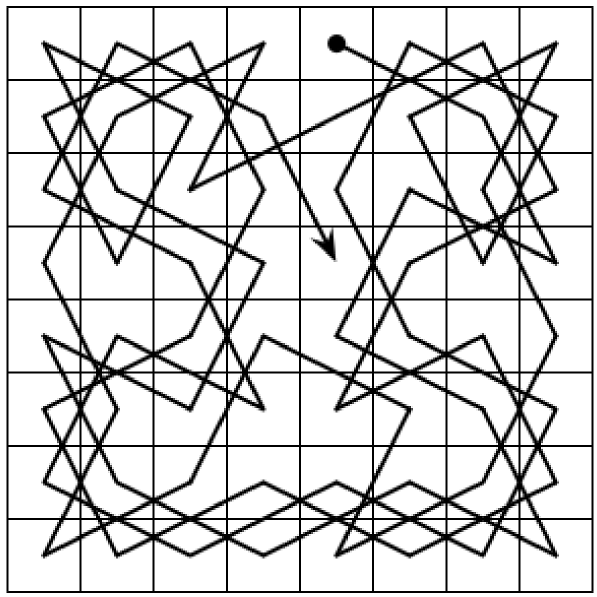
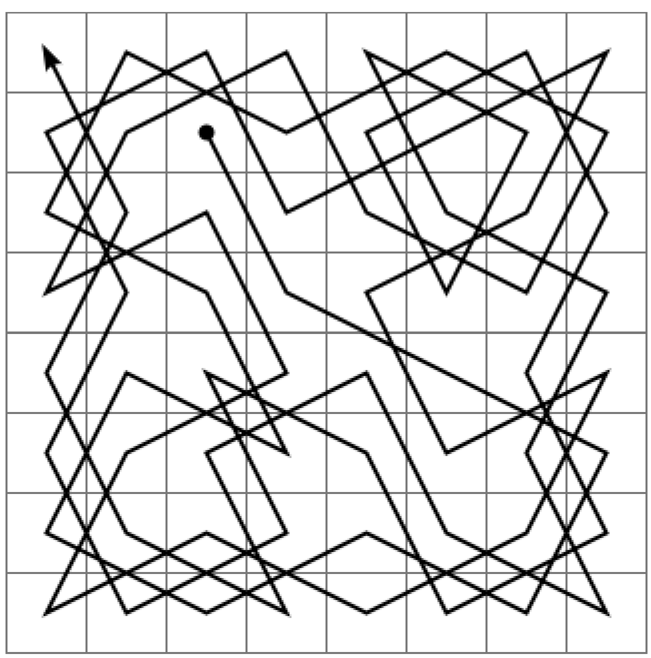
Knight's Tour



Open Knight's Tour



Closed Knight's Tour

Definition of a Knight’s Tour

Knight's tour is a sequence of knight move on a board which result in all board squares being visited by the knight(also known as an open knights tour).

A closed knight tour has the extra condition that the knight end it’s last move one move away from the start position.

Existence in 2D

Theorem 1: Schwenk proved that for any $m \times n$ board with $m \leq n$, a closed knight's tour is always possible unless one or more of these three conditions are met:

m and n are both odd

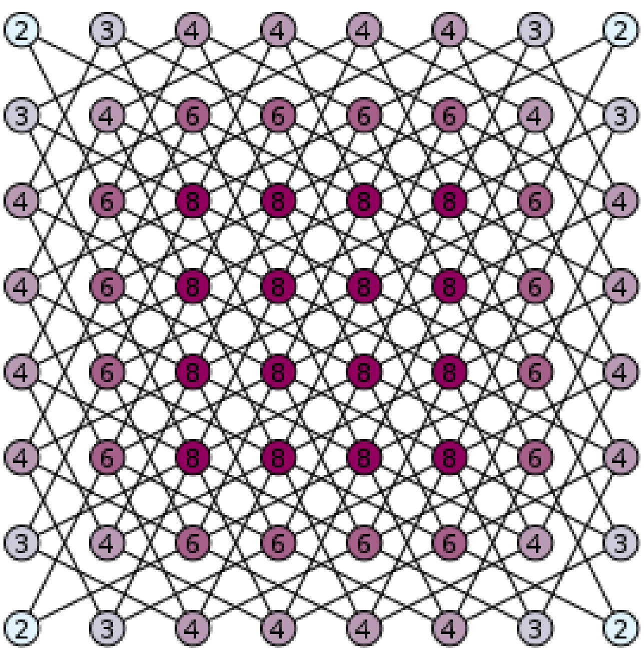
$m = 1, 2,$ or 4

$m = 3$ and $n = 4, 6,$ or 8.

Warnsdoff's Rule

Warnsdorff's rule is a heuristic for finding a knight's tour. By moving the knight so that it always proceeds to a square from which the knight will have the fewest future moves. When calculating the number of future moves for each possible square, only the unvisted moves count. It is possible to have a tie for least future moves between two or more choices. There are various methods for breaking such ties, including one devised by Pohl and another by Squirrel and Cull.

This rule may also more generally be applied to any graph. In graph-theoretic terms, each move is made to the adjacent vertex with the least degree.



number of moves from a position

Warnsdoff's Algorithm

Input: graph

Output: path that does not revisit a node (if complete, a Hamiltonian path)

For each node I (the Start Node):

Graph is the input graph;

Current Node is i;

Repeat until all nodes are reached or a dead-end:

Current Node is placed on the longest path;

From the Current Node select an Adjacent Node that has least degree;

If unique this becomes the new Current Node;

Else if there is more than one such node:

For each such node j

let tiebreak(j) be the least degree of an adjacent node of j

other than the Current Node;

Choose a j having smallest tiebreak(j) as the new Current Node

(if not unique then break tie arbitrarily);

Delete the old Current Node from the graph;

Output a longest path found

Expansion of a Board

this lemma shows the existence of a knight's tour in bigger boards and the construction of the tour.

If G(m, n) has a Hamiltonian cycle that includes the 10 edge

(1,n-1)-(3,n) (m-2,n-I)-(m,n) (m-1,l)-(m,3) (m-1,n-2)-(m,n) (4,n-1)-(2,n) (1,n)-(3,n-1)(m-2,n)-(m,n-1) (m,)-(m-1,3) (m,n-2)-(m-1,n)(m,2)-(m-1,4),

Then G(m,n+4) also has a Hamiltonian cycle including the corresponding 10 edge

(1,n+ 3)-(3,n +4) (m-2,n +3)-(m,n + 4) (m-1,1)-(m,3) (m-1,n + 2)-(m,n+4) (4,n+3)-(2,n+4)(1,n+4)-(3,n+3) (m-2,n+4)-(m,n+3) (m,1)-(m-1,3) (m,n+2)-(m-1,n+4) (m,2)-(m-1,4).

There exists base cases such that there explanation yields all board size that knights tour are possible on.

Existence in 3D

Theorem 2:(DeMaio and Mathew).Let $2<=n<=m<=p$. The $n×m×p$ chessboard has a closed knight tour if and only if the following conditions hold:

1.m,n, or p is even,

2.$m>3$,

3.$p>4$

Existence in n Dimensions

Theorem 3: Let $2<=n1 <= n2 <=...<=nr$, with $r>3$. The $n1 ×...× nr$ chessboard has a closed knight tour if and only if the following conditions hold:

1. Some ni is even,

2.$nr−1 > 3$,

3.$nr > 4$.

Sites and Proof for n Dimensions

sites are the key to n dimensional knights tours as they bridge one plane to the next. a site is a place in a 2D board where a knight changes to a new dimension and back after completing the remaining higher dimension or other plane one move ahead in the current dimension or plane. completing a higher dimension's or planes knight tour can also include jumping to another higher dimension or plane before returning and completing the first higher dimension or plane. it can be show that there exist enough sites to move up all 2D planes of a dimension and to new dimension and back down by the following proof.

Defn 1: a site is a place with the following pattern

$(|a1-c1),|a2-c2|) = (|b1-d1|,|b2-d2|) \in \{(0,2),(2,0)\}$ (3)

or if

$(|a1-d1|,|a2-d2|) = (|b1-c1|,|b2-c2|) \in \{(0,2),(2,0)\}$ (4)

Defn 2: it is said that a pair of edges{ (a,b),(c,d)} in K(n1,...,nr)is a site if there exist i1 not = i2 such that:

$aj = cj$ and $bj = dj$ for all $j \notin \{i1,i2\}$ and either

$(|a1-c1),|a2-c2|) = (|b1-d1|,|b2-d2|) \in \{(0,2),(2,0)\}$

or if

$(|a1-d1|,|a2-d2|) = (|b1-c1|,|b2-c2|) \in \{(0,2),(2,0)\}$ holds true.

Defn 3: A n1 ×...x nr tour, equivalently a Hamiltonian cycle on K(n1,...,nr), is called bi-sited if it contains two edge-disjoint sites

Defn 4: it is said two sites{(a,b),(c,d)}and{(e,f),(g,h)}in K (n1,...,nr) are adjacent if they have the same i1 and i2 in Defn 2 and if there exists j not∈{i1,i2} such that(a,b,c,d)∈(e,f,g,h)±(ej,ej,ej,ej).note that if there exist a n1×...×nr tour that contains a site [p] and I place two copies of the tour on top of each other, denoting by [p,i] the site in the i-Th copy, so as to cover a n1×...×nr×2 chessboard then [p,1] is adjacent to [p,2].

Defn 5:Take two adjacent sites [p] ={(a,b),(c,d)}and [q] ={(e,f),(g,h)}.

1. If [p] satisfies condition 3 (so does [q]), now delete the edges (a,b) and (g,h) and add the edges (a,g)and (b,h).

2. If [p] satisfies condition 4(so does [q]), now delete the edges (a,b) and (g,h) and add the edges (a,h) and (b,g). it is said that [p] and [q] are glued

Proposition 1:Given a bi-sited n1×...×nr tour and k >2 then there exists a bi-sited n1×...×nr×k tour.

Proof.

Start by taking k copies of the n1 ×... × nr tour and place them on top of each other, so as to cover the n1 ×...×nr×k chessboard. Lets denote by [p1,i] and[p2,i] the copies of the two sites in the i-Th copy of the tour, note that [pj,i] and [pj,i+ 1]are adjacent for all 1<=i<=k−1, 1<=j<=2. Now glue together [p1,1] and [p1,2] and then [p2,2] and [p2,3], continuing this way glue together [p1,i] and [p1,i+ 1] for i even and [p2,i] and [p2,i+ 1] for i odd. then start with k disjoint cycles covering K (n1,...,n2,k) and each time two sites are glued together two cycles become one. Hence at the end what is left is a Hamiltonian cycle on K (n1,...,nr,k), a n1×...×nr×k tour. It remains to check that the tour is bi-sited, and indeed there are two sites that haven't being used, [p2,1] and [p2,k] if k is even and [p2,1] and [p1,k] if k is odd. The plan is thus to construct bi-sited tours in of small dimension and use Proposition 1 to prove Theorem 3 inductively.

Proposition 2:Every closed tour on an n×m board is bi-sited.

Proof.

First, there is no closed tour on a 2×m and on 4×m boards.

i) Case 3×m, for m>10:Note that m is even. First one has (1,1) which is linked to (3,2) and to (2,3). The possible neighbours of (1,3) are (3,2),(2,1), (3,4), (2,5), As a cycle exists, exactly two of them are linked to (1,3) and three of them are part of site. Then, there is at least one site. Secondly repeat the argument for the upper right corner and get a new site. Since m>10, the two sites are edge disjoint.

ii) Case n×m, for n>5,and m>6: flip the board and consider m×n. repeat twice the first part of the point i) for the sub-board of size 3×n that contains the upper left corner and for the one that contains the lower left corner. The result is two disjoint sites. Combining this with Proposition 1 already gives us a large family of high dimensional tours.

Corollary 1:If an $n \times m$ tour exists then so does an $n \times m \times p1 \dots \times pr$ tour for any $p1,\dots,pr \in N \setminus \{0\}$.it is not proven an analogue to Proposition 2 for 3 dimensional boards, however for this paper purposes it is enough to prove the existence of specific bi-sited tours, for which this paper will rely on the construction of [3].

Theorem 4:Let $2<=n<=m<=p$. Then$\times m \times p$ chessboard has a bi-sited tour if and only if the following conditions hold:

1.n,m, or p is even,

2.$m>3$,

3.$p>4$.

Proof.

It is a simple, albeit lengthy check that the tours constructed in [3] are all bi-sited.

Theorem 5. Let $2<=n1<=n2<=\dots<=nr$, with $r>3$. The $n1\times\dots\times nr$ chessboard has a bi-sited tour if and only if the following conditions hold:

1. Some ni is even,

2.$nr-1>3$,

3.$nr>4$.

sufficient condition first.

Remark 1. note first if condition a) does not hold then, as K(n1,...,nr) is bipartite,no tour can exist by a simple parity consideration. Now if neither b) nor c) holds true,notice that (2,2,...,2) is not connected to (2,2,...,1).

Proof.

Keeping in mind the preceding remark, focusus on the necessary condition.

proceeding by induction on r. The case r = 3 is the content of Theorem 3. Given that the

result holds for all s < r and given 2<=n1<=...<=nr that satisfies the conditions of the

statement it's see that 2<=n2<=...<=nr is a nr−1 tuple which satisfies the conditions of

the statement and hence by the induction hypothesis a bi-sited n2×...×nr tour exists.

By Proposition 1 a bi-sited n2×...×nr×n1 tour exists, and by so Remark

1 a bi-sited n1×n2×...×nr tour exists. An immediate consequence of Theorem

5 and of Remark 1 is: Corollary 2: Let 2<=n1<=...<=nr, r>3. Suppose that some ni is even. Then the n1×...nr chessboard has a knight tour if and only if K(n1,...,nr) is connected.

Complexity of a Knights Tour

Although the Hamiltonian path problem is NP-hard in general, on many graphs that occur in practice this heuristic is able to successfully locate a solution in linear time. The knight's tour is a special case of the Hamiltonian path problem.

Interesting Related Other Points

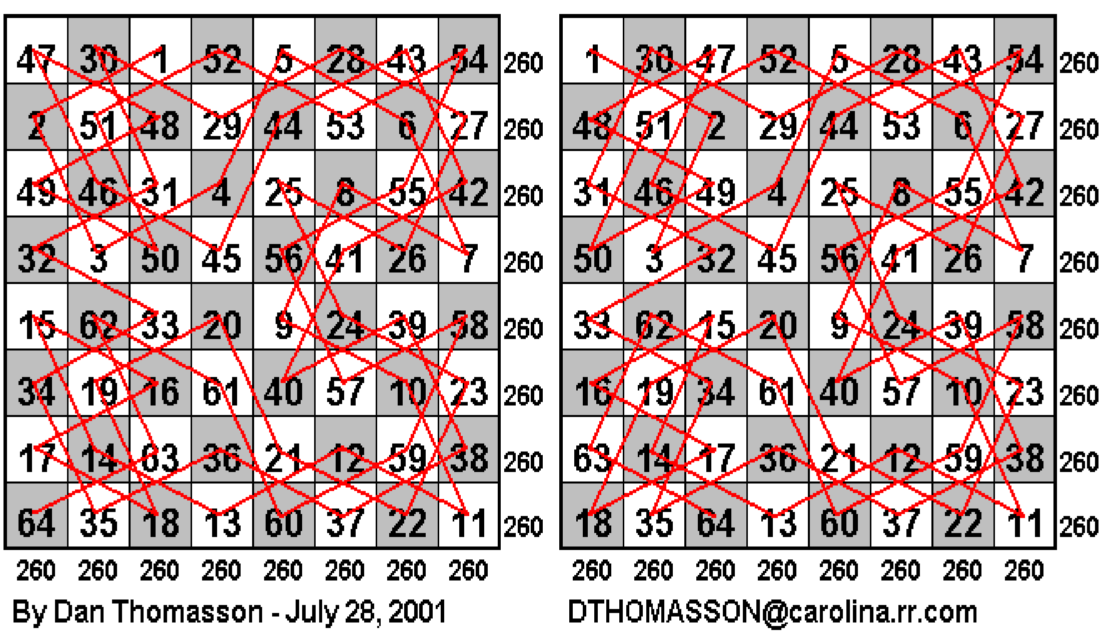
A knight’s tour can be done on other shape besides n dimensional boards

An open knights tour on an 8×8 board can produce a semi-magic square

used for ciphers

can be use to find primes

golden ratio



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