M390C: Geometry in Group Theory

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Abstract

Class taught by Daniel Allcock, notes taken by Reese Lance. The notes were taken by hand during class and typed later. Some of my own thoughts are interjected, but quite rarely. I initially thought to try to separate my thoughts from the professor's but it becomes too difficult. As such I will also try to expand on examples which are mentioned in passing in class, spell out proofs which are glossed over, and add insight where I think it is helpful. This helps to justify the existence of this set of notes, as opposed to live-texed notes. Especially because some of my own content is interspersed throughout these notes, any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu) or if you can find any other way to communicate with me, that is also fine. At the moment I'm trying to get the notes written, and worrying about making the format not look like trash later. I'm also not going to track theorem and lemma numbers, as I think that's mostly useless. If a proof somewhere says "Applying Theorem X", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers, which is what would have been the plan. Thanks to Arun Debray whose formatting choices inspired my own. A link for the homepage of the class.

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CHAPTER 1

Some Group Theory and Philosophy About the Class

Lecture 1, Jan 22. In standard undergraduate and even graduate group theory courses, the definition of a group, and a first encounter with it, tends to be highly unmotivated. The 'a group is a set with a binary operation satisfying blah blah blah' definition is the exact opposite of how mathematicians usually work with groups. The natural way to look at groups, Daniel argues, is to let them act on stuff. Almost all groups which are of interest in mathematical research arise in this way, as the symmetry group of some interesting object[1]. So in this class, we will almost always view groups as acting on something. In fact, we only ever need to engage with the abstract definition because sometimes groups do not have a nice faithful action.

If a group does not come with an action on something, we will attempt to find something which it does act on. If we cannot, we throw the group away[2]. One class of exceptions is groups which arise as a presentation, i.e. the form

$$G = \langle S \mid R \rangle$$

with a set of generators and relations. These sometimes find application in proving statements about logic[3]. Instead, we will usually see groups of the form

$$G = \mathrm{Aut}(\text{something cool})$$

If a group acts on an object, it usually will also act on 'derived objets'[4] such as functions on the object, for example. Roughly, we say we understand a group geometrically when we know several spaces which the group acts on, which are different enough to give distinct perspectives.

Example: We passed around a 3D dodecahedron/icosahedron, with certain vertices highlighted various colors. Upon inspecting the vertices, we see that vertices of a fixed colors (e.g. all red vertices) form a tetrahedron. There are 5 tetrahedra inside the dodecahedron, which

[1]Some exceptions include the space of line bundles over a space, which has an abelian group structure endowed by the tensor product.

[2]Daniel says this is only a slight exaggeration.

[3]I do not know what this is referring to because I don't know any pure logic.

[4]When he said 'derived objects,' I considered that I may have bitten off more than I could chew in this class. That still might be true, but 'derived' is being used in the colloquial sense here.

has symmetry group A_5 excluding the orientation reversing transformations. The tetrahedron group is A_4, since we can freely permute all 4 vertices. The 5 tetrahedra inside the dodecahedron, then correspond to the 5 A_4 subgroups inside of A_5, which are all conjugate. If we think of A_n as acting on n points, each subgroup $A_4 \subset A_5$ corresponds to fixing one of the vertices and acting on the remaining 4 vertices. So somehow, fixing one tetrahedron and permuting the rest corresponds to one of the A_4's inside A_5, so that fixing one tetrahedron should make the remaining transformations look like the tetrahedron group, but this is hard for me to visualize. I've never been very good at these visual arguments.

What is it about a space that makes it geometric? Somehow $\mathbb{R}^2$ is more geometric than the set of functions between two sets. Sophus Lie carried the philosophy that the thing that makes a space geometric is the fact that a group is acting on it. Under this view, $\mathbb{R}^2$ is not geometric because we can equip it with a distance metric, but because it is acted on by the group of translations and rotations.

Example: $SL_2(\mathbb{R}) \curvearrowright \mathbb{R}^2$ by the obvious action. Let V be a vector space of dimension 2, and define

$$\mathrm{Sym}^2(V^*) := \{\text{symmetric, bilinear forms on } V\}$$

If we choose a basis for V, $\{e_i, e_j\}$, then we can view $B \in \mathrm{Sym}^2(V^*)$ as a matrix B_{ij}:

$$B_{ij} = B(e_i, e_j)$$

We may also define

$$\det B := \det B_{ij}$$

However, this is of course basis dependent. If we pick a different basis, $f_i = 2e_i$, and define B_e as the matrix of B with respect to the e_i basis, and similarly for f, then if we compute the determinant again,

$$\begin{gathered}(B_f)_{ij} = B(f_i, f_j) = B(2e_i, 2e_j) = 4B(e_i, e_j) = 4(B_e)_{ij} \\ \Rightarrow \det B_f = 16 \det B_e\end{gathered}$$

Another way we could think of this is that $\mathrm{Sym}^2(V^*)$ acts on $V^2 \cong \mathbb{R}^4$. If we pick a basis $\{e_i\}$ of V, this gives a basis of V^2, $\{(e_1, 0), (e_2, 0), (0, e_1), (0, e_2)\}$. Picking the same $\{f_i\}$ as before similarly gives a basis on V^2, with change of basis matrix $A = \mathrm{diag}(2, 2, 2, 2)$. In such a case,

$$\begin{gathered}B_f = B_e A \\ \Rightarrow \det B_f = \det(B_e A) \\ \det B_f = \det(B_e)\det(A) \\ = 16\det(B_e)\end{gathered}$$

However, $SL_2(\mathbb{R})$ acts[5] on the set of bases of $\mathbb{R}^2$, and all bases in the same orbit will have $\det A = 1$, where A is the change of basis matrix. Now if we pick an $SL_2(\mathbb{R})$ orbit of the set of bases on V, we have a basis independent quadratic function

$$\det : \mathrm{Sym}^2(V^*) \to \mathbb{R}$$

[5] You may think we would care about $SL_4(\mathbb{R})$ and its action on $\mathbb{R}^4$, but the 4 dimensions are somewhat artificial. We're just looking at 2 copies of V, and as such only care about two dimensions.

which sends

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto ac - b^2$$

Which is just the double sided cone.

If G is a group, $G \curvearrowright G/H$. The action is boring if H is not a maximal subgroup[6], in a way which we will make more precise later. Broadly, we expect the space to break up into pieces which G permutes[7].

Example: $SL_2(\mathbb{R}) \curvearrowright \Big\{ \text{ det 0 hyperboloid } -\{0\} \Big\}$[8]. We remove the 0 point so that the action is transitive. Solving

$$\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$$
$$\Rightarrow a = 1, b = c = 0$$

So that the stabilizer[9] of a point is $\cong \mathbb{R}$. In $\mathbb{R}^2$, a singular quadratic form determines a line, $V^{\perp}$, and a non singular quadratic form on $V/V^{\perp}$. There's some more to this example, but I'm not understanding it, I'm going to go to office hours and come back and update this.

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[6] A subgroup $H \subset G$ is maximal if there is no subgroup $H' \subset G$ such that $H \subset H'$, where each containment is strict.

[7] I think the idea here is that the action on the individual pieces is more interesting. G is just many copies of the actual interesting thing inside it. Not sure what this has to do with maximality though. If I had to guess it would have had to do with normality, since if N is normal in G, then G permutes a bunch of N-invariant subspaces. Isn't that exactly what we're talking about?

[8] I think it acts on every level set?

[9] Daniel says that the stabilizer is $\mathbb{R} \rtimes \{\text{finite group}\}$ but I'm not sure why.

Generally, if $H < J < G$, and $G \curvearrowright G/H$, then

$$G \curvearrowright G/J$$

and

$$\begin{tikzcd} G/H \arrow[d, "G-equivariantly"] \\ G/J \end{tikzcd}$$

We want to understand the maximal subgroup, as the non-maximal subgroup often underlies a more interesting action. Returning to the icosahedron/dodecahedron, with 20 vertices, we have a partition into 5 sets of 4, the 5 tetrahedra. These correspond to the inclusions

$$\begin{tikzcd} & \mathbb{Z}/3 \times \mathbb{Z}/2 \arrow[dr, hook] & \\ \mathbb{Z}/3 \arrow[ur, hook] \arrow[dr, hook] & & A_5 \\ & A_4 \arrow[ur, hook] & \end{tikzcd}$$

I don't understand this diagram too well either.

CHAPTER 2

Idk the title yet

Lecture 2, Jan 24. Some books that might help: Thurston's 3D Geometry and Topology (book and online lecture notes), Conway and Sloane's Sphere packings, lattices, and groups (affectionately referred to as SPLAG).

Recall: Groups act on things. Every transitive action of G is equivalent to an action on G/H. I'm going to prove this because I forgot the proof in class:

Proof: Fix $x_0 \in X$. If the action is transitive, then every element $x \in X$ can be written as gx_0, for some $g \in G$. Then define $H = \mathrm{Stab}(x_0)$. Then we have the map $X \to G/H$ given by

$$x \mapsto gH$$

where g is the element such that $gx_0 = x$. To check well-defined-ness, pick some $gh \in gH$. Then $(gh)x_0 = gx_0 \mapsto gH$. This bijection is G-equivariant, which shows the result.

□

We saw last time that this is most interesting when H is maximal.

Definition: If $G \curvearrowright X$ is a transitive action, then the action is called <u>primitive</u> if there does not exist a G-invariant partition of X, except for the trivial ones.

Theorem: $G \curvearrowright G/H$ primitively iff H is maximal.

Proof: If the action were not primitive,