ldots, v_k \in \mathbb{C}^n$, we can define a map

$$G(k,n) \to \bigwedge^k \mathbb{C}^n$$
$$L(v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k$$

It is clear that changing the basis $v_1, \ldots, v_k$ only have the effect of multiplying $v_1 \wedge \cdots \wedge v_k$ by a constant, so we get a map on the projective space:

$$G(k,n) \to \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right) \cong \mathbb{C}P^{\binom{n}{k}-1}$$
$$L(v_1, \ldots, v_k) \mapsto [v_1 \wedge \cdots \wedge v_k]$$

There are still details to check, but this Plucker map is a holomorphic embedding, thus by Chow theorem, this manifold must be algebraic, given by the so-called Plucker relations: The easiest example is

$$G(2,4) \cong V(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}) \subset \mathbb{C}P^5$$

Finally, we discuss calculus on manifolds: If we take M to be a complex manifold, $T_{\mathbb{R}}M$ is the real tangent bundle, which is locally spanned by, taking local coordinates $z_i = x_i + iy_i$,

$$\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}\}$$

If we complexify this, we get the holomorphic vector bundle

$$T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C} \cong TM \oplus \overline{TM}$$

with local coordinates

$$\partial_{z_\mu} = \frac{1}{2}(\partial_{x_\mu} - i\partial_{y_\mu}) \quad \text{spans } TM$$
$$\partial_{\bar{z}_\mu} = \frac{1}{2}(\partial_{x_\mu} + i\partial_{y_\mu}) \quad \text{spans } \overline{TM}$$

It turns out that $T_{\mathbb{C}}M$ is a holomorphic vector bundle of rank n, the complex dimension of M, and the above is a local frame.
Dually, we have $T^*_{\mathbb{R}}M$, spanned by $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$, and we complexify

$$T^*_{\mathbb{C}}M = T^*M \otimes \mathbb{C} \cong T^*M \oplus \overline{T^*M} \equiv T^{1,0}M \oplus T^{0,1}M$$

Then the (p,q)-forms are defined as

$$T^{(p,q)}M = \bigwedge^p T^*M \otimes_{\mathbb{C}} \bigwedge^q \overline{T^*M}$$

This is different from real differential forms because there, all components are the same, i.e. we can define k-forms, whereas here p denotes the number of holomorphic components, and q denotes the number of anti holomorphic components. We can of course define integration of these forms by integrating the real and imaginary part separately.

'Definition': A <u>holomorphic vector bundle</u> is a vector bundle in the holomorphic category.
So π is a map of complex manifolds, and local trivializations are biholomorphic.

Example: The tautological line bundle is defined by taking a finite dimensional complex vector space, V, and noting $\mathbb{P}(V) = \{\ell \subset V \mid \dim \ell = 1\}$, we may define the tautological line bundle over $\mathbb{P}(V)$, $L \to \mathbb{P}(V)$, as

$$\begin{array}{ccc} L = \{(\ell, v) \subset \mathbb{P}(V) \times V \mid v \in \ell\} & \qquad & (\ell, v) \\ \downarrow & & \downarrow \\ \mathbb{P}(V) & & \ell \end{array}$$

Line bundles, of which the tautological line bundle is of course one, have the property that tensoring them together yields another line bundle, leading us to define the Picard group as the set of line bundles modulo isomorphism under the operation of tensor product. This is an abelian group, although $L \otimes L'$ need not be equal to $L' \otimes L$, they may not even type check, but they are always isomorphic. So the Picard group of a manifold, Pic(X), is the group of line bundles.
Proposition:
i) Pic($\mathbb{C}^n$) $= 0$
ii) Pic($\mathbb{C}P^n$) $\cong \mathbb{Z} \cong \langle L \rangle$
iii) *If E is an elliptic curve,* Pic(E) $\cong E \times \mathbb{Z}$

We will leave these unproved for now.

CHAPTER 12

Holomorphic vector bundles II

Lecture 12, Oct 6. Lots of administrative content discussed today so shorter math part.
If M is a complex manifold and $L_i \in \mathrm{Pic}(M)$, then $E = \bigoplus_i L_i$ gives us an easy way to generate higher rank vector bundles.
Theorem (Birkhoff-Grothendieck): *If $E \to \mathbb{C}P^1$ is a holomorphic vector bundle of rank r, then $E \cong \bigoplus L^{\otimes a_i}$ for $a_i \in \mathbb{Z}$ and $L \to \mathbb{C}P^1$ the tautological line bundle.*

Theorem (Atiyah): *If $M = \mathbb{C}/\Gamma$ is an elliptic curve, there exist indecomposable bundles $E_1, E_2, \ldots$ with rank $E_r = r$, such that if $E \to M$ is an indecomposable bundle, then for φ an automorphism of M,*

$$E \cong \varphi^* E_i$$

for some i.

So up to translation, there is only one indecomposable bundle.

We are learning that there aren't many holomorphic vector bundles. However, we do always have the tangent and cotangent bundles to a complex manifold. If Y is a complex submanifold of X, a complex manifold, there is an inclusion $TY \to TX$, and as in the real case, we can define the holomorphic normal bundle as the quotient $N := TX|_Y/TY$. Locally, if we say $Y = Z(z_{m+1}, \ldots, z_n)$, with local coordinates $z_1, \ldots, z_n$, then

$$TY = \langle \partial_{z_1}, \ldots, \partial_{z_m} \rangle \subset TX|_Y = \langle \partial_{z_1}, \ldots, \partial_{z_n} \rangle$$

If q is the quotient map $TX|_Y \to N$, then $q(\partial_{z_{m+1}}), \ldots, q(\partial_{z_n})$ form a frame for N.

There is an exact sequence

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow N \longrightarrow 0$$

Lemma: *If $0 \to E \to F \to G \to 0$ is an exact sequence of holomorphic vector bundles, then*

$$\det F = \det E \otimes \det G \quad \text{via a canonical isomorphism}$$

$\square$

Then the exact sequence for the normal bundle implies[1]

$$\kappa_Y = \kappa_{X|_Y} \otimes \det N$$

where if X is a complex manifold, κ_X denotes the determinant line bundle of its cotangent bundle. The above equation is known as the adjunction formula[2]

[1] is the order not wrong here?

[2] There are a lot of adjunction formulas. We even saw one in the other notes on symplectic topology

CHAPTER 13

Sheaf Theory

Lecture 13, Oct. 8. Our guiding example will be the situation: Let $E \to M$ be a holomorphic vector bundle. Then for every U open in M, we may assign an abelian group, $\xi(U) := \{s : U \to E|_U \mid s \text{ is a holomorphic section}\}$. If there is an open set $V \subset U$, the restriction map induces a restriction on the abelian group. This leads us to define
Definition: An (abelian group valued) presheaf on a topological space X is a functor $\mathcal{F} : Op(X) \to AbGrp$, where $Op(X)$ is the category of open sets of X, with morphisms given by inclusions, such that $F(\emptyset) = 0$ and the restriction $U \subset U$ induces the trivial restriction on abelian groups.

Definition: An (abelian group valued) sheaf on a topological space X is an (abelian group valued) pre-sheaf on X, such that $\forall\, U = \cup V_i,\ V_i$ open in X,
i) for $s \in F(U), \forall\, i, s|_{V_i} = 0 \Rightarrow s = 0$.
ii) for $s_i \in F(V_i), \forall\, i, j,\ \ s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} \Rightarrow \exists s \in F(U) \mid s|_{V_i} = s_i$.
The first condition says that checking that a global section is 0 is equivalent to checking it locally, and the second condition says that given local sections that agree on intersections, we can glue together a global section. These conditions together imply that the resulting glued section is unique.

Examples:
i) Sheaves of functions/sections: $\mathcal{O}_M, C^0_X, C^\infty_M, \xi$.
ii) Ideal sheaves: Take $A \subset M$ an analytic subset. Then $\mathcal{I}_A = \{f \in \mathcal{O}_M(U) \mid A \subset Z(f)\}$.
iii) Locally constant sheaves: If A is an abelian group, define $A_X(U) = \{U \to A \text{ locally constant}\}$.
The more naive approach to iii) is just assigning $A_X(U) = A$, and all the restriction maps to be the identity. This is a presheaf, but not a sheaf. The first axiom of checking zeros is satisfied: in fact, if a section vanishes on any subspace then it vanishes on the whole open set. The second axiom is not satisfied, however. If we have some U which is disoconnected, and $A \neq \{1\}$, then simply assign one component to 1 and one component to some $a \neq 1$, this defines a section on each component, but they do not glue together.

Of course, since a sheaf is defined as a certain type of functor, a sheaf morphism should be defined as a natural transformation:

Definition: A sheaf homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ is an assignment $\forall\ U,\ \exists\ \varphi_U : \mathcal{F}(U) \to$

$\mathcal{G}(U)$ such that $\forall\ V \supset U$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U) \\ \big\downarrow & & \big\downarrow \\ \mathcal{G}(V) & \longrightarrow & \mathcal{G}(U) \end{array}$$

where the horizontal maps are the restriction maps. So we have a set of objects (abelian sheafs) and a set of morphisms (natural transformations), which define a category, $Ab(X)$, the abelian sheafs over X.

To every $p \in X$, with a sheaf $\mathcal{F}$ over X, we can associate the abelian group $\mathcal{F}_p := \varinjlim_{U \ni p} \mathcal{F}(U)$, called the <u>stalk</u> at p.

Proposition: *The result of this colimit is an abelian group consisting of germs of sections near p.*
Proof: To see this, we show

$$\mathcal{F}_x := \{x_U \in \mathcal{F}(U) \mid x \in U\} \Big/ (x_U \sim x_V \iff \exists W \subset U \cap V \mid x_U|_W = x_V|_W)$$

along with the maps, for every U, $\psi_U : \mathcal{F}(U) \to \mathcal{F}_x$ sending a section x_U to its germ, $[x_U]$, satisfies the universal property of colimits. First we note that taking germs of sections makes this diagram commute with respect to the restriction morphisms. To do this, assume there is another object, Z, with maps $\varphi_U : \mathcal{F}(U) \to Z$, then we need to show there is a unique morphism (group homomorphism) $\mathcal{F}_x \to Z$. We may define

$$x_U \mapsto \varphi_U(x_U)$$

which type checks because $x_U \in \mathcal{F}(U)$ and $\varphi_U : \mathcal{F}(U) \to \mathcal{Z}$.

$\square$

For any sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$, we have a map[1] $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ sending

$$[x_U] \mapsto \varphi_U(x_U)$$

Proposition: *If $\phi : \mathcal{F} \to \mathcal{G}$ is a sheaf morphism, then φ is an isomorphism iff $\forall p$, φ_p is an isomorphism.*

$\square$

If $s \in \mathcal{F}(U)$, we can define $supp(s) = \{p \in U \mid s_p \neq 0\}$, s_p being the image of s in the stalk $\mathcal{F}_p$. Since $\mathcal{F}(U)$ is an abelian group, we can speak of a 0 element, hence this definition of support. In some settings we define the support to be the closure of the non-zero points, but here the support is automatically closed. To see this, take a point in

[1] This is part of the statement that "take the stalk at x" is a functor from the category of sheaves over X to the category of abelian groups.

the complement, i.e. for $U \ni p$, take $s' \in \mathcal{F}(U)$ such that $\psi_U(s') \equiv s'_p \equiv [s'] = [0]$. By definition of a germ, this means s' vanishes on some neighborhood of p, so being in the complement of the support is an open condition, so the support is closed.

In the same spirit, we can define the support of a sheaf, $supp(\mathcal{F}) := \{p \in X \mid \mathcal{F}_p \neq 0\}$, so the set of all non-zero stalks. In general, this is neither closed nor open. We will later see a class of sheaves called coherent sheaves, whose support is closed, and in fact analytic.

Now we will talk about sheaves of sets. The only thing that changes in the definition is that the target category is of course, Set, but also we need to replace our use of 0, since we are no longer in an abelian group. To do so, we replace the condition $F(\emptyset) = 0$ with the condition $F(\emptyset) = \emptyset$, and we also need to replace condtion i). The spirit of this condition was that it suffices to check that two sections are equivalent (s and 0) locally, so we replace this with a more general locality statement: If $\{U_i\}$ is a cover of $U \subset X$ and $s, s' \in \mathcal{F}(U)$, if $s|_{U_i} = s'|_{U_i}$, then $s = s'$. Everything else is the same.

If $\mathcal{F}$ is a presheaf of sets, we can form a topological space, the etale space,

$$\text{Ét}(\mathcal{F}) := \coprod_{p \in X} \mathcal{F}_x$$

and we can make it into a bundle with the projection

$$\begin{aligned} \pi : \text{Ét}(\mathcal{F}) &\to X \\ \mathcal{F}_x &\mapsto x \end{aligned}$$

How do we topologize the etale space? It's the largest topology making each section continuous. Spelling this out, we define $U \subset \text{Ét}(\mathcal{F})$ to be open iff $\forall\, V \subset X$ open, $\forall\, s \in \mathcal{F}(V)$, the set $\{p \in X \mid s_p \in U\}$[2] is open (in V and X).

Proposition: *$\mathcal{F}$ is a sheaf of sets $\Rightarrow$ $\mathcal{F}$ is iso to a sheaf of continuous sections of $\pi : \acute{E}t(X) \to X$. In fact, the isomorphism is canonical.*

This proposition describes what is known as <u>sheafification</u>: Given a presheaf $\mathcal{F}$ over X, produce a sheaf via the process:

$$\mathcal{F} \mapsto \text{Ét}(X) \to \text{ sheaf of sections }, \widetilde{\mathcal{F}}$$

Note: $\widetilde{\mathcal{F}}_p = \mathcal{F}_p$.

Another way to think of the sheafification is to define it directly: Given a presheaf $\mathcal{F}$ over X, we can define its sheafification, a presheaf, $\mathcal{F}^\#$ over X, so that for every $U \subset X$, we define

$$\mathcal{F}^\#(V) := \left\{ (s_u) \in \prod_{u \in U} \mathcal{F}_u \;\middle|\; (*) \right\}$$

[2]This doesn't type check. He introduced s_p without comment, so I assumed from context that it's the image of s in the stalk at p, ie the germ, so if s_p is not that, then I don't know what it's supposed to be.

where $(*)$ is the condition that $\forall\ u \in U,\ \exists\ V \subset U, u \in V$ such that $\exists \sigma \in \mathcal{F}(V)$ such that $\forall\ v \in V, s_v = (V, \sigma) \in \mathcal{F}_v$. Heuristically, we are adding in all the families of sections which could glue together, and identifying all the sections which agree on a restriction, effectively forcing this new presheaf to obey the gluing axioms. Then it's clear that if there is an inclusion $W \subset U$, the map

$$\prod_{W \in W} \mathcal{F}_w \to \prod_{u \in U} \mathcal{F}_u$$

$$(s_w)_{w \in W} \to (s_u)_{u \in U}$$

defines a restriction map, making $\mathcal{F}^{\#}$ a sheaf.

Sheafification obeys a universal property: for any morphism $\mathcal{F} \to \mathcal{G}$, there exists a unique morphism making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \widetilde{\mathcal{F}} \arrow[ur, dashed] & \end{tikzcd}$$

commute.

We have many further notions of sheaves:

i) Subsheaves: This just requires that $\mathcal{F}(U) \subset \mathcal{G}(U)$, respecting the structure of the sheaf, i.e. it's a subgroup if it's a group-valued sheaf, etc.

If $\varphi : \mathcal{F} \to \mathcal{G}$,
ii) ker: simply send $U \mapsto \ker\left(\mathcal{F}(U) \to \mathcal{G}(U)\right)$.
iii) im, coker: These fail to be a sheaf, so you have to sheafify.

iv) quotient sheaves
v) exact sequences

CHAPTER 14

Sheaf Cohomology

Lecture 14, Oct 13. As we discussed last time, the image and coker of a sheaf morphism need not result in a sheaf. To see this for the case of the image, we take the exponential map exp: $\mathcal{O}_X \to \mathcal{O}_X^*$, and two open subsets whose union is an annulus around the origin, and consider the function z. This is a global function with a local lift, namely picking a branch of the logarithm, but there is no section on the union that lifts. So taking the image doesn't obey the gluing axioms, hence you need to sheafify. For quotients, you also need to sheafify. We also had exact sequences, defined using the notions of kernel and image we had before. There is the usual result that for a SES of sheaf morphisms,

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$$

then $\mathcal{G} \cong$ the subsheaf $\varphi(\mathcal{G}) = \mathrm{im}\varphi \subset \mathcal{F}$ and $\mathcal{H} \cong \mathcal{F}/\mathcal{G}$, canonically. The convenience/power of exact sequences of sheaves is that you can always work stalkwise. The stalk is just an abelian group, so this is much less intimidating compared to working with the sheaves themselves. This slogan is captured by the proposition

Proposition: *We have the following canonical isomorphisms*

i) $(\ker \varphi)_p = \ker \varphi_p$

ii) $(\mathrm{im}\ \varphi)_p = \mathrm{im}\ \varphi_p$

iii) $(\mathrm{coker}\ \varphi)_p = \mathrm{coker}\ \varphi_p$

Example:

i) Take a complex manifold, X, and a complex submanifold, Z. We have the ideal sheaf, $\mathcal{I}_Z$, which is the sheaf of functions vanishing on Z, as a subsheaf of $\mathcal{O}_X$. There is also a restriction map $r : \mathcal{O}_X \to \mathcal{O}_Z$. Thus we may construct a sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

A priori this sequence looks weird because the first two sheaves are sheaves over X, while $\mathcal{O}_Z$ is a sheaf over Z. However, we can trivially view it as a sheaf over X by sending[1] $U \to \mathcal{O}_Z(U \cap Z)$. We want to show that this sequence is exact, and of course we do so stalkwise. That it is a complex is obvious: If a function is in the image of $\mathcal{I}_Z$, then it vanishes on Z, hence its restriction to Z is 0. To show it's exact, we assume $Z = \{z_{k+n} = \cdots = z_n = 0\}$

[1] More experienced mathematicians may recognize this as the direct image functor. More experienced than me, anyway.

locally, so that Z is k-dimesional. If $p \in Z$, then $r : (\mathcal{O}_X)_p = \mathbb{C}\{z_1, \ldots, z_n\} \to (\mathcal{O}_Z)_p = \mathbb{C}\{z_1, \ldots, z_k\}$ sends

$$f(z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_k, 0, \ldots, 0)$$

This is obviously a surjective map, and $\ker r = (z_{k+1}, \ldots, z_n) \subset \mathbb{C}\{z_1, \ldots, z_n\}$.

ii) Exponential sequence: Take M as a complex manifold, and associate to it the sheaf of holomorphic functions without zeros, $\mathcal{O}_M^*$ by the map $f \mapsto e^{2\pi i f}$. We view $\mathcal{O}_M^*$ as an abelian sheaf by multiplication: $\mathcal{O}_M^* : U \mapsto \{f \in \mathcal{O}_M \mid Z(f) = \emptyset\}$, and we are taking the addition as the abelian structure on $\mathcal{O}_M$, so we have the sequence

$$0 \to \mathbb{Z}_M \xrightarrow{\text{incl}} \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \to 0$$

This is a complex because if you have the germ of a function which has no zeros at a point, you can take the log so that the exponential map is surjective, with kernel $\mathbb{Z}$.

Note: If we take $M = \mathbb{C}^*$, then $\mathbb{Z}_M = \mathbb{Z}$, just unwinding the definition, and $\mathcal{O}_M = \{f : \mathbb{C}^* \to \mathbb{C}\}$. Then $z \in \mathcal{O}_M^*$, but it is not in the image of exp, because of our earlier discussion: it doesn't admit a logarithm, so exp is not surjective. This motivates the study of cohomology, as the study of obstructions to lifting.

Arguably the best approach is the derived functor approach, but we will take two more economical approaches, via flasque sheaves and cech cohomology. The first method is good for obtaining cohomology canonically, but is useless for computation, for which we use Cech cohomology.

Proposition: *If*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact, then $\forall\ U$*,*

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is also exact, i.e. the "global sections" functor is left exact.

Unfortunately as we saw, it may not be right exact.

Example: From our submanifold sequence, let's take $M = \mathbb{C}P^1$ and $Z = \{0, \infty\}$. Then the sequence becomes

$$0 \to \mathcal{I}_Z \to \mathcal{O}_M \to \mathcal{O}_Z \to 0$$

In this case, $\mathcal{O}_Z \cong \mathbb{C}_{\{0\}} \oplus \mathbb{C}_{\{1\}}$, it's the sum of skyscraper sheaves. Additionally, by the maximum principle, we know that $\mathbb{O}_M(M) = \mathbb{C}$, the constant functions. Also, $\mathcal{I}_Z = 0$ because it consists of constant functions which vanish at two points. Finally, $\mathcal{O}_Z = \mathbb{C}^2$, since the stalks don't talk to each other. The map $\mathcal{O}_M \to \mathcal{O}_Z$ is the diagonal map, since we're looking at a constant function which is restricted to two points, so obviously it takes on the same value. This is another example where surjectivity fails on the final map.

Definition: A sheaf is called flasque or flabby if $\forall\, U \subset X$ open, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Of course, this implies that all the restriction maps are surjective. Similarly, a sheaf is soft if for all A closed in X, the map

$$\mathcal{F}(X) \to \mathcal{F}(A) := \varinjlim_{U \supset A} \mathcal{F}(U)$$

is surjective.

Example: $\mathcal{C}^\infty_M$ is soft.

If $\mathcal{F}$ is a sheaf, then

$$\widetilde{\mathcal{F}} := \big(U \mapsto \{s : U \to \text{Ét}(\mathcal{F}),\ s \text{ a section}\}\big)$$

is a flasque sheaf, if we do not require the sections to be continuous. In this case, of course everything extends because we can just send everything not in the subspace to 0. Then $\widetilde{\mathcal{F}}$ is known as the canonical flasque sheaf which $\mathcal{F}$ embeds into, i.e. is a subsheaf of.

Proposition: *Given a SES,*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

if $\mathcal{F}'$ is flasque (soft), then

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is exact.

□

As discussed before, there is the global sections functor, $\Gamma : AbShf(X) \to AbGrp$, sending $\mathcal{F} \to \Gamma(\mathcal{F}) := \mathcal{F}(X)$. Per our proposition, this functor is left exact, but not exact in general.

Proposition: *Each $\mathcal{F} \in AbShf(X)$ has a canonical flasque resolution*

$$0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{\varphi^0} \mathcal{F}^1 \xrightarrow{\varphi^1} \mathcal{F}^2 \xrightarrow{\varphi^2} \dots$$

Here resolution means the sequence is exact, and flasque means each $\mathcal{F}^i$ is flasque[2].
Proof: To start, we already saw how to construct a flasque sheaf given a sheaf, so set $\mathcal{F}^0 := \widetilde{\mathcal{F}}$. We can take the sheaf of discountinous sections of $\mathcal{F}^i/\varphi(\mathcal{F}^{i-1})$. This may not be flasque, so again we take $\mathcal{F}^{i+1}$ to be the canonical flasque sheaf which the above defined sheaf embeds into.

□

[2]not necessarily $\mathcal{F}$ of course

So now we have an exact sequence, and after applying Γ we get a complex, and we take the cohomology of that complex, by defining

$$H^i(\mathcal{F}) := \ker(\Gamma\varphi^i)/\mathrm{im}(\Gamma\varphi^{i-1})$$

Relying on the fact that the resolution is canonical shows that this definition is well defined, but we don't even need to do that. If there are two flasque resolutions, you get a quasi morphism of chain complexes, which induces a unique isomorphism on cohomology, which also shows well defined-ness. While flasque-ness is not required to define the cohomology, it is required to show that it is well defined.

Definition: A sheaf $\mathcal{F}$ is acyclic (wrt Γ) if $\forall\, i > 0, H^i(\mathcal{F}) = 0$.

Note: $H^0(\mathcal{F}) = \Gamma(\mathcal{F}) = \mathcal{F}(X)$.

Proposition: *Flasque (soft) sheaves are acyclic.*

Proof: This is essentially by construction, since the resolution would just be

$$0 \to \mathcal{F} \to \mathcal{F} \to 0$$

$\square$

Proposition: *If*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact, then there is a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{F}') \arrow[r] & H^0(\mathcal{F}) \arrow[r] & H^0(\mathcal{F}'') \arrow[dll] \\
& H^1(\mathcal{F}') \arrow[r] & H^1(\mathcal{F}) \arrow[r] & H^1(\mathcal{F}'') \arrow[dll] \\
& H^3(\mathcal{F}') \arrow[r] & H^3(\mathcal{F}) \arrow[r] & H^3(\mathcal{F}'') \arrow[dll] \\
& {} & \vdots &
\end{tikzcd}$$

"Proof": As usual, the connecting homomorphism $H^i(\mathcal{F}'') \to H^{i+1}(\mathcal{F}')$ is defined via a diagram chase.

$\square$

As we noted, we may compute the cohomology groups from any resolution. In particular, we may consider only acyclic resolutions,

$$0 \to \mathcal{F} \to \mathcal{F}^\bullet$$

where $\mathcal{F}^\bullet$ is the sequence $\mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \dots$ connected via φ^i, such that $\mathcal{F}^\bullet$ are all acyclic.

Example (Singular Cohomology): If G is an abelian group, we define

$$\mathcal{F}^p := \text{sheafification of } \big(U \to \{\text{singular } p-\text{chains on } U \text{ with coefficients in } G\}\big)$$

Proposition: *If X is locally contractible[3] and paracompact[4], then*

$$0 \to G_X \to \mathcal{F}^\bullet$$

is an acyclic resolution.

Corollary:

$$H^i(X, G_X) = H^i(\Gamma\mathcal{F}^\bullet) = H^i(X, G)$$

where the LHS is sheaf cohomology and the RHS is singular.

Example (de Rham Theorem): If we have a smooth manifold, M, we have a soft sheaf of modules of C^∞ p-forms, $\mathcal{A}^\infty_M$.
Lemma (Poincare): *If $\alpha \in \mathcal{A}^p(\mathbb{R}^n)$, $d\alpha = 0 \Rightarrow \ \exists\, \beta \in \mathcal{A}^{p-1}(\mathbb{R}^n)$ such that $\alpha = d\beta$.*

Proposition: *The sequence*

$$0 \to \mathbb{R}_M \to \mathcal{A}^\bullet$$

is exact, where the maps on $\mathcal{A}^\bullet$ are the exterior differential.

Then applying the global sections functor to this exact sequences yields the de Rham complex. But because the global sections functor is soft, the resulting cohomology of the complex is equal to the sheaf cohomology. So we have the canonical isomorphisms:

$$H^i(M, \mathbb{R}) = H^i(\mathbb{R}_M) = H^i_{dR}(M)$$

[3] This gets rid of counterexamples such as the Hawaiian earring.
[4] "Tame at infinity"

CHAPTER 15

Cech Cohomology

Lecture 15, Oct 15. Cech cohomology is defined on a topological space X with an open cover, $\mathcal{U} = \cup U_i$. We fix an ordering of the U_i, but note that it won't matter what the ordering is, we just need one. For convenience, we also define, for $i_1, \dots, i_n$, $U_{i_1 \dots i_n} := U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_n}$. If $\mathcal{F} \in AbShf(X)$, the Cech cochains are given by

$$C^p(\mathcal{U}, X) := \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$$

e.g. if $p = 0$, there is one index. Then for any strictly increasing sequence $i_0, \dots, i_p$, we take a section over the intersection. So $C^0(\mathcal{U}, \mathcal{F})$ is just a section on each open subset, $C^1(\mathcal{U}, \mathcal{F})$ is a section on each pairwise intersection of open subsets, leaving out redundancy. We write the elements as $(s_{i_0 \dots i_p})_{i_0 < \dots < i_p}$. Sometimes we will omit the indexing.
We define the Cech differential, $\check{d} : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$, by

$$\check{d}(s_{i_0 \dots i_p}) := \left(\sum_{\mu=0}^{p+1} (-1)^\mu s_{\beta_0 \dots \hat{\beta}_\mu \dots \beta_p} \right)_{\beta_0 < \dots < \beta_{p+1}}$$

As usual, showing that the differential squares to 0 involves a shuffling around of indices.

Proposition: $\check{d}^2 = 0$.

Definition: $\check{H}^i(\mathcal{U}, \mathcal{F}) := H^i(C^\bullet(\mathcal{U}, \mathcal{F}), \check{d}^\bullet)$, and to get rid of the covering,
$\check{H}^i(\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$.

Note that if we have a refinemenet of $\mathcal{U}, \mathcal{U}' < \mathcal{U}$, i.e. $\forall\, U \in \mathcal{U}, \exists V \in \mathcal{U}' \mid V \subset U$, then we have restriction maps $\check{H}^i(\mathcal{U}, \mathcal{F}) \to \check{H}^i(\mathcal{U}', \mathcal{F})$. This depends on indices, i.e. for each U, you have to select a V, and then that defines the map. If you take the colimit over all such refinements, you get the result which depends only on X, which is $\check{H}^i(\mathcal{F})$.

Theorem: *If X is paracompact, then $H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{F})$.*
This is not very helpful because both sides of the equation are quite difficult to compute. To solve this problem, we introduce
Definition: A covering $\mathcal{U}$ is Leray if $\forall p, \forall i_0, \dots, i_p, \mathcal{F}|_{U_{i_0 \dots i_p}}$ is acyclic.

Theorem: *If $\mathcal{U}$ is Leray, then*

$$H^i(X, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F})$$

This theorem makes the previous theorem more applicable.
Example: i) If $U_{i_0 \dots i_p}$ is locally contractible, then $\mathcal{U}$ is Leray for locally constant sheaves.
ii) If M is a complex manifold, and $\mathcal{U} \ni U_i \cong$ a domain of holomorphy, i.e. an analytic subset of $\mathbb{C}^N \Rightarrow \mathcal{U}$ is Leray for any coherent $\mathcal{O}_M$-module.
Definition: A sheaf $\mathcal{F}$ of $\mathcal{O}_M$ modules[1] is called <u>coherent</u> if locally $\mathcal{F}|_U \cong$ coker $(\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus s})$. In english, that's if the sheaf is locally generated by finitely many sections, and has finitely many relations. The more compact notation is still quite explicit, though, because $(\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus s})$ is just an $s \times r$ matrix of holomorphic functions on U.

The guiding example should be the sheaf of sections of a holomorphic vector bundle, which is of course coherent, but there are other things like ideal sheaves.

There are some more standard sheaf constructions:
If $f : X \to Y$, $\mathcal{F} \in AbShf(X)$, the <u>direct image</u> sheaf is the sheaf, $f_*\mathcal{F}$, sending $Y \supset V \mapsto \mathcal{F}(f^{-1}(V))$. Note that $(f_*\mathcal{F})(Y) = \mathcal{F}(X)$.

Example: a) If $Z \subset Y$ and $z : Z \to Y$ is an inclusion, $\mathcal{F} \in AbShf(Z)$, then the push forward z_* turns the sheaf $\mathcal{F}$, a sheaf on Z, into a sheaf on Y. The support of $z_*\mathcal{F}$ is not Z, but $\overline{Z}$.
b) We get skyscraper sheaves by taking a) and letting $Z = \{*\}$.
c) If $f : X \to Y$ is a holomorphic surjection between two complex manifolds and f is proper, and the fibers are coonnected, then $f_*\mathcal{O}_X = \mathcal{O}_Y$. Note that taking Y to be a point recovers the famous consequence of the maximum principle.
d) $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$. Then $f_*\mathbb{Z}_{\mathbb{C}}$ has stalks $\mathbb{Z}$ at $z = 0$, and $\mathbb{Z}^2$ everywhere else.

[1] $\mathcal{F}(U)$ is an $\mathcal{O}_M$-module, modeled after vector bundles.

CHAPTER 16

Picard group and Divisors/Line Bundles

Lecture 15, Oct 15. If $\mathcal{U} = \cup U_i$ is an open cover of X, we obtain a homeomorphism

$$H^1(\mathcal{U}, \mathcal{O}_X^*) \to \operatorname{Pic}(X)$$

sending $h_{ij} \mapsto L_\xi :=$ holomorphic line bundle with transition functions h_{ij}, i.e.

$$\coprod_{i \in I} U_i \times \mathbb{C}/(z,u)\ (z, h_{ij}u)$$

Theorem: $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

We recall that $\mathcal{O}_X^*$ appears in the short exact exponential sequence:

$$0 \to \mathbb{Z}_M \to \mathcal{O}_M \to \mathcal{O}_M^* \to 0$$

Applying the global sections functor, we get

$$0 \to \Gamma(X, \mathbb{Z}_M) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X^*) \to 0$$

If X is connected, compact, then this sequence is just

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$$

which contains no information. Recall that this fits into the beginning terms of the LES for sheaf cohomology, because $H^0(X) = \Gamma(X) = \mathcal{F}(X)$. The remaining sequence is

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to H^2(X,\mathcal{O}_X^*) \to \dots$$

This is a very interesting sequence. The $H^i(X,\mathbb{Z})$ terms are just singular cohomology, i.e. purely topological. The $H^i(X,\mathcal{O}_X)$ terms are called Dolbeault groups, which we will soon encounter. They are the complex analogue of de Rham cohomology. $H^1(X,\mathcal{O}_X^*)$ is the Picard group, and $H^2(X,\mathcal{O}_X^*)$ is what is called the analytic Brauer group. Once we get into Kahler geometry and Hodge theory we can talk about these groups more.

Now onto divisors and line bundles:

Definition: A divisor, D, on a complex manifold X is a locally finite formal linear combination

$$D = \sum_{i \in I} a_i [Y_i]$$

where $Y_i \subset X$ is an irreducible, analytic hypersurface and $a_i \in \mathbb{Z}$. Further, D is effective if $a_i \geq 0$. There is also the divisor group, $\text{Div}(X) := \{D \text{ a divisor on } X\}$ with the group operation of formal addition.
Note: If X is compact, $\text{Div}(X)$ is the free abelian group generated by irreducible analytic hypersurfaces.

If $Y \subset X$ is a hypersurface, there is an associated divisor $[Y] := \sum[Y_i] \in \text{Div}(X)$, so you decompose your hypersurface into irreducible components and consider the sum as an effective divisor. Conversely, if you have a divisor $\sum a_i[Y_i]$, you get a hypersurface by taking $\cup_{a_i \neq 0} Y_i$, so a divisor is thought of as a hypersurface + multiplicities of irreducible components.

There is the notion of a sheaf of meromorphic functions, $\mathcal{K}_X$. It's not really a sheaf of functions because a meromorphic function is not a function, but an equivalence class of functions defined on open subsets, i.e. the complement of analytic subsets.

Definition: If X is a complex manifold, $(Y, x) \subset (X, x)$ an irreducible hypersurface germ, $Y = Z(g)$, g irreducible. Then we define

$$\text{ord}_{Y,x} : \mathcal{K}_{X,x} \setminus \{0\} \to \mathbb{Z}$$

sending f to a power of g in an irreducible decomposition of f. You do this by letting $f = g^k \cdot \pi g_i k^i$, then defining $\text{ord}_Y(f) = k$. If we have a global meromorphic function, you just pick any x and define the order of f to be the order at that point. It can be shown that the order is locally constant on Y, so that this definition is well defined.

Now we have the notion of a principal divisor: for $f \in \kappa(X)$, define

$$(f) := \sum_{Y \subset X \text{ irreducible hypersurface}} \text{ord}_Y(f) \cdot [Y] \in \text{ Div}(X)$$

this linear combination is locally finite. If you arrange this as a sum over a_i, $(f) = \sum_i a_i[Y_i]$, you can of course group the positive and negative coefficients:

$$(f) = \sum_{a_i > 0} a_i[Y_i] - \sum_{a_i < 0} (-a_i)[Y_i]$$

with the positive portion being called the zero divisor and the negative portion being the polar divisor.

CHAPTER 18

Kahler Geometry

Lecture 18, Oct 27. Start with Hermitian linear algebra: Take a complex VS, V, and define a Hermitian inner product as a sesquilinear[1] map $h : V \times V \to \mathbb{C}$ such that $\forall\, v, w \in V, h(v,w) = \overline{h(w,v)}$ and $\forall\, v \neq 0, h(v,v) > 0$. If we define $g := Re(h)$ and $\alpha = Im(h)$, then g defines an inner product and α defines a symplectic form[2], and we often denote $\langle v, w\rangle = h(v,w)$. If you have a complex structure, J, all these quantities are related

$$g(v,w) = \alpha(v, Jw) = g(Jv, Jw)$$

Conversely, if you have a real VS, V, with an inner product, g, then you can write down a product

$$h(v,w) = g(v,w) - ig(iv,w)$$

which is hermitian iff g is compatible with the complex structure $v \mapsto iv$, where compatibility means g is i-invariant. So a Hermitian manifold is a manifold equipped with a pointwise Hermitian product, i.e. at a point x, it is a map $T_xX \times T_xX \to \mathbb{C}$ sending $(v,w) \mapsto h(v,w)$. In general then, $h \in C^\infty(X, T^*X \otimes \overline{T^*X})$. Then defining $g = Re(h)$ gives a Riemannian metric, or a pointwise inner product, and $\alpha = Im(h)$ is a pointwise non-degenerate, symplectic, bilinear form, but we may not call it a global symplectic form because it need not be closed. Of course locally, the tangent spaces are generated by dz_i, so that locally h can be written as[3]

$$h = \sum h_{ij} dz_i \otimes d\bar{z}_j$$

In this case, $H := (h_{ij})$ is a pointwise Hermitian matrix.

[1] pulling out factors from the second entry comes with a bar

[2] Non-degenerate, alternating, bilinear form.

[3] Bernd notes that it's often convenient to denote any indices referring to anti holomorphic coordinates with a bar, when working with real computations, so for instance instead of the dot over the j, it should be written with a line over it, however Latex makes it look weird so the reader will have to manage without.

We associate a (1,1)-form to h, $\omega := \frac{i}{2}(h - \overline{h})$. Locally,

$$
\begin{aligned}
h &= \sum h_{ij} dz_i \otimes d\overline{z}_j \\
\Rightarrow \omega &= \frac{i}{2} \sum h_{ij} dz_i \wedge d\overline{z}_j \\
\omega \left(\partial_{z_i}, \partial_{\overline{z}_j}\right) &= \frac{i}{2} \left(h(\partial_{z_i}, \partial_{\overline{z}_j}) - \overline{h}(\partial_{z_i}, \partial_{\overline{z}_j})\right) = \frac{i}{2} h_{ij} \\
&= -\omega(\partial_{\overline{z}_j}, \partial_{z_i})
\end{aligned}
$$

ω is called the Kahler form, and is the anti-symmetrized hermitian form. From symplectic geometry, we recall that the symplectic form raised to the dimension of the manifold is a volume form, reflecting the fact that complex manifolds are always oriented.

CHAPTER 19

Kahler Geometry II

Lecture 19, Oct 27. We define the Lefschetz operator $L : \bigwedge^k T^*_{\mathbb{C}}X \to \bigwedge^{k+2} T^*_{\mathbb{C}}X$ sending $\alpha \to \alpha \wedge \omega$. We also define the Hodge star, $\star : \bigwedge^k T^*_{\mathbb{C}}X \to \bigwedge^{2n-k} T^*_{\mathbb{C}}X$ by $\beta \mapsto \star\beta$, where $\star\beta$ is uniquely determined by the constraint, for $\alpha, \beta \in (\bigwedge^k T^*_{\mathbb{C}})_p$:

$$\alpha \wedge \star\overline{\beta} = \langle \alpha, \beta \rangle \frac{1}{n!}\omega^{\wedge n} \equiv \langle \alpha, \beta \rangle \text{ vol}$$

Finally, we also define the dual Lefschetz operator,

$$\bigwedge : \star^{-1} \circ L \circ \star : \bigwedge^{k+2} T^*_{\mathbb{C}}X \to \bigwedge^k T^*_{\mathbb{C}}X$$

which we should just think of as some formal object for now.
To compute locally, take $\{e_i\}$ a unitary basis for $T^*_x X$ and $\{e^*_i\}$ a dual basis, i.e.

$$h(e^*_i, e^*_j) = \delta_{ij},\ \omega = \frac{i}{2}\sum e_i \wedge \overline{e}_i$$

Note that in general we can't write the e^*_i as dz_i, because we may have curvature. In such a case,

$$\frac{1}{n!}\omega^{\wedge n} = \left(\frac{i}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} e_1 \wedge \cdots \wedge \overline{e}_1 \wedge \cdots \wedge \overline{e}_n$$

We need to introduce multi index notation to work with calculations with the hodge star: If $I = \{i_1, \ldots, i_a\} \subset \{1, \ldots, n\}, i_1 < \cdots < i_a$, then write $e_I := e_{i_1} \wedge \ldots e_{i_a}$, and $|I| = a$. Then $\beta = e_I \wedge \overline{e}_J$ such that $|I| + |J| = k$ is a basis of

$$\bigwedge^k T^*_{\mathbb{C}}X = \oplus_{a=0}^k \bigwedge^a T^*X \otimes_{\mathbb{C}} \bigwedge^{k-a} \overline{T^*X}$$

In this basis,

$$\langle e_K \wedge \overline{e}_L, e_I \wedge \overline{e}_J \rangle = \delta_{KI}\delta_{LJ}$$

and the equation determining the Hodge star becomes

$$(e_K \wedge \overline{e}_L) \wedge \star\overline{(e_I \wedge \overline{e}_J)} = \delta_{KI}\delta_{LJ}\frac{1}{n!}\omega^{\wedge n}$$
$$\Rightarrow \star\overline{(e_I \wedge \overline{e}_J)} = \left(\frac{1}{2}\right)^n e_K \wedge \overline{e_L}$$

where K and L are complimentary to I and J, respectively.

Further properties:
i) $\langle \alpha, \star\beta \rangle = (-1)^k \langle \star\alpha, \beta \rangle$
ii) $\star 1 = \frac{\omega^n}{n!}$
iii) on $\bigwedge^k T^*X$, $\star^2 = (-1)^k$.
iv) Λ is adjoint to L: The defining equation is

$$\begin{aligned} \langle L\alpha, \beta \rangle \text{vol} &= L\alpha \wedge \star\overline{\beta} \\ &= \omega \wedge \alpha \wedge \star\overline{\beta} \\ &= \alpha \wedge \omega \wedge \star\overline{\beta} \\ &= \alpha \wedge L(\star\overline{\beta}) \\ &= \langle \alpha (\star^{-1} L \star)\beta \rangle \cdot \frac{\omega^n}{n!} \\ &= \langle \alpha, \Lambda\beta \rangle \frac{\omega^n}{n!} \end{aligned}$$

We define the adjoint operator to $d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$,

$$d^* = - \star d \star : \mathcal{A}^{k+1}(X) \to \mathcal{A}^k(X)$$

There is also the Laplace-Beltrami operator:

$$\Delta = d^*d + dd^* : \mathcal{A}^k(X) \to \mathcal{A}^k(X)$$

Using the complex structure, d and d^* decompose into

$$\begin{aligned} d &= \partial + \overline{\partial} \\ d^* &= \partial^* + \overline{\partial}^* \end{aligned}$$

where

$$\partial^* = - \star \overline{\partial} \star \;\text{ and }\; \overline{\partial}^* = - \star \partial \star$$

We know

$$\partial^2 = \overline{\partial}^2 = 0 \;\Rightarrow\; (\partial^*)^2 = (\overline{\partial}^*)^2 = 0$$

We also define $\Delta\partial := \partial^*\partial + \partial\partial^*$ and $\Delta_{\overline{\partial}} := \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$. It may seem like $\Delta = \Delta_\partial + \Delta_{\overline{\partial}}$, but that is not the case in general. It will be true for Kahler manifolds, however.

Definition: A Hermitian manifold (X, h) is a Kahler manifold if $d\omega = 0$, where ω is the anti-symmetrized hermitian form, and ω is called the Kahler form.

Note: there are equivalent definitions of the Kahler condition, i.e. parallel transport is $\mathbb{C}$-linear.

Proposition: *Let X be a compact, complex manifold. Then the set of Kahler forms on X is a (possibly empty) open, convex cone in*

$$C^{1,1}_{\mathbb{R}}(X) \equiv \{\alpha \in \mathcal{A}^{1,1}(X) \mid \overline{\alpha} = \alpha, d\alpha = 0\}$$

where the open condition is defined wrt the sup-norm.
Proof: First we show that Kahler forms are real: We bar the local form

$$\begin{aligned} \overline{\frac{1}{2}\sum h_{ij} dz_i \wedge d\overline{z}_j} &= -\frac{i}{2}\sum \overline{h_{ij}} d\overline{z}_i \wedge dz_j \\ &= \frac{i}{2}\sum h_{ji} dz_j \wedge d\overline{z}_j \end{aligned}$$

and end up with the same local expression, just swapped indices. The minus sign is cancelled by swapping the dz_j and dz_i, and h_{ij} is a Hermitian matrix, so the bar is equal to the transpose. It is open because being Kahler is an open condition on the determinant of h_{ij}, and convex because Hermitian forms are additive and Kahler forms are multiplicative by a positive constant.

CHAPTER 20

Fubini-Study Metric and the Kahler Identities

Lecture 20, Nov 3. The Fubini-Study metric on $\mathbb{C}P^{n+1}$ is the unique $U(n+1)$ invariant Kahler metric $\omega_{FS}(h_{FS})$ such that $\int_{\mathbb{C}P^1} \omega_{FS} = 1$.

Definition: On $U_i := \mathbb{C}P^n / V(z_j)$, define

$$\omega_i := \frac{1}{2\pi}\partial\overline{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_i}\right|^2\right)$$

One can show that these locally defined forms patch together to define a global Kahler form.

Lemma: *ω_{FS} is a Kahler form.*

□

Examples: More examples of Kahler manifolds:
i) Complex tori: $X = \mathbb{C}^n/\Gamma$, with $h_{ij} = \text{const.} \in M(n, \mathbb{C})$. $T_x X = \mathbb{C}^n$, canonically.
ii) Riemann surfaces Σ : Any Hermitian metric is Kahler. $d\omega \in \mathcal{A}^3(\Sigma) = 0$.
iii) Ball quotients: $B_1(0) \subset \mathbb{C}^n$, $\omega = \frac{i}{2}\partial\overline{\partial}\log(1 - \|z\|^2)$ is $SU(1, n)$-invariant.
Proposition: *If (X, ω) is a Kahler manifold and $Y \subset X$ is a complex submanifold, $(Y, \omega|_Y)$ is Kahler.*
Proof: If $\iota : Y \to X$ is the inclusion map, then $\omega|_Y = \iota^*\omega$, so non-degeneracy follows. By naturality, $d(\iota^*\omega) = i^*(d\omega) = \iota^*(0) = 0$.

□

Corrolary: *Complex projective manifolds are Kahler, as they are complex submanifolds of $\mathbb{C}^n$.*

Because the Kahler form is closed, it defines a class in $H^2_{dR}(X) \cap H^{1,1}(X)$.
Proposition[1]**:** *If (X, ω) is closed Kahler, $[\omega] \neq 0$.*

[1]Note this (and the subsequent corollary) is a feature of symplectic topology, not necessarily Kahler geometry.

Proof: There is a map $H^2_{dR}(X) \to \mathbb{R}$ sending

$$\alpha \mapsto \int_X \alpha$$

i.e.

$$\frac{1}{n!}[\omega^n] \mapsto \text{vol}(X) > 0$$

So $[\omega^n] \neq 0 \Rightarrow [\omega] \neq 0$.

$\square$

In fact, this shows every positive power of $[\omega]$ is non-zero:
Corollary: If X is Kahler, $b_{2k}(X) \neq 0 \; \forall \; k \in \mathbb{Z}$.

Example: $X \cong (\mathbb{C}^n \setminus \{0\}/\mathbb{Z} \cong S^1 \times S^{2n-1}$, a Hopf manifold, is not a Kahler manifold. If $n \geq 2$, then the Kunneth formula shows this has $H^2 = 0$.

Proposition (Kahler Identities): *Let (X, ω) be a Kahler manifold. Then*
i) $[\overline{\partial}, L] = [\partial, L] = 0$ and same for replacing ∂ by ∂^*.
ii) $[\overline{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\partial$, $[\Lambda, \overline{\partial}] = -i\partial^*$, and $[\Lambda, \partial] = i\overline{\partial}^*$
iii) $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$, and Δ commutes with $\star, L, \Gamma, \partial, \overline{\partial}, \partial^*$, and $\overline{\partial}^*$.

CHAPTER 21

Proof of Kahler Identities and Hodge theory

Lecture 21, Nov 5. We will now prove the Kahler identities: Recall
Proposition (Kahler Identities): *Let (X, ω) be a Kahler manifold. Then*
i) $[\overline{\partial}, L] = [\partial, L] = 0$ and same for replacing ∂ by ∂^*.
ii) $[\overline{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\partial$, $[\Lambda, \overline{\partial}] = -i\partial^*$, and $[\Lambda, \partial] = i\overline{\partial}^*$
iii) $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$, and Δ commutes with $\star, L, \Gamma, \partial, \overline{\partial}, \partial^*$, and $\overline{\partial}^*$.
Proof: i) $[\overline{\partial}, L](\alpha) \equiv \overline{\partial}(\omega \wedge \alpha) - \omega \wedge \overline{\partial}\alpha = (\overline{\partial}\omega) \wedge \alpha = 0$ (because $d\omega = 0$), and similarly for $[\partial, L]$, and you can star this whole process to get the remaining two.
ii)
iii) We will use a lemma, proved using ii): $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. We know $i(\partial\overline{\partial}^* + \overline{\partial}^*\partial = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0$. Then we can compute

$$
\begin{aligned}
\Delta_\partial &\equiv \partial^*\partial + \partial\partial^* = i[\Lambda, \overline{\partial}]\partial + i\partial[\Lambda, \overline{\partial}] \\
&= i\left(\Lambda\overline{\partial}\partial - (\overline{\partial}[\Lambda, \partial] + \overline{\partial}\partial\Lambda) + ([\Lambda, \partial]\overline{\partial} + \Lambda\partial\overline{\partial}) - \partial\overline{\partial}\Lambda\right) \\
&= i(\Lambda\overline{\partial}\partial - i\overline{\partial}\overline{\partial}^* - \overline{\partial}\partial\Lambda - i\overline{\partial}^*\overline{\partial} + \Lambda\partial\overline{\partial} + \partial\overline{\partial}^*\Lambda) \\
&\equiv \Delta_{\overline{\partial}}
\end{aligned}
$$

and the other proofs follow similarly.

□

To show Δ commutes with everything, we will just do one example, $[\Lambda, \Delta] = 0$. Define $d^c := -i(\partial - \overline{\partial})$ and $d^{c*} := - \star d^c \star$. Note $dd^c = 2i\partial\overline{\partial} = -d^c d$. Then we can compute

$$
\begin{aligned}
\Lambda\Delta &\equiv \Lambda dd^* + \Lambda d^* d \\
&= d\Lambda d^* - id^c d^* + d^*\Lambda d \\
&= dd^*\Lambda + id^*d^c + d^*d\Lambda - id^*d^c = 0.
\end{aligned}
$$

□

If (X, h) is a Hermitian manifold, then Δ, Δ_∂, and $\Delta_{\overline{\partial}}$ are elliptic partial differential operators, and X compact $\Rightarrow$ elliptic operators are Fredholm. In particular, their kernels are

all finite dimensional.
Definition:

$$\mathcal{H}^k(X,h) := \{\alpha \in \mathcal{A}^k(X) \mid \Delta\alpha = 0\}$$
$$\mathcal{H}^{p,q}(X,h) := \mathcal{H}^k(X,h) \cap \mathcal{A}^{p,q}(X)$$

and similarly for $\mathcal{H}_\partial, \mathcal{H}_{\overline{\partial}}$, and so on. Note (X,h) Kahler $\Rightarrow \mathcal{H}^{p,q} = \mathcal{H}^{p,q}_{\overline{\partial}} = \mathcal{H}^{p,q}_{\partial}$, reflecting iii) of the Kahler identities.
Lemma: *X compact and $\alpha \in \mathcal{A}^k(X)$, then*

$$\Delta_{\overline{\partial}}\alpha = 0 \iff \overline{\partial}\alpha = 0 \vee \overline{\partial}^*\alpha = 0$$

and analogously for Δ_∂.
Proof: $\Leftarrow$ is immediate. For $\Rightarrow$, we have

$$0 = (\Delta_{\overline{\partial}}\alpha, \alpha) = \int_X \Delta_{\overline{\partial}}\alpha \vee \star\overline{\alpha}$$
$$= (\overline{\partial}\alpha, \overline{\partial}\alpha) + (\overline{\partial}^*\alpha, \overline{\partial}^*\alpha)$$
$$\Rightarrow \overline{\partial}\alpha = 0 \vee \overline{\partial}^*\alpha = 0$$

$\square$

CHAPTER 22

Hodge theory II

Lecture 22, Nov 10. **Proposition:** *If X is Hermitian, then*

$$\mathcal{H}^k_{\bar\partial}(X,h) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar\partial}(X,h)$$

and analogously for $\mathcal{H}^k_{\partial}$.

$\square$

If (X,h) is a Hermitian manifold, then the Hodge star defines an isometry

$$\star : \mathcal{H}^{p,q}_{\bar\partial}(X,h) \to \mathcal{H}^{n-q,n-p}_{\partial}(X,h)$$

which depends on h. If X is compact, then there is a bilinear form

$$\mathcal{H}^{p,q}_{\bar\partial}(X,h) \times \mathcal{H}^{n-q,n-p}_{\bar\partial}(X,h) \to \mathbb{C}$$

sending

$$(\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

and it is non-degenerate. To see it's non-degenerate, take any non-zero form, α, and pair it with $\star\overline{\alpha}$. The result is

$$(\alpha,\overline{\alpha}) \mapsto \int_X \alpha \wedge \star\overline{\alpha} = \|\alpha\|^2 > 0$$

This of course induces an identification $\mathcal{H}^{p,q}_{\bar\partial}(X,h) = \mathcal{H}^{n-q,n-p}_{\bar\partial}(X,h)^*$, which is independent of h. This is a simple instance of a very general phenomena in algebraic geometry known as Serre duality.

There is a theorem for Riemannian manifolds called the Hodge Decomposition on forms, which we will make use of in the Kahler setting:

Theorem (Hodge Decomposition): *If (M,g) is a compact Riemannian manifold, then*

$$\mathcal{A}^k(X) = \mathcal{H}^k(M,g) \oplus d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k+1}(X)$$

Moreover, $dim_{\mathbb{R}}\mathcal{H}^k(M,g) < \infty$.

The Kahler version says
$$\mathcal{A}^{p,q}(X) = \mathcal{H}^{p,q}_{\partial}(X,h) \oplus \partial\mathcal{A}^{p-1,q}(X) \oplus \partial^*\mathcal{A}^{p+1}(X)$$
and analogously for $\overline{\partial}$.

This is useful when studying the Dolbeault cohomology groups:
Corollary: *If* (X,h) *is a compact Hermitian manifold, then there is an isomorphism*
$$\mathcal{H}^{p,q}_{\overline{\partial}}(X,h) \to H^{p,q}(X)$$
and similarly for $\mathcal{H}^{p,q}_{\partial}(X,h)$.

Recall that $H^{p,q}(X)$ are the $\overline{\partial}$ closed mod $\overline{\partial}$ exact (p,q) forms.
Proof: We showed in the previous lemma that $\overline{\partial}$ harmonic implies $\overline{\partial}$ closed, so this map is well defined. Define $K = ker\ \overline{\partial}$. By the Hodge Decomposition,
$$K = \overline{\partial}\mathcal{A}^{p,q-1}(X) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$
We know $\overline{\partial\partial}^*\beta = 0 \Rightarrow \overline{\partial}^*\beta = 0$, so that
$$H^{p,q}(X) = K/\overline{\partial}\mathcal{A}^{p,q-1}$$

□

Note that this does not require the Kahler condition.

Now we will see some background for the Hodge decomposition theorem: If V is a finite dimensional Hermitian vector space, with an endomorphism $d : V \to V$ such that $d^2 = 0$. Then we may define $\Delta = dd^* + d^*d$ and $\mathcal{H} = ker\ \Delta$. It is easily seen that $\mathcal{H}$, dV, and d^*V are mutually orthogonal. We note Δ induces an isomorphism $\Delta_{\mathcal{H}^\perp} : \mathcal{H}^\perp \to \mathcal{H}^\perp$. Then we define the block diagonal matrix
$$G := \begin{pmatrix} 0 & 0 \\ 0 & (\Delta_{\mathcal{H}^\perp})^{-1} \end{pmatrix}$$
and H to be the projection onto $\mathcal{H}$.
Lemma:
i) $G\Delta + H = I$
ii) $G, \Delta,$ *and* H *commute with each other.*
iii) d, d^* *commute with those in ii).*

□

Then we conclude $V \cong \mathcal{H} \oplus dV \oplus dV^*$.

I need to come back and complete the proof in the infinite dimensional case but it's a lot of functional analysis that I'm learning for the first time so it'll take a while.

CHAPTER 23

Hodge Theory III and $\partial\overline{\partial}$-Lemma

Lecture 23, Nov 12. We motivated the Hodge decomposition of forms last time to prove that the Dolbeault cohomology groups can be written in terms of the $\overline{\partial}$ harmonic p,q forms. There is another central result for compact Kahler manifolds
Theorem: *If (X,h) is compact Kahler, then*
i) $H^{p,q}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, *by the Hodge decomposition.*
ii) The isomorphism $\overline{H^{p,q}(X)} \cong \overline{\mathcal{H}^{p,q}_{\overline{\partial}}(X)} \cong \mathcal{H}^{p,q}_{\partial}(X) \cong H^{q,p}(X)$ *is induced by conjugation on* $H^k(X,\mathbb{C})$.
iii) $H^{p,q}(X) = H^{n-p,n-1}(X)^*$.

These conditions do not hold in general if (X,h) is just Hermitian, not Kahler.
Proof:
i) $H^k(X,\mathbb{C}) = \mathcal{H}^k(X,h) = \mathcal{H}^k_{\overline{\partial}}(X,h)$, by the Kahler condition. Note $H^k(X,\mathbb{C})$ could equivalently be replaced by de Rham cohomology. We continue

$$
\begin{aligned}
&= \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(X,h) \\
&= \bigoplus_{p+q=k} H^{p,q}(X) \\
\Rightarrow H^k(X,\mathbb{C}) &= \bigoplus_{p+q=k} H^{p,q}(X)
\end{aligned}
$$

We claim this equality is independent of h. Suppose h' is different Kahler metric. Then we have an isomorphism

$$\mathcal{H}^{p,q}(X,h) \cong H^{p,q}(X) \cong \mathcal{H}^{p,q}(X,h')$$

Because X is Kahler, we can drop the $\overline{\partial}$ and ∂. We need to show that if $\alpha \in \mathcal{H}^{p,q}(X,h)$, and α' its image through this isomorphism in $\mathcal{H}^{p,q}(X,h')$, then $[\alpha] = [\alpha']$, as elements of $H^k_{dR}(X,\mathbb{C})$. We know, for γ some $p,q-1$ form, that

$$
\begin{aligned}
\alpha' &= \alpha + \overline{\partial}\gamma \\
&= d\overline{\partial}\gamma = d\alpha' - d\alpha
\end{aligned}
$$

But as we saw earlier, a form is harmonic iff it is closed and co closed, so in particular their difference is co-closed, so

$$= 0$$

Applying the Hodge decomposition[1] for d, we have that there exists $\beta \mid \overline{\partial}\gamma = d\beta$, which shows i).
ii) This is true just by defn, because $\overline{\Delta_{\overline{\partial}}} = \Delta_{\partial}$.
iii) This follows from Serre duality on forms:

$$H^{p,q}(X) \cong \mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong \mathcal{H}^{n-p,n-q}_{\overline{\partial}}(X) \cong H^{n-p,n-q}(X)$$

□

From this, we define the Hodge numbers,

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega^p_X)$$

This leads to a diagram called the Hodge diamond, which displays the relationship between Hodge numbers.

Proposition ($\partial\overline{\partial}$-Lemma): *If (X, h) is compact Kahler, and $\alpha \in \mathcal{A}^{p,q}(X), d\alpha = 0$, then TFAE:*
i) $\exists\ \beta \in \mathcal{A}^{p-1,q}(X) \mid \alpha = d\beta$
ii) $\exists\ \beta \in \mathcal{A}^{p+q-1}(X) \mid \alpha = \partial\beta$
iii) $\exists\ \beta \in \mathcal{A}^{p,q-1}(X) \mid \alpha = \overline{\partial}\beta$
iv) $\exists\ \beta \in \mathcal{A}^{p-1,q-1}(X) \mid \alpha = \partial\overline{\partial}\beta$
v) *For any Kahler metric, $h', \alpha \perp \mathcal{H}^{p,q}(X, h')$.*

Corollary: *If (X, h) is compact, Kahler, then*
i) *If ω is the Kahler form, then locally, $\omega = i\partial\overline{\partial}f$, for some $f \in \mathcal{C}^\infty(U, \mathbb{R})$. f is known as the Kahler potential.*
ii) *If ω, ω' are Kahler forms on X with $[\omega] = [\omega'] \in H^2_{dR}(X)$, then $\omega = \omega' + i\partial\overline{\partial}f$ for some f.*
iii) *The Kahler Cone,*

$$\mathcal{K}_X := \{[\omega] \in H^{1,1}(X, \mathbb{R}) \mid \omega \text{ is a Kahler form}\}$$

is an open, convex cone. Moreover, it does not contain any line $[\omega] + \mathbb{R}\alpha$, and if $\alpha \in H^{1,1}(X, \mathbb{R})$, then $\alpha + \lambda[\omega] \in \mathcal{K}_X, \lambda \gg 0$.
Proof:
i) The Poincare lemma implies that locally, $\omega = d\alpha$ for some α, then $\big(\text{i}) \Rightarrow \text{iv})\big)$ of the $\partial\overline{\partial}$ Lemma gives the result.
ii) $[\omega] = [\omega'] \Rightarrow \exists\ \alpha \mid \omega - \omega' = d\alpha$. Applying the $\partial\overline{\partial}$ lemma again gives the result.
iii)

□

[1] Some confusion over whether this actually implies d-exactness. It follows from the $\partial\overline{\partial}$-lemma that we will see later, though.

CHAPTER 24

Hard Lefschetz Theorem

Lecture 24, Nov 17. **Theorem (Lefschetz Decomposition on Harmonic Forms):** *If (X,h) is Kahler, $n = \dim_{\mathbb{C}} X$, then for $0 \leq k = p+q \leq n$,*

$$L^{n-k} : \mathcal{H}^{p,q}(X,h) \to \mathcal{H}^{n-q,n-p}(X,h), \alpha \mapsto \alpha \wedge \omega^{n-k}$$

is an isomorphism.
This statement relies necessarily on being Kahler. This leads to the theorem
Theorem (Hard Lefschetz Theorem):*If (X,h) is compact Kahler, then*
i) $L^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R})$ is an isomorphism for $0 \leq k \leq n$.
ii) $H^k(X,\mathbb{R}) = \oplus_{i\geq 0} L^i H^{k-2i}(X,\mathbb{R})_p$, where

$$\begin{aligned} H^j(X,\mathbb{R})_p &= ker\big(\Lambda : H^j(X,\mathbb{R}) \to H^{j-2}(X,\mathbb{R})\big) \\ &= ker\big(L^{n-j+1} : H^i(X,\mathbb{R}) \to H^{2n-j+2}(X,\mathbb{R})\big) \end{aligned}$$

is called the primitive cohomology.

Theorem (Hodge-Riemann Bilinear Relations): *If (X,h) is compact Kahler, and ω is its Kahler form,*

$$\forall\, \alpha \in H^{p,q}(X)_p \setminus \{0\},\ i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-(p+q)} > 0$$

This theorem restricts the possible Hodge decompositions. Then he said some things about moduli spaces which went over my head, so I'm skipping that.

Theorem (Hodge Index Theorem): *If (X,h) is a compact Kahler *surface*, then the intersection form $H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to \mathbb{R}$, sending $(\alpha,\beta) \mapsto \int_X (\alpha \wedge \beta)$, has index $(2h^{2,0}(X) + 1, h^{1,1}(X) - 1)$. Restricted to $H^{1,1}(X)$, the index is $(1, h^{1,1}(X) - 1)$.*

In higher dimensions, if $\dim_{\mathbb{C}} X = 2m$, the signature of the intersection form on $H^{2m}(X,\mathbb{R})$ is

$$sgn(X) = \sum_{p,q=0}^{2m} (-1)^p h^{p,q}(X)$$

If (X, h) is compact Kahler, and $Y \subset X$ is a complex submanifold (or subvariety or analytic subset) with $k = \operatorname{codim}_{\mathbb{C}} Y$, the fundamental class $[Y]$ corresponds to, by Poincare duality, some $\alpha \in H^{k,k}(X) \cap H^{2k}(X, \mathbb{C})_{\mathbb{Z}}$. Indeed, $\beta \in H^{p,q} p \neq q, p + q = 2(n-k) \Rightarrow \int_Y \beta = 0$.
The **Hodge Conjecture** states that if X is projective, then $H^{k,k}(X) \cap H^{2k}(X, \mathbb{Q})$ is generated by such classes.

CHAPTER 25

Lefschetz Theorem on (1,1) classes, Formality, and Rational Homotopy Theory

Lecture 25, Nov 19. Theorem (Lefschetz Theorem on (1,1) Classes): *If X is compact Kahler, then Pic(X) $\to H^{1,1}(X)_{\mathbb{Z}}$ is surjective.*

To prove this we need a lemma
Lemma: *If X is compact Kahler, the following two maps agree:*

$$\varphi : H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X) = H^{0,k}(X)$$
$$\psi : H^k(X, \mathbb{C}) \to \bigoplus_{p+q=k} H^{p,q}(X) \to H^{0,k}(X)H^{0,k}(X)$$

Proof: Consider the commutative diagram of acyclic resolutions:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \underline{\mathbb{C}} \arrow[r, "d"] \arrow[d] & \mathcal{A}^0_{\mathbb{C}}(X) \arrow[r] \arrow[d, "="] & \mathcal{A}^1_{\mathbb{C}}(X) \arrow[r, "d"] \arrow[d, "\pi^{0,1}"] & \dots \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\bar\partial"] & \mathcal{A}^{0,0}(X) \arrow[r] & \mathcal{A}^{0,1}(X) \arrow[r, "\bar\partial"] & \dots
\end{tikzcd}$$

Because they are acyclic, they both compute sheaf cohomology. If $\xi \in H^k(X, \mathbb{C}) \Rightarrow \ \exists\, \alpha \in \mathcal{H}^k(X, h) \mid [\alpha] = \xi \Rightarrow \varphi(\xi) = [\pi^{0,k}(\alpha)] = \psi(\xi)$, as required.

$\square$

Proof of Theorem: The exponential sequence gives us a commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \arrow[r] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d] & H^2(X, \mathcal{O}_X) \\
& H^2(X, \mathbb{C}) \arrow[ur] &
\end{tikzcd}$$

where the factorization is induced by functoriality of sheaf cohomology. The diagonal map is given by just projecting onto the $0,2$ type/component. Then

$$\alpha \in \mathrm{im}(c_1) \iff \alpha^{0,2} = 0 \iff \alpha^{1,1} \in H^{1,1}(X)$$

$\square$

Onto Formality:
If M is a C^∞ manifold, then $(A^*(M), \wedge, d)$, is a graded commutative differential graded algebra (CDGA) over $\mathbb{R}$, meaning:
i) $A^\bullet = \bigoplus_{i\in\mathbb{N}} A^i$ is a graded K-algebra for K some field, and $K \subset A^0$.
ii) Graded commutative: $\alpha \wedge \beta = (-1)^{ij}\beta \wedge \alpha$, for $\alpha \in A^i, \beta \in A^j$.
iii) d satisfies a graded Leibniz rule: $d(\alpha \wedge \beta) = da \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \beta$

Then you can take the cohomology of the CDGA, resulting in another DGA, $H^i = \ker d \cap A^i / dA^{i-1})$, but now $d = 0$. The wedge product on differential forms induces the cup product on homology. Of course, a map of CDGAs, f, induces a map on their homology, $H^* f$. We define a quasi-isomorphism $f : A^\bullet \to B^\bullet$ as a homomorphism such that $H^* f$ is an isomorphism.

A fact from real homotopy theory tells us that if M is simply connected, then $A^*(M)$ knows everything about homotopy groups $\pi_n(M) \otimes_{\mathbb{Z}} \mathbb{R}$. In general terms, this is good because homology is not so hard, but homotopy theory is. However, $A^*(M)$ is an infinite dimensional vector space, so it can be difficult to work with.
Definition: A DGA $A^\bullet$ over K is called connected if $K \to A^\bullet$ induces an isomorphism[1] on H^0, and simply connected if it is connected and $H^1(A^\bullet) = 0$.

Example: The de Rham complex is connected $\iff$ M is connected, and simply connected $\iff$ $b_1(M) = 0$.

Definition: A DGA, $M^\bullet$, is called minimal if there exist $x_i \in M^{d_i}, 1 \le d_1 \le d_2 \le \dots$ such that
i) $M^\bullet = \langle x_i \rangle_{i\in\mathbb{N}} = \Lambda^* \bullet \left(\bigoplus k \cdot x_i\right)$, making it the free CDGA generated by x_i's.
ii) $\forall\, i, dx_i \in \langle x_1, \dots, x_{i-1}\rangle^+$ (the deg > 0 part)

Lemma: *If $M^\bullet$ is a minimal CDGA, then $M^\bullet$ is simply connected $\iff$ $M^1 = 0$.*

Definition: A minimal model of a DGA, $A^\bullet$, is a quasi-isomorphism $M^\bullet \to A^\bullet$, for some $M^\bullet$ a minimal DGA.

Proposition: *Every simply connected DGA has a minimal model which is unique up to isomorphism.*
Proof Idea: Build $M^\bullet$ from lifting generators of $H^*(A^\bullet)$, inductively over degree.

$\square$

So, for example, even though the DGA $A^*(M)$ is huge, there is a unique reduction of generators to those represented by cohomology, and a little more.

[1] This is not the same as being a quasi-iso.

Definition: A DGA, $A^\bullet$, is called <u>formal</u> if it is equivalent to a DGA, $B^\bullet$ with $d_B = 0$, and a smooth manifold, M, is called <u>formal</u> if $A^*(M)$ is formal, as a DGA.

Note: $A^\bullet$ is formal iff a minimal model $M^\bullet$ is formal.

Example: S^n is formal.
We are going to see later a big result that all compact Kahler manifolds are formal, which gives a nice obstruction to being Kahler.

CHAPTER 26

Formality of Kahler manifolds and Harmonic Theory on Vector Bundles

Lecture 26, Nov 24. **Proposition:** *Let (M, g) be a closed Riemannian manifold, and assume that products of harmonic forms are harmonic. Then M is formal.*
Proof: There is an inclusion $\mathcal{H}^\bullet(M, g) \to A^\bullet(M)$ which is a quasi-isomorphism of DGA's, but $d = 0$ on $\mathcal{H}^\bullet(M, g)$.

□

Theorem: *Kahler manifolds are formal.*
Proof (Sketch): Replace $A^*(X)$ by ker d^c, where $d^c = -i(\partial - \overline{\partial})$, using dd^c-Lemma, similar to the $\partial\overline{\partial}$-Lemma, i.e. d-exact forms are also dd^c exact. This shows that (ker $d^c, d) \to (A^*(X)$ is a DGA quasi-isomorphism. Further, (ker $d^c, d) \to H^*_{d^c}(X)$ is a DGA isomorphism, where $H^*_{d^c}(X)$ is the cohomology wrt d^c.

□

If $E \to M$ is a complex (not necessarily holomorphic) vector bundle with M real, a hermitian metric is a map $h : E \otimes_{\mathbb{C}} \overline{E} \to \mathbb{C}_M$, with (E_x, h_x) a Hermitian vector space.
Locally, if e_i is a local frame on U and e^i is a dual frame, then $h|_U = \sum h_{ij} e^i \otimes e^j$.
Proposition: *Any complex vector bundle has a Hermitian metric.*

The above follows from the standard partition of unity argument.

Now assume $M = X$ is a complex manifold, with Hermitian structure g. Then there is a Hodge star on (E, h),

$$\overline{\star}_E : \bigwedge^{p,q} X \otimes E \to \bigwedge^{n-p,n-q} X \otimes \overline{E}^*$$

satisfying

$$(\alpha, \beta) \star 1 = \alpha \wedge \overline{\star}_E \beta$$

Then Harmonic theory on $E \to M$ generalizes directly from (X, g):
i) $\overline{\partial}^*_E := -\overline{\star}_E \circ \overline{\partial}_E \circ \overline{\star}_E$
ii) $\Delta_E = \overline{\partial}^*_E \overline{\partial}_E + \overline{\partial}_E \overline{\partial}^*_E$
iii) Harmonic sections: $\mathcal{H}^{p,q}(X, E) = \{\alpha \in \mathcal{A}^{p,q}(E) \mid \Delta_E \alpha = 0\}$.

iv) Hodge Decomposition: If X is comapct, then

$$\mathcal{A}^{p,q}(X,E) = \overline{\partial}_E \mathcal{A}^{p,q-1}(X,E) \oplus \overline{\partial}_E^* \mathcal{A}^{p,q+1}(X,E) \oplus \mathcal{H}^{p,q}(X,E)$$

v) Serre duality: $H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \rightarrow \mathbb{C}$ is a non-degenerate pairing, and $H^q(X, \Omega^p \otimes E) \cong H^{n-q}(X, \Omega^{n-p}, E)$ by a canonical isomorphism.

If $E \rightarrow X$ is a holomorphic VB, then $\overline{\partial} : \mathcal{A}^{0,0}(E) \rightarrow \mathcal{A}^{0,1}(E)$ is well defined, but not ∂, nor d. To do this, you need a connection, i.e. a splitting of $TE \cong \pi^* TX \oplus \pi^* E$, or a splitting of the exact sequence

$$0 \longrightarrow \pi^*(E) \longrightarrow TE \longrightarrow \pi^*(TX) \longrightarrow 0$$

ie a section of one of the two middle maps.
Equivalently over $\mathbb{R}$, a map $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$, with $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$.

CHAPTER 27

Connections and Curvature

Lecture 27, Dec 1. Continuing from last time, if we have a connection, ∇, then $E = U \times \mathbb{R}^k$, and a section, $s = (s_1, \ldots, s_k) : U \to \mathbb{R}^k$, and $\nabla s = ds + A \otimes s = (ds_1, \ldots, ds_k) + A \cdot (s_1, \ldots, s_k)$, so locally connections look like $d + M$, where M is some 1-form valued matrix. In fact, $\nabla - d$ is $\mathcal{A}^0$-linear:
Proposition: *The space of connections on E is an affine space over* $\mathcal{A}^1(\text{End } E)$.

Note: For any ∇ and $x_0 \in X$, there always exists a local trivalization with $\nabla = d + A$ and $A(X_0) = 0$.

If we have two connections on 2 VB's, ∇_i on E_i, there are naturally induced bundles from just the data of E_i, namely Hom bundles, direct sums, tensor products, duals, etc. and all these have naturally induced connections as well, from ∇_i.

If (X, h) is a Hermitian manifold, denote

$$\langle s_1, s_2 \rangle_h := h(s_1 \otimes \bar{s}_2) = h(s_1, s_2)$$

Definition: A connection ∇ is called Hermitian if $\forall s$ a local section,

$$d\langle s_1, s_2 \rangle_h = \langle \nabla s_1, s_2 \rangle_h + \langle s_1, \nabla s_2 \rangle_h$$

Locally, if $\nabla = d + A$, then ∇ is Hermitian iff, denoting $E = U \times \mathbb{C}^r$, with constant sections e_i, and $H = \langle e_i, e_j \rangle_h$,

$$\begin{aligned} dH = (d\langle e_i, e_j \rangle_h)_{ij} &= \big(\langle \nabla e_i, e_j \rangle_h + \langle e_i, \nabla e_j \rangle_h\big)_{ij} \\ &= \big((Ae_i)^t H\bar{e}_j + e_j H\overline{Ae_j}\big)_{ij} = A^t H + H\overline{A} \end{aligned}$$

Proposition: *The set of Hermitian connections forms an affine space over* $\mathcal{A}^1(\text{End } (E, h))$, *the vector bundle of hermitian endomorphisms*. This is all for real vector bundles, but now we look at holomorphic vector bundles.
In this case, we can always decompose forms according to types, so

$$\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$$

Definition: A connection, ∇, is compatible with the complex structure if $\nabla^{0,1} = \overline{\partial}$.

Locally, this says that $\nabla = d + A = \partial + \overline{\partial} + A$. Of course, this only works for a holomorphic trivialization. One could always take a smooth trivialization that doesn't respect the type decomposition.

Proposition: *The space of connections compatible with the complex structure forms an affine space over* $\mathcal{A}^{1,0}(\text{End } E)$.

Proposition: If (E, h) is a holomorphic, Hermitian vector bundle, then there exists a unique ∇ compatible with h and with J.
Proof: It suffices to check this locally, where Hermitian implies $\nabla = d + A \Rightarrow dH = A^t H + H\overline{A}$. But ∇ is $\overline{\partial}$-compatible, so $\overline{\partial} H = H \cdot \overline{A}$, i.e. $A = \overline{H}^{-1} \cdot \partial \overline{H}$, which defines the connection.

$\square$

Curvature is the infimitesimal obstruction to the existence of a flat frame, i.e. $s_1, \ldots, s_k \in \mathcal{A}^0(U, E)$, fibrewise basis, $\nabla s_i = 0$.
Definition: The curvature of a connection, ∇, is

$$F_\nabla = \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \mathcal{A}^2(E)$$

Locally,

$$\begin{aligned} F_\nabla(s) = \nabla(ds + A \cdot s) &= \big(d(ds) + A \wedge ds\big) + \big(d(A \cdot s) + A \wedge (A \cdot s)\big) \\ &= (dA + A \wedge A)s \end{aligned}$$

So that

$$F_\nabla = dA + A \wedge A$$

is a 0th order differential operator.
Proposition:
i) F_∇ *is* $\mathcal{A}^0$*-linear*
ii) $A \in \mathcal{A}^1(\text{End } E)$, *then* $F_{\nabla + A} = F_\nabla + \nabla A + A \wedge A$.

CHAPTER 28

Chern-Weil Theory

Lecture 28, Dec 7. The recording for the lecture uploaded was a duplicate of the previous lecture and i wasn't there, so I can't do this yet.