COL106
Data Structures and Algorithms

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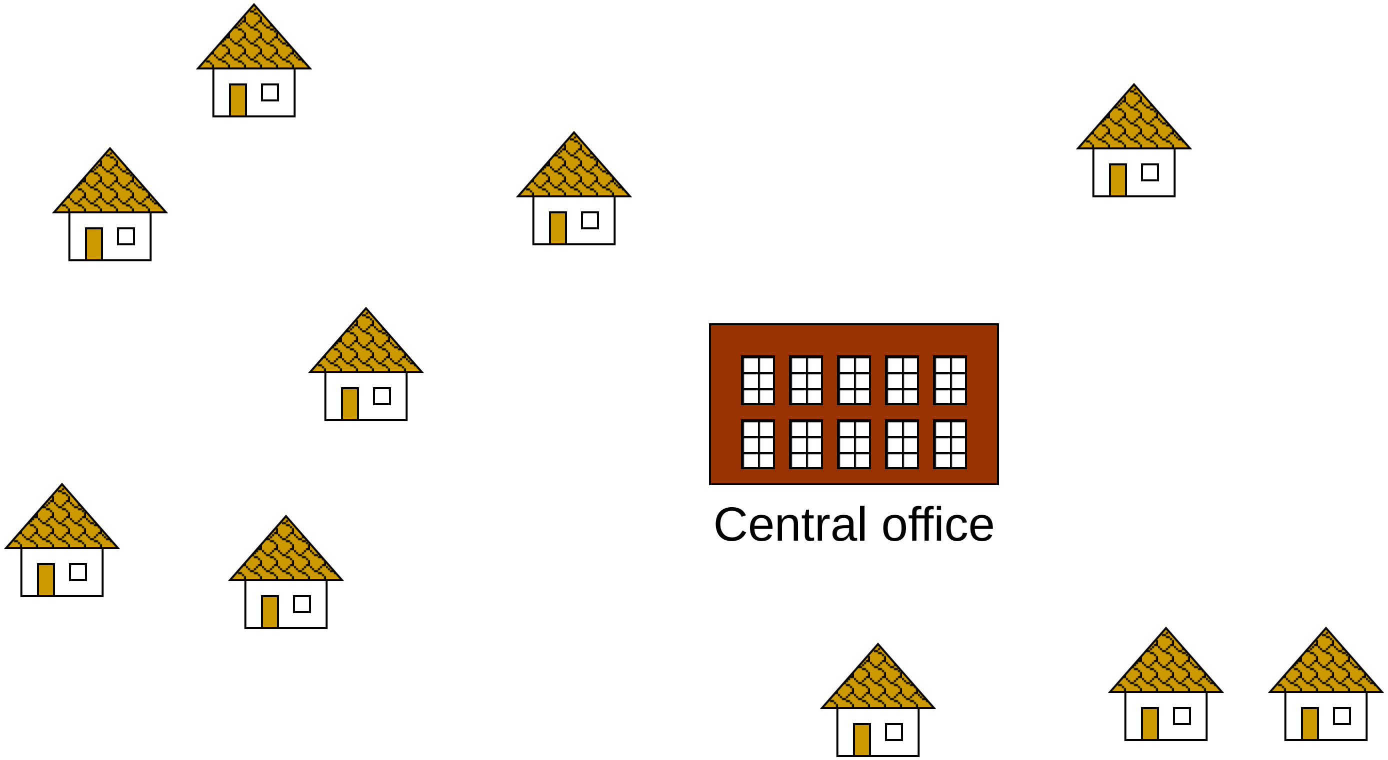
Announcements

- The lab attendance has gone down significantly
- All TAs will not be coming to the labs now
- Instructors will not be coming to the labs anymore
- Use the labs to discuss doubts and help with the assignments

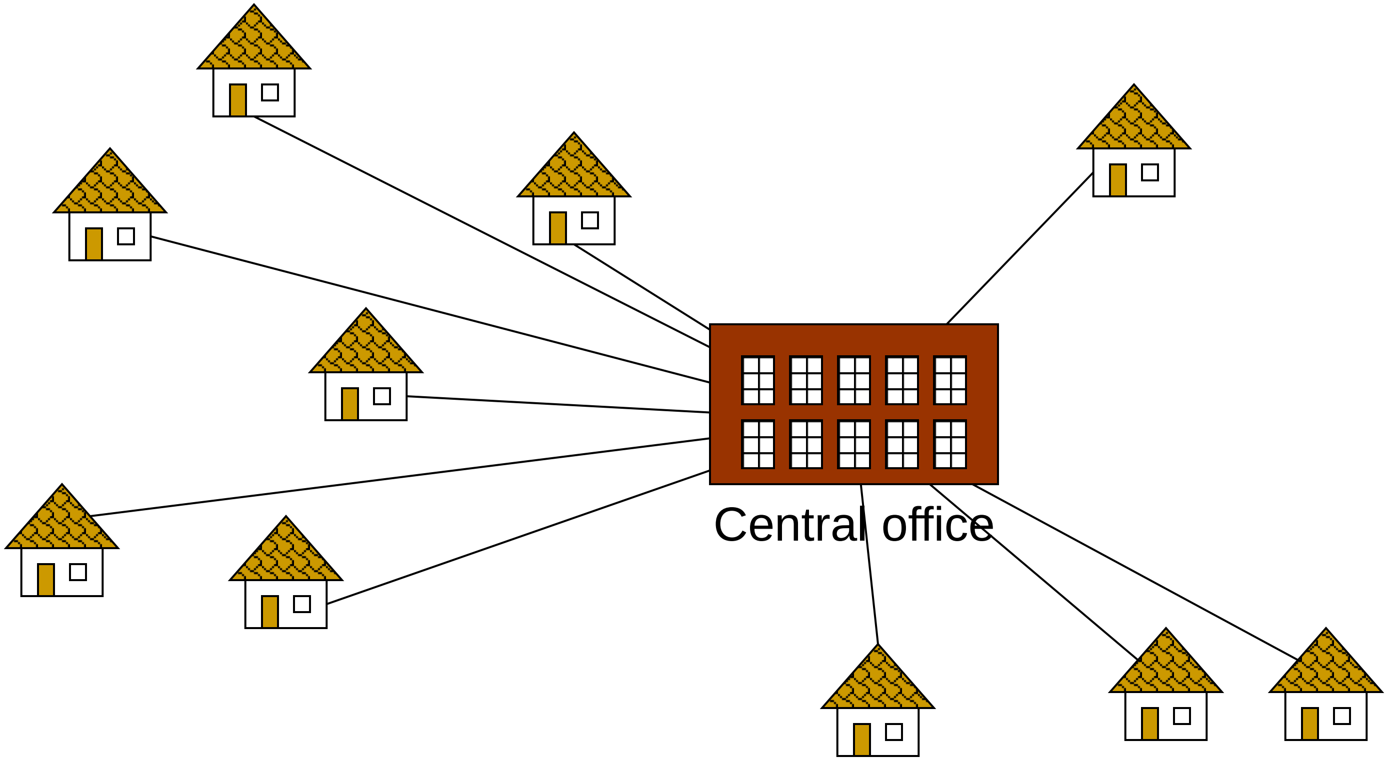
Minimum Spanning Trees

Based on slides by: Longin Jan Latecki, Temple University, George Bebis, University of Nevada, Reno (UNR)

Problem: Laying Telephone Wire

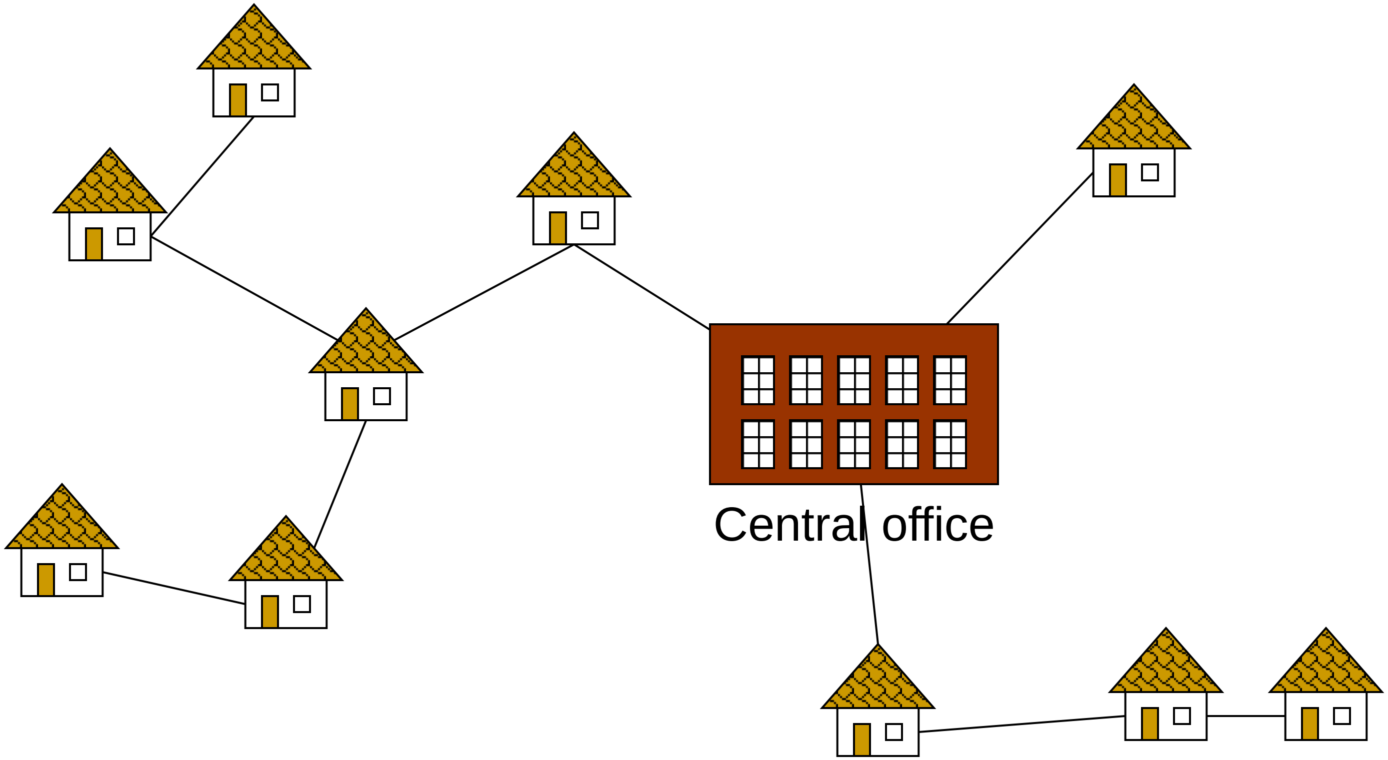


Wiring: Naive Approach



Expensive!

Wiring: Better Approach



Minimize the total length of wire connecting the customers

Definitions

- Given a graph G = (V, E)
- A subgraph of G is a graph G’ with a subset of vertices and a subset of edges
- G’ = (V’, E’) where $V' \subseteq V, E' \subseteq E$

Definitions

- Given a graph G = (V, E)
- A subgraph of G is a graph G' with a subset of vertices and a subset of edges
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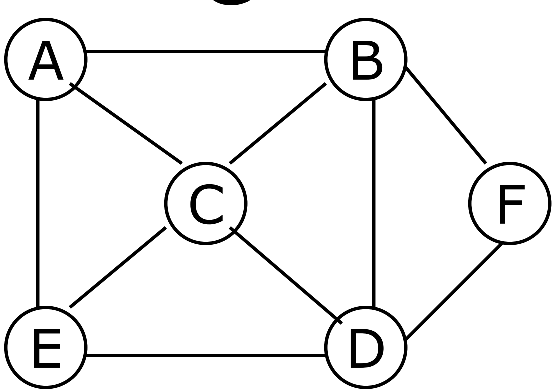
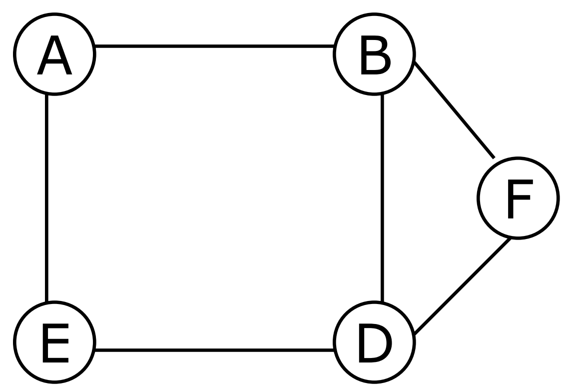
- Examples:

G



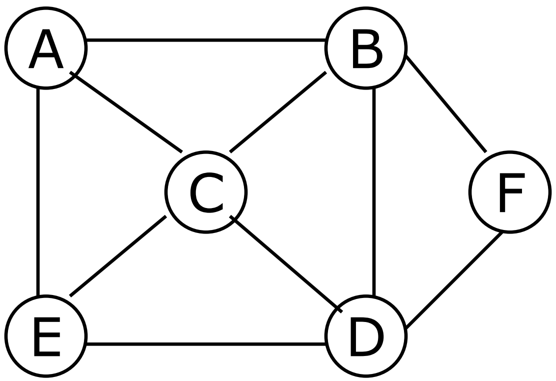
G'



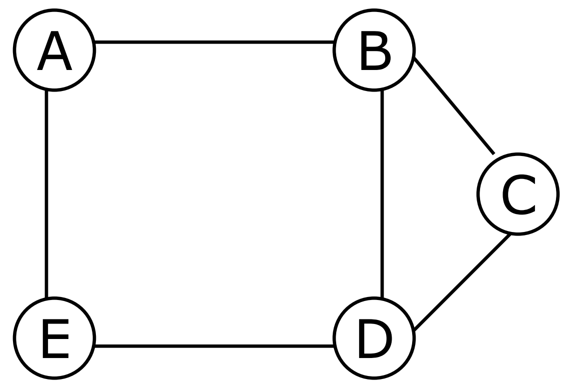
Which Graphs are Subgraphs of G?

G' = (V', E') is a subgraph of G=(V, E) if $V' \subseteq V, E' \subseteq E$

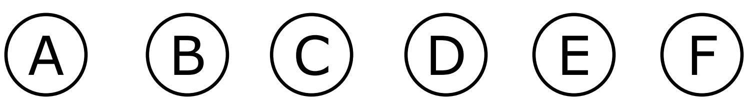
G



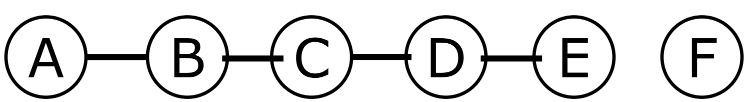
G1



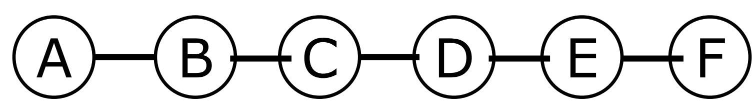
G2



G3



G4



Spanning Tree of a Graph

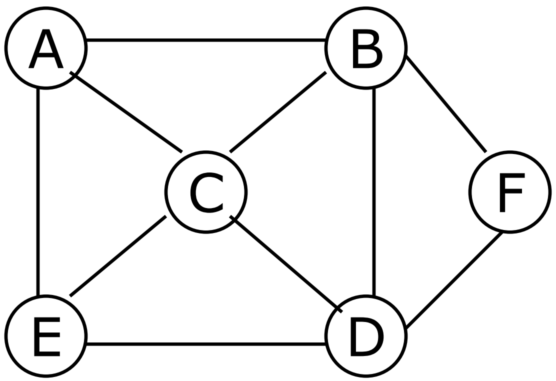
- T is a spanning tree of a graph G = (V, E) if
  - It is a subgraph of G
  - It has all the vertices V of G
  - All the vertices are connected
  - It has no cycles

Claim: If T is a spanning tree of G, then it has exactly |V| - 1 edges

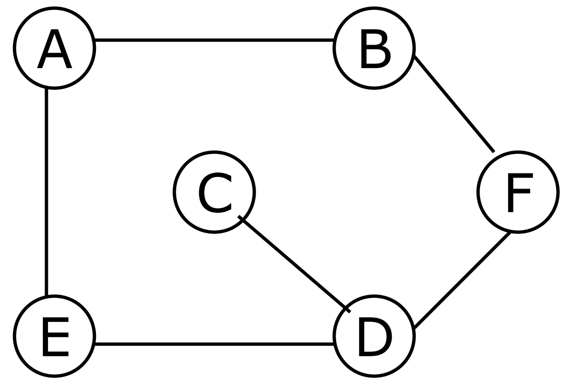
Proof: A tree with n vertices has n-1 edges

Which are Spanning Trees of G?

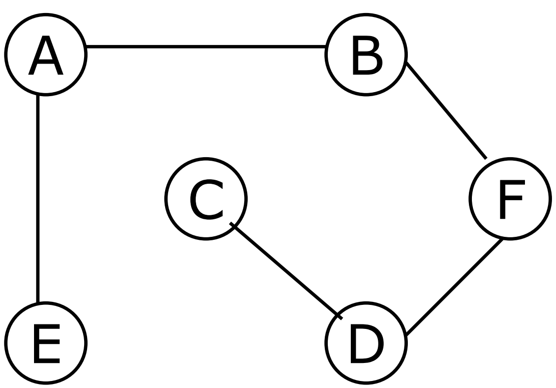
G



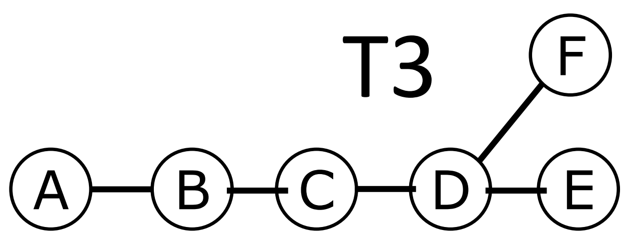
T1



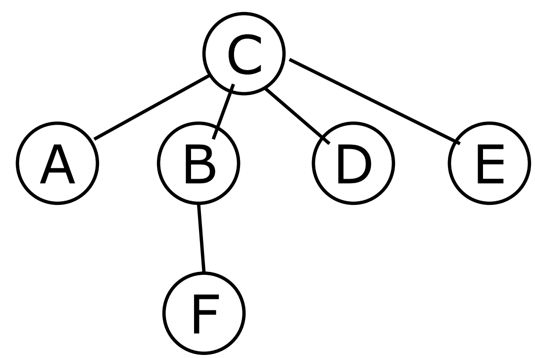
T2



T3



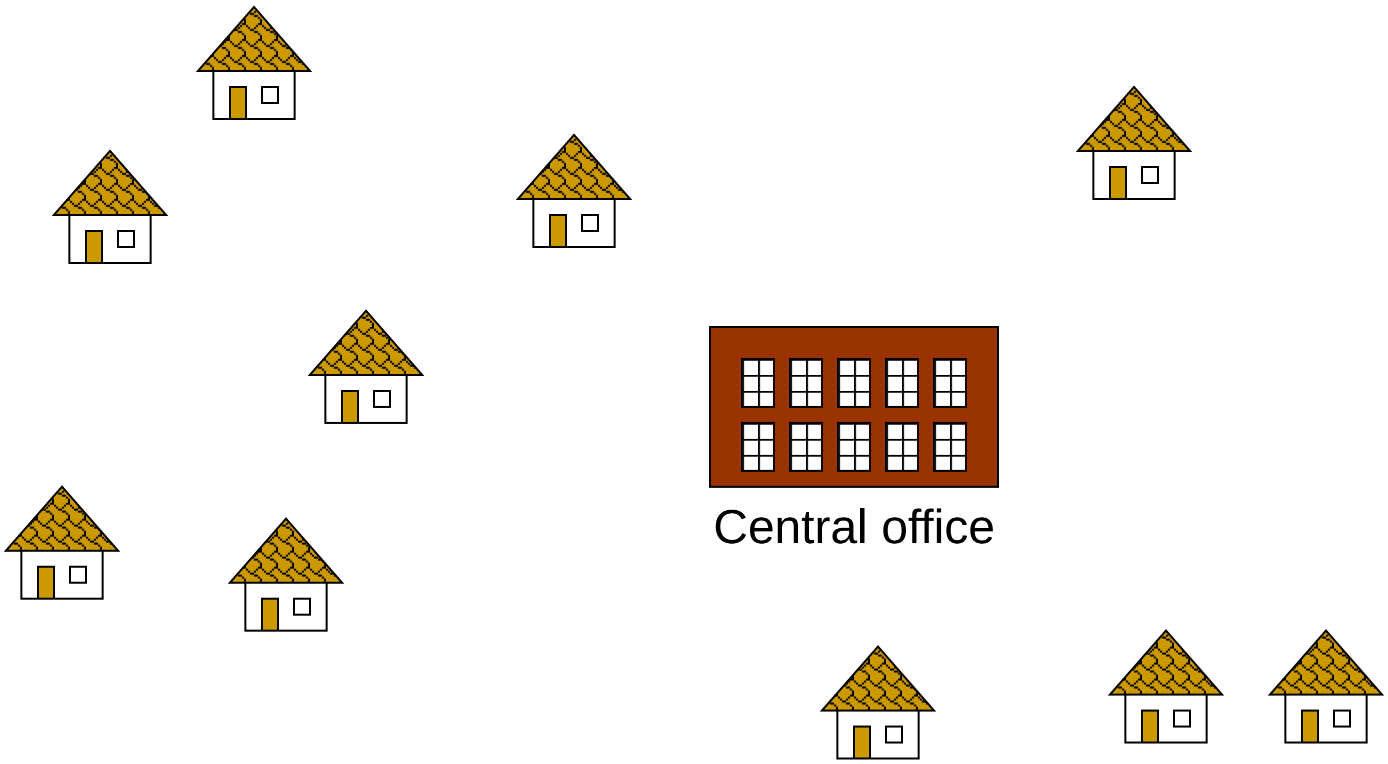
T4



Minimum Spanning Tree

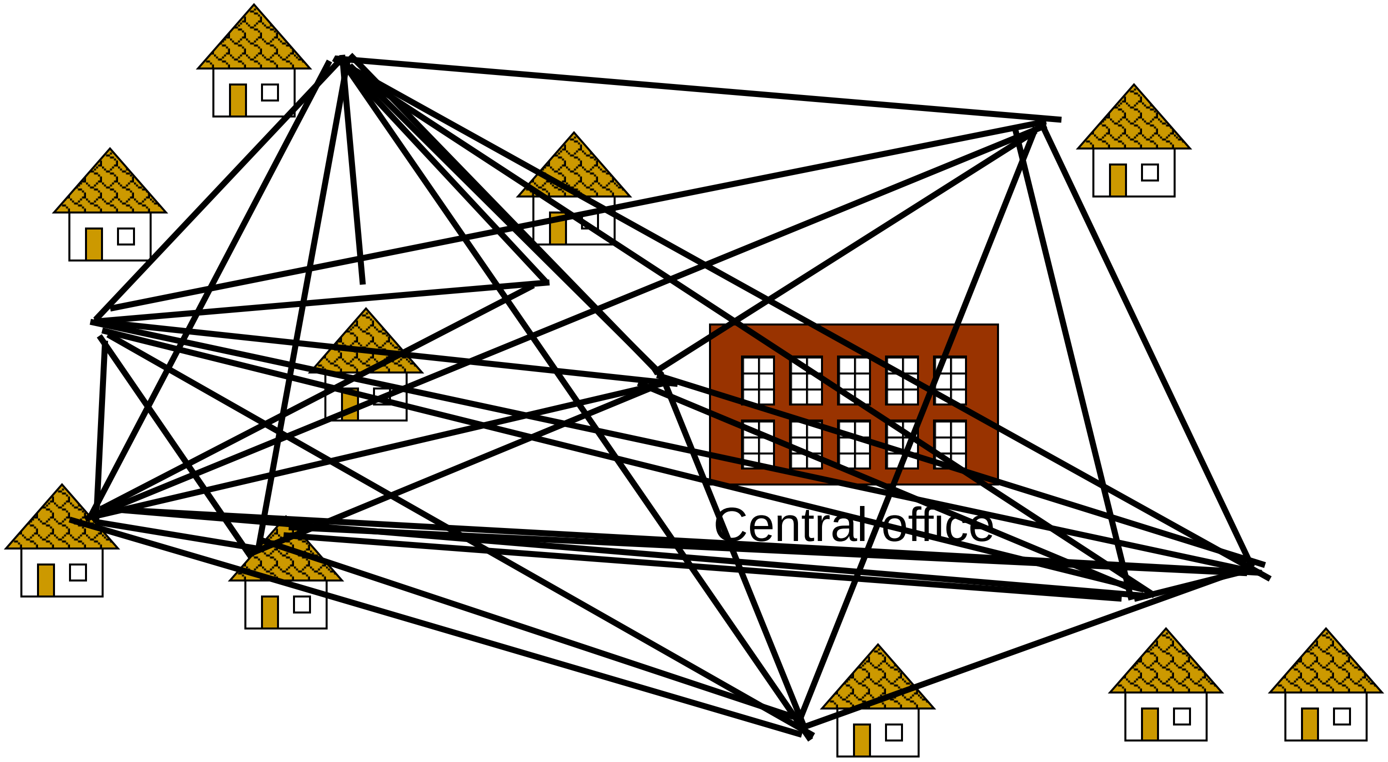
- Given a undirected weighted graph G = (V, E, W) where $W: E \rightarrow R^{+}$
- Cost of a spanning tree T = (V, E’) is given by $W(T) = \sum_{e \in E'} W(e)$
- T is the minimum cost spanning tree if and only iff T is a spanning tree of G and has the minimum cost

Problem: Laying Telephone Wire



Can it be formulated as a MST problem?
What will be the corresponding graph?

Problem: Laying Telephone Wire



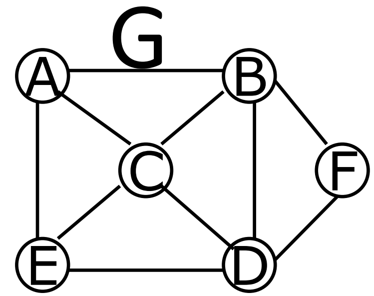
Can it be formulated as a MST problem?
What will be the corresponding graph?

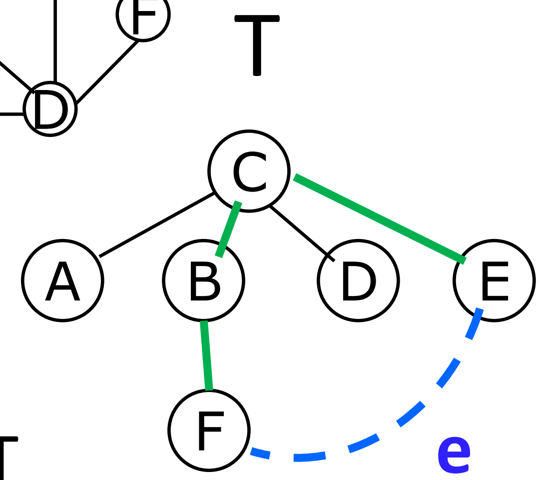
How to Find a MST?

- MST need not be unique
- If the graphs is unweighted, all spanning trees are MSTs
- Why?

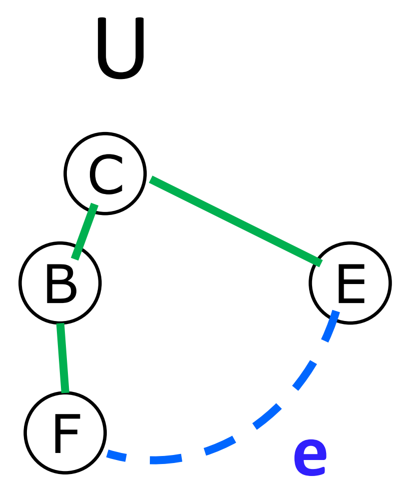
- Before trying to find MST let us examine some of their properties

Cycle Property





- Let **T** be a MST of a weighted undirected weighted graph G
- Let **e** be an edge of G that is not in T
- Let **U** be the cycle formed by adding e in T
- For every edge **f** of **U**, $W(f) \leq W(e)$
- Why?



Cycle Property

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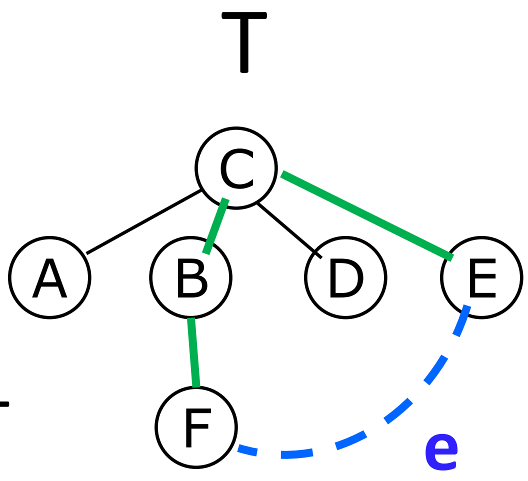
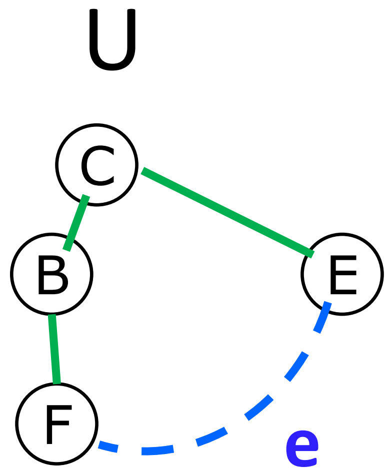
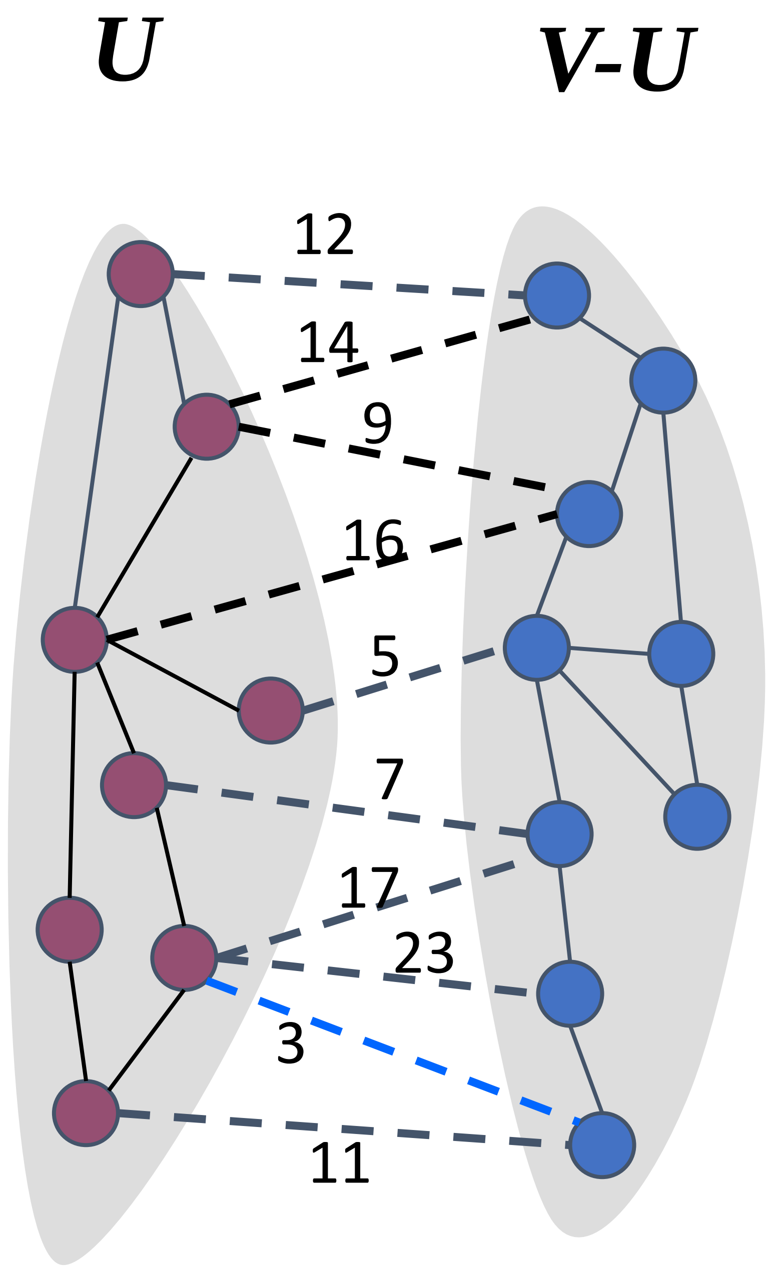
- For every edge **f** of **U**, $W(f) \leq W(e)$
- Why?
- Removing any single edge in U will make it a spanning tree again





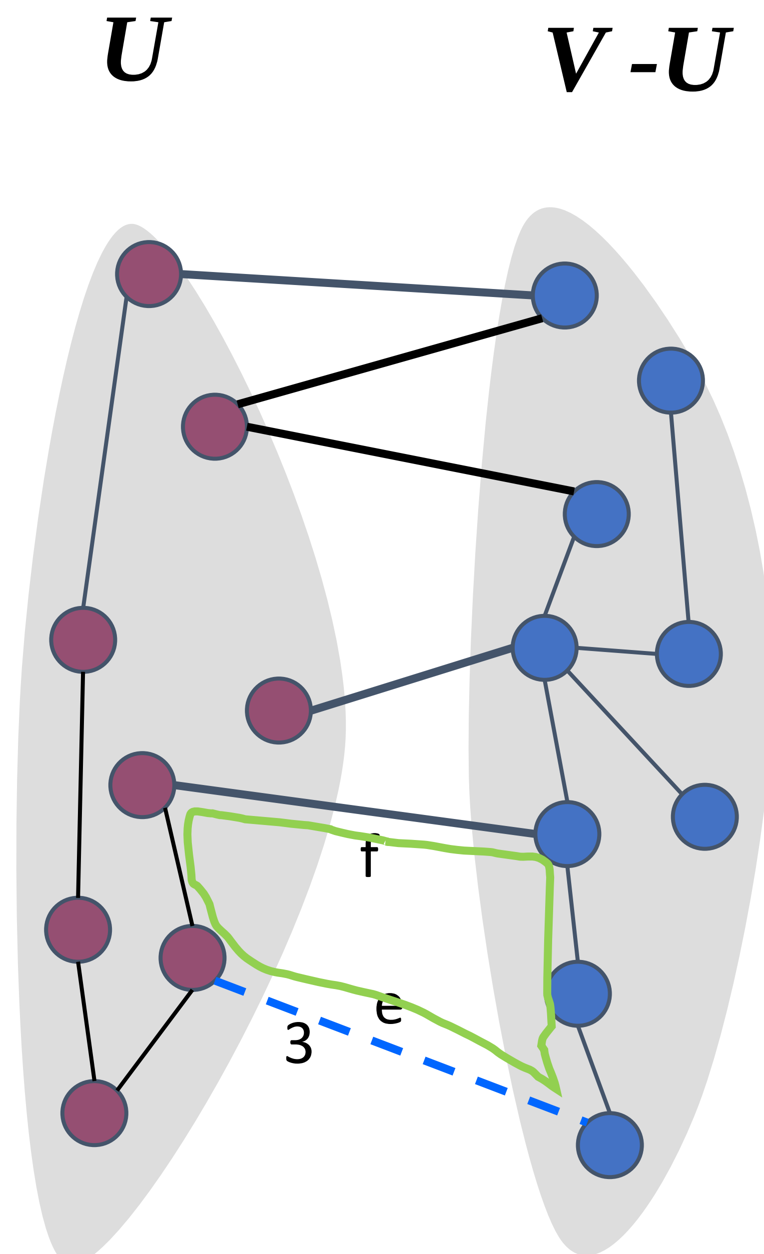
Cut Property

- Consider a partition of the vertices of G into subsets U and V-U
- Let e be an edge of minimum weight across (U, V-U)
- There is a minimum spanning tree of G containing edge e



Cut Property

- e: an edge of minimum weight across (U, V-U) cut
- *Claim:* There is a MST containing edge e
- *Proof:* Let T be an MST of G without e
- Consider the cycle C formed by adding e to T.
- Let f be an edge of C across the cut
- By the cycle property, $W(f) \leq W(e)$
- Thus, $W(f) = W(e)$
- We obtain another MST containing e by replacing f with e



Outline of Prim’s Algorithm for MST

- Start with U = a single vertex u
- Algorithm maintains
  - A connected set of vertices U
  - Using a subset of MST edges
- At very step add the smallest cost edge (x, y) in the cut (U, V-U) to the tree
- It cannot induce a cycle
- The newly added edge must be a part of a MST (using the cut property)
- Add the vertex y to U

Prim's MST Algorithm

Input: G = (V, E, W)

Output: MST T

Initialize: T = Ø; Pick an arbitrary $u \in$ V; U = {u}

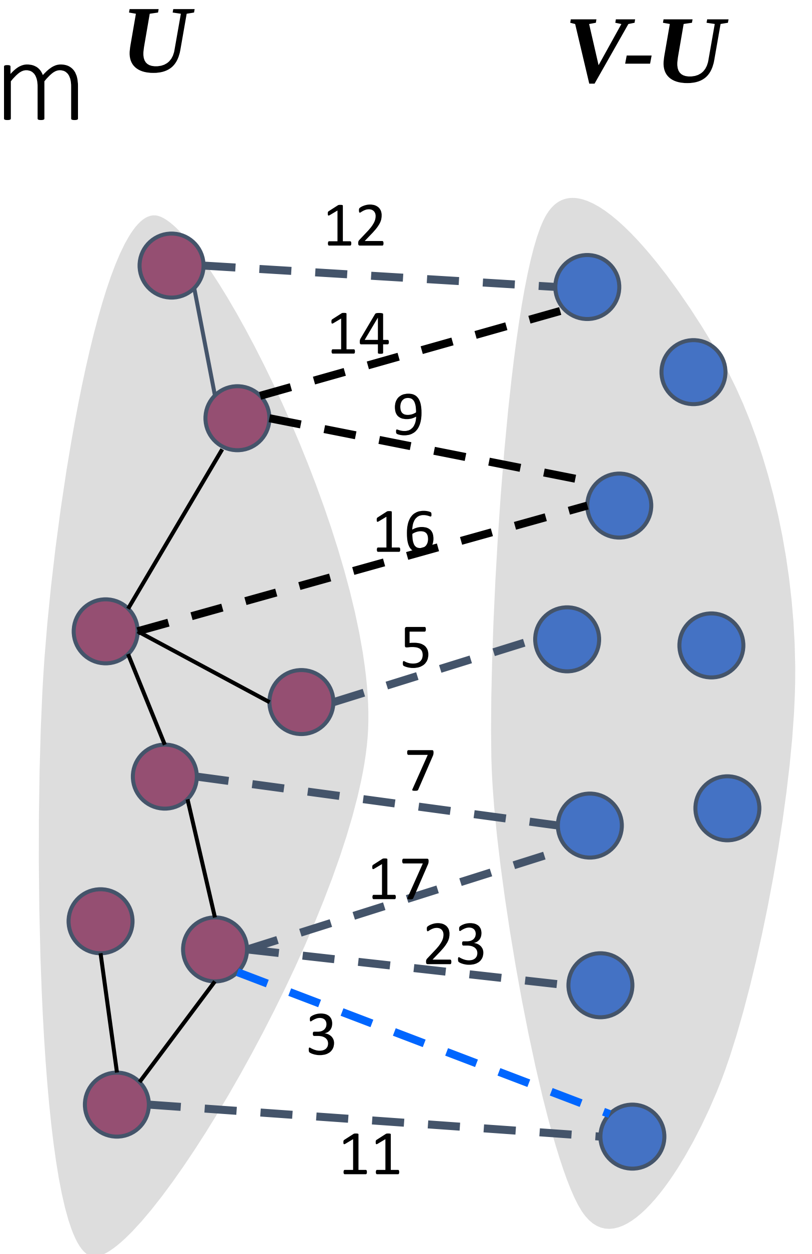
while (U $\neq$ V) do

  let (x, y) be smallest weight edge in (U, V-U)

  $U = U \cup \{ y \}$

  $T = T \cup (x, y)$

end while



Prim's MST Algorithm: Runtime

Input: G = (V, E, W)

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end while

O(|V|) Iterations

O(|E|) steps in a naive implementation

O(|E| |V|) time =O(n^3)

Can we do better?

Prim's MST Algorithm: Improved

Input: $G = (V, E, W)$

Output: MST T

Initialize: T = Ø; Pick an arbitrary $u \in$ V; U = {u}

while (U $\neq$ V) do

  let (x, y) be smallest weight edge in (U, V)

  $U = U \cup \{ y \}$

  $T = T \, U \, (x, y)$

end while

- For each vertex in V – U, store the min-cost edge to U
- Keep all the vertices in V-U in a priority queue
- Fetch the vertex with lowest edge cost to U
- How to maintain dynamically?

Prim’s MST: Improved

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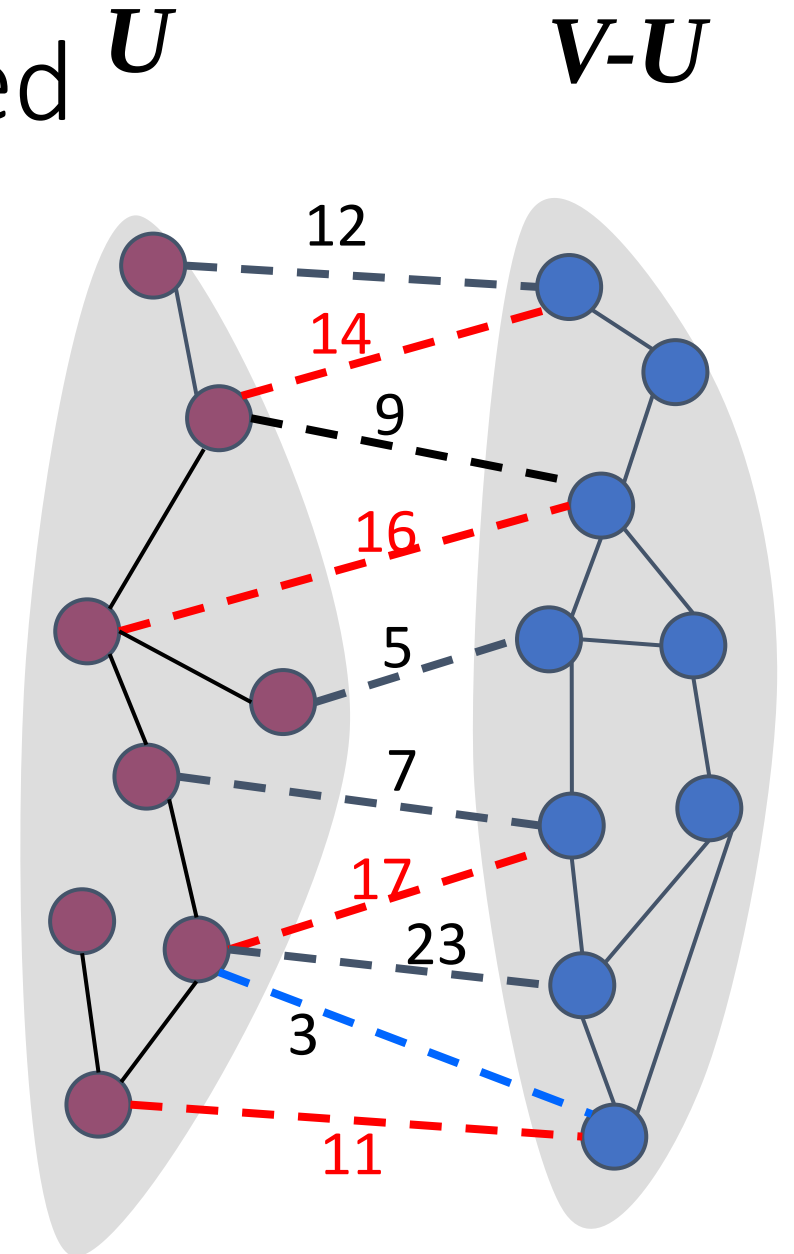

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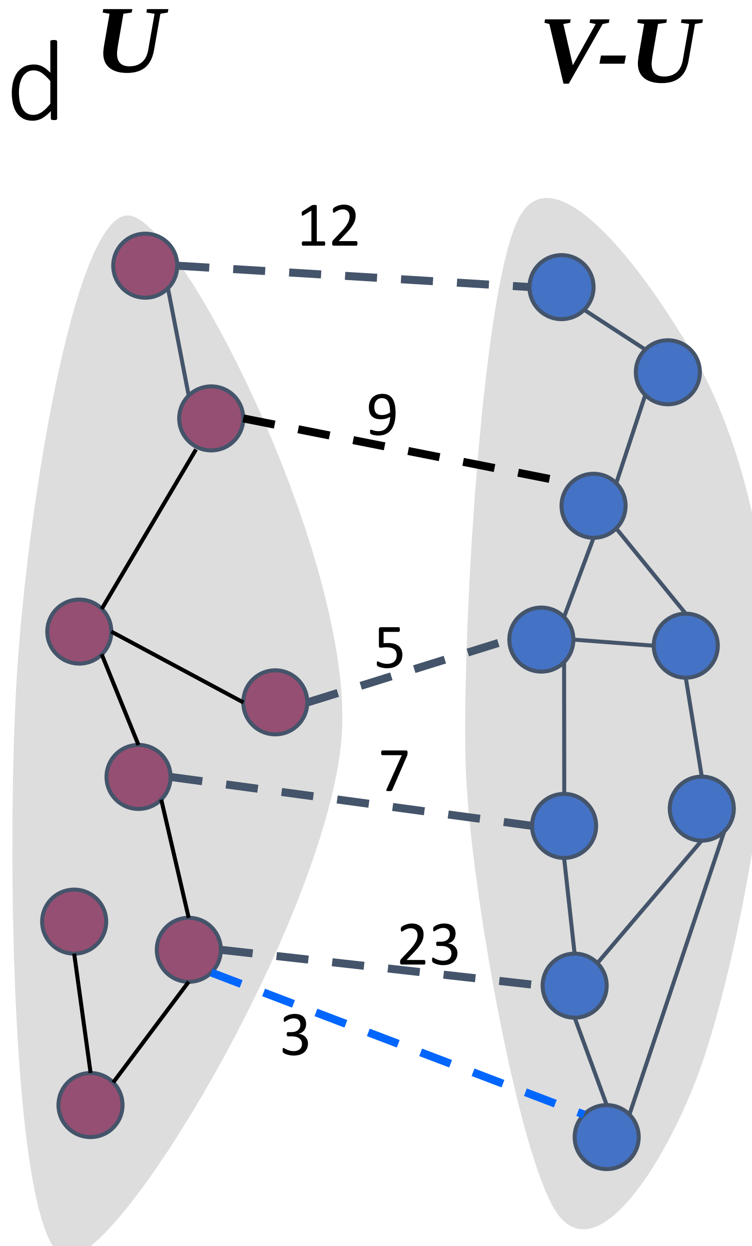

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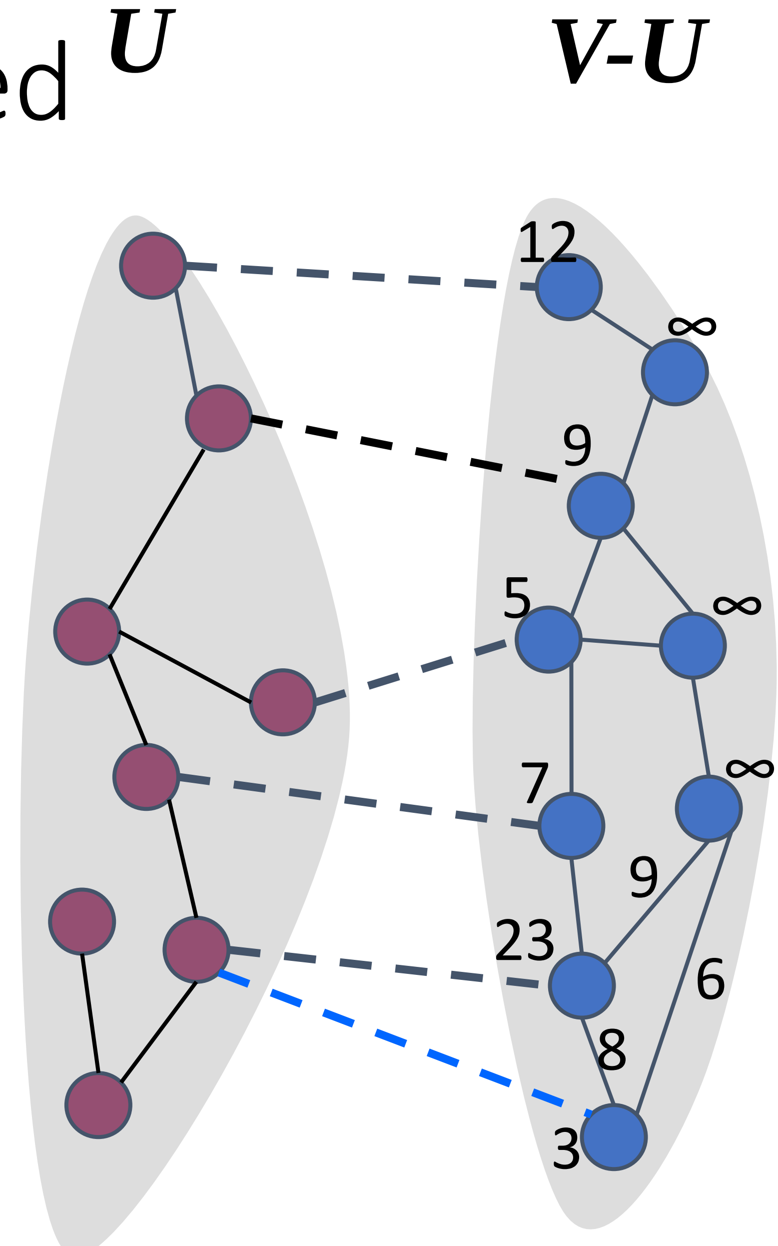

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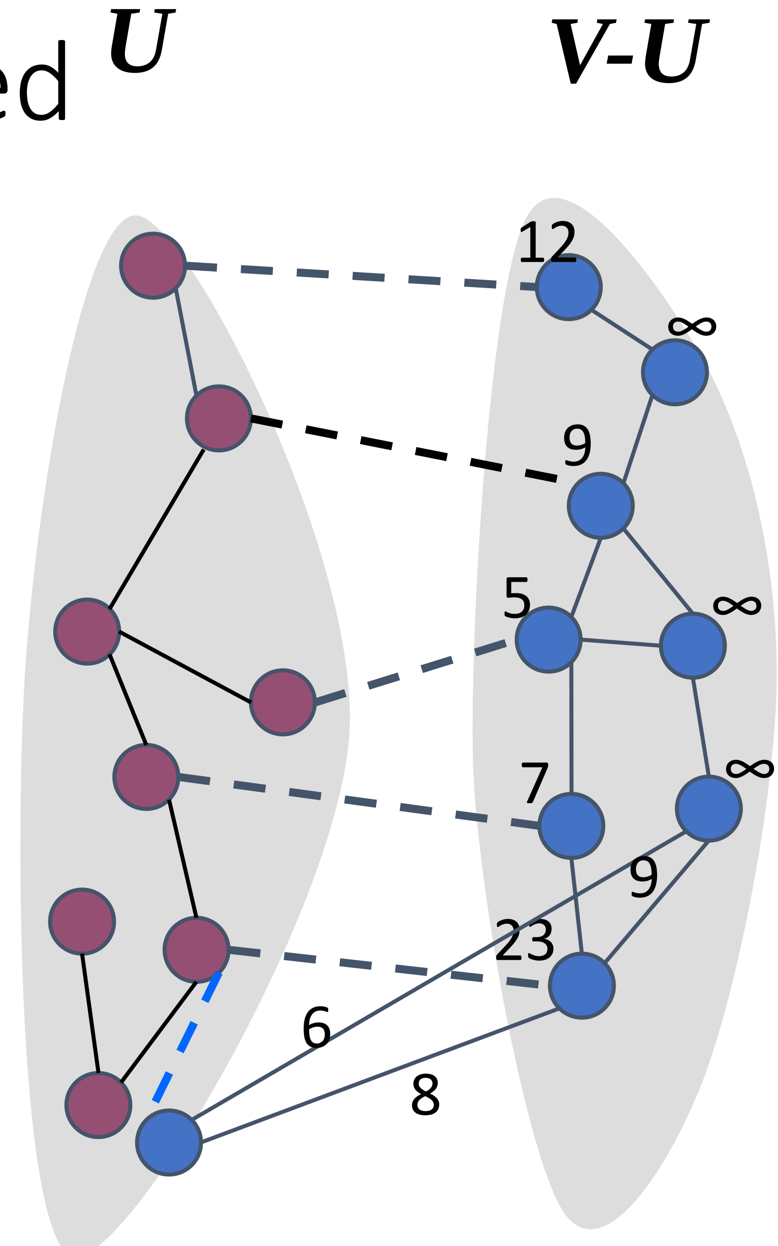

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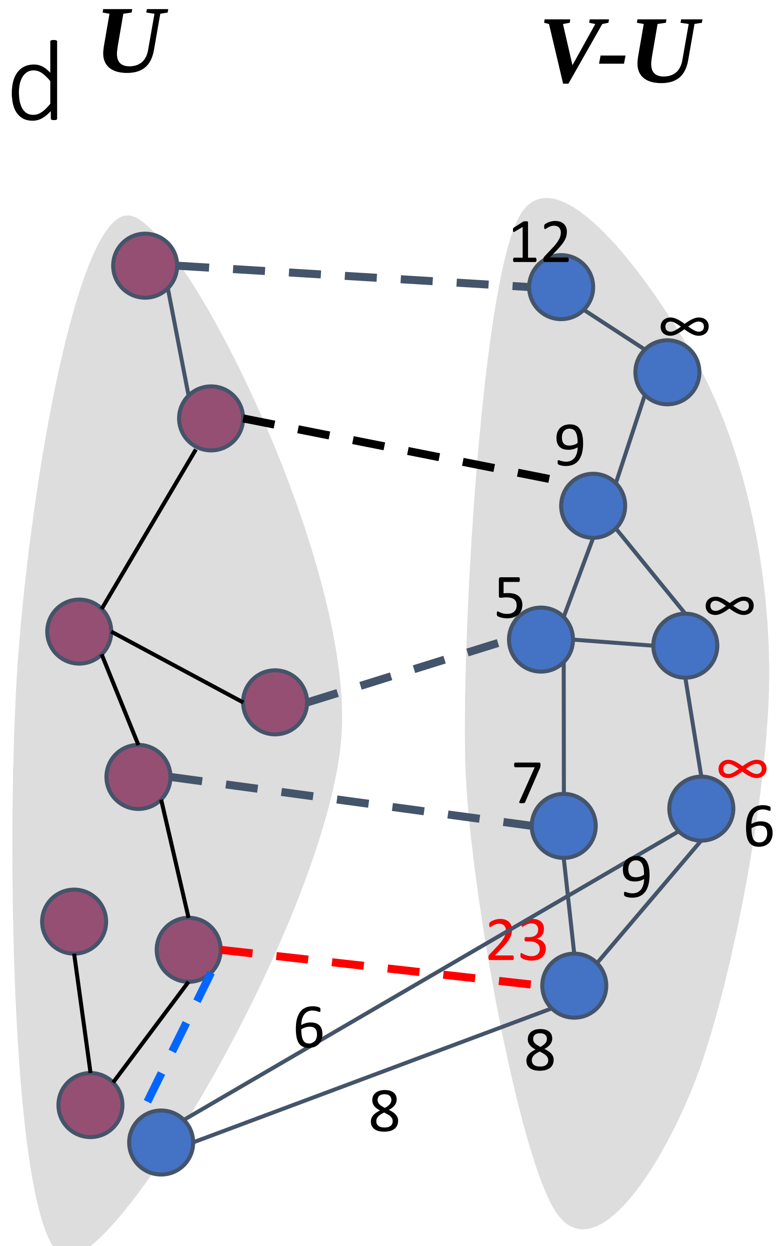
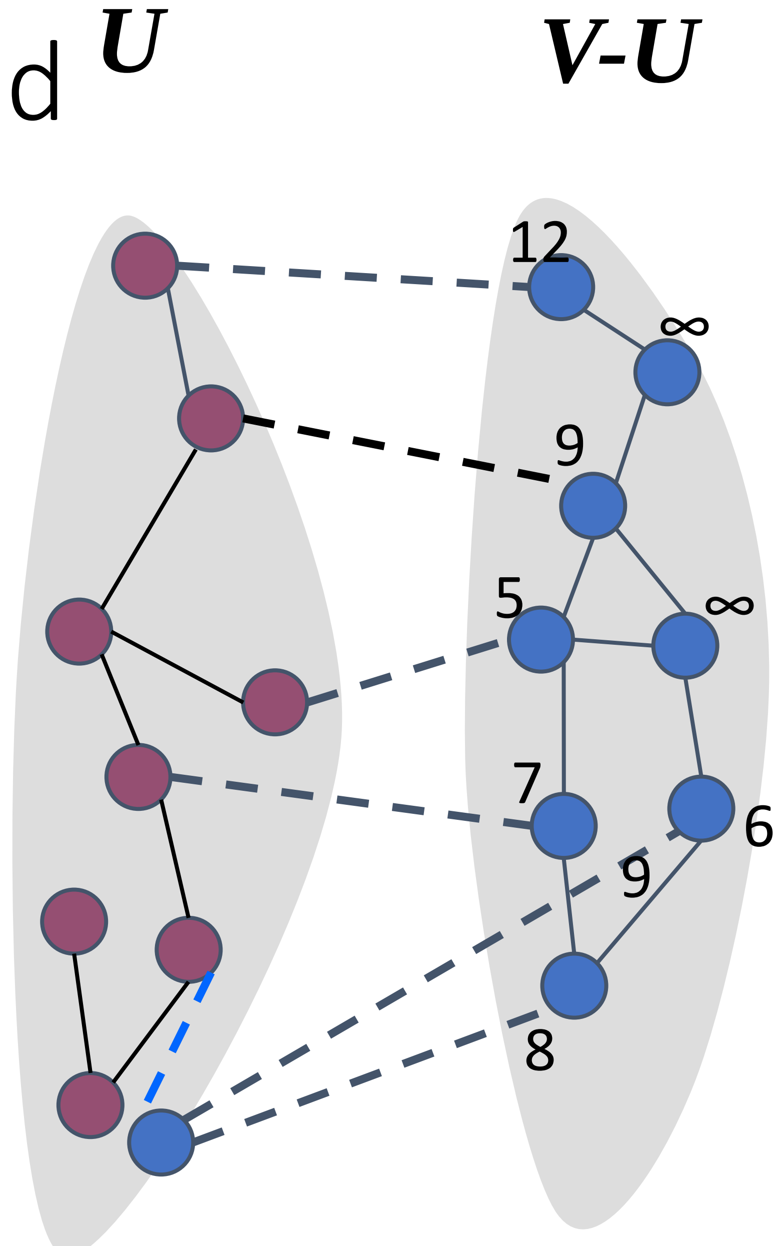


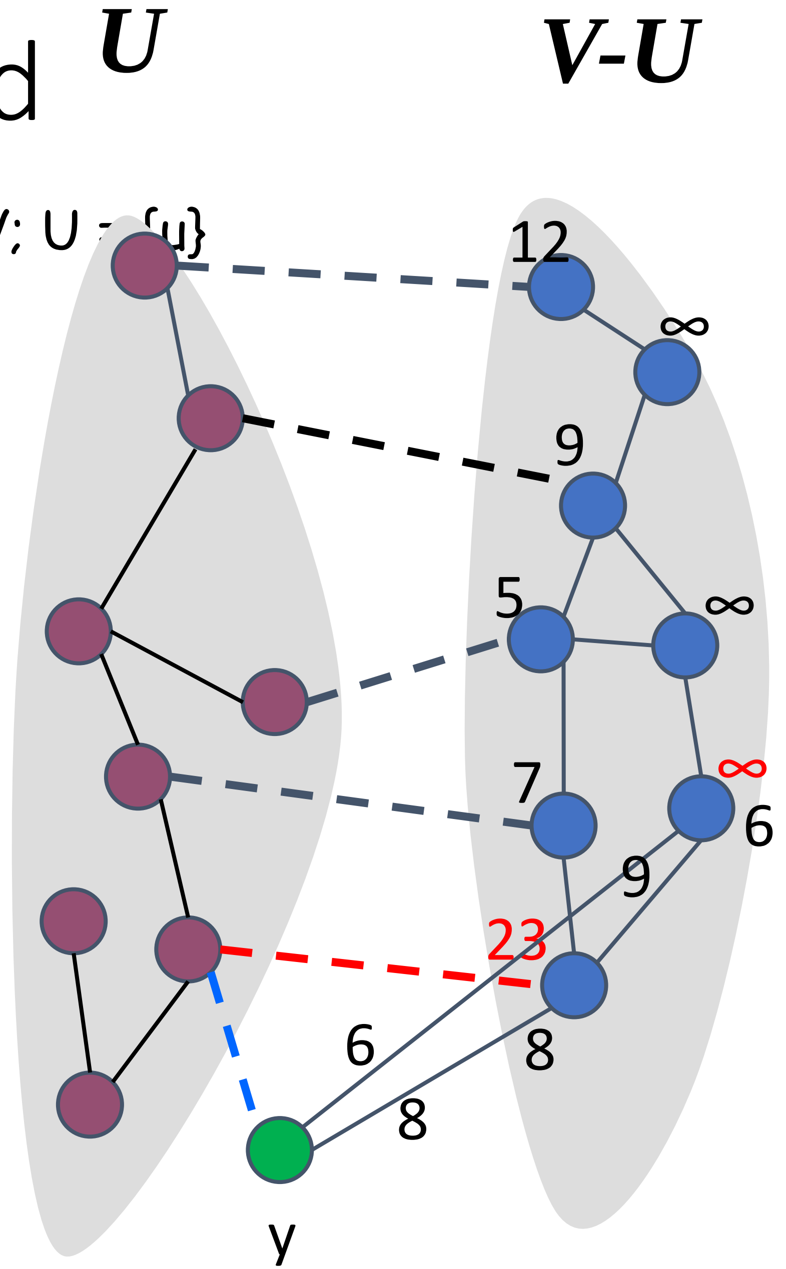
Prim’s MST: Improved

- Fetch the vertex with lowest edge cost to U
- Update the weights of all vertices in V-U
- How to maintain dynamically?
- Need the following in priority queues
  - InsertHeap
  - DeleteMin
  - DecreaseWeight



Prim's MST Improved

Initialize: T = Ø; Pick an arbitrary $u \in$ V; U = {u}
for all v: (u, v) in E do
    e = (u, v); D(v) = w(u,v)
    InsertHeap(w(u, v), e)
while (U $\neq$ V) do
   (x, y) = DeleteMin()
    $U = U \cup \{ y \}$
    $T = T \cup (x, y)$
    forall (y, w) in E st w in V-U do
        if w(y, w) < D(w)
            D(w) = w(y, w)
            DecreaseWeight(w)
        endif
    endfor
end while



Prim's MST Improved:Runtime

Initialize: T = Ø; Pick an arbitrary $u \in$ V; U = {u}
for all v: (u, v) in E do
    e = (u, v); D(v) = w(u,v)
    InsertHeap(w(u, v), e) ← O(|V| log(|V|) total
while (U ≠ V) do
   (x, y) = DeleteMin() ← O(|V| log(|V|) total
    $U = U \cup \{ y \}$
    $T = T\ U\ (x, y)$
    forall (y, w) in E st w in V-U do ← O(|E|) iterations total
        if w(y, w) < D(w)
            D(w) = w(y, w)
            DecreaseWeight(w)
        endif
    endfor
end while

O(|E|) iterations total
O(log(V)) per iteration

Total runtime: O(|V| log(|V|) + |E| log(|V|))

Using Fibonacci Heaps

Depending on the heap implementation, running time could be improved!

	EXTRACT-MIN	DECREASE-KEY	Total
binary heap	O(lgV)	O(lgV)	O(ElgV)
Fibonacci heap	O(lgV)	O(1)	O(VlgV + E)

From $O(|E|\log(|V|)$ to $O(|V|\log|V| + |E|)$

Thank You