

## 6.4 Reach Set Approximation

**Motivation:** *Reach set* is strong tool for *Obstacle Avoidance* because it contains all possible *avoidance maneuvers*. The current implementations (sec. ??) have following flaws:

1. *Realistic approximation* - *nonlinear systems* or *heavily constrained systems* cannot be approximated well by *linear continuous-time Reach Sets*.
2. *Finite count of possibilities* - continuous-time *Reach Set* contains infinite possibilities for *avoidance maneuvers*, the SAA system demands conflict resolution in finite time.
3. *Computationally feasible data structures* - binding related properties seems problematic, because *continuous-time reach sets* does not have unique identifier of maneuver, trajectory nor segment.

**Proposed Solution Features:** Our Reach set Estimation method will provide following features:

1. *System Control Interface* - implemented via *Movement Automaton*, requiring only *discrete command chain* to approximate system behaviour.
2. *Finite count of possibilities* - finite number of elements in *Reach set* will enable *scalable* calculation.
3. *Computationally feasible data structures* - approximation of Reach set as a set of trajectories, each trajectory can be split into finite number of segments. Each element will have unique identifier enabling both-side property binding.
4. *Computationally feasible data-structures* - some specific behavior, like horizontal/vertical separation, or maneuver shape can be encoded into a different types of the reach set approximation algorithms.

### 6.4.1 Trajectory Set Approximation of Reach Set

**Discretization of Reach set:** There is a need for a discrete finite *Reach Set approximation* to enable *Avoidance Strategy Evaluation* in finite time. Replacing *Continuous Control Set Inputs(t)* by *Movement Automaton* is feasible:

**Definition 1** (Reach set Approximation by Movement Automaton). A trajectory (def. ??) for system  $state = f(time, state, input)$  under control of the movement automaton  $\mathcal{MA}$  is given as execution of movement buffer (def. ??) with initial state of system  $state_0$ . Therefore notation  $Trajectory(state_0, buffer)$  is used.

**The Complete Reach Set** (6.1) for system with initial state  $state_0$  with existing control strategy  $control(time) \in Controls(time)$ . for time  $\tau > time_0$ .

$$ReachSet(\tau, time_0, state_0) = \bigcup \{state(s) : control(s) \in Controls(s), s \in (time_0, \tau]\} \quad (6.1)$$

**The Reach Set Approximation by Movement Automaton** (6.2) of the system under the control of the movement automation  $\mathcal{MA}$  consist from the set of trajectories  $Trajectory(state_0, Buffer)$ , which are executed in constrained time  $\tau > time_0$ .

$$ReachSet(\tau, time_0, state_0) = \left\{ Trajectory(state_0, buffer) : \begin{array}{l} duration(buffer) \\ \leq \\ (time_0 - \tau) \end{array} \right\} \quad (6.2)$$

*Note.* *Reach Set Approximation* (def. 1) is subset of *Full Reach Set* (def. ??) in continuous space  $\mathbb{R}^n$  it inherits all important properties, like *Invariance* [1].

*Discretization* of *Reach Set* have been achieved leaving us with *finite count* of *Trajectories*, instead of *Infinite subspace* or  $\mathbb{R}^N$

**Approximated Reach Set Containment:** The *Approximated Reach Set* introduced in (def. 1) is constrained only by *future expansion time*  $\tau$ . UAS makes space assessment in *Avoidance Grid*. There is no point to consider *Trajectories* outside of *Avoidance Grid*

**Definition 2** (Contained Aproximated Reach Set). For pair  $(state_0, AvoidanceGrid_0)$  at time  $time_0$  and prediction horizon  $\tau = \infty$  there is Contained Reduced Reach Set:

$$ReachSet \left( \begin{array}{c} time_0, \\ state_0, \\ AvoidanceGrid_0 \end{array} \right) = \left\{ \begin{array}{c} Trajectory(\dots) \\ \in \\ ReachSet(6.2) \end{array} : \begin{array}{l} \forall segment \in AvoidanceGrid_0, \\ segment \in Trajectory(\dots) \end{array} \right\} \quad (6.3)$$

**Properties:** Container Aproximated Reach Set contains only trajectories where all segments belongs to Avoidance Grid, there are following functions:

1. Membership function for any Trajectory in Constrained Reduced Reach set returns Ordered Set of Passing Cells.
2. Cost function for any Trajectory Portion in Constrained Reduced Reach Set return Cost of Execution

**Passing cell:** Cell of Avoidance Grid which has some intersection with Trajectory.

*Note.* *Contained Reduced Reach Set* (eq. 6.3) which is contained in *Avoidance Grid* and have an *Membership Function* enable Property transition between Reach set and *Avoidance grid*.

*Example:* Visibility from cells along *Trajectory* can be gathered to calculate *Trajectory's* feasibility.

**Reach Set Pruning:** There is a need to implement *Set Difference* between *Reach Set* and *Constraint Set*. *Constraint Set* can be *Obstacle Set* from *Known World* (sec. ??) and other different constraints.

**Reach Set Trajectory Tree:** (6.4) Any *Reach Set* where *Control Strategy Constraint* is implemented as *Movement Automaton*, with defined *Movements* set and for single initial  $state_0$ . The *Reach Set* is given as discrete tree with root  $Trajectory(state_0, \emptyset)$ .

$$ReachSet(state_0, \dots) = \left\{ Trajectory(state_0, buffer) : \begin{array}{l} buffer \in Movements^i, \\ i \in \{1, \dots, k\} \end{array} \right\} \quad (6.4)$$

For each *Trajectory Segment*, there exists *intersection function* which evaluates as true if there exists at least one point in *Segment* which belongs to *Constraint Set*. Formally:

$$intersection(segment, Set) : \begin{cases} \exists point \in segment, & : true \\ point \in Set & \\ Otherwise & : false \end{cases} \quad (6.5)$$

**Definition 3** (Pruned Reach Set). For *Reach set represented as Trajectory Tree* (eq. 6.4) and some *constraint set (Set)* where exist *intersection function* (eq. 6.5). The *Pruned Reach set* is given as follows:

$$Prune(ReachSet, Set) = \left\{ Trajectory(\dots) : \begin{array}{l} \forall segment \in Trajectory, \\ \neg intersection(segment, Set) \end{array} \right\} \quad (6.6)$$

*Note.* *Pruning*(def. 3) [2] is applicable multiple times for various *Constraints Set*.

Example of *Approximated Reach set Calculation* (def. 1), *Reach Set Containment* (def. 2), and, *Pruning* is given in [3].

### 6.4.2 Distinctive Properties of the Trajectories

**Motivation:** The need to Make *Reach Set* scalable approach. This may be a problem due the *Expansion rate*. *Reach set* represented as a *Trajectory Tree* (eq. 6.4) for *Avoidance Grid* with *layerCount* and *Movement automaton* with *movementCount*, the *Node count* is given as:

$$1 + \left( \sum_{i \in \{1 \dots layerCount\}} (movementCount)^i \right) \quad (6.7)$$

This scaling is not feasible for *Avoidance Grid* with many layers ( $< 10$ ) or *Movement Set* with many movements ( $< 9$ ). There is need for *Reduced Reach set calculation*.

**Performance Criteria:** The scaling factor (eq. 6.7) shows that there are going to be many trajectories. The main point is that not every trajectory in *Reach Set* are giving us *maneuverability advantage*. Our expectations lies in following *Performance Requirements*:

1. *Reach set* must *Cover* maximum of the *possible unique maneuvers* in *Avoidance Grid*.
2. *Trajectories* in *Reach Set* should be smoothest possible to prevent cargo damage / UAS wear.

**Trajectory footprint:** Discrete space of *Avoidance Grid* is organized in cells. *Cell* is minimal space portion accessible by *property binding*. There is need to know if two trajectories contribution to *Maneuverability* in this environment.

Each trajectory passes through space in *Avoidance Grid*. If there exists a method to extract unique identifier for each *trajectory passed cells*, we can compare two trajectories *Coverage* in *Avoidance Grid*.

**Definition 4** (Trajectory footprint). *For Trajectory from Reach set (def. 2) defined for Avoidance Grid has membership function. Membership Function returns ordered set of passing cells:*

$$footprint \left( \begin{array}{c} Trajectory, \\ AvoidanceGrid \end{array} \right) = \left\{ \begin{array}{c} cell \in AvoidanceGrid : \\ isMember(trajjectory, cell) \end{array} \right\} \quad (6.8)$$

Then we can define equality function for *Trajectory<sub>1</sub>* and *Trajectory<sub>2</sub>*, as comparison of their footprints in common *Avoidance Grid* as follow:

$$isEqual \left( \begin{array}{c} Trajectory_1, \\ Trajectory_2, \\ AvoidanceGrid \end{array} \right) : \left\{ \begin{array}{c} \left( footprint(Trajectory_1, \dots) \right) \\ = \\ \left( footprint(Trajectory_2, \dots) \right) \\ Otherwise \end{array} \right. \begin{array}{c} : true \\ \\ : false \end{array} \quad (6.9)$$

*Note.* Depending on *Movement Automaton's* movement set and *Avoidance Grid* parameters, there can be multiple *trajectories* which are equal.

**Coverage set:** Now it is possible to create set of unique *trajectory footprints* due to *footprint function* (eq. 6.8). Similarly there is a possibility to create *Reach set skeleton* containing unique trajectories, by using *equality function* (eq. 6.9). *Coverage set* is sufficient for now.

**Definition 5** (Coverage Set). Coverage set (6.10) is defined for Avoidance Grid and Reach Set pair as set of unique Trajectory footprints:

$$CoverageSet \left( \begin{matrix} AvoidanceGrid, \\ ReachSet \end{matrix} \right) = \left\{ footprint \left( \begin{matrix} Trajectory, \\ AvoidanceGrid \end{matrix} \right) : \begin{matrix} \forall Trajectory \\ \in ReachSet \end{matrix} \right\} \quad (6.10)$$

**Coverage set properties:** Trajectory footprint (eq. 6.8) is not *bijection*, neither *injection* for  $ReachSet \rightarrow CoverageSet$ . This implies following properties:

1. Equal *Reach Sets* in same *Avoidance Grid* have equal *Coverage Sets*.
2. Equal *Coverage Sets* does not imply *Reach Set* equality.
3. For two *Coverage Sets* there is a possibility to compare their member count to create coverage ratio.

The second *Property* gives us a preposition that there is a possibility of *Reach Set Reduction* without losing *Coverage*.

**Definition 6** (Coverage Ratio). Coverage Ratio is a ratio of Coverage Set Member Count between two Reach Sets. *Reach set* with lesser count of unique Trajectories is considered as *Reduced Reach Set*. *Reach set* with greater Count of unique Trajectories is considered as *Reference Reach Set*.

$$\begin{aligned} referenceCoverage &= |CoverageSet(ReferenceReachSet, AvoidanceGrid)| \\ reducedCoverage &= |CoverageSet(ReducedReachSet, AvoidanceGrid)| \\ CoverageRatio &= \frac{reducedCoverage}{referenceCoverage} \in [0, 1] \end{aligned} \quad (6.11)$$

*Note.* *Reference Reach Set* is usually *Full Reach Set* containing all possible trajectories in space contained by *Avoidance Grid*. In case *Full Reach Set* cannot be computed, *Avoidance Grid* is too large, most complex *Reach Set* is used as *Reference Reach Set*.

**Trajectory smoothness:** Trajectory other than straight line have some changes in *UAS* heading.

The goal is to minimize *Maneuvering* of *UAS*, because:

1. *Every Heading Change* needs to be reported to *UTM*.

2. *Sharp Maneuvering* can damage cargo/wear UAS.
3. *Often course changes* makes *Intruder prediction* harder for other Civil General Aviation.

For this purpose *Smoothness Metric* needs to be applied for *Reach Set* or *Trajectory*. In case of *Movement Automaton Control* two distinguish *Movement Sets* can be introduced: *Smooth* nad *Chaotic* movements set with following properties:

$$\begin{aligned}
 \text{MovementSet} &= \text{SmoothMovements} \cup \text{ChaoticMovements} \\
 \text{SmoothMovements} \cap \text{ChaoticMovements} &= \emptyset \\
 |\text{SmoothMovements}| > 0, \quad |\text{ChaoticMovements}| > 0
 \end{aligned} \tag{6.12}$$

Then *Smoothnes clasificator* for *Trajectory(initialState,buffer)* can be defined as *isSmooth* and *Smooth Movement Counter* function as *smoothCount* like follow:

$$\begin{aligned}
 \text{isSmooth}(\text{movement}) &= \begin{cases} \text{movement} \in \text{SmoothMovements} & : 1 \\ \text{movement} \in \text{ChaoticMovements} & : 0 \end{cases} \\
 \text{smoothCount}(\text{Trajectory}(\dots, \text{buffer})) &= \sum_{\forall \text{movement} \in \text{Buffer}} \text{isSmooth}(\text{movement}),
 \end{aligned} \tag{6.13}$$

**Definition 7** (Smoothness Rating for Trajectory). Smoothness for trajectory generated by Movement Automaton for some Initial State with some Movement Buffer, under assumption of Smooth and Chaotic Movement Set split (eq. 6.12), with existing classification and counter functionals (eq. 6.13) is given as follows:

$$\text{Smoothness}(\text{Trajectory}(\dots, \text{buffer})) = \frac{\text{isSmooth}(\text{Trajectory})}{\text{movementCount}(\text{Trajectory})} \in [0, 1] \tag{6.14}$$

For Trajectory with  $\text{buffer} = \emptyset$  Smoothness is given as 1.

### 6.4.3 Heuristic Trajectory Tree Building

**Motivation:** *Purpose of Navigation* is to move forward to *Goal Waypoint* in *Mission*. *Structure of Avoidance Grid* is designed to enable *forward* and *turning* maneuvers. The *Avoidance Grid* is organized in *Layers* characteristic by same distance from *Avoidance Grid Origin*.

Survey of motion planning algorithm was given in [4]. The ideal candidate for propagation algorithm is *Wave-front* algorithm propagating *Trajectory tree* through *Layers*. Due the *Avoidance Grid* onion like layers, there is possibility to implement turn maneuver through layers iterative and effectively .

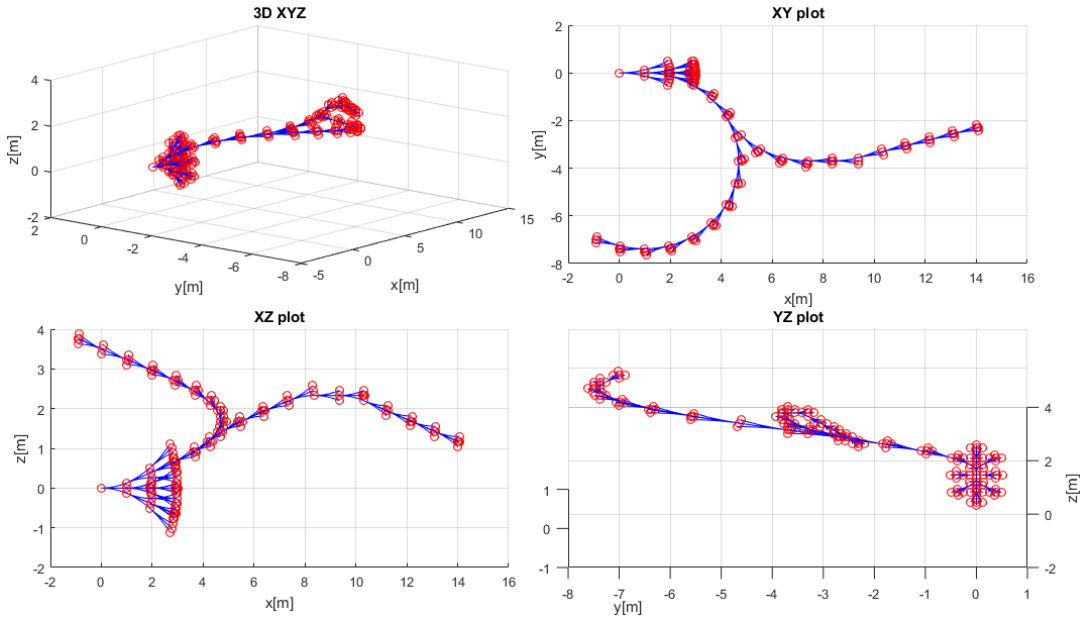


Figure 6.1: *Rapid Exploration tree as result of Constrained trajectory expansion.*

**Rapid Exploration Tree** (fig. 6.1) was selected, because it enables *Movement Automaton Utilization* and *Property Binding*. Similar approach was used for space exploration [5].

**The example** (fig. 6.1) shows a *Rapid Exploration Tree* in *Free Space* containing *Waypoint Navigation Path* and *Turn Away Path*. Both paths are starting in same *Root Node* (red circle) which was expanded with simple *Movement Automaton* (bunch of nodes originating from one node are showing way of expansion). The connection (blue line) between two nodes (red circles) represents *Trajectory portion* for *Executed Movement*.

**Rapid Exploration Tree Node** will contain following information:

1. *Initial state* - root entry point, used in state evolution calculation.
2. *Trajectory (state evolution)* - trajectory passing through *state space* in local coordinate frame of *Avoidance Grid*.
3. *Buffer* (applied movements) - ordered list of *executed movements* applied on *initial state* to obtain *state evolution*.
4. *Cost* - calculated for *state evolution* based on *predefined cost function*.
5. *Footprint* - ordered set of *passing cells* in *Avoidance Grid*.
6. *Parent Node Reference* - tree reference for parent node, not in case of *root node*.
7. *Other Bounded Properties* - value list of other properties, depending on *Expansion Constraints* and *Reachability* evaluation algorithm.

**Wave-front propagation of Rapid Exploration Tree** is given in (alg. 6.1).

The *Avoidance Grid* have UAS with *position*  $\in$  *Initial State* at the *origin*. The *Grid Layer* is a column ordered set of cells with same *Mean distance* from origin. *Grid Layers* are indexed from origin starting with 1, there is maximum of  $i \geq 1$  layers.

**Step: Initialization** contains base structure preparation like follows:

1. *Avoidance Grid* - Space containing *Reach set* (def. 2).
2. *Movement Automaton* - Used as *Predictor*, consuming *buffer* containing *Movements* to generate *Trajectory*(*initialState*, *buffer*).
3. *Reach Set* - tree consisting from *Wave-frontNodes* representing the end point of *Trajectory*(*initialState*, *buffer*) where each *Edge* represents *one Movement application*. The root is set as node containing *Initial State*.

Function *initializeReachSet*(*root*, *stack*, *grid*, *automaton*) will take the root and enforces *full wavefornt propagation* to *First Layer*.

**Step: Wave-front Propagation** is forced propagation of trajectories from layer  $i$  to layer  $i + 1$ . The process goes as follows:

1. *Selection of Feasible candidates* - function *[candidates, leftovers] = ExpansionConstraints.select(stack)* for working layer, row and cell selects *feasible trajectory nodes* ordered by *Cost function*. The *Example of Cost Function* can be *Trajectory Smoothness* (def. 7).
2. *Expansion of Candidates* - for each *candidate* function *candidate.expandNode (automaton)* is invoked. This function will expand *Candidate Node structure* by appending *Full Trajectory Tree Evolution* until each *Leaf Trajectory* reaches *Next Layer*. Simply put *Parent Node Node(initialState, buffer, cost, footprint)* *buffer* is appended by movements until the next layer is reached.
3. *Leftovers purge* - function *reachSet.purge(leftovers)* removes unexpanded *Nodes* leading to cell, effectively removing trajectories which does not lead to *next layer*.
4. *Append Reach Set* - function *reachSet.append(leafs)* puts newly created *Nodes (Trees)* into *Reach Set* structure. The *Wave-front Propagation* for one cell is finished.

**Step: After Layer Propagation Purge** is covered by function *reachSet.purgeSameFootprint()* which takes trajectories with same footprint and keeps some of them based on *Selection criteria*, more in (sec. 6.4.4, 6.4.5). *Pruning methods* over *Large Decision Trees* are *fast* and *viable* [6].



*Note.* *Reach Set* is usually computed *Prior the Flight* for some *Initial State* in *Local Coordinate Frame* in *right had coordinate frame* with  $X^+$  used as *main axis*.

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**Algorithm 6.1:** *Wave-front propagation of Rapid Exploration Tree* to form *Reach Set*.

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```

Input : Node(initialState,buffer= $\emptyset$ ,cost=0,footprint= $\emptyset$ ), AvoidanceGrid,
        ExpansionConstraints, MovementAutomaton(movementSet)
Output: ReachSet(AvoidanceGrid)

# Initialization Sequence;
grid=AvoidanceGrid, automaton=MovementAutomaton, root = Node;
reachSet = initializeReachSet(root,stack,grid,automaton);

# Main Expansion through, layers (i), rows (j), cells(k);
for layer(1...i) in grid do
    for row(1...j) in layer do
        for cell(1...k) in row do
            # apply selection criteria ;
            [candidates,leftovers] = ExpansionConstraints.select(stack);

            # collect expansions ;
            leafs = [];
            for candidate in Candidates do
                | leafs= [leafs, candidate.expandNode(automaton)];
            end
            reachSet.purge(leftovers);
            reachSet.append(leafs);
        end
    end
    reachSet.purgeSameFootprint();
end

```

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#### 6.4.4 Coverage-Maximizing Reach Set Aproximation

**Motivation:** Design of calculation method for *Reach Set Approximation* guarantying high *Maneuverability*.

**Background:** There is *Coverage Ratio* property of *Reach Set* (def. 6). It has been shown that creating *Reach Set* via *greedy approach* is not feasible due the *Scaling Factor. Contracted Expansion* (sec. 6.4.3) is enabling to apply selection criteria while building *Reach Set* in given *Cell*.

The *Cell*  $cell_{i,j,k}$  has a center and walls from UAS viewpoint: front wall , back wall (for  $layer > 1$ ), top wall, left wall, right wall, bottom wall. It is expected that trajectory leading close to one cell walls will continue to different cell, increasing chance to obtain

more *Unique Footprints*.

**Expansion Constraint Function Implementation** (alg. 6.2) is based on simple principle: *Select candidate Nodes which are closest to outer walls of Cell, with unique footprint.*

**Tuning Parameters** : *Proximity to Cell outer wall* gives good chances to break into other rows or columns in *Avoidance Grid*. *Unique footprint* guarantees future *Unique Footprint* after appending Trajectory by *Movement application*.

1. *Considered Footprint Length* - how much last cells in footprint should be considered in unique path track, minimal value 1, default value 3, maximal value  $\infty$ . If you want to generate non redundant trajectories use  $\infty$ , it will consider full footprint.
2. *Spread Limit* - upper limit of candidates which are going to be select for further expansion, minimal value 1, default value *Count of unique Moves in Movement set*, maximal value  $\infty$ . If more than default values is selected the algorithm will generate *redundant trajectories*. If less is selected then some trajectories are omitted and *Coverage Rate* decreases sharply.

**Step: Initialization** initialization of *candidate* array (return value), *leftovers* array (return Value). Node array *passing* is populated with *Nodes* which represents *end node of Trajectory* and the tip of *trajectory is constrained in*  $cell_{i,j,k}$ .

**Step: Evaluate best trajectories with unique Footprints** following steps are executed:

1. *Best Performance Map* is created with *footprint* as key set element to ensure footprint uniqueness.
2. *Wall distance for test node* is calculated as a closest trajectory portion distance to *top, bottom, left, right* wall of cell  $cell_{i,j,k}$
3. *Footprint for test node* is created with maximal length given by *Footprint Length* tuning parameter.
4. *Existence and Performance Test* is executed to ensure that best performing node is selected. If there is not key entry in the *Best Performance Map*, then new entry for *Test Node* is created. If there is key entry, the performance of *Old Node* and *Test Node* is compared and better is stored.

**Step: Select candidates** is executed on *Best Performance Map* records using *Wall distance* as pivot parameter, ordering by closest proximity and limited by *Search Limit* tuning parameter. The *Leftovers* are difference set between *Passing Nodes* and *Candidate Nodes*.

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**Algorithm 6.2:** Expansion Constraint function for *Chaotic Reach Set Approximation*

---

```

Input : Node[] stack, Cell celli,j,k
Tuning Parameters: int+ footprintLength, int+ spreadLimit
Output : Node[] candidates, Node[] leftovers

# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = celli,j,k.getFinishingTrajectories(stack);

# Select best performing trajectories with unique footprint;
Map<Footprint,Node> bestPerformanceMap;
for Node test ∈ passing do
    wallDistance= test.minimalDistanceToWall(celli,j,k);
    footPrint = test.getFootprint(lastCells = footprintLength);
    if bestPerformanceMap.contains(footPrint) then
        old = bestPerformanceMap.getByKey(footprint);
        oldPerformance= old.minimalDistanceToWall(celli,j,k);
        if oldPerformance > wallDistance then
            bestPerformanceMap.setByKey(footprint,test);
        end
    else
        bestPerformanceMap.setByKey(footprint,test);
    end
end

# Select best performing nodes up to spreadLimit count;
candidates = bestPerformanceMap.select(count =
    spreadLimit).orderBy('wallDistance','Ascending');
leftovers = passing - candidates;
return [candidates,leftovers]

```

---

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.2). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*.

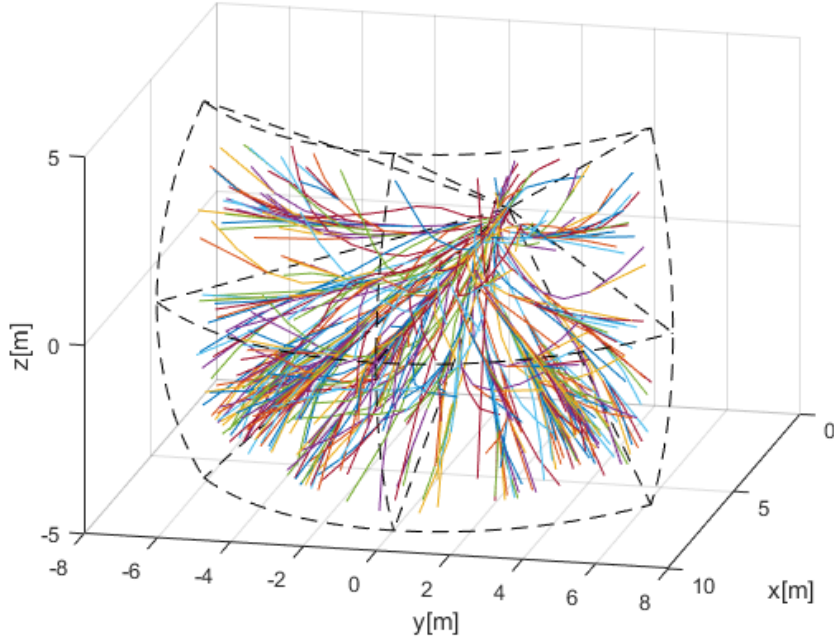


Figure 6.2: *Chaotic reach set approximation.*

**Pros and Cons:** It can be seen from example (fig. 6.2) that *Chaotic Reach Set Approximation Method* (alg. 6.2) generates a lot of *turning and shaky trajectories*.

*High Coverage Ratio* ( $\sim 0.9$ ) is provided, while keeping *medium node count*. The calculation complexity scales linearly with grid size. The *upper limit of trajectories* is given as follow:

$$\begin{aligned} \text{countTrajectories}(\text{ReachSet}) \leq & \text{layerCellCount} \times \text{spreadLimit} \\ & \times \text{size}(\text{Movements}) \quad (6.15) \end{aligned}$$

The *upper limit of nodes* is given as follow:

$$\begin{aligned} \text{countNodes}(\text{ReachSet}) \leq & \text{layerCount} \times \text{layerCellCount} \\ & \times \text{size}(\text{Movements}) \times \text{spreadLimit} \quad (6.16) \end{aligned}$$

*Absence of Smooth Trajectories* disqualifies *Chaotic Reach Set Approximation* to be used for *Navigation*. This type of reach set is feasible for *Avoidance*, because it contains variety of maneuvers.

#### 6.4.5 Turn-Minimizing Reach Set Approximation

**Motivation:** Imagine having an *Avoidance Grid* like (fig. ??). There is a need of *Reach Set Approximation* which will have *Smooth Trajectories* (def. 7) going nearby *cell centers*.

**Background:** The *Smoothness Rating for Trajectory* (def. 7) uses two distinct sets *Smooth Movements* and *Chaotic Movements* (eq. 6.12) which are defined for our *Movement Automaton* (sec. ??) like follow:

$$\begin{aligned} \text{SmoothMovements} &= \{\text{Straight}\} \\ \text{ChaoticMovements} &= \text{Movements} - \text{SmoothMovements} \end{aligned} \quad (6.17)$$

*Smooth Movements* contains only *Straight* movement, because others are considered as extreme turning movements. *Smooth Movements* should contain only direct flight movements or slight heading correction. *Chaotic Movements* set is supplement of *Movement Automaton's Movement Set*.

The *Avoidance Grid* (fig. ??) cell centers for fixed indexes  $j_{fix}$ ,  $k_{fix}$  are linearly aligned with *initial state*. That means that cell centers of cells  $cell_{1,j_{fix},k_{fix}}, \dots, cell_{i,j_{fix},k_{fix}}$ , where  $i$  is count of *layers* lies on one line. If the trajectory can achieve *cell center* on some *layer* only minor trajectory corrections are required to stay on given line. This type of trajectory gives us following advantages:

1. *Minimal steering at beginning* - the minimal steering is advantageous in *Controlled Airspace* because is diminishing the amount of communication to *UTM Service*.
2. *Additional safe space in Linear segment* - once the *center of cell* is reached, *Trajectory* sticks to line between cell centers. Each point on this line has *maximal distance* to outer walls of cell. This gives us extra space given as minimum of distance between *UAS position* and *Outer cell walls*.

**Expansion Constraint Function Implementation** (alg. 6.3) is based on simple principle: *Select candidate Nodes which are closest to Cell center, with unique footprint.*

*Note.* *Cell center* can be closely reached by *smooth movement* from previous cell or *chaotic movement* from neighbouring cell from current or previous layer. These trajectories are usually equivalent in *Smoothness*.

**Tuning Parameter:** *Proximity to Cell Center* gives a good chance to keep trajectory smooth or *smooth after one correction maneuver*. It has been mentioned that *Cell Center* can be reached by various trajectories. In this method full footprint length is always considered, therefore only one tuning parameter can be offered:

1. *Spread Limit* - upper limit of candidates which are going to be selected for further expansion, minimal value 1, default value *Count of unique Moves in Movement set*, maximal value  $\infty$ . If maximal value  $\infty$  is selected, algorithm will generate skeleton of *Reach Set* with full Coverage and with the smoothest *Trajectories*.

**Step: Initialization** sets candidate *Nodes* as empty set, leftover *Nodes* as empty set. and selects all *Nodes* from *Stack* which represents *Finishing Trajectories* in working cell

$cell_{i,j,k}$ .

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**Algorithm 6.3:** Expansion Constraint function for *Harmonic Reach Set Approximation*

---

```

Input : Node[] stack, Cell celli,j,k
Tuning Parameters: int+ spreadLimit
Output : Node[] candidates, Node[] leftovers

# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = celli,j,k.getFinishingTrajectories(stack);

# Select unique smoothest trajectories;
Map<Buffer,Node> bestPerformanceMap;
for Node test ∈ passing do
    centerDistance= test.getPerformance(celli,j,k);
    footprint = test.getFootprint();
    if bestPerformanceMap.contains(footprint) then
        old = bestPerformanceMap.getByKey(footprint);
        oldPerformance= old.getPerformance(celli,j,k);
        if oldPerformance > centerDistance then
            bestPerformanceMap.setByKey(footprint,test);
        end
    else
        bestPerformanceMap.setByKey(footprint,test);
    end
end

# Select best performing nodes up to spreadLimit count;
candidates = bestPerformanceMap.select(count =
    spreadLimit).orderBy('cellCenterDistance','Ascending');
leftovers = passing - candidates;
return [candidates,leftovers]

```

---

**Step: Evaluate smoothest trajectories with unique Footprints** is implemented as *multi-criteria filtration*.

*First criterion* is *distance to Cell Center* which is penalized by trajectory *smoothness rate* implemented in method *Node.getPerformance(Cell cell<sub>i,j,k</sub>)* defined as follow.

$$getPerformance(Node, Cell) = \frac{distance(Node.Trajectory, Cell.Center)}{SmoothnessRate(Node.Trajectory)} \quad (6.18)$$

Distance of *Trajectory* is *enumerator*, because its considered as *base value* and is defined in interval  $[0, maximalWallDistance]$ . The *Smoothness Rate* is in denominator, because

it is a penalization coefficient defined in interval  $[0, 1]$ .

*Second criterion is trajectory uniqueness* This is provided by *Best Performance Map*, where best performing *Node* belongs to one unique *trajectory footprint*. The implementation is identical to *chaotic reach set expansion* (alg. 6.2).

**Step: Select candidates** is executed on *Best Performance Map* records using *Penalized Cell Center Distance* as pivot parameter, ordered in ascending order and limited by *Spread Limit* tuning parameter. The *Leftovers* are difference set between *Passing Nodes* and *Candidate Nodes*.

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.3). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*. The *Spread Limit* in this case was set to 9 which is *Size of the Movement Set*.

*Note.* Please note *Trajectories* are organized in bundles going around *Cell Centers* smoothly. Most of the steering maneuvers are executed at the *beginning* of *Avoidance Grid*.

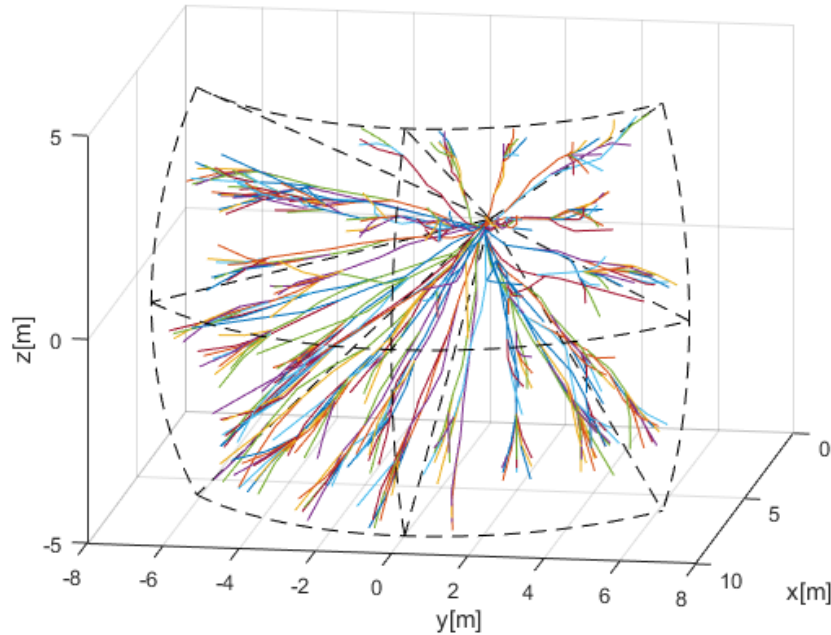


Figure 6.3: *Harmonic reach set approximation.*

**Pros and Cons:** It can be seen from example (fig. 6.3) that *Harmonic Reach Set Approximation Method* (alg. 6.3) generates *smooth evenly spread trajectories*.

High smoothness ratio ( $\geq 0.9$ ) is provided, while keeping low node count for UAS systems. The calculation complexity scales linearly with grid size. The upper limit of trajectories is given as follow:

$$\begin{aligned} \text{countTrajectories}(\text{ReachSet}) \leq \text{layerCellCount} \times \text{spreadLimit} \\ \times \text{size}(\text{Movements}) \end{aligned} \quad (6.19)$$

The *upper limit of nodes* is given as follow:

$$\text{countNodes}(\text{ReachSet}) \leq \text{layerCount} \times \text{layerCellCount} \times \text{spreadLimit} \quad (6.20)$$

Absence of *High Coverage Ratio* disqualifies *Harmonic Reach Set Approximation* to be used for *Emergency Avoidance*. This type of *Reach Set* is feasible for *Open Space Navigation* or *Controlled Airspace Navigation*. Its low turning rate in contained *Trajectories* are desired for such tasks.

#### 6.4.6 ACAS-X like Reach Set Approximation

**Motivation:** The implementation of *ACAS-Xu* behavior in DAA system will be mandatory for *National Airspace System Integration* in United spaces [7].

Implementation of ACAS-Xu like behaviour increase usability of approach, if it can be achieved without major concept changes.

**Background:** The *ACAS-Xu* system on operational level has been described in [8]. The *Policy for Collision Avoidance* proposal has been given in [9].

Some behavioural patterns can be encoded into *Reach Set*. ACAS-Xu navigation part is basically *Look-up table of Maneuvers for Allowed Separations*.

The *Evasive Maneuver* selection process in ACAS-Xu is similar to our approach: *Select most energy efficient maneuver in compliance with space-time constraints*. ACAS-Xu intruder model is similar to our *Body Volume Intersection Model* (app. ??). The *ACAS-Xu* defines following base separations:

1. *Horizontal* - movements on *Horizontal Plane* in *Global Coordinate System*.
2. *Vertical* - movements on *Vertical Plane* in *Global Coordinate System*.

There are allowed custom separations which can be used, for further experimentation:

1. *Slash* - movement on  $+45^\circ$  *Tilted Plane to Horizontal Plane* in *Global Coordinate System*.
2. *Backslash* - movement on  $-45^\circ$  *Tilted Plane to Horizontal Plane* in *Global Coordinate System*.



For given *Movement Automaton* implementation (sec. ??) the separations are given as follow:

$$\begin{aligned}
 Horizontal &= \{Straight, Left, Right\} \\
 Vertical &= \{Straight, Up, Down\} \\
 Slash &= \{Straight, UpLeft, DownRight\} \\
 Backslash &= \{Straight, UpRight, DownLeft\}
 \end{aligned} \tag{6.21}$$

For each  $Node(\dots, buffer)$  and each *separation* there is a evaluation function *isSeparation* which decides, if *Trajectory* defined by node buffer is made up only from *Separation* movements. The function *isSeparation*(...) is defined like:

$$isSeparation(buffer, separation) = \begin{cases} \forall movement \in buffer, & : true \\ movement \in separation & \\ otherwise & : false \end{cases} \tag{6.22}$$

Following *Separation Modes* can be defined with given *separations*:

1. *Horizontal* (ACAS-X defined mode) containing *horizontal* separation.
2. *Vertical* (ACAS-X defined mode) containing *vertical* separation.
3. *Horizontal-Vertical* (ACAS-X defined mode) containing *horizontal*, *vertical* separations.
4. *Full* (custom defined mode) containing all *Separation Modes*.

*Note.* Every separation modes generates 2D trajectories set on *Respective plane*. There is no need for *Tuning parameters* for further *Expansion Constraint*.

**Expansion Constraint Function Implementation** (alg. 6.4) is based on simple principle: *Select only candidate Nodes which Trajectories have at least one desired Separation Mode.*

**Step: Initialization** sets candidate *Nodes* as empty set, leftover *Nodes* as empty set, and, select all nodes form *stack* which represents *Finishing Trajectories* in working  $cell_{i,j,k}$ ,

**Step: Candidate Selection Process** is evaluated for each *test Node* from *passing Node Set*.

For each *applicable separation*, given as input parameter *separations*, The test function *isSeparation* (eq. 6.22) is applied:

1. If *test Node* trajectory belongs to at least one allowed separation it is added to candidates set.

2. Else is added to *Leftovers*.

*Note.* Separation sets (eq. 6.21) are not *exclusive sets* in *Movement Automaton* domain. One *Trajectory* contained by Node can belong to multiple *Separations*.

---

**Algorithm 6.4:** Expansion Constraint function for *ACAS-like Reach Set Approximation*

---

**Input** : Node[] stack, Cell cell<sub>*i,j,k*</sub>, Separation[] separations

**Tuning Parameters:** *None* :  $\emptyset$

**Output** : Node[] candidates, Node[] leftovers

# Initialize structures;

Node[] candidates = [], Node[] leftovers=[];

Node[] passing = cell<sub>*i,j,k*</sub>.getFinishingTrajectories(stack);

# Select nodes containing trajectories with usable separations;

**for** Node test  $\in$  passing **do**

**for** separation  $\in$  separations **do**

        # Get separations for Node;

        Separations[] nodeSeparations = test.getSeparations();

        # If trajectory given by buffer is on Separation plane;

**if** *isIn(isSeparation(test.buffer, separation)(6.22))* **then**

            candidates.append(test);

**end**

**end**

    # If there was no applicable separation, throw Node away;

**if** test  $\notin$  candidates **then**

        leftovers.append(test);

**end**

**end**

# Return results;

**return** [candidates, leftovers]

---

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.4). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*.

*Full* separation mode given in (fig. 6.4a). *Horizontal-Vertical* separation mode, used in original *ACAS-Xu* testing [8], given in (fig. 6.4b). *Horizontal* separation mode given in (fig. 6.4c) is usually used by planes. *Vertical* separation mode given in (fig. 6.4d) is usually used by copters.

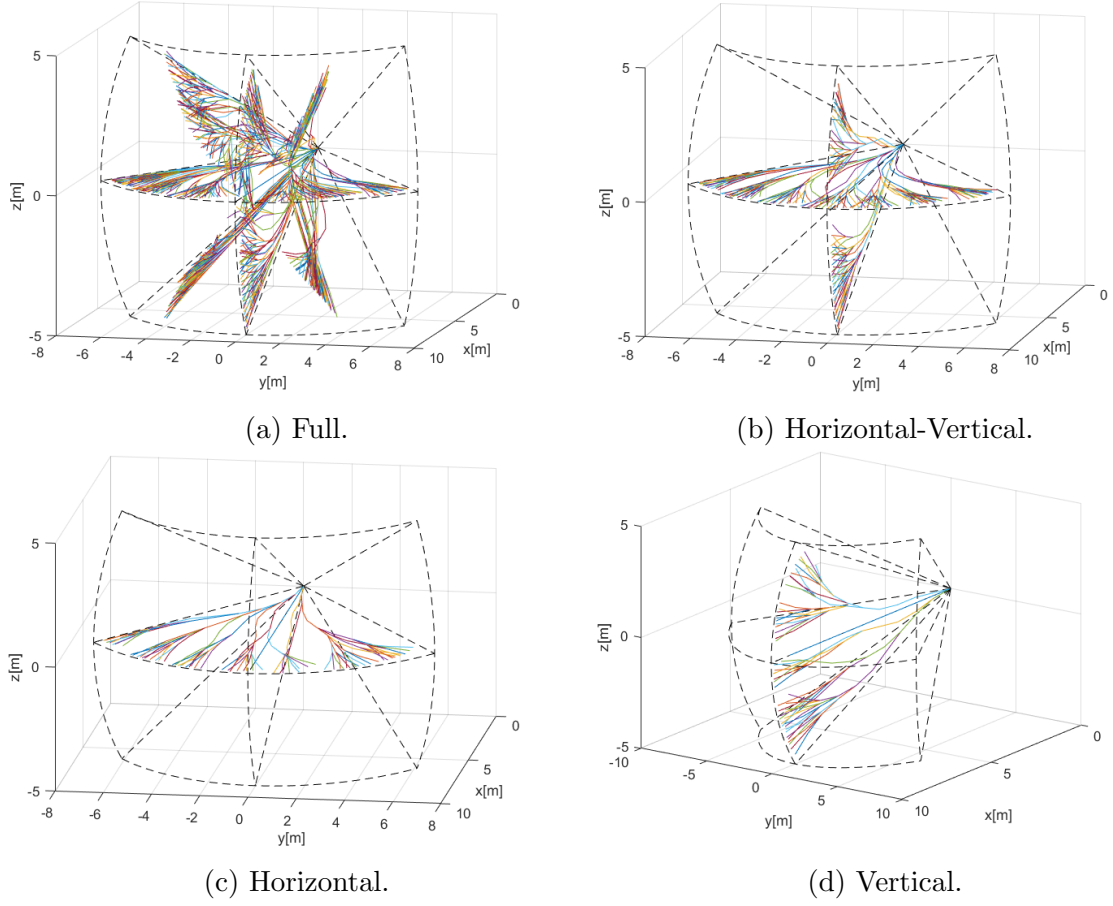


Figure 6.4: ACAS-X imitation *reach set* approximation for various *separation modes*.

**Pros and Cons:** It can be seen from examples (fig. 6.4) that *ACAS-like Reach Set Approximation Method* (alg. 6.4) generates full reach set for 2D plane located in 3D space.

The *Reach Set* contains trajectories with *high coverage ratio* and *high smoothness rating* for selected 2D separation plane. Overall performance compared to full 3D reach sets (sec. 6.4.4, 6.4.5 6.4.7) is poor.

The *node* and *trajectory* count boundary was not implemented. It is common knowledge that *2D* avoidance sets does not require scaling [8]. Otherwise trajectory footprint mechanism like in *Harmonic Reach Set Approximation* (alg. 6.3) can be introduced.

This reach set implements *Planar-Separation* as native feature, it can be used for both *navigation* and *avoidance* tasks in *Controlled Airspace*. For *Non-controlled Airspace* there are far more superior *Combined Reach Set* (sec. 6.4.7).

### 6.4.7 Combined Reach Set Approximation - Tree Merge

**Motivation:** Harmonic reach set (sec. 6.4.5) is *efficient* for *Navigation in Controlled Airspace*. Chaotic reach set (sec. 6.4.4) is good for *Emergency avoidance*. The need to differentiation between *Navigation* and *Emergency Avoidance* mode is necessary in *Controlled Airspace*. but not in *Non-controlled Airspace*. The combination of *Harmonic* and *Chaotic* reach sets is obvious solution.

*Automatic mode switch* can be provided by combination of *Navigation Reach Set* and *Avoidance Reach Set* with elevated cost function. Overall having a method to merge multiple trees would be beneficial.

**Background:** If two *Reach Set Approximation* were calculated for same *Avoidance Grid* and *Initial State*, using same *Movement Automaton* and *UAS model* are possible to merge.

The *Reach Set Approximation* is tree with *Root Node* in *initail state* with movement buffer =  $\emptyset$ . The *movement buffer* in each node can be used as *route trace* during merging procedure. The example two reach set merge can be given as follow, where only *latest* applied movement is taken into account.

$$\left[ \begin{array}{l} \text{First Reach Set} \\ \emptyset \rightarrow \left\langle \begin{array}{l} \text{left} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \\ \emptyset \end{array} \right. \\ \text{Second Reach Set} \\ \emptyset \rightarrow \left\langle \begin{array}{l} \emptyset \\ \text{right} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \end{array} \right] \rightarrow \left[ \begin{array}{l} \text{Combined Reach Set} \\ \emptyset \rightarrow \left\langle \begin{array}{l} \text{left} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \\ \text{right} \rightarrow \left\langle \begin{array}{l} \text{left} \\ \text{right} \end{array} \end{array} \right. \end{array} \right] \quad (6.23)$$

*First Reach Set* contains two trajectories given by buffers  $\{\text{left}, \text{left}\}$  and  $\{\text{left}, \text{right}\}$ . *Second Reach Set* contains two trajectories given by buffers  $\{\text{right}, \text{left}\}$  and  $\{\text{right}, \text{right}\}$ . The *Combined Reach Set* contains all four trajectories.

*Note.* The combined tree [10] does not need to have combined amount of original *Reach Sets* trajectories. There can be *Duplicity* which means that any bounded property like *Cost* must be *calculated* again.

**Combined Reach Set Calculation Function** (alg. 6.5) is implemented as function *NodecombinedReachSet(...)* which takes root Node with *initial State*, *Avoidance Grid* and respective parameters for each calculation method. *Harmonic spread* for *Harmonic Reach set calculation* and *Chaotic Spread, Footprint Length* for *Chaotic Reach set calculation*.

*Separate Reach Sets* are calculated using *Wave-front propagation* (alg. 6.1) using respective *Constrained Expansion* functions for *Harmonic* (alg. 6.3) and *Chaotic* (alg. 6.2) reach sets.

*Combined Reach Set* is created using *Node mergeTree(...)* function. Because different cost function or *Bounded Parameters Calculation* may be applied on *Original Reach Sets*.

*Cost* for *each node* needs to be recalculated due to original reach sets disparity. Function *combined.applyCostFunction()* will recalculate the new cost for each node.

The Goal is to have penalization for *Chaotic behaviour*, implementation of *Automatic Mode Switch* can be done like follows:

1. *Calculate Normal Cost* for Node  $Cost(Node)$  for associated trajectory:  
 $Cost(Node.Trajectory)$ .
  
2. *Calculate Penalization for Chaotic Behaviour*, calculate *Smoothness Rating for Trajectory* (def. 7) in interval  $[0, 1]$ , introduce penalization with base 100%.

The final  $Cost(Node)$  function is applied on each *Combined Reach Set Node* and look like follows:

$$Cost(Node) = Cost(Node.Trajectory) \times \dots \times (1 + (1 - SmoothnessRate(Node.Trajectory))) \quad (6.24)$$

**Tree Merge Function**  $mergeTree(\dots)$  implements *Outer Join* operation on two trees. Example was given in (eq. 6.23). Function is applied on *root Node* iterating over

*Movements in Movement Set, because Movement is pivot.*

---

**Algorithm 6.5:** Reach Set Merge Function and Combined Reach Set calculation

---

```

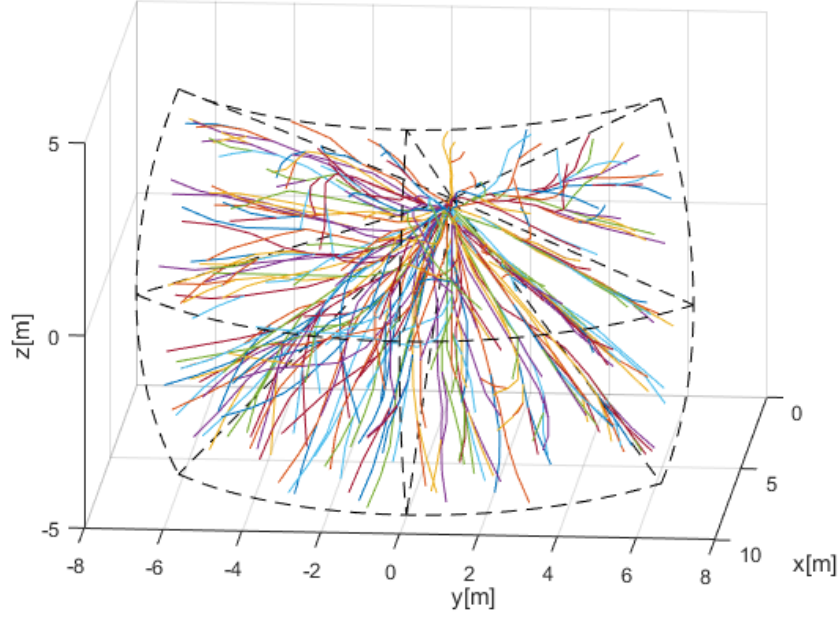
# Tree merge function;
Node mergeTree(Node firstNode, Node secondNode)
|
|   # Try to copy reference node or return null;
|   Node referenceNode = (firstNode?:(secondNode?: return null));
|   Node merged = new Node(referenceNode);
|   merged.leafs= [];
|
|   # Try to fetch movement nodes if exist in any sub tree;
|   for movement ∈ Movements do
|       |   firstLeaf = firstNode.getLeafFor(movement);
|       |   secondLeaf = secondNode.getLeafFor(movement);
|       |   newLeaf = mergeTree(firstLeaf,secondLeaf);
|       |   if newLeaf ~= null then
|       |       |   merged.leafs.append(newLeaf);
|       |   end
|   end
|   return merged
|
# Combined Reach Set calculation function;
Node combinedReachSet(Node root, AvoidanceGrid grid,int+ chaoticSpread,
|   int+ harmonicSpread, int+ footprintLength)
|   |   Node chaotic = chaoticReachSet(root,grid, footprintLength,chaoticSpread);
|   |   Node harmonic = harmonicReachSet(root,grid, harmonicSpread);
|   |   Node combined = mergeTree(chaotic,harmonic);
|   |   combined.applyCostFunction();
|   return combined

```

---

**Example:** for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range*  $[-45^\circ, +45^\circ]$ , *Horizontal Cell Count 7*, *Vertical range*  $[-30^\circ, +30^\circ]$ , and *Vertical Cell Count 5*. Is given in (fig. 6.5). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*. The *Chaotic Spread* was set to 8, *Footprint Length* to 3 and *Harmonic Spread* to 1.

*Note.* Notice there are typical trajectories from both *Harmonic* (fig. 6.3) and *Chaotic* (fig. 6.2) *Reach Set Approximations*.

Figure 6.5: *Combined reach set approximation.*

**Pros and Cons:** It can be seen from example (fig. 6.5) that *Combined Reach Set Approximation* (alg. 6.5) contains both types of maneuvers. *Cheaper Smooth* for navigation and *More Expensive Chaotic* for *Emergency Avoidance*. The upper limit of trajectories is given as follow:

$$\begin{aligned} \text{countTrajectories}(\text{ReachSet}) \leq & \text{countTrajectories}(\text{Chaotic}) \\ & + \text{countTrajectories}(\text{Harmonic}) \end{aligned} \quad (6.25)$$

The *upper limit of nodes* is given as follow:

$$\text{countNodes}(\text{ReachSet}) \leq \text{countNodes}(\text{Chaotic}) + \text{countNodes}(\text{Harmonic}) \quad (6.26)$$

*Harmonic Reach Set* is ideal for *Non-controlled Airspace* missions, because it contains *Automatic Mode Switch* between *Navigation* and *Emergency Avoidance*.





# Bibliography

- [1] Franco Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.
- [2] John Birmingham and Peter Kent. Tree-searching and tree-pruning techniques. In *Computer chess compendium*, pages 123–128. Springer, 1988.
- [3] Alojz Gomola, João Borges de Sousa, Fernando Lobo Pereira, and Pavel Klang. Obstacle avoidance framework based on reach sets. In *Iberian Robotics conference*, pages 768–779. Springer, 2017.
- [4] Chad Goerzen, Zhaodan Kong, and Bernard Mettler. A survey of motion planning algorithms from the perspective of autonomous uav guidance. In *Selected papers from the 2nd International Symposium on UAVs, Reno, Nevada, USA June 8–10, 2009*, pages 65–100. Springer, 2009.
- [5] Catherine Plaisant, Jesse Grosjean, and Benjamin B Bederson. Spacetree: Supporting exploration in large node link tree, design evolution and empirical evaluation. In *Information Visualization, 2002. INFOVIS 2002. IEEE Symposium on*, pages 57–64. IEEE, 2002.
- [6] John Mingers. An empirical comparison of pruning methods for decision tree induction. *Machine learning*, 4(2):227–243, 1989.
- [7] Jay Shively. Uas integration in the nas: Detect and avoid. 2018.
- [8] Mike Marston and Gabe Baca. Acas-xu initial self-separation flight tests, 2015.
- [9] Kyle D Julian, Jessica Lopez, Jeffrey S Brush, Michael P Owen, and Mykel J Kochenderfer. Policy compression for aircraft collision avoidance systems. In *Digital Avionics Systems Conference (DASC), 2016 IEEE/AIAA 35th*, pages 1–10. IEEE, 2016.
- [10] Patrick O’Neil, Edward Cheng, Dieter Gawlick, and Elizabeth O’Neil. The log-structured merge-tree (lsm-tree). *Acta Informatica*, 33(4):351–385, 1996.