M392C: Orderability/3-Manifold Groups

Reese Lance

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Abstract

Class taught by Cameron Gordon, notes taken by Reese Lance. The notes were taken by hand during class and typed later. Some of my own thoughts are interjected, but quite rarely. I initially thought to try to separate my thoughts from the professor's but it becomes too difficult. As such I will also try to expand on examples which are mentioned in passing in class, spell out proofs which are glossed over, and add insight where I think it is helpful. This helps to justify the existence of this set of notes, as opposed to live-texed notes (which I believe there are for this class, thanks to Jackson van Dyke), which are probably slightly better for a faithful representation of what is being taught in the classroom. Especially because some of my own content is interspersed throughout these notes, any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu) or if you can find any other way to communicate with me, that is also fine. At the moment I'm trying to get the notes written, and worrying about making the format not look like trash later. I'm also not going to track theorem and lemma numbers, though Cameron does, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked. Also I always forget to write down the number. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers, which is what would have been the plan. Also thanks to Arun Debray whose formatting choices inspired my own.

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CHAPTER 1

Overview and Preliminary Group Theory

1 Lecture 1, Jan 21.

The book we will be following is "Ordered Groups and Topology" by Clay and Rolfsen, a pdf of a draft of which can be found here. Orderability was originally developed as a group theoretic concept, and only (somewhat) recently has topology entered the picture. In fact, some of the bare bones definitions can be made entirely in Set:

Definition: Given a set X, a strict, total order (STO) on X is a binary relation $<$, satisfying

$$i) \text{ Transitivity} : \forall\ x, y, z \in X, x < y, y < z \Rightarrow x < z$$
$$ii) \text{ Totality: } \forall\ x, y \in X, \text{exactly one of the following holds:}$$
$$x < y, \quad y < x, \quad \text{or } x = y$$

For convenience, we may also define the STO $>$ as induced by an STO:

$$x > y \iff x \neq y \text{ and } x \not< y$$

If we upgrade to Grp, we can use the multiplication structure to talk about invariance:
Definition: Given a group, G, and an STO, $<$, on the underlying set, $<$ is called a left order (LO) on G if

$$g < h \Rightarrow fg < fh\ \forall\ f \in G$$

and we call G left orderable if there exists a left ordering of G. Similarly, we can define right-orderability in the obvious way. One might ask how different the concepts of left and right orders are. It is clear that for abelian groups, they are one and the same, that is: all left orders are also right orders. In general this may not hold. If it does,
Definition: Given an LO, $<$, on G, it is called a bi order (BO) if it is also an RO, and we say G is bi orderable if it admits a bi order.
To return to the above question, given an LO $<$, we can induce an R, $\prec$ O in the following way:

$$g \prec h \iff h^{-1} < g^{-1}$$

Lemma: *The above binary relation is an RO on G.*
Proof: Transitivity and totality of $\prec$ follow immediately from $<$ being an STO. To show it

is a right order, consider elements g, h, f of G such that

$$g \prec h$$

Then

$$\begin{gathered} h^{-1} < g^{-1} \\ \Rightarrow f^{-1}h^{-1} < f^{-1}g^{-1} \\ \Rightarrow (f^{-1}h^{-1})^{-1} \succ (f^{-1}g^{-1})^{-1} \\ \Rightarrow gf \prec hf \end{gathered}$$

$\square$

So every left order induces a right order, i.e. all left orderable groups are also right orderable. Note that this does not imply that all left orders are bi orders. That would require that all left orders are **themselves** right orders. All we have shown is that a left order **induces** a right order. Indeed, it is not the case: There are groups which are left orderable but not bi orderable. We obviously can restrict the LO to an LO on a subgroup $H < G$ (denoting subgroup here), just as an STO can be restricted to a subset $A \subset X$.

Example: $(\mathbb{R}, +)$ is bi ordered, with the usual ordering. Its subgroups $\mathbb{Z}, \mathbb{Q}$, etc. can inherit this ordering.

Here is a series of elementary lemmas:

Lemma: *Given $<$ an LO on G,*

$$\begin{gathered} i)\ g > 1, h > 1 \Rightarrow gh > 1 \\ ii)\ g > 1 \Rightarrow g^{-1} < 1 \\ iii)\ < \text{ is BO} \iff g < h \Rightarrow f^{-1}gf < f^{-1}hf\ \forall\ f \in G \\ (\text{invariant under conjugation}) \end{gathered}$$

Proof:

$$\begin{gathered} i)\ h > 1, \Rightarrow gh > g > 1 \\ ii)\ g > 1 \Rightarrow g^{-1}g > g^{-1} \Rightarrow 1 > g^{-1} \\ iii) \text{ Assume } < \text{ is also an RO. Then we can clearly apply left invariance on } f^{-1} \\ \text{and right invariance on } f \text{ to obtain the result. If } < \text{ is conjugation invariant, then} \\ g < h \Rightarrow f^{-1}gf < f^{-1}hf \\ gf < hf \end{gathered}$$

Using left invariance in the final line.

$\square$

Lemma: *Given $<$ a BO on G,*

$$\begin{gathered} i)\ g < h \Rightarrow g^{-1} > h^{-1} \\ ii)\ g_1 < h_1, g_2 < h_2 \Rightarrow g_1g_2 < h_1h_2 \end{gathered}$$

Proof:

$$i)\ g < h \Rightarrow g^{-1}g < g^{-1}h$$
$$1 < g^{-1}h \Rightarrow h^{-1} < g^{-1}hh^{-1}$$
$$\Rightarrow h^{-1} < g^{-1}$$
$$ii)\ g_1 < h_1 \Rightarrow g_1g_2 < h_1g_2 < h_1h_2$$

The final inequality follows because $g_2 < h_2$, applying left invariance of h_1.

□

Statement *i)* above need not hold for an LO on a group, although we showed that it always holds in the case that $h = 1$. So we've only seen one example of an orderable group, the real numbers. We'd like to see some more, but explicit orderings can be very difficult to write down, as we'll see later. For now, we can give an important class of non examples via the following:

Lemma: *If G is left orderable, it is torsion-free.*

Proof: Given $g \in G/\{1\}$, wlog assume $g > 1$. By left invariance,

$$g^n > g^{n-1} > \cdots > 1$$

For any n, this series does not terminate, since the inequalities are strict.

□

In particular, this rules out all finite groups from being left orderable. Another class of not-necessarily-examples is quotient groups. Because subgroups inherit orders, we might guess that quotient groups also do. By simply quotienting something orderable to get something finite, we see that this is not the case. For example, the quotient

$$\mathbb{Z} \to \mathbb{Z}_n$$

does not transport an order.

Remark: If $\{G_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of groups, the direct product of G_λ is the set

$$\prod_{\lambda\in\Lambda} G_\lambda$$

with multiplication defined componentwise, i.e. the elements of the product are λ[1]-tuples, where multiplication is performed in the individual components where we know how to multiply.

Definition: A well order (WO) is an STO on a set X such that for every nonempty $A \subset X$, A has a least element, that is $\exists\ x_0 \in A$ such that $x_0 < x\ \forall\ x \in A$. We might want to know which sets can be well ordered. It might seem like all sets can do this, and indeed that statement is equivalent to the axiom of choice [2]

[1] I'm not actually sure if the appropriate notation is to use λ or Λ here.

[2] I only understand this on an intuitive level, since I've never had any formal contact with the axiom of choice.

Theorem: *The family G_λ is $\binom{LO}{BO}$*[3] *iff the direct product is* $\binom{LO}{BO}$

Proof: The $\Leftarrow$ implication is easy because we realize each G_λ as a subgroup of the direct product, so that it inherits any ordering. In the other direction, assume we have an $\binom{LO}{BO}$ ordering $<_\lambda$ on each group G_λ. To construct an $\binom{LO}{BO}$ ordering on the direct product, take two elements, $(g_\lambda), (h_\lambda)$. Assuming AoC, we can well order the indexing set, Λ. Choose a well ordering. Assuming $(g_\lambda) \neq (h_\lambda)$, the set of indices where $g_i \neq h_i$ is a nonempty subset of Λ, and thus has a least element, λ_0. We define

$$(g_\lambda) < (h_\lambda) \iff g_i <_{\lambda_0} h_i$$

Intuitively, this is the same as comparing numbers. Start at their largest digit, compare. If they're the same, go to the next digit, and continue until they're not equal, and compare there. To show this is a transitive relation, suppose $f < g, g < h$. Then for each relation, we have a least index where they differ, λ_0, μ_0. Suppose $\lambda_0 \leq \mu_0$.

[3] Meaning each individual G_λ is _____orderable