M392C: Symplectic Topology

Reese Lance

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Abstract

Class taught by Timothy Perutz at the University of Texas at Austin, notes taken by Reese Lance. The notes are not live texed. They are post-mortem-texed, that is, taken by hand during class and typed later. Some of my own thoughts are interjected, but rarely. As such I will also try to expand on examples which are mentioned in passing in class and spell out proofs which are glossed over. This helps to justify the existence of this set of notes, as opposed to live-texed notes (which are often available for classes at this university), which are probably slightly better for a faithful representation of what is being taught in the classroom. Any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu). At the moment I'm trying to get the notes written, and worrying about the format later, possibly never. I'm also not going to track theorem and lemma numbers, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers next to things, which is what would have been the plan. Thanks to Arun Debray, whose formatting choices inspired my own.

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CHAPTER 1

Introduction to Symplectic Vector Spaces and Manifolds

Lecture 1, Aug 27. We begin in a simple setting where we can speak about the notion of "symplectic forms".

Definition: A symplectic vector space (SVS), (V, β) is a vector space over $\mathbb{R}$, equipped with a bilinear, skew-symmetric, non-degenerate form $\beta : V \times V \to \mathbb{R}$.

Recall: Skew symmetric forms are those which satisfy the condition $\beta(x, y) + \beta(y, x) = 0$[1]. Non-degeneracy is the condition that the map $V \to V^*$ sending $v \mapsto \beta(v, -)$ is an isomorphism. More concretely[2], β is non-degenerate if, fixing

$$v \in V,\ \beta(v, y) = 0\ \forall\, y \in V \Rightarrow v = 0$$

If we pick a basis $\{e_i\}$ for V, then

$$\beta_{ij} = \beta(e_i, e_j)$$

is an invertible, skew-symmetric matrix.

Exercise: If (V, β) is a SVS, show that $\dim V = 2n$ for some n, and admits a symplectic basis, $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, i.e. one which satisfies the equations:

$$\beta(e_i, e_j) = 0 = \beta(f_i, f_j), \qquad \text{and} \qquad \beta(e_i, f_j) = \delta_{ij}$$

Proof: We note the first statement is implied by the second. To prove the second statement, pick a non-zero vector, v, and set $e_1 = v$. Because β is non-degenerate, there exists $w \in V$ such that $\beta(v, w) \neq 0$. Note that this implies w and v are linearly independent. Set $f_1 = \frac{w}{\beta(v,w)}$. We see that

$$\beta(e_1, f_1) = 1$$

If $\text{span}(e_1, f_1) = V$, we are done. If not, assume that we have repeated this step $n - 1$ times, that is, we have basis vectors $\{e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}\}$ (call E the span of these vectors) such that $\beta(e_i, e_j) = 0 = \beta(f_i, f_j)$ and $\beta(e_i, f_j) = \delta_{ij}$. If we assume $E \neq V$, then we recall from linear algebra that $V = E \oplus E^\perp$, i.e. there is some vector $e_n \in V$ which is not in the span of the $2n - 2$ basis vectors, and which pairs to 0 with all the $\{e_i\}$ and $\{f_i\}$.

[1] When not working over a field of characteristic 2, this is equivalent to the condition that $\beta(x, x) = 0$

[2] These are only equivalent in finite dimensional vector spaces

Because $e_n \neq 0$, there is some vector $u \neq 0$, such that $\beta(e_n, u) \neq 0$. In particular, u and e_n are linearly independent. Now set $f_n = \frac{u}{\beta(e_n,u)}$. Either $f_n \in E$ or $f_n \in E^{\perp}$. Assume $f_n \in E$. Then

$$\begin{aligned}
f_n &= \sum_i \lambda_i e_i + \sum_j \omega_j f_j \\
\Rightarrow \boxed{\beta\left(e_n, f_n\right)} &= \beta\left(e_n, \sum_i \lambda_i e_i + \sum_j \omega_j f_j\right) \\
&= \sum_i \lambda_i \beta(e_n, e_i) + \sum_j \omega_j \beta(e_n, f_j) = \boxed{0} \qquad \perp
\end{aligned}$$

so $f_n \in E^{\perp}$, i.e. it pairs to 0 with every basis vector except e_n, which completes the induction proof. Because V is f.d., this process will terminate eventually, and dim $V = 2n$ for some n, with the symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$.

Technically, we have proven the first statement already, but I wanted to prove it independently just for fun. For the first result, I'm going to try to work with the matrix β_{ij} instead. I initially tried to show that you could do elementary matrix operations to any skew symmetric, invertible matrix to get it into the form

$$\beta_{ij} = \begin{pmatrix} 0_n & I \\ -I & 0_n \end{pmatrix}$$

which should suffice, but I wasn't able to[3]. So I came up with the following:
Pick any basis of V. Then β_{ij} is an invertible, skew symmetric matrix. But we know

$$\det \beta_{ij}^T = \det\ -\beta_{ij}$$

Multiplying the matrix by a scalar has the effect of multiplying each of the n rows by that scalar, which gives applies n factors of (-1) to the determinant, i.e.

$$\det\ -\beta_{ij} = (-1)^n \det \beta_{ij}$$

From which it follows that if n is odd, β_{ij} is singular[4], a contradiction. Thus n is even.

□

We know[5] $\beta \in \bigwedge^2(V^*)$. If we pick a (symplectic) basis of V, and $\{e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*\}$ is the dual basis[6], then

$$\beta = \sum_i e_i^* \wedge f_i^*$$

[3] I think it should be true though? I think it's equivalent to the problem statement

[4] again, assuming infinite characteristic

[5] We recall the exterior algebra is a quotient of the tensor algebra by elements of the form $x \otimes x$, so the exterior algebra consists of antisymmetric tensors. The wedge product is the "image of the tensor product" through this quotient. So we are saying that β is an anti-symmetric 2-tensor, i.e. it is an object which eats 2 vectors and anti-symmetrically returns an element of the base field

[6] The dual basis is the basis $\{v^i\}$ of V^* such that $v^i \cdot v_j = \delta_{ij}$

we will refer to this as the standard symplectic pairing on $\mathbb{R}^{2n}$, β_0. The matrix represented by β_0 is

$$(\beta_0)_{ij} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0 \end{pmatrix}$$

This corresponds to the standard almost complex structure[7] on $\mathbb{C}^n$, multiplication by i.

Definition: An <u>isomorphism of symplectic vector spaces $(V, \beta) \to (V', \beta')$</u> is a linear isomorphism $\varphi : V \to V'$ which is compatible with the symplectic pairing:

$$\beta'(\varphi v, \varphi w) = \beta(v, w)$$

or equivalently, that the diagram

$$\begin{tikzcd} V \times V \arrow[r, "\varphi \times \varphi"] \arrow[d, "\beta"'] & V' \times V' \arrow[dl, "\beta'"] \\ \mathbb{R} & \end{tikzcd}$$

commutes. In fancier language, if

$$\varphi^* : \bigwedge^2(V'^*) \to \bigwedge^2(V^*)$$
$$\varphi^*(\beta') = \beta$$

I believe the induced map here is given by taking the isomorphism $V \to V^*$ provided by β, then inducing a map on the exterior powers $\bigwedge^2(\varphi)$, given by applying φ to each component of the wedge. Then we my re-interpret our definition of a symplectic basis as an isomorphism $b : (V, \beta) \to (\mathbb{R}^{2n}, \beta_0)$.
Definition: The <u>symplectic linear group, Sp(V, β)</u>, is the group of isomorphisms $(V, \beta) \to (V, \beta)$.

We typically abbreviate Sp$(\mathbb{R}^{2n}, \beta_0)$ as Sp$(\mathbb{R}^{2n})$.
Exercise: Let $A \in GL_{2n}(\mathbb{R})$ and $\{e_i, f_i\}$ a symplectic basis for $\mathbb{R}^{2n}$. Then

$$A \in \mathrm{Sp}(\mathbb{R}^{2n}) \iff A^T J_0 A = J_0$$

where $J_0 = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$.
Interpret this condition in terms of block matrices.
Proof: ($\Rightarrow$:) If $A \in \mathrm{Sp}(\mathbb{R}^{2n})$, then A commutes with the standard symplectic pairing on $\mathbb{R}^{2n}$, i.e.

$$(\beta_0)(Av, Aw) = \beta_0(v, w)$$

still need to come back and do this

□

[7] An almost complex structure is a linear endomorphism on each tangent space of the manifold, varying smoothly from point to point, which squares to -1

Let (V, β) have determinant line $\det V^* := \bigwedge^{2n} V^*$. Then claim $\beta^n := \overbrace{\beta \wedge \cdots \wedge \beta}^{n \text{ times}}$ is a non-zero element of the determinant line. To show this,

$$\beta_0 = \sum_i e_i^* \wedge f_i^*$$
$$\Rightarrow \beta_0^n = n!(e_i^* \wedge f_i^*) \wedge \cdots \wedge (e_n^* \wedge f_n^*) \neq 0 \in \det \mathbb{R}^{2n*}$$

Then because $(V, \beta) \cong (\mathbb{R}^{2n}, \beta_0)$, we are done.
Then, $\forall\ A \in \mathrm{Sp}(V, \beta)$,

$$A^*\beta = \beta \Rightarrow A^*\beta^n = \beta^n$$

and

$$A^*(\beta^n) = (\det A)\beta^n \Rightarrow \det A = 1$$
$$\Rightarrow \mathrm{Sp}(V, \beta) \subset \mathrm{SL}(V)$$

we can interpret the first equality as the definition of $\det A$. So symplectic isomorphisms are volume preserving.

Onto symplectic manifolds.
Definition: A symplectic manifold is a smooth (C^∞) manifold equipped with a closed, non-degenerate 2-form, ω.

We think of ω as a skew-symmetric pairing on each tangent space which varies smoothly from tangent space to tangent space. Here, non-degeneracy implies

$$T_pM \to T_p^*M$$
$$v \mapsto \omega_p(v, -) \text{ is an isomorphism}$$

Definition: A diffeomorphism $\varphi(M, \omega_M) \to (N, \omega_N)$ is called a symplectomorphism if it pulls back ω_N to ω_M, i.e.

$$\varphi^*(\omega_N) = \omega_M$$

We interpret φ^* as the pull-back of tensor fields, since ω is a $(0, 2)$ tensor.
Definition: $\mathrm{Aut}(M, \omega) = \Big\{\text{symplectomorphisms } (M, \omega) \to (M, \omega)\Big\}$.

The prototype symplectic vector space is $\mathbb{R}^{2n}, \omega_0$. If we choose coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, then

$$\omega_0 = \sum_i dx_i \wedge dy_i$$

In this case, the coordinate vector fields,

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)$$

form a symplectic basis[8] for $T_{(x,y)}\mathbb{R}^{2n}$.

Theorem (Darboux): *For all ω on M (dimension $2n$) and $\forall p \in M$, $\exists$ neighborhoods $U \ni p$ and $U' \ni 0 \in \mathbb{R}^{2n}$ such that $\phi : U \to U'$ is a symplectomorphism.*
Proof: We will come back and prove this later, once we have more machinery.

In spirit, this theorem says that you can always find local coordinates in which the coordinate vector fields form symplectic bases for the tangent space of any (even dimension) manifold, as we were able to do globally in the case of $\mathbb{R}^{2n}$.
As a consequence, we may re-interpret the definition of a symplectic manifold as an equivalence class of symplectic atlases:

$$A = \left\{(U_i, \phi_i) \mid \phi_j \circ \phi_i^{-1} \text{ is a symplectomorphism}\right\}$$

where two atlases are considered equivalent if their union is a symplectic atlas.
Question: Does there exist a symplectomorphism of $\mathbb{R}^{2n}$ which carries the open unit ball $B^{2n}(0;1)$ to a ball of strictly smaller radius?
Answer: No, symplectomorphisms preserve the standard volume form on $\mathbb{R}^{2n}$. As a precursor to this, we also saw that in the context of vector spaces, isomorphisms have determinant 1. For the manifold computation,

$$\omega_0^n = n!\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$
$$\phi^*(\omega_0) = \omega_0 \Rightarrow \phi^*(\omega_0^n) = \omega_0^n$$

So ϕ preserves the standard volume form, and volume is calculated by integrating against the volume form:

$$\text{volume}(B) = \int_B \text{vol}$$

$\square$

Question: Does there exist a symplectomorphism of $\mathbb{R}^{2n}$ which carries the open ball $B^{2n}(0;1)$ into the cylinder $B^2(0;r) \times \mathbb{R}^{2n-2}$ for $r < 1$?
Answer: No. This result is Gromov's non-squeezing theorem, which we will also prove later.

[8] wouldn't this be more appropriately called something like a symplectic frame of (M, ω)?

CHAPTER 2

Lagrangian Submanifolds

Lecture 2, Sep 1. There are many important types of submanifolds of symplectic manifolds that emerge in the study of symplectic topology, but the most ubiquitous example is that of the Lagrangian submanifold.
Definition: A Lagrangian immersion is an immersion[1] $i : L \to (M, \omega)$ such that $\dim L = \dim M/2$ and $i^*\omega = 0$.

i.e. $\forall u, v \in TL,\ \omega(u, v) = 0$. Similarly we may define Lagrangian embeddings.
Definition: A Lagrangian submanifold is the image of a Lagrangian embedding.

Recall that symplectomorphisms preserve the symplectic form, so they also carry Lagrangian submanifolds to Lagrangian submanifolds.
Example: Define $C(r_1, \dots, r_n) \subset (\mathbb{R}^{2n}, \omega_0) = \underbrace{\mathbb{R}^2 \times \cdots \times \dots \mathbb{R}^2}_{\text{n times}}$ by

$$C(r_1, \dots, r_n) := C(r_1) \times \cdots \times C(r_n)$$

where $C(r)$ is the circle of radius r in $\mathbb{R}^2$. This is a Lagrangian submanifold of $\mathbb{R}^2$. In fact, every curve $\gamma : I \to \mathbb{R}^2$ is Lagrangian because TI is one-dimensional, so $\gamma^*(\omega_0) = 0$. The product of Lagrangian submanifolds is also Lagrangian[2], so $C(r_1, \dots, r_n)$ is a Lagrangian submanifold in $\mathbb{R}^{2n}$.
Corollary: *Every symplectic manifold has a Lagrangian submanifold.*
Proof: First we need a lemma:
Lemma: *If $\varphi : (M, \omega) \to (N, \omega')$ is a symplectomorphism, and $i : L \to N$ is a Lagrangian submanifold, then $\varphi^{-1} \circ i(L) \subset M$ is also a Lagrangian submanifold.*
Proof: Because φ is a diffeomorphism, we have $2\dim L = \dim M$. Further,

$$(\varphi^{-1} \circ i)^*(\omega) = i^* \circ (\varphi^{-1})^*(\omega) = i^*(\omega') = 0$$

□

Returning to the Corollary, by Darboux's theorem, we can take any chart (U, φ) where φ

[1] An immersion is a smooth mapping such that the derivative is injective at all points
[2] If $L_1 \subset M_1$ and $L_2 \subset M_2$ are Lagrangian, then $L_1 \times L_2$ is Lagrangian in $M_1 \times M_2$, otherwise the dimensions don't work out

is a symplectomorphism to an open neighborhood of $0 \in \mathbb{R}^{2n}$, and define a Clifford torus of sufficiently small radius in $\varphi(U) \subset \mathbb{R}^{2n}$, $C(\mathbf{r})$. Then by the previous lemma, $\varphi^{-1}(C(\mathbf{r}))$ is a Lagrangian submanifold of M.

$\square$

Question: Which n-manifolds can appear as Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_0)$? Or which admit a Lagrangian immersion?
Simpler question: Which n-manifolds can be immersed smoothly into $\mathbb{R}^{2n}$? Answer, all of them, by Whiteny embedding. What about embeddings? Answer, most of them, but not all, by strong Whitney embedding.
Proposition: *If $L \subset (\mathbb{R}^{2n}, \omega)$ is a closed, orientable, Lagrangian submanifold, then*

$$\chi(L) = 0$$

and if L is non-orientable, then $\chi(L)$ is even.
Proof: Recall that $J_0 = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$ describes multiplication by i on $\mathbb{C}^n$. Then

$$L \text{ Lagrangian} \quad \Rightarrow \quad T_p\mathbb{R}^{2n} \cong T_pL \oplus J_0 T_pL$$

We will take this isomorphism for granted now, and will be proved later. Given that, J_0 gives an isomorphism

$$J_0|_{TL} : TL \to N_L$$

the normal bundle to L. Then the self-intersection number, $L \cdot L =$Euler #(N_L), the signed count of zeros of a suitably transverse section of N_L.

$$\begin{aligned} &= \text{Euler } \#(TL) \\ &= \chi(L) \end{aligned}$$

by Poincare-Hopf theorem. But the self intersection number of $L \subset \mathbb{R}^{2n} = 0$.

$\square$

Remark: If n is odd, this result is not interesting, since the Euler characteristic is already 0.
Theorem (Gromov): *If $L \subset \mathbb{R}^{2n}$ is compact and Lagrangian, then $b_1(L) > 0$, where b_1 is the first betti number.*

$\square$

Corollary: *S^n, $\mathbb{R}P^n$ cannot emerge as Lagrangian manifolds.*

$\square$

So many manifolds do not admit Lagrangian embeddings. This is an example of symplectic rigidity. If we turn to the immersion question, we have an example of symplectic flexibility, via the theorem
Theorem (Gromov, Lees): *Every n-manifold admits a Lagrangian immersion $L \to \mathbb{R}^{2n}$.*

This interplay of rigidity and flexibility is what makes symplectic topology so interesting. Returning to Clifford tori:
Question: For what $\mathbf{r}$ does there exist $\varphi \in \mathrm{Aut}(\mathbb{R}^{2n}, \beta_0)$ such that

$$\varphi(C(1, \ldots, 1)) = C(r_1, \ldots, r_n)$$

Answer: φ exists only when $\mathbf{r} = (1, \ldots, 1)$. Long, hard proof that I didn't understand.

$\square$

We will now talk about complex projective space. $S^2 \subset \mathbb{R}^3$ admits an $SO(3)$-invariant symplectic form, ω, defined in the following exercise:
Exercise: *For $\omega = i(e) dx_1 \wedge dx_2 \wedge dx_3$, where $e = \frac{1}{2}\nabla|x|^2$, show $\int_{S^2} \omega = 4\pi$.*
Proof: Expanding out, this is

$$\int_{S^2} x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2$$

By Stokes' theorem

$$\begin{aligned} &= \int_{D^3} d\Big(x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2 \Big) \\ &= \int_{D^3} dx_1 \wedge dx_2 \wedge dx_3 + dx_2 \wedge dx_1 \wedge dx_3 + dx_3 \wedge dx_1 \wedge dx_2 \\ &= 3 \int_{D^3} dx_1 \wedge dx_2 \wedge dx_3 = 3\left(\frac{4}{3}\pi R^3\right) = 4\pi \end{aligned}$$

$\square$

We may think of $\mathbb{C}P^n$ as

$$\mathbb{C}P^n = S^{2n+1}/U(1)$$

This space admits a unique symplectic form, τ_n, such that i) τ_n is invariant under the action of $PU(n+1)$, and ii) $\int_{\mathbb{C}P^n} \tau_n = 1$, called the Fubini-Study form.
To expand on condition i), $U(n+1) \curvearrowright \mathbb{C}^{n+1}$, preserving S^{2n+1}. Thus it induces an action $U(n+1) \curvearrowright S^{2n+1}/U(1)$, and the scalars in $U(n+1)$ act trivially, so there is an induced action of $PU(n+1) = \frac{U(n+1)}{U(1)}$.

CHAPTER 3

Symplectic topology of $\mathbb{C}P^n$

Lecture 3, Sept 3. Thinking of $\mathbb{C}^{n+1}$ as a complex vector space, with the standard hermitian product $h_0(z,w) = \sum_j z_j \overline{w}_j$,

$$h_0 = g_0 + i\beta_0$$

where g_0 is the dot product, and β_0 is the standard symplectic pairing. Recall J_0 is the endomorphism of multiplication by i. A couple of facts:
Proposition:

$$i)\beta_0(u, J_0 v) = g_0(u,v)$$
$$ii)\beta_0(v, J_0 v) = g(v,v) > 0$$
$$iii)\beta_0(J_0 v, J_0 v) = \beta_0(v,v)$$

Proof: just plug in

□

Think of $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ as a symplectic manifold with $\omega_0 = \sum_j dx_j \wedge dy_j$, and define

$$\eta := \omega_0|_{S^{2n+1}}$$

Note η is invariant under $U(n+1)$. By definition, $U(n+1)$ acts on $\mathbb{C}^{n+1}$ and preserves S^{2n+1}, but that doesn't guarantee this invariance. Instead, $U(n+1)$ is defined by fixing the hermition form h_0. In particular, then, it fixes its real and imaginary parts, so it fixes ω, and thus η.
We think of $\mathbb{C}P^n = S^{2n+1}/U(1)$. Then

$$T_z S^{2n+1} = T_z(U(1)\cdot z) \oplus z^{\perp}$$

where the orthogonal complement is taken wrt h_0, so $z^{\perp}$ has complex dimension n, and $T_z(U(1)\cdot z)$ has real dimension 1. We have the quotient map

$$\rho : S^{2n+1} \to \mathbb{C}P^n$$

known as the Hopf fibration. The derivative restricts to an isomorphism

$$D_z\rho : z^{\perp} \to T_{[z]}\mathbb{C}P^n$$

because it kills off the direction generated by $U(1)$, and is an isomorphism on the rest of the space. This is a real isomorphism, but of course, $\mathbb{C}P^n$ can be viewed as a complex manifold with complex charts and tangent spaces. In this case, $D_z\rho$ becomes a complex linear isomorphism. Then we claim there exists a unique 2-form τ on $\mathbb{C}P^n$ such that

$$\rho^*\tau = \frac{1}{\pi}\eta$$

So define

$$\tau_{[z]}(u,v) = \frac{1}{\pi}\eta_z(u^\natural, v^\natural)$$

where $x^\natural$ denotes the lift via the isomorphim $D_z\rho$.
Exercise: Show that τ is well defined, and why is it symplectic? Why is it a (1,1)-form, i.e. $\tau(Ju, Jv) = \tau(u,v)$, and why is $\tau_n|_{\mathbb{C}P^{n-1}} = \tau_{n-1}$?

Proposition:

$$\int_{\mathbb{C}P^1} \tau_n = 1$$

Proof: In class notes.

□

Proposition: *Define* $\phi : \mathbb{C} \to \mathbb{C}P^1$, $\phi(z) = [z:1]$. *Then*

$$\phi^*(\tau_1) = \frac{dx \wedge dy}{\pi(1-|z|^2)^2}$$

Proof: In class notes.
Then if f is a function on $\mathbb{C}^n$, define the 1-form $d^c f$ as

$$\begin{gathered} d^c f(v) = df(J_0 v) \\ = \sum_k (\partial_{y_k} f)dx_k - (\partial_{x_k} f)dy_k \\ \Rightarrow \phi^*\tau_1 = d(d^c K) \\ \text{where } K = -\frac{1}{4\pi}\log(1+|z|^2) \end{gathered}$$

K is known as a Kahler potential.
Remark: In the case of $(\mathbb{C}^n, \omega_0)$,

$$\omega_0 = dd^c\left(\frac{|z|^2}{2}\right)^2$$

so there is no log business.
Onto the symplectic topology of $\mathbb{C}P^n$, mostly $n = 2$. We will state some results which show off the symplectic topology, some of which we will prove later in the course, some of which we won't prove at all because their proofs involve the geometric analysis of Seiberg-Witten theory.

Theorem (McDuff): *If M is a closed symplectic 4-manifold, and $b_2(M) = 1$. Suppose $S \subset M$ is an embedded 2-sphere, $S \cdot S = 1$, S is a symplectic manifold, and $\int_S \omega = 1$. Then there exists a symplectomorphism*

$$(M, \omega) \to (\mathbb{C}P^2, \tau_2)$$

This result was improved on later,
Theorem (Taubes): *If ω is a symplectic form on $\mathbb{C}P^2$ with $\int_{\mathbb{C}P^1} \omega = 1$, then $\exists$ a self-diffeomorphism ϕ of $\mathbb{C}P^2$ such that $\phi^*\omega = \tau_2$.*

Theorem (Gromov): *Aut($\mathbb{C}P^2$), τ_2 deformation retracts to its subgroup $PU(3)$.*

Theorem (Kronheimer–Mrowka) *Let S_1 and S_2 be closed, connected, orientable surfaces embedded in $\mathbb{C}P^2$, in the same homology class. If S_1 is a τ-symplectic surface, then the genus of S_2 is at least that of S_1.*
So symplectic surfaces minimize the genus of their homology class.
Theorem (Seidel): *If $V \cong S^n \subset M$ is a Lagrangian submanifold of (M, ω), then $\exists$ a symplectic automorphism, τ_V, supported in a tubular neighborhood of V, which is the antipodal map when thinking of V as S^n.*
This is known as a generalized Dehn twist, and becomes the usual Dehn twist when $V = S^1$.
Theorem (Seidel): *If $n = 2$, τ_V^2 is smoothly isotopic to the identity, but for certain examples, it is not isotopic to id_M in $Aut(M, \omega)$.*

CHAPTER 4

Symplectic linear algebra

Lecture 4, Sept 8. We return to symplectic linear algebra
Definition: $\beta \in \bigwedge^2(V^*)$ is symplectic if it is non-degenerate.

Defintion: If (V, β) and (V', β') are symplectic vector spaces, a linear map $\alpha : V \to V'$ is a symplectic linear map if $\alpha^*\beta' = \beta$.

Remarks: i) Being symplectic is an open condition on β, because the non-degeneracy condition is equivalent to a determinant being non-zero, which is an open condition. ii) Symplectic vector spaces form a category. iii) Symplectic maps are injective. To see this, suppose $\varphi : (V, \beta) \to (V', \beta')$ is a symplectic linear map. Then for $v, w \neq 0 \in V$,

$$\begin{gathered}\varphi(v) = \varphi(w) \\ \Rightarrow \beta(v, w) = \beta'(\varphi(v), \varphi(w)) = 0 \\ \Rightarrow \beta(v, w) = 0 \Rightarrow v = w\end{gathered}$$

This easy result already constricts the space of symplectic maps. There are no symplectic maps going from $V \to V'$ if dim $V' <$ dim V, and if you want to go up in dimension, the map has to be injective. iv) If β is degenerate, then the induced map on $V \setminus \ker\beta^\#$ is a symplectic form. v) $GL(n)$ acts on the symplectic pairings by $A \cdot \beta = A^*\beta$. In this case, if β is represented by the matrix S, then $A^*\beta$ is represented by $A^T SA$.
Now we will talk about symplectic subspaces. If $U \subset V$ is a subspace of (V, β) a symplectic vector space, we can speak of the symplectic complement,

$$U^\beta := \{v \in V \mid \beta(u, v) = 0 \;\forall\, u \in U\}$$

Further,

$$\begin{gathered}\beta^\# : V \to V^* \\ U^\beta \mapsto \operatorname{ann}(U) := \{a \in V^* \mid a(u) = 0 \;\forall\, u \in U\}\end{gathered}$$

However, ann(U) is defined without reference to the symplectic form, so we have dim $U^\beta =$ dim ann$(U) = \operatorname{codim}_V(U)$. It's clear that $U \subset U^{\beta\beta}$, and since they have the same dimension, (taking the codimension twice), they are equal.
Recall a symplectic subspace is a vector subspace which is also a symplectic vector space

by the restriction of the symplectic form. This requirement is equivalent to intersecting trivially with its symplectic complement, so that if U is a symplectic subspace, then

$$V = U \oplus U^\beta$$

By applying induction, we can conclude that each symplectic vector space is the direct sum of many 2-dimensional subspaces.
Onto complex structures[1]. We are working with the standard hermitian form, h_0. If (e_i) is the standard complex basis of $\mathbb{C}^n$, then $\mathbb{R}^{2n}$ has a standard basis $e_1, \ldots, e_n, f_1, \ldots, f_n)$, where $f_j = i \cdot e_j$, in which case the hermitian pairing can be written $h_0 = g_0 + i\beta_0$, where g_0 is the standard dot product and β_0 is the standard symplectic pairing. Let $J_0 \in \mathrm{End}(\mathbb{R}^{2n})$ represent multiplication by i. Then i) $J_0^2 = -1$, ii) $h_0(J_0 z, J_0 w) = h_0(z, w)$, i.e. J_0 is unitary. Also, $g_0(w, v) = \beta_0(w, J_0 v)$, and $g_0(J_0 w, v) = \beta_0(w, v)$.
Definition: If (V, β) is a symplectic vector space, a <u>compatible complex structure</u>, J, is a complex structure, i.e. $J \in \mathrm{End}(V)$ such that $J^2 = -1$, and

$$i)\ \beta(Ju, Jv) = \beta(u, v)$$
$$ii)\ \beta(u, Ju) > 0, \quad u \neq 0$$

In such a case, defining $g(u, v) := \beta(u, Jv)$ defines an inner product. We note that complex structures always exist, because every symplectic vector space is isomorphic to $\mathbb{R}^{2n}$, which has a compatible complex structure. J makes V a complex, hermitian vector space, $i = J$, $h = g + i\beta$.
Definition: A subspace $I \subset (V, \beta)$ is <u>isotropic</u> if $\beta|_I = 0$, or equivalently, $I \subset I^\beta$. $C \subset (V, \beta)$ is called <u>coisotropic</u> if $C^\beta \subset C$.

It follows that if I is isotropic, I^β is coisotropic.
Example: Given a symplectic basis, (e_i, f_j),

$$I_k := \mathrm{span}(e_1, \ldots, e_k) \text{ is isotropic}$$
$$C_k := I_k^{\beta_0} = \mathrm{span}(e_1, \ldots, e_n, f_{k+1}, \ldots, f_n) \text{ is coistropic}$$

Definition: A subspace, L, is called <u>Lagrangian</u> if $L^\beta = L$.

Exercise: If $L \subset (V, \beta)$, the following are equivalent:
i) L is Lagrangian
ii) L is maximal isotropic
iii) L is minimal coisotropic
iv) $\exists$ a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ such that L =span$(e_1, \ldots, e_n)$
v) dim $L = \frac{1}{2}$ dim V, and L is either isotropic or coisotropic

[1] We took an aside for the Pfaffian, but I'm already behind in the notes and it's mostly irrelevant for the course, so I'm skipping over it.

CHAPTER 5

Symplectic linear algebra II

Lecture 5, Sept 10. We note that the exercise from the last lecture characterizes how isotropic and coisotropic subspaces look.
Example: If $\alpha : (V, \beta) \to (V', \beta')$ is a symplectic linear isomorphsim, then $\Gamma_\alpha \subset (V \oplus V', -\beta + \beta')$ is Lagrangian.
Proof: We must show $-\beta + \beta' \equiv 0$ on Γ_α. For any $(v_1, v_1'), (v_2, v_2') \in \Gamma_\alpha$

$$(-\beta + \beta')\Big((v_1, v_1'), (v_2, v_2')\Big) \equiv -\beta(v_1, v_2) + \beta'(v_1', v_2')$$
$$= -\beta(v_1, v_2) + \beta'\Big(\alpha(v_1), \alpha(v_2)\Big) \text{ by definition of } \Gamma_\alpha$$
$$= -\beta(v_1, v_2) + \beta(v_1, v_2) \text{ because } \alpha \text{ is symplectic}$$
$$= 0$$

□

Suppose L is a vector space. Then $L \oplus L^*$ carries the symplectic pairing β_{can},

$$\beta_{can}\Big((v, \alpha), (v', \alpha')\Big) := \alpha'(v) - \alpha(v')$$

Definition: A polarization of (V, β) is a pair (L, L') of transverse Lagrangian subspaces. Here transverse means $V = L + L'$, which is equivalent to their intersection being trivial.

If you have a polarization, then $\beta^\#$ provides an isomorphism $L' \cong L^*$, so that $V \cong L \oplus L^*$ via $\mathrm{Id} \otimes \beta^\#$, with the symplectic form β_{can}. In such a case, there exists a symplectic basis such that $L = \text{span } (e_1, \ldots, e_n)$ and $L' = \text{span}(f_1, \ldots, f_n)$.

If we take a Lagrangian subspace $\bigwedge \subset (L \oplus L^*, \beta_{can})$, if $\bigwedge \pitchfork L^*$, we can think of $\bigwedge$ as the graph of a uniquely determined linear map $q : L \to L^*$. There is a bijection

$$\mathrm{Hom}(L, L^*) \to \mathrm{Hom}(L \times L, \mathbb{R})$$

where the first set is linear maps and the second set is bilinear maps.

Lemma: Γ_q *is Lagrangian iff* $\widetilde{q}$ *is symmetric, where* $\widetilde{q}$ *is the image of* q *through the above isomorphism.*
Proof: We must show $\beta_{can} \equiv 0$ on Γ_q:

$$\begin{aligned}\beta_{can}\Big((u, q(u)), (v, q(v))\Big)& \\ \equiv q(v)(u) - q(u)(v)& \\ = \widetilde{q}(v, u) - \widetilde{q}(u, v)&\end{aligned}$$

which equals 0 $\forall\ u, v$ iff $\widetilde{q}$ is symmetric.

$\square$

Corollary: Γ_q *is* $i) - v)$ *from the previous lecture's exercise iff* $\widetilde{q}$ *is symmetric.*

Corollary: $\Gamma_q \pitchfork L$ *iff* $\widetilde{q}$ *is non-degenerate.*

Exercise: If $L_0 \pitchfork L_1$ are Lagrangian in (V, β), then they can be identified with $\mathbb{R}^n \subset \mathbb{C}^n$ via a symplectic isomorphism. However, not any triple of pairwise transverse Lagrangians can be standardized in this way. We can identify $V \cong L_0 \oplus L_1$, and L_2 as the graph Γ_q for some $q : L \to L^*$, so that $\widetilde{q}$ is non-degenerate. We may define an invariant $\tau(L_0, L_1, L_2) \in \mathbb{Z}$ as the signature of $\widetilde{q} \in \{-n, -n+2, \ldots, n-2, n\}$. Show that τ is a complete invariant, i.e. it characterizes (L_0, L_1, L_2) up to symplectic linear isomorphism. τ is often referred to as the Kashiwara index.

We recall $\mathrm{Sp}(V)$ is a closed subgroup of $\mathrm{GL}(V)$, and as such is a Lie group. To examine the associated Lie algebra, we note[1]

$$\begin{gathered}\mathrm{Sp}(V, \beta) = \varphi^{-1}(0) \\ \varphi(A) = A^*\beta - \beta \\ \Rightarrow \mathfrak{sp}(V, \beta) = \mathrm{Ker}\ D_I\varphi \\ = \{\xi \in \mathrm{End}(V) \mid \beta(u, \xi v) + \beta(\xi u, v) = 0\} \\ = J \cdot \mathrm{symm}_g(V)\end{gathered}$$

where we choose a complex structure J, and define $g = \beta(\cdot, J\cdot)$, and $\mathrm{symm}_g(V)$ denotes the V-endomorphisms which are g-self-adjoint. In the case of $V = \mathbb{R}^{2n}$, this reduces to J being multiplication by i and $\mathrm{symm}_g(V)$ being the set of symmetric matrices. This grants us a large number of symplectic matrices, by simply taking σ any symmetric matrix, so that you have the 1-parameter subgroup

$$t \mapsto \exp(tJ_0\sigma)$$

though we note this map is not always surjective. Note if λ is an eigenvalue of a symplectic matrix, so is λ^{-1}, leading to a simply characterization of diagonal symplectic matrices.

[1] I really need to get a hold for how these calculations work

Lemma: *If $A \in Sp(V, \beta)$, and λ an eigenvalue, then λ^{-1} is an eigenvalue of the same multiplicity, the multiplicities of* 1 *and* -1 *are even, and*

$$\beta(E_\lambda, E_{\lambda'}) = 0$$

unless $\lambda = \lambda'^{-1}$.
Proof: Follow your nose, except the first part. To prove the first part, note $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \beta_0)$ iff $A^T J_0 A = A$ ie

$$A^T = J_0 A^{-1} J_0^{-1}$$

so that $A^T,\ A^{-1}$, and A share the same characteristic polynomial.

$\square$

Suppose we fix a compatible complex structure, (V, β, J), so that $g = \beta(\cdot, J\cdot)$ is the inner product, and $h = g - i\beta$ is the standard hermitian product. Then we have $\mathrm{Sp}(V, \beta)$, $\mathrm{GL}(V, J)$ and $O(V, J)$ sitting inside $GL(V)$, which respect the structures β, J, g respectively. Then if A respects two of these structures, then it respects all 3. Stated formally,
Lemma:

$$\begin{aligned} U(V, h) &= \mathrm{Sp}(V, \beta) \cap \mathrm{Gl}(V, J) \\ &= \mathrm{Gl}(V, J) \cap O(V, g) \\ &= \mathrm{Sp}(V, \beta) \cap O(V, g) \end{aligned}$$

Theorem: *$U(n)$ is a deformation retract of $Sp(V, \mathbb{R}^{2n})$, and this deformation retraction is equivariant wrt conjugation on $U(n)$. Further, $U(n)$ is a maximal compact subgroup, and any other maximal compact subgroup is conjugate to $U(n)$.*
Of course, this implies that they share homotopy groups.
Sketch of Proof: We consider the "Cartan involution", $\Theta : GL(\mathbb{R}^d) \to GL(\mathbb{R}^d)$:

$$\Theta(A) = (A^T)^{-1} = (A^{-1})^T$$

because both inversion and transposition are anti-involutions[2], Θ is indeed a homomorphism, and it is an involution. We note $(GL(\mathbb{R}^d))^\Theta = O(d)$, which will end up being a maximal compact subgroup. If $\theta = D_i\Theta : \mathfrak{gl}(\mathbb{R}^d) \to \mathfrak{gl}(\mathbb{R}^d)$, sending $x \to -x^T$, the $+1$ eigenspace is $\mathfrak{o}(d)$, but it also has a -1 eigenspace, the symmetric matrices. We also note

$$[\mathrm{Symm}(d), \mathfrak{o}(d)] \subset \mathrm{Symm}(d)$$

so there is an exponential map

$$\exp : \mathrm{Symm}(d) \to S \equiv \{\text{pos def matrices}\}$$

which turns out to be a diffeomorphism, so that we may define its inverse, the logarithm, by diagonalizing and taking log of the diagonals, and S is invariant under conjugation by

[2] Do not make the mistake i did: anti-involution here means it reverses the order of group multiplication, not that it squares to -1. Since we we flip the group structure twice, ...

$O(d)$.
Lemma (Polar Decomposition for $\mathbf{GL}(\mathbb{R}^d)$): *The following map is a diffeomorphism:*

$$\begin{aligned}\Phi : \mathrm{Symm}(d) \times O(d) &\to \mathrm{GL}(\mathbb{R}^d)\\ (\xi, O) &\mapsto \exp(\xi) \cdot O\end{aligned}$$

Note this implies $O(n)$ is a deformation retract of $\mathrm{GL}(n)$, because vector spaces, (Symm(d)) are contractible.
Proof of this lemma and the theorem in the next lemma.

CHAPTER 6

Polar Decomposition and the Grassmannian(s)

Lecture 6, Sept 15. First we prove the lemma:
Lemma (Polar Decomposition for GL($\mathbb{R}^d$)): *The following map is a diffeomorphism:*

$$\Phi : \mathrm{Symm}(d) \times O(d) \to \mathrm{GL}(\mathbb{R}^d)$$
$$(\xi, O) \mapsto \exp(\xi) \cdot O$$

Proof: If $Q \in \mathrm{End}(\mathbb{R}^d)$ is diagonalizable, and has real, non-negative eigenvalues, then we can take powers of Q via

$$Q = A\ \mathrm{diag}(\lambda_1, \ldots, \lambda_n) A^{-1}$$
$$Q^\alpha = A\ \mathrm{diag}(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) A^{-1}$$

Assume (V, g) is a d-dimensional inner product space, and $Z \in \mathrm{GL}(V)$. Then

$$g(Z^* u, v) = g(u, Zv)$$

Z^*Z is self adjoint and positive semi-definite, thus it has a unique positive semi-definite (thus, symmetric) square root, $P := P_z = (Z^*Z)^{\frac{1}{2}}$. We set $O = O_z = P_z^{-1}Z$, so that

$$Z = PO$$

Let

$$P = \exp \xi$$

defined via the logarithm, so $Z = \exp(\xi) \cdot O$, so that Φ is surjective. For injectivity, say $Z = PO$, for P positive semi definite and self adjoint, while O is orthogonal. Then

$$ZZ^* = POO^*P^*$$
$$= PP^* = P^2$$

So P is the unique positive definite square root of Z^*Z, so O is also unique. Clearly Φ is smooth, as is its inverse.

□

Now, for the proof of the theorem. Recall we were considering the map

$$\begin{aligned}\Theta : \mathrm{Sp}(\mathbb{R}^{2n}) &\to \mathrm{Sp}(\mathbb{R}^{2n})\\ A &\mapsto (A^{-1})^T\end{aligned}$$

and $\theta = D_I\Theta$, sending $\xi \mapsto -\xi^T$. Then

$$\begin{aligned}\mathrm{Sp}(\mathbb{R}^{2n})^\Theta &= \mathrm{Sp}(\mathbb{R}^{2n}) \cap O(2n) \quad \text{(transpose the usual defn of } O(2n)\text{)}\\ &= U(n)\end{aligned}$$

The second line coming from our 2-out-of-3 rule[1]. Let

$$\begin{aligned}\mathfrak{p} &= E_{-1} \text{ wrt } \theta\\ &= \mathfrak{sp}(\mathbb{R}^{2n}) \cap \mathrm{Symm}(\mathbb{R}^{2n})\\ &= J_0\mathrm{Symm}(\mathbb{R}^{2n}) \cap \mathrm{Symm}(\mathbb{R}^{2n})\end{aligned}$$

i.e. matrices which are symmetric, and remain symmetric after applying J_0. These have the form

$$\begin{pmatrix} R & -S \\ S & R \end{pmatrix} \text{ for } R, S \in \mathrm{Symm}(\mathbb{R}^{2n})$$

We also note

$$[\mathfrak{p}, U(n)] \subset \mathfrak{p}$$

We can exponentiate $P := \exp(\mathfrak{p}) = \{\text{pos. def. symmetric, symplectic matrices}\}$.

Proposition (Polar Decomposition for Sp): *The following map is a diffeomorphism*

$$\begin{aligned}\Psi : U(n) \times \mathfrak{p} &\to \mathrm{Sp}(\mathbb{R}^{2n})\\ (U, \xi) &\mapsto U \cdot \exp(\xi)\end{aligned}$$

and is equivariant wrt conjugation in $U(n)$.

Proof: We note that $\Psi = \Phi|_{U(n)\times\mathfrak{p}}$, so we have injectivity for free. If Z is symplectic, then its polar part, P_z is also symplectic[2].

□

Theorem: *$U(n)$ is a deformation retract of $Sp(V, \mathbb{R}^{2n})$, and this deformation retraction is equivariant wrt conjugation on $U(n)$. Further, $U(n)$ is a maximal compact subgroup, and any other maximal compact subgroup is conjugate to $U(n)$.*

Proof: The first claim is done by the polar decomposition. The second and third group require knowledge of measure theory, which I do not possess, so I skipped this.

[1] Which I still have not internalized

[2] All the parts except the square root are obvious, idk why that should preserve being symplectic

Exercise: Suppose we consider the complex symplectic group, i.e. those complex matrices which preserve the standard symplectic form. Then claim $\mathrm{Sp}(\mathbb{C}^{2n})$ deformation retracts to $\mathrm{Sp}(n)$, the compact symplectic group, defined by

$$\mathrm{Sp}(n) := \{B \in GL(\mathbb{H}^n)\}$$

Proof hint: You can use the same strategy of looking at the Cartan involution.

Definition: Define $\mathcal{J}(V,\beta) := \{J \in \mathrm{End}(V) \mid J^2 = -Id$, and is compatible with $\beta\}$, the space of β-compatible complex structures on V.

So $\mathrm{Sp}(V,\beta) \curvearrowright \mathcal{J}(V,\beta)$ transitively, by conjugation. So if we pick a reference complex structure, J_0, by the characterization of transitive actions, we have

$$\mathcal{J}(V,\beta) \cong \frac{\mathrm{Sp}(V,\beta)}{(J_0)^{\mathrm{Sp}(V,\beta)}}$$

sending

$$ZJ_0Z^{-1} \longleftarrow [Z]$$

What is the stabilizer of J_0? It is complex linear matrices which are also symplectic, $GL(V,\beta) \cap \mathrm{Sp}(V,\beta)$. By our 2-out-of-3 rule[3], that is equal to $U(V,\beta)$, so we are looking at cosets of the unitary group. But by polar decomposition, $\mathrm{Sp}(\mathbb{R}^{2n}) \cong U(n) \times \mathfrak{p}$, so

$$\mathcal{J}(V,\beta) \cong \mathfrak{p}$$

so it is contractible, for example, which is a very important fact which we will use when studying psuedo-holomorphic curves.
Definition: $\mathcal{L}(V,\beta)$ is the space of Lagrangian subspaces of (V,β).

We can study this in 2 ways: fix a complex structure J.
i) $\mathcal{L}(V,\beta) \subset \mathcal{G}(V,\beta)$, the grassmanian. If $L \in \mathcal{L}(V,\beta)$, (L, JL) is a polarization, in which case we may view $(V,\beta) \cong (L \oplus L^*, \beta_{can})$. We have a map

$$\mathrm{Hom}(L,L^*) = \mathrm{Bil}(L \times L, \mathbb{R}) \to \mathcal{G}(V,n)$$
$$\theta \mapsto \Gamma_\theta$$

We recall the graph of a map in $\mathrm{Hom}(L \times L, \mathbb{R})$ is Lagrangian iff the bilinear form is symmetric, so the above map restricts to

$$\mathrm{Symm}^2(L^*) \to \mathcal{L}(V,n)$$

Thus we deduce the dimension is $\frac{1}{2}n(n+1)$.
ii) Pick J, L. Then $U(V) \curvearrowright \mathcal{L}(V)$ by $L \mapsto U(L)$, which is transitive, so that

$$\mathcal{L}(V) \cong U(V) \setminus L^{U(V)}$$
$$= U(V) \setminus O(L)$$

[3] think im starting to get the hang of this thing

We may also consider the oriented Grassmannian,

$$\widetilde{\mathcal{G}}(V, \beta) := \{\text{oriented set of } n \text{ planes}\}$$

which is a 2-fold cover of the un-oriented Grassmannian. This restricts to a 2-fold covering of the Lagrangian Grassmannian,

$$U(n) \setminus SO(L) \cong \mathcal{L}^2(V) \to \mathcal{L}$$

Example: $\mathcal{L}(\mathbb{R}^2)$ is the set of Lagrangian subspaces of $\mathbb{R}^2$, but all 1-d subspaces are Lagrangian, so $\mathcal{L}(\mathbb{R}^2) \cong \mathbb{R}P^1$, which has 2-fold covering of $\mathcal{L}^2(\mathbb{R}^2) \cong S^1$.

Example:

$$\left(\widetilde{\mathcal{G}}(\mathbb{R}^4, 2), \mathcal{L}^2(\mathbb{R}^4)\right) \cong \left((S^2 \times S^2), (S^1 \times S^2)\right)$$

CHAPTER 7

Some Lie Theory and Differential Geometry

Lecture 7, Sept 17. The determinant map

$$\det_{\mathbb{C}} : U(n) \to U(1) = S^1$$

induces an isomorphism on the fundamental group. This is proved by the long exact sequence of homotopy groups for fiber bundles, with the fiber bundle

$$SU(n) \to U(n) \to U(1)$$

thus we have an isomorphism

$$\begin{tikzcd} U(n) \arrow[r] \arrow[dr, dashed, bend right, "c_1"'] & U(1) \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

we name the composition c_1, representing the first chern class[1]. Sometimes it is also referred to as the Maslov index.
In general,

$$\begin{aligned} H^1(X) &\cong \operatorname{Hom}(\pi_1(X), \mathbb{Z}) \\ &= \operatorname{Hom}\big((\pi_1(X))_{Ab}, \mathbb{Z}\big) \\ &= \operatorname{Hom}(H_1(X), \mathbb{Z}) \end{aligned}$$

So c_1 is a generator for $H^1(U(n)) = \mathbb{Z}$. Because $\mathrm{Sp}(\mathbb{R}^{2n})$ deformation retracts onto $U(n)$, they share homotopy groups. We can do a similar process for the Lagrangian Grassmannian. We have

$$\mathcal{L}(\mathbb{R}^{2n}) \cong \frac{U(n)}{O(n)}$$

and we want to map it to $U(1)$ via the determinant. But that is not well defined because $O(n)$ can change the sign, but taking $\det^2$ is well defined. In fact,
Lemma: *det^2 is a fiber bundle, and the fibers are connected and simply connected.*

[1] no idea what this means

Consequently, we have an isomorphism

$$\begin{tikzcd}
\pi_1(\mathcal{L}(\mathbb{R}^{2n})) \arrow[r] \arrow[dr, dashed, bend right, "\mu"'] & \pi_1(U(1)) \arrow[d] \\
& \mathbb{Z}
\end{tikzcd}$$

where $\mu \in H^1(\mathcal{L}(\mathbb{R}^{2n})$ is called the Maslov index.

We can also consider the universal cover

$$\mathcal{L}^\infty(\mathbb{R}^{2n}) \to \mathcal{L}(\mathbb{R}^{2n})$$

which is the pull back by $\det^2$ of the universal cover of the circle, resulting in the fibered product

$$\mathcal{L}(\mathbb{R}^{2n}) \times_{\det^2} \mathbb{R}$$

There is also the universal cover of

$$\begin{tikzcd}
\mathrm{Sp}^\infty(V,\beta) \arrow[d] \\
\mathrm{Sp}(V,\beta)
\end{tikzcd}$$

which should be a Lie group

$$\mathrm{Sp}^\infty(V,\beta) = \left\{(A,\widetilde{A}) \mid A \in \mathrm{Sp}(V,\beta),\ \widetilde{A} : \mathcal{L}^\infty(V,\beta) \to \mathcal{L}^\infty(V,\beta)\right\}$$

with $\widetilde{A}$ being $\mathbb{Z}$ equivariant.

Now onto actual symplectic manifolds.
The general setting is a symplectic manifold, (M,ω), with $\omega \in \Omega^2(M)$, and $d\omega = 0$ and non-degenerate, and we are interested in symplectomorphisms $\phi : (M,\omega) \to (M',\omega')$ such that ϕ is a diffeo and

$$\phi^*\omega' = \omega$$

Each condition on ω gives rise to basic invariants, which also end up being constraints. Because ω is closed, $[\omega] \in H^2_{DR}(M)$ is an invariant in the sense that $\phi^*([\omega']) = [\omega]$. We note $TM \to M$ is a symplectic vector bundle, because each tangent space has a non-degenerate skew pairing. We know

$$\begin{tikzcd}
H^2_{dR}(M) \arrow[r, "\sim"] \arrow[dr, dashed] & H^2_{sing}(M;\mathbb{R}) \arrow[d, "\wr"] \\
& \mathrm{Hom}(H_2(M;\mathbb{R}),\mathbb{R})
\end{tikzcd}$$

, all arrows being isomorphisms. We know any homology class in degree 2 can be represented by a closed, oriented surface Σ and an embedding $f : \Sigma \to H_2(M)$ realizing that class. Then given $[\eta] \in H^2_{dR}(M)$, let $\varphi([\eta])$ be the element of $\mathrm{Hom}(H_2(M), \mathbb{R})$ sending

$$[\Sigma, f] \mapsto \int_\Sigma f^* \eta$$

The de Rham theorem states that φ is an isomorphism. So given a form, pull it back and integrate over a surface. If we consider the case of $(\mathbb{R}^{2n}, \omega_0)$, then $[\omega_0 = \sum_i dx_i \wedge dy_i] = 0$, making $(\mathbb{R}^{2n}, \omega_0)$ an exact symplectic manifold. It must be 0 since $H^2(\mathbb{R}^{2n}) = 0$, but we can also show that directly, since

$$\omega_0 = d \sum_i x_i \wedge dy_i$$

Given an exact symplectic manifold $(\omega = d\theta)$ and a map of a surface $f : \overline{\Sigma} \to M$, one has

$$\int_\Sigma f^* \omega = \int_\Sigma f^* d\theta$$
$$= \int_\Sigma d(f^* \theta) = 0$$

by Stokes theorem. So there does not exist a 2-dim compact, symplectic submanifold Σ, because ω will restrict to be degenerate. OTOH, if (M, ω) is compact, then

$$\int_M [\omega]^n = \int_M \omega^n > 0$$

This implies $[\omega]^n \neq 0 \Rightarrow [\omega] \neq 0 \Rightarrow M$ is not exact.
As a result, for $n > 1$, S^{2n} does not admit any symplectic forms, because $H^2(S^{2n}) = 0$, so any symplectic form would make S^{2n} exact, but S^{2n} is compact.
Example: Take $S^p \times S^q$, $p + q$ even. Then when does it admit symplectic forms?
Well, if both p and q are greater than 2, then $H^2 = 0$, so there are no symplectic structures. If $p = 2, q > 2$, then $H^2 = \mathbb{R} \cdot x$, but $x^2 = 0$, so that can't work either. We have (basically) shown that the only cases that work are $p = q = 2$, or $p = q = 1$. In the first case, we have the product symplectic form, and in the second, we restrict the standard symplectic form from $\mathbb{R}^2$ onto the torus.
Example: $\overline{\mathbb{C}P^{2n}}$ is the complex projective space with the opposite orientation. There are no symplectic structures inducing this orientation:

$$H^n(\mathbb{C}P^{2n}) = \frac{\mathbb{R}[h]}{(h^{2n+1})}$$

where h has degree 2. So

$$[\omega] = c \cdot h$$
$$[\omega]^{2n} = c^{2n} \cdot h^{2n}$$

i.e. is always positive. The symplectic class also acts as a constraint on symplectomorphisms.
Example: $(\mathbb{C}P^n, \tau_n)$:
Say $\phi : \mathbb{C}P^n \to \mathbb{C}P^n$ is a diffeomorphism. Then claim $\phi^*[\tau_n] = \pm\tau_n$. This is true because the total volume of M wrt ω must be equal to the volume of M wrt ω', i.e. total volume is a symplectic invariant.
Exercise: Let (M, ω) be compact and symplectic, and η a closed 2-form. Then for small $\epsilon > 0, \omega + \epsilon\eta$ is symplectic, but if $[\eta] \neq 0$, $(M, \omega + \epsilon\eta)$ is not symplectomorphic to (M, ω).

Example: For which nonzero a, b does there exist a symplectomorphism $(S^2 \times S^2, a\omega \oplus b\omega) \cong (S^2 \times S^2, \omega \oplus \omega)$, where ω is a symplectic form on S^2 such that $\int_{S^2} \omega = 1$? Partial answer: $H^2(S^2 \times S^2) \cong \mathbb{Z}x \oplus \mathbb{Z}y$. If p_x, p_y denote the projection onto the first and second component,

$$p_1^*[\omega] = x, \qquad p_2^*[\omega] = y$$

Then

$$\int_{S^2 \times S^2} x \cdot y = 1, \quad x^2 = y^2 = 0$$

$$\text{then vol}(\omega \oplus \omega) = \int (x+y)^2 = 2$$

while $a\omega \oplus b\omega$ has volume $2ab$, so we need $ab = 1$. However, any self diffeomorphism preserves the integer lattice $H^2(S^2 \times S^2, \mathbb{Z}) \subset H^2(S^2 \times S^2, \mathbb{R})$. So if $\phi^*(\omega \oplus \omega) = a\omega \oplus b\omega$, then a and b must be integers, i.e. $a = b = 1$ or $a = b = -1$ are the only solutions. It turns out the latter case is not possible, and we will see how next lecture.

CHAPTER 8

Vector Bundles and Chern Classes

Lecture 8, Sept 22. **Definition:** A symplectic vector bundle over M is a real vector bundle $V \to M$, with a smooth section β of $\bigwedge^2 V^* \to M$, defining a symplectic pairing on each fiber. Similarly, a symplectic map of symplectic vector bundles is a bundle map covering the identity which restricts to a symplectic linear map on each vector space.

Example: The main example is (TM, ω), where (M, ω) is a symplectic manifold. The fact that ω, realizing the definition of M being a symplectic manifold, is closed is not relevant, only its non degeneracy.

Symplectic vector bundles are locally trivial, i.e. β can be made standard. Further, there are sections $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ over U open forming a symplectic basis for V_x for every $x \in U$. The proof of this is the exact same spirit as when we proved there existed local symplectic bases for vector spaces.

Theorem: Every symplectic vector bundle $(V \to M, \beta)$ admits compatible complex structures. Moreover, the space $\mathcal{J}(V, \beta)$ of compatible complex structures is contractible.
We will discuss exactly what we mean by "space" here later.
Nevertheless, proof: View M as a cell complex (we can do this by morse theory), and denote M^k as the set of k-cells. We will construct a compatible complex structure J^k on the k-skeleton, and proceed by induction. In the $k = 0$ case, we have a disjoint union of points, in which case we only need the existence of compatible complex structures on vector spaces, which we proved previously. For the inductive step, assume we have a compatible complex structure on the k-skeleton. Each $k+1$ cell C is a $\overline{D^{k+1}}$ in M attached to M^k via $S^k \to M$. So $V|_C \to C$ is a vector bundle over $\overline{D^{k+1}}$, and we have a complex structure over ∂C. Then we can trivialize $V|_C \to C$, so that the given complex structure is a map

$$J_{\partial C} : \partial C \to \mathcal{J}(\mathbb{R}^{2n})$$

Then we wish to extend this to a map $C \to \mathcal{J}(\mathbb{R}^{2n})$, but this is possible because $\mathcal{J}(\mathbb{R}^{2n})$ is contractible, so each map on the boundary of a disk extends to a map on the disk itself. So we may extend J^k over J^{k+1}, and the induction is finished. This is only a continuous extension, but it can be smoothed out, but we won't talk about the details of that. So we've shown there exists one compatible complex structure. We also claimed the space of such is contractible. Say we have

$$J \in \mathcal{J}(V, \beta) = \{\text{sections of the fiber bundle } \mathbb{J}(V, \beta) \to M \text{ with fibers } \mathcal{J}(V_x, \beta_x)\}$$

But a choice of $J_x \in \mathbb{J}(V, \beta)$ identifies $\mathcal{J}(V_x, \beta_x) \cong \mathfrak{p}_x$, so $\mathbb{J}(V, \beta)$ can be identified with a vector bundle $\mathfrak{p} \to M$. But the sections of a vector bundle are contractible, so we are done.

□

The conclusion is that every symplectic vector bundle can be viewed as a complex vector bundle, which is itself determined up to complex VB isomorphism. Further, symplectic VB's are isomorphic iff they are iso as complex VB's, so the theory of symplectic VB's is basically isomorphic to the theory of complex VB's, which is generally well understood.

So we get a constraint: A manifold admits symplectic forms only if it admits almost complex structures. We also get an invariant, i.e. TM, with its complex structure. Concretely, if we have a symplectomorphism $\phi : (M, \omega) \to (M', \omega')$, and a compatible almost complex structure $J' \in \mathcal{J}(M', \omega')$, then

$$\phi^* J' \mathcal{J}(M, \omega)$$

so Aut(M, ω) preserves $\mathcal{J}(M, \omega)$.

If we have any cts. complex vector bundle $E \to M$ of rank n, there are associated chern classes

$$c_i(E),\ i \in \{0, 1, \ldots, n\}$$
$$c_k(E) \in H^{2k}(X; \mathbb{Z})$$

and $c_0 = 1 \in H^0$.
Theorem: *There exists exactly one way to assign c_i such that the following hold*

$$i) \text{ for } f : Y \to X,\ c_i(f^*E) = f^*(c_i(E))$$
$$ii) \text{ the } \underline{\text{total sum}},\ c(E) = \sum_i c_i(E) \in H^\bullet(X).$$

$$\text{Then for VB's } E, E' \text{ over } X,\ c(E \oplus E') = c(E) \smile c(E')$$

ie

$$c_k(E \oplus E') = \sum_{i+j=k} c_i(E) \smile c_j(E')$$

and

$$iii) \text{ the tautological line bundle, } \ell, \text{ has } \langle c_1(\ell), [\mathbb{C}P^1] \rangle = -1$$

Example: $c_1(\ell_n \to \mathbb{C}P^n)$. We may restrict the tautological line bundle to $\mathbb{C}P^1$,

$$\begin{array}{ccc} \ell_n & \text{to} & \ell_1 \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & & \mathbb{C}P^1 \end{array}$$

We recall the cohomology of $\mathbb{C}P^n$:

$$H^k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k \text{ is even and } 0 \leq k \leq 2n \\ 0 & k > 2n \end{cases}$$

with cohomology ring

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \frac{\mathbb{Z}[x]}{x^{n+1}}$$

where x is the generator of the second degree cohomology, and x^i is a generator of $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$. If we consider the inclusion $i : \mathbb{C}P^1 \to \mathbb{C}P^n$, we have that

$$i^* x_n = x_1$$

where x_i is the generator of $H^2(\mathbb{C}P^i;\mathbb{Z})$. Then we may compute

$$\boxed{i^* c_1(\ell_n)} = c_1 i^*(\ell_n) = c_1(\ell_1) = -x_1 = \boxed{-i^* x_n}$$

But because i^* is an isomorphism on H^2, we may apply $(i^*)^{-1}$ to both sides of the boxed equations to conclude

$$c_1(\ell_n) = -x_n$$

□

Example: $c(T\mathbb{C}P^n)$: We know the tangent space looks like, thinking of $\mathbb{C}^{n+1} \setminus \lambda$ as $\lambda^\perp$,

$$T_\lambda(\mathbb{C}P^n) = \mathrm{Hom}(\lambda, \mathbb{C}^{n+1} \setminus \lambda) = \lambda^* \otimes \mathbb{C}^{n+1} \setminus \lambda$$

where the first equality is realized by taking graphs. So globally, the tangent bundle looks like

$$T(\mathbb{C}P^n) = \ell_n^* \otimes \left(\frac{\mathbb{C}^{n+1}}{\ell_n} \right)$$

So we get a SES of vector bundles, where the underline indicates a trivial bundle

$$0 \to \ell_n \to \underline{\mathbb{C}^{n+1}} \to \underline{\mathbb{C}^{n+1}} \setminus \ell_n \to 0$$

tensoring with ℓ_n^* gives the new SES

$$0 \to \underline{\mathbb{C}} \to \ell_n^{* \oplus n+1} \to \ell_n^* \otimes \underline{\mathbb{C}^{n+1}} \setminus \ell_n = T\mathbb{C}P^n \to 0$$

The sequence splits, so that

$$\ell_n^{* \oplus n+1} \cong \underline{\mathbb{C}} \oplus T\mathbb{C}P^n$$

Lemma: *If $E \to X$ is a trivial vector bundle, then $c(E) = 1$.*
Proof: If $E \to X$ is a trivial vector bundle, i.e. $E = X \times \mathbb{C}^k$ for some k, we may realize it as the pullback of the diagram

$$\begin{array}{ccc} & & \mathbb{C}^K \\ & & \downarrow \pi \\ X & \xrightarrow{f} & \{\mathrm{pt}\} \end{array}$$

in which case $c_i(E) = c_i(f^*\mathbb{C}^k) = f^*(c_i(\mathbb{C}^k))$. But $H^n(\{\text{pt}\})$ is $\mathbb{Z}$ for degree 0 and 0 otherwise, so $c_0 = 1$ and $c_i = 0$ for else, so that

$$c(E) = 1$$

□

Returning to the example, this shows that[1]

$$c(\ell_n^{*\oplus n+1}) = c(\ell_n^*)^{n+1} = c(\underline{\mathbb{C}}) \cdot c(T\mathbb{C}P^n) = c(T\mathbb{C}P^n)$$
$$\Rightarrow c(T\mathbb{C}P^n) = (1 + x_n)^{n+1}$$

□

Definition: The Picard group, $\text{Pic}(X)$, of a space X, is

$$\text{Pic}(X) = \{\text{iso. classes of line bundles over } X\}$$

with the group operation of tensor product, i.e. $L^{-1} = L^*$.

Then we have a map $c_1 : \text{Pic}(X) \to H^2(X)$.
Proposition:
i) c_1 is an isomorphism.
ii) If M is compact, and s is a section of a line bundle $L \to M$, vanishing transversely, then

$$c_1(L) = PD(S^{-1}(0))$$

Under these hypotheses, $S^{-1}(0)$ is a codim 2 submanifold of M. For example, this theorem implies that if S is a Riemann surface, then

$$\langle c_1(TS), [S] \rangle = \chi(S)$$

So concretely, to compute $c_1(E)$, if $E = \bigoplus L_i$, for line bundles L_i, then

$$c_1(E) = \sum_i c_1(L_i)$$

OTOH, if we take the top exterior power, we have

$$\det E = \bigwedge^n E \cong L_1 \otimes \cdots \otimes L_n$$
$$c_1(\det E) = c_1(L_1 \otimes \cdots \otimes L_n)$$
$$= \sum_i c_1(L_i) = c_1(E)$$

This fact is also true in general. One can drop the assumption that E is a direct sum of line bundles, and still

$$c_1(E) = c_1(\det E)$$

Then to compute $c_1(E)$, let s be a section of $\det E$. Then $c_1(E) = \text{PD}\,[S^{-1}(0)]$.

[1] I don't really know how the cup product turned into regular multplication, but that's because i only really know the formal definition of a cup product and i've never done any examples.

CHAPTER 9

Almost Complex Structures and Chern Classes II

Lecture 9, Sept 24. Last lecture we saw the theory of chern classes for complex vector bundles. There are other, a priori different theories of chern classes. There is, for example, the theory over line bundles, which happens to coincide with the theory over complex vector bundles. We have the Picard group, $P(X)$ and an isomorphism $c_1 : P(X) \to H^2(X;\mathbb{Z})$, and we can require that it be natural wrt pull backs, and it turns out to be equivalent. Recall from the previous lecture we found

$$\begin{aligned} c(T\mathbb{C}P^n) &= (1+h)^{n+1} \\ &= 1 + (n+1)h + \binom{n+1}{2} h^2 + \cdots + (n+1)h^n \end{aligned}$$

Given (M, ω), we can define

$$c_i(M) := c_i(TM, J)$$

for J a compatible almost complex strcutre. If you have a homotopy J_t from J_0 to J_1, it turns out all the (TM, J_t) are isomorphic, so that M completely determines the chern classes.

Definition: The anti-canonical line bundle is defined as

$$k_M^{-1} = \det_{\mathbb{C}}(TM, J) = \bigwedge_{\mathbb{C}}^{n}(TM)$$

As we proved last lecture, for a vector bundle E, $c_1(E) = c_1(\det E)$, which is PD to a zero set of a section of $\det E$, thus

$$c_1(TM, \omega) = c_1(k_M^{-1})$$

Then a trivialization of k_M is a nowhere vanishing complex volume form $\Omega \in \Gamma(k_M)$, so that $\Omega_x \in \bigwedge_{\mathbb{C}}^n (T_x^* M)$, so $c_1(M, \omega)$ is the obstruction to trivializing k_M, or the obstruction to finding a nowhere vanishing complex volume form.

To see a useful application of chern classes, we return to the question of $S^2 \times S^2$, with p_i projecting onto S^i, and defining

$$\omega_{a,b} = p_1^*\omega + p_2^*\omega$$

and asking for which a, b is

$$(S^2 \times S^2, \omega_{a,b}) \cong (S^2 \times S^2, \omega_{1,1})$$

Last time, we showed that this is only possible if $a = b = \pm 1$. We will now rule out the case $a = b = -1$. Given a compact Riemann surface, Σ, and a complex line bundle $L \to \Sigma$, then $c_1(L) = PD(S^{-1}(0))$, so $c_1(T\Sigma)$, which is a line bundle bc Σ is a Riemann surface, satisfies

$$\langle c_1(T\Sigma), [\Sigma] \rangle = \# S^{-1}(0)$$

where the RHS is the signed count of zeros. This is the number of zeros of a vector field, which, by Poincare Hopf, is the euler char of Σ. In particular,

$$\langle c_1(TS^2), [S^2] \rangle = 2$$

If we examine $S^2 \times S^2$, we know

$$T(S^2 \times S^2) = p_1^* TS^2 \oplus p_2^* TS^2$$

Then we may compute c_1:

$$\begin{aligned} c_1(T(S^2 \times S^2)) &= c_1(p_1^* TS^2) \cdot 1 + c_1(p_2^* TS^2) \cdot 1 \\ &= p_1^* c_1(TS^2) + p_2^* c_1(TS^2) \\ &= 2x + 2y \end{aligned}$$

where $x = p_1^*[\omega]$, and $y = p_2^*[\omega]$, and ω generates $H^2(S^2)$. We note that $c_1(S^2 \times S^2, \omega_{a,b})$ is independent of a, b. If $\phi : (S^2 \times S^2, \omega_{a,b}) \to (S^2 \times S^2, \omega_{1,1})$, then

$$\begin{aligned} &\phi^*(2x + 2y) = 2x + 2y \\ \iff &\phi^*(x + y) = x + y \end{aligned}$$

But $x + y = [\omega_{1,1}]$, so we have found that ϕ must preserve $\omega_{1,1}$, so that $a = b = 1$ only. For example, this also differentiates between the forms $\omega_{1,4}$ and $\omega_{2,2}$, despite them having the same volume.

Why did we say the chern class was independent of the symplectic form? If ω_t is a path of symplectic forms, and J_t is a corresponding path of compatible almost complex structures, then

$$(TM, J_0) \cong (TM, J_1)$$

i.e. you cannot deform the isomorphism class of a complex vector bundle. In particular, their chern classes will be equal. We are not in a position to prove this at the moment, but it should make intuitive sense: the chern class is an element of the discrete cohomology group: it shouldn't be changed by this sort of continuous variation.

If (M, ω) is compact, and $S \subset M$ is a codimension 2 symplectic submanifold. We want to compute $c_1(S, \omega|_S)$, in terms of $c_1(M, \omega)$. We will use a fact: You can always choose a complex structure J on M such that it preserves TS, in other words, pick a complex structure on S and extend it to one on M. Then there is a SES of complex VB's on S,

$$0 \longrightarrow TS \longrightarrow TM|_S \longrightarrow N_{S\backslash M} \longrightarrow 0$$

where N is the normal bundle, defined as the quotient. This is a SES of complex vector bundles because we picked J such that it would restrict to a complex structure on S, so the quotient is also a complex bundle. Then

$$TM|_S \cong TS \oplus N_{S\backslash M}$$

so that

$$c_1(TM|_S) = c_1(TM)|_S = c_1(TS) + c_1(N_{S\backslash M})$$

The <u>adjunction formula</u> is the equation

$$c_1(M, \omega)|_S = c_1(S, \omega|_S) + c_1(N_{S\backslash M})$$

Exercise: If (M, ω) is compact, symplectic of dimension 4, and $S \subset M$ a symplectic surface with Euler characteristic χ. Then

$$\langle c_1(M), [S] \rangle = \chi(S) + S \cdot S$$

hence the genus of S is determined by its fundamental class. Compute the genus of a symplectic surface in $\mathbb{C}P^2$ representing d times the hyperplane class, $d > 0$, and the genus of a surface in $(S^2 \times S^2, \omega_{1,1})$ representing $mx + ny$ $m, n \geq 0$.

Example: Take $X \subset \mathbb{C}P^n$ to be a smooth, complex hypersurface in $\mathbb{C}P^n$, i.e. the 0 set of a homogeneous polynomial of degree d. Then the adjunction formula says

$$c_1(X, \tau|_X) = (d - n - 1) \cdot h|_x$$

where h is the hyperplane class. We note that $f \in \Gamma\big((\ell^*)^{\otimes d}\big)$. Then it follows that $N_{X\backslash\mathbb{C}P^n} \cong (\ell^*)^{\otimes d}|_X$, so

$$c_1(N_{X\backslash\mathbb{C}P^n}) = d \cdot h|_X$$
$$c_1(\mathbb{C}P^n, \tau) = (n+1)h$$

then the adjunction formula proves the result. In particular, if $d = n + 1$, $c_1(X) = 0$.

$\square$

We will now discuss the application of obstruction theory to symplectic topology. Say X is a closed, oriented, 4-manifold, and simply connected. Then $H^2(X; \mathbb{Z})$ is a finite rank, free abelian group. It carries the intersection form induced by the cup product. From the viewpoint of de Rham cohomology,

$$\text{singular cohomology: } (x, y) \mapsto \langle x \smile y, [M] \rangle$$
$$\text{de Rham cohomology: } ([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

This form turns out to be unimodular over the integers by Poincare duality, i.e.

$$\begin{aligned} H^2(X) &\to \mathrm{Hom}(H^2(X), \mathbb{Z}) \\ x &\mapsto x \cdot - \end{aligned}$$

Also, $H^2(X;\mathbb{R}) = H^2(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, so the intersection form extends, so that

$$H^2(X;\mathbb{R}) = H^+ \oplus H^-$$

where H^+ is the positive eigenspace, and vice versa. In this case, we define $\tau(X)$ to be the signature of its intersection form, which is an invariant of the manifold. There is also a mod 2 version of the cup product, namely by looking at cohomology with $\mathbb{Z}_2$ coefficients. There is a $\mathbb{Z}_2$-linear map

$$\begin{aligned} H^2(X;\mathbb{Z}_2) &\to \mathbb{Z}_2 \\ x &\mapsto x \cdot x \end{aligned}$$

This form is again non-degenerate, so that there exists a unique $w_2 \in H^2(X;\mathbb{Z}_2)$ such that

$$x \cdot x = w_2 \cdot x$$

As usual, $H^2(X;\mathbb{Z}_2) = \frac{H^2(X;\mathbb{Z})}{2H^2(X;\mathbb{Z})}$. Then a lift of w_2 to $H^2(X;\mathbb{Z})$ is called a <u>characteristic vector</u>, c, if it satisfies $\forall\ x$, $x \cdot x = c \cdot x \bmod 2$.

Theorem (Wu): *If J is an almost complex structure on X, then $c_1(TX, J)$ is a characteristic vector.*

Theorem (Wu, Hirzebruch-Hopf): *If X is a simply connected, closed, oriented 4-manifold, then a characteristic vector $c \in H^2(X;\mathbb{Z})$ is the first chern class of an almost complex structure inducing an orientation iff $c^2 = 3\tau(X) + 2\chi(X)$.*

Example: Take $X = \mathbb{C}P^2 \# \mathbb{C}P^2$. Claim this is not almost complex, thus not symplectic. Recall $H^2(X;\mathbb{Z}) = H^2(\mathbb{C}P^2;\mathbb{Z}) \oplus H^2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}^2$. The intersection form is the 2 x 2 identity matrix, so $\tau(X) = 2$, and $\chi(X) = 4$, so that $3\tau(X) + 2\chi(X) = 14$. However, one can check that the characteristic vectors are of the form $c = (2m+1, 2n+1)$. Then one can compute

$$\begin{aligned} c \cdot c &= 4m(m+1) + 4n(n+1) + 2 \\ &\equiv 2 \bmod 8 \end{aligned}$$

but $14 \equiv -2 \bmod 8$.

$\square$

CHAPTER 10

Vector Bundles and Chern Classes

Lecture 10, Sept 29. Recall the theorem from last lecture:
Theorem (Wu): *If X is a closed, oriented, simply connected, 4-manifold, then given $c \in H^2(X;\mathbb{Z})$, there exists an almost complex structure, J, on X inducing the given orientation and such that $c_1(TM, J) = c$ iff c is a characteristic vector, and $c \cdot c = 3\tau(X) + 2\chi(X)$.*
Exercise: $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ is almost complex, with 24 different possible chern classes. Nevertheless, it is not symplectic (not part of the exercise).

In order to prove Wu's theorem (or give a proof sketch), we will briefly go over Stiefel-Whitney classes, which are the real analogue of chern classes.
Definition: If $E \to B$ is a real vector bundle of rank n, $w_i(E) \in H^i(X;\mathbb{Z}_2)$, $w_0 = 1$, are called the Stiefel-Whitney classes of E. We write $w(E) = \sum_i w_i(E) \in H^*(B;\mathbb{Z}_2)$.

Proposition: *We have the analogous properties from chern classes:*

$$i) \text{ for } f: Y \to X,\ w_i(f^*E) = f^*(w_i(E))$$
$$ii) \text{ the total sum, } w(E) = \sum_i w_i(E) \in H^\bullet(X).$$

$$\text{Then for VB's } E, E' \text{ over } X,\ w(E \oplus E') = w(E) \smile w(E')$$

ie
$$w_k(E \oplus E') = \sum_{i+j=k} w_i(E) \smile w_j(E')$$

and
$$iii) \text{ the tautological line bundle, } \ell \to \mathbb{R}P^1, \text{ has } w_1(\ell) \neq 0 \in H^2(\mathbb{R}P^1;\mathbb{Z})$$

in this case, the third requirement completely identifies $w_1(\ell)$. Again we have an existence and uniqueness theorem which we don't prove.

For a complex vector bundle, $\overline{c_i(E)} = w_{2i}(E)$, where the line indications a reduction mod 2.

We also have the Pontryagin class of a real vector bundle $V \to X$, given by

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X;\mathbb{Z})$$

Finally there is the Euler class: If $V \to X$ is an oriented VB of rank n, $e_(V) \in H^n(X;\mathbb{Z})$. If $V \to X$ is a smooth vector bundle over X, a compact, oriented smooth manifold, then

$$e(V) = PD[S^{-1}(0)]$$

for S some transversely vanishing section. This is also natural wrt pullbacks. To prove Wu's theorem, it is easier to prove a more general theorem. We think of the tangent bundle to a 4-manifold, but allow for something more general:

Theorem (Wu): *Let X be a CW complex of dimension ≥ 4, and $V \to X$ an oriented real VB of rank 4. Then V admits a complex structure J inducing the orientation of V with $c_1(V) = c$ iff $\bar{c} = w_2(V)$ and $c^2 = p_1(V) + 2e(V)$.*

How does this lead to the original formulation? We take $V = TM$, for M a 4-manifold. In this case, $\langle p_1(V), [X]\rangle = 3\tau(X)$, by the Hirzebruch signature theorem. On the other hand,

$$\langle e(TX), [X]\rangle = \chi(X)$$

by Poincare-Hopf, so that

$$\langle c^2, [X]\rangle = 3\tau(X) + 2\chi(X)$$

as required. Further, if X is simply connected,

$$\bar{c} = w_2(TX) \iff c \text{ is characteristic}$$

this is another famous result, called Wu's formula.

$\square$

Proof: ($\Longrightarrow$): If $V \to X$ is as described, we must show i) $\bar{c} = w_2$ and $c^2 = p_1(V) + 2e(V)$. i) just comes from general chern theory, so we won't prove it. It's just always[1] true that $\overline{c_1(E)} = w_2(E)$. For ii), if E is a complex VB, with complex structure J, then

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \overline{E}$$

where J acts as $-J$ on the second factor in the sum. Then

$$\begin{aligned} p_1(E \otimes_{\mathbb{R}} \mathbb{C}) &= -c_2(E \oplus \overline{E}) \\ &= -c_2(E) - c_2(\overline{E}) - c_1(E) \smile c_1(\overline{E}) \end{aligned}$$

But we claim $c_k(\overline{E}) = (-1)^k c_k(E)$. To see this[2], we simply note that the assignment[3] $E \mapsto (-1)^k c_k(\overline{E})$ satisfies the axioms of a theory of chern classes, thus by the uniqueness theorem, we have the equality. Then we can continue with the computation:

$$\begin{aligned} p_1(E) &= -2c_2(E) + c_1(E)^2 \\ \Rightarrow c_1(V)^2 &= p_1(V) + 2c_2(V) \\ &= p_1(V) + 2e(V) \end{aligned}$$

[1] As you might hope.
[2] what a neat proof strategy.
[3] I think you bar both sides to get the result.

where we have used a general fact that for a rank n complex vector bundle, $c_n(E) = e(V)$. $\square$

Suppose we have a fiber bundle $Z \to Y$, with fibers F, where F is simple, i.e. path connected, has abelian fundamental group, and $\pi_1(F, x)$ acting on $\pi_n(F, x)$ is trivial. This condition basically ensures that there is no need to track base points. Then we begin with X a CW complex, with k skeleton X^k and a map $f : X \to Y$, which we want to lift to Z:

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

Say we have a lift over $f : X^k \to Y$, $\widetilde{f}$, and we want to extend this to X^{k+1}. The $k+1$ cells are given by $e_\alpha^{k+1} \cong \overline{D}^{k+1}$, with attaching maps $\varphi_\alpha : S^k = \partial e_\alpha^{k+1} \to X^k$. We also have the inclusion $i_\alpha : e_\alpha^{k+1} \to X$. Over each cell, we have a fiber bundle by pulling back

$$\begin{tikzcd} (f \circ i_\alpha)^* Z \arrow[d] \\ e_\alpha^{k+1} \end{tikzcd}$$

But fiber bundles over disks are always trivial, so there is a global trivialization

$$\tau_\alpha : (f \circ i_\alpha)^* Z \to F$$

Then we have the maps[4]

$$\tau_\alpha \circ \widetilde{f} \circ \varphi_\alpha : S^k \to F$$

so that for each α, $[\tau_\alpha \circ \widetilde{f} \circ \varphi_\alpha] \in \pi_k(F)$, and we can forget base points by the simplicity condition. If we let α vary, we have a map $\{k+1\text{-cells}\} \to \pi_k(F)$, i.e.

$$c_{\widetilde{f}} \in C^{k+1}_{\text{CELL}}(Z; \pi_k(F))$$

This cochain is the obstruction to extending $\widetilde{f}$ over X^{k+1}, i.e. a lift exists iff $c_{\widetilde{f}} = 0$, but this is not a very useful criteria. Obstruction theory says that $c_{\widetilde{f}}$ actually defines a cellular cocycle, i.e. it vanishes on cellular boundaries. Further,
Theorem: *There is a lift $\widetilde{f}' : X^{k+1} \to Z$ of f agreeing with $\widetilde{f}$ over the X^{k-1} iff $[c_{\widetilde{f}}] = 0]$.*

Example: If $V \to X$ is an oriented real VB with inner product, we want to look for a section of its unit sphere bundle $S(V) \to X$, i.e. a nowhere vanishing section. We could view this as trying to lift over the identity map

$$\begin{tikzcd} & S(V) \arrow[d] \\ X \arrow[ur, dashed, "s"] \arrow[r, "id"] & X \end{tikzcd}$$

[4] this doesn't type check does it?

Of course, the fiber is $F = S^{n-1}$, so that $\pi_i(F) = 0$ for $i < n-1$. So obstruction theory tells us we can find a section of $S(V)|_{X^{n-1}}$, which is unique up to homotopy over X^{n-2}. However, $\pi_{n-1}(F) \cong \mathbb{Z}$, so there is an obstruction $o(V) \in H^n(X; \pi_{n-1}(F)) = H^n(X; \mathbb{Z})$. You can check that $o(V)$ is natural with respect to pullbacks, so that it defines a characteristic class. Then we define the Euler class to be $e(V) = o(V)$, so it is the (only) obstruction to finding a non-vanishing section of $V|_{X^n}$. So the problem we were considering, which is finding a complex structure $V \to X$, a rank 4 vector bundle, can be cast as an obstruction theory problem by:

$$\begin{tikzcd} & BU(2) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & BSO(4) \end{tikzcd}$$

Here the fibers are $F \cong \frac{SO(4)}{U(2)} \cong S^2$, so there are obstructions in $H^3(X; \mathbb{Z})$ and $H^4(X; \mathbb{Z})$. So the Hirzebruch-Hopf approach is to factorize the problem into two simpler problems, each of which has only a single obstruction. To describe the steps, if (V, g) is an oriented inner product space, then there is a hodge star operation $\star : \bigwedge^2(V^*) \to \bigwedge^2(V^*)$ characterized by the property

$$\lambda \wedge \star\mu = g(\lambda, \mu)\mathrm{vol}_g$$

If $\{e_i\}$ is an oriented orthonormal basis for V, then $\mathrm{vol}_g = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$, and g is the inner product on $\bigwedge^2(V^*)$ such that $e_i^* \wedge e_j^*$ forms an orthonormal basis for $i < j$. The upshot is then that

$$\bigwedge^2(V^*) \cong \lambda^+(V^*) \oplus \bigwedge^-(V^*)$$

where $\star$ acts by 1 on the first factor and -1 on the second. If we denote

$$\eta_{ij;kl} := \frac{1}{\sqrt{2}}(e_i^* \wedge e_j^* + e_k^* \wedge e_l^*)$$

then a basis for $\bigwedge^+(V^*)$ is given by

$$\{\eta_{12;34},\ \eta_{13;42},\ \beta_{14;23}\}$$

The action of $SO(V)$ on $\bigwedge^2(V^*)$ preserves $\bigwedge^+(V^*)$, so defines a map

$$\lambda^* : SO(V) \to O\left(\bigwedge^+(V^*)\right)$$
$$SO(4) \to SO(3)$$

There is a diffeomorphism $\mathcal{J}(V, g) \equiv \{J \in SO(V) \mid J^2 = -1\} \to S^2 \subset \bigwedge^+(V^*)$ by

$$J \mapsto \frac{1}{\sqrt{2}}\beta_J$$

where β_J is the 2-form $g(J\cdot, \cdot)$. So for a bundle $V \to X$, we can recast the problem of finding J to finding a self-dual unit length 2-form $\beta \in \Gamma\left(S(\bigwedge^+(V^*)) \to X\right)$. How can we do this? By finding a rank 2 Hermitian VB, $E \to X$, and an isomorphism $\mathbb{P}E \cong S(\bigwedge^+(V^*))$, where $\mathbb{P}E$ is the projective bundle of E. This is a particular case of a spin^C-structure. Further, we want to choose E such that it admits a nowhere vanishing section, σ, i.e. a section of the projective bundle of E, which by the chosen isomorphism gives a section of $S(\bigwedge^+(V^*))$, which gives a complex structure on V. So the challenge is to construct such an E admitting a non-vanishing section. Using obstruction theory, one can show that E exists so long as $w_2(E)$ admits an integral lift, and there is an obstruction to admitting a section, $c_2(E)$, which can be removed exactly when the conditions in Wu's theorem are met.

CHAPTER 11

Differential Geometry and Hamiltonians

Lecture 11, Oct 1. **Definition:** Let X_t be a time dependent vector field on M. The flow, ϕ_t, on some t-interval, $I \ni 0$, is a map $\phi_t : M \times I \to M$ such that

$$\phi_0 = Id_M, \quad \frac{d\phi_t}{dt} = X_t \circ \phi_t$$

Definition: For a differential form, η, the Lie derivative along X_t at $t = 0$ of η is

$$\mathcal{L}_{X_0}\eta := \frac{d}{dt}\Big|_{t=0}(\phi_t^*\eta)$$

We may define the Lie derivative for any time t by the lemma:
Lemma: $\forall$ *differential forms* η,

$$\frac{d}{dt}(\phi_t^*\eta) = \phi_t^*(\mathcal{L}_{X_t}\eta)$$

Proof: It suffices to check on functions: In this case,

$$\begin{aligned}\frac{d}{dt}(\phi_t^* f) &= \frac{d}{dt}(f \circ \phi_t)\\ &= \frac{df}{dt} \circ \frac{d\phi_t}{dt}\\ &= \frac{df}{dt}(X_t \circ \phi_t)\\ &= \phi_t^*(\mathcal{L}_{X_t}(f))\end{aligned}$$

where the last equality follows because the Lie derivative on functions reduces to the differential.

$\square$

Lemma: *If X_t is a time dependent vector field on (M, ω), a symplectic manifold, and ϕ_t its flow, TFAE, for $t \in I$:*
i) ϕ_t is a symplectomorphism, i.e. $\phi_t^*\omega = \omega$.
ii) $\mathcal{L}_{X_t}\omega = 0$
iii) $da_t = 0$, where $a_t = \iota_{X_t}\omega$

Proof: Becuase $\phi_0 = Id_M$, i) is equivalent to stating $\frac{d}{dt}(\phi_t^*\omega) = 0$. By the previous lemma, this occurs iff $\phi_t^*(\mathcal{L}_{X_t}\omega = 0$, iff ii).
We recall the famous Cartan formula:

$$\mathcal{L}_{X_t}\omega = d(\iota_X\omega) + \iota_X d\omega \equiv \{d, \iota_X\}\omega$$

In our case, ω is symplectic, so that

$$\begin{aligned}\mathcal{L}_{X_t}\omega &= d(\iota_{X_t}\omega) \\ &\equiv da_t\end{aligned}$$

so that ii) $\iff$ iii)

$\square$

Definition: A time dependent vector field satisfying i), ii) or iii) is called a symplectic vector field.

Then we can produce a path of symplectomorphisms ϕ_t, as the flow of a symplectic vector field, by i). OTOH, iii) provides a convenient way of getting symplectic vector fields: Take a path of closed 1-forms, a_t, then the condition

$$\iota_{X_t}\omega = a_t$$

defines X_t, by non-degeneracy of ω. So a path of closed 1-forms gives a family of symplectomorphisms. A key case is when a_t is an exact 1-form, so we take $H : I \times M \to \mathbb{R}$, a time dependent smooth function, and define X_t by the condition

$$\iota_{X_t}\omega = dH_t$$

ϕ_t^H is called a Hamiltonian flow, H is a Hamiltonian (energy function), X_t^H is a Hamiltonian vector field, and ϕ_1^H is a Hamiltonian symplectomorphism. It is convenient to allow H_t to be continuous and piecewise smooth wrt t, so that concatenations of Hamiltonians are Hamiltonians, so we get a group structure.

Proposition: *If the setup is as above, and J is a compatible, complex structure on M, $g = \omega(\cdot, J\cdot)$ a Riemannian metric, and ∇ the gradient, characterized by*

$$g(\nabla_g f, v) = df(v)$$

then

$$X_t^H = -J \cdot \nabla_g H_t$$

Proof: By the previous lemma, we need to check

$$\iota_{X_t^H}\omega = dH_t$$

$$\begin{aligned}\iota_{X_t^H}\omega \equiv \omega(X_t^H, \cdot) &= \omega(-J \cdot \nabla_g H_t, \cdot) \\ &= \omega(-J^2 \cdot \nabla_g H_t, J\cdot) \\ &= \omega(\nabla_g H_t, J\cdot) \\ &= g(\nabla_g H_t, \cdot) = dH_t\end{aligned}$$

□

In the time-independent case, H is often referred to as "autonomous". Then

$$\mathcal{L}_{X^H} H = dH(X^H)$$
$$= \omega(X^H, X^H) = 0$$

so that $\phi^* H$ preserves H.
Example: Take $\mathbb{R}^{2n}$, with standard coordinates, so that $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, and consider the translation map $T_{(a,b)}(x,y) = (x+a, y+b)$. This is a symplectomorphism, and claim $T_{(a,b)}$ is a Hamiltonian vector field, generated by the Hamiltonian $H(x,y) = \sum_{i=1}^n -b_i x_i + a_i y_i$. To check, we have the standard J_0 and standard Riemannian metric, so that

$$\nabla H = \sum -b_i \frac{\partial}{\partial x_i} + a_i \frac{\partial}{\partial y_i}$$
$$\Rightarrow -J_0 \nabla H = \sum b_i \frac{\partial}{\partial y_i} + a_i \frac{\partial}{\partial x_i} = (a,b)$$

which generates the flow $\phi_t(x,y) = (x+ta, y+tb)$. Then $T_{(a,b)}$ is the time 1 flow of the hamiltonian H, so it is a Hamiltonian vector field.

Example (Linear symplectic group): Take $\xi \in \mathfrak{sp}(\mathbb{R}^{2n})$, and $\alpha(x) = \exp(\xi) \cdot x$, so $\alpha \in \text{Aut}(\mathbb{R}^{2n})$. Claim α is the time 1 flow for the Hamiltonian $H(z) = \omega_0(z, \xi z) = -g_0(z, J\xi z)$. Again we check with the formula

$$\nabla H = -\sum_i (J\xi z)_i \frac{\partial}{\partial z_i}$$
$$\Rightarrow -J_0 \nabla H = (\xi z)_i \frac{\partial}{\partial z_i}$$

□

Example: Say $(M, \omega = d\theta)$ is an exact symplectic manifold, and ϕ_t a path of self-diffeomorphsisms with $\phi_0 = Id$. Show i) if $\phi_t^* \theta = \theta$, then ϕ_t is a Hamiltonian flow, and ii) more generally, if $\phi_t^* \theta = \theta + df_t$ for some f_t, then the same conclusion applies.

Example: Consider the 2-sphere inside the cylinder $S^1 \times [-1,1]$, with the map $F : S^2 \setminus \{(0,0,\pm 1)\} \to S^1 \times [-1,1]$ sending a point horizontally out to the cylinder, i.e.

$$(x,y,z) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right)$$

in spherical coordinates and cylindrical coordinates,

$$(\theta, \varphi) \mapsto (\theta, z)$$

then F is a symplectomorphism, when S^2 has the form ω is SO(3) invariant, with total volume 2π, and the cylinder has the form $d\theta \wedge dz$. Then the claim that F is a symplectomorphism boils down to an argument originally put forward by Archimedes, stating that the surface area of the cap $z \geq z_0$ is given by $\pi\rho^2$, where ρ is the distance from the north pole to the edge of the cap. From this fact, you can see that the symplectic forms agree, so F is a symplectomorphism. Then taking (S^2, ω) and R_α to be the rotation around the z-axis by an angle α, for $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. Then R_α is a Hamiltonian flow for the Hamiltonian function $H = \left(\frac{\alpha}{2\pi}\right) z$.

Example: Take a lattice in $\mathbb{R}^{2n}, \Lambda$, and consider $(M = \mathbb{R}^{2n}/\Lambda, \omega_0)$, the torus, and the translation map

$$[x, y] \mapsto [x + a, y + b]$$

similar to the translation example on $\mathbb{R}^{2n}$. This map is isotopic to the identity through symplectomorphisms, but not Hamiltonian.

CHAPTER 12

Poisson Bracket and Cotangent Bundle

Lecture 12, Oct 6. Consider a symplectic manifold (M, ω) of dimension $2n$, and its automorphism group $\mathrm{Aut}(M, \omega)$. For convenience, we only consider those automorphisms which are compactly supported, so that flows of compactly supported vector fields exist for all time. If one were to topologize $\mathrm{Aut}(M, \omega)$, we could talk about its identity component, $\mathrm{Aut}_0(M, \omega)$, but we don't want to go to the work of topologizing it, so we simply define the identity component to be those symplectomorphisms, ϕ, such that $\phi = \phi_1$, for some ϕ_t, a path starting at $\phi_0 = Id$. It is immediate to check that this subset forms a normal subgroup. In this subgroup, we have a smaller normal subgroup, $\mathrm{Ham}(M, \omega)$ of Hamiltonian automorphisms, i.e. those which are the time 1 flows generated by some Hamiltonian vector field, i.e. those which satisfy $\iota_{X_t}\omega = \omega(X_t, \cdot) = dH_t$. It's not as easy, but still true, that this is a normal subgroup in the whole symplectic automorphism group.

Heuristically[1], we think of (compactly supported) $\mathrm{Diff}(M)$ as a Lie group. Then the lie algebra should consist of infinitesimal diffeomorphisms, i.e. $\mathfrak{X}(M)$. The exponential map in this case is

$$\begin{aligned} \mathfrak{X}(M) &\to \mathrm{Diff}(M) \\ X &\mapsto \phi_1 \end{aligned}$$

where ϕ_t is the flow induced by the vector field X. Here is where the compact support hypothesis comes in, guaranteeing that the flow exists for all time. We also have a Lie bracket, given by the bracket of vector fields. We have a "Lie subgroup", $\mathrm{Aut}(M, \omega) \subset \mathrm{Diff}(M)$, and its Lie algebra is the symplectic vector fields, i.e. if u, v are symplectic vector fields, then $[u, v]$ is too:

$$\mathcal{L}_{[u,v]}\omega = (\mathcal{L}_u \circ \mathcal{L}_v - \mathcal{L}_v \circ \mathcal{L}_u)\omega = 0$$

We have a SES of vector spaces[2]:

$$0 \longrightarrow C^\infty(M)/\mathbb{R} \longrightarrow \mathfrak{X}_C(M) \longrightarrow H^1(M; \mathbb{R}) \longrightarrow 0$$

[1] This can be made precise, but we're not going to do it right now.

[2] Assuming M is connected, so that locally constant = constant.

where $C^\infty(M)/\mathbb{R}$ is the space of Hamiltonian functions, $\mathfrak{X}_C(M)$ is the space of symplectic, compactly supported vector fields, so the maps are:

$$H \longmapsto X^H \longmapsto \iota_X\omega = dH$$

where X^H is the Hamiltonian vector field induced by the Hamiltonian H, i.e. the one which satisfies $\iota_X\omega = dH$. $\iota_X\omega$ forms a de Rham class because it is symplectic, thus closed, by the lemma from the previous lecture, and the sequence is exact because $\iota_X\omega$ is an exact form if X is Hamiltonian.

Lemma: *The bracket of Hamiltonian vector fields, X^H, X^K, is again Hamiltonian, and*

$$[X^H, X^K] = X^{\{H,K\}}$$

where

$$\{H, K\} := \omega(X^H, X^K)$$

Proof:

$$\{H, K\} = \iota_{X^K}\iota_{X^H}\omega = \iota_{X^K}dH \boxed{=} \mathcal{L}_{X^K}H$$

The boxed equality follows from the Cartan formula:

$$\mathcal{L}_{X^K}H = d(\iota_{X^K}H) + \iota_{X^K}dH$$

but by convention, the interior product with a 0-form, H, is 0, thus the equality follows.

$$d\{H, K\} = d\mathcal{L}_{X^K}H = \mathcal{L}_{X^K}dH$$

Note that d commutes with the Lie derviative since it commutes with pull backs. Then

$$\iota_{[X^H,X^K]}\omega = \iota_{(\mathcal{L}_{X^K}X^H)}\omega$$

The Lie derivative also obeys a form of a Leibniz rule when acting on a contraction, so

$$= \mathcal{L}_{X^K}(\iota_{X^H}\omega) - \cancel{\iota_{X^H}\mathcal{L}_{X^K}\omega} = \mathcal{L}_{X^K}dH$$
$$\Rightarrow \iota_{[X^H,X^K]}\omega = d\{H, K\}$$

□

so compactly support Hamiltonain vector fields are a Lie subalgebra of compactly supported vector fields, i.e. $H \to X^H$ is a linear iso, and this map intertwines the Poisson bracket and Lie bracket in the sense that the following diagram commutes:

$$\begin{array}{ccc}
(C^\infty(M)/\mathbb{R})^2 & \longrightarrow & (\mathrm{Ham}(M))^2 \\
\downarrow & & \downarrow \\
C^\infty(M)/\mathbb{R} & \longrightarrow & \mathrm{Ham}(M)
\end{array}$$

sending

$$\begin{array}{ccc} (H,K) & \longrightarrow & (X^H, X^K) \\ \downarrow & & \downarrow \\ \{H,K\} & \longrightarrow & X^{\{H,K\}} = [X^H, X^K] \end{array}$$

In particular, this makes $C^\infty(M)/\mathbb{R}$ into a Lie algebra, with the Poisson bracket. This also makes $C^\infty(M)$ a Lie algebra via the following argument: Extend the definition of $\{\cdot,\cdot\}$ to $C^\infty(M)$. To show it is a Lie algebra, we must show skew symmetry, bilinearity, and the Jacobi identity. The first two are manifestly satisfied by the definition of ω. To check the Jacobi identity, recognize that ω is a local object, i.e. the value of ω near x depends only on H and K near x. Then take three functions, and consider their Jacobiator, the expression which is supposed to be equal to 0 to satisfy the Jacobi identity. By locality, we can simply find some point far away from x, and wiggle the three functions until the Jacobiator is 0. But becuase the Jacobiator is a constant function, we are done.

Definition: A Poisson manifold is a manifold with a Poisson structure, that is a Lie algebra structure, $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibniz property $\forall f,g,h \in C^\infty(M)$,

$$\{fg,h\} = f\{g,h\} + g\{f,h\}$$

Example: The Poisson bracket defined above is a Poisson structure.

Example: In $M = \mathbb{R}^{2n}$, and coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$,

$$\{x_i,x_j\} = \{y_i,y_j\} = 0,\ \{x_i,y_j\} = \delta_{ij}$$

matching the structure of our symplectic basis.

Proposition: *If $(H_1,\ldots,H_m)$ are Poisson commuting Hailtonians, and if 0 is a regular value of $H := (H_1,\ldots,H_m)$, then $H^{-1}(0)$ is isotropic.*
Proof: $T_x(H^{-1}(0) = \text{span}(v_1,\ldots,v_m)$, where $v_i = X^{H_i}$. Then $\omega(v_i,v_j) = \{H_i,H_j\} = 0$. $\square$

So there are at most n commuting Hamiltonians whose vector fields are linearly independent.

If L is a smooth mfld of dimension n, then T^*L carries a tautological 1-form, θ_L, such that $\omega_L = d\theta_L$ makes T^*L into an exact symplectic manifold. To define this 1-form, for any $\lambda \in T^*L$ and $v \in T_\lambda T^*L$, we define

$$\langle \theta_L, v\rangle_\lambda = \langle \lambda, d\pi(v)\rangle_{\pi(\lambda)}$$

In a sense, the 1-form at λ is just evaluate with λ. This construction is intrinsic to L, i.e. if $f : L \to M$ is a diffeomorphism, then d^*f respects the tautological 1-forms.

If we take coordinates $(q_1, \ldots, q_n)$ on $U \subset L$. Then $T^*U \to L$ is isomorphic to a trivial bundle, $\mathbb{R}^n \times U$, with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$, via the isomorphism

$$(\vec{p}, \vec{q}) \mapsto \sum p_i dq_i$$

In this case, $\theta_L|_{T^*U} = \sum p_i dq_i$, and we often informally write $\theta_L = \vec{p}d\vec{q}$, reflecting the fact that, at a point λ, applying theta just means "pair with λ". Then

$$\omega_L \equiv d\theta_L = \sum dp_q \wedge dq_i = d\vec{p} \wedge d\vec{q}$$

i.e. the standard symplectic form on $\mathbb{R}^{2n}$. So this gives a nice invariant of any symplectic manifold. Another type of invariant exists for exact symplectic manifolds: If $\omega = d\theta$), there is a unique vector field, Z, such that

$$\iota_Z \omega = \omega(Z, \cdot) = \theta$$

this comes because on an exact symplectic manifold, we have a canonical choice of 1-form. The uniqueness comes from ω being pointwise non-degenerate. By the Cartan formula,

$$\begin{aligned} \mathcal{L}_Z \omega &= d(\iota_Z \omega) + \cancel{\iota_Z d\omega} \\ &= d(\omega(Z, \cdot)) = d(\theta) = \omega \end{aligned}$$

so the Lie derivative along Z conformally expands ω, it scales ω up. On T^*L, using our previous coordinates, the Louisville vector field is $Z = \sum p_i \partial_{p_i}$, the radial Euler vector field.

In classical dynamics, we often model phase spaces as cotangent bundles. E.g. an object in $\mathbb{R}^3$ has position $q \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. But the coordinate transformation changes of physics demand that we choose a covector to represent the momentum rather than a vector. Similarly if a system has n objects in $\mathbb{R}^3$, the momentum of the system is stored in $T^*\mathbb{R}^{3n}$.