

## 6.4 Reach Set Approximation

**Summary:** There is a need to have a tool with a finite count of trajectories, which has enough variability to support avoidance task. The reach set covers all possible trajectories, but it is not countable. Trajectories represented as tree originating in the same initial state can be considered as a skeleton of the reach set. There is a need to compare trajectories regarding avoidance capability. To achieve this, it is necessary to distinguish them based on measurable criteria. Different approximation based on the measurable criteria is introduced to cover different avoidance behaviors.

**Motivation:** *Reach set* is a strong tool for *Obstacle Avoidance* because it contains all possible *avoidance maneuvers*. The current implementations (sec. ??) have the following flaws:

1. *Realistic approximation - nonlinear systems or heavily constrained systems* cannot be approximated well by *linear continuous-time Reach Sets*.
2. *Finite count of possibilities - continuous-time Reach Set* contains infinite possibilities for *avoidance maneuvers*; the DAA system demands conflict resolution in finite-time.
3. *Computationally feasible data structures - binding related properties* seem problematic because *continuous-time reach sets* do not have a unique identifier of maneuver, trajectory nor segment.

**Proposed Solution Features:** Our Reach set Estimation method will provide the following features:

1. *System Control Interface - implemented via Movement Automaton*, requiring only a *discrete command chain* to approximate system behavior.
2. *Finite count of possibilities - finite number of elements in Reach set* will enable *scalable* calculation.
3. *Computationally feasible data structures - approximation of Reach set as a set of trajectories*, each trajectory can be split into a finite number of segments. Each element will have a unique identifier enabling both-side property binding.
4. *Computationally feasible data-structures - some specific behavior*, like horizontal/vertical separation, or maneuver shape can be encoded into different types of reach set approximation algorithms.

### 6.4.1 Trajectory Set Approximation

**Summary:** To achieve finite representation, it is necessary to establish a general relationship between the reach sets and trajectory trees, built from a set of prototypical movements.

**Discretization of Reach set:** There is a need for a discrete finite *Reach Set approximation* to enable *Avoidance Strategy Evaluation* in finite-time. Replacing *Continuous Control Set Inputs(t)* by *Movement Automaton* is feasible:

**Definition 1** (Reach set Approximation by Movement Automaton). A trajectory (def. ??) for system  $\dot{state} = f(time, state, input)$  under control of the movement automaton  $\mathcal{MA}$  is given as execution of movement buffer (def. ??) with an initial state of system  $state_0$ . Therefore notation  $Trajectory(state_0, buffer)$  is used.

**The Complete Reach Set** (6.1) for system with initial state  $state_0$  with existing control strategy  $control(time) \in Controls(time)$ . for time  $\tau > time_0$ .

$$ReachSet(\tau, time_0, state_0) = \bigcup \{state(s) : control(s) \in Controls(s), s \in (time_0, \tau]\} \quad (6.1)$$

**The Reach Set Approximation by Movement Automaton** (6.2) of the system under the control of the movement automation  $\mathcal{MA}$  consist from the set of trajectories  $Trajectory(state_0, Buffer)$ , which are executed in constrained time  $\tau > time_0$ .

$$ReachSet(\tau, time_0, state_0) = \left\{ Trajectory(state_0, buffer) : \begin{array}{l} \text{duration}(buffer) \\ \leq \\ (time_0 - \tau) \end{array} \right\} \quad (6.2)$$

*Note.* *Reach Set Approximation* (def. 1) is a subset of *Full Reach Set* (def. ??) in continuous space  $\mathbb{R}^n$  it inherits all important properties, like *Invariance* [1].

*Discretization* of *Reach Set* have been achieved leaving us with a *finite count* of *Trajectories*, instead of *Infinite subspace* or  $\mathbb{R}^N$

**Approximated Reach Set Containment:** The *Approximated Reach Set* introduced in (def. 1) is constrained only by *future expansion time*  $\tau$ . UAS makes space assessment in *Avoidance Grid*. There is no point to consider *Trajectories* outside of *Avoidance Grid*

**Definition 2** (Contained Approximated Reach Set). For a pair  $(state_0, AvoidanceGrid_0)$

at time  $time_0$  and prediction horizon  $\tau = \infty$  there is Contained Reduced Reach Set:

$$ReachSet \left( \begin{array}{c} time_0, \\ state_0, \\ AvoidanceGrid_0 \end{array} \right) = \left\{ \begin{array}{c} Trajectory(\dots) \\ \in \\ ReachSet(6.2) \end{array} : \begin{array}{l} \forall segment \in AvoidanceGrid_0, \\ segment \in Trajectory(\dots) \end{array} \right\} \quad (6.3)$$

**Properties:** Container Approximated Reach Set contains only trajectories where all segments belong to Avoidance Grid, there are following functions:

1. The membership function for any Trajectory in Constrained Reduced Reach set returns Ordered Set of Passing Cells.
2. The cost function for any Trajectory Portion in Constrained Reduced Reach Set return Cost of Execution

**Passing cell:** Cell of Avoidance Grid which has some intersection with Trajectory.

*Note.* Contained Reduced Reach Set (eq. 6.3) which is contained in the Avoidance Grid and have a Membership Function enable Property transition between Reach set and Avoidance grid.

*Example:* Visibility from cells along Trajectory can be gathered to calculate Trajectory's feasibility.

**Reach Set Pruning:** There is a need to implement Set Difference between Reach Set and Constraint Set. Constraint Set can be an Obstacle Set from Known World (sec. ??) and other different constraints.

**Reach Set Trajectory Tree:** (6.4) Any Reach Set where Control Strategy Constraint is implemented as Movement Automaton, with defined Movements set and for single initial  $state_0$ . The Reach Set is given as discrete tree with root  $Trajectory(state_0, \emptyset)$ .

$$ReachSet(state_0, \dots) = \left\{ Trajectory(state_0, buffer) : \begin{array}{l} buffer \in Movements^i, \\ i \in \{1, \dots, k\} \end{array} \right\} \quad (6.4)$$

For each Trajectory Segment, there exists intersection function which evaluates as true if there exists at least one point in Segment which belongs to Constraint Set. Formally:

$$intersection(segment, Set) : \begin{cases} \exists point \in segment, & : true \\ point \in Set & \\ Otherwise & : false \end{cases} \quad (6.5)$$

**Definition 3** (Pruned Reach Set). For Reach set represented as Trajectory Tree (eq. 6.4) and some constraint set (*Set*) where exist intersection function (eq. 6.5). The Pruned Reach set is given as follows:

$$Prune(ReachSet, Set) = \left\{ Trajectory(\dots) : \begin{array}{l} \forall segment \in Trajectory, \\ \neg intersection(segment, Set) \end{array} \right\} \quad (6.6)$$

*Note.* Pruning(def. 3) [2] is applied multiple times for various Constraints Set.

Example of *Approximated Reach set Calculation* (def. 1), *Reach Set Containment* (def. 2), and, *Pruning* is given in [3].

### 6.4.2 Distinctive Properties of the Trajectories

**Summary:** A characterization of trajectories is provided to support the selection of feasible representatives to build a skeleton of the reach set.

**Motivation:** The need to Make *Reach Set* scalable approach. This may be a problem due to the *Expansion rate*. *Reach set* represented as a *Trajectory Tree* (eq. 6.4) for Avoidance Grid with *layer – count* and Movement automaton with *movement – count*, the *Node count* is given as:

$$1 + \left( \sum_{i \in \{1 \dots layerCount\}} (movementCount)^i \right) \quad (6.7)$$

*This scaling* is not feasible for *Avoidance Grid* with many layers ( $< 10$ ) or *Movement Set* with many movements ( $< 9$ ). There is a need for *Reduced Reach set calculation*.

**Performance Criteria:** The scaling factor (eq. 6.7) shows that there are going to be many trajectories. The main point is that not every trajectory in *Reach Set* is giving us *maneuverability advantage*. Our expectations lie in following *Performance Requirements*:

1. *Reach set* must *Cover* maximum of the *possible unique maneuvers* in *Avoidance Grid*.
2. *Trajectories* in *Reach Set* should be smoothest possible to prevent cargo damage / UAS wear.

**Trajectory footprint:** Discrete space of *Avoidance Grid* is organized in cells. The *cell* is a minimal space portion accessible by *property binding*. There is a need to know if two trajectories contribution to *Maneuverability* in this environment.

Each trajectory passes through space in *Avoidance Grid*. If there exists a method to extract unique identifier for each *trajectory passed cells*, we can compare two trajectories *Coverage* in *Avoidance Grid*.

**Definition 4** (Trajectory footprint). *For Trajectory from Reach set (def. 2) defined for Avoidance Grid has membership function. Membership Function returns ordered set of passing cells:*

$$\text{footprint} \left( \begin{array}{c} \text{Trajectory}, \\ \text{AvoidanceGrid} \end{array} \right) = \left\{ \begin{array}{c} \text{cell} \in \text{AvoidanceGrid} : \\ \text{isMember}(\text{trajectory}, \text{cell}) \end{array} \right\} \quad (6.8)$$

*Then we can define equality function for Trajectory<sub>1</sub> and Trajectory<sub>2</sub>, as the comparison of their footprints in common Avoidance Grid as follow:*

$$\text{isEqual} \left( \begin{array}{c} \text{Trajectory}_1, \\ \text{Trajectory}_2, \\ \text{AvoidanceGrid} \end{array} \right) : \left\{ \begin{array}{c} \left( \text{footprint}(\text{Trajectory}_1, \dots) \right) \\ = \\ \left( \text{footprint}(\text{Trajectory}_2, \dots) \right) \\ \text{Otherwise} \end{array} \right. \begin{array}{c} : \text{true} \\ \\ : \text{false} \end{array} \quad (6.9)$$

*Note.* Depending on *Movement Automaton's* movement set and *Avoidance Grid* parameters, there can be multiple *trajectories* which are equal.

**Coverage set:** Now it is possible to create a set of unique *trajectory footprints* due to *footprint function* (eq. 6.8). Similarly, there is a possibility to create *Reach set skeleton* containing unique trajectories, by using *equality function* (eq. 6.9). *Coverage set* is sufficient for now.

**Definition 5** (Coverage Set). *Coverage set (6.10) is defined for Avoidance Grid and Reach Set pair as a set of unique Trajectory footprints:*

$$\text{CoverageSet} \left( \begin{array}{c} \text{AvoidanceGrid}, \\ \text{ReachSet} \end{array} \right) = \left\{ \text{footprint} \left( \begin{array}{c} \text{Trajectory}, \\ \text{AvoidanceGrid} \end{array} \right) : \begin{array}{c} \forall \text{Trajectory} \\ \in \text{ReachSet} \end{array} \right\} \quad (6.10)$$

**Coverage set properties:** Trajectory footprint (eq. 6.8) is not a *bijection*, neither *injection* for  $\text{ReachSet} \rightarrow \text{CoverageSet}$ . This implies the following properties:

1. Equal *Reach Sets* in same *Avoidance Grid* have equal *Coverage Sets*.
2. Equal *Coverage Sets* does not imply *Reach Set* equality.
3. For two *Coverage Sets*, there is a possibility to compare their member count to create coverage ratio.

The second *Property* gives us a proposition that there is a possibility of *Reach Set Reduction* without losing *Coverage*.

**Definition 6** (Coverage Ratio). Coverage Ratio is a ratio of Coverage Set Member Count between two Reach Sets. Reach set with a lesser count of unique Trajectories is considered as Reduced Reach Set. Reach set with greater Count of unique Trajectories is considered as Reference Reach Set.

$$\begin{aligned}
referenceCoverage &= |CoverageSet(ReferenceReachSet, AvoidanceGrid)| \\
reducedCoverage &= |CoverageSet(ReducedReachSet, AvoidanceGrid)| \\
CoverageRatio &= \frac{reducedCoverage}{referenceCoverage} \in [0, 1]
\end{aligned} \tag{6.11}$$

*Note.* Reference Reach Set is usually Full Reach Set containing all possible trajectories in space contained by Avoidance Grid. In case Full Reach Set cannot be computed, Avoidance Grid is too large, most complex Reach Set is used as Reference Reach Set.

**Trajectory smoothness:** Trajectory other than straight line have some changes in UAS heading.

The goal is to minimize *Maneuvering* of UAS, because:

1. *Every Heading Change* needs to be reported to UTM.
2. *Sharp Maneuvering* can damage cargo/wear UAS.
3. *Often course changes* make *Intruder prediction* harder for other Civil General Aviation.

For this purpose, *Smoothness Metric* needs to be applied for *Reach Set* or *Trajectory*. In the case of *Movement Automaton Control*, two distinguish *Movement Sets* are introduced: *Smooth* and *Chaotic* movements set with the following properties:

$$\begin{aligned}
MovementSet &= SmoothMovements \cup ChaoticMovements \\
SmoothMovements \cap ChaoticMovements &= \emptyset \\
|SmoothMovements| &> 0, \quad |ChaoticMovements| > 0
\end{aligned} \tag{6.12}$$

Then *Smoothness classifier* for *Trajectory(initialState, buffer)* can be defined as *isSmooth* and *Smooth Movement Counter* function as *smoothCount* like follow:

$$\begin{aligned}
isSmooth(movement) &= \begin{cases} movement \in SmoothMovements & : 1 \\ movement \in ChaoticMovements & : 0 \end{cases} \\
smoothCount(Trajectory(\dots, buffer)) &= \sum_{\forall movement \in Buffer} isSmooth(movement),
\end{aligned} \tag{6.13}$$

**Definition 7** (Smoothness Rating for Trajectory). Smoothness for trajectory generated by Movement Automaton for some Initial State with some Movement Buffer, under the assumption of Smooth and Chaotic Movement Set split (eq. 6.12), with existing classification and counter functionals (eq. 6.13) is given as follows:

$$\text{Smoothness}(\text{Trajectory}(\dots, \text{buffer})) = \frac{\text{isSmooth}(\text{Trajectory})}{\text{movementCount}(\text{Trajectory})} \in [0, 1] \quad (6.14)$$

For Trajectory with  $\text{buffer} = \emptyset$  Smoothness is given as 1.

### 6.4.3 Heuristic Trajectory Tree Building

**Summary:** There is a need for building tool to create trajectory set based reach set approximation iteratively. The iterative procedure is based on wave-front expansion algorithm. The node selection criteria for trajectory construction is used to build performance-based reach set approximations.

**Motivation:** Purpose of Navigation is to move forward to Goal Waypoint in Mission. Structure of Avoidance Grid is designed to enable forward and turning maneuvers. The Avoidance Grid is organized in Layers characteristic by the same distance from Avoidance Grid Origin.

Survey of motion planning algorithm was given in [4]. The ideal candidate for propagation algorithm is Wave-front algorithm propagating Trajectory tree through Layers. Due to the Avoidance Grid onion-like layers, there is a possibility to implement turn maneuver through layers iterative and effectively.

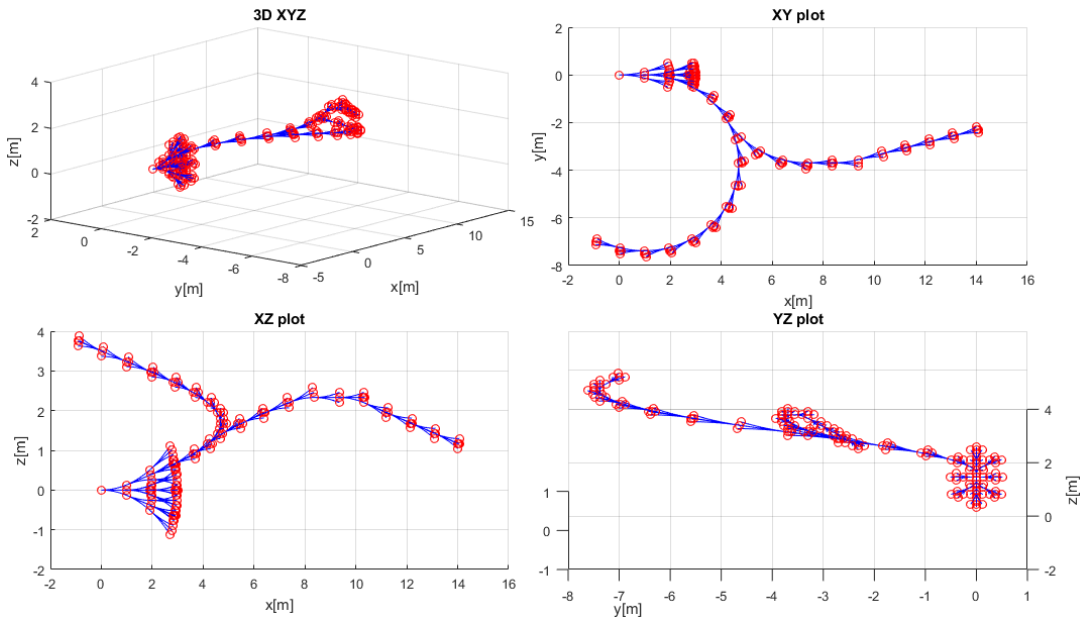


Figure 6.1: Rapid Exploration tree as a result of Constrained trajectory expansion.

**Rapid Exploration Tree** (fig. 6.1) was selected because it enables *Movement Automaton Utilization* and *Property Binding*. A similar approach was used for space exploration [5].

**The example** (fig. 6.1) shows a *Rapid Exploration Tree* in *Free Space* containing *Waypoint Navigation Path* and *Turn Away Path*. Both paths are starting in same *Root Node* (red circle) which was expanded with simple *Movement Automaton* (a bunch of nodes originating from one node is showing the way of expansion). The connection (blue line) between two nodes (red circles) represents *Trajectory portion* for *Executed Movement*.

**Rapid Exploration Tree Node** will contain the following information:

1. *Initial state* - root entry point, used in state evolution calculation.
2. *Trajectory (state evolution)* - trajectory passing through *state space* in the local coordinate frame of *Avoidance Grid*.
3. *Buffer* (applied movements) - ordered list of *executed movements* applied on the *initial state* to obtain *state evolution*.
4. *Cost* - calculated for *state evolution* based on a *predefined cost function*.
5. *Footprint* - ordered set of *passing cells* in *Avoidance Grid*.
6. *Parent Node Reference* - tree reference for the parent node, not in case of the *root node*.
7. *Other Bounded Properties* - value list of other properties, depending on *Expansion Constraints* and *Reachability* evaluation algorithm.

**Wave-front propagation of Rapid Exploration Tree** is given in (alg. 6.1).

The *Avoidance Grid* have UAS with *position*  $\in$  *Initial State* at the *origin*. The *Grid Layer* is a column ordered set of cells with same *Mean distance* from the origin. *Grid Layers* are indexed from origin starting with 1; there is a maximum of  $i \geq 1$  layers.

**Step: Initialization** contains base structure preparation like follows:

1. *Avoidance Grid* - space containing *Reach set* (def. 2).
2. *Movement Automaton* - Used as *Predictor*, consuming *buffer* containing *Movements* to generate *Trajectory(initialState, buffer)*.
3. *Reach Set* - tree consisting from *Wave-frontNodes* representing the endpoint of *Trajectory(initialState, buffer)* where each *Edge* represents *one Movement application*. The root is set as a node containing *Initial State*.



Function *initializeReachSet(root,stack,grid,automaton)* will take root and enforces *full wavefront propagation* to *First Layer*.

**Step: Wave-front Propagation** is forced propagation of trajectories from layer  $i$  to layer  $i + 1$ . The process goes as follows:

1. *Selection of Feasible candidates* - function  $[candidates, leftovers] = ExpansionConstraints.select(stack)$  for working layer, row and cell selects *feasible trajectory nodes* ordered by *Cost function*. The *Example of Cost Function* can be *Trajectory Smoothness* (def. 7).
2. *Expansion of Candidates* - for each *candidate* function *candidate.expandNode (automaton)* is invoked. This function will expand *Candidate Node structure* by appending *Full Trajectory Tree Evolution* until each *Leaf Trajectory* reaches *Next Layer*. Simply put *Parent Node Node(initialState, buffer, cost, footprint)* buffer is appended by movements until the next layer is reached.
3. *Leftovers purge* - function *reachSet.purge(leftovers)* removes unexpanded *Nodes* leading to cell, effectively removing trajectories which do not lead to the *next layer*.
4. *Append Reach Set* - function *reachSet.append(leafs)* puts newly created *Nodes (Trees)* into *Reach Set* structure. The *Wave-front Propagation* for one cell is finished.

**Step: After Layer Propagation Purge** is covered by function *reachSet.purgeSame-Footprint()* which takes trajectories with the same footprint and keeps some of them based on *Selection criteria*, more in (sec. 6.4.4, 6.4.5). *Pruning methods over Large Decision Trees* are *fast* and *viable* [6].

*Note.* *Reach Set* is usually computed *Prior the Flight* for some *Initial State* in *Local Coordinate Frame* in *right had coordinate frame* with  $X^+$  used as *main axis*.

---

**Algorithm 6.1:** *Wave-front propagation of Rapid Exploration Tree* to form *Reach Set*.

---

**Input** : Node(initialState,buffer= $\emptyset$ ,cost=0,footprint= $\emptyset$ ), AvoidanceGrid,  
ExpansionConstraints, MovementAutomaton(movementSet)

**Output:** ReachSet(AvoidanceGrid)

# Initialization Sequence;

grid=AvoidanceGrid, automaton=MovementAutomaton, root = Node;

reachSet = initializeReachSet(root,stack,grid,automaton);

# Main Expansion through, layers (i), rows (j), cells(k);

**for** layer(1...i) in grid **do**

**for** row(1...j) in layer **do**

**for** cell(1...k) in row **do**

            # apply selection criteria ;

            [candidates,leftovers] = ExpansionConstraints.select(stack);

            # collect expansions ;

            leafs = [];

**for** candidate in Candidates **do**

                | leafs= [leafs, candidate.expandNode(automaton)];

**end**

            reachSet.purge(leftovers);

            reachSet.append(leafs);

**end**

**end**

    reachSet.purgeSameFootprint();

**end**

---

#### 6.4.4 Coverage-Maximizing Reach Set Approximation

**Summary:** This procedure will produce trajectories maximizing aggressive maneuvering, ideal for avoidance tasks.

**Motivation:** Design of calculation method for *Reach Set Approximation* guarantying high *Maneuverability*.

**Background:** There is *Coverage Ratio* property of *Reach Set* (def. 6). It has been shown that creating *Reach Set* via *greedy approach* is not feasible due to the *Scaling Factor*. *Contracted Expansion* (sec. 6.4.3) is enabling to apply selection criteria while building *Reach Set* in given *Cell*.

The *Cell*  $cell_{i,j,k}$  has a center and walls from UAS viewpoint: a front wall, back wall (for  $layer > 1$ ), a top wall, left wall, right wall, bottom wall. It is expected that trajectory

leading close to one cell walls will continue to a different cell, increasing the chance to obtain more *Unique Footprints*.

**Expansion Constraint Function Implementation** (alg. 6.2) is based on the simple principle: *Select candidate Nodes which are closest to outer walls of Cell, with a unique footprint.*

**Tuning Parameters:** *Proximity to Cell outer wall* gives good chances to break into other rows or columns in the *Avoidance Grid*. *Unique footprint* guarantees future *Unique Footprint* after appending Trajectory by *Movement application*.

1. *Considered Footprint Length* - how much last cells in footprint should be considered in unique path track, minimal value 1, default value 3, maximal value  $\infty$ . If there is a need to generate non-redundant trajectories use  $\infty$ , it will consider full footprint.
2. *Spread Limit* - the upper limit of candidates which are going to be select for further expansion, minimal value 1, default value *Count of unique Moves in Movement set*, maximal value  $\infty$ . If more than default values are selected, the algorithm will generate *redundant trajectories*. If less is selected, then some trajectories are omitted, and *Coverage Rate* decreases sharply.

**Step: Initialization** initialization of *candidate* array (return value), *leftovers* array (return Value). Node array *passing* is populated with *Nodes* which represents *end node of Trajectory*, and the tip of the *trajectory is constrained in the cell<sub>i,j,k</sub>*.

**Step: Evaluate best trajectories with unique Footprints** following steps are executed:

1. *Best Performance Map* is created with a *footprint* as a key set element to ensure footprint uniqueness.
2. *Wall distance* for the *test node* is calculated as a closest trajectory portion distance to the *top, bottom, left, right* wall of the *cell<sub>i,j,k</sub>*
3. The *Footprint* for the *test node* is created with the maximal length given by *Footprint Length* tuning parameter.
4. *Existence and Performance Test* is executed to ensure that the best performing node is selected. If there is no key entry in the *Best Performance Map*, then a new entry for *Test Node* is created. If there is a key entry, the performance of *Old Node* and *Test Node* is compared, and better is stored.

**Step: Select candidates** is executed on *Best Performance Map* records using *Wall distance* as pivot parameter, ordering by closest proximity and limited by *Search Limit* tuning parameter. The *Leftovers* are difference set between *Passing Nodes* and *Candidate Nodes*.

---

**Algorithm 6.2:** Expansion Constraint function for *Coverage-Maximizing Reach Set Approximation*

---

```

Input                : Node[] stack, Cell celli,j,k
Tuning Parameters: int+ footprintLength, int+ spreadLimit
Output              : Node[] candidates, Node[] leftovers

# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = celli,j,k.getFinishingTrajectories(stack);

# Select best performing trajectories with unique footprint;
Map<Footprint,Node> bestPerformanceMap;
for Node test  $\in$  passing do
    wallDistance= test.minimalDistanceToWall(celli,j,k);
    footprint = test.getFootprint(lastCells = footprintLength);
    if bestPerformanceMap.contains(footprint) then
        old = bestPerformanceMap.getByKey(footprint);
        oldPerformance= old.minimalDistanceToWall(celli,j,k);
        if oldPerformance > wallDistance then
            bestPerformanceMap.setByKey(footprint,test);
        end
    else
        bestPerformanceMap.setByKey(footprint,test);
    end
end

# Select best performing nodes up to spreadLimit count;
candidates = bestPerformanceMap.select(count =
    spreadLimit).orderBy('wallDistance','Ascending');
leftovers = passing - candidates;
return [candidates,leftovers]

```

---

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.2). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*.

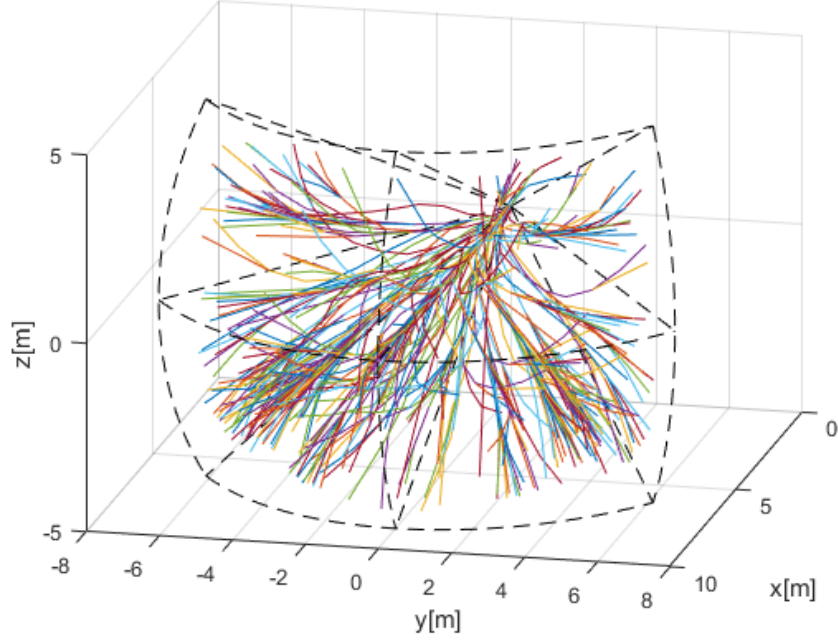


Figure 6.2: *Coverage-Maximizing reach set approximation.*

**Pros and Cons:** It can be seen from example (fig. 6.2) that *Coverage-Maximizing Reach Set Approximation Method* (alg. 6.2) generates much *turning* and *shaky trajectories*.

*High Coverage Ratio* ( $\sim 0.9$ ) is provided while keeping *medium node count*. The calculation complexity scales linearly with grid size. The *upper limit of trajectories* is given as follow:

$$\begin{aligned} \text{countTrajectories}(\text{ReachSet}) \leq & \text{layerCellCount} \times \text{spreadLimit} \\ & \times \text{size}(\text{Movements}) \quad (6.15) \end{aligned}$$

The *upper limit of nodes* is given as follow:

$$\begin{aligned} \text{countNodes}(\text{ReachSet}) \leq & \text{layerCount} \times \text{layerCellCount} \\ & \times \text{size}(\text{Movements}) \times \text{spreadLimit} \quad (6.16) \end{aligned}$$

The *absence of Smooth Trajectories* disqualifies *Coverage Maximizing -RSA* to be used for *Navigation*. This type of reach set is feasible for *Avoidance* because it contains a variety of maneuvers.

### 6.4.5 Turn-Minimizing Reach Set Approximation

**Summary:** This procedure will produce trajectories minimizing aggressive maneuvering, ideal for navigation tasks.

**Motivation:** Imagine having an *Avoidance Grid* like (fig. ??). There is a need of *Reach Set Approximation* which will have *Smooth Trajectories* (def. 7) going nearby *cell* centers.

**Background:** The *Smoothness Rating for Trajectory* (def. 7) uses two distinct sets *Smooth Movements* and *Chaotic Movements* (eq. 6.12) which are defined for our *Movement Automaton* (sec. ??) like the following:

$$\begin{aligned} \text{SmoothMovements} &= \{\text{Straight}\} \\ \text{ChaoticMovements} &= \text{Movements} - \text{SmoothMovements} \end{aligned} \tag{6.17}$$

*Smooth Movements* contains only *Straight* movement because others are considered as extreme turning movements. *Smooth Movements* should contain only direct flight movements or slight heading correction. *Chaotic Movements* set is a supplement of *Movement Automaton's Movement Set*.

The *Avoidance Grid* (fig. ??) cell centers for fixed indexes  $j_{fix}, k_{fix}$  are linearly aligned with the *initial state*. That means that cell centers of cells  $cell_{1,j_{fix},k_{fix}}, \dots, cell_{i,j_{fix},k_{fix}}$ , where  $i$  is a count of *layers* lie on one line. If the trajectory can achieve *cell center* on some *layer*, only minor trajectory corrections are required to stay on the given line. This type of trajectory gives us the following advantages:

1. *Minimal steering at the beginning* - the minimal steering is advantageous in *Controlled Airspace* because is diminishing the amount of communication to *UTM Service*.
2. *Additional safe space in the linear segment* - once the *center of the cell* is reached, *Trajectory* sticks to the line between cell centers. Each point on this line has the *maximal distance* to outer walls of the cell. This gives us extra space given as minimum of distance between *UAS position* and *Outer cell walls*.

**Expansion Constraint Function Implementation** (alg. 6.3) is based on the simple principle: *Select candidate Nodes which are closest to Cell center, with a unique footprint*.

*Note.* *Cell center* can be closely reached by *smooth movement* from a previous cell or *chaotic movement* from a neighboring cell from the current or previous layer. These trajectories are usually equivalent in *Smoothness*.

**Tuning Parameter:** *Proximity to Cell Center* gives a good chance to keep trajectory smooth or *smooth after one correction maneuver*. It has been mentioned that *Cell Center*

can be reached by various trajectories. In this method full footprint length is always considered; therefore only one tuning parameter can be offered:

1. *Spread Limit* - the upper limit of candidates which are going to be selected for further expansion, minimal value 1, default value *Count of unique Moves in Movement set*, the maximal value  $\infty$ . If maximal value  $\infty$  is selected, the algorithm will generate the skeleton of *Reach Set* with full Coverage and with the smoothest *Trajectories*.

**Step: Initialization** sets candidate *Nodes* as empty set, leftover *Nodes* as empty set. and selects all *Nodes* from *Stack* which represents *Finishing Trajectories* in working cell  $cell_{i,j,k}$ .

---

**Algorithm 6.3:** Expansion Constraint function for *Turn-Minimizing Reach Set Approximation*

---

```

Input                : Node[] stack, Cell celli,j,k
Tuning Parameters: int+ spreadLimit
Output              : Node[] candidates, Node[] leftovers

# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = celli,j,k.getFinishingTrajectories(stack);

# Select unique smoothest trajectories;
Map<Buffer,Node> bestPerformanceMap;
for Node test  $\in$  passing do
    centerDistance= test.getPerformance(celli,j,k);
    footprint = test.getFootprint();
    if bestPerformanceMap.contains(footPrint) then
        old = bestPerformanceMap.getByKey(footprint);
        oldPerformance= old.getPerformance(celli,j,k);
        if oldPerformance > centerDistance then
            bestPerformanceMap.setByKey(footprint,test);
        end
    else
        bestPerformanceMap.setByKey(footprint,test);
    end
end

# Select best performing nodes up to spreadLimit count;
candidates = bestPerformanceMap.select(count =
    spreadLimit).orderBy('cellCenterDistance','Ascending');
leftovers = passing - candidates;
return [candidates,leftovers]

```

---

**Step: Evaluate smoothest trajectories with unique Footprints** is implemented as *multi-criteria filtration*.

The *first criterion* is the *distance to Cell Center* which is penalized by trajectory *smoothness rate* implemented in method  $Node.getPerformance(Cell\ cell_{i,j,k})$  defined as follow.

$$getPerformance(Node, Cell) = \frac{distance(Node.Trajectory, Cell.Center)}{SmoothnessRate(Node.Trajectory)} \quad (6.18)$$

Distance of *Trajectory* is *enumerator* because its considered as the *base value* and is defined in the interval  $[0, maximalWallDistance]$ . The *Smoothness Rate* is the denominator, because it is a penalization coefficient defined in the interval  $[0, 1]$ .

The *second criterion* is *trajectory uniqueness*. This is provided by *Best Performance Map*, where best performing *Node* belongs to one unique *trajectory footprint*. The implementation is identical to *coverage-maximizing set expansion* (alg. 6.2).

**Step: Select candidates** is executed on *Best Performance Map* records using *Penalized Cell Center Distance* as pivot parameter, ordered in ascending order and limited by *Spread Limit* tuning parameter. The *Leftovers* are difference set between *Passing Nodes* and *Candidate Nodes*.

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.3). The UAS is at *Back-side* of *Figure* (the initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its color and ends at *Front-side* of *Avoidance Grid Boundary*. The *Spread Limit*, in this case, was set to 9 which is *Size of the Movement Set*.

*Note.* Please note *Trajectories* are organized in bundles going around *Cell Centers* *smoothly*. Most of the steering maneuvers are executed at the *beginning* of the *Avoidance Grid*.



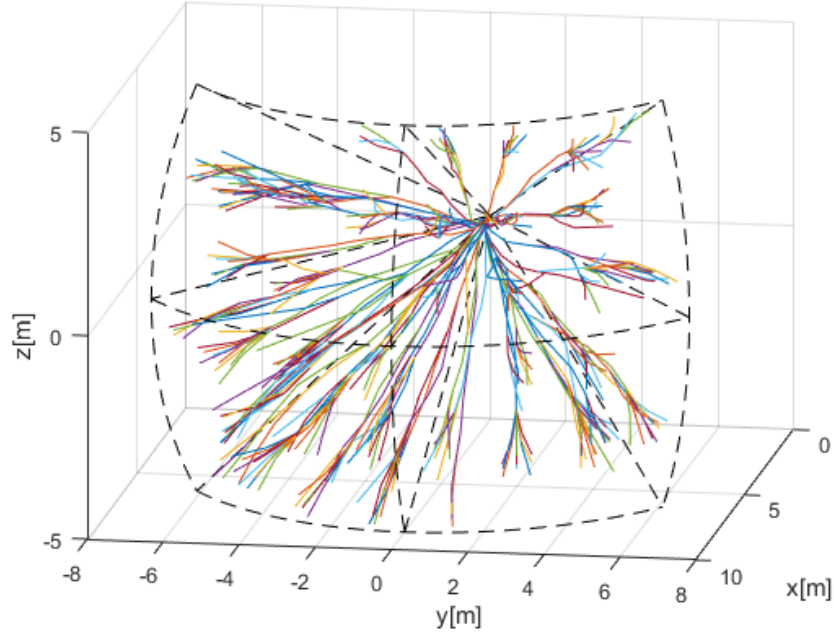


Figure 6.3: *Turn-minimizing reach set approximation.*

**Pros and Cons:** It can be seen from example (fig. 6.3) that *Turn-Minimizing Reach Set Approximation Method* (alg. 6.3) generates *smooth evenly spread trajectories*.

High smoothness ratio ( $\geq 0.9$ ) is provided while keeping low node count for UAS systems. The calculation complexity scales linearly with grid size. The upper limit of trajectories is given as follow:

$$\text{countTrajectories}(\text{ReachSet}) \leq \text{layerCellCount} \times \text{spreadLimit} \times \text{size}(\text{Movements}) \quad (6.19)$$

The *upper limit of nodes* is given as follow:

$$\text{countNodes}(\text{ReachSet}) \leq \text{layerCount} \times \text{layerCellCount} \times \text{spreadLimit} \quad (6.20)$$

The absence of *High Coverage Ratio* disqualifies *Turn-Minimizing Reach Set Approximation* to be used for *Emergency Avoidance*. This type of *Reach Set* is feasible for *Open Space Navigation* or *Controlled Airspace Navigation*. Its low turning rate in contained *Trajectories* are desired for such tasks.

### 6.4.6 ACAS-X like Reach Set Approximation

**Summary:** This procedure will produce trajectories encoding ACAS-X separation modes, ideal for controlled airspace navigation.

**Motivation:** The implementation of *ACAS-Xu* behavior in DAA system will be mandatory for *National Airspace System Integration* in United spaces [7].

Implementation of ACAS-Xu like behavior increase usability of approach, if it can be achieved without major concept changes.

**Background:** The *ACAS-Xu* system on the operational level has been described in [8]. The *Policy for Collision Avoidance* proposal has been given in [9].

Some behavioral patterns can be encoded into *Reach Set*. ACAS-Xu navigation part is basically *Look-up table of Maneuvers for Allowed Separations*.

The *Evasive Maneuver* selection process in ACAS-Xu is similar to our approach: *Select most energy efficient maneuver in compliance with space-time constraints*. ACAS-Xu intruder model is similar to our *Body Volume Intersection Model* (app. ??). The *ACAS-Xu* defines following base separations:

1. *Horizontal* - movements on a *Horizontal Plane* in *Global Coordinate System*.
2. *Vertical* - movements on a *Vertical Plane* in *Global Coordinate System*.

There are allowed custom separations which can be used, for further experimentation:

1. *Slash* - movement on  $+45^\circ$  *Tilted Plane to Horizontal Plane* in *Global Coordinate System*.
2. *Backslash* - movement on  $-45^\circ$  *Tilted Plane to Horizontal Plane* in *Global Coordinate System*.

For given *Movement Automaton* implementation (sec. ??) the separations are given as follow:

$$\begin{aligned}
 \textit{Horizontal} &= \{\textit{Straight}, \textit{Left}, \textit{Right}\} \\
 \textit{Vertical} &= \{\textit{Straight}, \textit{Up}, \textit{Down}\} \\
 \textit{Slash} &= \{\textit{Straight}, \textit{UpLeft}, \textit{DownRight}\} \\
 \textit{Backslash} &= \{\textit{Straight}, \textit{UpRight}, \textit{DownLeft}\}
 \end{aligned} \tag{6.21}$$

For each *Node*(..., *buffer*) and each *separation* there is a evaluation function *isSeparation* which decides, if *Trajectory* defined by node buffer is made up only from *Separation* movements. The function *isSeparation*(...) is defined like:

$$isSeparation(buffer, separation) = \begin{cases} \forall movement \in buffer, & : true \\ movement \in separation & \\ otherwise & : false \end{cases} \quad (6.22)$$

Following *Separation Modes* can be defined with given *separations*:

1. *Horizontal* (ACAS-X defined mode) containing *horizontal* separation.
2. *Vertical* (ACAS-X defined mode) containing *vertical* separation.
3. *Horizontal-Vertical* (ACAS-X defined mode) containing *horizontal*, *vertical* separations.
4. *Full* (custom defined mode) containing all *Separation Modes*.

*Note.* Every separation modes generate 2D trajectories set on *Respective plane*. There is no need for *Tuning parameters* for further *Expansion Constraint*.

**Expansion Constraint Function Implementation** (alg. 6.4) is based on the simple principle: *Select only candidate Nodes which Trajectories have at least one desired Separation Mode.*

**Step: Initialization** sets candidate *Nodes* as the empty set, leftover *Nodes* as the empty set, and, select all nodes to form a *stack* which represents *Finishing Trajectories* in working  $cell_{i,j,k}$ ,

**Step: Candidate Selection Process** is evaluated for each *test Node* from *passing Node Set*.

For each *applicable separation*, given as input parameter *separations*, The test function *isSeparation* (eq. 6.22) is applied:

1. If *test Node* trajectory belongs to at least one allowed separation it is added to candidates set.
2. Else is added to *Leftovers*.

*Note.* *Separation sets* (eq. 6.21) are not *exclusive sets* in *Movement Automaton* domain. One *Trajectory* contained by *Node* can belong to multiple *Separations*.

---

**Algorithm 6.4:** Expansion Constraint function for *ACAS-like Reach Set Approximation*


---

**Input** : Node[] stack, Cell cell<sub>*i,j,k*</sub>, Separation[] separations

**Tuning Parameters:** *None* :  $\emptyset$

**Output** : Node[] candidates, Node[] leftovers

# Initialize structures;

Node[] candidates = [], Node[] leftovers=[];

Node[] passing = cell<sub>*i,j,k*</sub>.getFinishingTrajectories(stack);

# Select nodes containing trajectories with usable separations;

**for** Node test  $\in$  passing **do**

**for** separation  $\in$  separations **do**

        # Get separations for Node;

        Separations[] nodeSeparations = test.getSeparations();

        # If trajectory given by buffer is on Separation plane;

**if** isIn(isSeparation(test.buffer, separation)(6.22) **then**

            candidates.append(test);

**end**

**end**

    # If there was no applicable separation, throw Node away;

**if** test  $\notin$  candidates **then**

        leftovers.append(test);

**end**

**end**

# Return results;

**return** [candidates, leftovers]

---

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.4). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*.

*Full* separation mode is given in (fig. 6.4a). *Horizontal-Vertical* separation mode, used in original *ACAS-Xu* testing [8], given in (fig. 6.4b). *Horizontal* separation mode given in (fig. 6.4c) is usually used by planes. *Vertical* separation mode given in (fig. 6.4d) is usually used by copters.

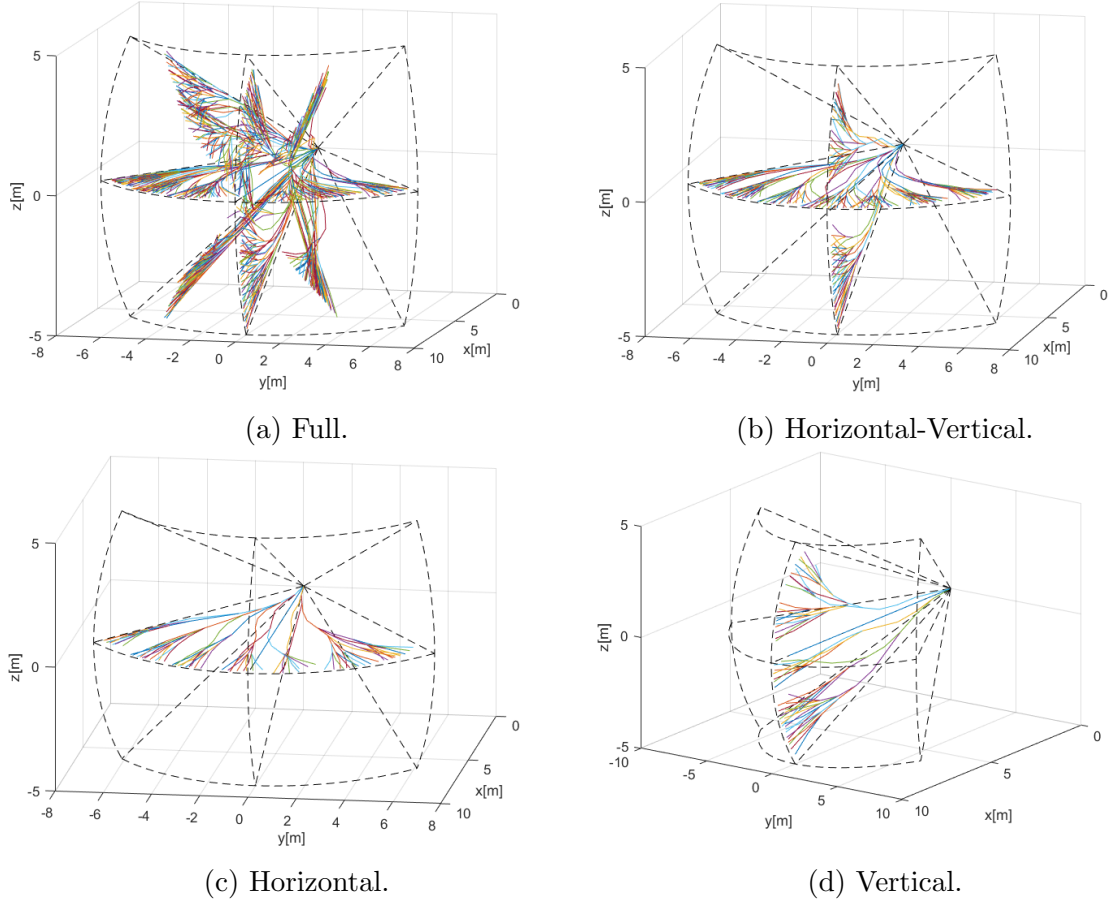


Figure 6.4: ACAS-X imitation *reach set* approximation for various *separation modes*.

**Pros and Cons:** It can be seen from examples (fig. 6.4) that *ACAS-like Reach Set Approximation Method* (alg. 6.4) generates a full reach set for 2D plane located in 3D space.

The *Reach Set* contains trajectories with *high coverage ratio* and *high smoothness rating* for selected 2D separation plane. Overall performance compared to full 3D reach sets (sec. 6.4.4, 6.4.5 6.4.7) is poor.

The *node* and *trajectory* count boundary was not implemented. It is common knowledge that *2D* avoidance sets do not require scaling [8]. Otherwise, trajectory footprint mechanism like in *Turn-Minimizing Reach Set Approximation* (alg. 6.3) can be introduced.

This reach set implements *Planar-Separation* as a native feature, it can be used for both *navigation* and *avoidance* tasks in *Controlled Airspace*. For *Non-controlled Airspace*, there are far more superior *Combined Reach Set* (sec. 6.4.7).

### 6.4.7 Combined Reach Set Approximation - Tree Merge

**Motivation:** Turn-Minimizing Reach Set Approximation (sec. 6.4.5) is *efficient* for *Navigation* in *Controlled Airspace*. Coverage-Maximizing Reach Set Approximation (sec. 6.4.4) is good for *Emergency avoidance*. The need for the differentiation between *Navigation* and *Emergency Avoidance* mode is necessary for *Controlled Airspace*, but not

for *Non-controlled Airspace*. The combination of *Turning-Minimizing* and *Coverage-Maximizing* reach set approximations is an obvious solution.

*Automatic mode switch* can be provided by a combination of *Navigation Reach Set* and *Avoidance Reach Set* with an elevated cost function. Overall having a method to merge multiple trees would be beneficial.

**Background:** If two *Reach Set Approximation* were calculated for the same *Avoidance Grid* and *Initial State*, using same *Movement Automaton* and *UAS model* are possible to merge.

The *Reach Set Approximation* is a tree with *Root Node* in *initial state* with movement buffer =  $\emptyset$ . The *movement buffer* in each node can be used as *route trace* during the merging procedure. The example two reach set merge can be given as follow, where only the *latest* applied movement is taken into account.

$$\begin{array}{c} \text{First Reach Set} \\ \emptyset \rightarrow \left\langle \begin{array}{l} \text{left} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \\ \emptyset \end{array} \right. \\ \text{Second Reach Set} \\ \emptyset \rightarrow \left\langle \begin{array}{l} \emptyset \\ \text{right} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \end{array} \right. \end{array} \rightarrow \begin{array}{c} \text{Combined Reach Set} \\ \emptyset \rightarrow \left\langle \begin{array}{l} \text{left} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \\ \text{right} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \end{array} \right. \quad (6.23)$$

*First Reach Set* contains two trajectories given by buffers  $\{\text{left}, \text{left}\}$  and  $\{\text{left}, \text{right}\}$ . *Second Reach Set* contains two trajectories given by buffers  $\{\text{right}, \text{left}\}$  and  $\{\text{right}, \text{right}\}$ . The *Combined Reach Set* contains all four trajectories.

*Note.* The combined tree [10] does not need to have combined amount of original *Reach Sets* trajectories. There can be *Duplicity* which means that any bounded property like *Cost* must be *calculated* again.

**Combined Reach Set Calculation Function** (alg. 6.5) is implemented as function *NodecombinedReachSet(...)* which takes root Node with *initial State*, *Avoidance Grid* and respective parameters for each calculation method. *turn-minimizing spread* for *Turn-Minimizing Reach set calculation* and *coverage spread*, *Footprint Length* for *Coverage-Maximizing Reach Set Approximation*.

*Separate Reach Sets* are calculated using *Wave-front propagation* (alg. 6.1) using respective *Constrained Expansion* functions for *Turn-Minimizing* (alg. 6.3) and *Coverage-Maximizing* (alg. 6.2) reach sets.

*Combined Reach Set* is created using *Node mergeTree(...)* function, because different cost function or *Bounded Parameters Calculation* may be applied on *Original Reach Sets*.

*Cost* for *each node* needs to be recalculated due to original reach sets disparity. Function *combined.applyCostFunction()* will recalculate the new cost for each node.

The Goal is to have a penalization for *non-turn-minimizing behavior*, implementation of *Automatic Mode Switch* can be done like follows:

1. *Calculate Normal Cost* for Node  $Cost(Node)$  for the associated trajectory:  
 $Cost(Node.Trajectory)$ .
2. *Calculate Penalization for additional maneuvering*, calculate *Smoothness Rating for Trajectory* (def. 7) in the interval  $[0, 1]$ , introduce penalization with base 100%.

The final  $Cost(Node)$  function is applied to each *Combined Reach Set Node* and look like follows:

$$Cost(Node) = Cost(Node.Trajectory) \times \dots \times (1 + (1 - SmoothnessRate(Node.Trajectory))) \quad (6.24)$$

**Tree Merge Function** *mergeTree(...)* implements *Outer Join* operation on two trees. Example was given in (eq. 6.23). Function is applied on *root Node* iterating over *Movements in Movement Set*, because *Movement is pivot*.

---

**Algorithm 6.5:** Reach Set Merge Function and Combined Reach Set calculation

---

```
# Tree merge function;
Node mergeTree(Node firstNode, Node secondNode)
|
| # Try to copy reference node or return null;
| Node referenceNode = (firstNode?:(secondNode?: return null));
| Node merged = new Node(referenceNode);
| merged.leafs= [];
|
| # Try to fetch movement nodes if exist in any sub tree;
| for movement ∈ Movements do
| | firstLeaf = firstNode.getLeafFor(movement);
| | secondLeaf = secondNode.getLeafFor(movement);
| | newLeaf = mergeTree(firstLeaf,secondLeaf);
| | if newLeaf ~ = null then
| | | merged.leafs.append(newLeaf);
| | end
| end
| return merged

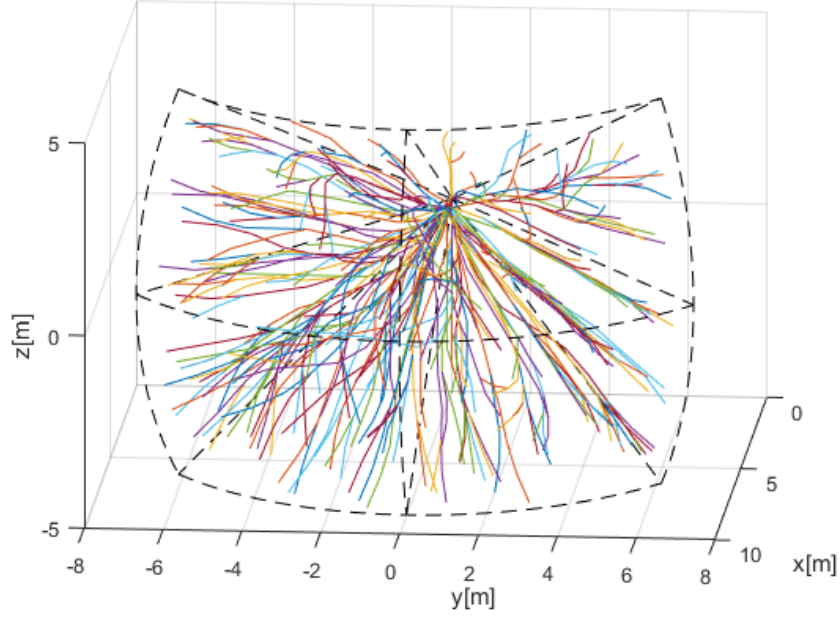
# Combined Reach Set calculation function;
Node combinedReachSet(Node root, AvoidanceGrid grid,int+ coverageSpread,
int+ turnSpread, int+ footprintLength)
|
| Node cmrsa = chaoticReachSet(root,grid, footprintLength,coverageSpread);
| Node tmrsa = harmonicReachSet(root,grid, turnSpread);
| Node combined = mergeTree(cmrsa,tmrsa);
| combined.applyCostFunction();
| return combined
```

---

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.5). The UAS is at *Back-side of Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side of Avoidance Grid Boundary*. The *Coverage-Maximizing Spread* was set to 8, *Footprint Length* to 3 and *Turn-Minimizing Spread* to 1.

*Note.* Notice there are typical trajectories from both *Turn-Minimizing* (fig. 6.3) and *Coverage-Maximizing* (fig. 6.2) *Reach Set Approximations*.



Figure 6.5: *Combined reach set approximation.*

**Pros and Cons:** It can be seen from example (fig. 6.5) that *Combined Reach Set Approximation* (alg. 6.5) contains both types of maneuvers. *Cheaper turn-minimizing* for navigation and *More Expensive Coverage-Maximizing* for *Emergency Avoidance*. The upper limit of trajectories is given as follow:

$$\begin{aligned} \text{countTrajectories}(\text{ReachSet}) \leq & \text{countTrajectories}(\text{CM} - \text{RSA}) \\ & + \text{countTrajectories}(\text{TM} - \text{RSA}) \quad (6.25) \end{aligned}$$

The *upper limit of nodes* is given as follow:

$$\text{countNodes}(\text{ReachSet}) \leq \text{countNodes}(\text{CM} - \text{RSA}) + \text{countNodes}(\text{TM} - \text{RSA}) \quad (6.26)$$

*Turn-Minimizing Reach Set* is ideal for *Non-controlled Airspace* missions because it contains *Automatic Mode Switch* between *Navigation* and *Emergency Avoidance*.



# Bibliography

- [1] Franco Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.
- [2] John Birmingham and Peter Kent. Tree-searching and tree-pruning techniques. In *Computer chess compendium*, pages 123–128. Springer, 1988.
- [3] Alojz Gomola, João Borges de Sousa, Fernando Lobo Pereira, and Pavel Klang. Obstacle avoidance framework based on reach sets. In *Iberian Robotics conference*, pages 768–779. Springer, 2017.
- [4] Chad Goerzen, Zhaodan Kong, and Bernard Mettler. A survey of motion planning algorithms from the perspective of autonomous uav guidance. In *Selected papers from the 2nd International Symposium on UAVs, Reno, Nevada, USA June 8–10, 2009*, pages 65–100. Springer, 2009.
- [5] Catherine Plaisant, Jesse Grosjean, and Benjamin B Bederson. Spacetree: Supporting exploration in large node link tree, design evolution and empirical evaluation. In *Information Visualization, 2002. INFOVIS 2002. IEEE Symposium on*, pages 57–64. IEEE, 2002.
- [6] John Mingers. An empirical comparison of pruning methods for decision tree induction. *Machine learning*, 4(2):227–243, 1989.
- [7] Jay Shively. Uas integration in the nas: Detect and avoid. 2018.
- [8] Mike Marston and Gabe Baca. Acas-xu initial self-separation flight tests, 2015.
- [9] Kyle D Julian, Jessica Lopez, Jeffrey S Brush, Michael P Owen, and Mykel J Kochenderfer. Policy compression for aircraft collision avoidance systems. In *Digital Avionics Systems Conference (DASC), 2016 IEEE/AIAA 35th*, pages 1–10. IEEE, 2016.
- [10] Patrick O’Neil, Edward Cheng, Dieter Gawlick, and Elizabeth O’Neil. The log-structured merge-tree (lsm-tree). *Acta Informatica*, 33(4):351–385, 1996.