# COL106
# Data Structures and Algorithms

Subodh Sharma and Rahul Garg

# Announcements

# Exam Syllabus

- The syllabus is all that has been covered in the class
- The major and lab tests may not have questions from all the topics
- But you are expected to study all the topics covered in the classes

# Exam Syllabus

- The syllabus is all that has been covered in the class
- The major and lab tests may not have questions from all the topics
- But you are expected to study all the topics covered in the classes
- Some of the topics introduced will be covered in more depth in subsequent courses
- Use this as a learning opportunity

# Union-Find

Based on slides by:  Goodrich and Tamasia, CSE 373 University of Washington, Kong Lwang, Xzou, Purdue University Indianapolis,

# Applications

- Finding connected components in a Graph
- Example: A social network
- Given a set, S, of n people.
- A social network for S comprises a set E, of edges or ties between pairs of people (such as in a friendship network, like Facebook, or tagging network by twitter)
- A **connected component** in a network is a subset, T of S such that:
  - Every two people in T are connected through a sequence of friendship relationships
  - No one in T is friends with anyone outside of T.

# Union-Find Operations

- A **partition** or **union-find** structure is a data structure supporting a collection of disjoint sets subject to the following operations:
- makeSet(e): Create a singleton set containing the element e and return the position storing e in this set
- union(A,B): Return the set A U B, naming the result “A” or “B”
- find(e): Return the set containing the element e

# Connected Components Algorithm

**Algorithm** UFConnectedComponents$(S, E)$:

***Input:*** A set, $S$, of $n$ people and a set, $E$, of $m$ pairs of people from $S$ defining pairwise relationships

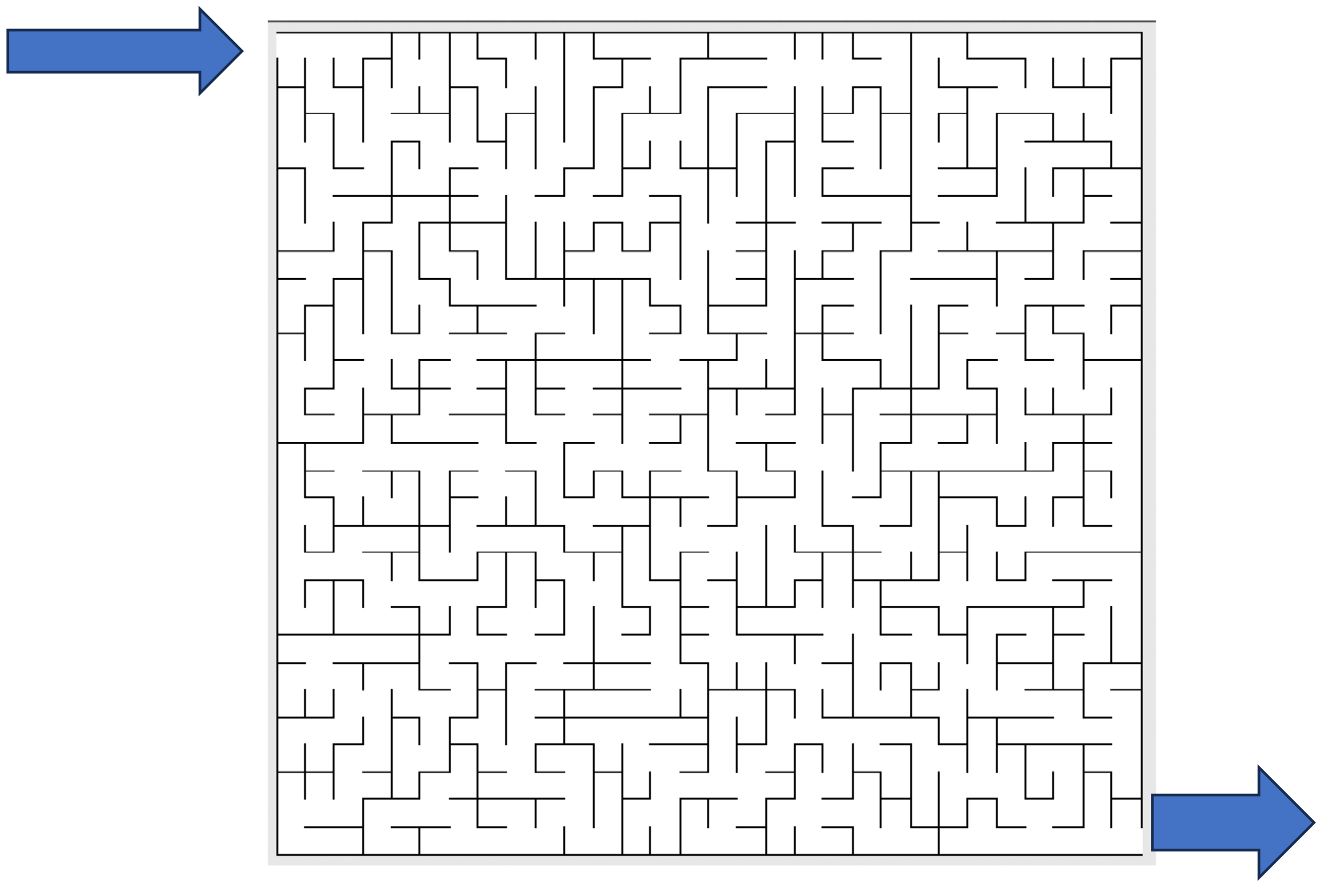
***Output:*** An identification, for each $x$ in $S$, of the connected component containing $x$

**for** each $x$ in $S$ **do**
    makeSet$(x)$
**for** each $(x, y)$ in $E$ **do**
    **if** find$(x) \neq$ find$(y)$ **then**
        union(find$(x)$, find$(y)$)
**for** each $x$ in $S$ **do**
    Output "Person $x$ belongs to connected component" find$(x)$

- The running time of this algorithm is O(t(n,n+m)), where t(j,k) is the time for k union-find operations starting from j singleton sets.

# Maze Construction

- Problem: Construct a good maze that connects entry point to the exit point



# How to Construct a Maze?

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- Hint: Model it as a connectivity problem in a Graph

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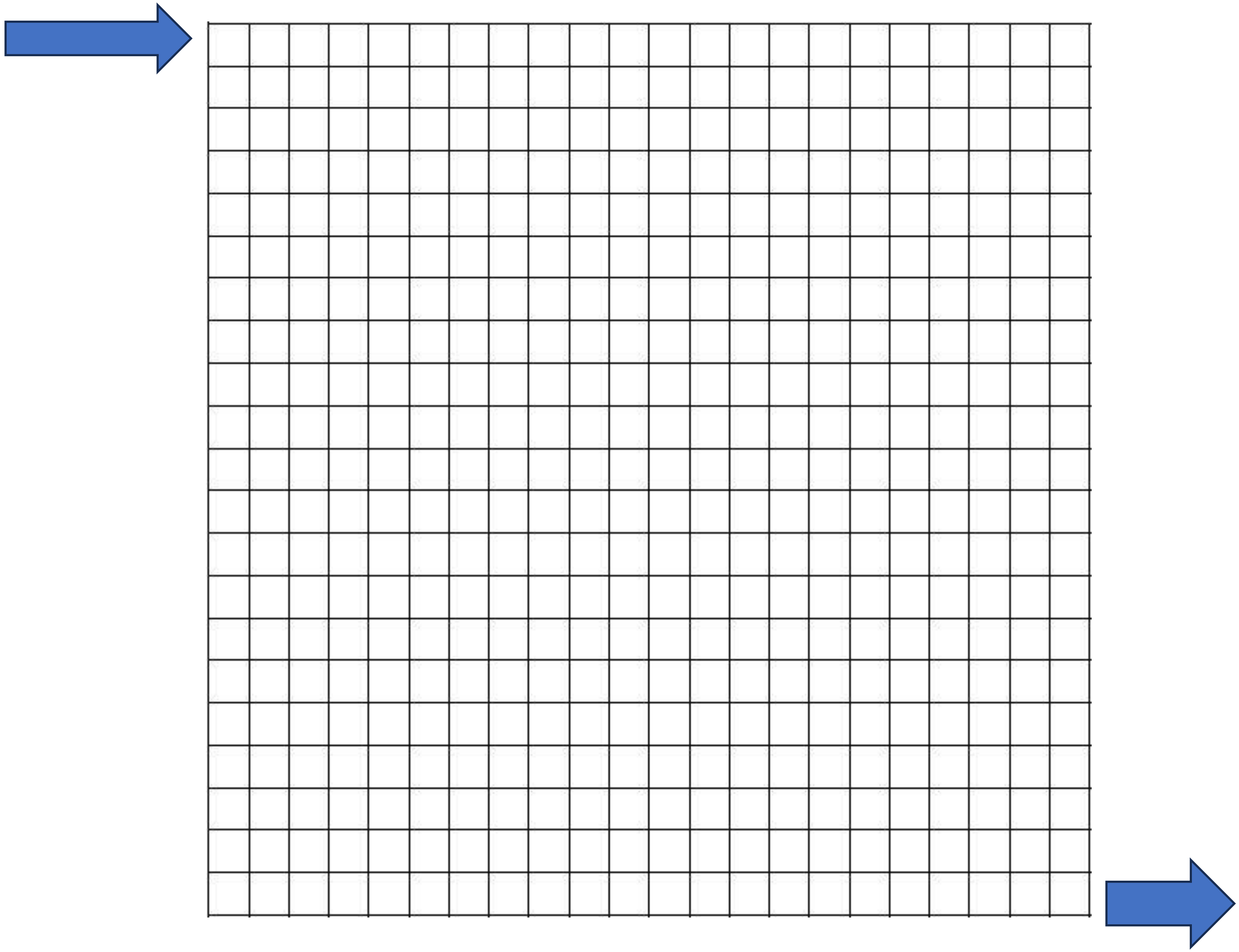
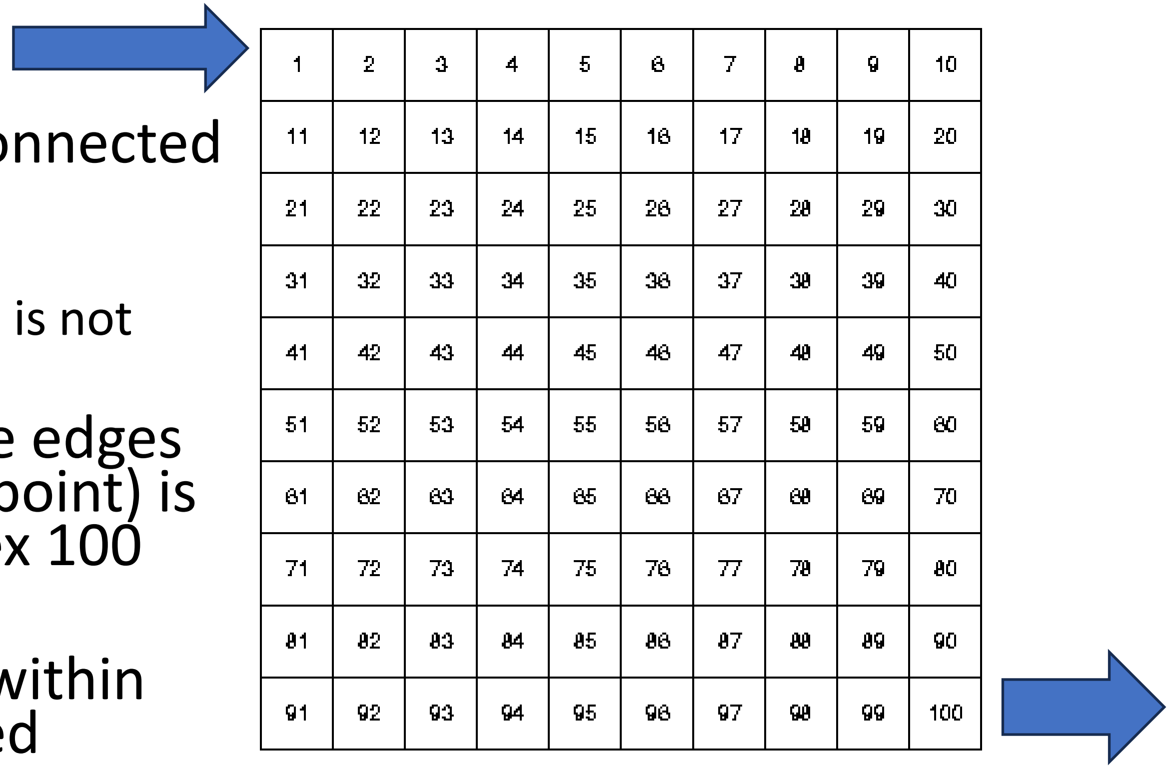
- Hint: Model it as a connectivity problem in a Graph
- What are the vertices?
- What are the edges?



| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

# How to Construct a Maze?

- Vertices are cells
- Two vertices are connected by an edge if:
  - They are adjacent
  - The adjoining wall is not there
- Keep on adding the edges till vertex 1 (entry point) is connected to vertex 100 (exit point)
- Do not add edges within the same connected components
- Construct a spanning tree



# A Maze Generator

**Algorithm** MazeGenerator$(G, E)$:

***Input:*** A grid, $G$, consisting of $n$ cells and a set, $E$, of $m$ "walls," each of which divides two cells, $x$ and $y$, such that the walls in $E$ initially separate and isolate all the cells in $G$

***Output:*** A subset, $R$ of $E$, such that removing the edges in $R$ from $E$ creates a maze defined on $G$ by the remaining walls

**while** $R$ has fewer than $n - 1$ edges **do**

  Choose an edge, $(x, y)$, in $E$ uniformly at random from among those previously unchosen

  **if** find$(x) \neq$ find$(y)$ **then**

    union(find$(x)$, find$(y)$)

    Add the edge $(x, y)$ to $R$

**return** $R$

# Data Structures for Union Find

- Linked lists?
- Trees?
- Others?

# Disjoint Sets with Linked Lists

- Approach 1: Create a linked list for each set.
  - last/first element is representative
  - cost of union? find?

- Approach 2: Create linked list for each set. Every element has a reference to its representative.
  - last/first element is representative
  - cost of union? find?

# Disjoint Sets with Linked Lists

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- Union:

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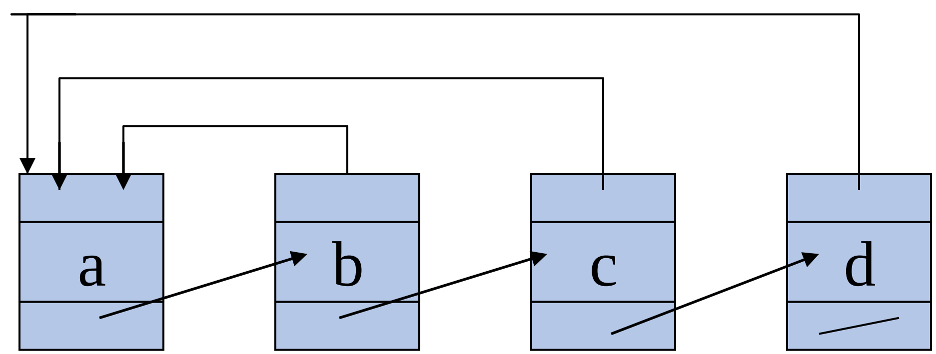
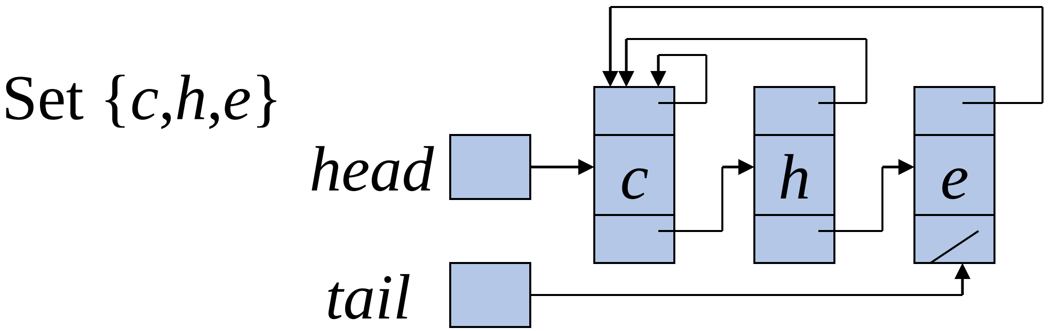
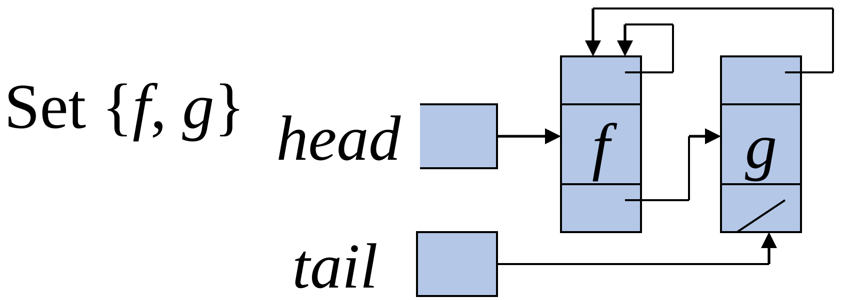
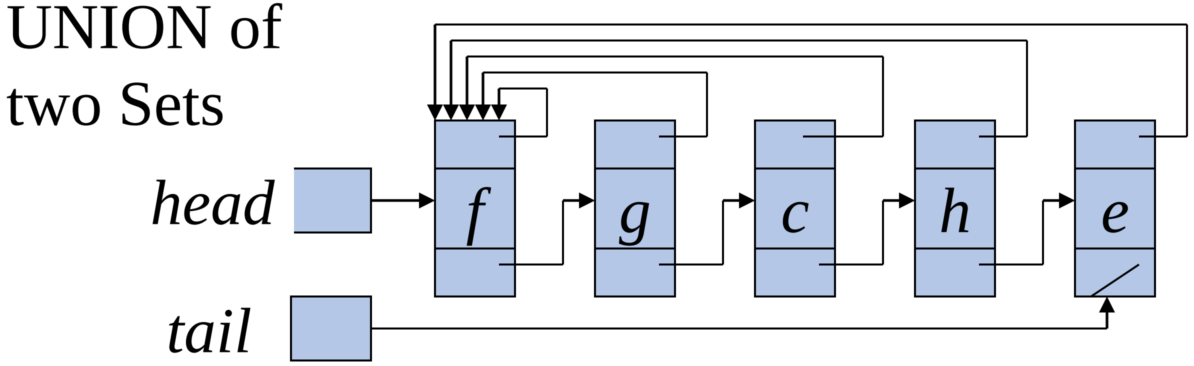
- Approach 2: Create linked list for each set. Every element has a reference to its representative.
  - Last/first element is representative
  - Cost of union? find?



# Linked-lists for two sets



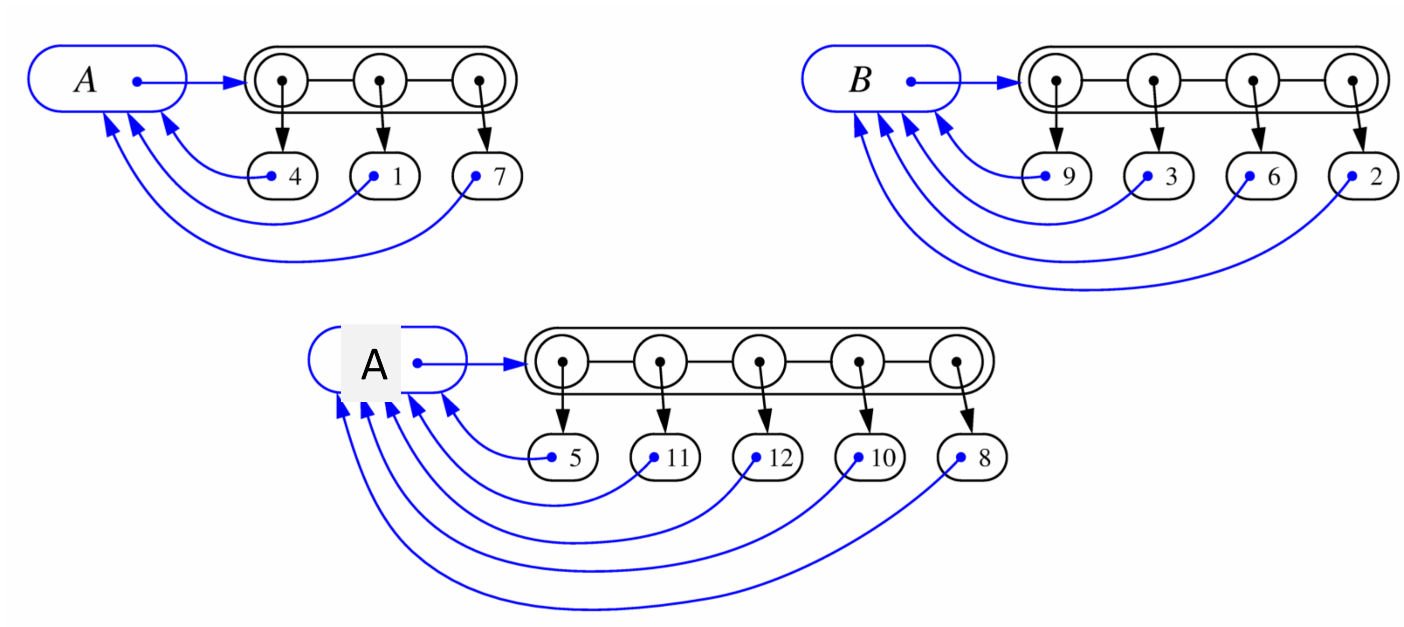



# UNION Implementation

- A simple implementation: UNION($x$,$y$) just appends $x$ to the end of $y$, updates all back-to-representative pointers in $x$ to the head of $y$.
- Each UNION takes time linear in the $x$'s length.
- Suppose $n$ MAKE-SET($x_i$) operations ($O$(1) each) followed by $n$-1 UNION
  - UNION($x_1$, $x_2$), $O$(1),
  - UNION($x_2$, $x_3$), $O$(2),
  - …..
  - UNION($x_{n-1}$, $x_n$), $O$($n$-1)
- The UNIONs cost 1+2+…+$n$-1=$\Theta(n^2)$
- So 2$n$-1 operations cost $\Theta(n^2)$, average $\Theta(n)$ each.
- Not good!! How to solve it ???

# Linked-lists: Approach 2b

- For union, keep the identity of larger set
- Modify the elements of the smaller set to point to the larger set's ID



# Analysis of Approach 2b

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and m finds ??

# Analysis of Approach 2b

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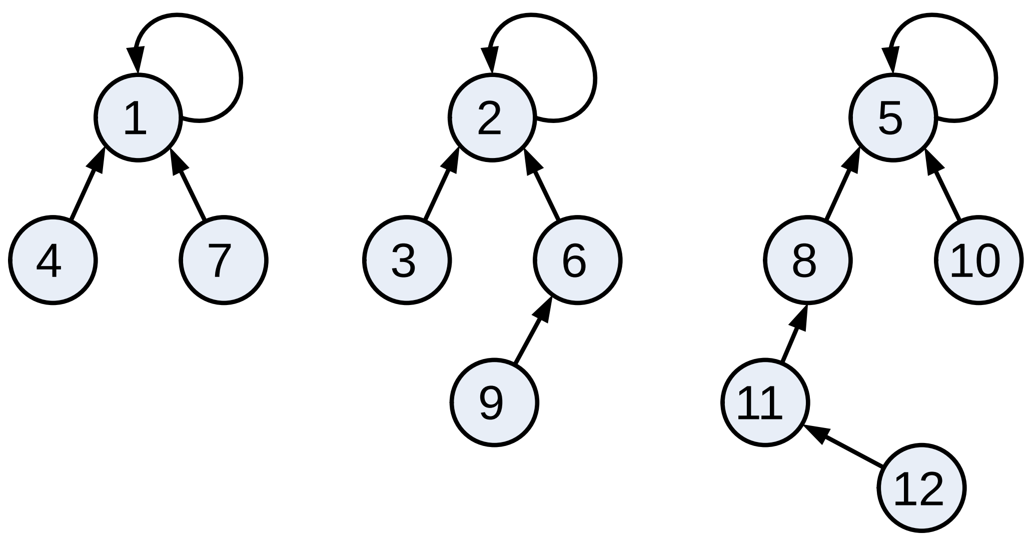
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and m finds is O(n log n + m).

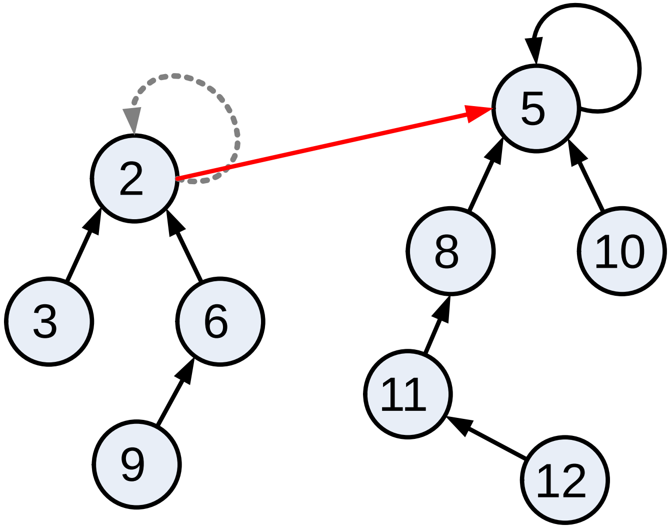
# Tree-based Implementation

- Each element is stored in a node, which contains a pointer to parent
- A node v whose parent pointer points back to v is also a set name
- Each set is a tree, rooted at a node with a self-referencing set pointer
- For example: The sets “1”, “2”, and “5”:

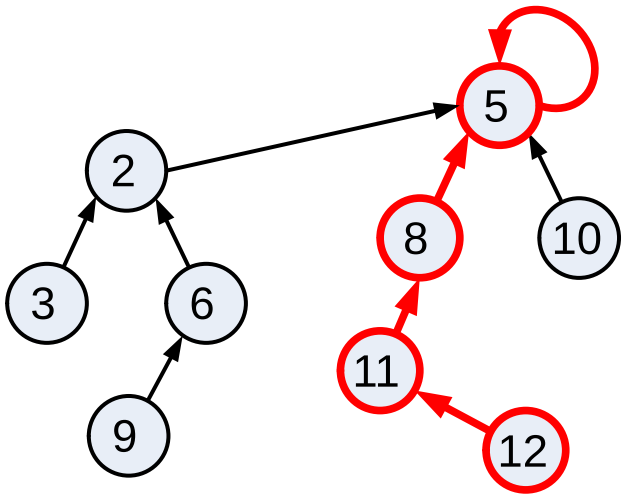


# Union-Find Operations

- To do a union, simply make the root of one tree point to the root of the other

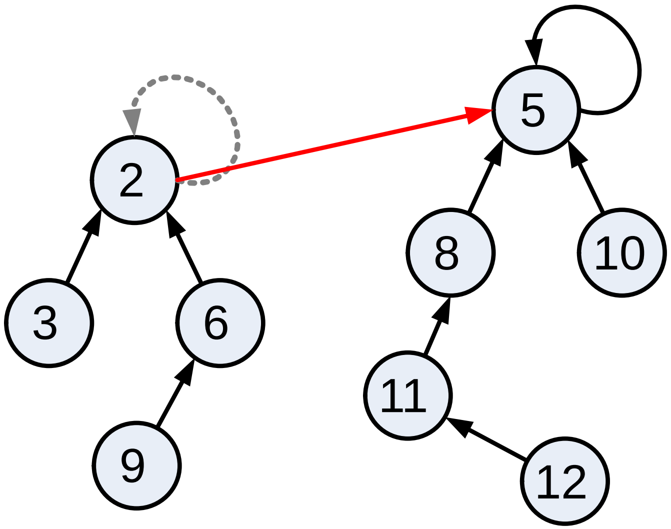



- To do a find, follow set-name pointers from the starting node until reaching a node whose parent pointer refers back to itself



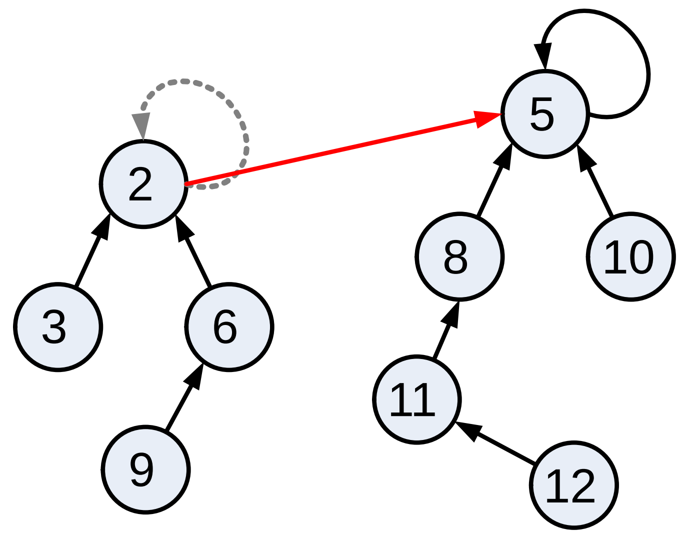
# Union-Find Heuristic 1a

- **Union by size**:
  - Maintain a variable size at every node
  - When performing a union, make the root of smaller tree point to the root of the larger
  - Update the size of the root
- Implies O(n log n) time for performing n union-find operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most O(log n) pointers for any find.



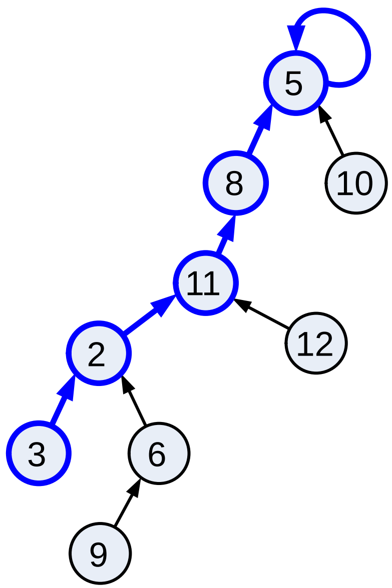
# Union-Find Heuristic 1b

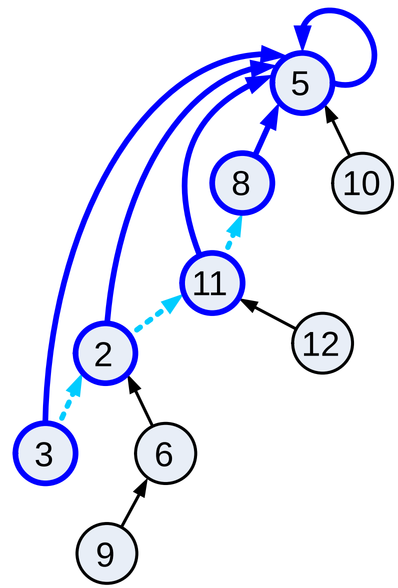
- **Union by rank**:
  - Maintain a rank at every node initialized to zero
  - When performing a union, make the root of smaller-rank tree point to the root of the larger rank three
  - If rank of both the trees are the same, add one to the rank of the root
- Implies O(n log n) time for performing n union-find operations:
  - Why?



# Union-Find Heuristic 2

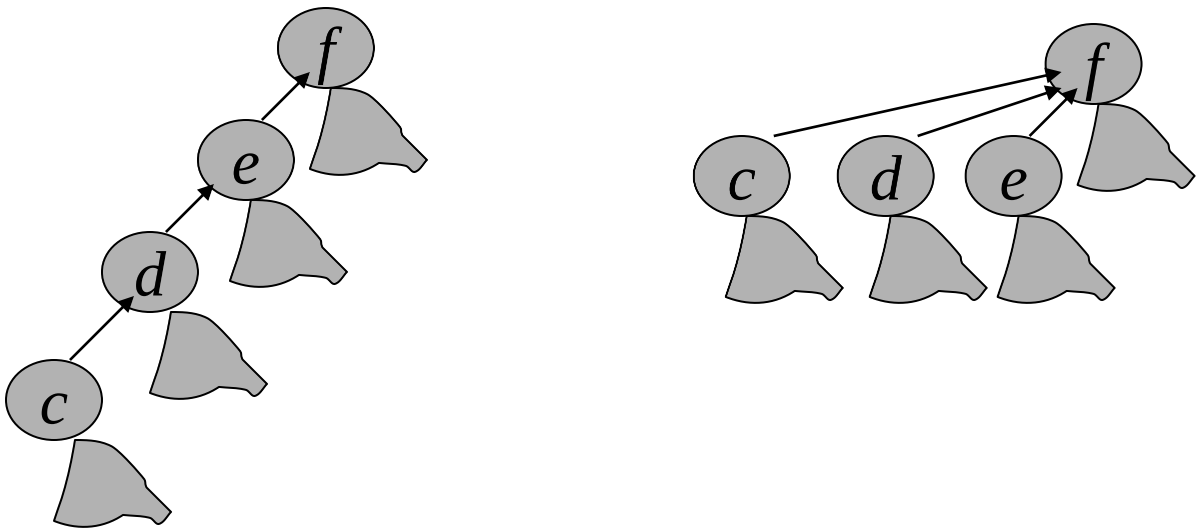
- **Path compression**:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root




- Implies a fast "almost linear" time for n union-find operations.

# Path Compression



# Algorithm for Disjoint-Set Forest

MAKE-SET($x$)
1. $p[x] \leftarrow x$
2. $rank[x] \leftarrow 0$

UNION($x$,$y$)
1. LINK(FIND-SET($x$),FIND-SET($y$))

LINK($x$,$y$)
1. **if** $rank[x]>rank[y]$
2. **then** $p[y] \leftarrow x$
3. **else** $p[x] \leftarrow y$
4. **if** $rank[x]=rank[y]$
5. **then** $rank[y]$++

FIND-SET($x$)
1. **if** $x \neq p[x]$
2. **then** $p[x] \leftarrow$ FIND-SET($p[x]$)
3. **return** $p[x]$

Worst case running time for $m$ MAKE-SET, UNION, FIND-SET operations is: $O(m\alpha(n))$ where $\alpha(n)\leq 4$. So nearly linear in $m$.

# Analysis of Union by Rank with Path Compression (by amortized analysis)

- Discuss the following:
  - A very quickly growing function and its very slowly growing inverse
  - Properties of Ranks
  - Proving time bound of $O(m\alpha(n))$ where $\alpha(n)$ is a very slowly growing function.

# A Fun Exercise

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What is the largest number you can imagine represented using less than 100 characters?

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Start thinking algorithmically

# A Fun Exercise

What is the largest number you can imagine represented using less than 100 characters?

Start thinking algorithmically and non-linearly

# Largest number you can imagine

# Ackermann Function

Let **4**

$$A_0(x) = x + 1$$ **10**

$$A_{i+1}(x) = A_i^{(x)}(x)$$ **17**

where, f(k) denotes k applications of the function f to x

$$f^{(1)}(x) = f(x)$$ **13**

$$f^{(k)}(x) = f\left(f^{(k-1)}(x)\right)$$ **21**

$$A_9(99)$$ **7**

4 + 10+17+13+21+7

**Total: 72 characters**

# Ackermann Function

The version of the Ackermann function we use is based on an indexed function, $A_i$, which is defined as follows, for integers $x \geq 0$ and $i > 0$:

$$\begin{aligned} A_0(x) &= x+1 \\ A_{i+1}(x) &= A_i^{(x)}(x), \end{aligned}$$

where $f^{(k)}$ denotes the $k$-fold composition of the function $f$ with itself. That is,

$$\begin{aligned} f^{(0)}(x) &= x \\ f^{(k)}(x) &= f(f^{(k-1)}(x)). \end{aligned}$$

So, in other words, $A_{i+1}(x)$ involves making $x$ applications of the $A_i$ function on itself, starting with $x$. This indexed function actually defines a progression of functions, with each function growing much faster than the previous one:

- $A_0(x) = x + 1$, which is the increment-by-one function
- $A_1(x) = 2x$, which is the multiply-by-two function
- $A_2(x) = x2^x \geq 2^x$, which is the power-of-two function
- $A_3(x) \geq 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (with $x$ number of 2's), which is the tower-of-twos function
- $A_4(x)$ is greater than or equal to the tower-of-tower-of-twos function
- and so on.

# Ackermann Function

Let

$$A_0(x) = x + 1$$

$$A_{i+1}(x) = A_i^{(x)}(x)$$

$A_1(x) = A_0^{(x)}(x)$

$=$

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$$A_{i+1}(x) = A_i^{(x)}(x)$$

$A_1(x) = A_0^{(x)}(x)$

$= 2x$

# Ackermann Function

Let

$$A_0(x) = x + 1$$

$$A_{i+1}(x) = A_i^{(x)}(x)$$

$$A_2(x) = A_1^{(x)}(x)$$

$$= A_1^{(x-1)}(A_1(x))$$

$$= A_1^{(x-1)}(2x)$$

$$= A_1^{(x-2)}(2^2x)$$

$$= x2^x$$

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$$\geq 2^x$$

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$$A_{i+1}(x) = A_i^{(x)}(x)$$

$A_3(x) = A_2^{(x)}(x)$

$= A_2^{(x-1)}(A_2(x))$

$\geq A_2^{(x-1)}(2^x)$

$\geq$

$\geq$

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- and so on.

# Ackermann Function

We then define the **Ackermann function** as

$$A(x) = A_x(2),$$

which is an incredibly fast-growing function.

- To get some perspective, note that $A(3) = 2048$ and $A(4)$ is greater than or equal to a tower of 2048 twos, which is much larger than the number of subatomic particles in the universe.

Likewise, its inverse, which is pronounced “alpha of n”,

$$\alpha(n) = \min\{x: A(x) \geq n\},$$

is an incredibly slow-growing function. Even though $\alpha(n)$ is indeed growing as n goes to infinity, for all practical purposes, $\alpha(n) \leq 4$.

# Thank You