6.4 Reach Set Approximation

Motivation: *Reach set* is a strong tool for *Obstacle Avoidance* because it contains all possible *avoidance maneuvers*. The current implementations (sec. **??**) have the following flaws:

1. *Realistic approximation* - *nonlinear systems* or *heavily constrained systems* cannot be approximated well by *linear continuous-time Reach Sets*.
2. *Finite count of possibilities* - continuous-time *Reach Set* contains infinite possibilities for *avoidance maneuvers*; the DAA system demands conflict resolution in finite-time.
3. *Computationally feasible data structures* - binding related properties seem problematic because *continuous- time reach sets* do not a have unique identifier of maneuver, trajectory nor segment.

Proposed Solution Features: Our Reach set Estimation method will provide the following features:

1. *System Control Interface* - implemented via *Movement Automaton*, requiring only a *discrete command chain* to approximate system behavior.
2. *Finite count of possibilities* - finite number of elements in *Reach set* will enable *scalable* calculation.
3. *Computationally feasible data structures* - approximation of Reach set as a set of trajectories, each trajectory can be split into a finite number of segments. Each element will have a unique identifier enabling both-side property binding.
4. *Computationally feasible data-structures* - some specific behavior, like horizontal/vertical separation, or maneuver shape can be encoded into different types of reach set approximation algorithms.

6.4.1 Trajectory Set Approximation of Reach Set

Discretization of Reach set: There is a need for a discrete finite *Reach Set approximation* to enable *Avoidance Strategy Evaluation* in finite-time. Replacing *Continuous Control Set Inputs(t)* by *Movement Automaton* is feasible:

Definition 1 (Reach set Approximation by Movement Automaton)**.** A trajectory *(def.* **??***) for system* $\dot{state} = f(time, state, input)$ *under control of the movement automaton* $\mathcal{MA}$ *is given as execution of movement buffer (def.* **??***) with an initial state of system* $state_0$*. Therefore notation* $Trajectory(state_0, buffer)$ *is used.*

The Complete Reach Set *(6.1) for system with initial state* $state_0$ *with existing* control strategy $control(time) \in Controls(time)$. *for time* $\tau > time_0$.

$$ReachSet(\tau, time_0, state_0) = \bigcup \{state(s) : control(s) \in Controls(s), s \in (time_0, \tau]\} \tag{6.1}$$

The Reach Set Approximation by Movement Automaton *(6.2) of the system under the control of the movement automation* $\mathcal{MA}$ *consist from the set of trajectories* $Trajectory$ ($state_0$, $Buffer$), *which are executed in constrained time* $\tau > time_0$.

$$ReachSet(\tau, time_0, state_0) = \left\{ Trajectory(state_0, buffer) : \begin{matrix} duration(buffer) \\ \leq \\ (time_0 - \tau) \end{matrix} \right\} \tag{6.2}$$

Note. Reach Set Approximation (def. 1) is a subset of *Full Reach Set* (def. **??**) in continuous space $\mathbb{R}^n$ it inherits all important properties, like *Invariance* [1].

Discretization of *Reach Set* have been achieved leaving us with a *finite count* of *Trajectories*, instead of *Infinite subspace or* $\mathbb{R}^N$

Approximated Reach Set Containment: The *Approximated Reach Set* introduced in (def. 1) is constrained only by *future expansion time* τ. UAS makes space assessment in *Avoidance Grid*. There is no point to consider Trajectories outside of *Avoidance Grid*

Definition 2 (Contained Approximated Reach Set)**.** *For a pair (*$state_0$, $AvoidanceGrid_0$*) at time* $time_0$ *and* prediction horizon $\tau = \infty$ *there is* Contained Reduced Reach Set:

$$ReachSet \begin{pmatrix} time_0, \\ state_0, \\ AvoidanceGrid_0 \end{pmatrix} = \left\{ \begin{matrix} Trajectory(\dots) \\ \in \\ ReachSet(6.2) \end{matrix} : \begin{matrix} \forall segment \in AvoidanceGrid_0, \\ segment \in Trajectory(\dots) \end{matrix} \right\} \tag{6.3}$$

Properties: Container Approximated Reach Set *contains only trajectories where all segments belong to* Avoidance Grid, *there are following functions:*

1. *The* membership function *for any* Trajectory *in* Constrained Reduced Reach set *returns* Ordered Set *of* Passing Cells.

2. *The* cost function *for any* Trajectory Portion *in* Constrained Reduced Reach Set *return* Cost of Execution

Passing cell: Cell *of* Avoidance Grid *which has some intersection with Trajectory.*

Note. Contained Reduced Reach Set (eq. 6.3) which is contained in the *Avoidance Grid* and have a *Membership Function* enable Property transition between Reach set and *Avoidance grid*.

Example: Visibility from cells along *Trajectory* can be gathered to calculate *Trajectory's* feasibility.

Reach Set Pruning: There is a need to implement *Set Difference* between *Reach Set* and *Constraint Set*. Constraint Set can be an *Obstacle Set* from *Known World* (sec. **??**) and other different constraints.

Reach Set Trajectory Tree: (6.4) *Any Reach Set* where *Control Strategy Constraint* is implemented as *Movement Automaton*, with defined *Movements* set and for single initial $state_0$. The *Reach Set* is given as discrete tree with root $Trajectory(state_0, \varnothing)$.

$$ReachSet(state_0, \dots) = \left\{ Trajectory(state_0, buffer) : \begin{matrix} buffer \in Movements^i, \\ i \in \{1, \dots, k\} \end{matrix} \right\} \quad (6.4)$$

For each *Trajectory Segment*, there exists *intersection function* which evaluates as true if there exists at least one point in *Segment* which belongs to *Constraint Set*. Formally:

$$intersection(segment, Set) : \begin{cases} \exists point \in segment, \\ \ point \in Set & : true \\ Otherwise & : false \end{cases} \quad (6.5)$$

Definition 3 (Pruned Reach Set)**.** *For* Reach set *represented as* Trajectory Tree *(eq. 6.4) and some constraint set (Set) where exist* intersection function *(eq. 6.5). The* Pruned Reach set *is given as follows:*

$$Prune(ReachSet, Set) = \left\{ Trajectory(\dots) : \begin{matrix} \forall segment \in Trajectory, \\ \neg intersection(segment, Set) \end{matrix} \right\} \quad (6.6)$$

Note. Pruning(def. 3) [2] is applied multiple times for various *Constraints Set*.

Example of *Approximated Reach set Calculation* (def. 1), *Reach Set Containment* (def. 2), and, *Pruning* is given in [3].

6.4.2 Distinctive Properties of the Trajectories

Motivation: The need to Make *Reach Set* scalable approach. This may be a problem due to the *Expansion rate*. *Reach set* represented as a *Trajectory Tree* (eq. 6.4) for Avoidance Grid with $layer - count$ and Movement automaton with $movement - count$, the *Node count* is given as:

$$1 + \left(\sum_{i \in \{1 \ldots layerCount\}} (movementCount)^i \right) \tag{6.7}$$

This scaling is not feasible for *Avoidance Grid* with many layers (< 10) or *Movement Set* with many movements (< 9). There is a need for *Reduced Reach set calculation.*

Performance Criteria: The scaling factor (eq. 6.7) shows that there are going to be many trajectories. The main point is that not every trajectory in *Reach Set* is giving us *maneuverability advantage.* Our expectations lie in following *Performance Requirements*:

1. *Reach set* must *Cover* maximum of the *possible unique maneuvers* in *Avoidance Grid.*

2. *Trajectories* in *Reach Set* should be smoothest possible to prevent cargo damage / UAS wear.

Trajectory footprint: Discrete space of *Avoidance Grid* is organized in cells. The *cell* is a minimal space portion accessible by *property binding.* There is a need to know if two trajectories contribution to *Maneuverability* in this environment.

Each trajectory passes through space in *Avoidance Grid.* If there exists a method to extract unique identifier for each *trajectory passed cells*, we can compare two trajectories *Coverage* in *Avoidance Grid.*

Definition 4 (Trajectory footprint)**.** *For* Trajectory *from* Reach set *(def. 2) defined for* Avoidance Grid *has membership function.* Membership Function *returns* ordered set of passing cells*:*

$$footprint \begin{pmatrix} Trajectory, \\ AvoidanceGrid \end{pmatrix} = \left\{ \begin{matrix} cell \in AvoidanceGrid : \\ isMember(trajectory, cell) \end{matrix} \right\} \tag{6.8}$$

Then we can define equality function for $Trajectory_1$ *and* $Trajectory_2$*, as the comparison of their footprints in common* Avoidance Grid *as follow:*

$$isEqual \begin{pmatrix} Trajectory_1, \\ Trajectory_2, \\ AvoidanceGrid \end{pmatrix} : \begin{cases} \begin{pmatrix} footprint(Trajectory_1, \ldots) \\ = \\ footprint(Trajectory_2, \ldots) \end{pmatrix} & : true \\ Otherwise & : false \end{cases} \tag{6.9}$$

Note. Depending on *Movement Automaton's* movement set and *Avoidance Grid* parameters, there can be multiple *trajectories* which are equal.

Coverage set: Now it is possible to create a set of unique *trajectory footprints* due to *footprint function* (eq. 6.8). Similarly, there is a possibility to create *Reach set skeleton* containing unique trajectories, by using *equality function* (eq. 6.9). *Coverage set* is sufficient for now.

Definition 5 (Coverage Set)**.** Coverage set *(6.10) is defined for* Avoidance Grid *and* Reach Set *pair as a set of unique* Trajectory footprints*:*

$$CoverageSet\begin{pmatrix} AvoidanceGrid, \\ ReachSet \end{pmatrix} = \left\{ footprint\begin{pmatrix} Trajectory, \\ AvoidanceGrid \end{pmatrix} : \begin{matrix} \forall Trajectory \\ \in ReachSet \end{matrix} \right\} \tag{6.10}$$

Coverage set properties: Trajectory footprint (eq. 6.8)is not a *bijection*, neither *injection* for $ReachSet \rightarrow CoverageSet$. This implies the following properties:

1. Equal *Reach Sets* in same *Avoidance Grid* have equal *Coverage Sets*.
2. Equal *Coverage Sets* does not imply *Reach Set* equality.
3. For two Coverage Sets, there is a possibility to compare their member count to create coverage ratio.

The second *Property* gives us a proposition that there is a possibility of *Reach Set Reduction* without losing *Coverage*.

Definition 6 (Coverage Ratio)**.** Coverage Ratio *is a ratio of* Coverage Set Member Count *between two* Reach Sets*. Reach set with a* lesser count of unique Trajectories *is considered as* Reduced Reach Set*. Reach set with* greater Count of unique Trajectories *is considered as* Reference Reach Set*.*

$$\begin{aligned} referenceCoverage &= |CoverageSet(ReferenceReachSet, AvoidanceGrid)| \\ reducedCoverage &= |CoverageSet(ReducedReachSet, AvoidanceGrid)| \\ CoverageRatio &= \frac{reducedCoverage}{referenceCoverage} \in [0,1] \end{aligned} \tag{6.11}$$

Note. Reference Reach Set is usually *Full Reach Set* containing all possible trajectories in space contained by *Avoidance Grid*. In case *Full Reach Set* cannot be computed, Avoidance Grid is too large, most complex *Reach Set* is used as *Reference Reach Set*.

Trajectory smoothness: Trajectory other than straight line have some changes in *UAS* heading.

The goal is to minimize *Maneuvering* of UAS, because:

1. *Every Heading Change* needs to be reported to *UTM*.

2. *Sharp Maneuvering* can damage cargo/wear UAS.

3. *Often course changes* make *Intruder prediction* harder for other Civil General Aviation.

For this purpose, *Smoothness Metric* needs to be applied for *Reach Set* or *Trajectory*. In the case of *Movement Automaton Control*, two distinguish *Movement Sets* are introduced: *Smooth* and *Chaotic* movements set with the following properties:

$$\begin{gathered} MovementSet = SmoothMovements \cup ChaoticMovements \\ SmoothMovements \cap ChaoticMovements = \varnothing \\ |SmoothMovements| > 0, \quad |ChaoticMovements| > 0 \end{gathered} \tag{6.12}$$

Then *Smoothnes clasificator* for $Trajectory(initialState, buffer)$ can be defined as $isSmooth$ and *Smooth Movement Counter* function as $smoothCount$ like follow:

$$\begin{aligned} isSmooth(movement) &= \begin{cases} movement \in SmoothMovements & : 1 \\ movement \in ChaoticMovements & : 0 \end{cases} \\ smoothCount(Trajectory(\ldots, buffer)) &= \sum isSmooth(movement), \\ &\qquad \forall movement \in Buffer \end{aligned} \tag{6.13}$$

Definition 7 (Smoothness Rating for Trajectory)**.** Smoothness *for trajectory generated by* Movement Automaton *for some* Initial State *with some* Movement Buffer*, under the assumption of* Smooth and Chaotic Movement Set *split (eq. 6.12), with existing* classification *and* counter *functionals (eq. 6.13) is given as follows:*

$$Smoothness(Trajectory(\ldots, buffer)) = \frac{isSmooth(Trajectory)}{movementCount(Trajectory)} \in [0, 1] \tag{6.14}$$

For Trajectory *with* $buffer = \varnothing$ Smoothness *is given as 1.*

6.4.3 Heuristic Trajectory Tree Building

Motivation: *Purpose* of *Navigation* is to move forward to *Goal Waypoint* in *Mission*. *Structure* of *Avoidance Grid* is designed to enable *forward* and *turning* maneuvers. The *Avoidance Grid* is organized in *Layers* characteristic by the same distance from *Avoidance Grid Origin*.

Survey of motion planning algorithm was given in [4]. The ideal candidate for propagation algorithm is *Wave-front* algorithm propagating *Trajectory tree* through Layers. Due to the *Avoidance Grid* onion-like layers, there is a possibility to implement turn maneuver through layers iterative and effectively.

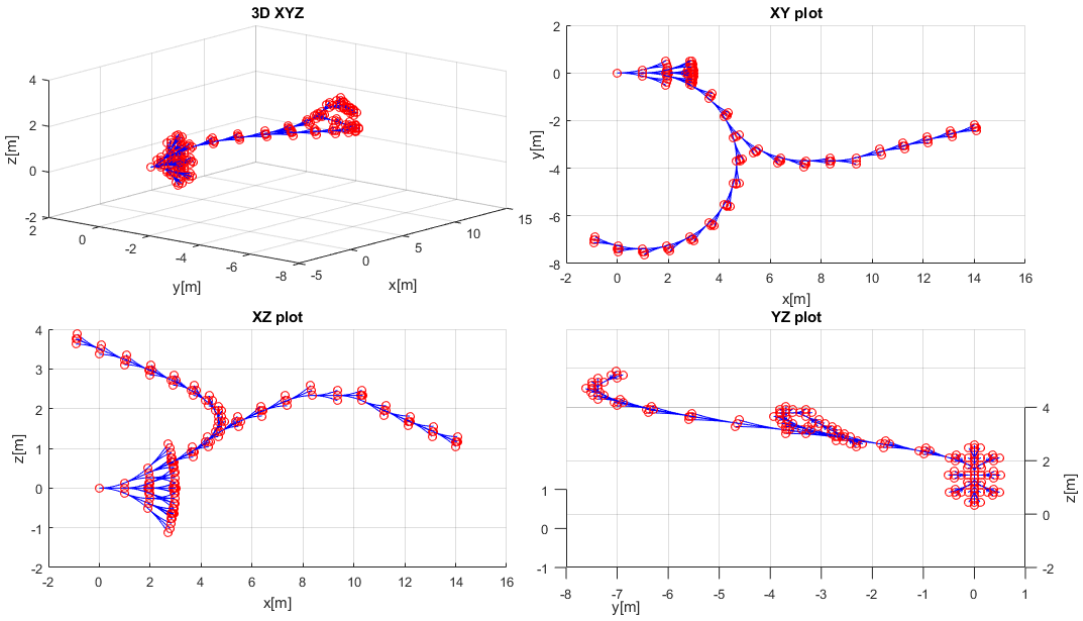


Figure 6.1: *Rapid Exploration tree as a result of* Constrained trajectory expansion.

Rapid Exploration Tree (fig. 6.1) was selected because it enables *Movement Automaton Utilization* and *Property Binding*. A similar approach was used for space exploration [5].

The example (fig. 6.1) shows a *Rapid Exploration Tree* in *Free Space* containing *Waypoint Navigation Path* and *Turn Away Path*. Both paths are starting in same *Root Node* (red circle) which was expanded with simple *Movement Automaton* (a bunch of nodes originating from one node is showing the way of expansion). The connection (blue line) between two nodes (red circles) represents *Trajectory portion* for *Executed Movement*.

Rapid Exploration Tree Node will contain the following information:

1. *Initial state* - root entry point, used in state evolution calculation.
2. *Trajectory (state evolution)* - trajectory passing through *state space* in the local coordinate frame of *Avoidance Grid*.
3. *Buffer* (applied movements) - ordered list of *executed movements* applied on the *initial state* to obtain *state evolution*.
4. *Cost* - calculated for *state evolution* based on a *predefined cost function*.
5. *Footprint* - ordered set of *passing cells* in *Avoidance Grid*.
6. *Parent Node Reference* - tree reference for the parent node, not in case of the *root node*.
7. *Other Bounded Properties* - value list of other properties, depending on *Expansion Constraints* and *Reachability* evaluation algorithm.

Wave-front propagation of Rapid Exploration Tree is given in (alg. 6.1).

The *Avoidance Grid* have UAS with *position* $\in$ *Initial State* at the *origin*. The *Grid Layer* is a column ordered set of cells with same *Mean distance* from the origin. *Grid Layers* are indexed from origin starting with 1; there is a maximum of $i \geq 1$ layers.

Step: Initialization contains base structure preparation like follows:

1. *Avoidance Grid* - space containing *Reach set* (def. 2).
2. *Movement Automaton* - Used as *Predictor*, consuming *buffer* containing *Movements* to generate $Trajectory(initialState, buffer)$.
3. *Reach Set* - tree consisting from *Wave-frontNodes* representing the endpoint of $Trajectory(initialState, buffer)$ where each *Edge* represents *one Movement application*. The root is set as a node containing *Initial State*.

Function *initializeReachSet(root,stack,grid,automaton)* will take root and enforces *full wavefront propagation* to *First Layer*.

Step: Wave-front Propagation is forced propagation of trajectories from layer i to layer $i + 1$. The process goes as follows:

1. *Selection of Feasible candidates* - function *[candidates,leftovers] = ExpansionConstraints.select(stack)* for working layer, row and cell selects *feasible trajectory nodes* ordered by *Cost function*. The *Example of Cost Function* can be *Trajectory Smoothness* (def. 7).
2. *Expansion of Candidates* - for each *candidate* function *candidate.expandNode (automaton)* is invoked. This function will expand *Candidate Node structure* by appending *Full Trajectory Tree Evolution* until each *Leaf Trajectory* reaches *Next Layer*. Simply put *Par rent Node Node(initialState, buffer, cost, footprint)* buffer is appended by movements until the next layer is reached.
3. *Leftovers purge* - function *reachSet.purge(leftovers)* removes unexpanded *Nodes* leading to cell, effectively removing trajectories which do not lead to the *next layer*.
4. *Append Reach Set* - function *reachSet.append(leafs)* puts newly created *Nodes (Trees)* into *Reach Set* structure. The *Wave-front Propagation* for one cell is finished.

Step: After Layer Propagation Purge is covered by function *reachSet.purgeSameFootprint()* which takes trajectories with the same footprint and keeps some of them based on *Selection criteria*, more in (sec. 6.4.4, 6.4.5). *Pruning methods* over *Large Decision Trees* are *fast* and *viable* [6].

Note. Reach Set is usually computed *Prior the Flight* for *some Initial State* in *Local Coordinate Frame* in *right had coordinate frame* with X^+ used as *main axis*.

Algorithm 6.1: *Wave-front propagation* of *Rapid Exploration Tree* to form *Reach Set*.

Input : Node(initialState,buffer=∅,cost=0,footprint=∅), AvoidanceGrid, ExpansionConstraints, MovementAutomaton(movementSet)

Output: ReachSet(AvoidanceGrid)

```
# Initialization Sequence;
grid=AvoidanceGrid, automaton=MovementAutomaton, root = Node;
reachSet = initializeReachSet(root,stack,grid,automaton);

# Main Expansion through, layers (i), rows (j), cells(k);
for layer(1...i) in grid do
    for row(1...j in layer) do
        for cell(1...k) in row do
            # apply selection criteria ;
            [candidates,leftovers] = ExpansionConstraints.select(stack);
            # collect expansions ;
            leafs = [];
            for candidate in Candidates do
                leafs= [leafs, candidate.expandNode(automaton)];
            end
            reachSet.purge(leftovers);
            reachSet.append(leafs);
        end
    end
    reachSet.purgeSameFootprint();
end
```

6.4.4 Coverage-Maximizing Reach Set Aproximation

Motivation: Design of calculation method for *Reach Set Approximation* guarantying high *Maneuverability*.

Background: There is *Coverage Ratio* property of *Reach Set* (def. 6). It has been shown that creating *Reach Set* via *greedy approach* is not feasible due to the *Scaling Factor*. *Contracted Expansion* (sec. 6.4.3) is enabling to apply selection criteria while building *Reach Set* in given *Cell*.

The *Cell* $cell_{i,j,k}$ has a center and walls from UAS viewpoint: a front wall, back wall (for $layer > 1$), a top wall, left wall, right wall, bottom wall. It is expected that trajectory leading close to one cell walls will continue to a different cell, increasing the chance to

obtain more *Unique Footprints*.

Expansion Constraint Function Implementation (alg. 6.2) is based on the simple principle: *Select candidate Nodes which are closest to outer walls of Cell, with a unique footprint*.

Tuning Parameters: *Proximity to Cell outer wall* gives good chances to break into other rows or columns in the *Avoidance Grid*. *Unique footprint* guarantees future *Unique Footprint* after appending Trajectory by *Movement application*.

1. *Considered Footprint Length* - how much last cells in footprint should be considered in unique path track, minimal value 1, default value 3, maximal value ∞. If there is a need to generate non-redundant trajectories use ∞, it will consider full footprint.

2. *Spread Limit* - the upper limit of candidates which are going to be select for further expansion, minimal value 1, default value *Count of unique Moves in Movement set*, maximal value ∞. If more than default values are selected, the algorithm will generate *redundant trajectories*. If less is selected, then some trajectories are omitted, and *Coverage Rate* decreases sharply.

Step: Initialization initialization of *candidate* array (return value), *leftovers* array (return Value). Node array *passing* is populated with *Nodes* which represents *end node of Trajectory*, and the tip of the *trajectory is constrained in the* $\text{cell}_{i,j,k}$.

Step: Evaluate best trajectories with unique Footprints following steps are executed:

1. *Best Performance Map* is created with a *footprint* as a key set element to ensure footprint uniqueness.

2. *Wall distance* for the *test node* is calculated as a closest trajectory portion distance to the *top, bottom, left, right* wall of the $cell_{i,j,k}$

3. The *Footprint* for the *test node* is created with the maximal length given by *Footprint Length* tuning parameter.

4. *Existence and Performance Test* is executed to ensure that the best performing node is selected. If there is no key entry in the *Best Performance Map*, then a new entry for *Test Node* is created. If there is a key entry, the performance of *Old Node* and *Test Node* is compared, and better is stored.

Step: Select candidates is executed on *Best Performance Map* records using *Wall distance* as pivot parameter, ordering by closest proximity and limited by *Search Limit* tuning parameter. The *Leftovers* are difference set between *Passing Nodes* and *Candidate Nodes*.

Algorithm 6.2: Expansion Constraint function for *Coverage-Maximizing Reach Set Approximation*

Input : Node[] stack, Cell $\text{cell}_{i,j,k}$
Tuning Parameters: int^+ footprintLength, int^+ spreadLimit
Output : Node[] candidates, Node[] leftovers

```
# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = cell_{i,j,k}.getFinishingTrajectories(stack);

# Select best performing trajectories with unique footprint;
Map<Footprint,Node> bestPerformanceMap;
for Node test ∈ passing do
    wallDistance= test.minimalDistanceToWall(cell_{i,j,k})];
    footPrint = test.getFootprint(lastCells = footprintLength);
    if bestPerformanceMap.contains(footPrint) then
        old = bestPerformanceMap.getByKey(footprint);
        oldPerformance= old.minimalDistanceToWall(cell_{i,j,k});
        if oldPerformance > wallDistance then
            bestPerformanceMap.setByKey(footprint,test);
        end
    else
        bestPerformanceMap.setByKey(footprint,test);
    end
end

# Select best performing nodes up to spreadLimit count;
candidates = bestPerformanceMap.select(count =
 spreadLimit).orderBy('wallDistance','Ascending');
leftovers = passing - candidates;
return [candidates,leftovers]
```

Example: for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range* $[-45°, +45°]$, *Horizontal Cell Count 7*, *Vertical range* $[-30°, +30°]$, and *Vertical Cell Count 5*. Is given in (fig. 6.2). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*.

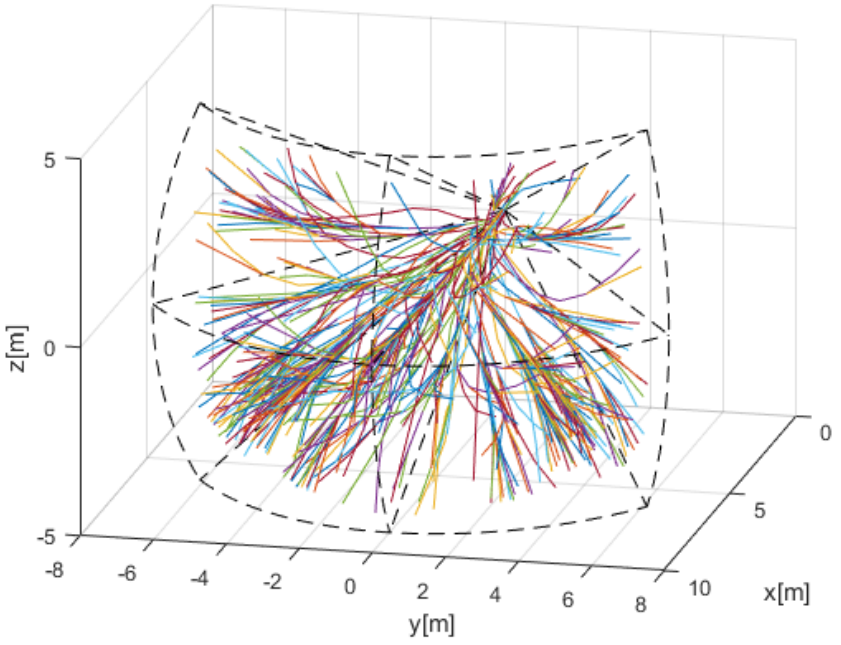


Figure 6.2: *Coverage-Maximizing* reach set *approximation.*

Pros and Cons: It can be seen from example (fig. 6.2) that *Coverage-Maximizing Reach Set Approximation Method* (alg. 6.2) generates much *turning* and *shaky trajectories.*

High Coverage Ratio ($\sim$ 0.9) is provided while keeping *medium node count.* The calculation complexity scales linearly with grid size. The *upper limit of trajectories* is given as follow:

$$countTrajectories(ReachSet) \leq layerCellCount \times spreadLimit \times size(Movements) \quad (6.15)$$

The *upper limit of nodes* is given as follow:

$$countNodes(ReachSet) \leq layerCount \times layerCellCount \times size(Movements) \times spreadLimit \quad (6.16)$$

The *absence* of *Smooth Trajectories* disqualifies *Coverage Maximizing -RSA* to be used for *Navigation.* This type of reach set is feasible for *Avoidance* because it contains a variety of maneuvers.

6.4.5 Turn-Minimizing Reach Set Approximation

Motivation: Imagine having an *Avoidance Grid* like (fig. **??**). There is a need of *Reach Set Approximation* which will have *Smooth Trajectories* (def. 7) going nearby *cell*

centers.

Background: The *Smoothness Rating for Trajectory* (def. 7) uses two distinct sets *Smooth Movements* and *Chaotic Movements* (eq. 6.12) which are defined for our *Movement Automaton* (sec. **??**) like the following:

$$\begin{aligned} SmoothMovements &= \{Straight\} \\ ChaoticMovements &= Movements - SmoothMovements \end{aligned} \tag{6.17}$$

Smooth Movements contains only *Straight* movement because others are considered as extreme turning movements. *Smooth Movements* should contain only direct flight movements or slight heading correction. *Chaotic Movements* set is a supplement of *Movement Automaton's Movement Set.*

The *Avoidance Grid* (fig. **??**) cell centers for fixed indexes j_{fix}, k_{fix} are linearly aligned with the *initial state.* That means that cell centers of cells $cell_{1,j_{fix},k_{fix}}, \ldots, cell_{i,j_{fix},k_{fix}}$, where i is a count of *layers* lie on one line. If the trajectory can achieve *cell center* on some *layer*, only minor trajectory corrections are required to stay on the given line. This type of trajectory gives us the following advantages:

1. *Minimal steering at the beginning* - the minimal steering is advantageous in *Controlled Airspace* because is diminishing the amount of communication to *UTM Service.*

2. *Additional safe space in the linear segment* - once the *center of the cell* is reached, *Trajectory* sticks to the line between cell centers. Each point on this line has the *maximal distance* to outer walls of the cell. This gives us extra space given as minimum of distance between *UAS position* and *Outer cell walls.*

Expansion Constraint Function Implementation (alg. 6.3) is based on the simple principle: *Select candidate Nodes which are closest to Cell center, with a unique footprint.*

Note. Cell center can be closely reached by *smooth movement* from a previous cell or *chaotic movement* from a neighboring cell from the current or previous layer. These trajectories are usually equivalent in *Smoothness.*

Tuning Parameter: *Proximity to Cell Center* gives a good chance to keep trajectory smooth or *smooth after one correction maneuver.* It has been mentioned that *Cell Center* can be reached by various trajectories. In this method full footprint length is always considered; therefore only one tuning parameter can be offered:

1. *Spread Limit* - the upper limit of candidates which are going to be selected for further expansion, minimal value 1, default value *Count of unique Moves in Movement set*, the maximal value ∞. If maximal value ∞ is selected, the algorithm will generate the skeleton of *Reach Set* with full Coverage and with the smoothest *Trajectories.*

Step: Initialization sets candidate *Nodes* as empty set, leftover *Nodes* as empty set. and selects all *Nodes* from *Stack* which represents *Finishing Trajectories* in working cell $cell_{i,j,k}$.

Algorithm 6.3: Expansion Constraint function for *Turn-Minimizing Reach Set Approximation*

Input : Node[] stack, Cell $cell_{i,j,k}$
Tuning Parameters: int^+ spreadLimit
Output : Node[] candidates, Node[] leftovers

```
# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = cell_{i,j,k}.getFinishingTrajectories(stack);

# Select unique smoothest trajectories;
Map<Buffer,Node> bestPerformanceMap;
for Node test ∈ passing do
    centerDistance= test.getPerformance(cell_{i,j,k})];
    footPrint = test.getFootprint();
    if bestPerformanceMap.contains(footPrint) then
        old = bestPerformanceMap.getByKey(footprint);
        oldPerformance= old.getPerformance(cell_{i,j,k});
        if oldPerformance > centerDistance then
            bestPerformanceMap.setByKey(footprint,test);
        end
    else
        bestPerformanceMap.setByKey(footprint,test);
    end
end

# Select best performing nodes up to spreadLimit count;
candidates = bestPerformanceMap.select(count =
 spreadLimit).orderBy('cellCenterDistance','Ascending');
leftovers = passing - candidates;
return [candidates,leftovers]
```

Step: Evaluate smoothest trajectories with unique Footprints is implemented as *multi-criteria filtration.*

The *first criterion* is the *distance to Cell Center* which is penalized by trajectory *smoothness rate* implemented in method *Node.getPerformance(Cell* $cell_{i,j,k}$*)* defined as follow.

$$getPerformance(Node, Cell) = \frac{distance(Node.Trajectory, Cell.Center)}{SmoothnessRate(Node.Trajectory)} \tag{6.18}$$

Distance of *Trajectory* is *enumerator* because its considered as the *base value* and is defined in the interval $[0, maximalWallDistance]$. The *Smoothness Rate* is the denominator, because it is a penalization coefficient defined in the interval $[0, 1]$.

The *second criterion* is *trajectory uniqueness*. This is provided by *Best Performance Map*, where best performing *Node* belongs to one unique *trajectory footprint*. The implementation is identical to *coverage-maximizing set expansion* (alg. 6.2).

Step: Select candidates is executed on *Best Performance Map* records using *Penalized Cell Center Distance* as pivot parameter, ordered in ascending order and limited by *Spread Limit* tuning parameter. The *Leftovers* are difference set between *Passing Nodes* and *Candidate Nodes*.

Example: for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range* $[-45°, +45°]$, *Horizontal Cell Count 7*, *Vertical range* $[-30°, +30°]$, and *Vertical Cell Count 5*. Is given in (fig. 6.3). The UAS is at *Back-side* of *Figure* (the initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its color and ends at *Front-side* of *Avoidance Grid Boundary*. The *Spread Limit*, in this case, was set to 9 which is *Size of the Movement Set*.

Note. Please note *Trajectories* are organized in bundles going around *Cell Centers smoothly*. Most of the steering maneuvers are executed at the *beginning* of the *Avoidance Grid*.

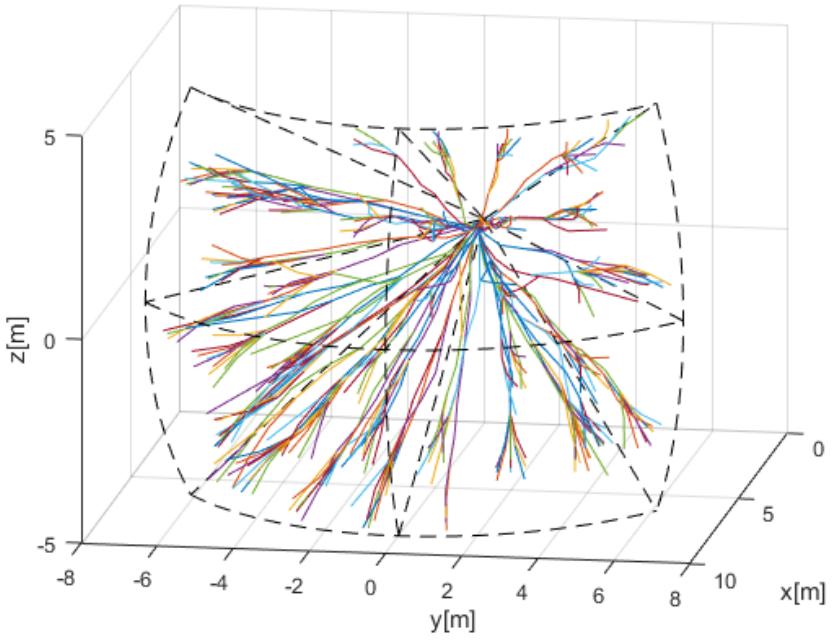


Figure 6.3: *Turn-minimizing* reach set *approximation*.

Pros and Cons: It can be seen from example (fig. 6.3) that *Turn-Minimizing Reach Set Approximation Method* (alg. 6.3) generates *smooth evenly spread trajectories*.

High smoothness ratio (≥ 0.9) is provided while keeping low node count for UAS systems. The calculation complexity scales linearly with grid size. The upper limit of trajectories is given as follow:

$$countTrajectories(ReachSet) \leq layerCellCount \times spreadLimit \times size(Movements) \quad (6.19)$$

The *upper limit of nodes* is given as follow:

$$countNodes(ReachSet) \leq layerCount \times layerCellCount \times spreadLimit \quad (6.20)$$

The absence of *High Coverage Ratio* disqualifies *Turn-Minimizing Reach Set Approximation* to be used for *Emergency Avoidance*. This type of *Reach Set* is feasible for *Open Space Navigation* or *Controlled Airspace Navigation*. Its low turning rate in contained *Trajectories* are desired for such tasks.

6.4.6 ACAS-X like Reach Set Approximation

Motivation: The implementation of *ACAS-Xu* behavior in DAA system will be mandatory for *National Airspace System Integration* in United spaces [7].

Implementation of ACAS-Xu like behavior increase usability of approach, if it can be achieved without major concept changes.

Background: The *ACAS-Xu* system on the operational level has been described in [8]. The *Policy for Collision Avoidance* proposal has been given in [9].

Some behavioral patterns can be encoded into *Reach Set*. ACAS-Xu navigation part is basically *Look-up table of Maneuvers for Allowed Separations*.

The *Evasive Maneuver* selection process in ACAS-Xu is similar to our approach: *Select most energy efficient maneuver in compliance with space-time constraints*. ACAS-Xu intruder model is similar to our *Body Volume Intersection Model* (app. **??**). The *ACAS-Xu* defines following base separations:

1. *Horizontal* - movements on a *Horizontal Plane* in *Global Coordinate System*.
2. *Vertical* - movements on a *Vertical Plane* in *Global Coordinate System*.

There are allowed custom separations which can be used, for further experimentation:

1. *Slash* - movement on $+45°$ *Tilted Plane to Horizontal Plane* in *Global Coordinate System*.

2. *Backslash* - movement on $-45°$ *Tilted Plane to Horizontal Plane* in *Global Coordinate System.*

For given *Movement Automaton* implementation (sec. **??**) the separations are given as follow:

$$\begin{aligned} Horizontal &= \{Straight, Left, Right\} \\ Vertical &= \{Straight, Up, Down\} \\ Slash &= \{Straight, UpLeft, DownRight\} \\ Backslash &= \{Straight, UpRight, DownLeft\} \end{aligned} \tag{6.21}$$

For each $Node(\dots, buffer)$ and each *separation* there is a evaluation function *isSeparation* which decides, if *Trajectory* defined by node buffer is made up only from *Separation* movements. The function $isSeparation(\dots)$ is defined like:

$$isSeparation(buffer, separation) = \begin{cases} \forall movement \in buffer, & \\ \quad movement \in separation & : true \\ otherwise & : false \end{cases} \tag{6.22}$$

Following *Separation Modes* can be defined with given *separations*:

1. *Horizontal* (ACAS-X defined mode) containing *horizontal* separation.

2. *Vertical* (ACAS-X defined mode) containing *vertical* separation.

3. *Horizontal-Vertical* (ACAS-X defined mode) containing *horizontal, vertical* separations.

4. *Full* (custom defined mode) containing all *Separation Modes.*

Note. Every separation modes generate 2D trajectories set on *Respective plane.* There is no need for *Tuning parameters* for further *Expansion Constraint.*

Expansion Constraint Function Implementation (alg. 6.4) is based on the simple principle: *Select only candidate Nodes which Trajectories have at least one desired Separation Mode.*

Step: Initialization sets candidate *Nodes* as the empty set, leftover *Nodes* as the empty set, and, select all nodes to form a *stack* which represents *Finishing Trajectories* in working $cell_{i,j,k}$,

Step: Candidate Selection Process is evaluated for each *test Node* from *passing Node Set*.

For each *applicable separation*, given as input parameter *separations*, The test function *isSeparation* (eq. 6.22) is applied:

1. If *test Node* trajectory belongs to at least one allowed separation it is added to candidates set.

2. Else is added to *Leftovers*.

Note. Separation sets (eq. 6.21) are not *exclusive sets* in *Movement Automaton* domain. One *Trajectory* contained by Node can belong to multiple *Separations*.

Algorithm 6.4: Expansion Constraint function for *ACAS-like Reach Set Approximation*

Input : Node[] stack, Cell $\text{cell}_{i,j,k}$, Separation[] separations
Tuning Parameters: *None* : $\varnothing$
Output : Node[] candidates, Node[] leftovers

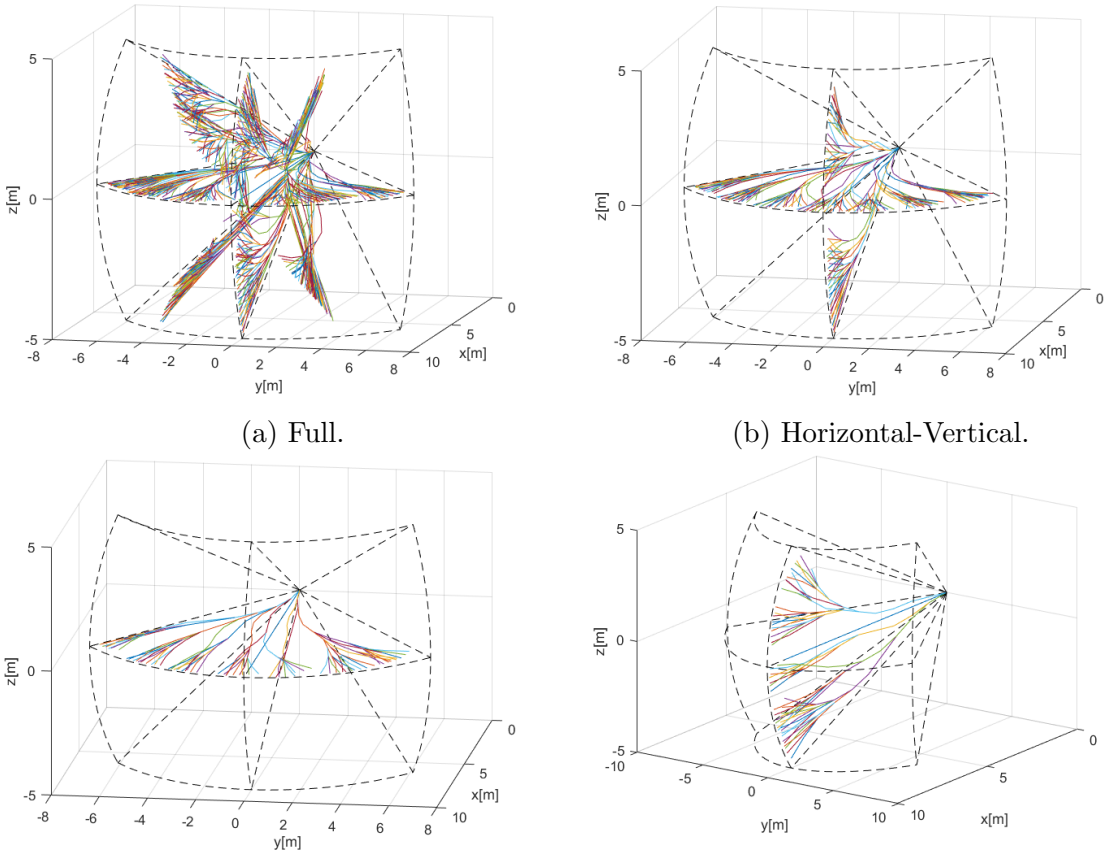
```
# Initialize structures;
Node[] candidates = [], Node[] leftovers=[];
Node[] passing = cell_{i,j,k}.getFinishingTrajectories(stack);

# Select nodes containing trajectories with usable separations;
for Node test ∈ passing do
    for separation ∈ separations do
        # Get separations for Node;
        Separations[] nodeSeparations = test.getSeparations();
        # If trajectory given by buffer is on Separation plane;
        if isIn(isSeparation(test.buffer,separation)(6.22) then
            candidates.append(test);
        end
    end
    # If there was no applicable separation, throw Node away;
    if test ∉ candidates then
        leftovers.append(test);
    end
end
# Return results;
return [candidates,leftovers]
```

Example: for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range* $[-45°, +45°]$, *Horizontal Cell Count 7*, *Vertical range* $[-30°, +30°]$, and *Vertical Cell Count 5*. Is given in (fig. 6.4). The UAS is at *Back-side* of *Figure* (initial state is at all

Trajectory Origins). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*.

Full separation mode is given in (fig. 6.4a). *Horizontal-Vertical* separation mode, used in original *ACAS-Xu* testing [8], given in (fig. 6.4b). *Horizontal* separation mode given in (fig. 6.4c) is usually used by planes. *Vertical* separation mode given in (fig. 6.4d) is usually used by copters.



(a) Full. (b) Horizontal-Vertical.

(c) Horizontal. (d) Vertical.

Figure 6.4: ACAS-X imitation *reach set* approximation for various *separation modes*.

Pros and Cons: It can be seen from examples (fig. 6.4) that *ACAS-like Reach Set Approximation Method* (alg. 6.4) generates a full reach set for 2D plane located in 3D space.

The *Reach Set* contains trajectories with *high coverage ratio* and *high smoothness rating* for selected 2D separation plane. Overall performance compared to full 3D reach sets (sec. 6.4.4, 6.4.5 6.4.7) is poor.

The *node* and *trajectory* count boundary was not implemented. It is common knowledge that *2D* avoidance sets do not require scaling [8]. Otherwise, trajectory footprint mechanism like in *Turn-Minimizing Reach Set Approximation* (alg. 6.3) can be introduced.

This reach set implements *Planar-Separation* as a native feature, it can be used for both *navigation* and *avoidance* tasks in *Controlled Airspace*. For *Non-controlled Airspace*, there are far more superior *Combined Reach Set* (sec. 6.4.7).

6.4.7 Combined Reach Set Approximation - Tree Merge

Motivation: Turn-Minimizing Reach Set Approximation (sec. 6.4.5) is *efficient* for *Navigation in* Controlled Airspace. Coverage-Maximizing Reach Set Approximation (sec. 6.4.4) is good for *Emergency avoidance*. The need for the differentiation between *Navigation* and *Emergency Avoidance* mode is necessary for *Controlled Airspace*, but not for *Non-controlled Airspace*. The combination of *Turning-Minimizing* and *Coverage-Maximizing* reach set approximations is an obvious solution.

Automatic mode switch can be provided by a combination of *Navigation Reach Set* and *Avoidance Reach Set* with an elevated cost function. Overall having a method to merge multiple trees would be beneficial.

Background: If two *Reach Set Approximation* were calculated for the same *Avoidance Grid* and *Initial State*, using same *Movement Automaton* and *UAS model* are possible to merge.

The *Reach Set Approximation* is a *tree* with *Root Node* in *initial state* with movement buffer = $\varnothing$. The *movement buffer* in each node can be used as *route trace* during the merging procedure. The example two reach set merge can be given as follow, where only the *latest* applied movement is taken into account.

$$\left[\begin{array}{l}\texttt{First Reach Set}\\ \quad \varnothing \rightarrow \left\langle \begin{array}{l} left \rightarrow \left\langle \begin{array}{l} left \\ right \end{array}\right. \\ \varnothing \end{array}\right. \\ \texttt{Second Reach Set}\\ \quad \varnothing \rightarrow \left\langle \begin{array}{l} \varnothing \\ right \rightarrow \left\langle \begin{array}{l} left \\ right \end{array}\right. \end{array}\right. \end{array}\right] \rightarrow \left[\begin{array}{l}\texttt{Combined Reach Set}\\ \quad \varnothing \rightarrow \left\langle \begin{array}{ll} left & \rightarrow \left\langle \begin{array}{l} left \\ right \end{array}\right. \\ right & \rightarrow \left\langle \begin{array}{l} left \\ right \end{array}\right. \end{array}\right. \end{array}\right] \tag{6.23}$$

First Reach Set contains two trajectories given by buffers {*left,left*} and {*left,right*}. *Second Reach Set* contains two trajectories given by buffers {*right,left*} and {*right,right*}. The *Combined Reach Set* contains all four trajectories.

Note. The combined tree [10] does not need to a have combined amount of original *Reach Sets* trajectories. There can be *Duplicity* which means that any bounded property like *Cost* must be *calculated* again.

Combined Reach Set Calculation Function (alg. 6.5) is implemented as function $NodecombinedReachSet(\dots)$ which takes root Node with *initial State*, *Avoidance Grid* and respective parameters for each calculation method. *turn-minimizing spread* for *Turn-Minimizing Reach set calculation* and *coverage spread*, *Footprint Length* for *Coverage-Maximizing Reach Set Approximation*.

Separate Reach Sets are calculated using *Wave-front propagation* (alg. 6.1) using respective *Constrained Expansion* functions for *Turn-Minimizing* (alg. 6.3) and *Coverage-Maximizing* (alg. 6.2) reach sets.

Combined Reach Set is created using *Node mergeTree(...)* function, because different cost function or *Bounded Parameters Calculation* may be applied on *Original Reach Sets*.

Cost for *each node* needs to be recalculated due to original reach sets disparity. Function *combined.applyCostFunction()* will recalculate the new cost for each node.

The Goal is to have a penalization for *non-turn-minimizing behavior*, implementation of *Automatic Mode Switch* can be done like follows:

1. *Calculate Normal Cost* for Node $Cost(Node)$ for the associated trajectory: $Cost(Node.Trajectory)$.

2. *Calculate Penalization for* additional maneuvering, calculate *Smoothness Rating for Trajectory* (def. 7) in the interval $[0, 1]$, introduce penalization with base 100%.

The final $Cost(Node)$ function is applied to each *Combined Reach Set Node* and look like follows:

$$Cost(Node) = Cost(Node.Trajectory) \times \ldots \\ \cdots \times (1 + (1 - SmoothnessRate(Node.Trajectory))) \quad (6.24)$$

Tree Merge Function $mergeTree(\ldots)$ implements *Outer Join* operation on two trees. Example was given in (eq. 6.23). Function is applied on *root Node* iterating over

Movements in Movement Set, because *Movement is pivot*.

Algorithm 6.5: Reach Set Merge Function and Combined Reach Set calculation

```
# Tree merge function;
Node mergeTree(Node firstNode, Node secondNode)
    # Try to copy reference node or return null;
    Node referenceNode = (firstNode?:(secondNode?: return null));
    Node merged = new Node(referenceNode);
    merged.leafs= [];
    # Try to fetch movement nodes if exist in any sub tree;
    for movement ∈ Movements do
        firstLeaf = firstNode.getLeafFor(movement);
        secondLeaf = secondNode.getLeafFor(movement);
        newLeaf = mergeTree(firstLeaf,secondLeaf);
        if newLeaf ∼= null then
            merged.leafs.append(newLeaf);
        end
    end
    return merged
# Combined Reach Set calculation function;
Node combinedReachSet(Node root, AvoidanceGrid grid,int+ coverageSpread,
 int+ turnSpread, int+ footprintLength)
    Node cmrsa = chaoticReachSet(root,grid, footprintLength,coverageSpread);
    Node tmrsa = harmonicReachSet(root,grid, turnSpread);
    Node combined = mergeTree(cmrsa,tmrsa);
    combined.applyCostFunction();
    return combined
```

Example: for *Avoidance Grid* with *Distance 10 m*, *Layer count 10*, *Horizontal range* $[-45°, +45°]$, *Horizontal Cell Count 7*, *Vertical range* $[-30°, +30°]$, and *Vertical Cell Count 5*. Is given in (fig. 6.5). The UAS is at *Back-side* of *Figure* (initial state is at all *Trajectory Origins*). The *black dashed line* marks *Avoidance Grid* space boundary. Each trajectory has its own color and ends at *Front-side* of *Avoidance Grid Boundary*. The *Coverage-Maximizing Spread* was set to 8, *Footprint Length* to 3 and *Turn-Minimizing Spread* to 1.

Note. Notice there are typical trajectories from both *Turn-Minimizing* (fig. 6.3) and *Coverage-Maximizing* (fig. 6.2) *Reach Set Approximations*.

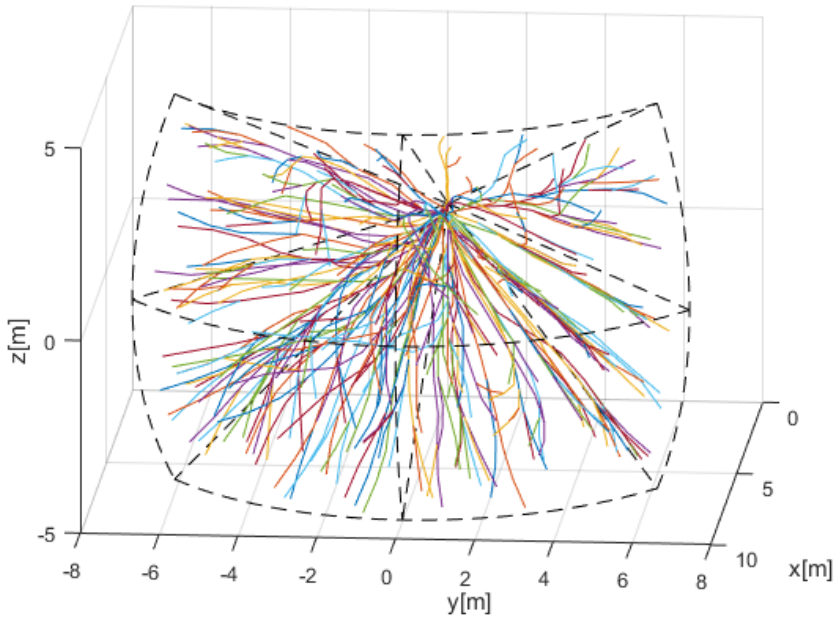


Figure 6.5: *Combined* reach set *approximation.*

Pros and Cons: It can be seen from example (fig. 6.5) that *Combined Reach Set Approximation* (alg. 6.5) contains both types of maneuvers. *Cheaper turn-minimizing* for navigation and *More Expensive Coverage-Maximizing* for *Emergency Avoidance.* The upper limit of trajectories is given as follow:

$$countTrajectories(ReachSet) \leq countTrajectories(CM - RSA) + countTrajectories(TM - RSA) \quad (6.25)$$

The *upper limit of nodes* is given as follow:

$$countNodes(ReachSet) \leq countNodes(CM - RSA) + countNodes(TM - RSA) \quad (6.26)$$

Turn-Minimizing Reach Set is ideal for *Non-controlled Airspace* missions because it contains *Automatic Mode Switch* between *Navigation* and *Emergency Avoidance.*

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