M392C: Orderability/3-Manifold Groups

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Abstract

Class taught by Cameron Gordon, notes taken by Reese Lance. The notes are note live texed. They are post-mortem-texed, that is, taken by hand during class and typed later. Some of my own thoughts are interjected, but rarely. I initially thought to try to separate my thoughts from the professor's but it becomes too difficult. As such I will also try to expand on examples which are mentioned in passing in class, spell out proofs which are glossed over, and add insight where I think it is helpful. This helps to justify the existence of this set of notes, as opposed to live-texed notes (which I believe there are for this class, thanks to Jackson van Dyke), which are probably slightly better for a faithful representation of what is being taught in the classroom. Especially because some of my own content is interspersed throughout these notes, any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu) or if you can find any other way to communicate with me, that is also fine. At the moment I'm trying to get the notes written, and worrying about making the format not look like trash later. I'm also not going to track theorem and lemma numbers, though Cameron does, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers, which is what would have been the plan. Thanks to Arun Debray whose formatting choices inspired my own.

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CHAPTER 1

Overview and Preliminary Group Theory

Lecture 1, Jan 21. The book we will be following is "Ordered Groups and Topology" by Clay and Rolfsen, a pdf of a draft of which can be found here. Orderability was originally developed as a group theoretic concept, and only (somewhat) recently has topology entered the picture. In fact, some of the bare bones definitions can be made entirely in Set:

Definition: Given a set X, a strict, total order (STO) on X is a binary relation $<$, satisfying

$$i)\ \text{Transitivity}: \forall\ x, y, z \in X, x < y, y < z \Rightarrow x < z$$
$$ii)\ \text{Totality: } \forall\ x, y \in X, \text{exactly one of the following holds:}$$
$$x < y,\ \ y < x,\ \ \text{or } x = y$$

For convenience, we may also define the STO $>$ as induced by an STO:

$$x > y \iff x \neq y \text{ and } x \not< y$$

If we upgrade to Grp, we can use the multiplication structure to talk about invariance:
Definition: Given a group, G, and an STO, $<$, on the underlying set, $<$ is called a left order (LO) on G if

$$g < h \Rightarrow fg < fh\, \forall\, f \in G$$

and we call G left orderable if there exists a left ordering of G. Similarly, we can define right-orderability in the obvious way. One might ask how different the concepts of left and right orders are. It is clear that for abelian groups, they are one and the same, that is: all left orders are also right orders. In general this may not hold. If it does,
Definition: Given an LO, $<$, on G, it is called a bi order (BO) if it is also an RO, and we say G is bi orderable if it admits a bi order.
To return to the above question, given an LO $<$, we can induce an R, $\prec$ O in the following way:

$$g \prec h \iff h^{-1} < g^{-1}$$

Lemma: *The above binary relation is an RO on G.*
Proof: Transitivity and totality of $\prec$ follow immediately from $<$ being an STO. To show it is a right order, consider elements g, h, f of G such that

$$g \prec h$$

Then

$$\begin{gathered} h^{-1} < g^{-1} \\ \Rightarrow f^{-1}h^{-1} < f^{-1}g^{-1} \\ \Rightarrow (f^{-1}h^{-1})^{-1} \succ (f^{-1}g^{-1})^{-1} \\ \Rightarrow gf \prec hf \end{gathered}$$

□

So every left order induces a right order, i.e. all left orderable groups are also right orderable. Note that this does not imply that all left orders are bi orders. That would require that all left orders are **themselves** right orders. All we have shown is that a left order **induces** a right order. Indeed, it is not the case: There are groups which are left orderable but not bi orderable. We obviously can restrict the LO to an LO on a subgroup $H < G$ (denoting subgroup here), just as an STO can be restricted to a subset $A \subset X$.

Example: $(\mathbb{R}, +)$ is bi ordered, with the usual ordering. Its subgroups $\mathbb{Z}, \mathbb{Q}$, etc. can inherit this ordering.

Here is a series of elementary lemmas:

Lemma: *Given $<$ an LO on G,*

$$\begin{gathered} i)\ g > 1, h > 1 \Rightarrow gh > 1 \\ ii)\ g > 1 \Rightarrow g^{-1} < 1 \\ iii)\ < \text{ is BO} \iff g < h \Rightarrow f^{-1}gf < f^{-1}hf\ \forall f \in G \\ (\text{invariant under conjugation}) \end{gathered}$$

Proof:

$$\begin{gathered} i)\ h > 1, \Rightarrow gh > g > 1 \\ ii)\ g > 1 \Rightarrow g^{-1}g > g^{-1} \Rightarrow 1 > g^{-1} \end{gathered}$$

$iii)$ Assume $<$ is also an RO. Then we can clearly apply left invariance on f^{-1} and right invariance on f to obtain the result. If $<$ is conjugation invariant, then

$$\begin{gathered} g < h \Rightarrow f^{-1}gf < f^{-1}hf \\ gf < hf \end{gathered}$$

Using left invariance in the final line.

□

Lemma: *Given $<$ a BO on G,*

$$\begin{gathered} i)\ g < h \Rightarrow g^{-1} > h^{-1} \\ ii)\ g_1 < h_1, g_2 < h_2 \Rightarrow g_1g_2 < h_1h_2 \end{gathered}$$

Proof:

$$i)\; g < h \Rightarrow g^{-1}g < g^{-1}h$$
$$1 < g^{-1}h \Rightarrow h^{-1} < g^{-1}hh^{-1}$$
$$\Rightarrow h^{-1} < g^{-1}$$
$$ii)\; g_1 < h_1 \Rightarrow g_1g_2 < h_1g_2 < h_1h_2$$

The final inequality follows because $g_2 < h_2$, applying left invariance of h_1.

□

Statement $i)$ above need not hold for an LO on a group, although we showed that it always holds in the case that $h = 1$. So we've only seen one example of an orderable group, the real numbers. We'd like to see some more, but explicit orderings can be very difficult to write down, as we'll see later. For now, we can give an important class of non examples via the following:

Lemma: *If G is left orderable, it is torsion-free.*

Proof: Given $g \in G/\{1\}$, wlog assume $g > 1$. By left invariance,

$$g^n > g^{n-1} > \cdots > 1$$

For any n, this series does not terminate, since the inequalities are strict.

□

In particular, this rules out all finite groups from being left orderable. Another class of not-necessarily-examples is quotient groups. Because subgroups inherit orders, we might guess that quotient groups also do. By simply quotienting something orderable to get something finite, we see that this is not the case. For example, the quotient

$$\mathbb{Z} \to \mathbb{Z}_n$$

does not transport an order.

Remark: If $\{G_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of groups, the direct product of G_λ is the set

$$\prod_{\lambda\in\Lambda} G_\lambda$$

with multiplication defined componentwise, i.e. the elements of the product are λ[1]-tuples, where multiplication is performed in the individual components where we know how to multiply.

Definition: A well order (WO) is an STO on a set X such that for every nonempty $A \subset X$, A has a least element, that is $\exists\, x_0 \in A$ such that $x_0 < x \;\forall\, x \in A$. We might want to know which sets can be well ordered. It might seem like all sets can do this, and indeed that statement is equivalent to the axiom of choice [2]

[1] To be pedantic, these should be labelled as Card(Λ)-tuples

[2] I only understand this on an intuitive level, since I've never had any formal contact with the axiom of choice.

Theorem: *The family G_λ is* $\begin{pmatrix} LO \\ BO \end{pmatrix}$[3] *iff the direct product is* $\begin{pmatrix} LO \\ BO \end{pmatrix}$

Proof: The $\Leftarrow$ implication is easy because we realize each G_λ as a subgroup of the direct product, so that it inherits any ordering. In the other direction, assume we have an $\begin{pmatrix} LO \\ BO \end{pmatrix}$ ordering $<_\lambda$ on each group G_λ. To construct an $\begin{pmatrix} LO \\ BO \end{pmatrix}$ ordering on the direct product, take two elements, $(g_\lambda), (h_\lambda)$. Assuming AoC, we can well order the indexing set, Λ. Choose a well ordering, $\prec$. Assuming $(g_\lambda) \neq (h_\lambda)$, the set of indices where $g_i \neq h_i$ is a nonempty subset of Λ, and thus has a least element, λ_0. We define

$$(g_\lambda) < (h_\lambda) \iff g_i <_{\lambda_0} h_i$$

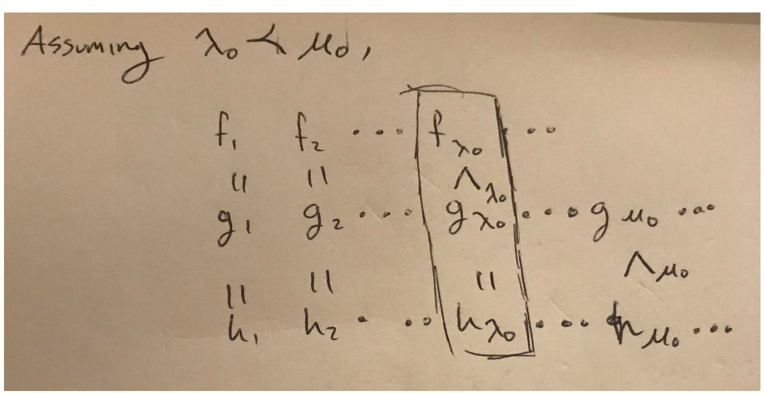
Intuitively, this is the same as comparing numbers. Start at their first digit, compare. If they're the same, go to the next digit, and continue comparing until they're not equal, and compare there. To show this is a transitive relation, suppose $f < g, g < h$. Then for each relation, we have a least index where the elements being compared differ λ_0, μ_0. If it happens that $\lambda_0 = \mu_0$, then we have $f_{\lambda_0} <_{\lambda_0} g_{\lambda_0}$ and $g_{\lambda_0} <_{\lambda_0} h_{\lambda_0}$, and $f_i = g_i = h_i$ for $i \prec \lambda_0$. But $<_{\lambda_0}$ was an STO so by transitivity, $f_{\lambda_0} <_{\lambda_0} h_{\lambda_0} \Rightarrow f < h$. If $\lambda_0 \prec \mu_0$, again $f_i = g_i = h_i$ for $i \prec \lambda_0$. We know g and h agree on all indices $j \prec \mu_0$, in particular at λ_0, i.e.

$$g_{\lambda_0} = h_{\lambda_0}$$

Combining this with the fact that $f < g$, and they disagree at λ_0, we have

$$f_{\lambda_0} <_{\lambda_0} g_{\lambda_0} = h_{\lambda_0}$$
$$\Rightarrow f < h$$

The case where $\mu_0 \prec \lambda_0$ follows in the same manner. During class, I remember being completely confused by this, and getting lost in symbols, and had to revisit after class to understand it, so I drew a little picture that may help put things in place[4]:



[3]Meaning each individual G_λ is _____orderable

[4]After having made the "diagram" I realized how simple the idea was, and probably nobody had any problem understanding this proof, but I went to the trouble of drawing it so I may as well use it.

So this is a transitive binary relation. It is totally ordered because the $<_\lambda$ are. It is left(right) invariant because multiplciation doesn't change the index λ_0. To belabor the point, if we have $g < h$, then $g_i = h_i$ for $i \prec \lambda_0$ and $g_{\lambda_0} <_{\lambda_0} h_{\lambda_0}$. Thus, for $i \prec \lambda_0$,

$$(fg)_i = f_i g_i = f_i h_i \text{ because } g_i = h_i$$
$$(fg)_{\lambda_0} = f_{\lambda_0} g_{\lambda_0} <_{\lambda_0} f_{\lambda_0} h_{\lambda_0}$$
$$\Rightarrow fg < fh$$

which shows $<$ is an LO on the direct product.

$\square$

We now know that we can build orderable groups out of known orderable groups by simply taking their direct product.

Definition: Given a family of indexed groups $\{G_\lambda\}_{\lambda\in\Lambda}$, the direct sum, $\sum_{\lambda\in\Lambda} G_\lambda$ is the subgroup of the direct product $\prod_{\lambda\in\Lambda} G_\lambda$ consisting of only those elements such that all but finitely many coordinates are 1.

Corollary: *G_λ is $\begin{pmatrix} LO \\ BO \end{pmatrix}$ iff the direct sum is $\begin{pmatrix} LO \\ BO \end{pmatrix}$.*

Proof: The direct sum is a subgroup of the direct product, so it inherits the order.

$\square$

Corollary: *Free abelian groups are bi orderable.*

Proof: The free abelian group on a set Λ is just a direct sum over $\mathbb{Z}$:

$$Free(\Lambda)_{Ab} = \sum_{\lambda\in\Lambda} \mathbb{Z}$$

$\square$

Definition: Let $<$ be an LO on G. The positive cone, $P_<$ is the set of positive elements:

$$P_\lambda = \{g \in G |\, g > 1\}$$

Lemma:

$$i)\ P_\lambda \ is\ a\ sub\ semi-group.$$
$$ii)\ G = P_\lambda \sqcup P_\lambda^{-1} \sqcup 1$$
$$iii)\ < is\ a\ BO\ iff\ f^{-1}P_\lambda f \subset P_\lambda\ \forall f \in G$$

Proof: We have already proven all the facts needed for this lemma. The only interseting part is that P_λ is a sub *semi*-group rather than a subgroup. Recall the distinguishing features of a semi group is it is a group that need not have an identity element, and elements need not have multiplicative inverses. Indeed that is the case here, there are neither. The identity element does not satisfy $1 < 1$, and if $g \in P_\lambda$, i.e. $g > 1$, then $g^{-1} < 1$, i.e. $g^{-1} \notin P_\lambda$.

$\square$

Definition: $P \subset G$ is a positive cone if it satisfies properties $i)$ and $ii)$ above.

Lemma: *If $P \subset G$ is a positive cone, then $g < h \iff g^{-1}h \in P$ defines an LO on G, with*

$P_< = P$.
Combining the previous two lemmas, we have a correspondence

$$\{LO's\ on\ G\} \Longleftrightarrow \{positive\ cones\ in\ G\}$$
$$\{BO's\ on\ G\} \Longleftrightarrow \{conjugation\ invariant\ positive\ cones\ on\ G\}$$

This correspondence is the essence of where topology comes into the discussion. Left and right orderability are algebraic concepts. Positive cones are nice subsets of G, i.e. elements of the power set, which can be topologized. So the space of left orders corresponds to a subspace of a topological space. Let's revisit the topology on a power set[5]. Given a set, X, its power set can be viewed as $P(X) = \text{Hom}(X, \{0,1\})$. We can always view functions $X \to Y$ as a nice[6] subset of $X \times Y$, so if we give $\{0,1\}$ the discrete topology $P(X) = \text{Hom}(X, \{0,1\}) \subset \{0,1\}^{\text{Card}(X)}$ which is given the product topology[7]. So the space of left orders on a group is a topological space, and in some cases it may look odd. It could be a cantor set, or other weird things. The cardinalities of these spaces is interesting. Some groups admit only finitely many LO's. In such a case, the order is always 2^n, for some n. If a group does admit infinitely many LO's, it is always uncountably many. Following Tychonoff's theorem, the space of orders is compact, and we will see a proof of this later.
Theorem: Free(2) *is LO.*
Proof: An explicit ordering was constructed, but I didn't pay close enough attention. It was gross. Look at Jackson's notes for a discussion of this.
Corollary: Any free group on a countable number of generators is also orderable.
Proof: Any free group on a countable number of generators is isomorphic to a subgroup of Free(2). This comes from covering space theory[8]. We'll end with a theorem that might give insight to how this will arise in the future:
Theorem: *Given an SES,*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

i) If H,Q are LO $\Rightarrow$ G is LO
ii) If Q is LO and H has a conjugation invariant BO, then G is BO.

[5]Because I forgot it already.
[6]Include conditions here so that you ensure you actually get functions which are well defined. This is not a note to myself, I'm not going to come back and do this.
[7]A nice way to think of the product topology is that it is the coarsest topology which makes projections continuous, projections being the main structure you get when you have a product of spaces. This is a nice way to think of a lot of 'derived' topologies, like quotients, products, etc. What topology makes the important thing continuous? That's the natural topology to give the 'derived' object.
[8]I know the general idea of this statement, but I'd like to come back and flesh this out. I've forgotten too much alg top already.

CHAPTER 2

Extensions, Torsion, and Orderability

Lecture 2, Jan 23. So last time, we stated the theorem related to extensions of G. We will now prove it:
Theorem: *Given an SES,*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

i) If H,Q are LO $\Rightarrow$ G is LO
ii) If Q is LO and H has a conjugation invariant BO, then G is BO.
Proof: i) To define a left order on G, we will specify the positive elements. Let $H = \text{Ker}\,\varphi < G$, and let P_H, P_Q be positive cones on H and Q respectively. Define

$$P = \varphi^{-1}(P_Q) \sqcup P_H$$

To show it is actually a positive cone, we need to show two things: That it is closed under multiplication, and $G = P \sqcup P^{-1} \sqcup 1$. For i), take $g, h \in P$, then there are 3 cases corresponding to where g and h live:

1. If $g, h \in P_H$, then $gh \in P_H \subset P$ becuase P_H is a positive cone.
2. If, $g, h \in \varphi^{-1}(P_Q)$, then $\varphi(g), \varphi(h) \in P_Q \Rightarrow \varphi(gh) = \varphi(g)\varphi(h) \in P_Q \Rightarrow gh \in \varphi^{-1}(P_Q)$.
3. WLOG, if $g \in \varphi^{-1}(P_Q), h \in P_H)$, then $\varphi(gh) = \varphi(g)\varphi(h) = \varphi(g)e = \varphi(g) \in P_Q \Rightarrow gh \in \varphi^{-1}(P_Q)$.

ii) By exactness,

$$G = H/\{1\} \sqcup \varphi^{-1}(Q/\{1\}) \sqcup \{1\}$$

But

$$\varphi^{-1}(Q/\{1\}) = \varphi^{-1}(P_Q) \sqcup \varphi^{-1}(P_Q^{-1}) \text{ and } H/\{1\} = P_H \sqcup P_H^{-1}$$

So substituting into the first equation gives the result.

$\square$

Example: Let X^2 be the Klein Bottle, and

$$K = \langle a, b \mid b^{-1}ab = a^{-1} \rangle$$

be its fundamental group. Then there is an SES

$$1 \longrightarrow \langle a \rangle = \mathbb{Z} \longrightarrow K \longrightarrow \mathbb{Z} \longrightarrow 1$$

So K is LO. However, K is not BO. If it was, then

$$\begin{aligned} a > 1 &\iff b^{-1}ab > 1 \text{ by LO} \\ &\Rightarrow a^{-1} > 1 \end{aligned}$$

which is a contradiction. Recall $\mathbb{Z}$ has only two orders, so there are 4 choices of LO on K.
Theorem: *These are the only left orders on K.*
Proof: It suffices to show that each determines a unique LO[1]. We first note that all elements of K can be written as $a^m b^n$. Take an LO on K, $<$. If $a > 1$, $b > 1$, then $a^k < b \ \forall\, k \in \mathbb{Z}$.
Claim: $a^m b^n > 1 \iff n > 0$ or $n = 0$ and $m > 0$. This defines a positive cone. The $n = 0$ case is trivial. If $n > 0$, then $a^{-k} b > 1 \ \forall\, k$, by the above claim. Then $b^n > b$ so $a^{-k} b^n > a^{-k} b > 1$. If $n < 0$, then $a^m b^n = b^n a^{\pm m} = (a^{\mp m} b^{-n})^{-1}$. By the first statement, this is > 1, so its inverse is less than 1, which shows the result.

In general, if $<$ is an LO on G, and $\alpha : G \to G$ is an automorphism, $(<, \alpha)$ induces an LO on G by $g <_\alpha h \iff \alpha(g) < \alpha(h)$. There exist automorphisms of K, α_1, α_2, such that

$$\alpha_1(a) = a, \quad \alpha_1(b) = b^{-1}$$

and

$$\alpha_2(a^{-1}) = a, \quad \alpha_2(b) = b^{-1}$$

So $<_{(i)}$ is the unique LO on K determined by i). The LO $<_{(ii)}$ is determined by $(<_{(i)}, \alpha_1)$. $<_{(iii)}$ is determined by $(<_{(i)}, \alpha_2)$, and $<_{(iv)}$ is determined by $(<_{(i)}, \alpha_1\alpha_2)$.

□

As we mentioned before, if a group has finitely many left orders, the number of left orders is 2^n for some n. K is an example of that fact, which has 2^2 left orders, each factor of two coming from a choice on $\mathbb{Z}$. As an exercise, you can show that $\forall\, n > 0$, there is a group with 2^n left orders[2].
Corollary: *For any LO on K, $h \in \langle a \rangle$, $g \in K/\langle a \rangle$, $g > 1 \Rightarrow g > h$.*
Suppose M is a closed, orientable 3-fold. We may ask "Is $\pi_1(M)$ LO?" The only strong statement we have is that it must be torsion free if we hope for the answer to be yes. So already we have examples of spaces whose fundamental group is not LO. If $M = L(p, q)$ is a Lens space, we know $\pi_1(M) = \mathbb{Z}_p$, so we have an infinite class of non examples. If $M = M_1 \# M_2$, then we know $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, e.g. $X = M \# L(p, q) \Rightarrow \pi_1(X)$ is not LO. However, $T^3 = S^1 \times S^1 \times S^1$ is LO, since the property "being LO" commutes with direct product. $M = \#_n(S^1 \times S^2) \cong F_n$ is also LO.

[1] I don't understand this remark.
[2] Hint: examine the exact sequence we just used and iterate.

Until now, we might be led to believe that being torsion free implies that a group is LO. This is not the case:

Example: Take X^2 the Klein Bottle, $K = \pi_1(X^2)$ and consider $X = X^2 \times I$. This is a non orientable 3-fold. Take the 2-fold cover

$$p : T^2 \to X^2$$

so that

$$p_*(\pi_1(T^2)) = \langle a, b^2 \rangle \cong \mathbb{Z} \times \mathbb{Z} < K$$

and let N be the mapping cylinder of p, i.e.

$$N = (T^2 \times I) \sqcup X^2 \Big/ \Big\{ (x, 0) \sim p(x) \; \forall \, x \in T^2 \Big\}$$

and N_1, N_2 two copies of N. Then of course, $\pi_1(N_i) = K \equiv \langle a_i, b_i \mid \dots \rangle$, since there is a strong deformation retraction $N \to X^2$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Then

$$M = N_1 \sqcup_\varphi N_2$$

is a closed, orientable 3-fold, and $\pi_1(M) = K_1 *_{\mathbb{Z}\times\mathbb{Z}} K_2$, which is torsion free.

Theorem: $H_1(M)$ is finite, so $\pi_1(M)$ is not LO

Proof: φ is determined by, up to isotopy, $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$, and we denote $H_1(\partial N_i) = \langle a_i, 2b_i \rangle$. Then φ_* is a matrix, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, so that

$$\varphi_*(a_1) = pa_2 + 2qb_2$$
$$\varphi_*(2b_1) = ra_2 + 2sb_2$$

Denote

$$H_1(N_i) = \langle a_i, b_i \mid a_i^2 \rangle$$

Then $H_1(M)$ is presented by

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{pmatrix}$$

so that $H_1(M)$ is finite $\iff$ $\det(A) \neq 0$ $\iff$ $q \neq 0$ $\iff$ $\varphi_*(a_1) \neq \pm a_2$. Suppose $\pi_1(M)$ were LO, then we would induce an LO on the subsets ∂N_i. There are only 4 LO's on $\pi_1(N_i)$.

Corollary: for any LO on $\pi_1(N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$ summand of $\pi_1(\partial N) = \langle a, b^2 \rangle$ such that $h \in \langle a \rangle$ and $g \in \pi_1(\partial N)/\{1\}, g > 1$, then $g > h$.

$$\Rightarrow \varphi_*(\pm a_2)$$

which shows the result.