M375T: Dynamical Systems

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Spring 2020

Abstract

Class taught by Lewis Bowen, notes taken by Reese Lance. The notes were taken by hand during class and typed later. Some of my own thoughts are interjected, but quite rarely. I initially thought to try to separate my thoughts from the professor's but it becomes too difficult. As such I will also try to expand on examples which are mentioned in passing in class, spell out proofs which are glossed over, and add insight where I think it is helpful. This helps to justify the existence of this set of notes, as opposed to live-texed notes. Especially because some of my own content is interspersed throughout these notes, any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu) or if you can find any other way to communicate with me, that is also fine. At the moment I'm trying to get the notes written, and worrying about making the format not look like trash later. I'm also not going to track theorem and lemma numbers, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers, which is what would have been the plan. Thanks to Arun Debray whose formatting choices inspired my own.

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CHAPTER 1

Overview and a little bit of history

1 Lecture 1, Jan 21. Given a set Ω and a map $f : \Omega \to \Omega$, there is not much we can say about this scenario. If we embed $\Omega \subset \mathbb{R}^n$, we gain a lot more structure, in particular there is a smooth structure, so that we can speak about smoothness of f and do calculus if Ω is a nice enough submanifold. In this class, we will often consider $\Omega = I = [0, 1]$. Often in real world practice, studying dynamical systems necessarily involve large dimensions[1], but to really understand the mathematics of dynamical systems, we consider one dimension often. One of the main questions of dynamical systems is this:

$$\text{Given } f : \Omega \to \Omega, \text{ what can we say about } f^n(x) \text{ when } n \text{ becomes large}$$

Here $f^n(x)$ denotes the process of applying f n times. At least my first instinct was "nothing, that is too general". It seems like dynamical systems is broadly the study of 'systems' $f : \Omega \to \Omega$ such that the above question has an interesting answer. Here is one such example:

Example: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then we know

$$\lim_{n\to\infty} f^n(x) = \begin{cases} \infty & |x| > 1 \\ 1 & |x| = 1 \\ 0 & |x| < 1 \end{cases}$$

Looking at the positive half of the real line, points greater than 1 diverge to infinity, the point 1 itself is a fixed point of f, and points less than 1 converge to 0, and the negative half mirrors the positive half.

Example: Define $S^1 = \{e^{i\theta} \mid 0 \le \theta \le 2\pi\}$, the unit circle in the complex plane. Then for any $\alpha \in \mathbb{R}$, we define R_α, the rotation map by an angle α:

$$\begin{aligned} R_\alpha : S^1 &\to S^1 \\ e^{i\theta} &\mapsto e^{i(\theta+\alpha)} \end{aligned}$$

this has the effect of rotating the entire unit circle by an angle θ. This map is an isometry, that is, it preserves the distance between any two points, taking the usual metric by embedding

[1] For example in classical mechanics, the phase space of a system generally has $6n$ dimensions for n particles, three position and three velocity degrees of freedom.

$S^1 \subset \mathbb{R}^2$. But what can we say about its long term behavior? We should probably guess that it depends on α. For example, if $\alpha = \frac{2\pi p}{q}$, for some $p, q \in \mathbb{Z}$, then $R_\alpha^q = Id$, because

$$\begin{aligned} R_\alpha \circ \cdots \circ R_\alpha(e^{i\theta}) &= R_\alpha \circ \cdots \circ R_\alpha(e^{i(\theta+\alpha)}) \\ &= R_\alpha \circ \cdots \circ R_\alpha(e^{i(\theta+2\alpha)}) \\ &\vdots \\ &= e^{i(\theta+q\alpha)} \\ &= e^{i(\theta+q\frac{2\pi p}{q})} = e^{i\theta}e^{2\pi i p} = e^{i\theta} \end{aligned}$$

If α is not a rational multiple of 2π, i.e. it is irrational, then there will not be a fixed point. We denote the orbit[2] of an element $x = e^{i\theta}$ as

$$O_x = \{f^n(e^{i\theta}) \mid n \in \mathbb{Z}\}$$

The orbit of any point under this action is dense, in the following sense: Take any point in S^1. Then any neighborhood of this point[3] has infinite intersection with O_x. This is a little more difficult to see, but eventually, the distribution becomes equidistributed, meaning that a point in the orbit, given large enough n, is equally likely to be in a certain arc of the circle as any other, provided they have the same arc length.

Example: Define

$$\begin{aligned} D : S^1 &\to S^1 \\ e^{i\theta} &\mapsto e^{2i\theta} \end{aligned}$$

This is known as the doubling map, as it doubles the angle of any given point in the circle. What does this map actually do to the unit circle? It maps the upper half to all of S^1, for example, and the first quartile to the upper half, and so on. We will now show some elementary properties of this map:

Lemma: *For any $p \in \mathbb{R}$, there is a point in S^1 which has period p.*

Proof: The point 1 trivially satisfies this condition, but the point is that there is a non-trivial point which does this. In fact, there will be infinitely many points in the circle which satisfy this condition: If $p \in \mathbb{R}$ is the desired period, then consider the point, for any $k \in \mathbb{Z}$,

$$e^{i\theta}, \qquad \theta = \frac{\pi k}{2^{p-1} - \frac{1}{2}}$$

Compute

$$D^p(e^{i\theta}) = e^{i\frac{2^p \pi k}{2^{p-1}-\frac{1}{2}}} = e^{\frac{i\pi k}{2^{p-1}-\frac{1}{2}}} = e^{i\theta}$$

[2]Bowen states that this nomenclature comes from studying the orbits of celestial objects. It made me think of group actions. Indeed it is used in the same spirit as in group actions.

[3]Which can be thought of as an open arc on the circle containing the point in question.

as required.

$\square$

For example, if we set $p=5$, then $\theta = \frac{2\pi}{31}$, and

$$D^5(e^{i\theta}) = e^{\frac{i64\pi}{31}} = e^{\frac{i62\pi}{31}} e^{\frac{i2\pi}{31}} = e^{\frac{i2\pi}{31}}$$

There are also points which may not be periodic, but after applying D a couple times, they become periodic. For a trivial example, the point -1 is not periodic, but

$$D(-1) = 1$$

which is periodic. There are also pairs of distinct points whose distance goes to 0 as $n \to \infty$.
Example: Let $\Sigma_2 = \left\{(x_i)_{i=1}^{\infty} \mid x_i \in \{0,1\}\right\}$ be the set of sequences of points in $\{0,1\}$, and define the shift map $\sigma : \Sigma_2 \to \Sigma_2$ by

$$\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

Also define

$$\Phi : \Sigma_2 \to S^1$$

via

$$\Phi(x_i) = e^{2\pi i \sum_i x_i 2^{-i}}$$

Lemma: *There is a commutative diagram*

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "\sigma"] \arrow[d, "\Phi"'] & \Sigma_2 \arrow[d, "\Phi"] \\
S^1 \arrow[r, "D"'] & S^1
\end{tikzcd}$$

Proof: We check

$$(D \circ \Phi)(x_i) = D(e^{2\pi i \sum_i x_i 2^{-i}}) = (e^{2\pi i \sum_i x_i 2^{-i+1}})$$
$$e^{2\pi i (x_1 + \frac{x_2}{2} + \frac{x_3}{4} + \dots)}$$
$$= e^{2\pi i (\frac{x_2}{2} + \frac{x_3}{4} + \dots)} e^{2\pi i x_1} = e^{2\pi i (\frac{x_2}{2} + \frac{x_3}{4} + \dots)}$$

Similarly,

$$(\Phi \circ \sigma)(x_i) = \Phi(x_2, x_3, \dots)$$
$$= e^{2\pi i (\frac{x_2}{2} + \frac{x_3}{4} + \dots)}$$

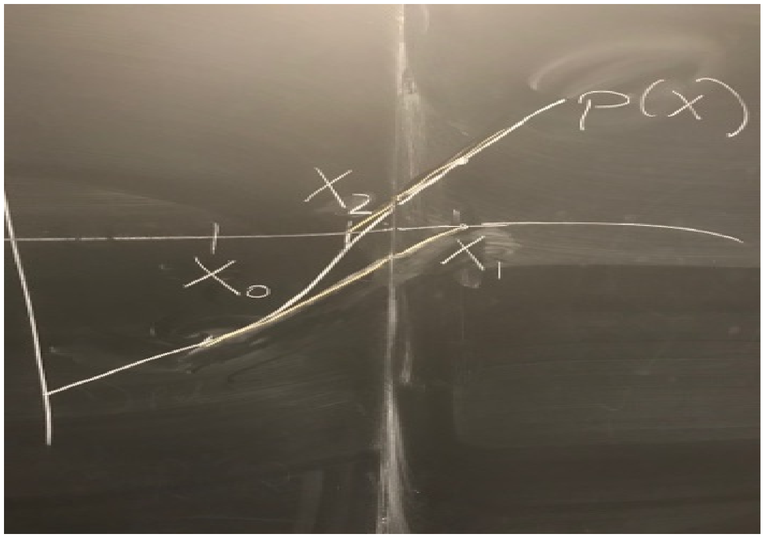
as required.

$\square$

Example: Newton's Method: Let

$$p(x) = \sum_{n=0}^{N} a_n x^n$$

Suppose you want to find a root of this polynomial. Then Newton's method says to make an initial guess for the root, x_0. Then we want to make better and better guesses, until they finally converge to a root. How do we make a better guess? Take the tangent line at x_0. If x_0 is not a critical point, then the tangent line will intersect the x-axis at a point, call it x_1. Then take the tangent line at x_1, and its intersection with the x-axis will be x_2, and so on.



How can we be sure this process converges? The picture should give intuition: Each successive choice is closer to the desired root. For an actual proof, we will set

$$x_{i+1} = x_i - \frac{p(x_i)}{p'(x_i)}$$

and $N(x) = x - \frac{p(x)}{p'(x)}$. Then want to show

$$\lim_{n\to\infty} N^n(x) = p \quad \text{and} \quad P(p) = 0$$

Theorem: *There exists a root p of P and $\epsilon > 0$, such that*

$$|x_0 - p| < \epsilon \Rightarrow \lim_{n\to\infty} N^n(x_0) = p$$

□

So Newton's Method works! And can be used to find roots of polynomials[4].
Now for a little bit of history: We all know the famous equation $F = ma$. This is a differential equation, so if you write down all the forces in a given system, and manage to solve the differential equation, if you have suitable initial conditions,[5] you can describe the time evolution of your system. For example, if one were to write down force diagrams for our solar system, there would need to be 54 initial conditions to describe the evolution of our system, since there are 3 position coordinates, 3 velocity coordinates, and 1 sun + 8 planets[6]. In general, the phase space of an n-body system in 3D will be $\mathbb{R}^{6n}$.
Poincare's viewpoint: We should study $\Omega \subset \mathbb{R}^n$ and all possible trajectories in the phase space. In classical physics, we view dynamics as deterministic: If we have the initial state of the system, and the differential equation describing its dynamics is known, then the future state of the system is completely determined. Poincare held that determinism in

[4]This is in general an impossible task, as we learn via Galois theory for degree 5 or higher polynomials.
[5]In this case, you need 2 initial conditions to uniquely specify a solution.
[6]RIP Pluto.

this sense is wrong because dynamics are inherently chaotic: If there is a small error in the measurement of the initial conditions, as there must be because there is no such thing as a perfect measurement, then that error can propagate exponentially when predicting the future states of the system, i.e. a small error can become large in a chaotic system. For example, there are many animations online of the double pendulum which show close initial conditions that become chaotic very quickly.

Historical aside over.

Here are a series of definitions of objects we will see frequently in the course:

Definition: Let I, J be intervals in $\mathbb{R}$. A homeomorphism from I to J is a continuous, invertible map whose inverse is also continuous.

Definition: Given a set $M \subset \mathbb{R}$, then

$$C^r(M) := \big\{ f : M \to \mathbb{R} \mid f \text{ is } r \text{ times continuously differentiable} \big\}$$

ie the $f'^{(r)}(x)$ exists and is continuous. A map $f : M \to \mathbb{R}$ is said to be C^r smooth, or simply C^r if it lies in $C^r(M)$.

Definition: A C^r diffeomorphism is an invertible C^r map such that its inverse is also C^r.

Example: $f(x) = x^3$ is invertible, with inverse $f^{-1}(x) = x^{\frac{1}{3}}$. As such, it is a homeomorphism, since both x^3 and $x^{\frac{1}{3}}$ are continuous. However, it is not a C^1 diffeomorphism because $(f^{-1}(x))' = \frac{1}{3}x^{\frac{-2}{3}}$, which is not even defined at the origin, and is thus not differentiable at the origin. Away from the origin, however, this function is a C^∞ diffeomorphism.

Theorem: *Let $I = [a, b] \subset \mathbb{R}$, and $f : I \to I$. Then f has a fixed point, i.e. a point $p \in I$ such that $f(p) = p$.*

Proof: We offered a visual proof which relies on showing the plot $y = x$ intersects the plot of f. It can be formalized using the intermediate value theorem: **Theorem:** *If $f : [a, b] \to \mathbb{R}$ is continuous and $z \in [f(a), f(b)]$, then $\exists c \in [a, b]$ such that $f(c) = z$.*

If we apply IVT to the function $g(x) := f(x) - x$, then $g(a) < 0, g(b) > 0$, so by IVT, there must be a point c such that $g(c) = 0$, which shows the result. c is the fixed point.

$\square$