# M392C: Orderability/3-Manifold Groups

Reese Lance

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**Abstract**

Class taught by Cameron Gordon, notes taken by Reese Lance. The notes are not live texed. They are post-mortem-texed, that is, taken by hand during class and typed later. Some of my own thoughts are interjected, but rarely. I initially thought to try to separate my thoughts from the professor's but it becomes too difficult. As such I will also try to expand on examples which are mentioned in passing in class, spell out proofs which are glossed over, and add insight where I think it is helpful. This helps to justify the existence of this set of notes, as opposed to live-texed notes (which I believe there are for this class, thanks to Jackson van Dyke), which are probably slightly better for a faithful representation of what is being taught in the classroom. Especially because some of my own content is interspersed throughout these notes, any corrections, questions, comments, suggestions, etc., can be sent via email (reese.lance@utexas.edu) or if you can find any other way to communicate with me, that is also fine. At the moment I'm trying to get the notes written, and worrying about making the format not look like trash later. I'm also not going to track theorem and lemma numbers, though Cameron does, as I think that's mostly useless. If a proof somewhere says "Applying Theorem $X$", it can usually be determined from context what theorems need to be invoked, and if the reader doesn't find it readily apparent, then searching for the theorem in question will be a valuable experience. Also I always forget to write down the numbers. Also as I revisit and add in more stuff, the numbering becomes involved and I'd have to actually figure out how to number properly instead of just manually putting numbers, which is what would have been the plan. Thanks to Arun Debray whose formatting choices inspired my own.

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CHAPTER 1

# Overview and Preliminary Group Theory

**Lecture 1, Jan 21.** The book we will be following is "Ordered Groups and Topology" by Clay and Rolfsen, a pdf of a draft of which can be found here. Orderability was originally developed as a group theoretic concept, and only (somewhat) recently has topology entered the picture. In fact, some of the bare bones definitions can be made entirely in Set:

**Definition:** Given a set $X$, a strict, total order (STO) on $X$ is a binary relation $<$, satisfying

$$i)\text{ Transitivity : } \forall\, x, y, z \in X, x < y, y < z \Rightarrow x < z$$
$$ii)\text{ Totality: } \forall\, x, y \in X, \text{exactly one of the following holds:}$$
$$x < y,\ \ y < x,\ \text{ or } x = y$$

For convenience, we may also define the STO $>$ as induced by an STO:

$$x > y \iff x \neq y \text{ and } x \not< y$$

If we upgrade to Grp, we can use the multiplication structure to talk about invariance:
**Definition:** Given a group, $G$, and an STO, $<$, on the underlying set, $<$ is called a left order (LO) on $G$ if

$$g < h \Rightarrow fg < fh \,\forall\, f \in G$$

and we call $G$ left orderable if there exists a left ordering of $G$. Similarly, we can define right-orderability in the obvious way. One might ask how different the concepts of left and right orders are. It is clear that for abelian groups, they are one and the same, that is: all left orders are also right orders. In general this may not hold. If it does,
**Definition:** Given an LO, $<$, on $G$, it is called a bi order (BO) if it is also an RO, and we say $G$ is bi orderable if it admits a bi order.
To return to the above question, given an LO $<$, we can induce an R, $\prec$ O in the following way:

$$g \prec h \iff h^{-1} < g^{-1}$$

**Lemma:** *The above binary relation is an RO on G.*
*Proof:* Transitivity and totality of $\prec$ follow immediately from $<$ being an STO. To show it is a right order, consider elements $g, h, f$ of $G$ such that

$$g \prec h$$

Then

$$
\begin{gathered}
h^{-1} < g^{-1} \\
\Rightarrow f^{-1}h^{-1} < f^{-1}g^{-1} \\
\Rightarrow (f^{-1}h^{-1})^{-1} \succ (f^{-1}g^{-1})^{-1} \\
\Rightarrow gf \prec hf
\end{gathered}
$$

□

So every left order induces a right order, i.e. all left orderable groups are also right orderable. Note that this does not imply that all left orders are bi orders. That would require that all left orders are **themselves** right orders. All we have shown is that a left order **induces** a right order. Indeed, it is not the case: There are groups which are left orderable but not bi orderable. We obviously can restrict the LO to an LO on a subgroup $H < G$ (denoting subgroup here), just as an STO can be restricted to a subset $A \subset X$.

**Example:** $(\mathbb{R}, +)$ is bi ordered, with the usual ordering. Its subgroups $\mathbb{Z}, \mathbb{Q}$, etc. can inherit this ordering.

Here is a series of elementary lemmas:

**Lemma:** *Given $<$ an LO on G,*

$$
\begin{gathered}
i)\ g > 1, h > 1 \Rightarrow gh > 1 \\
ii)\ g > 1 \Rightarrow g^{-1} < 1 \\
iii)\ < \ is\ BO \iff g < h \Rightarrow f^{-1}gf < f^{-1}hf \ \forall f \in G \\
(invariant\ under\ conjugation)
\end{gathered}
$$

*Proof:*

$$
\begin{gathered}
i)\ h > 1, \Rightarrow gh > g > 1 \\
ii)\ g > 1 \Rightarrow g^{-1}g > g^{-1} \Rightarrow 1 > g^{-1}
\end{gathered}
$$

$iii)$ Assume $<$ is also an RO. Then we can clearly apply left invariance on $f^{-1}$ and right invariance on $f$ to obtain the result. If $<$ is conjugation invariant, then

$$
\begin{gathered}
g < h \Rightarrow f^{-1}gf < f^{-1}hf \\
gf < hf
\end{gathered}
$$

Using left invariance in the final line.

□

**Lemma:** *Given $<$ a BO on G,*

$$
\begin{gathered}
i)\ g < h \Rightarrow g^{-1} > h^{-1} \\
ii)\ g_1 < h_1, g_2 < h_2 \Rightarrow g_1g_2 < h_1h_2
\end{gathered}
$$

*Proof:*

$$
\begin{gathered}
i)\ g < h \Rightarrow g^{-1}g < g^{-1}h \\
1 < g^{-1}h \Rightarrow h^{-1} < g^{-1}hh^{-1} \\
\Rightarrow h^{-1} < g^{-1} \\
ii)\ g_1 < h_1 \Rightarrow g_1g_2 < h_1g_2 < h_1h_2
\end{gathered}
$$

The final inequality follows because $g_2 < h_2$, applying left invariance of $h_1$.

$\square$

Statement $i)$ above need not hold for an LO on a group, although we showed that it always holds in the case that $h = 1$. So we've only seen one example of an orderable group, the real numbers. We'd like to see some more, but explicit orderings can be very difficult to write down, as we'll see later. For now, we can give an important class of non examples via the following:

**Lemma:** *If G is left orderable, it is torsion-free.*

*Proof:* Given $g \in G/\{1\}$, wlog assume $g > 1$. By left invariance,

$$g^n > g^{n-1} > \cdots > 1$$

For any $n$, this series does not terminate, since the inequalities are strict.

$\square$

In particular, this rules out all finite groups from being left orderable. Another class of not-necessarily-examples is quotient groups. Because subgroups inherit orders, we might guess that quotient groups also do. By simply quotienting something orderable to get something finite, we see that this is not the case. For example, the quotient

$$\mathbb{Z} \to \mathbb{Z}_n$$

does not transport an order.

**Remark:** If $\{G_\lambda\}_{\lambda \in \Lambda}$ is an indexed family of groups, the <u>direct product</u> of $G_\lambda$ is the set

$$\prod_{\lambda \in \Lambda} G_\lambda$$

with multiplication defined componentwise, i.e. the elements of the product are $\lambda$[1]-tuples, where multiplication is performed in the individual components where we know how to multiply.

**Definition**: A <u>well order (WO)</u> is an STO on a set $X$ such that for every nonempty $A \subset X$, $A$ has a least element, that is $\exists\, x_0 \in A$ such that $x_0 < x \;\forall\, x \in A$. We might want to know which sets can be well ordered. It might seem like all sets can do this, and indeed that statement is equivalent to the axiom of choice [2]

[1] To be pedantic, these should be labelled as Card($\Lambda$)-tuples

[2] I only understand this on an intuitive level, since I've never had any formal contact with the axiom of choice.

**Theorem**: *The family $G_\lambda$ is* $\begin{pmatrix} LO \\ BO \end{pmatrix}$[3] *iff the direct product is* $\begin{pmatrix} LO \\ BO \end{pmatrix}$

*Proof*: The $\Leftarrow$ implication is easy because we realize each $G_\lambda$ as a subgroup of the direct product, so that it inherits any ordering. In the other direction, assume we have an $\begin{pmatrix} LO \\ BO \end{pmatrix}$ ordering $<_\lambda$ on each group $G_\lambda$. To construct an $\begin{pmatrix} LO \\ BO \end{pmatrix}$ ordering on the direct product, take two elements, $(g_\lambda), (h_\lambda)$. Assuming AoC, we can well order the indexing set, $\Lambda$. Choose a well ordering, $\prec$. Assuming $(g_\lambda) \neq (h_\lambda)$, the set of indices where $g_i \neq h_i$ is a nonempty subset of $\Lambda$, and thus has a least element, $\lambda_0$. We define

$$(g_\lambda) < (h_\lambda) \iff g_i <_{\lambda_0} h_i$$

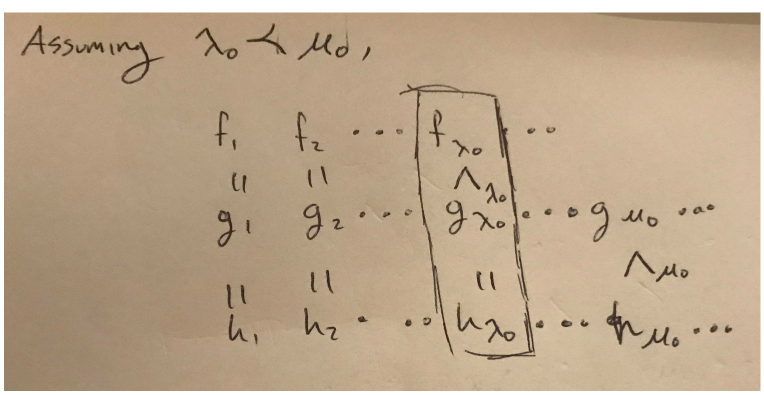
Intuitively, this is the same as comparing numbers. Start at their first digit, compare. If they're the same, go to the next digit, and continue comparing until they're not equal, and compare there. To show this is a transitive relation, suppose $f < g, g < h$. Then for each relation, we have a least index where the elements being compared differ $\lambda_0, \mu_0$. If it happens that $\lambda_0 = \mu_0$, then we have $f_{\lambda_0} <_{\lambda_0} g_{\lambda_0}$ and $g_{\lambda_0} <_{\lambda_0} h_{\lambda_0}$, and $f_i = g_i = h_i$ for $i \prec \lambda_0$. But $<_{\lambda_0}$ was an STO so by transitivity, $f_{\lambda_0} <_{\lambda_0} h_{\lambda_0} \Rightarrow f < h$. If $\lambda_0 \prec \mu_0$, again $f_i = g_i = h_i$ for $i \prec \lambda_0$. We know $g$ and $h$ agree on all indices $j \prec \mu_0$, in particular at $\lambda_0$, i.e.

$$g_{\lambda_0} = h_{\lambda_0}$$

Combining this with the fact that $f < g$, and they disagree at $\lambda_0$, we have

$$f_{\lambda_0} <_{\lambda_0} g_{\lambda_0} = h_{\lambda_0}$$
$$\Rightarrow f < h$$

The case where $\mu_0 \prec \lambda_0$ follows in the same manner. During class, I remember being completely confused by this, and getting lost in symbols, and had to revisit after class to understand it, so I drew a little picture that may help put things in place[4]:




[3] Meaning each individual $G_\lambda$ is ____orderable

[4] After having made the "diagram" I realized how simple the idea was, and probably nobody had any problem understanding this proof, but I went to the trouble of drawing it so I may as well use it.

So this is a transitive binary relation. It is totally ordered because the $<_\lambda$ are. It is left(right) invariant because multiplciation doesn't change the index $\lambda_0$. To belabor the point, if we have $g < h$, then $g_i = h_i$ for $i \prec \lambda_0$ and $g_{\lambda_0} <_{\lambda_0} h_{\lambda_0}$. Thus, for $i \prec \lambda_0$,

$$(fg)_i = f_i g_i = f_i h_i \text{ because } g_i = h_i$$
$$(fg)_{\lambda_0} = f_{\lambda_0} g_{\lambda_0} <_{\lambda_0} f_{\lambda_0} h_{\lambda_0}$$
$$\Rightarrow fg < fh$$

which shows $<$ is an LO on the direct product.

$\square$

We now know that we can build orderable groups out of known orderable groups by simply taking their direct product.

**Definition:** Given a family of indexed groups $\{G_\lambda\}_{\lambda \in \Lambda}$, the <u>direct sum, $\sum_{\lambda \in \Lambda} G_\lambda$</u> is the subgroup of the direct product $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of only those elements such that all but finitely many coordinates are 1.

**Corollary:** *$G_\lambda$ is $\begin{pmatrix} LO \\ BO \end{pmatrix}$ iff the direct sum is $\begin{pmatrix} LO \\ BO \end{pmatrix}$.*

**Proof:** The direct sum is a subgroup of the direct product, so it inherits the order.

$\square$

**Corollary:** *Free abelian groups are bi orderable.*

**Proof:** The free abelian group on a set $\Lambda$ is just a direct sum over $\mathbb{Z}$:

$$Free(\Lambda)_{Ab} = \sum_{\lambda \in \Lambda} \mathbb{Z}$$

$\square$

**Definition:** Let $<$ be an LO on $G$. The <u>positive cone, $P_<$</u> is the set of positive elements:

$$P_\lambda = \{g \in G |\, g > 1\}$$

**Lemma:**

$$i)\ P_\lambda \ is\ a\ sub\ semi-group.$$
$$ii)\ G = P_\lambda \sqcup P_\lambda^{-1} \sqcup 1$$
$$iii)\ < is\ a\ BO \iff f^{-1} P_\lambda f \subset P_\lambda\ \forall\, f \in G$$

**Proof:** We have already proven all the facts needed for this lemma. The only interseting part is that $P_\lambda$ is a sub *semi*-group rather than a subgroup. Recall the distinguishing features of a semi group is it is a group that need not have an identity element, and elements need not have multiplicative inverses. Indeed that is the case here, there are neither. The identity element does not satisfy $1 < 1$, and if $g \in P_\lambda$, i.e. $g > 1$, then $g^{-1} < 1$, i.e. $g^{-1} \notin P_\lambda$.

$\square$

**Definition:** $P \subset G$ is a <u>positive cone</u> if it satisfies properties $i)$ and $ii)$ above.

**Lemma:** *If $P \subset G$ is a positive cone, then $g < h \iff g^{-1}h \in P$ defines an LO on $G$, with*

$P_< = P$.
Combining the previous two lemmas, we have a correspondence

$$\{LO's\ on\ G\} \Longleftrightarrow \{positive\ cones\ in\ G\}$$
$$\{BO's\ on\ G\} \Longleftrightarrow \{conjugation\ invariant\ positive\ cones\ on\ G\}$$

This correspondence is the essence of where topology comes into the discussion. Left and right orderability are algebraic concepts. Positive cones are nice subsets of $G$, i.e. elements of the power set, which can be topologized. So the space of left orders corresponds to a subspace of a topological space. Let's revisit the topology on a power set[5]. Given a set, $X$, its power set can be viewed as $P(X) = \text{Hom}(X, \{0,1\})$. We can always view functions $X \to Y$ as a nice[6] subset of $X \times Y$, so if we give $\{0,1\}$ the discrete topology $P(X) = \text{Hom}(X, \{0,1\}) \subset \{0,1\}^{\text{Card}(X)}$ which is given the product topology[7]. So the space of left orders on a group is a topological space, and in some cases it may look odd. It could be a cantor set, or other weird things. The cardinalities of these spaces is interesting. Some groups admit only finitely many LO's. In such a case, the order is always $2^n$, for some $n$. If a group does admit infinitely many LO's, it is always uncountably many. Following Tychonoff's theorem, the space of orders is compact, and we will see a proof of this later.
**Theorem:** Free(2) *is LO.*
**Proof:** An explicit ordering was constructed, but I didn't pay close enough attention. It was gross. Look at Jackson's notes for a discussion of this.
**Corollary:** Any free group on a countable number of generators is also orderable.
**Proof:** Any free group on a countable number of generators is isomorphic to a subgroup of Free(2). This comes from covering space theory[8]. We'll end with a theorem that might give insight to how this will arise in the future:
**Theorem:** *Given an SES,*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*i) If H,Q are LO $\Rightarrow$ G is LO*
*ii) If Q is LO and H has a conjugation invariant BO, then G is BO.*

[5]Because I forgot it already.

[6]Include conditions here so that you ensure you actually get functions which are well defined. This is not a note to myself, I'm not going to come back and do this.

[7]A nice way to think of the product topology is that it is the coarsest topology which makes projections continuous, projections being the main structure you get when you have a product of spaces. This is a nice way to think of a lot of 'derived' topologies, like quotients, products, etc. What topology makes the important thing continuous? That's the natural topology to give the 'derived' object.

[8]I know the general idea of this statement, but I'd like to come back and flesh this out. I've forgotten too much alg top already.

CHAPTER 2

# Extensions, Torsion, and Orderability

**Lecture 2, Jan 23.** So last time, we stated the theorem related to extensions of $G$. We will now prove it:
**Theorem:** *Given an SES,*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*i) If H,Q are LO $\Rightarrow$ G is LO*
*ii) If Q is LO and H has a conjugation invariant BO, then G is BO.*
**Proof:** i) To define a left order on $G$, we will specify the positive elements. Let $H = \text{Ker } \varphi < G$, and let $P_H, P_Q$ be positive cones on $H$ and $Q$ respectively. Define

$$P = \varphi^{-1}(P_Q) \sqcup P_H$$

To show it is actually a positive cone, we need to show two things: That it is closed under multiplication, and $G = P \sqcup P^{-1} \sqcup 1$. For i), take $g, h \in P$, then there are 3 cases corresponding to where $g$ and $h$ live:

1. If $g, h \in P_H$, then $gh \in P_H \subset P$ becuase $P_H$ is a positive cone.
2. If, $g, h \in \varphi^{-1}(P_Q)$, then $\varphi(g), \varphi(h) \in P_Q \Rightarrow \varphi(gh) = \varphi(g)\varphi(h) \in P_Q \Rightarrow gh \in \varphi^{-1}(P_Q)$.
3. WLOG, if $g \in \varphi^{-1}(P_Q), h \in P_H)$, then $\varphi(gh) = \varphi(g)\varphi(h) = \varphi(g)e = \varphi(g) \in P_Q \Rightarrow gh \in \varphi^{-1}(P_Q)$.

ii) By exactness,
$$G = H/\{1\} \sqcup \varphi^{-1}(Q/\{1\}) \sqcup \{1\}$$
But
$$\varphi^{-1}(Q/\{1\}) = \varphi^{-1}(P_Q) \sqcup \varphi^{-1}(P_Q^{-1}) \text{ and } H/\{1\} = P_H \sqcup P_H^{-1}$$
So substituting into the first equation gives the result.

$\square$

**Example:** Let $X^2$ be the Klein Bottle, and

$$K = \langle a, b \mid b^{-1}ab = a^{-1} \rangle$$

be its fundamental group. Then there is an SES

$$1 \longrightarrow \langle a \rangle = \mathbb{Z} \longrightarrow K \longrightarrow \mathbb{Z} \longrightarrow 1$$

So $K$ is LO. However, $K$ is not BO. If it was, then

$$\begin{aligned} a > 1 &\iff b^{-1}ab > 1 \text{ by LO} \\ &\Rightarrow a^{-1} > 1 \end{aligned}$$

which is a contradiction. Recall $\mathbb{Z}$ has only two orders, so there are 4 choices of LO on $K$.
**Theorem:** *These are the only left orders on K.*
**Proof:** It suffices to show that each determines a unique LO[1]. We first note that all elements of $K$ can be written as $a^m b^n$. Take an LO on $K$, $<$. If $a > 1$, $b > 1$, then $a^k < b \,\forall\, k \in \mathbb{Z}$.
Claim: $a^m b^n > 1 \iff n > 0$ or $n = 0$ and $m > 0$. This defines a positive cone. The $n = 0$ case is trivial. If $n > 0$, then $a^{-k}b > 1 \,\forall\, k$, by the above claim. Then $b^n > b$ so $a^{-k}b^n > a^{-k}b > 1$. If $n < 0$, then $a^m b^n = b^n a^{\pm m} = (a^{\mp m} b^{-n})^{-1}$. By the first statement, this is $> 1$, so its inverse is less than 1, which shows the result.

In general, if $<$ is an LO on $G$, and $\alpha : G \to G$ is an automorphism, $(<, \alpha)$ induces an LO on $G$ by $g <_\alpha h \iff \alpha(g) < \alpha(h)$. There exist automorphisms of $K$, $\alpha_1, \alpha_2$, such that

$$\alpha_1(a) = a, \quad \alpha_1(b) = b^{-1}$$

and

$$\alpha_2(a^{-1}) = a, \quad \alpha_2(b) = b^{-1}$$

So $<_{(i)}$ is the unique LO on $K$ determined by i). The LO $<_{(ii)}$ is determined by $(<_{(i)}, \alpha_1)$. $<_{(iii)}$ is determined by $(<_{(i)}, \alpha_2)$, and $<_{(iv)}$ is determined by $(<_{(i)}, \alpha_1\alpha_2)$.

□

As we mentioned before, if a group has finitely many left orders, the number of left orders is $2^n$ for some $n$. $K$ is an example of that fact, which has $2^2$ left orders, each factor of two coming from a choice on $\mathbb{Z}$. As an exercise, you can show that $\forall\, n > 0$, there is a group with $2^n$ left orders[2].
**Corollary:** *For any LO on K, $h \in \langle a \rangle$, $g \in K/\langle a \rangle$, $g > 1 \Rightarrow g > h$.*
Suppose $M$ is a closed, orientable 3-fold. We may ask "Is $\pi_1(M)$ LO?" The only strong statement we have is that it must be torsion free if we hope for the answer to be yes. So already we have examples of spaces whose fundamental group is not LO. If $M = L(p, q)$ is a Lens space, we know $\pi_1(M) = \mathbb{Z}_p$, so we have an infinite class of non examples. If $M = M_1 \# M_2$, then we know $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, e.g. $X = M \# L(p, q) \Rightarrow \pi_1(X)$ is not LO. However, $T^3 = S^1 \times S^1 \times S^1$ is LO, since the property "being LO" commutes with direct product. $M = \#_n(S^1 \times S^2) \cong F_n$ is also LO.

[1] I don't understand this remark.
[2] Hint: examine the exact sequence we just used and iterate.

Until now, we might be led to believe that being torsion free implies that a group is LO. This is not the case:
**Example:** Take $X^2$ the Klein Bottle, $K = \pi_1(X^2)$ and consider $X = X^2 \times I$. This is a non orientable 3-fold. Take the 2-fold cover

$$p : T^2 \to X^2$$

so that

$$p_*(\pi_1(T^2)) = \langle a, b^2 \rangle \cong \mathbb{Z} \times \mathbb{Z} < K$$

and let $N$ be the mapping cylinder of $p$, i.e.

$$N = (T^2 \times I) \sqcup X^2 \Big/ \Big\{ (x, 0) \sim p(x) \; \forall \, x \in T^2 \Big\}$$

and $N_1, N_2$ two copies of $N$. Then of course, $\pi_1(N_i) = K \equiv \langle a_i, b_i \mid \ldots \rangle$, since there is a strong deformation retraction $N \to X^2$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Then

$$M = N_1 \sqcup_\varphi N_2$$

is a closed, orientable 3-fold, and $\pi_1(M) = K_1 *_{\mathbb{Z}\times\mathbb{Z}} K_2$, which is torsion free.
**Theorem:** *$H_1(M)$ is finite, so $\pi_1(M)$ is not LO.*
**Proof:** $\varphi$ is determined by, up to isotopy, $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$, and we denote $H_1(\partial N_i) = \langle a_i, 2b_i \rangle$. Then $\varphi_*$ is a matrix, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, so that

$$\varphi_*(a_1) = pa_2 + 2qb_2$$
$$\varphi_*(2b_1) = ra_2 + 2sb_2$$

Denote

$$H_1(N_i) = \langle a_i, b_i \mid a_i^2 \rangle$$

Then $H_1(M)$ is presented by

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{pmatrix}$$

so that $H_1(M)$ is finite $\iff$ $\det(A) \neq 0$ $\iff$ $q \neq 0$ $\iff$ $\varphi_*(a_1) \neq \pm a_2$. Suppose $\pi_1(M)$ were LO, then we would induce an LO on the subsets $\partial N_i$. There are only 4 LO's on $\pi_1(N_i)$.
**Corollary:** *For any LO on $\pi_1(N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$ summand of $\pi_1(\partial N) = \langle a, b^2 \rangle$ such that $h \in \langle a \rangle$ and $g \in \pi_1(\partial N)/\{1\}, g > 1$ then $g > h$.*

$$\Rightarrow \varphi_*(\pm a_2)$$

which shows the result.

CHAPTER 3

# An Equivalent Condition to LO

**Lecture 3, Jan 28.** **Theorem:** *$A, B$ torsion free groups $\Rightarrow A *_C B$ is torsion free.*

$\square$

**Lemma:** *Let $G = Homeo_+(\mathbb{R}) := \{$Orientation preserving homeomorphisms of $\mathbb{R}\}$. Then $G$ is LO.*
The proof of this lemma follows from a more general fact:
**Definition:** If $<$ is an STO on a set $X$, $\underline{\mathcal{B}(X,<)}$ is the set of set of order preserving bijections $\varphi : X \to X$ (i.e. $x < y \Rightarrow \varphi(x) < \varphi(y)$). This is a group under composition, and in fact is a subgroup of the group of the automorphism group.
**Theorem:** $\mathcal{B}(X,<)$ *is* LO.
**Proof:** Let $\prec$ be a well ordering on $X$[1], and let

$$f \neq g \in \mathcal{B}(X,<)$$

Then define $[f \neq g] := \{x \in X \mid f(x) \neq g(x)\}$ as the set of points on which $f$ and $g$ disagree. This set is nonempty because $f \neq g$, so let $x_0$ be its $\prec$-least element. Then define $f < g \iff f(x_0) < g(x_0)$. This is an LO on $\mathcal{B}(X,<)$.

$\square$

From this, the above lemma follows: If we let $<$ be the standard ordering on $\mathbb{R}$, then the set of order preserving bijections is equal to the set of orientation preserving homeomorphisms, so the above theorem applies.
**Remark:** Let $\prec_x$ be a well ordering on $\mathbb{R}$ such that $x$ is the least element on $\mathbb{R}$[2]. Then $<_x$ is the induced LO on $Homeo_+(\mathbb{R})$, and is distinct from all other orders induced by a different $x' \in \mathbb{R}$, so there are uncountably many LO's on $\mathcal{B}(\mathbb{R},<)$.
Given $x, y \in \mathbb{R}, \exists g \in Homeo_+(\mathbb{R})$ s.t.

$$g(x) > x, g(y) < y$$

Then $g >_x 1$ and $g <_y 1$ iff $<_x \neq <_y$.
**Corollary:** *$G$ is LO iff $G$ acts faithfully on an STO'd set $(X,<)$.*

[1]It's still not clear to me how much of a problem it is that we are making a choice of WO in these constructions. It's clear that it isn't a problem for these existence proofs, but there is no uniqueness property satsified here. I don't know how much of a concern that is. The mere property of being left orderable is already interesting enough, so maybe it doesn't matter that these aren't natural constructions.

[2]It's not clear to me that this can be done for any $x \in \mathbb{R}$.

**Proof:** One direction is the previous proof, the other direction is seen from the fact that $G$ acts faithfully on itself by left multiplication.
**Corollary:** *Any subgroup of $Homeo_+(\mathbb{R})$ is LO.*
In fact, this characterizes countable LO groups:
**Theorem:** *If G is countable, G is LO iff G is isomorphic to a subgroup of $Homeo_+(\mathbb{R})$.*
**Proof:** For the forward direction, we will show something more general:
**Theorem:** *If $(G, <)$ is countable, there exists an LO on $Homeo_+(\mathbb{R})$, and an order preserving injection homomorphism*

$$(G, <) \to (Homeo_+(\mathbb{R}), <)$$

**Sketch of Proof:** If $G \neq \{1\}$, if it is infinite, let $g_1, g_2, \dots$ be some indexing of the elements of $G$. Define an embedding

$$e : G \to \mathbb{R}$$

via

$$e(g_1) = 0, \quad \text{and}$$

$$g_{n+1} \begin{pmatrix} > \\ < \end{pmatrix} g_i \,\forall\, 1 \leq i \leq n \Rightarrow e(g_{n+1}) = \begin{pmatrix} \max\{e(g_i) |\, 1 \leq i \leq n+1\} \\ \min\{e(g_i) |\, 1 \leq i \leq n-1\} \end{pmatrix}$$

If the first condition doesn't hold, then let $g_l = \max\{g_i, 1 \leq i \leq n, g_i < g_{n+1}\}$, and similarly for $g_r$ defined via the min. Then set

$$e(g_{n+1}) = \frac{e(g_l) + e(g_r)}{2}$$

So if $g_{n+1}$ is greater or less than everything before it, just stick it on the end, if not, stick it in the middle. We note that $e$ is order preserving by construction. Also if the first condition does hold, then the first $n+1$ elements are integers. In fact, $e(g_{n+1})$ is an integer iff the first condition holds. We also know

$$\begin{gathered} g > 1 \Rightarrow g^2 > g > 1 \text{ and } g^{-1} < g \\ g < 1 \Rightarrow g^2 < g < 1 \text{ and } g^{-1} > g \\ \Rightarrow \mathbb{Z} \subset e(G) \equiv \Gamma \end{gathered}$$

Then $G \curvearrowright \Gamma$ via $g(e(a)) = e(ga)$, which is order preserving.
We want to extend this to an action on $\mathbb{R}$. This is a continuous action, so it extends to an action when we throw in limit points, i.e. $G \curvearrowright \overline{\Gamma}$. But $\mathbb{R}/\overline{\Gamma}$ is a countable disjoint union of open intervals, $\sqcup_i (a_i, b_i)_i$. So $G$ is defined on the set of $\{a_i, b_i\}$, and we may extend affinely to the intervals $(a_i, b_i)$.

$\square$

To define an LO on $Homeo_+(\mathbb{R})$ that restricts to the LO on $\Gamma$ from $G$, let $\gamma \in \Gamma$. Then

$$g \begin{pmatrix} > 1 \\ < 1 \end{pmatrix} \iff g(\alpha) \begin{pmatrix} > \gamma \\ < \gamma \end{pmatrix}$$

Then let $\prec$ be a WO on $\mathbb{R}$ such that $\gamma$ is the $\prec$-least element of $\mathbb{R}$, and let $\ll$ be the LO on $Homeo_+(\mathbb{R})$ induced by $\prec$. Then

$$g\begin{pmatrix} > 1 \\ < 1 \end{pmatrix} \iff g\begin{pmatrix} \gg 1 \\ \ll 1 \end{pmatrix}$$

For prime 3-manifold groups, we can eliminate the faithful condition, so you need only act on $\mathbb{R}$. This will have important implications for things like foliations in the future.

Now we will talk about an application of orderability, in fact one which really got people interested in orderability in the first place.
**Definitions:** If $R$ is a unital ring, $a \in R$ is a
i) unit if $\exists b \in R$ such that $ab = ba = 1$.
ii) 0-divisor if $a \neq 0$ and $\exists b \neq 0 \in R$ such that $ab = 0$ or $ba = 0$.
iii) non-trivial idempotent if $a \notin \{0, 1\}$ and $a^2 = a$.
**Definition:** If $G$ is a group and $R$ a ring, the group ring is defined as

$$RG := \left\{ \sum_g r_g g \mid g \in G, r_g \in R, r_g = 0 \text{ for all but finitely many } g \right\}$$

Then one can check that $RG$ is a ring. Furthermore, if $g \in G$, $r$ is a unit in $R$, then $rg$ is a unit in $RG$.
**Definition:** A unit in $RG$ is non-trivial if it is not of the above form.
Group rings always have trivial units if $R$ has units, but the question of when they have non trivial units can be interesting.
**Theorem:** *Suppose $G$ has torsion and $K$ is a field of characteristic 0. Then*
*i) $KG$ has 0-divisors*
*ii) $KG$ has non-trivial units*
*iii) $KG$ has non-trivial idempotents.*
**Proof:** i) Let $g$ have order $n \geq 2$. Then

$$\sigma := 1 + g + g^2 + \cdots + g^{n-1} \in KG$$
$$\Rightarrow g\sigma = \sigma \Rightarrow (1-g)\sigma = 0 \Rightarrow \sigma \text{ is a 0-divisor.}$$

ii)

$$\sigma^2 = 1\sigma + g\sigma + g^2\sigma + \cdots + g^{n-1}\sigma = n\sigma$$
$$\Rightarrow (1-\sigma)\left(1 - \frac{1}{n-1}\sigma\right) = 1 \Rightarrow (1-\sigma) \text{ is a non-trivial unit}$$

We note that if $n = 2$, $(1-\sigma)$ is a trivial unit, in which case we may also compute

$$(1-2\sigma)\left(1 - \frac{2}{3}\sigma\right) = 1$$

iii)

$$\left(\frac{1}{n}\sigma\right)^2 = \frac{1}{n^2}\sigma^2 = \frac{1}{n}\sigma$$

which gives nontrivial idempotents.

$\square$

The upshot of this theorem is that when $G$ has torsion, we have a bunch of "bad stuff" in the group ring. However, if $G$ is torsion-free,

**Theorem:** (Kaplansky Conjecture) *If $G$ is a torsion-free group and $K$ is a field, then $KG$ does not have any non-trivial units, 0-divisors, or non-trivial idempotents.*

It was later proven that all conditions are equivalent, yet it is still not known if any of them are true. Since orderable groups are torsion free, they obey the Kaplansky Conjecture, which historically spawned interest in orderable groups.

# CHAPTER 4

# More Applications of Orderability and $\mathbb{Z} \times \mathbb{Z}$

**Lecture 4, Jan 30.** We will now directly prove that left orderable groups satisfy the Kaplansky Conjecture, without using the fact that LO groups are torsion-free:
**Proof:** i) Note:

$$\left(\sum_{i=1}^{m} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j h_j\right) = \sum_{i}^{n}\sum_{j}^{m} \alpha_i \beta_j (g_i h_j)$$

and assume, WLOG, $g_i < g_{i+1}, h_i < h_{i+1}$. Let $g_k h_l$ be the minimal element of

$$S := \{g_i h_j\} \subset G$$

Then $h_1 < h_j \; \forall \, j > 1$, so $gh_1 < gh_j \Rightarrow l = 1$. Also, $gh_1 = g'h_1 \Rightarrow g = g'$, so $g_k h_1$ is unique. We similarly define a unique maximal element, $g_r h_n$. These two elements are distinct, so that there are at least two terms on the RHS of the equation. If we supposed that the equation above is equal to 1, this shows that there are no non-trivial units, and if we suppose it were equal to 0, this shows that there are no non-trivial 0-divisors. It is a general phenomenon that ii) implies iii), so we are done.

□

Another related application is the
**Conjecture:** (Isomorphism Conjecture) *If $G$ is torsion-free, then*

$$\mathbb{Z}G \cong \mathbb{Z}H \Rightarrow G \cong H$$

Note that there is $G$ finite such that the conjecture is false[1]
**Corollary:** *$G$ LO $\Rightarrow$ $G$ satisfies the Isomorphism Conjecture.*
**Proof:** Define $U_{\mathbb{Z}G} := \{\textit{units in } \mathbb{Z}G\} \cong \mathbb{Z}_2 \times G$. It is easily seen that this has a group structure. Sppose $\mathbb{Z}G \cong \mathbb{Z}H$. If $\mathbb{Z}G$ has no 0-divisors, then $\mathbb{Z}H$ has no 0-divisors, so $H$ is torsion free , by a previous theorem[2]. Then $H < U_{\mathbb{Z}H} \cong U_{\mathbb{Z}G} \cong \mathbb{Z}_2 \times G$, so $H$ can be realized as a subgroup of $Z_2 \times G$, which is LO, so $H$ is LO, becuase subgroups inherit orderings. But LO groups do not have 0 divisors, as they are torsion free, so

$$U_{\mathbb{Z}H} \cong \mathbb{Z}_2 \times H$$
$$\Rightarrow \mathbb{Z}_2 \times H \cong \mathbb{Z}_2 \times G \Rightarrow G \cong H$$

[1] Hertweck 2001, "A counterexample to the isomorphism problem for integral group rings".
[2] We proved the contrapositive of this statement earlier.

There are details to check about cancelling the factors of $\mathbb{Z}_2$. It certainly doesn't hold in generality, but it does in this trivial case[3].

$\square$

We will now discuss Bi-Orders on $\mathbb{Z} \times \mathbb{Z}$. Take a line in $\mathbb{R}^2$ through the origin with irrational slope $\alpha$. This gives an LO via the positive cone which declares every point in the integer lattice above the line is positive, i.e.

$$P = \{(m,n) \mid n > m\alpha\}$$

For each irrational slope, we have a unique order, thus $\mathbb{Z} \times \mathbb{Z}$ has uncountably many LO's. If $\alpha$ is rational, then we can define an LO on the integer lattice as long as we decide consistently what to do with points that lie on the line. Points which lie on the line are a subset of $\mathbb{Z}$, so we may pick a $\mathbb{Z}$ LO on there, $P_0$. Then

$$P = P_0 \sqcup \{(m,n) \mid n > m\alpha\}$$

is an LO. It can be shown that these are all of the LO's on $\mathbb{Z} \times \mathbb{Z}$. Each $\alpha$ yields 2 LO's if it is irrational and 4 LO's if it is rational, and this generalizes in the obvious way to the $n$-dimensional integer lattice.

We will now make the complete argument for topologizing the space of LO's of a group. We recall some point set topology results and definitions:
**Lemma:** *$X_\lambda$ Hausdorff $\Rightarrow \prod X_\lambda$ Hausdorff.*
**Definition:** $X$ is totally disconnected if the only non-empty connected subspaces are the singletons.
**Lemma:** *$X_\lambda$ totally disconnected $\Rightarrow \prod X_\lambda$ totally disconnected.*
This implies that Pow$(X)$ is compact, Hausdorff, and totally disconnected for any space $X$. For $x \in X$, we may define the projection $\pi_x : Pow(X) \to \{0,1\} \subset \{0,1\}^X$ via

$$\pi_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

and define the sets in Pow$(X)$.

$$U_x = \pi_x^{-1}(1), \qquad V_x = \pi_x^{-1}(0)$$

It is readily seen that $U_x = V_x^c$. These $U_x$ and $V_x$ then form a basis for Pow$(X)$, if we allow them to vary over $x$. This coincides with the product topology.

[3]Just for the sake of synthesis, if the Kaplansky Conjecture were true, the theorem would be "Torsion free groups satisfy IC", and its proof would not reference orderability or topology.