Data Structures & Algorithms

Week 8 - Priority Queues (Binary Heaps, Skew Heaps, Applications)

Subodh Sharma, Rahul Garg

{svs,rahulgarg}@iitd.ac.in

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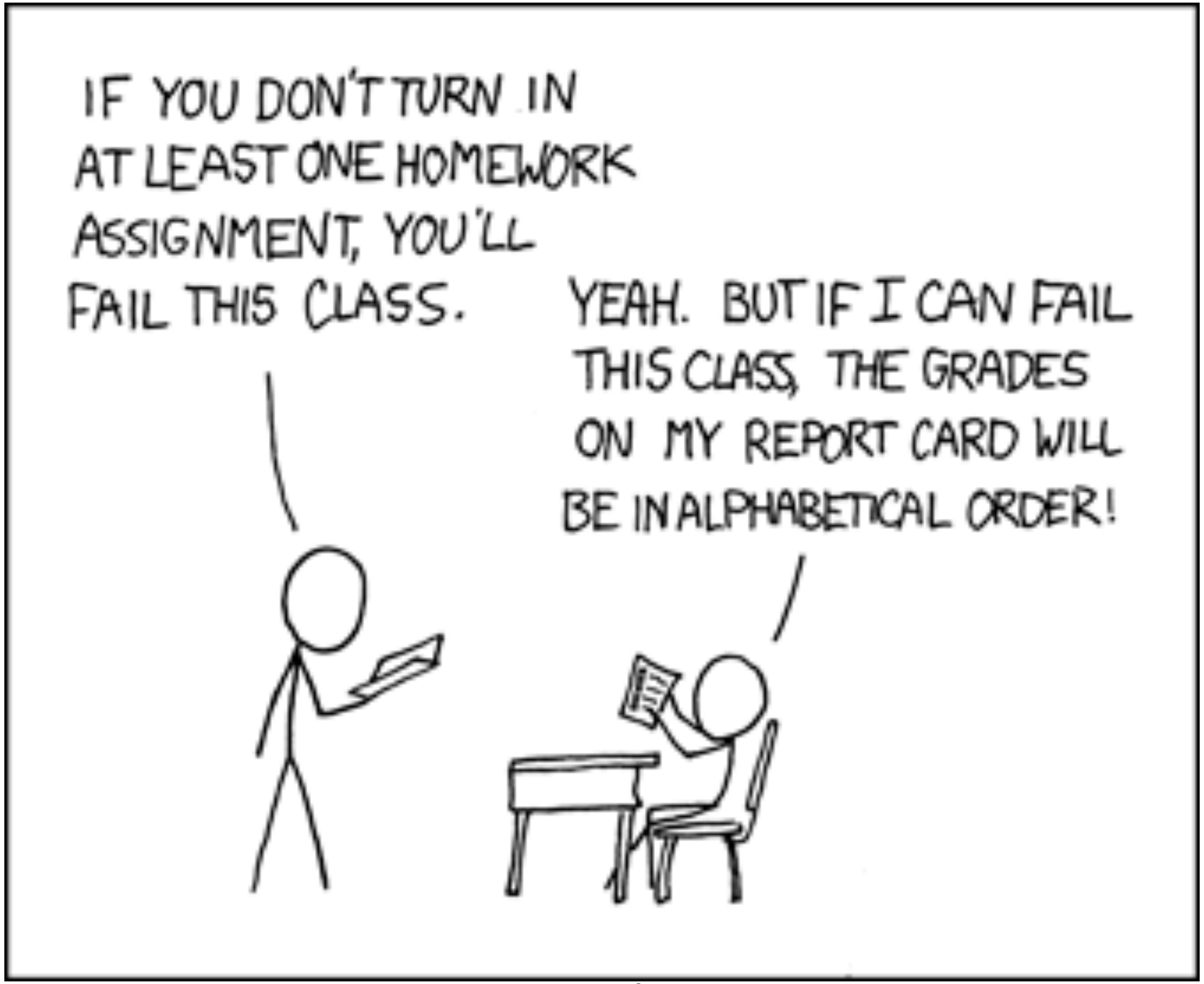
- **Applications:**
  - Scheduling, Algorithmic efficiency (Spanning Trees, Shortest Paths etc.), Simulation Systems (Discrete Event Simulation, etc.), Network Traffic Mgmt. (routing pkts with different service reqs.), E-commerce, Load balancing, etc.

On Priorities



Courtesy: XKCD Comics

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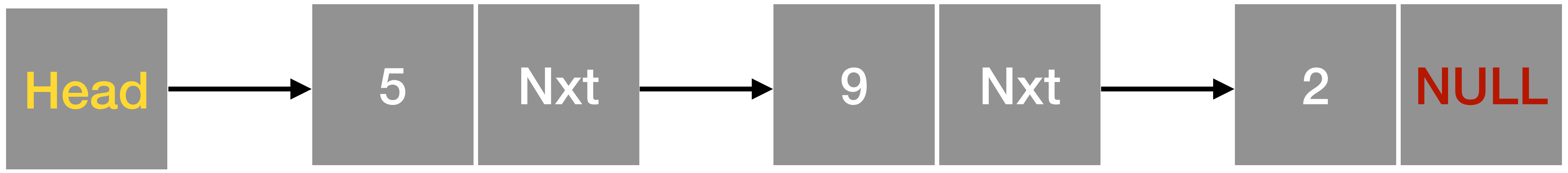

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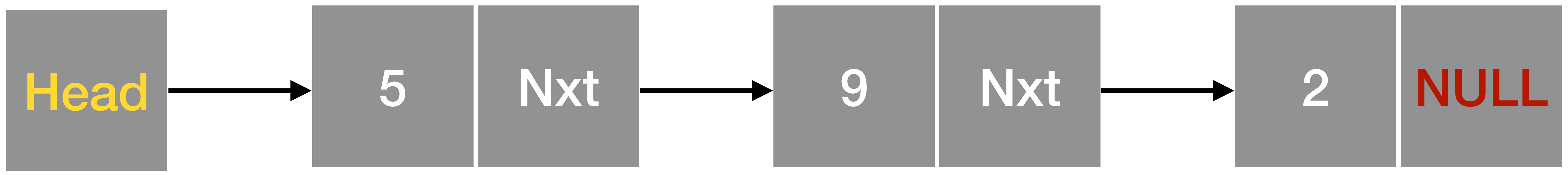

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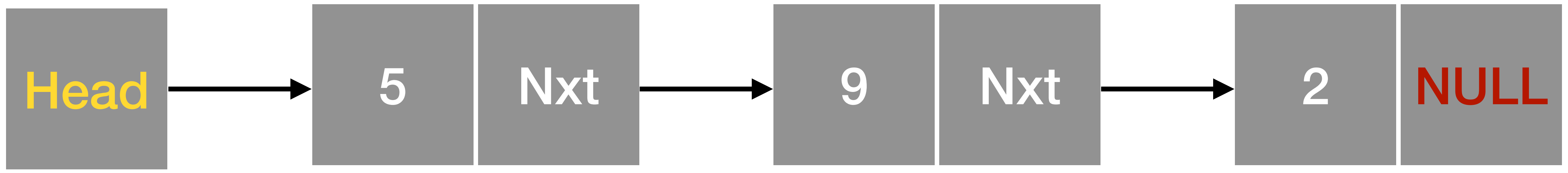

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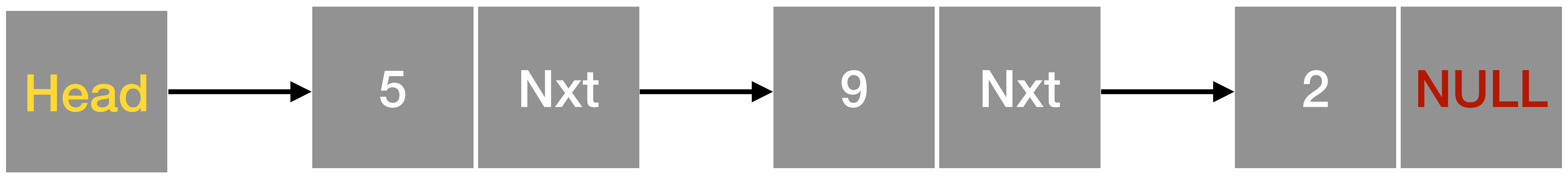

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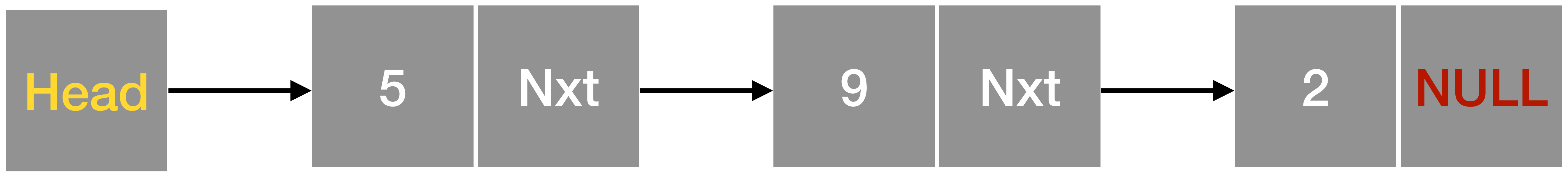

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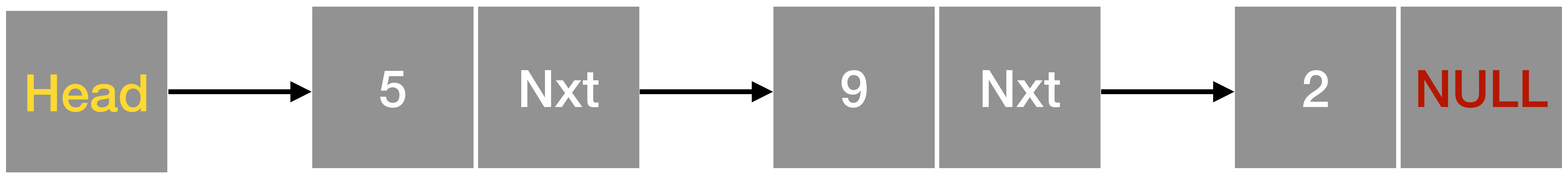

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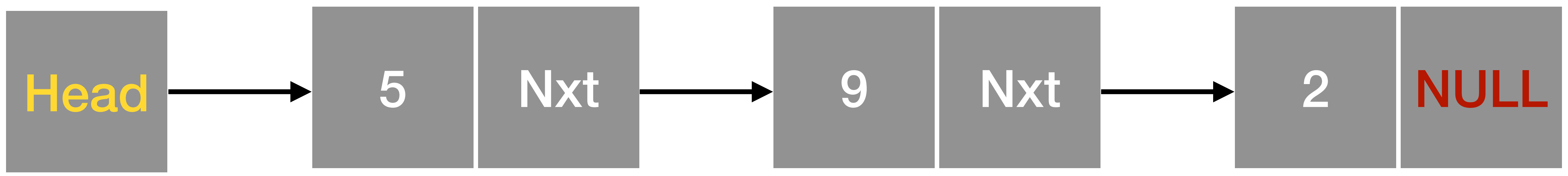

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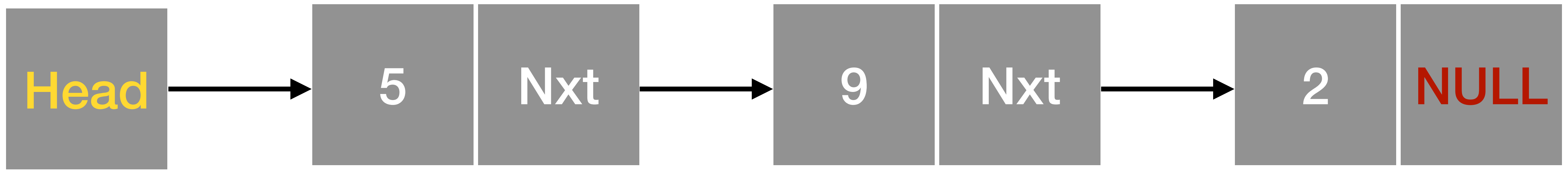

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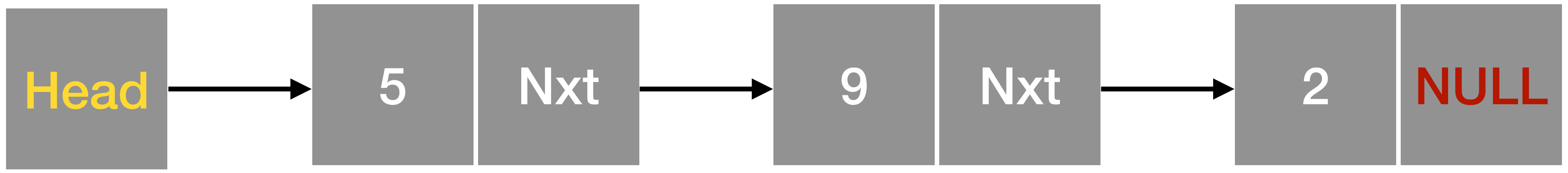

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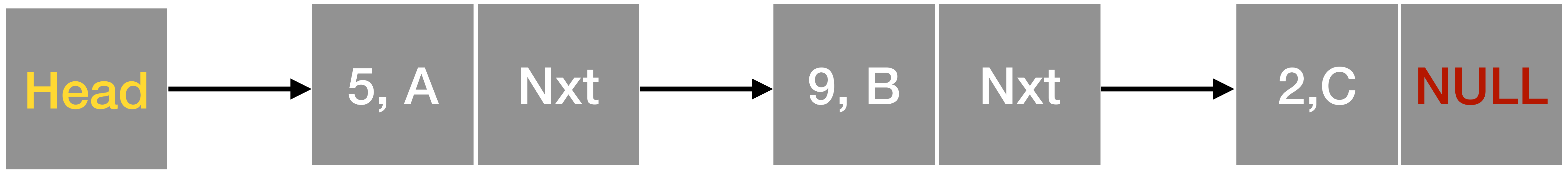

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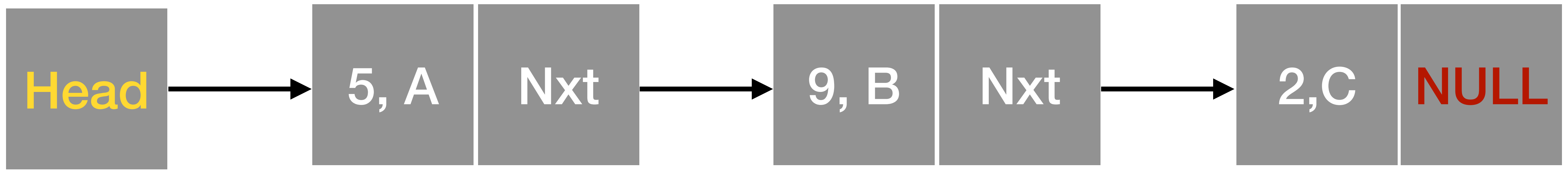

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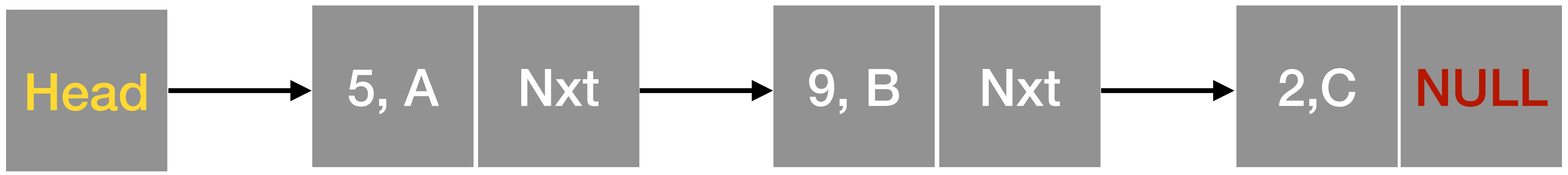

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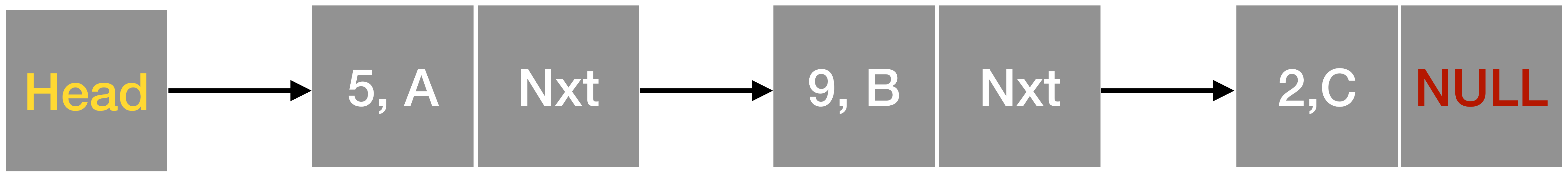

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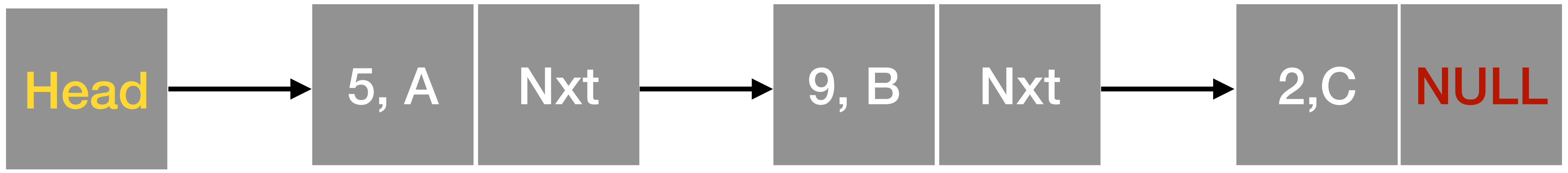

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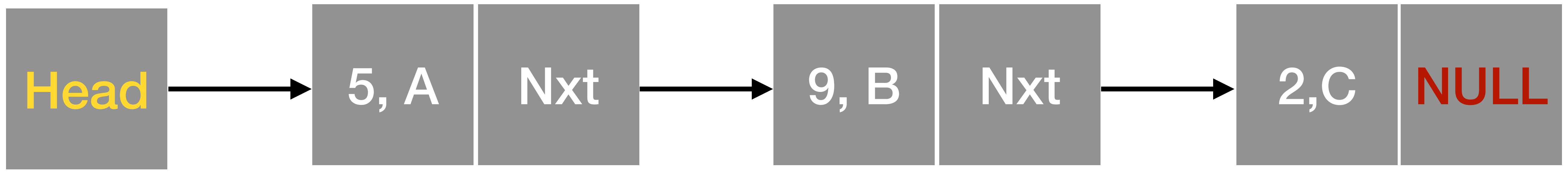

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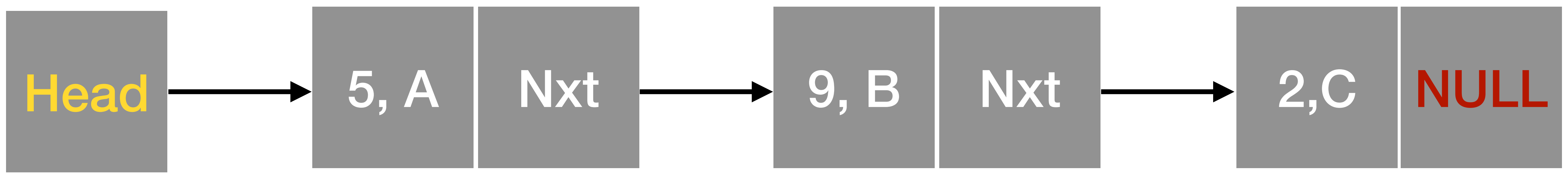

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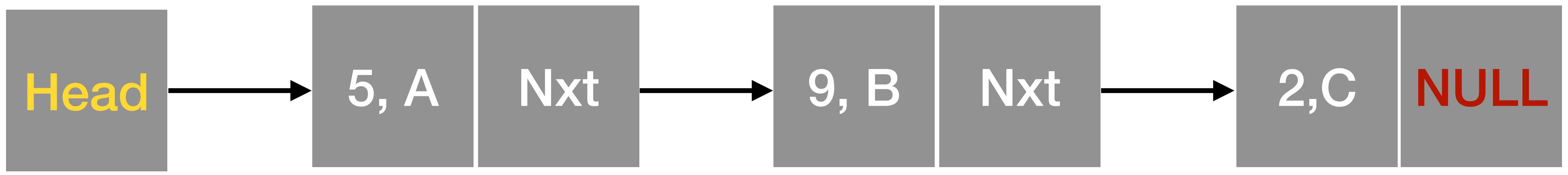

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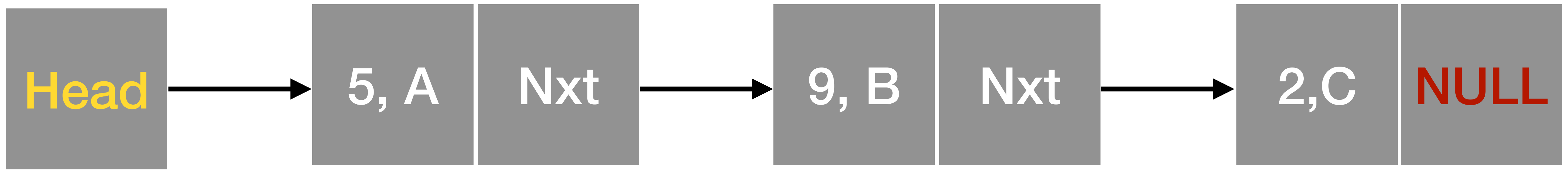

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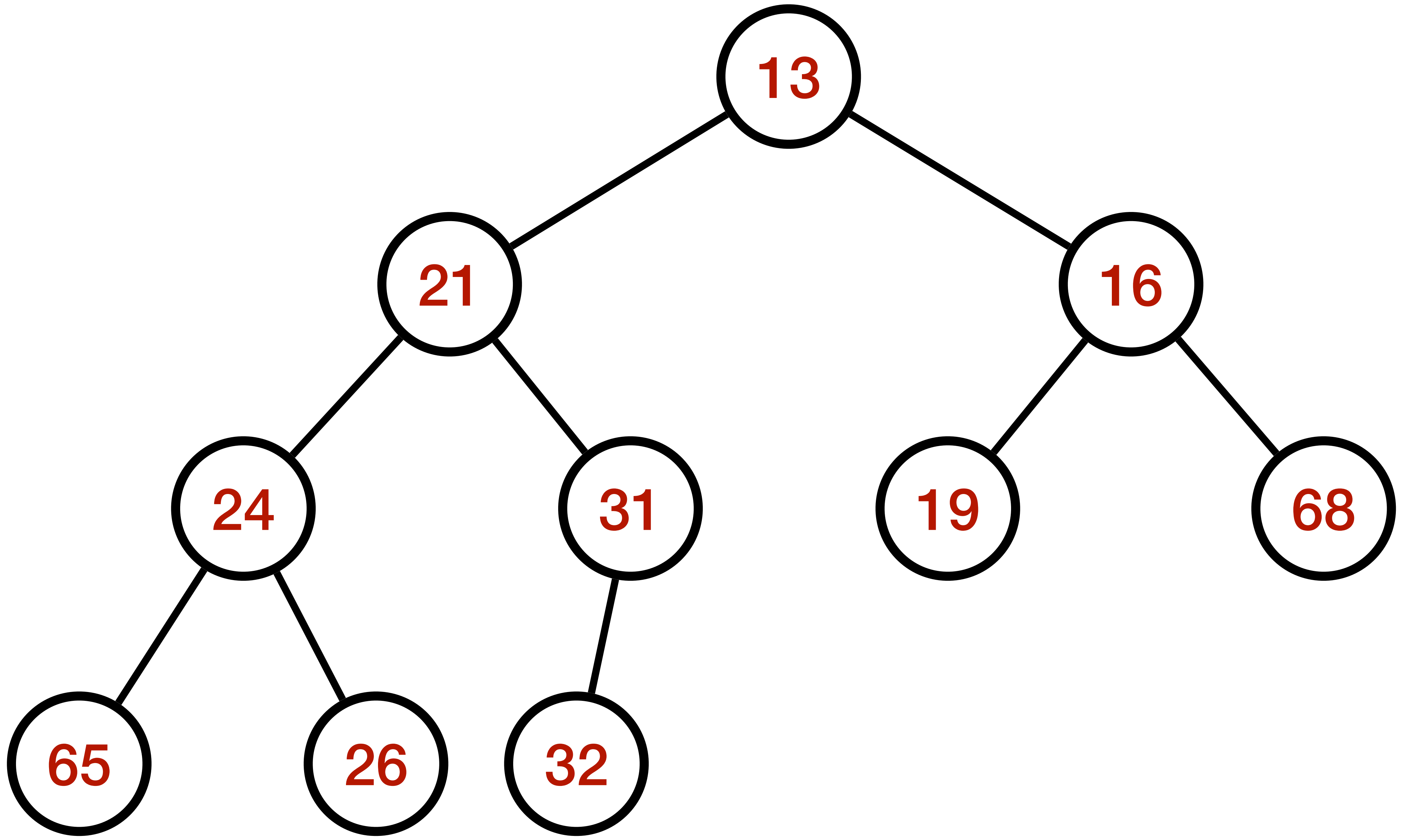

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- Heaps have two important properties:
  - **Structure Property:** Heap is completely filled with the exception of the last level.
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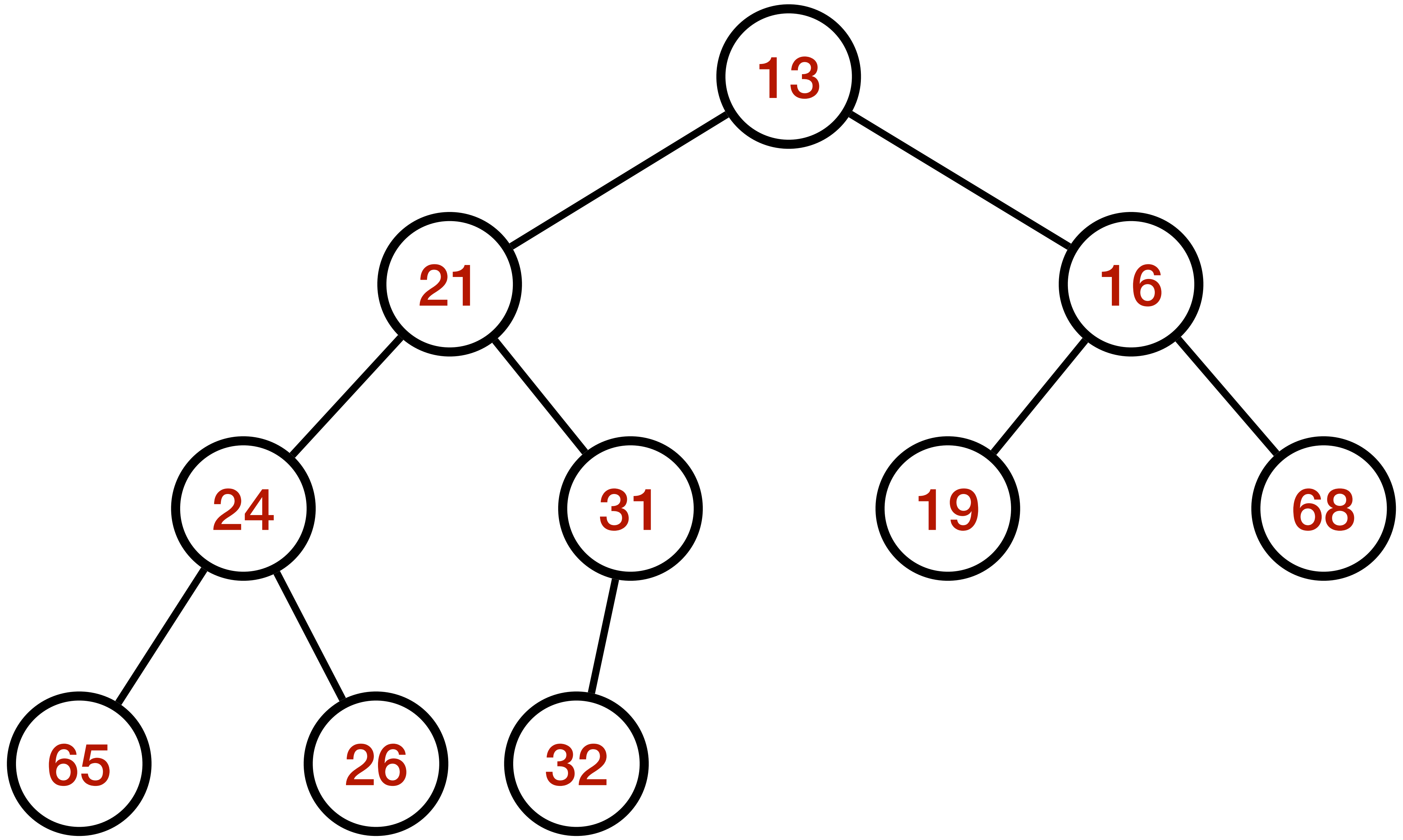
  - **Order Property**: Every node should be smaller than all of its descendants



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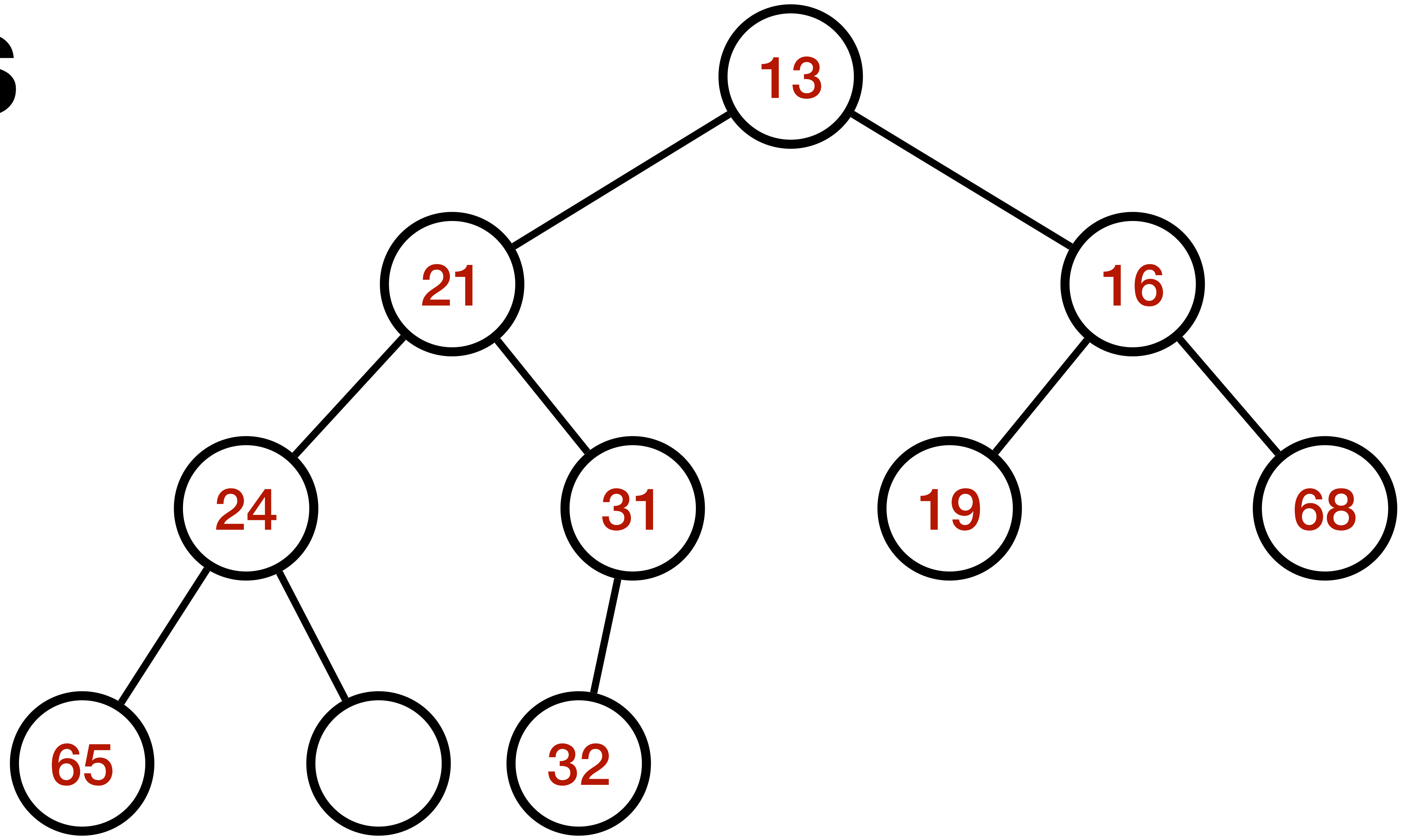

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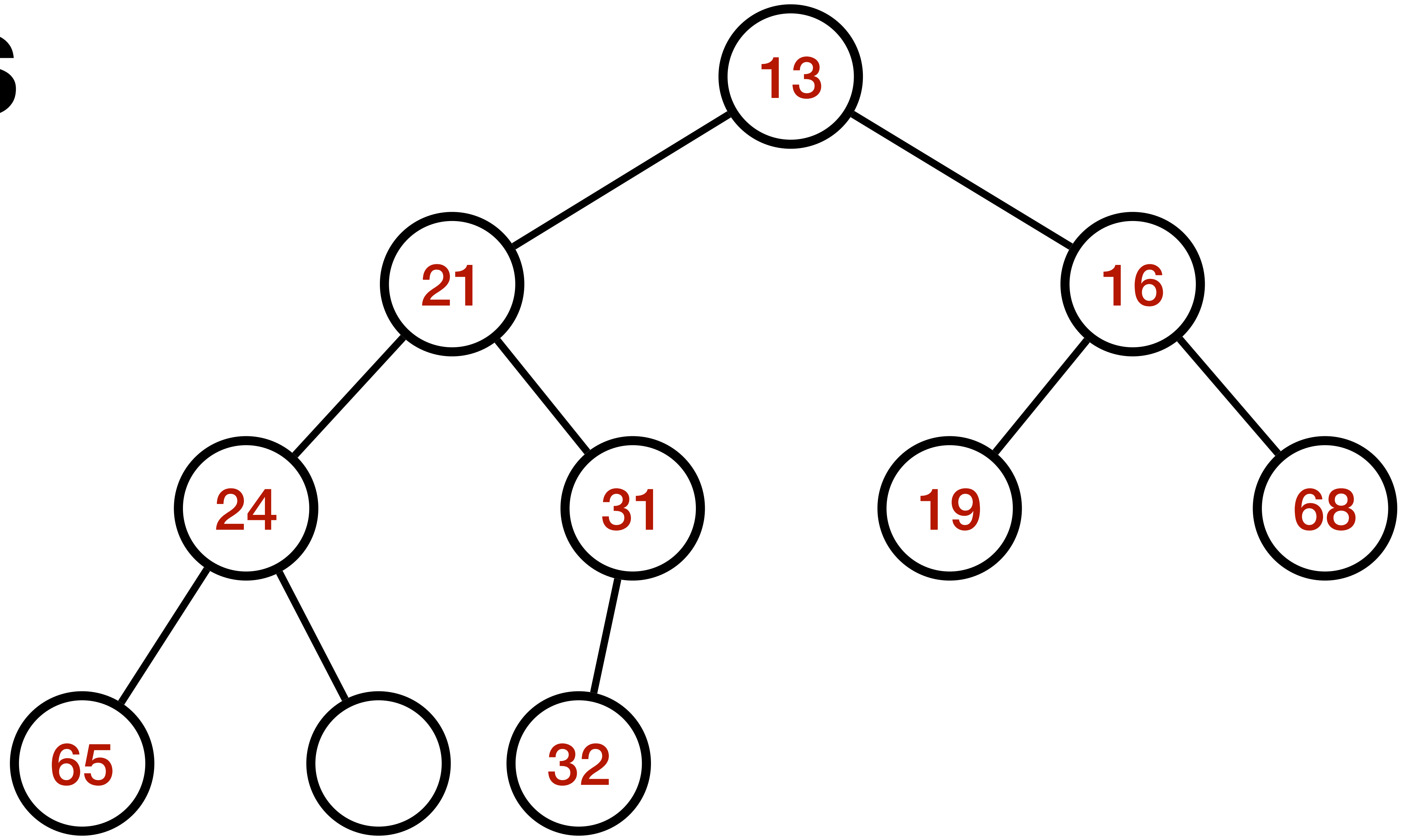

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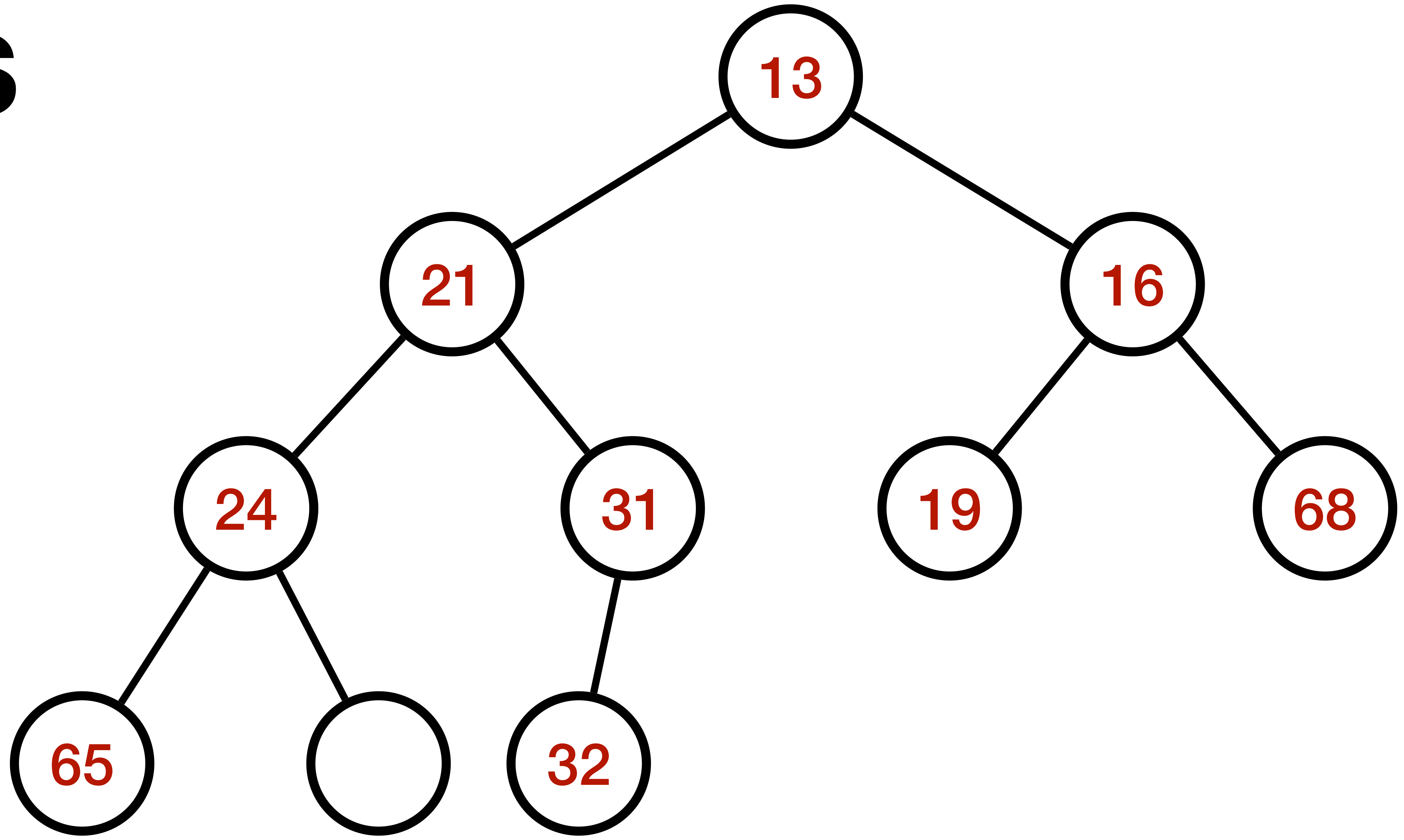

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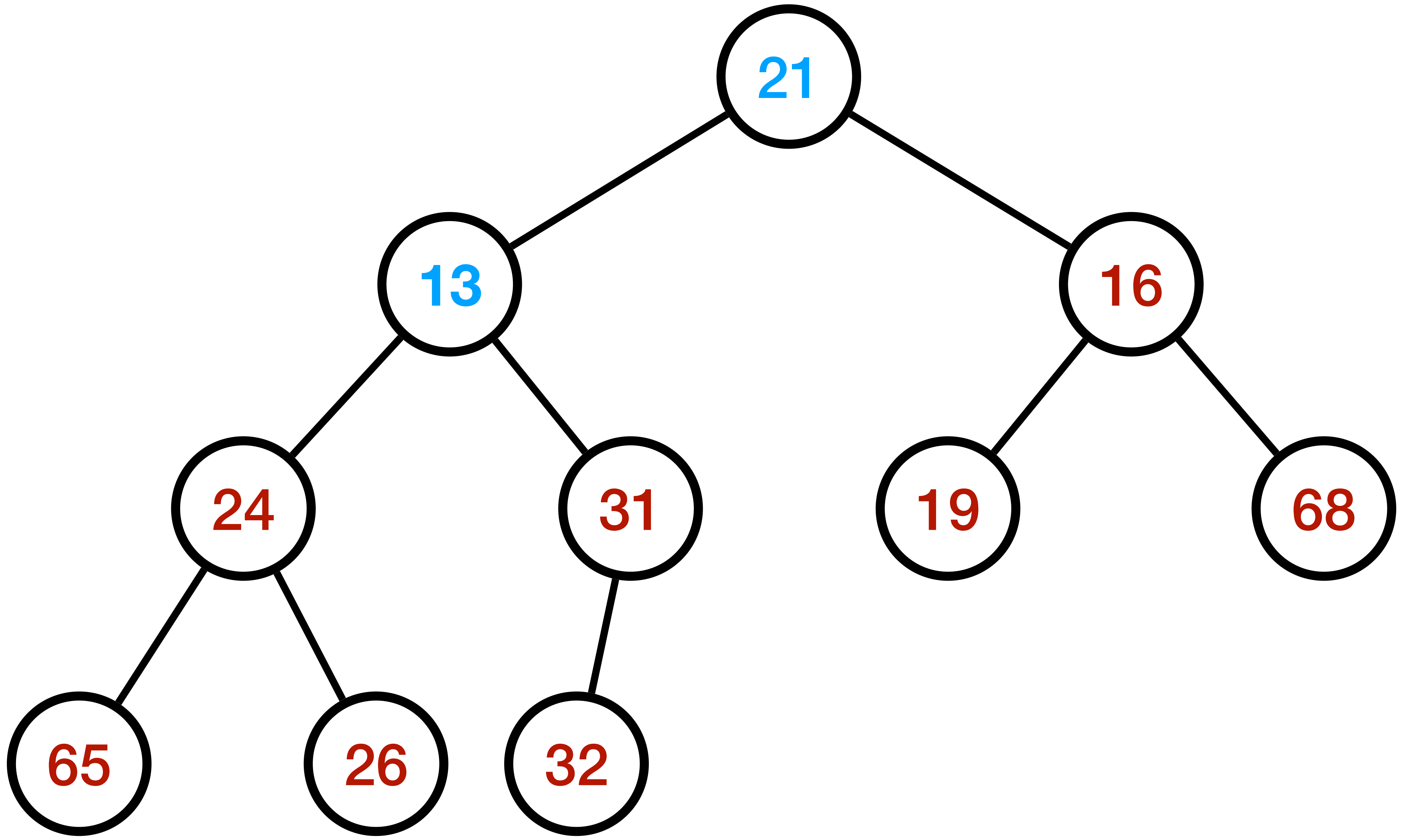

Where is the minimum element?

- Minimum element sits at the top of the heap — root of the heap!
- Why?
  - Otherwise the heap order property would be violated.
- Thus, finding the minimum element can be accomplished in O(1) time.

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- Thus, $h = \lfloor log_2 n \rfloor$

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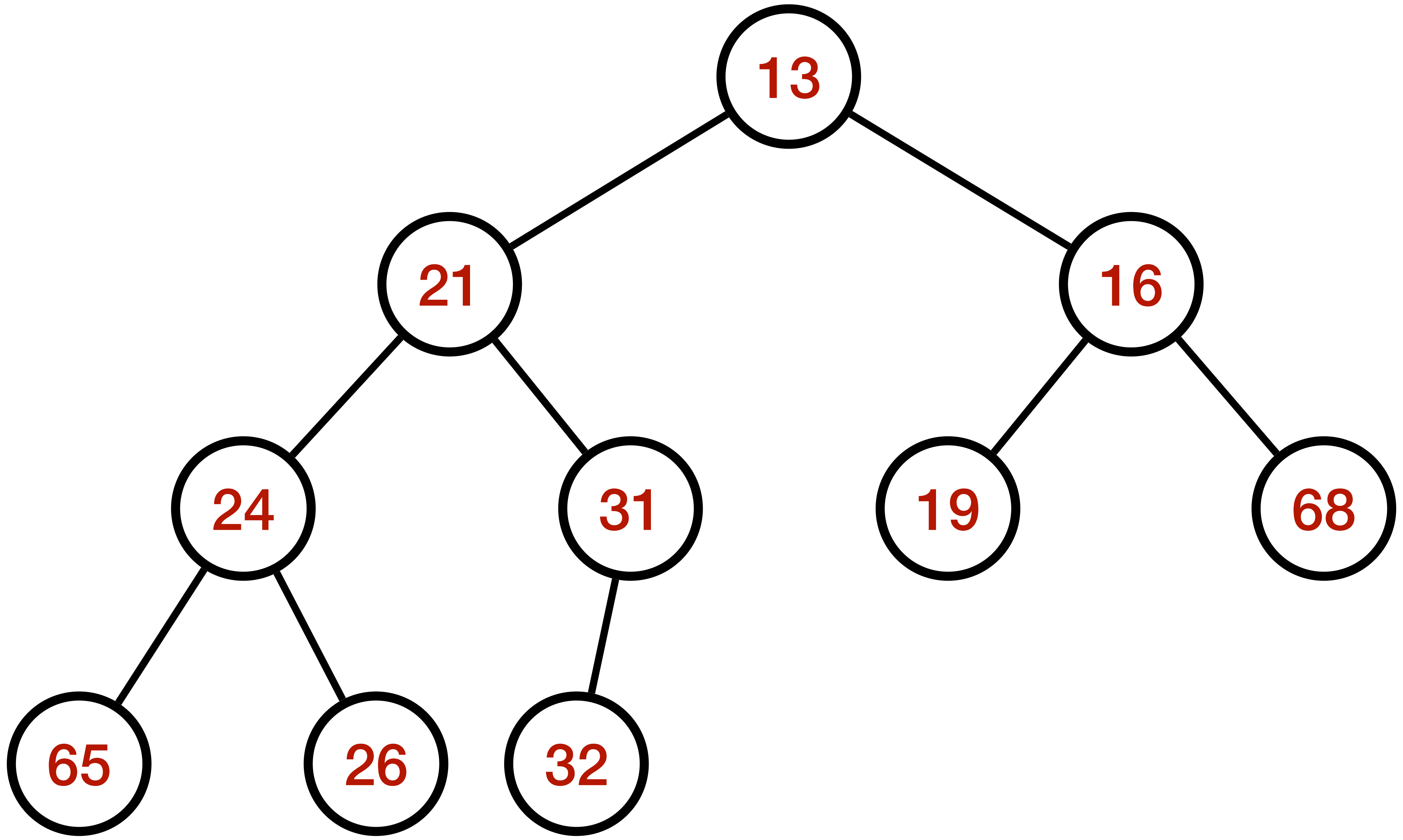

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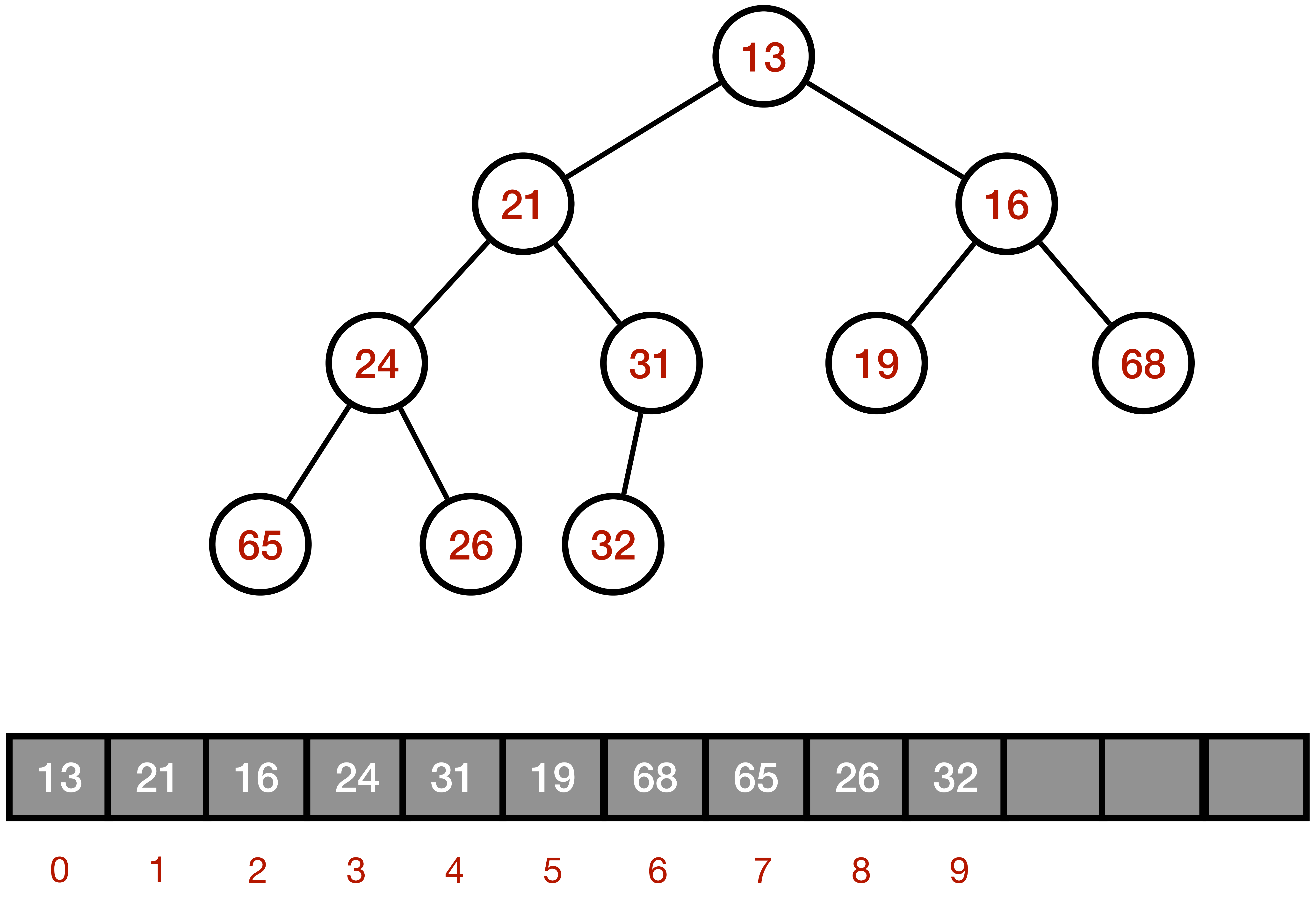

Implementing Heaps

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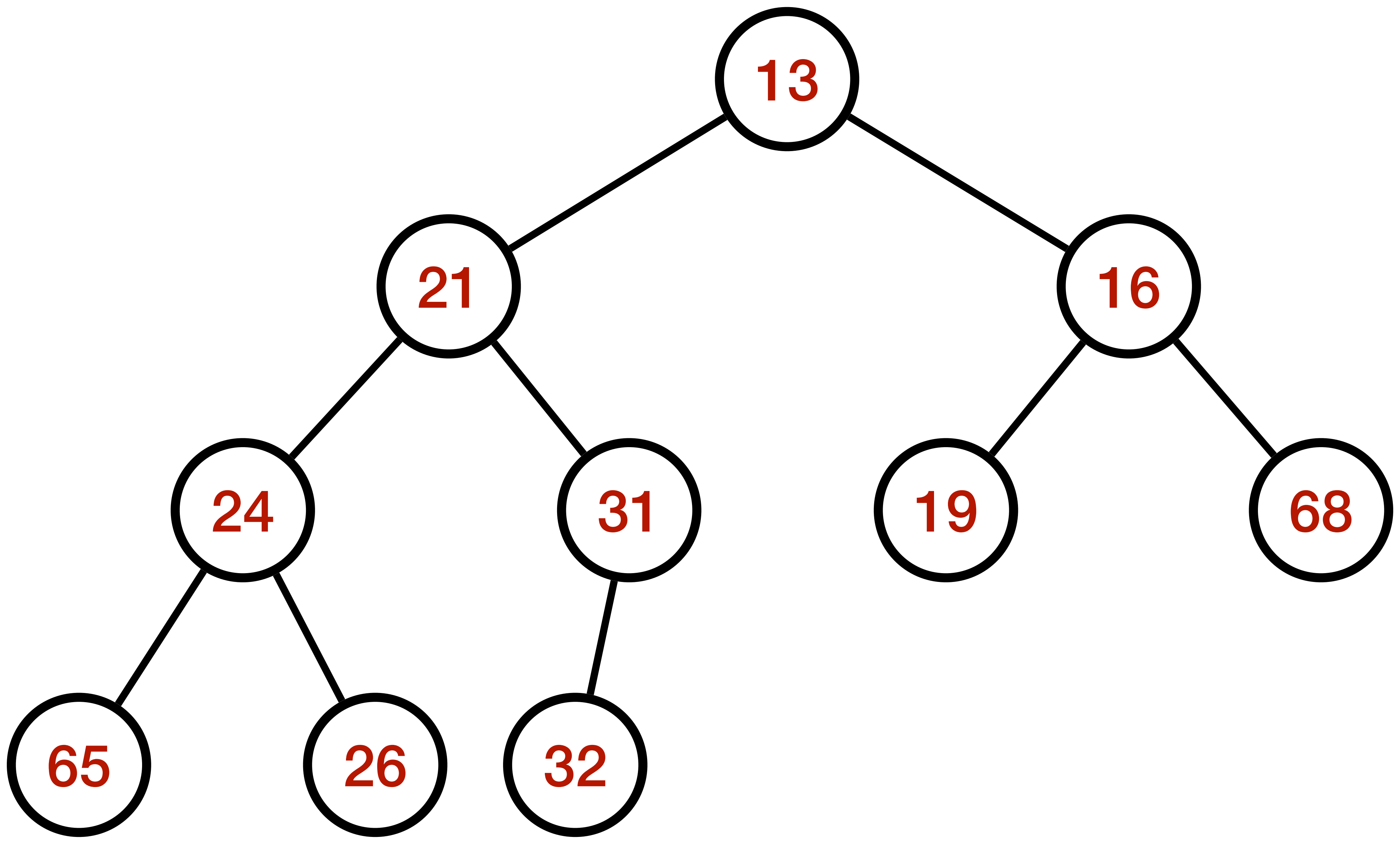
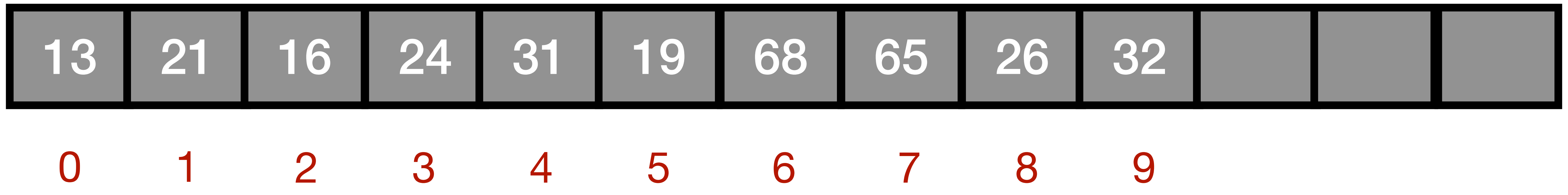
```
int getParentIndex(int i) {
    return (i - 1) / 2;
}

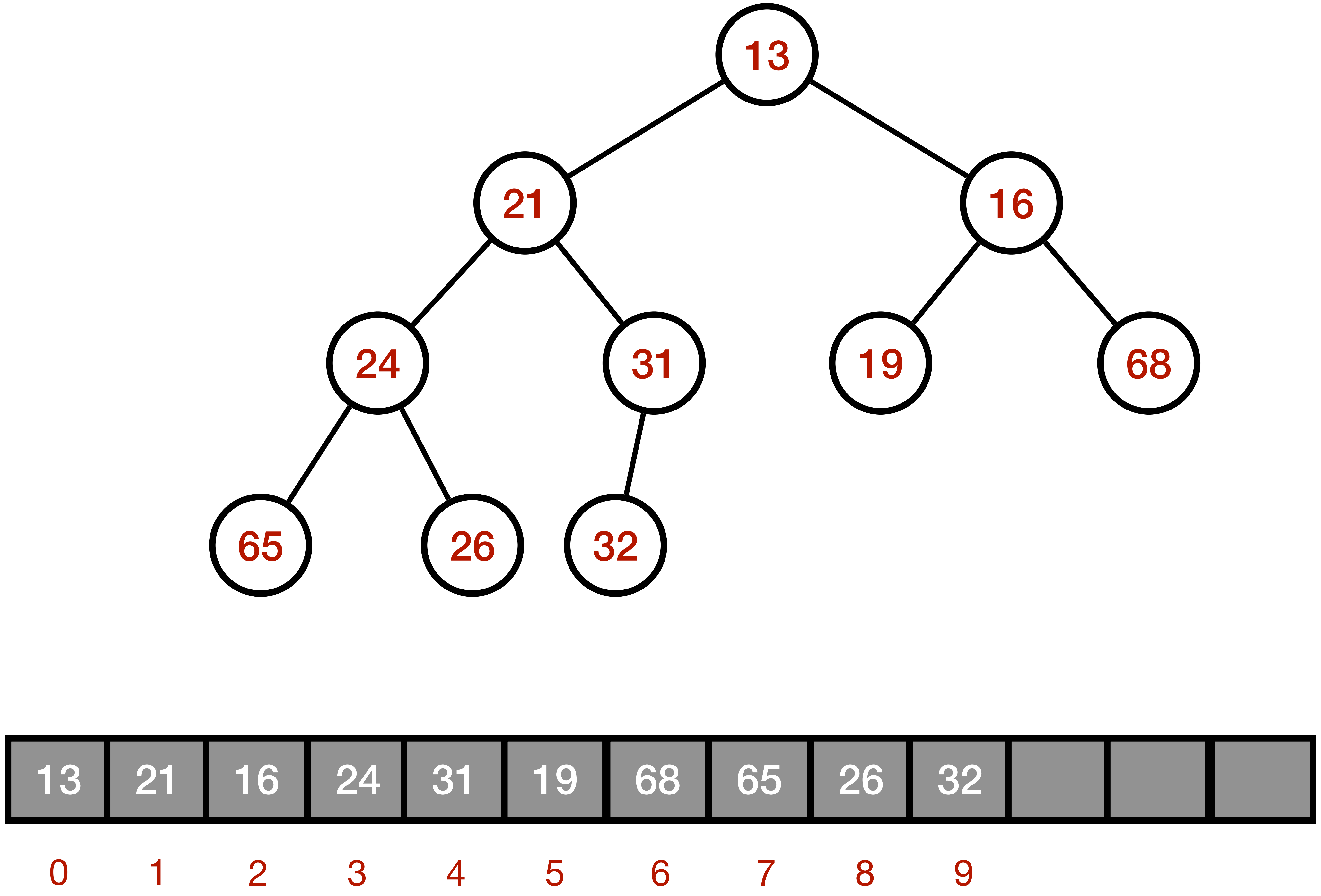
int getLeftChildIndex(int i) {
    return 2 * i + 1;
}

int getRightChildIndex(int i) {
    return 2 * i + 2;
}
```



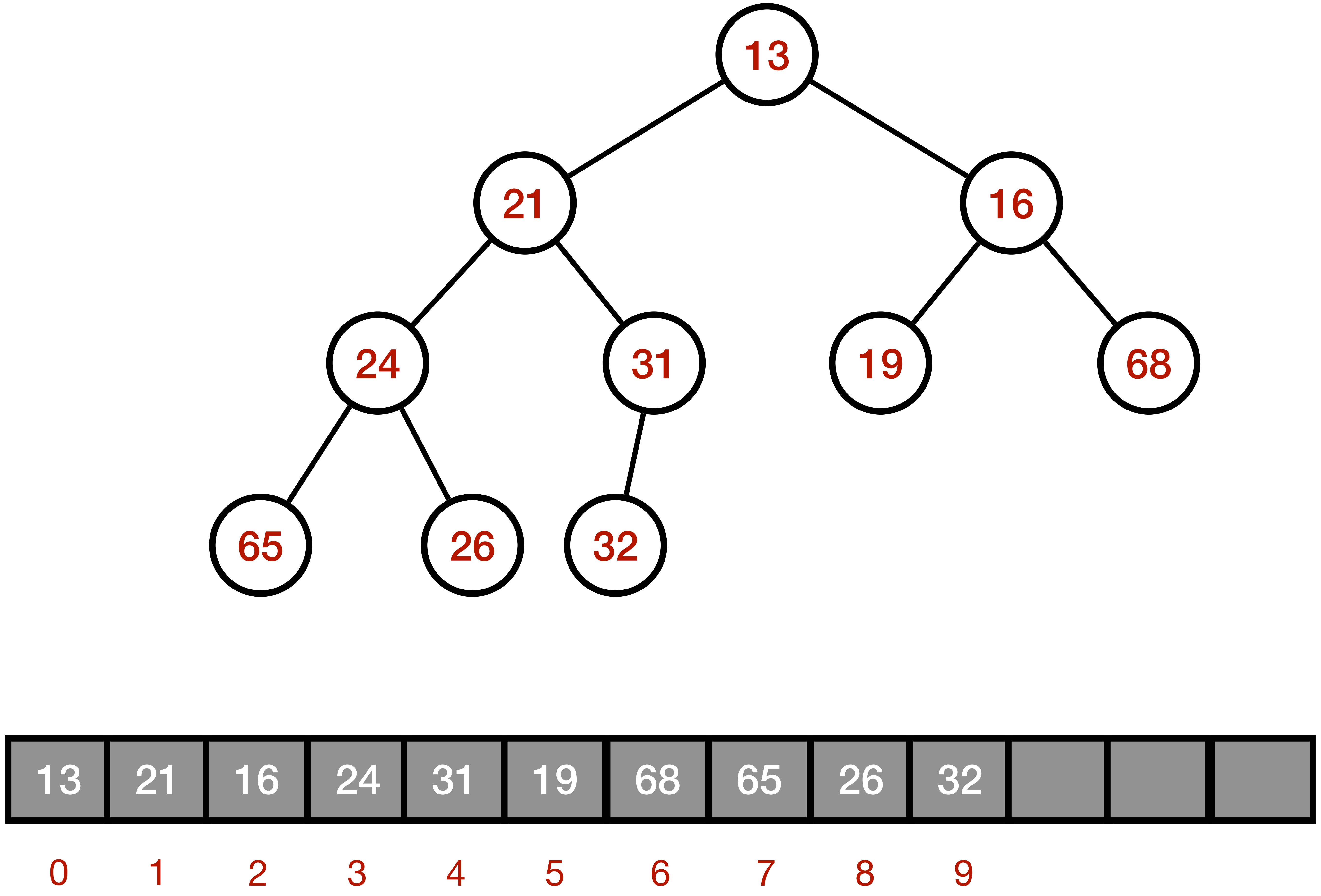


Implementing Heaps: Efficiency



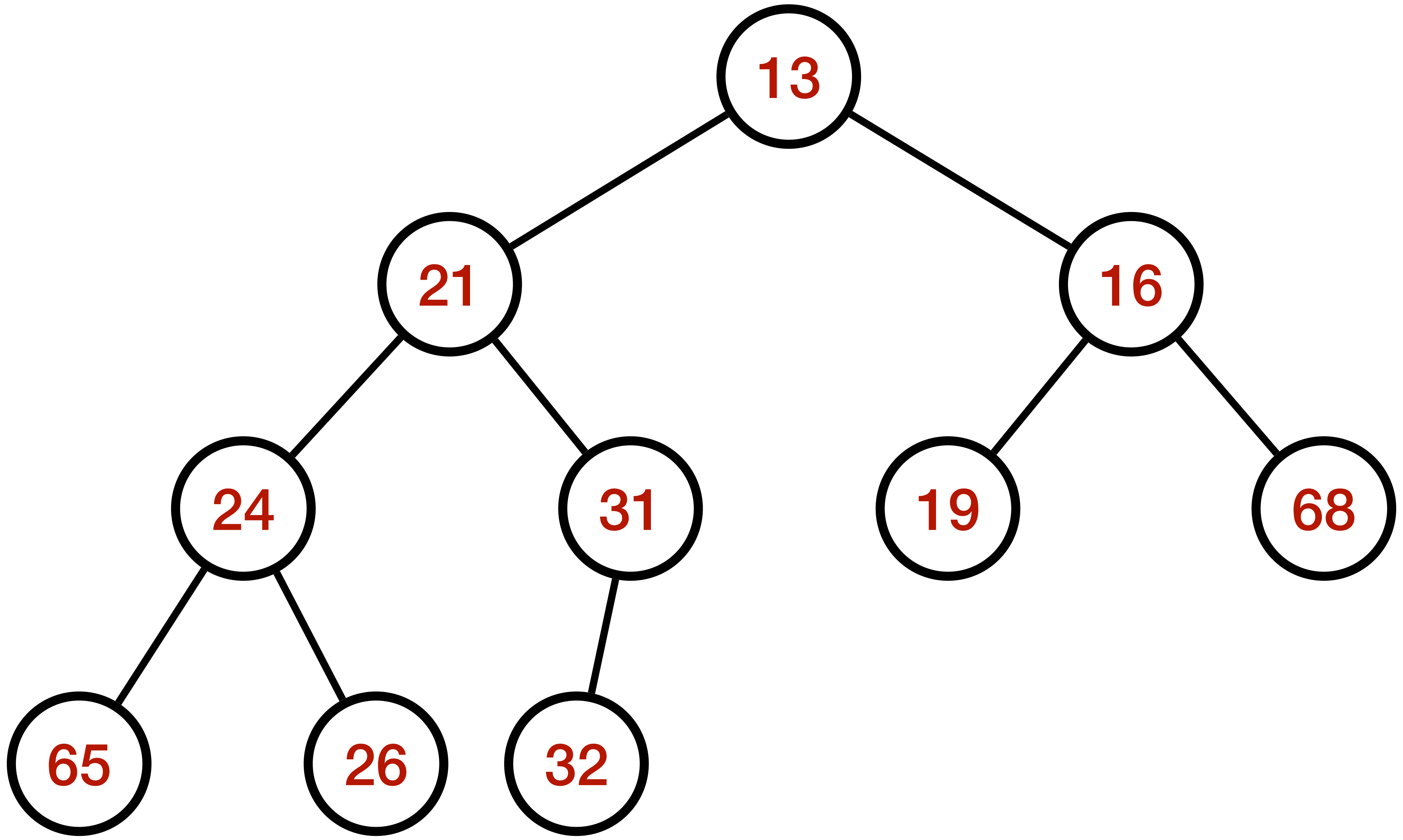
Implementing Heaps: Efficiency

- **Observation:** In binary representation, **multiplication by 2 is a left shift** and FMA instructions to multiply and add (adding 1 to the lowest bit)



Heap Insertion

- Insert 12



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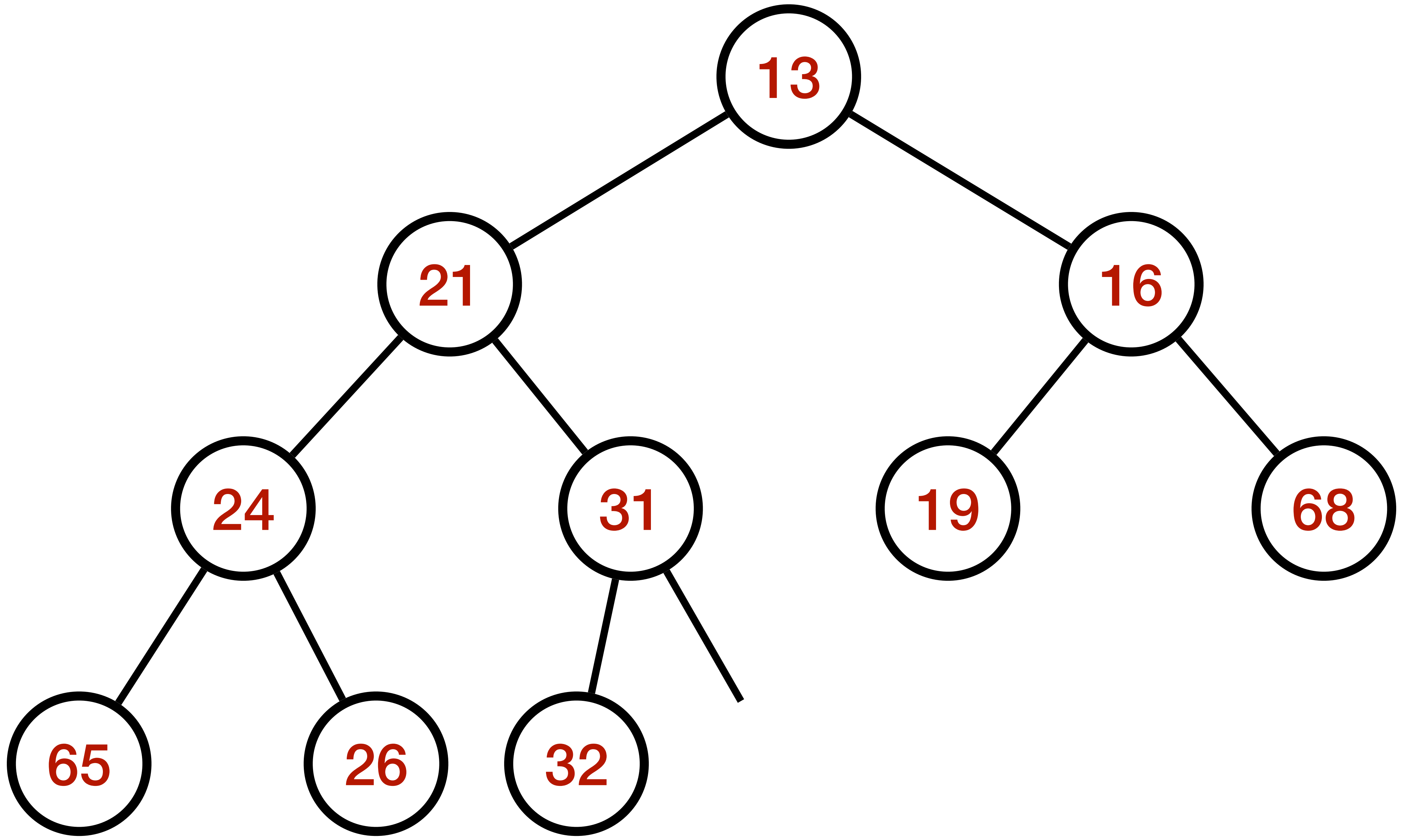

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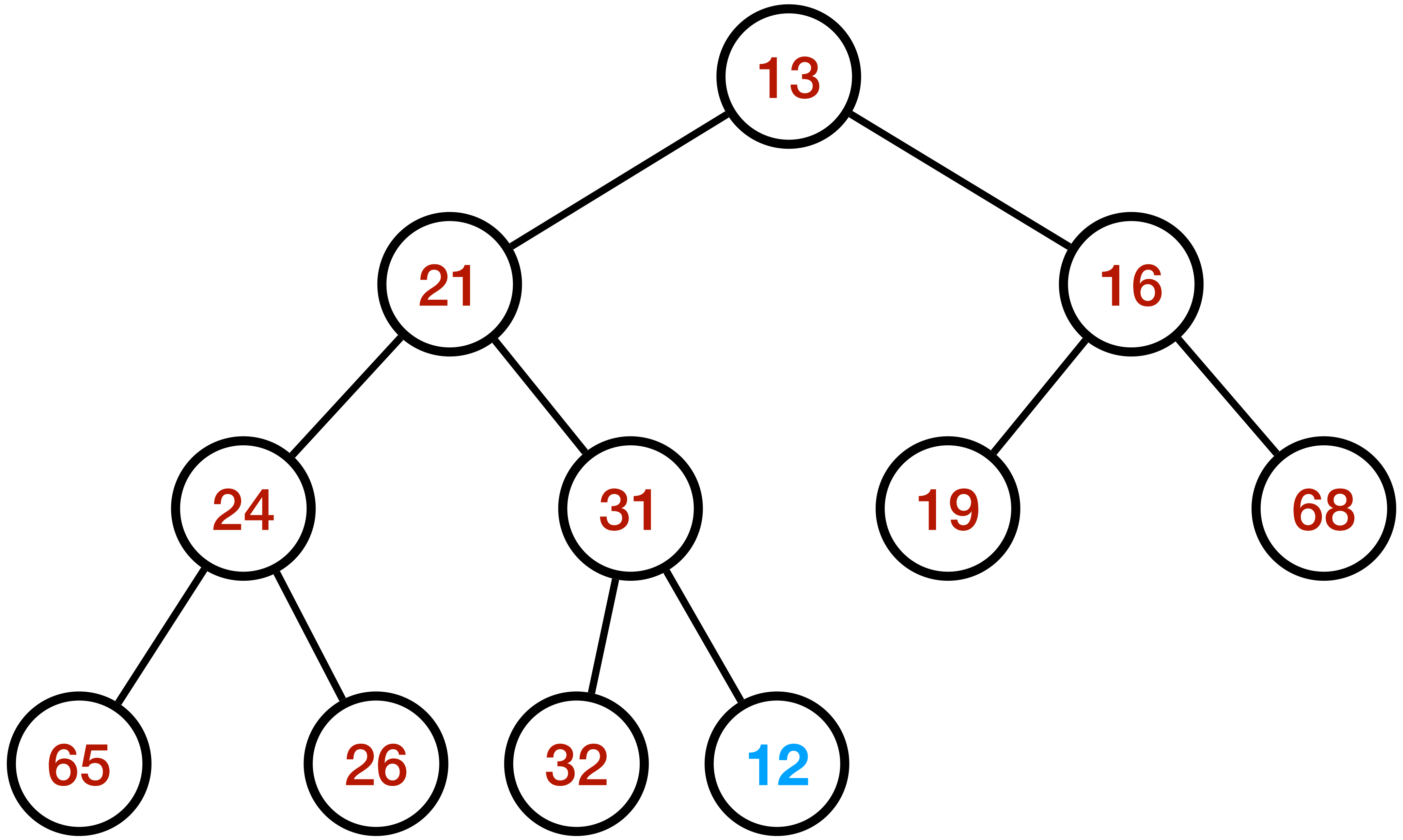

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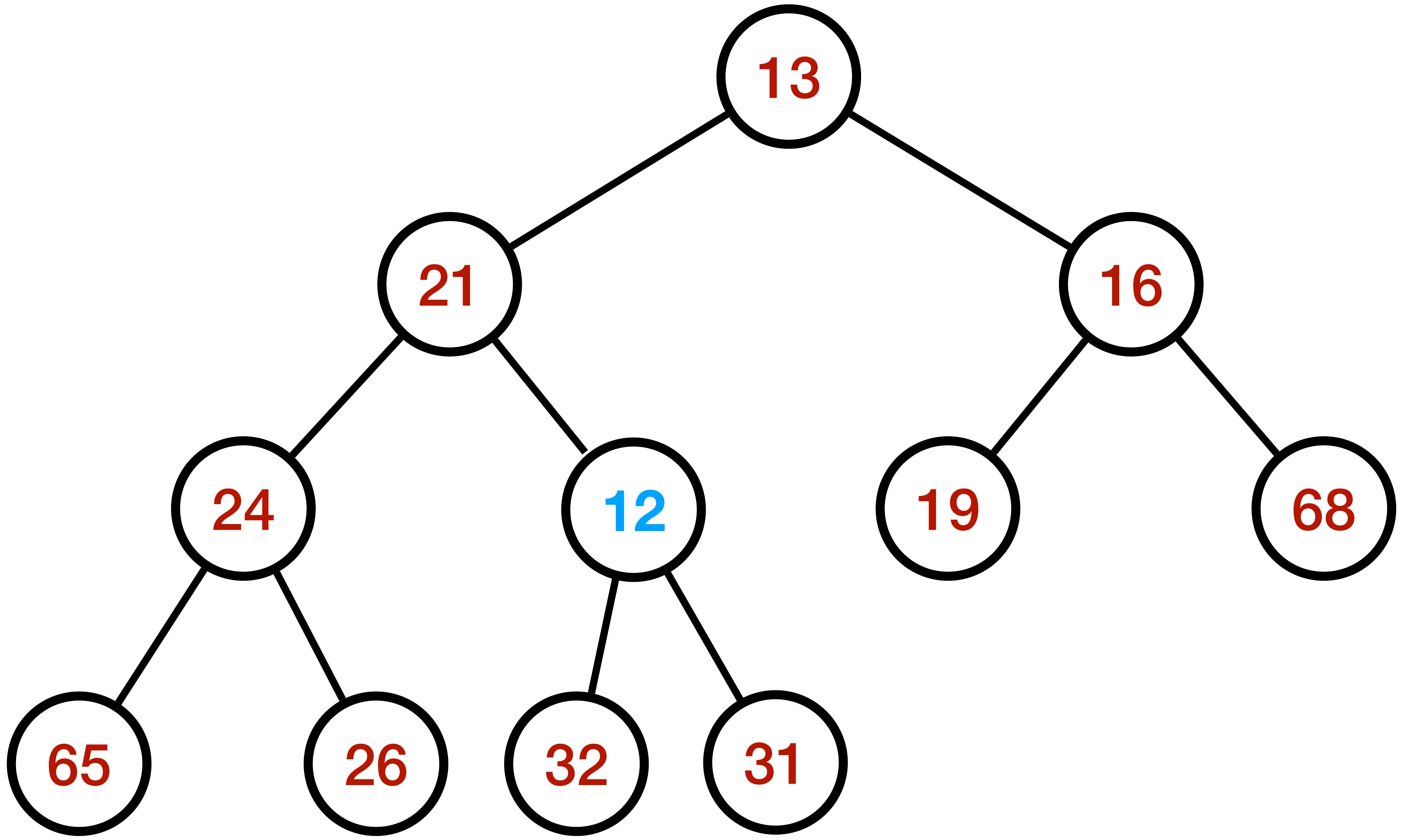

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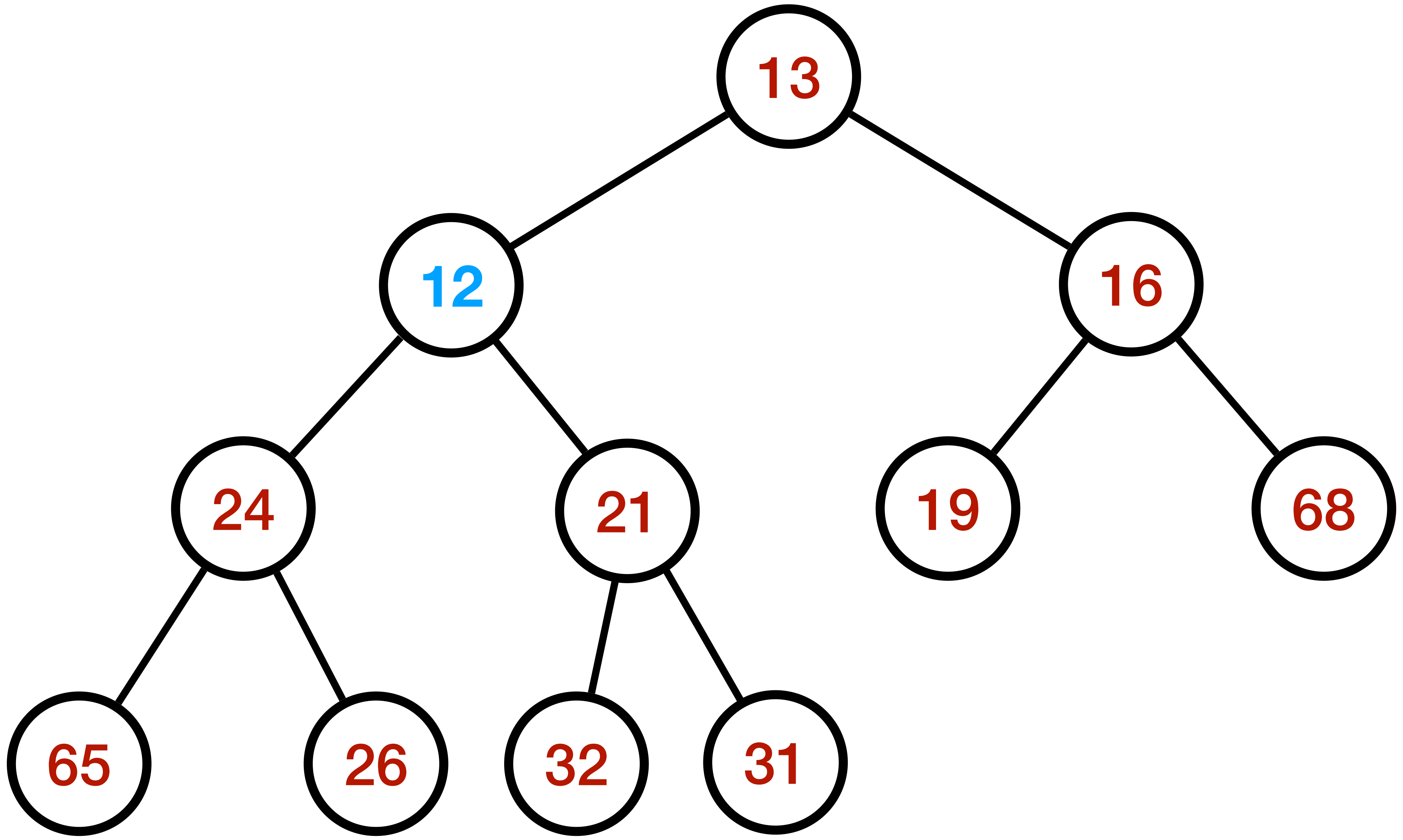

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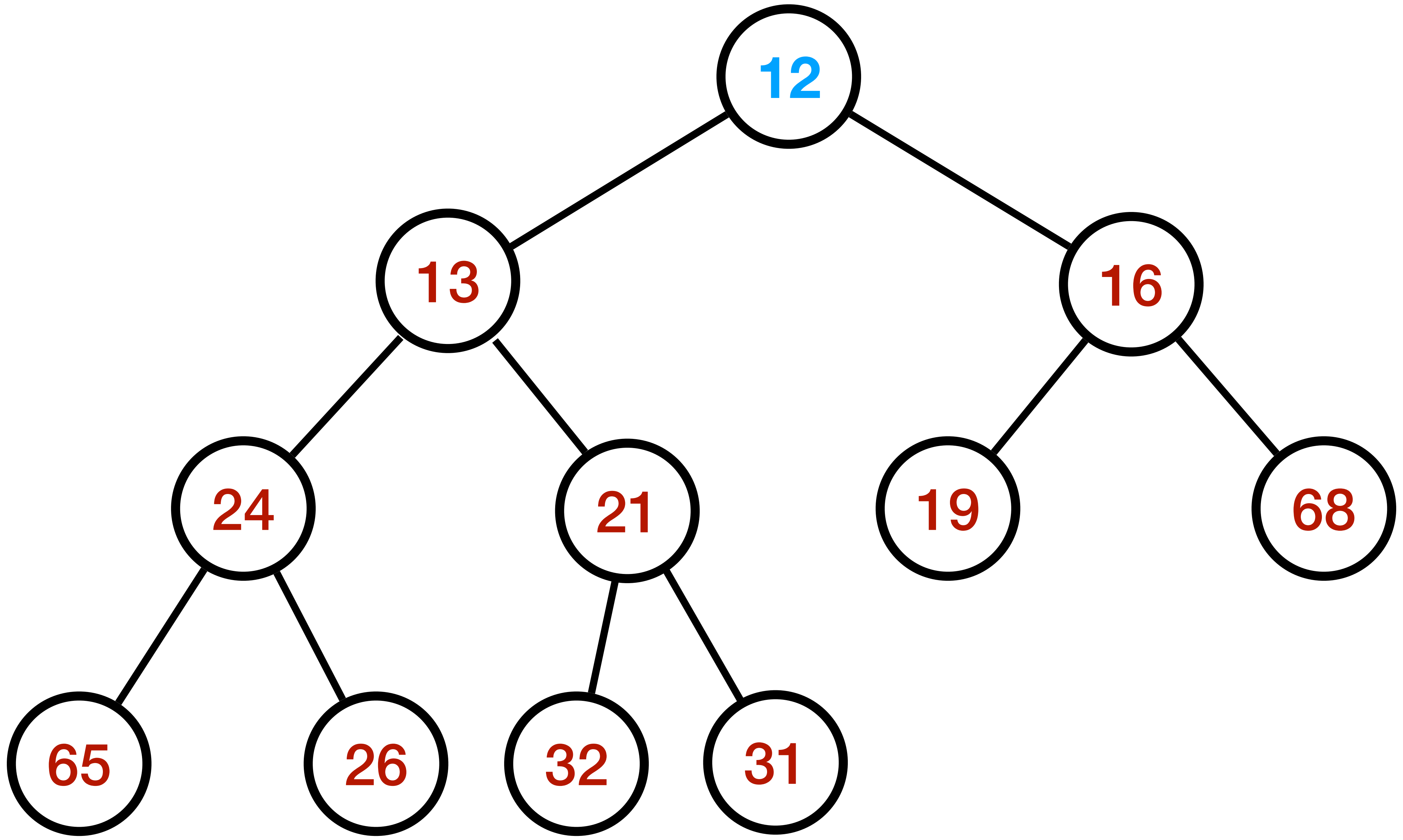

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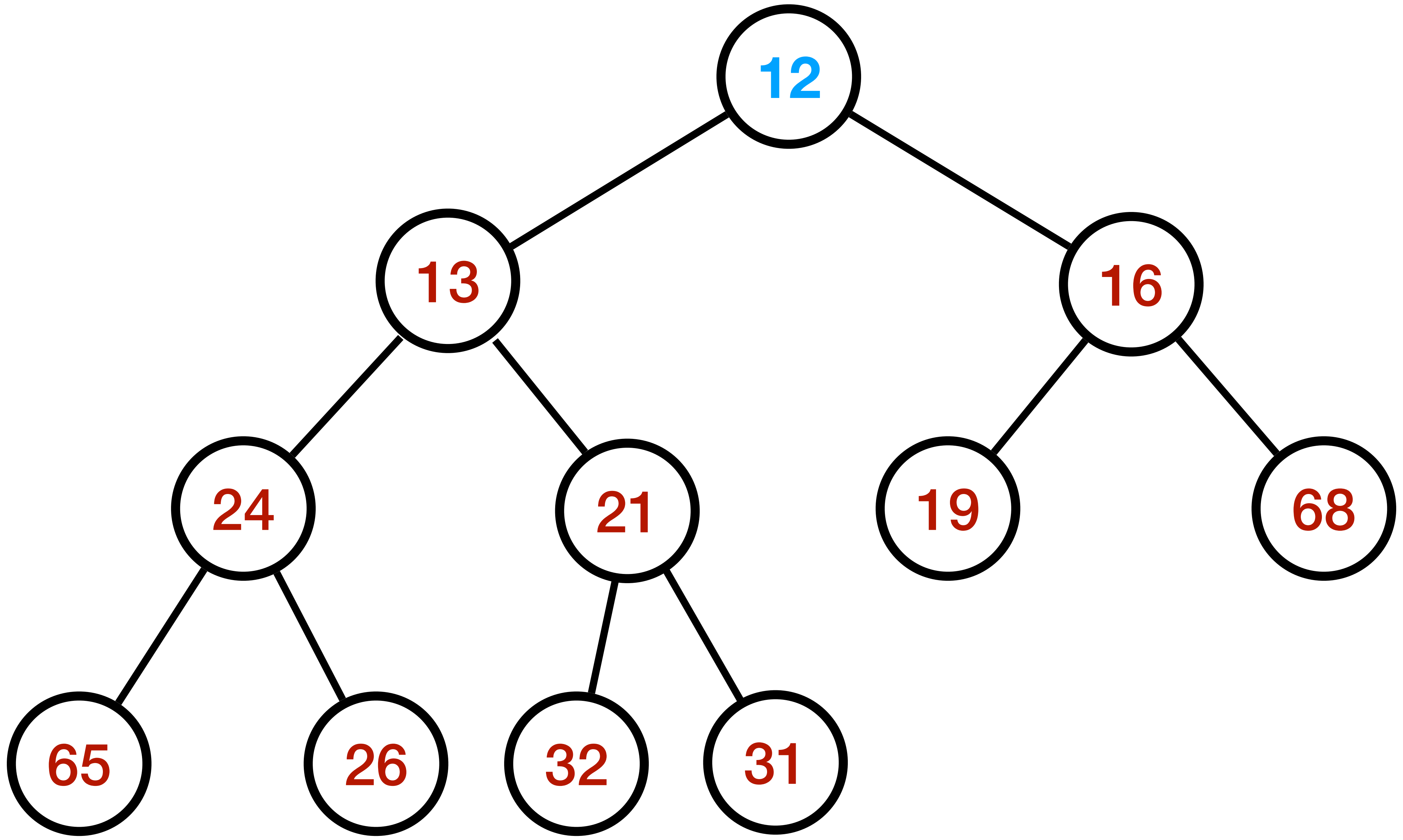

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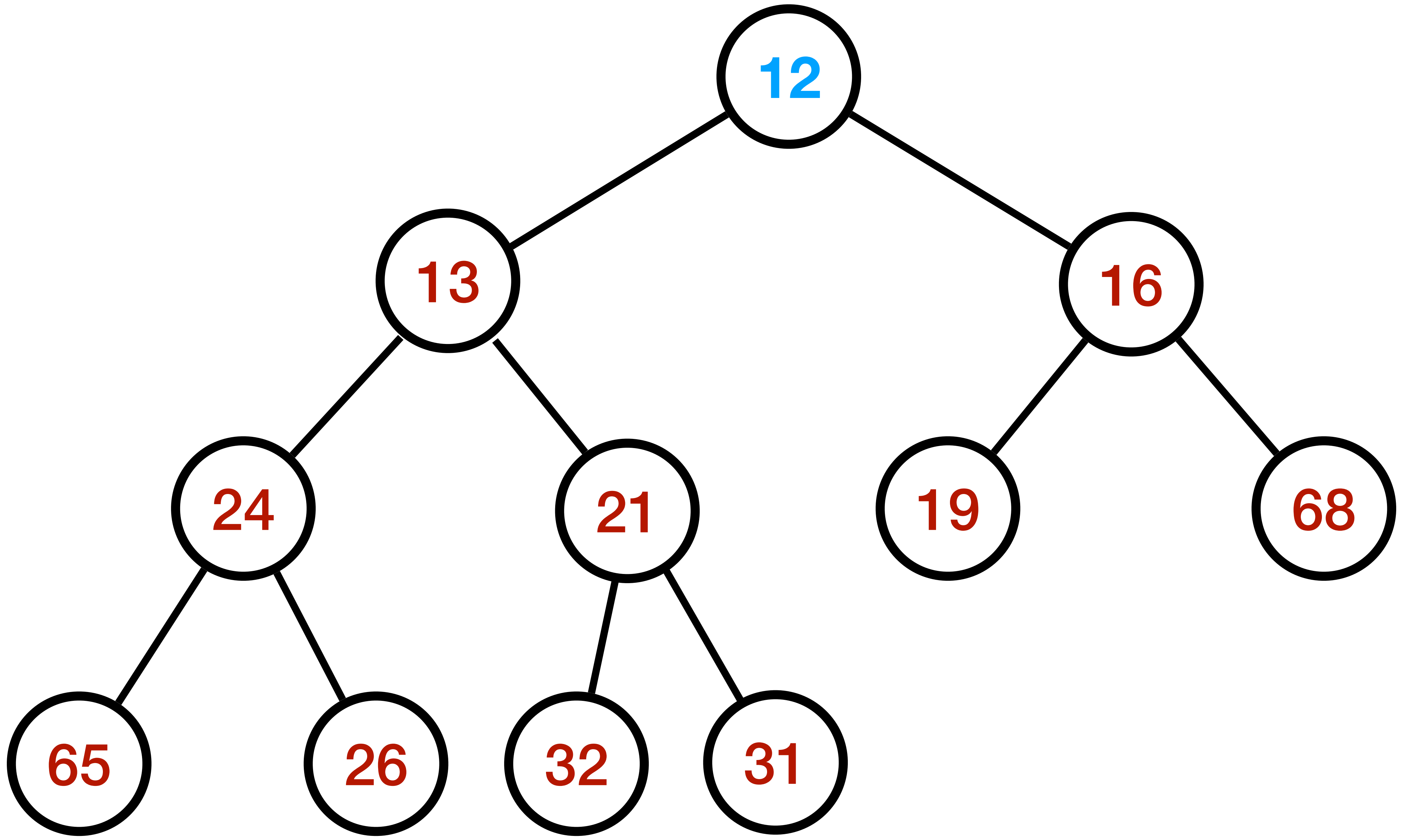
```
void insert(int val) {
  heap.push_back(val);
  heapifyUp(heap.size() - 1);
}

void heapifyUp(int index) {
  if (index == 0) return;

  int parentIndex = getParentIndex(index);

  if (heap[parentIndex] > heap[index]) {
    swap(heap[parentIndex], heap[index]);
    heapifyUp(parentIndex);
  }
}
```



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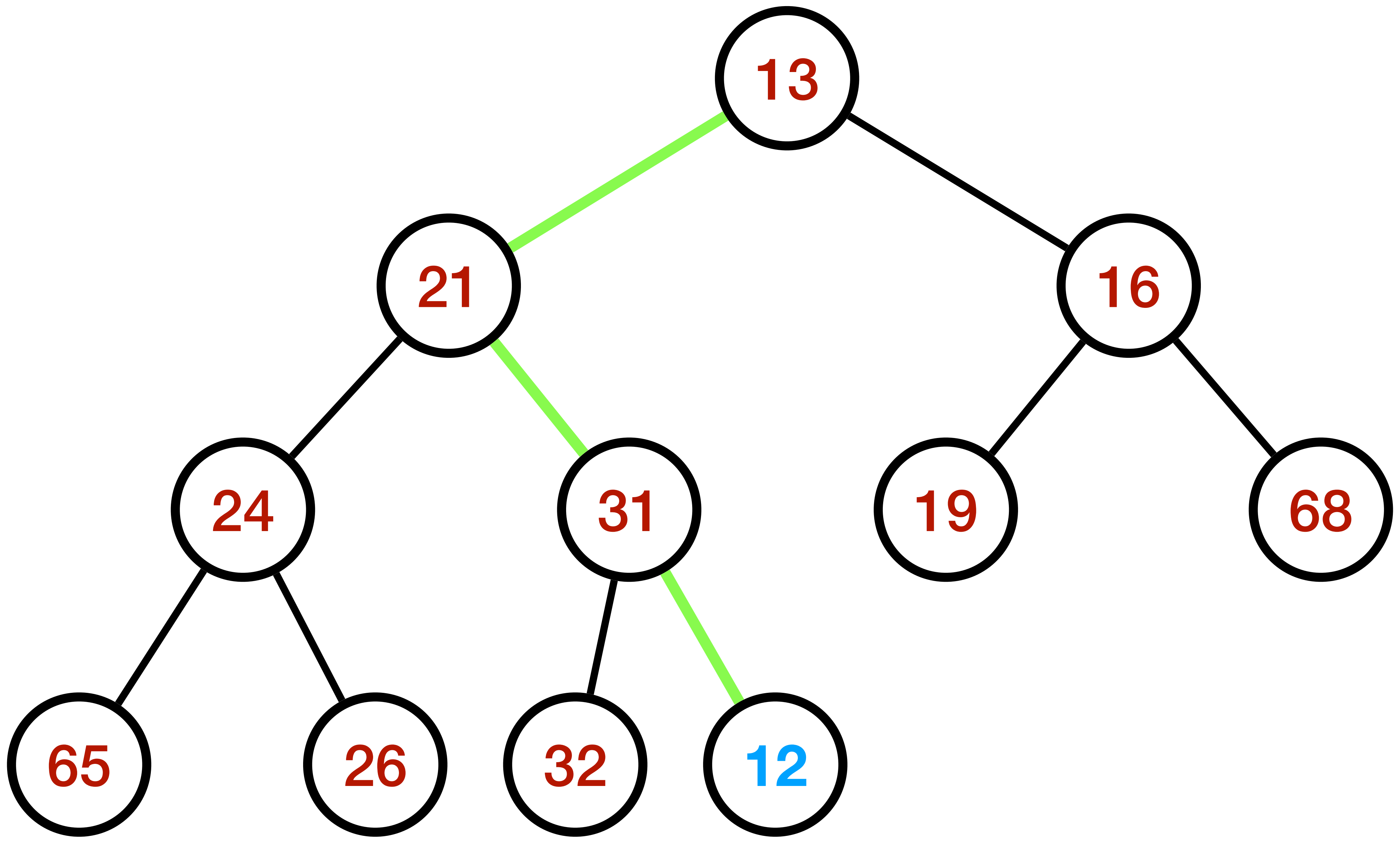

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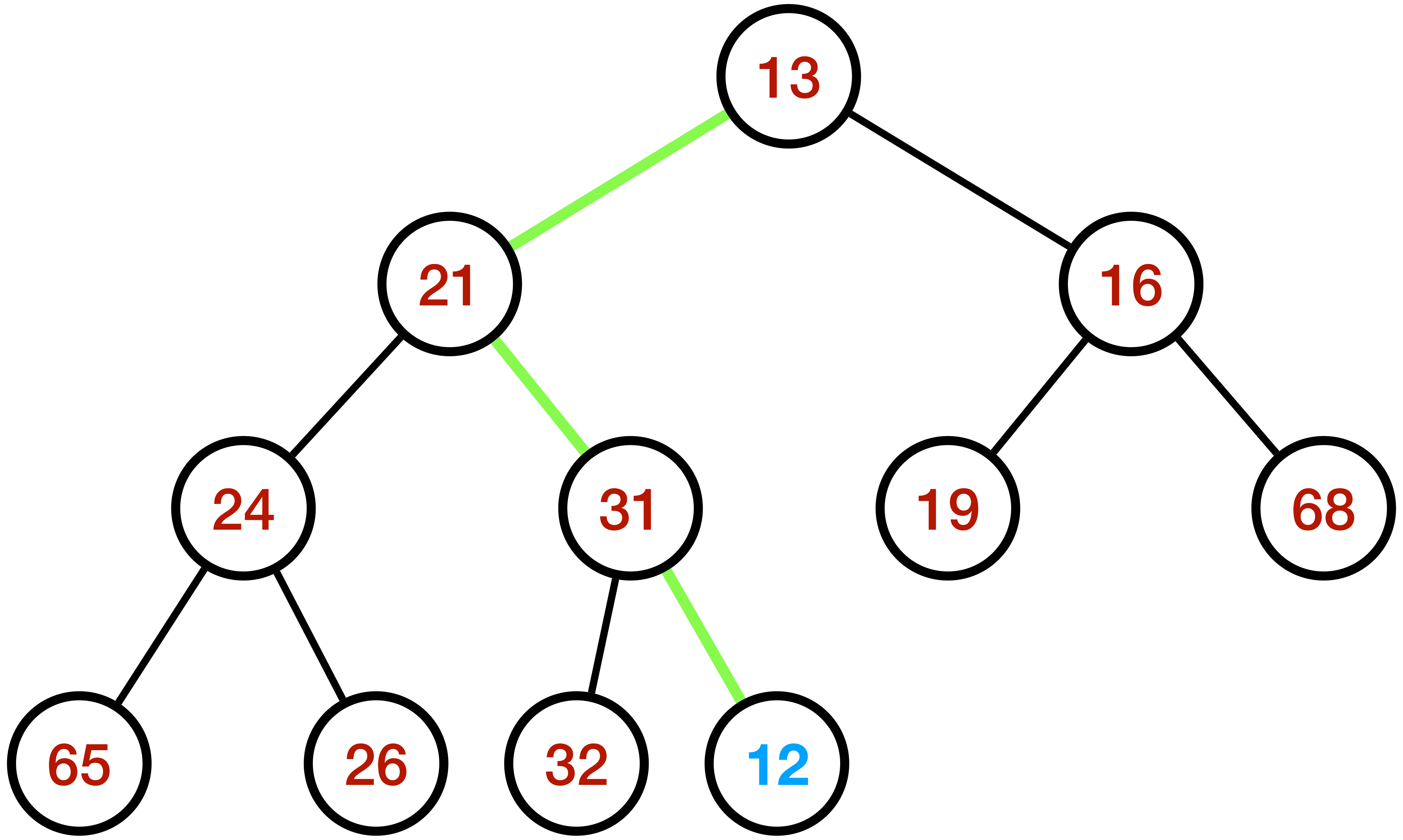

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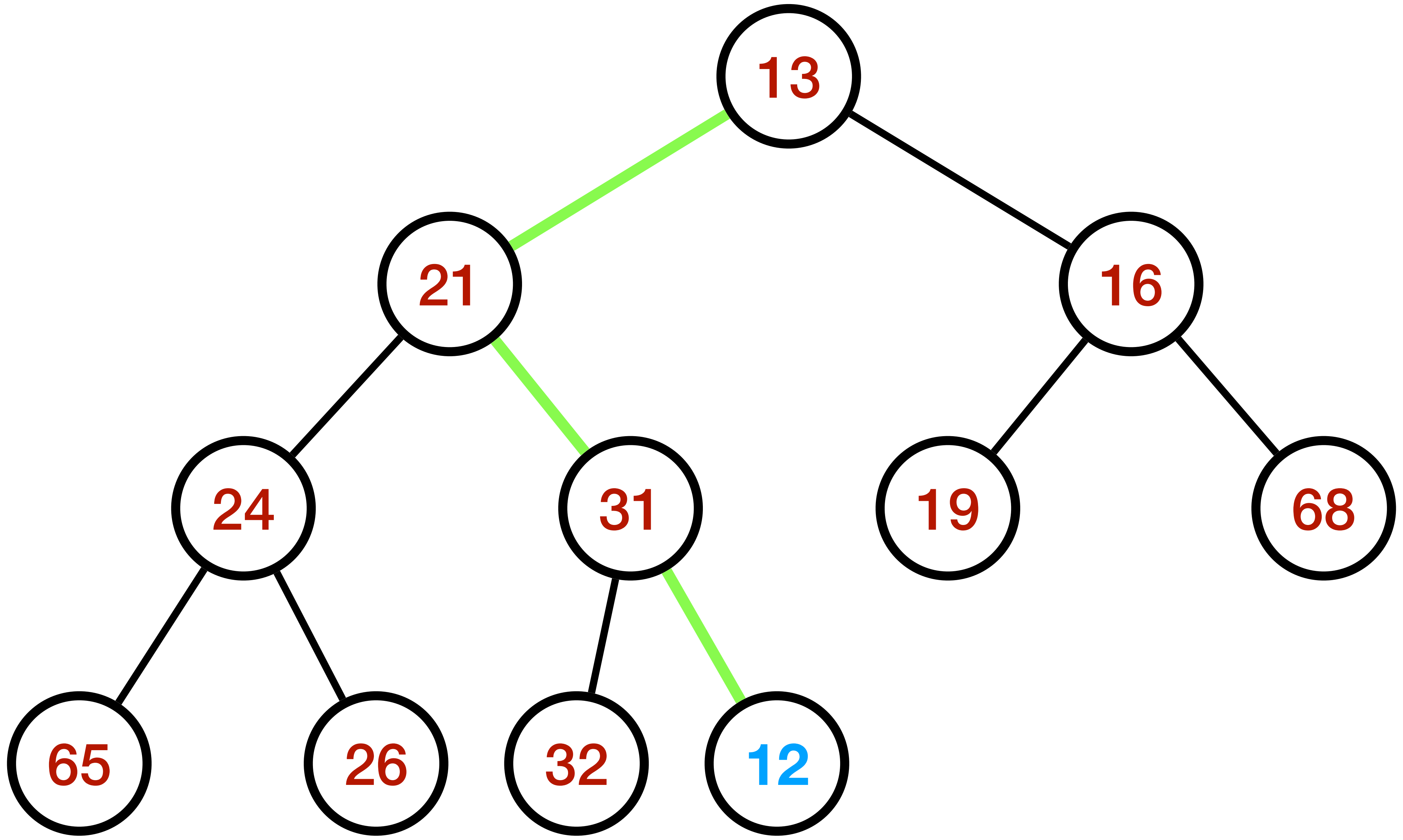

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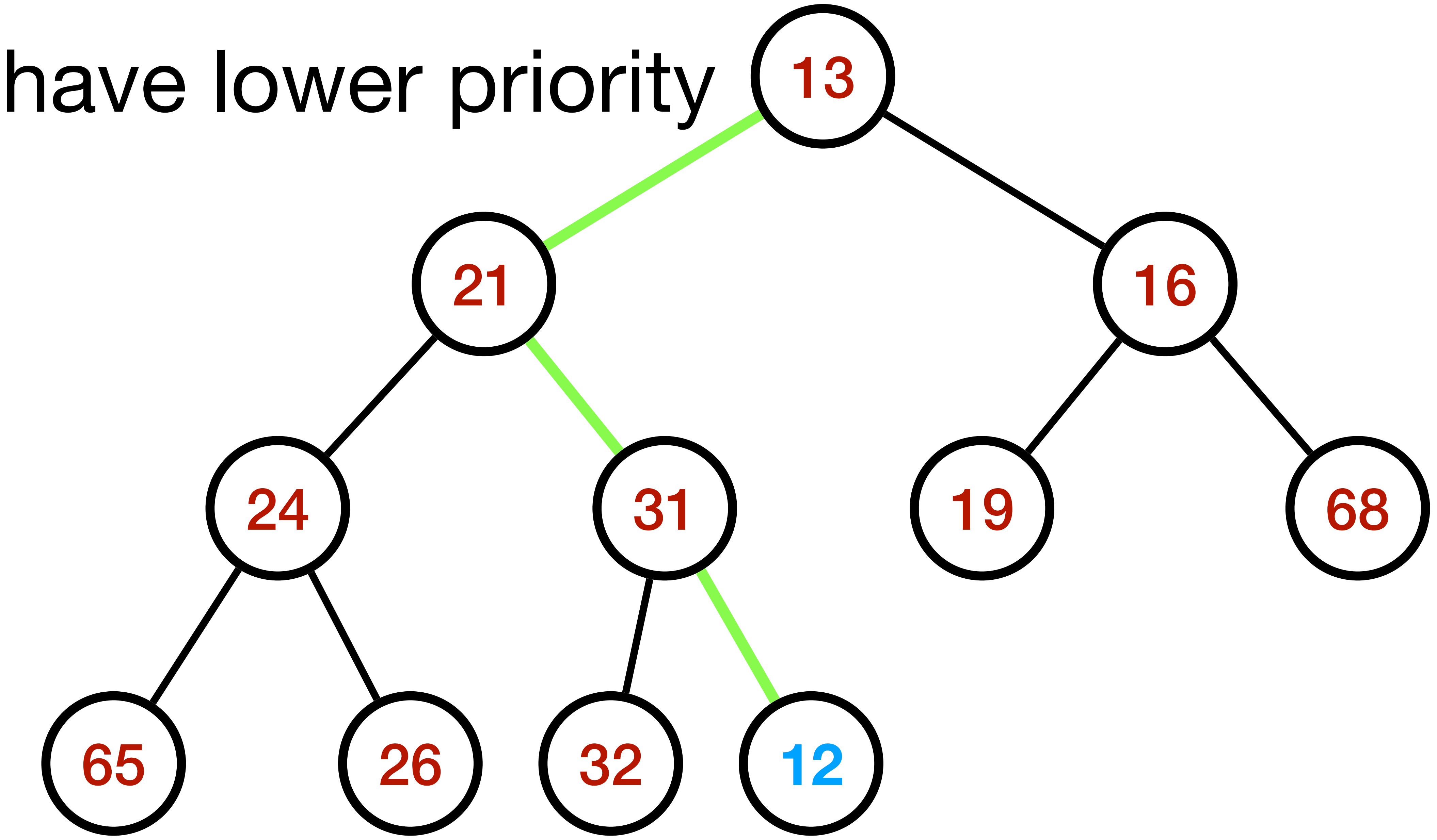

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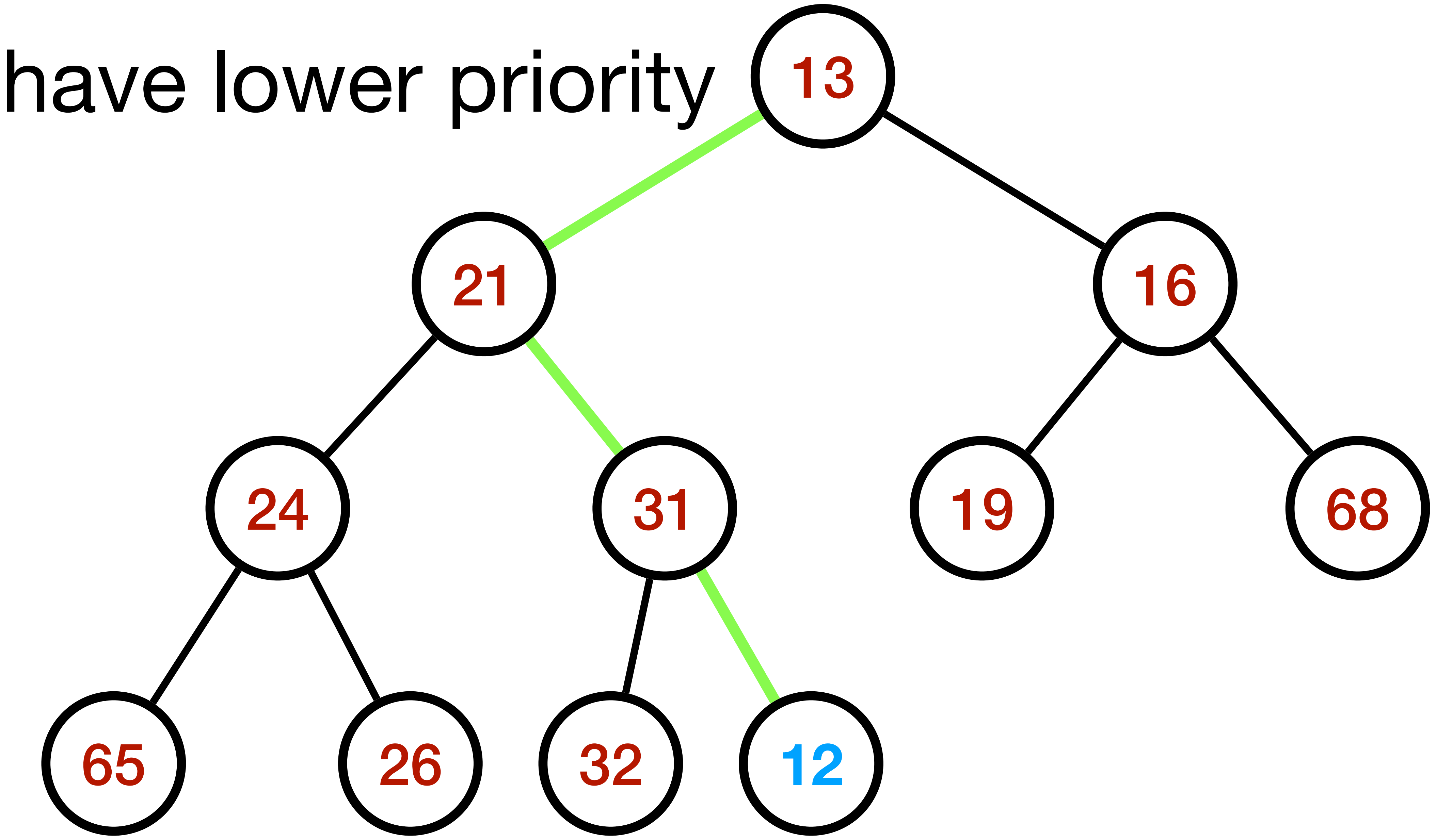
- Thus, it is correct!



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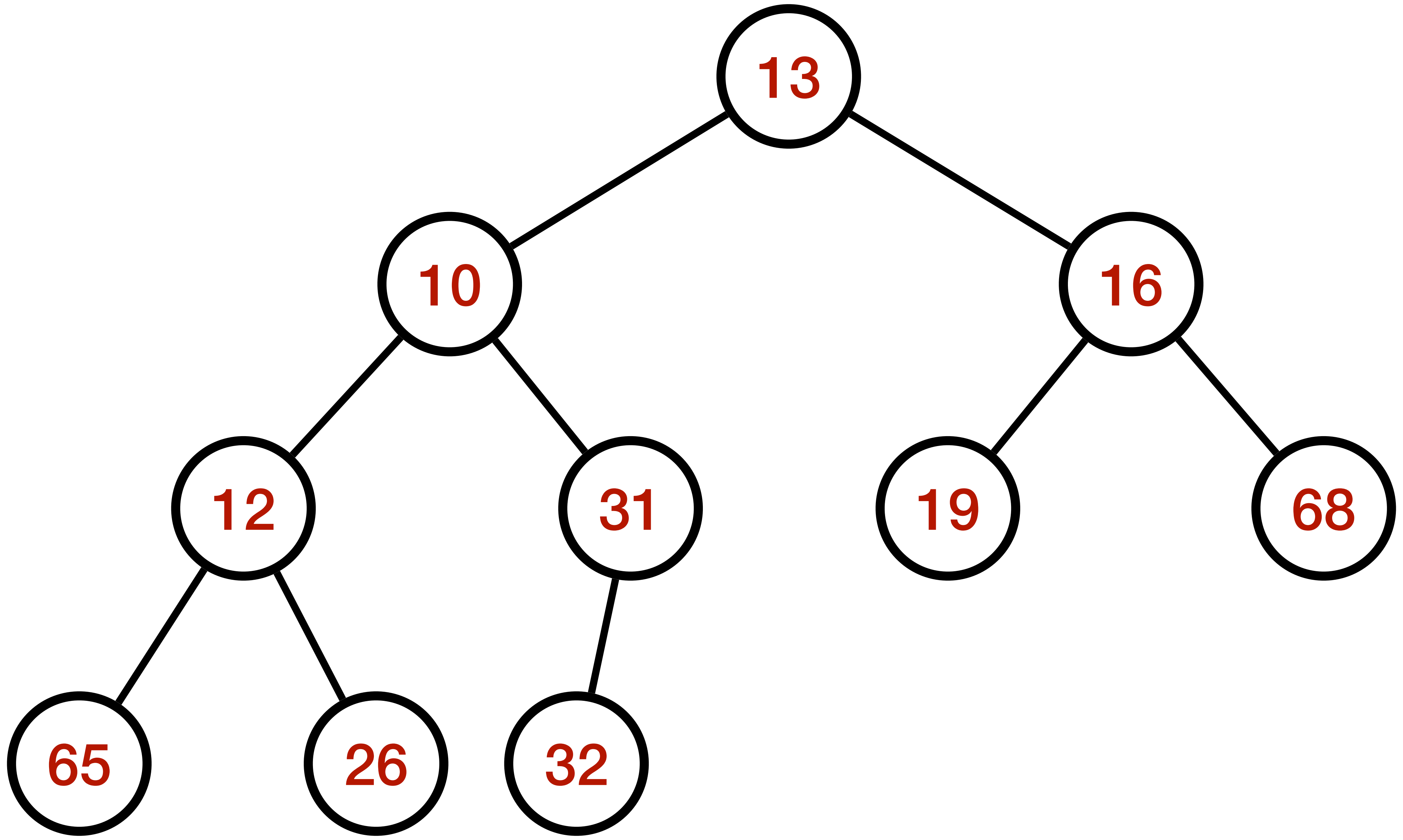
- Heap order property violated at index 0
- The subtrees rooted at index 1 and 2 are valid heaps
  - This is an important point — Heapify would work only when this observation holds
- heapifyDown(0)
- **ToDo — Prove the correctness of HeapifyDown**



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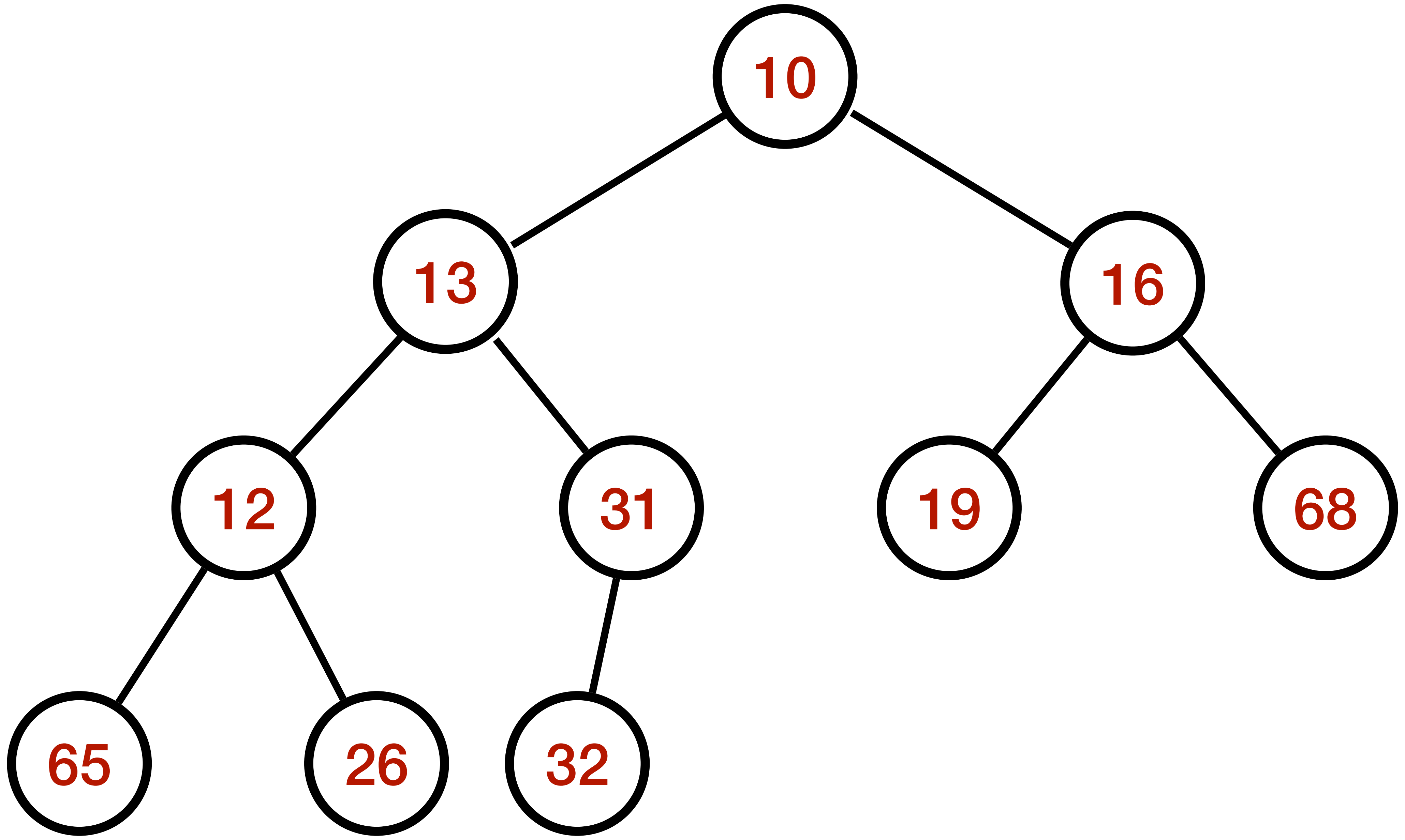
- **ToDo — Prove the correctness of HeapifyDown**



Another View of Heapify

- Heap order property violated at index 0
- The subtrees rooted at index 1 and 2 are valid heaps
  - This is an important point — Heapify would work only when this observation holds
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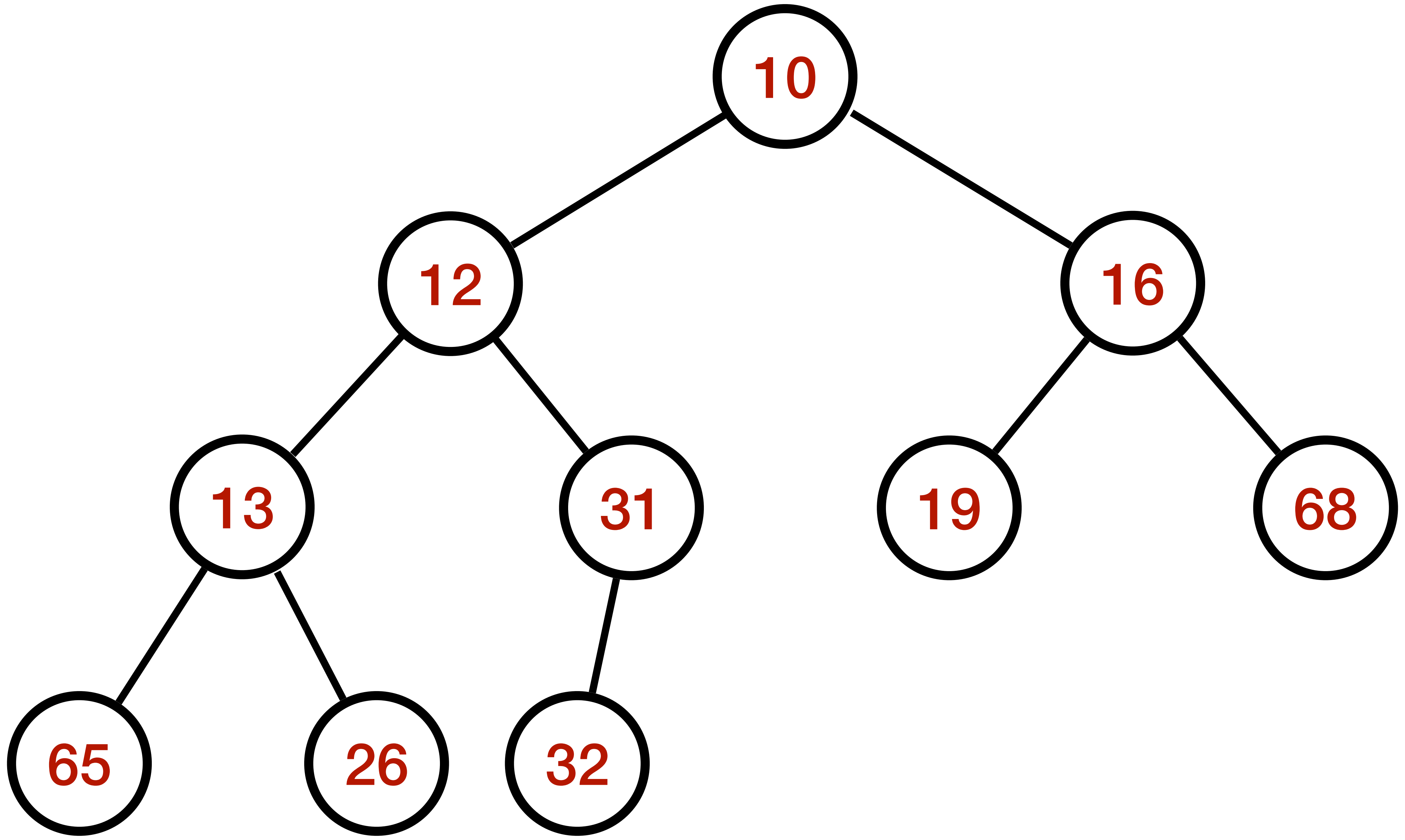
- **ToDo — Prove the correctness of HeapifyDown**



HeapifyDown

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void heapifyDown(int index) {
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  if (leftChild >= heap.size()) return; // No children

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  if (rightChild < heap.size() && heap[minIndex] > heap[rightChild]) {
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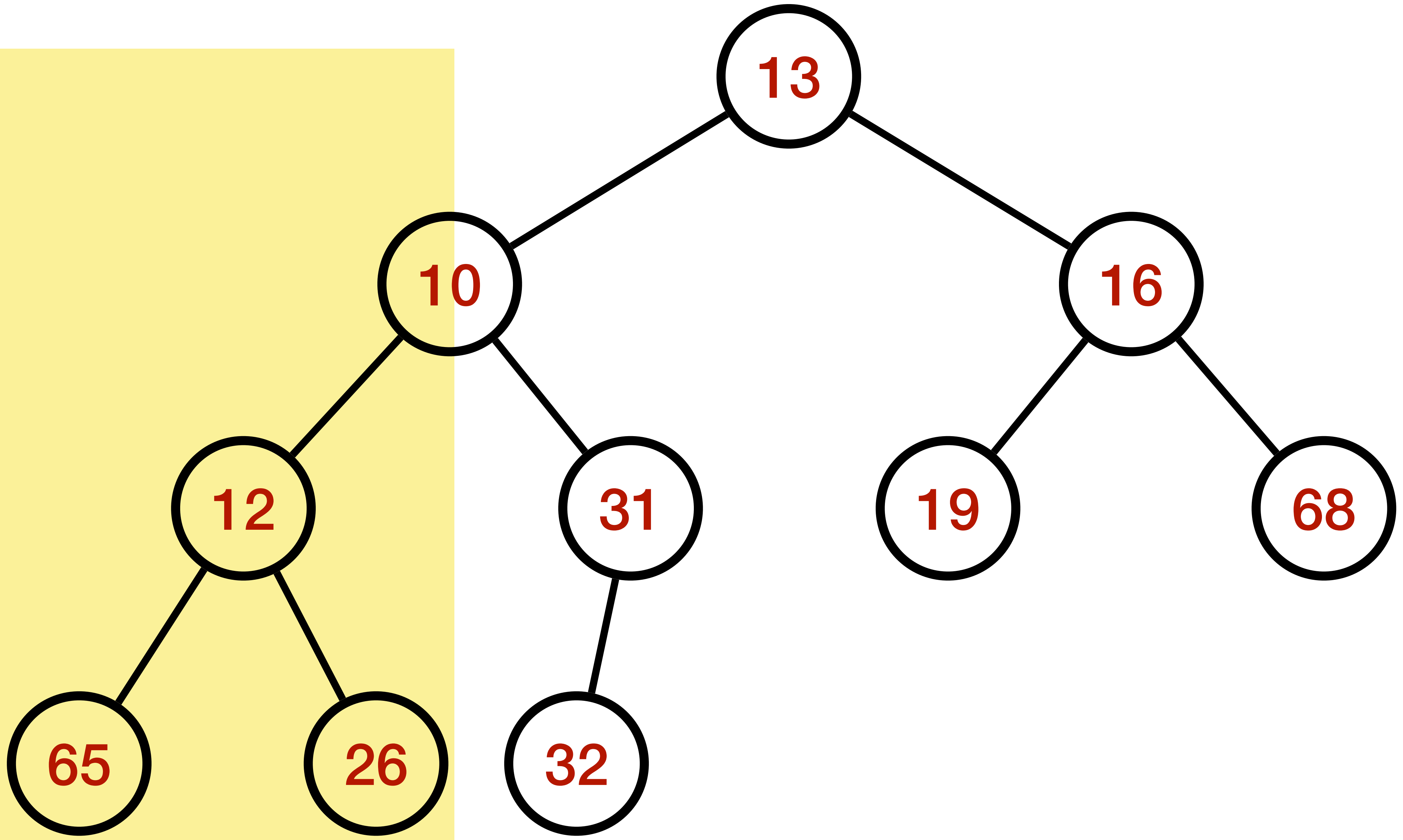

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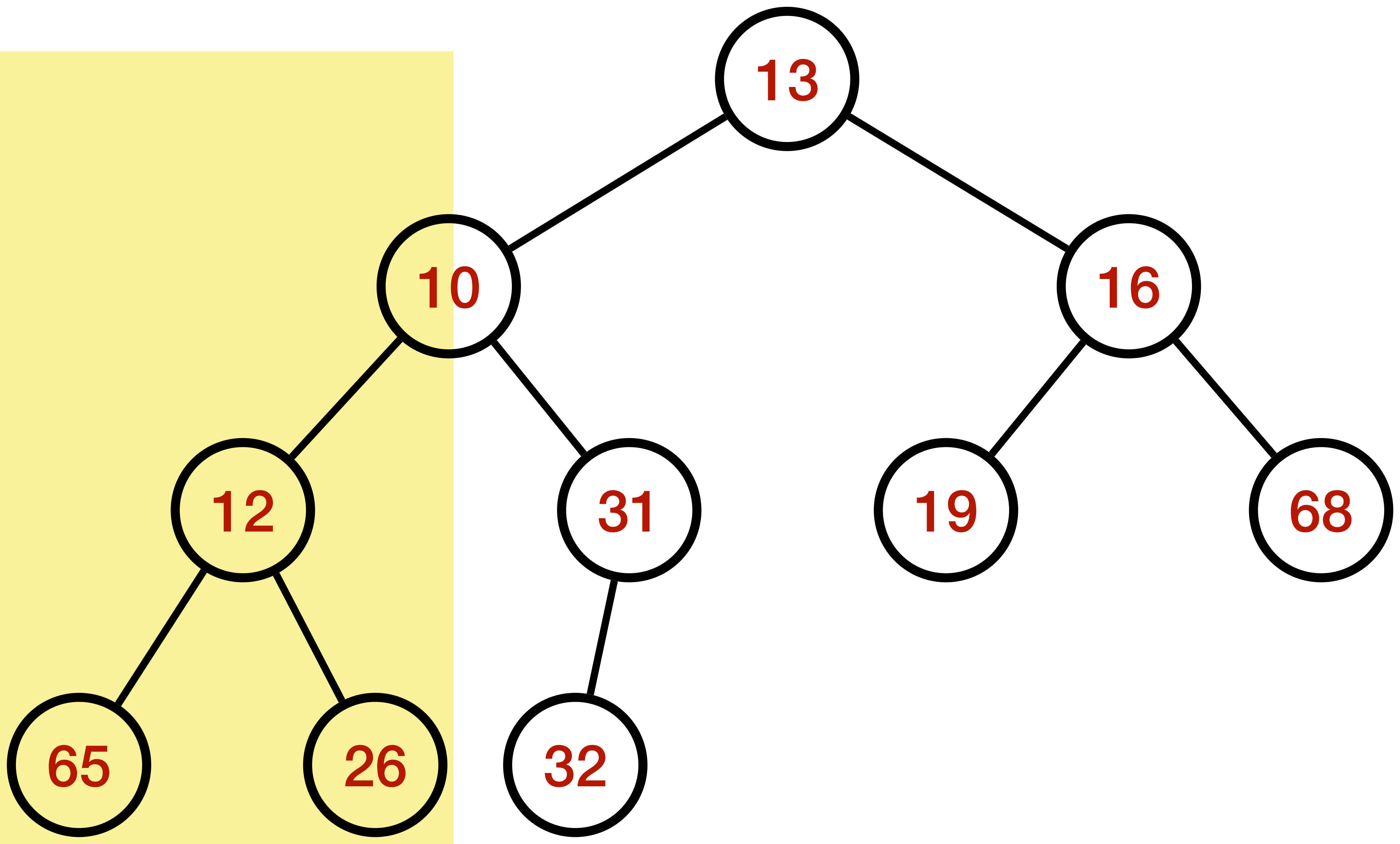

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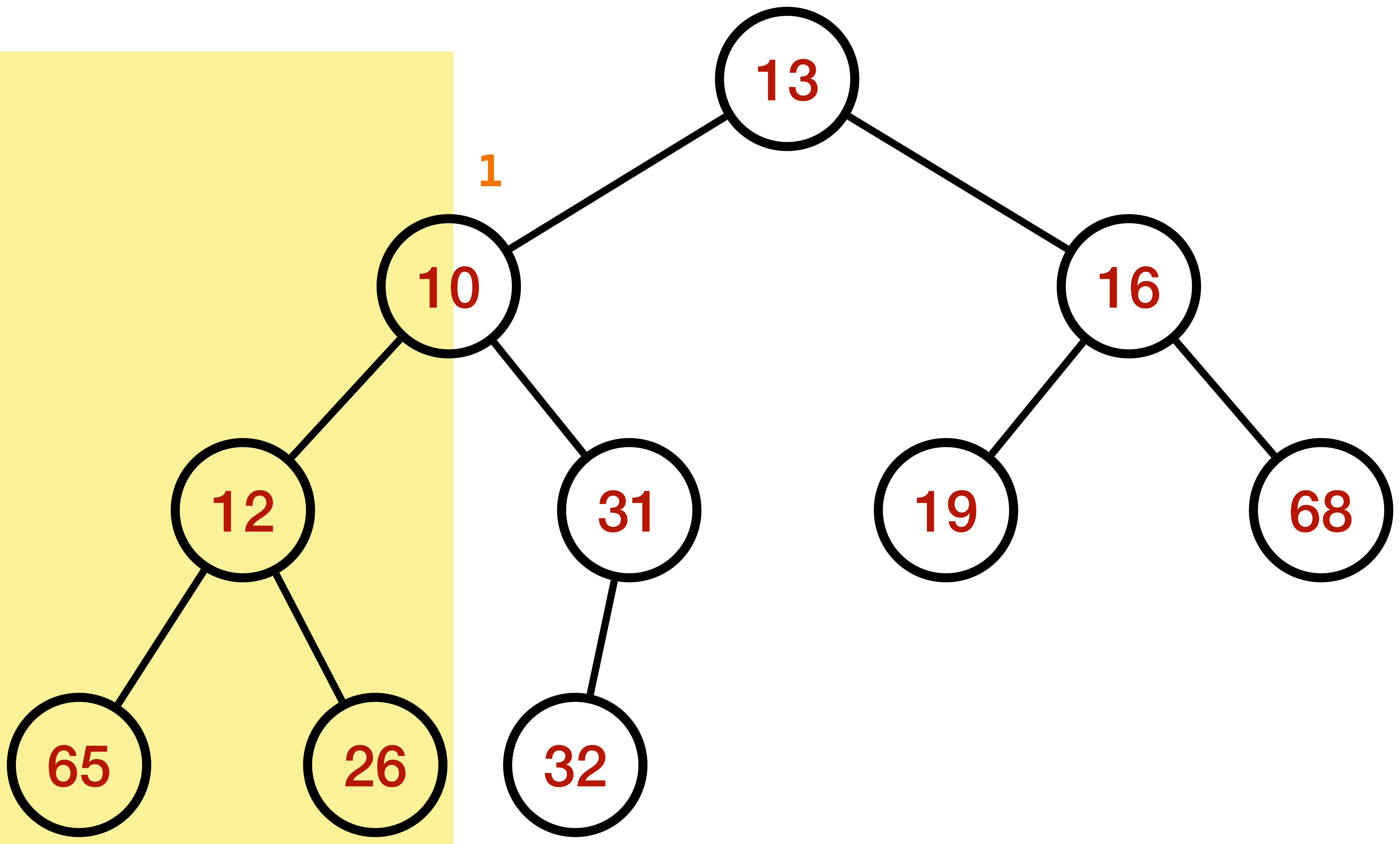

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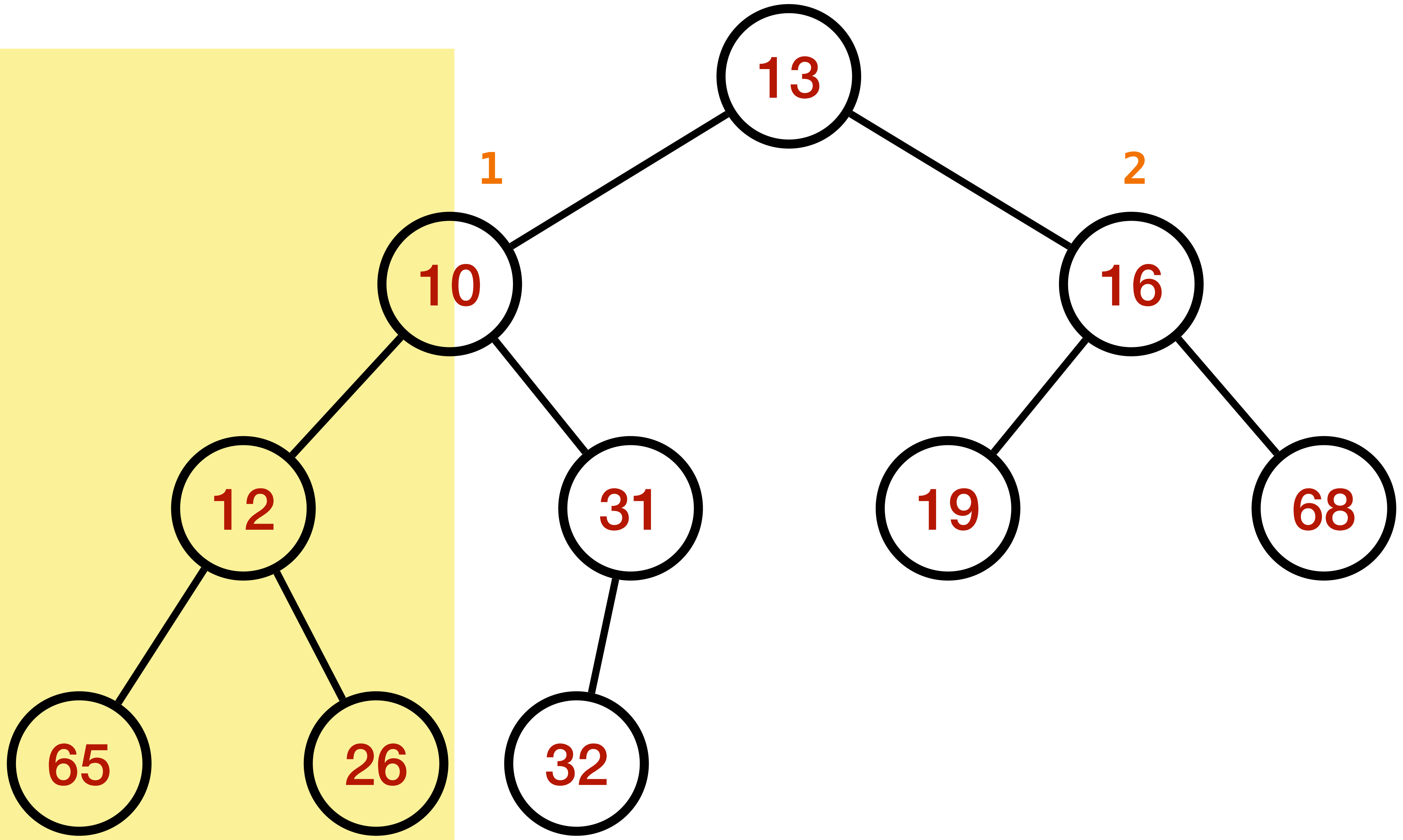

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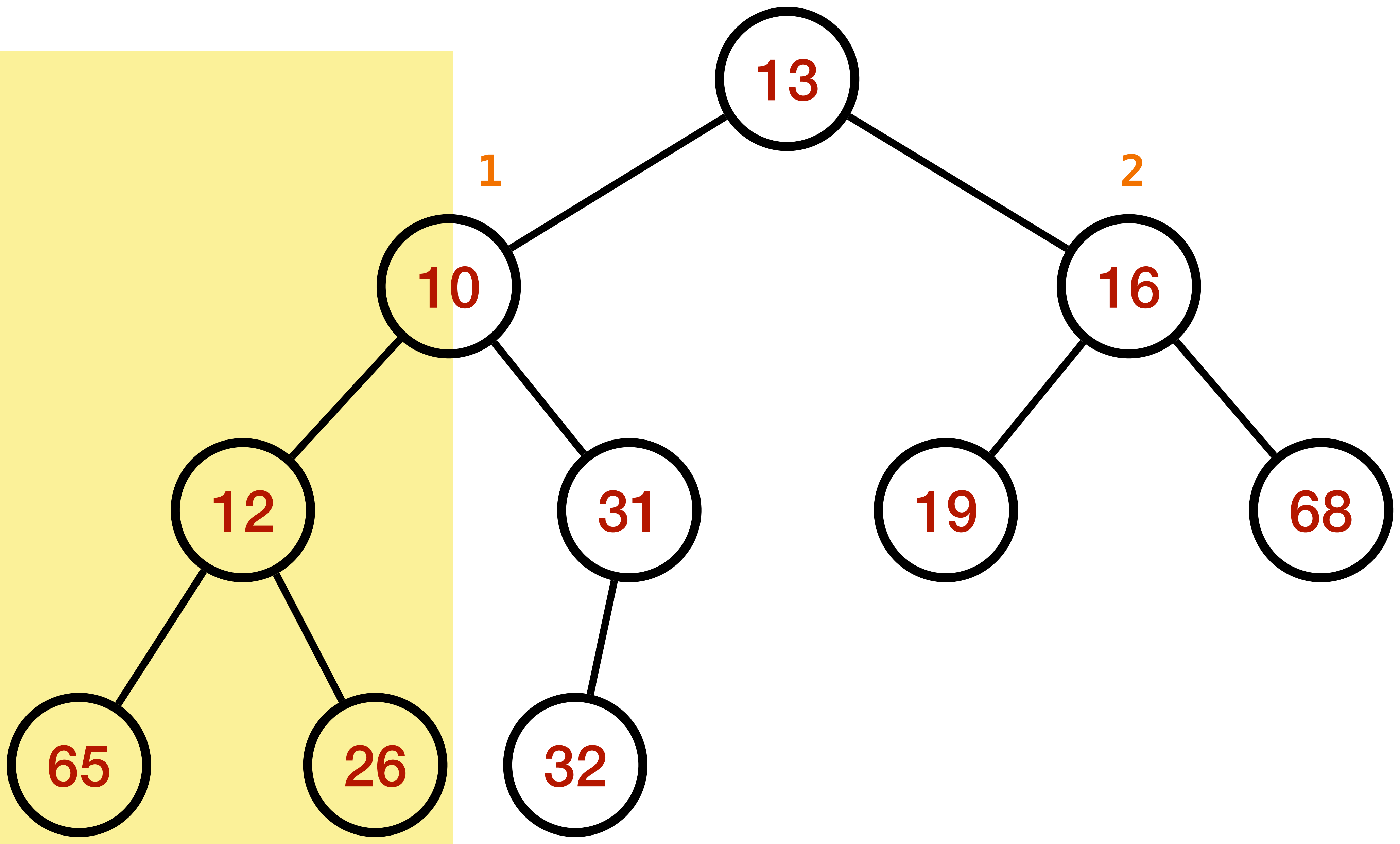

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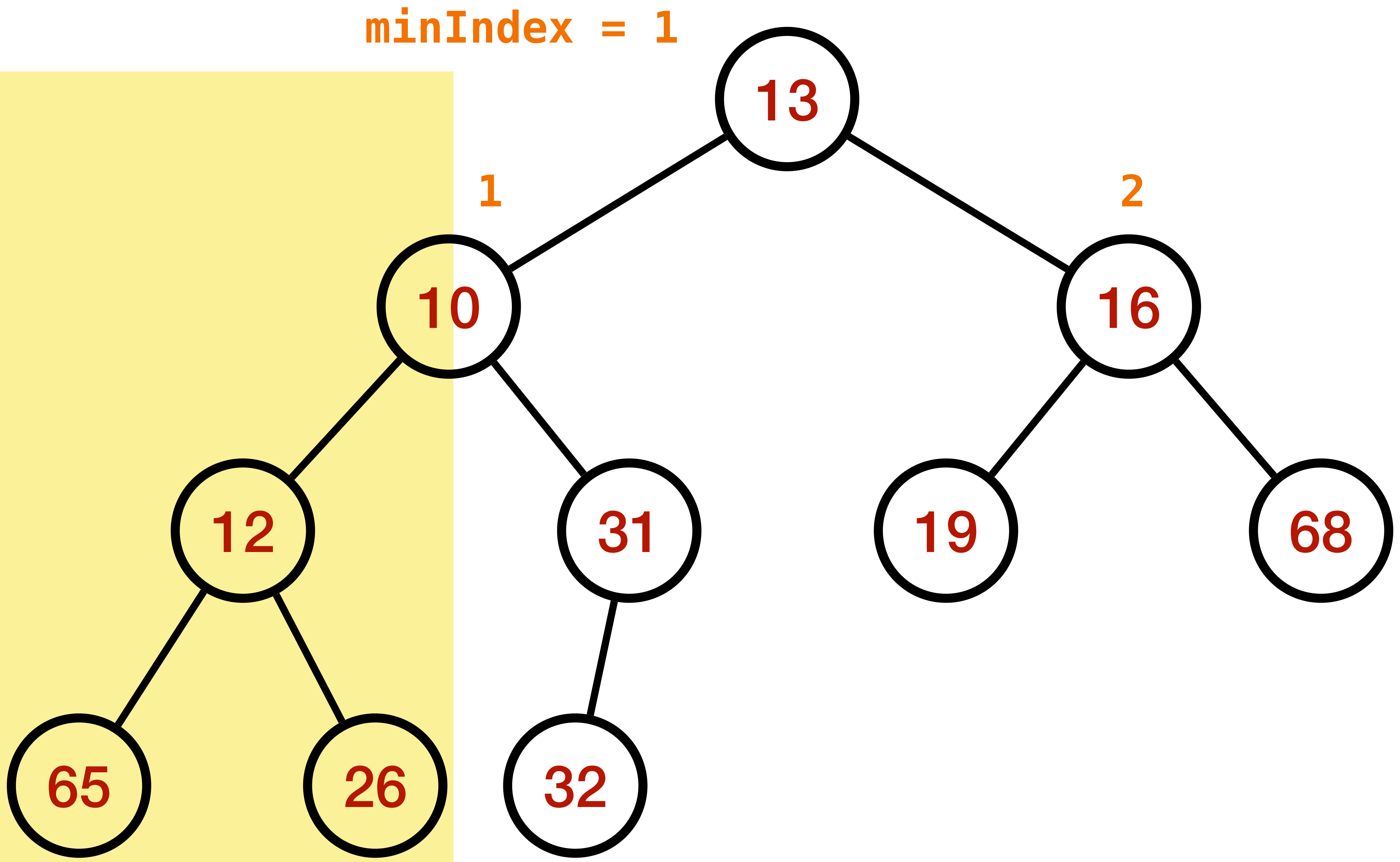

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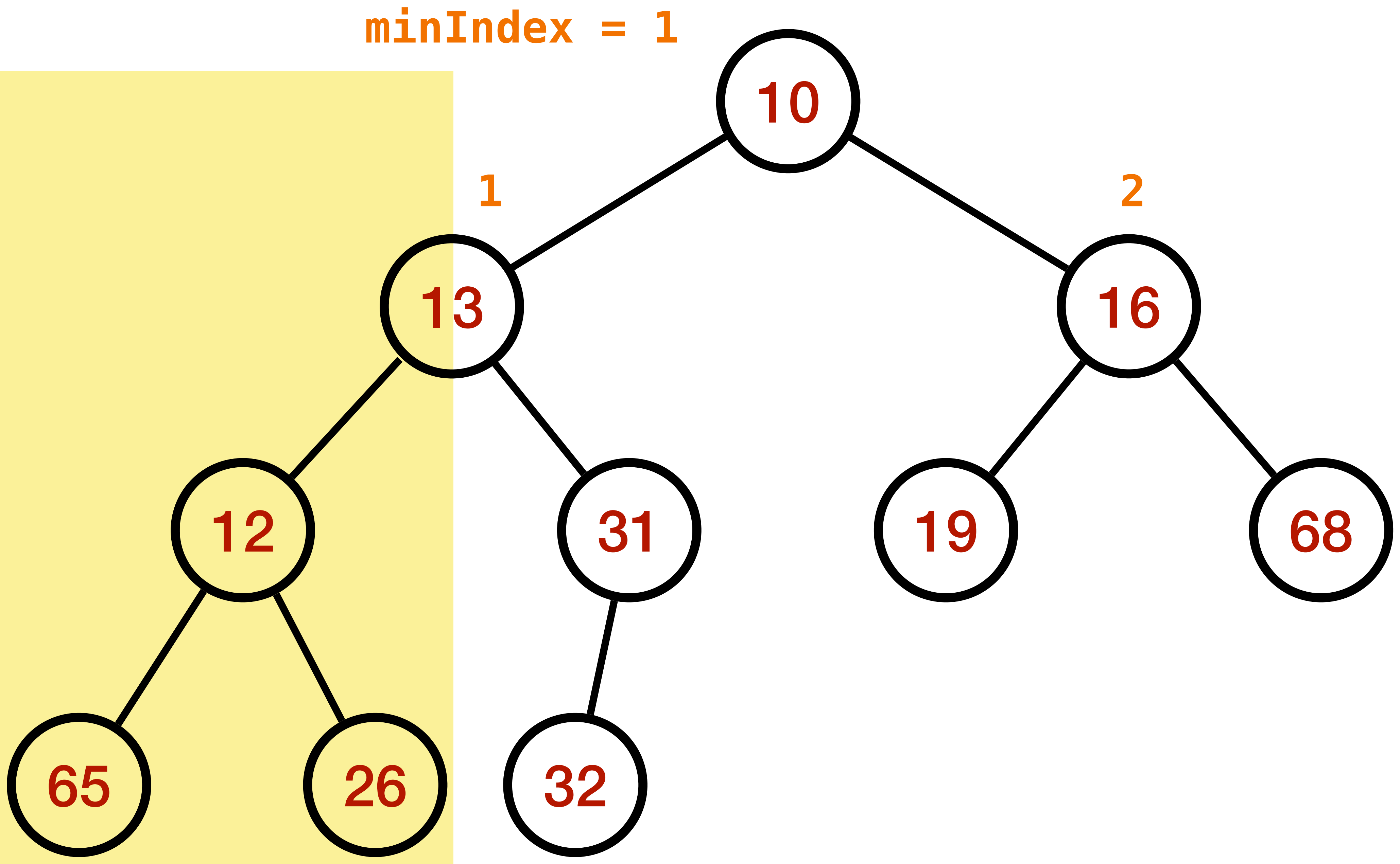
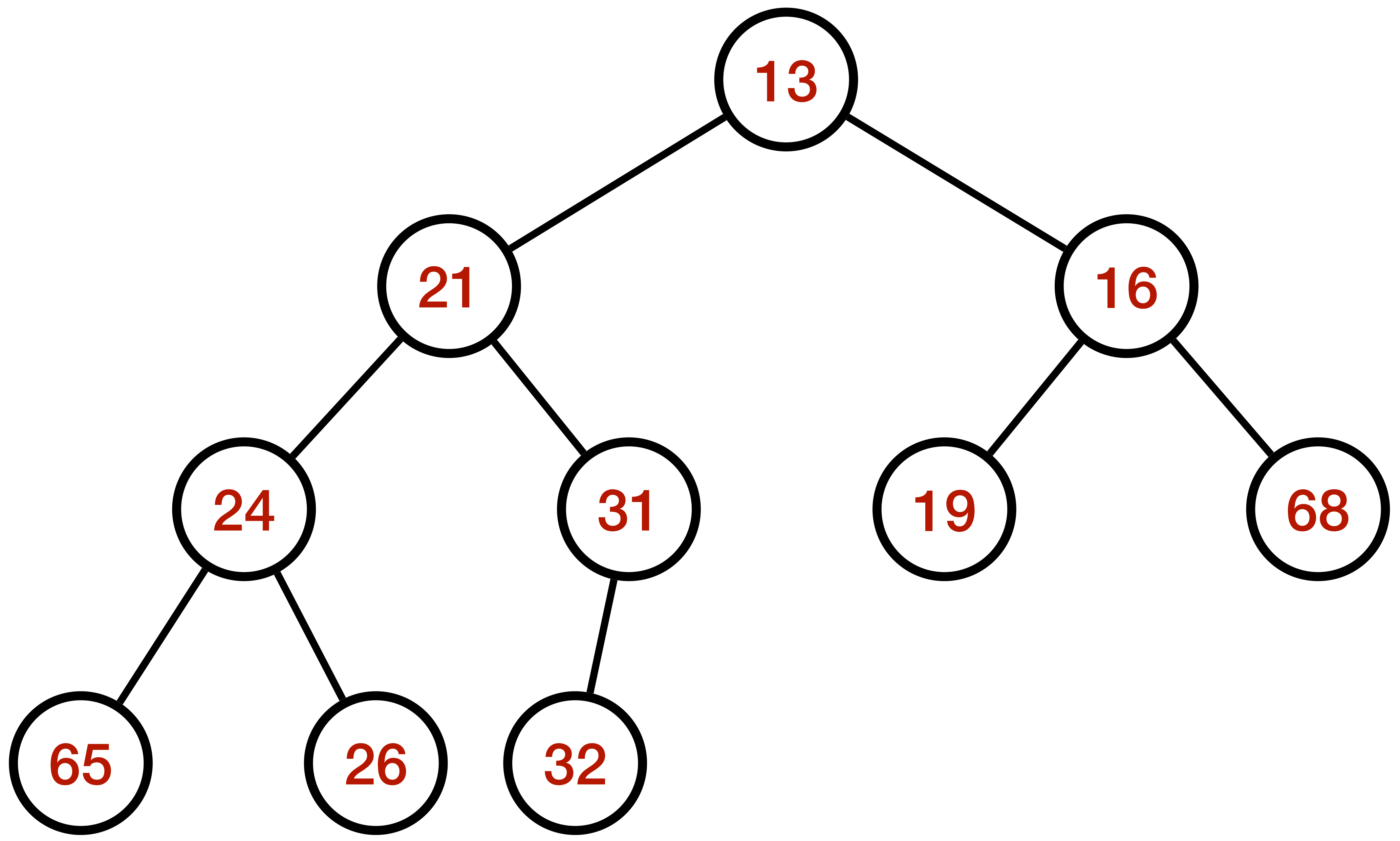


DeleteMin Operation

- Remember that the minimum element is at the root of the heap
  - We can delete this and move one of its children to fill the space!
    - Empty location moves down the tree
    - Resulting tree may not be **left-filled**

DeleteMin Operation: Attempt 1

- Delete 13
- 16 moves up
- 19 moves up
- Not left-filled



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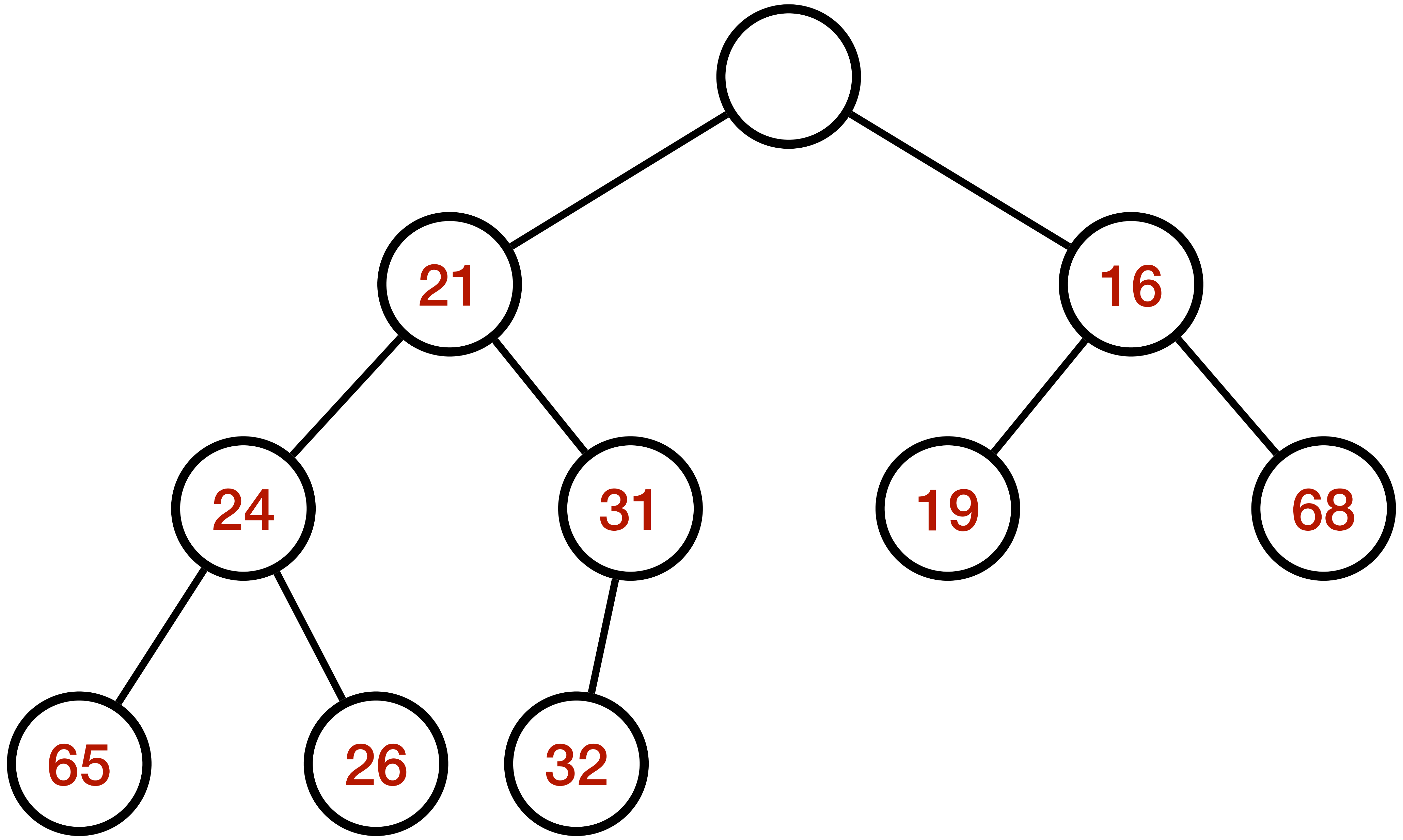

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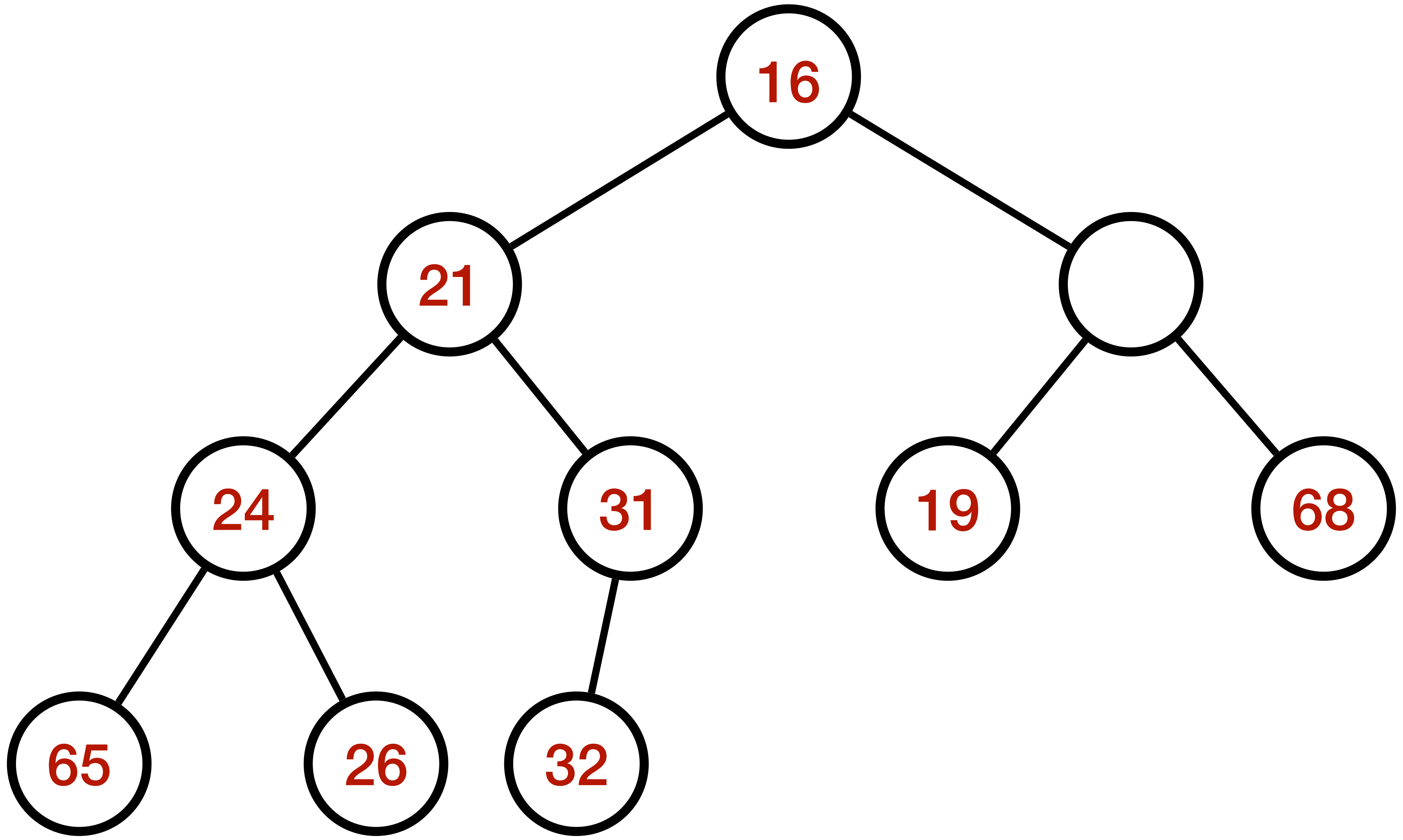

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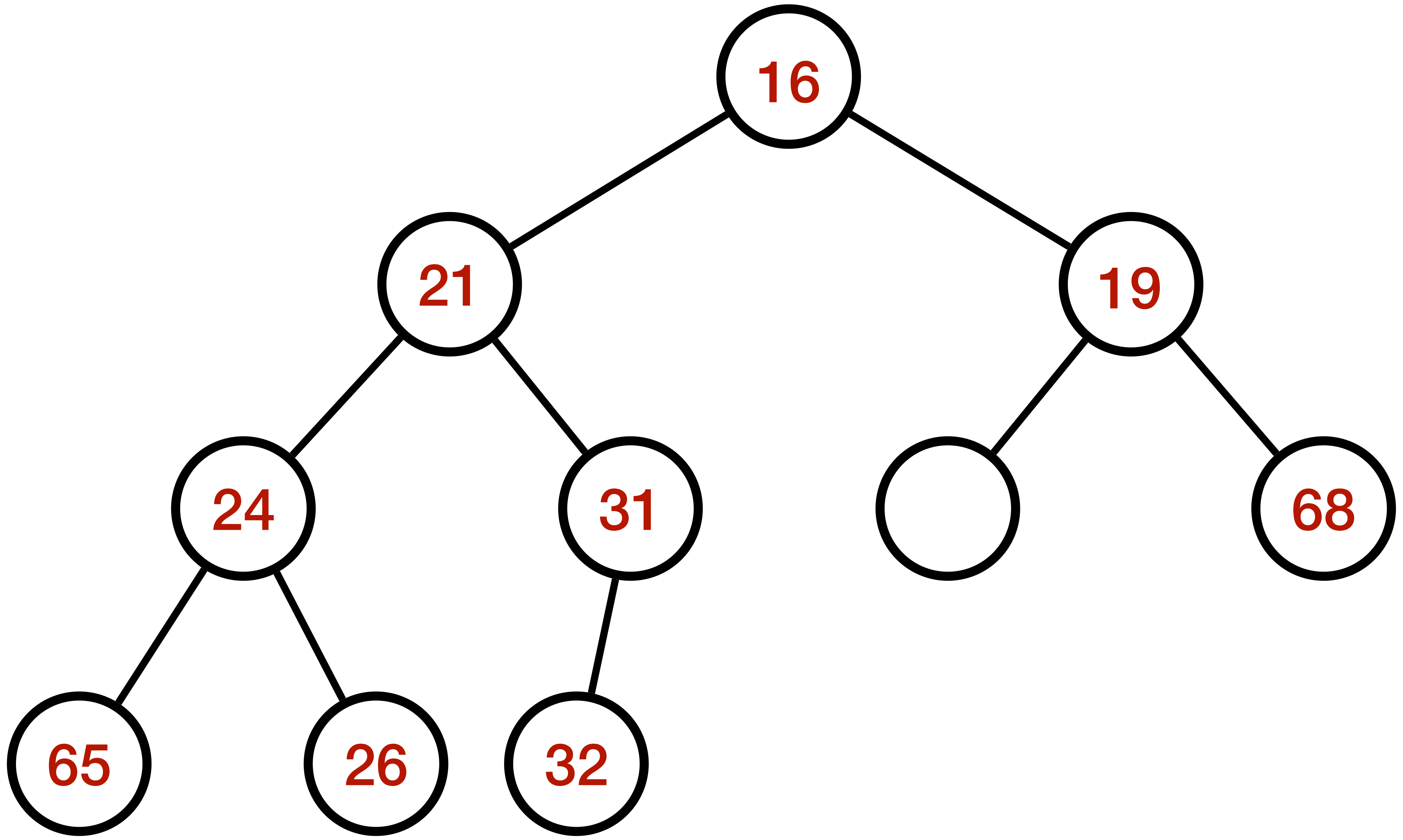
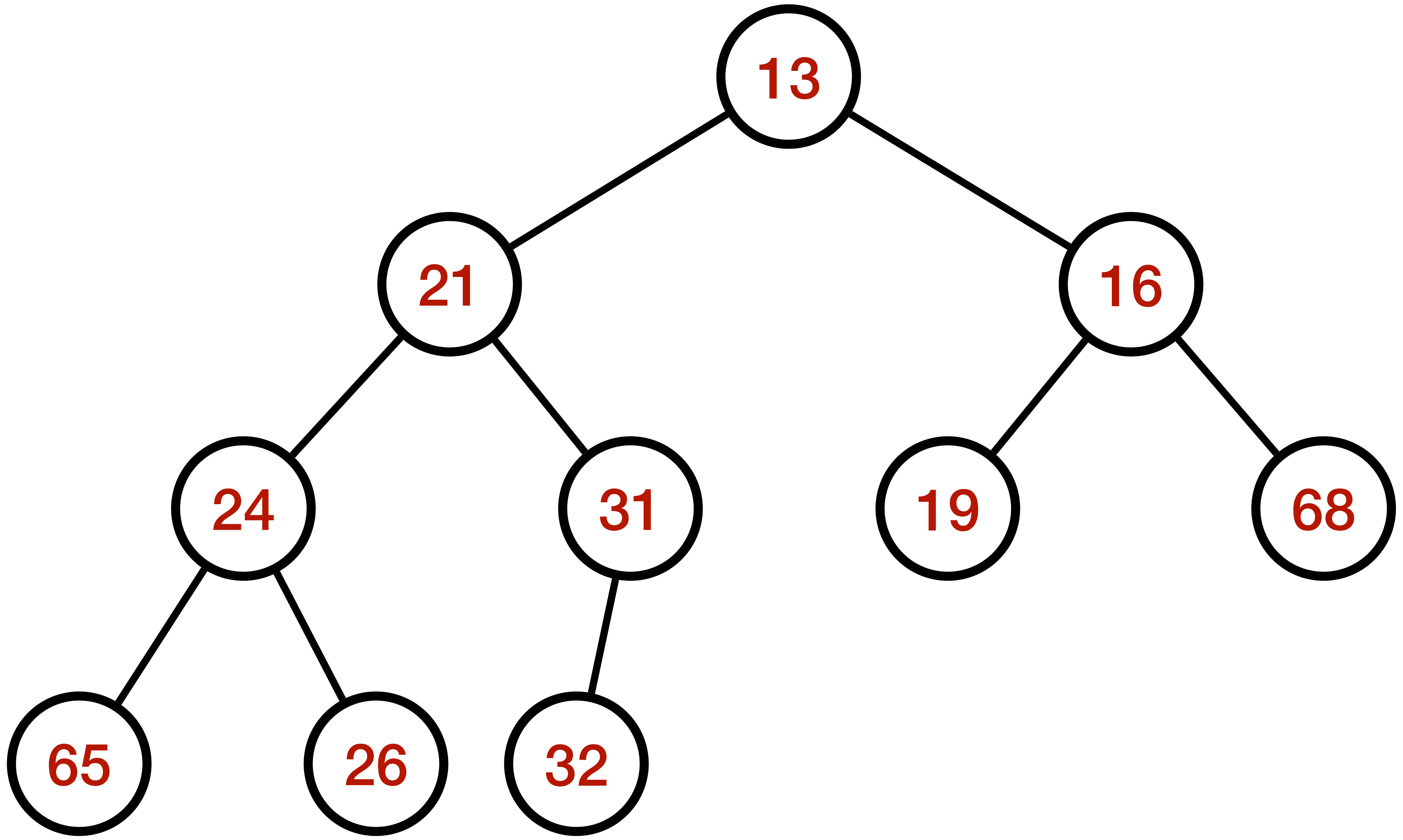
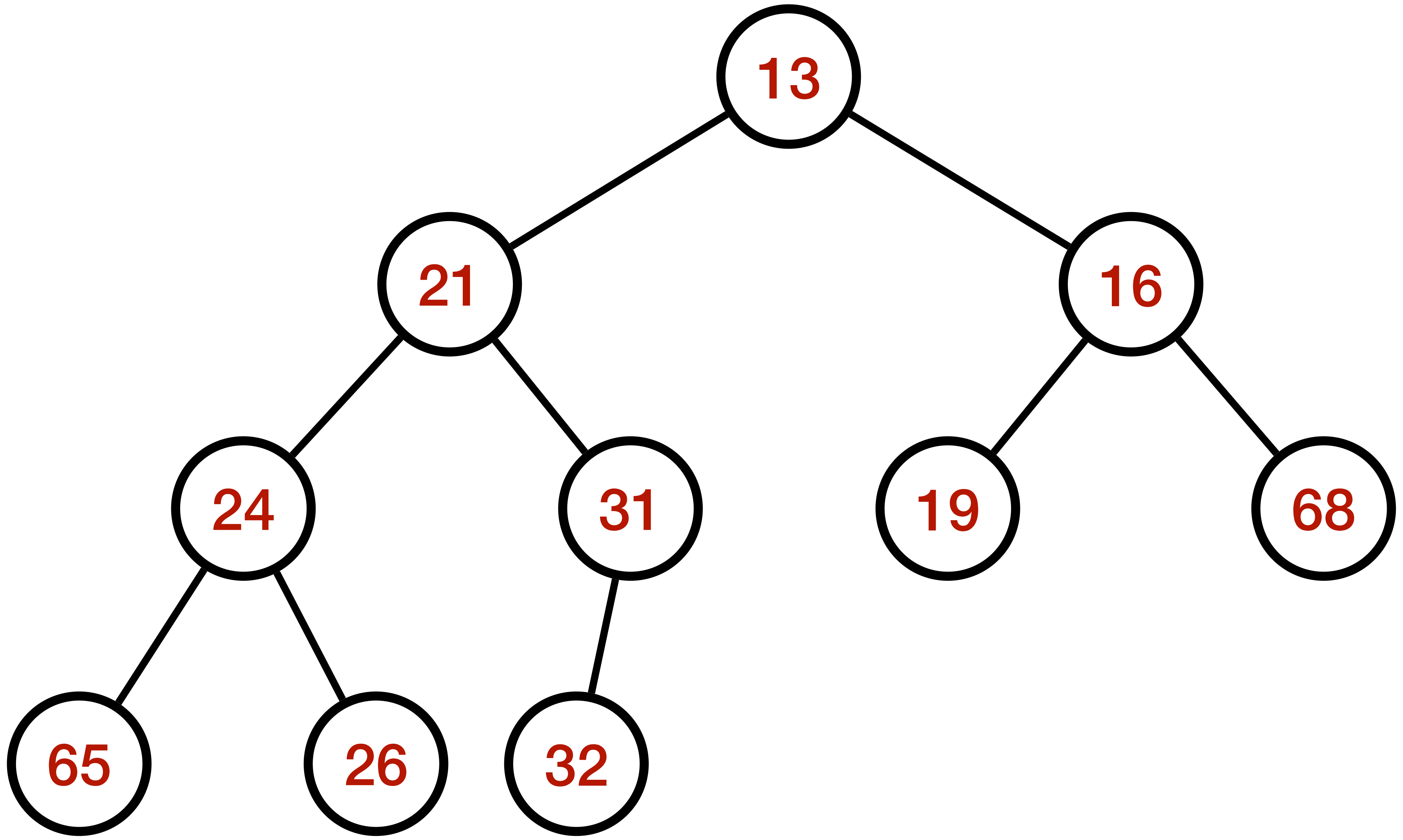


DeleteMin Operation: Attempt 2



DeleteMin Operation: Attempt 2

- Replace root element with the last element of the heap



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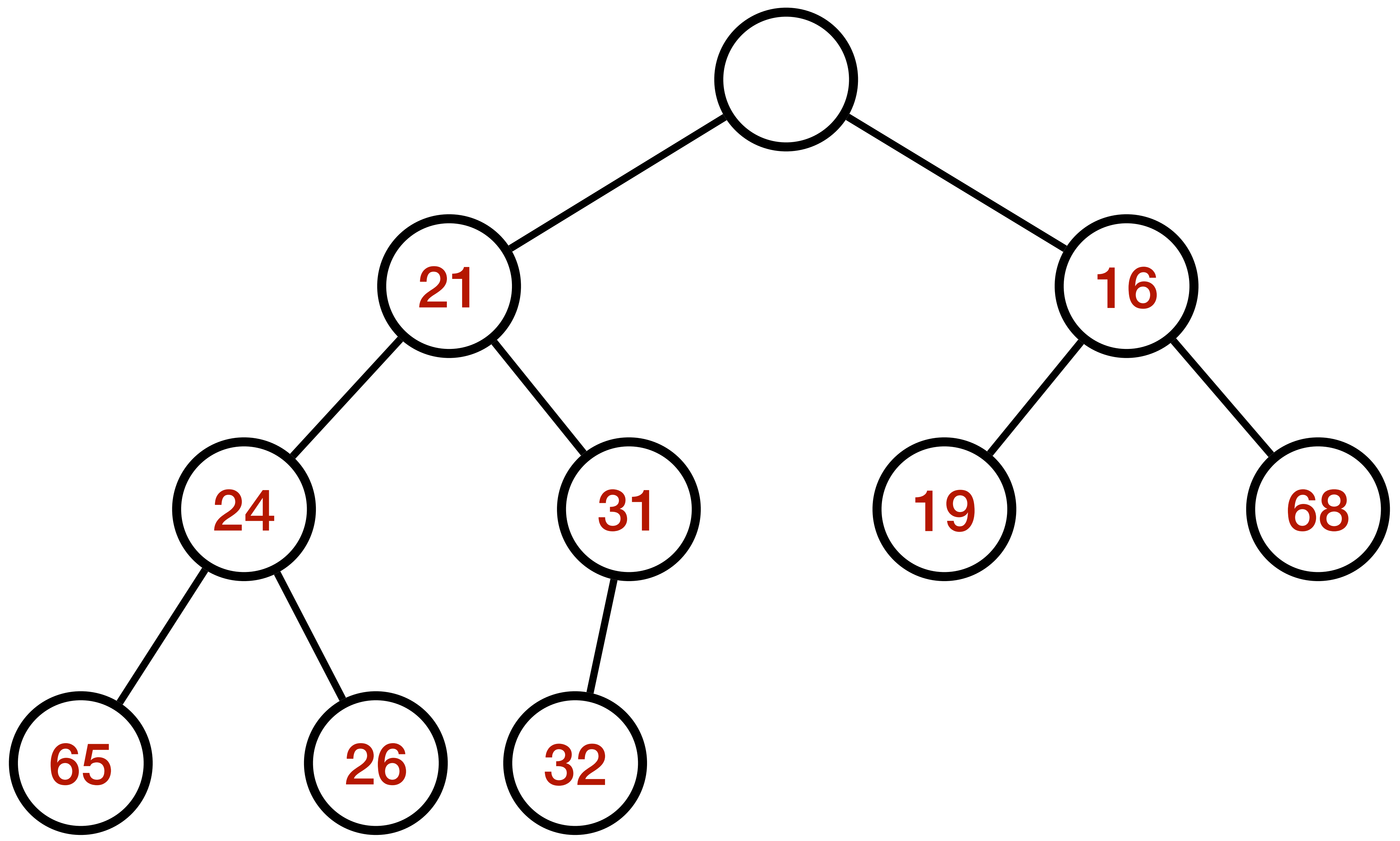

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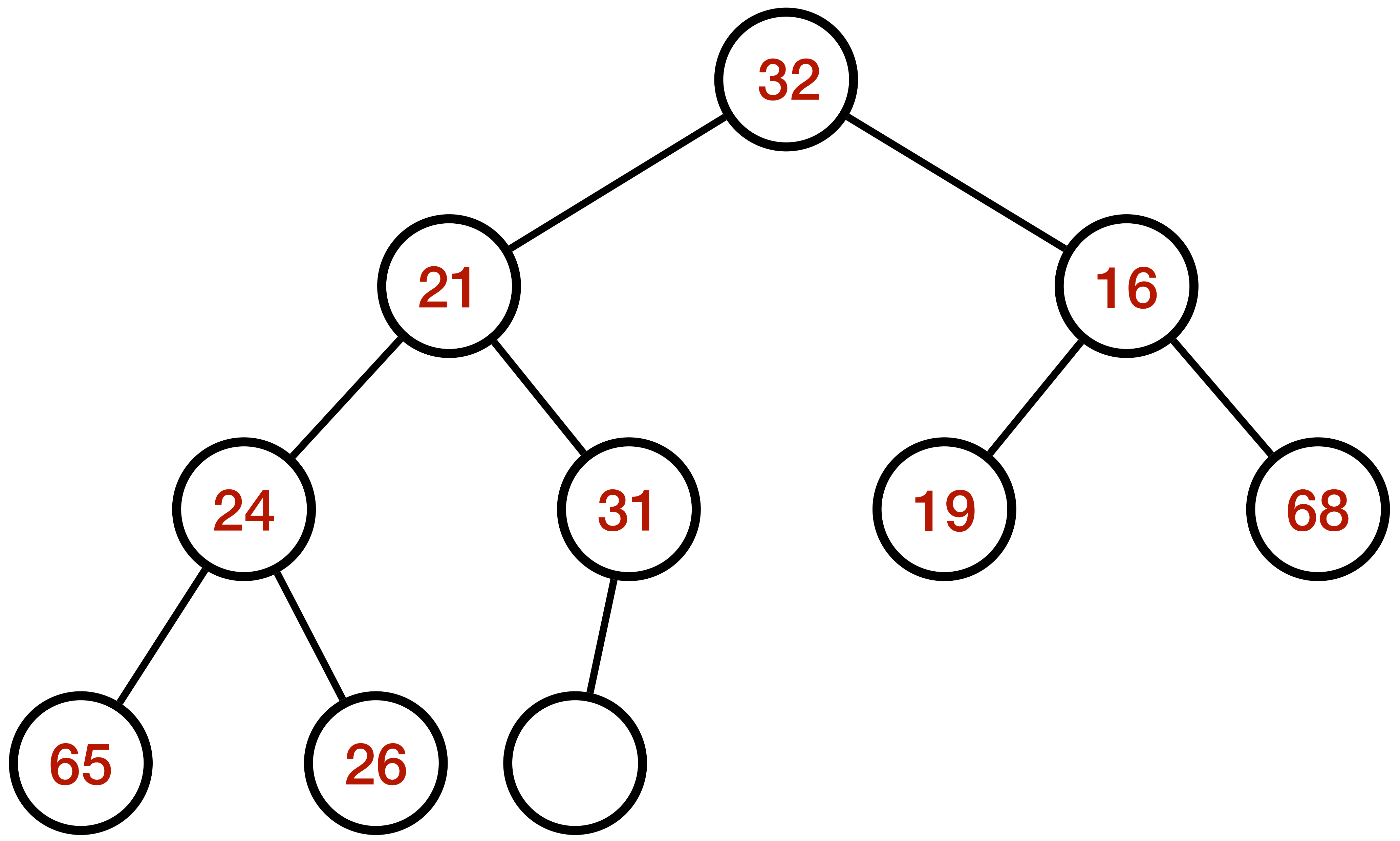

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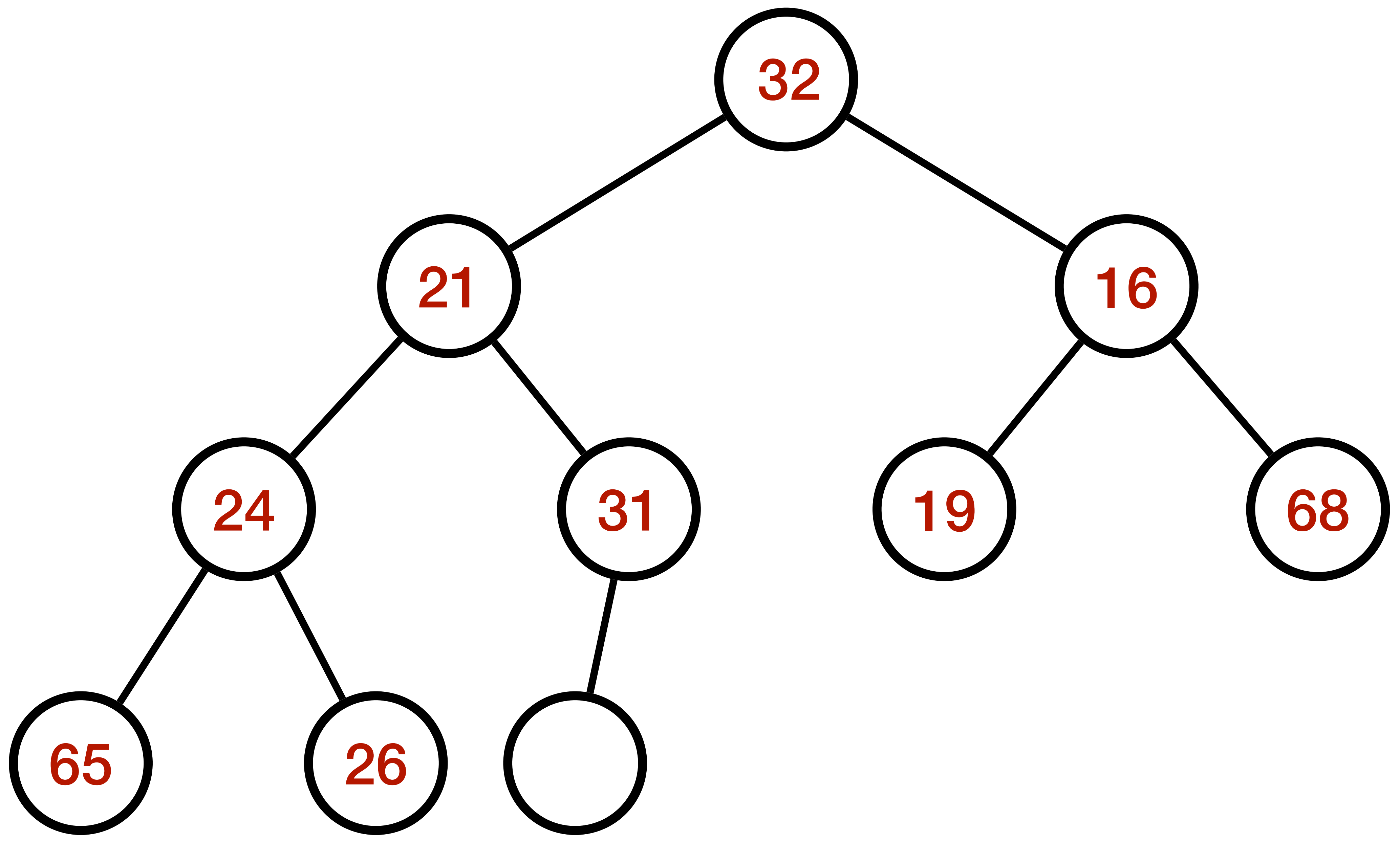

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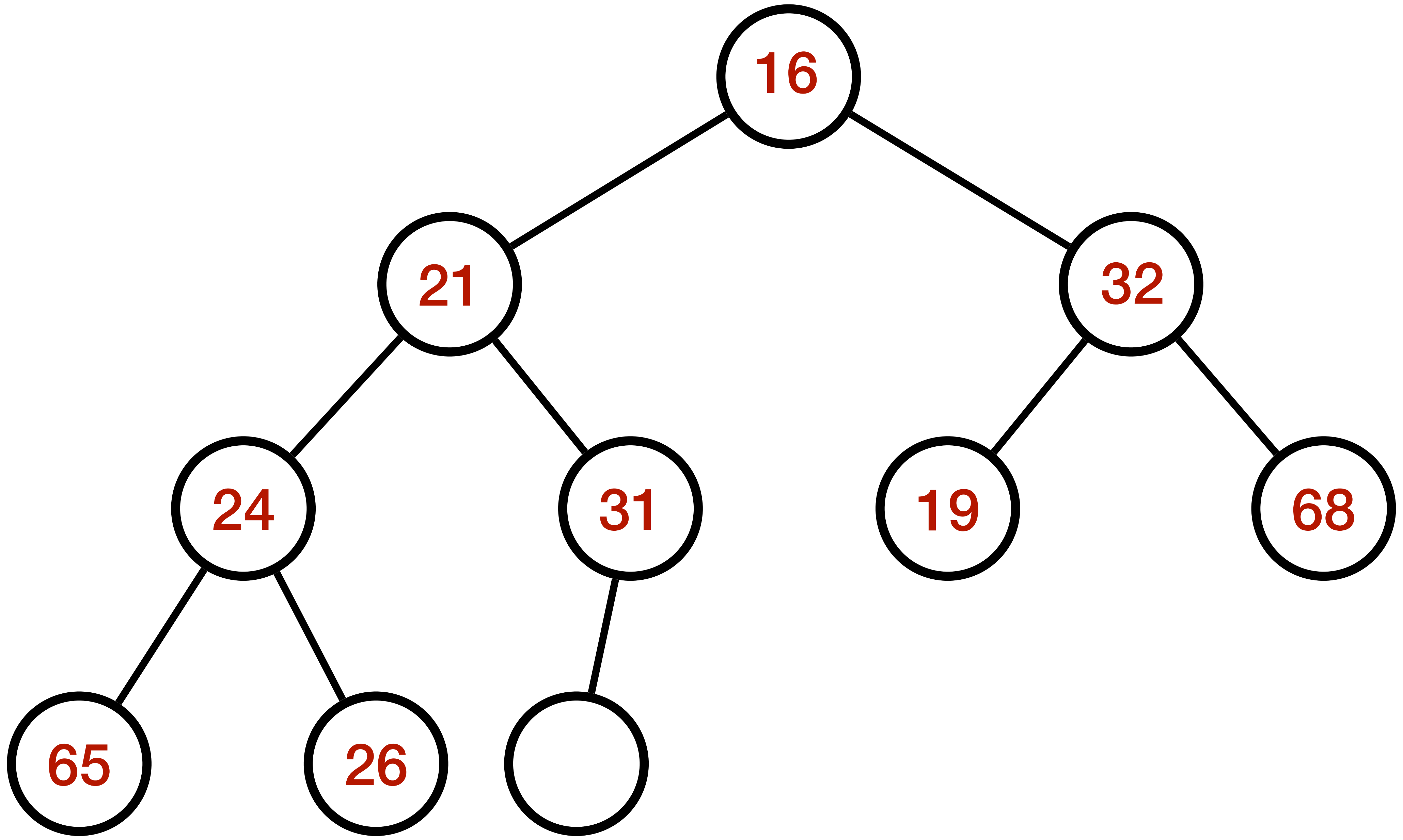

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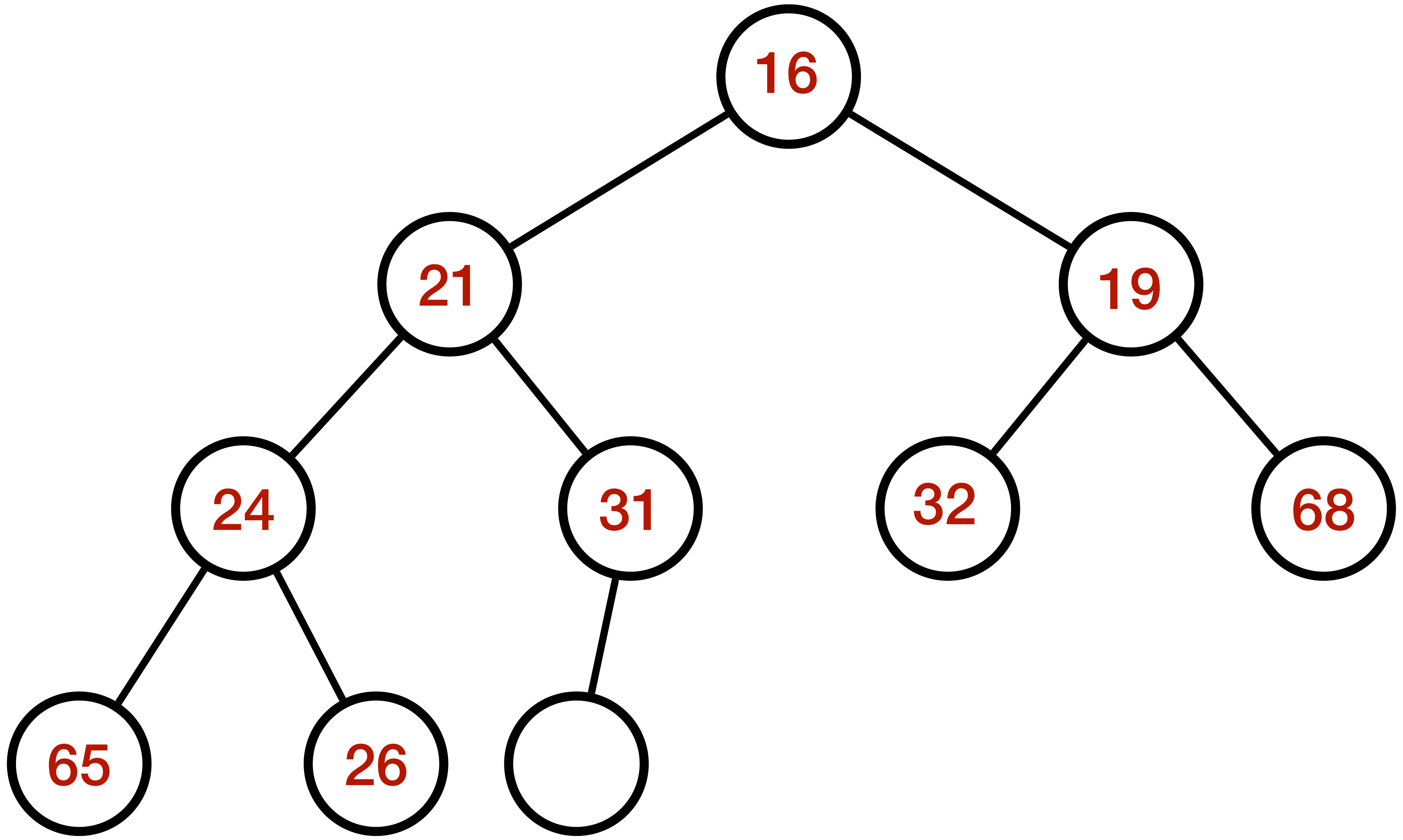

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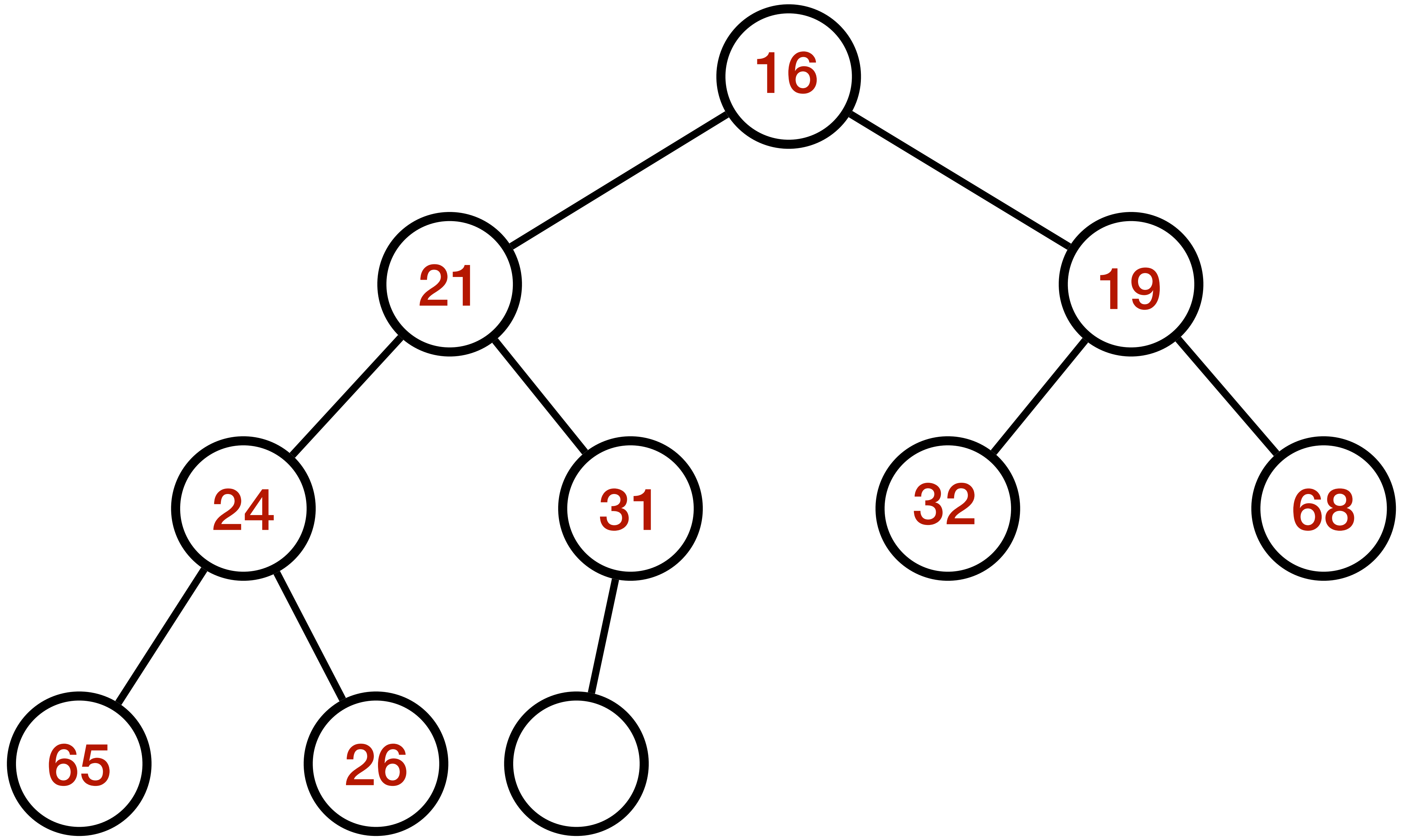
```
void deleteMin() {
  if (heap.empty()) {
    std::cout << "Heap is empty!" << std::endl;
    return;
  }

  heap[0] = heap.back();
  heap.pop_back();

  heapifyDown(0);
}
```



Building Heap

- Simple method — Repeatedly call insert method
  - Time complexity: $\Sigma_{i=1}^{n} log\, i = O(log\, n!) = O(nlog\, n)$
- Better solution: We start from the bottom and move up
- All leaves are heaps (inductive construction)

```
void buildHeap(const std::vector<int> &arr) {
  heap = arr;
  int n = heap.size();

  for (int i = n / 2 - 1; i >= 0; i--) {
    heapifyDown(i);
  }
}
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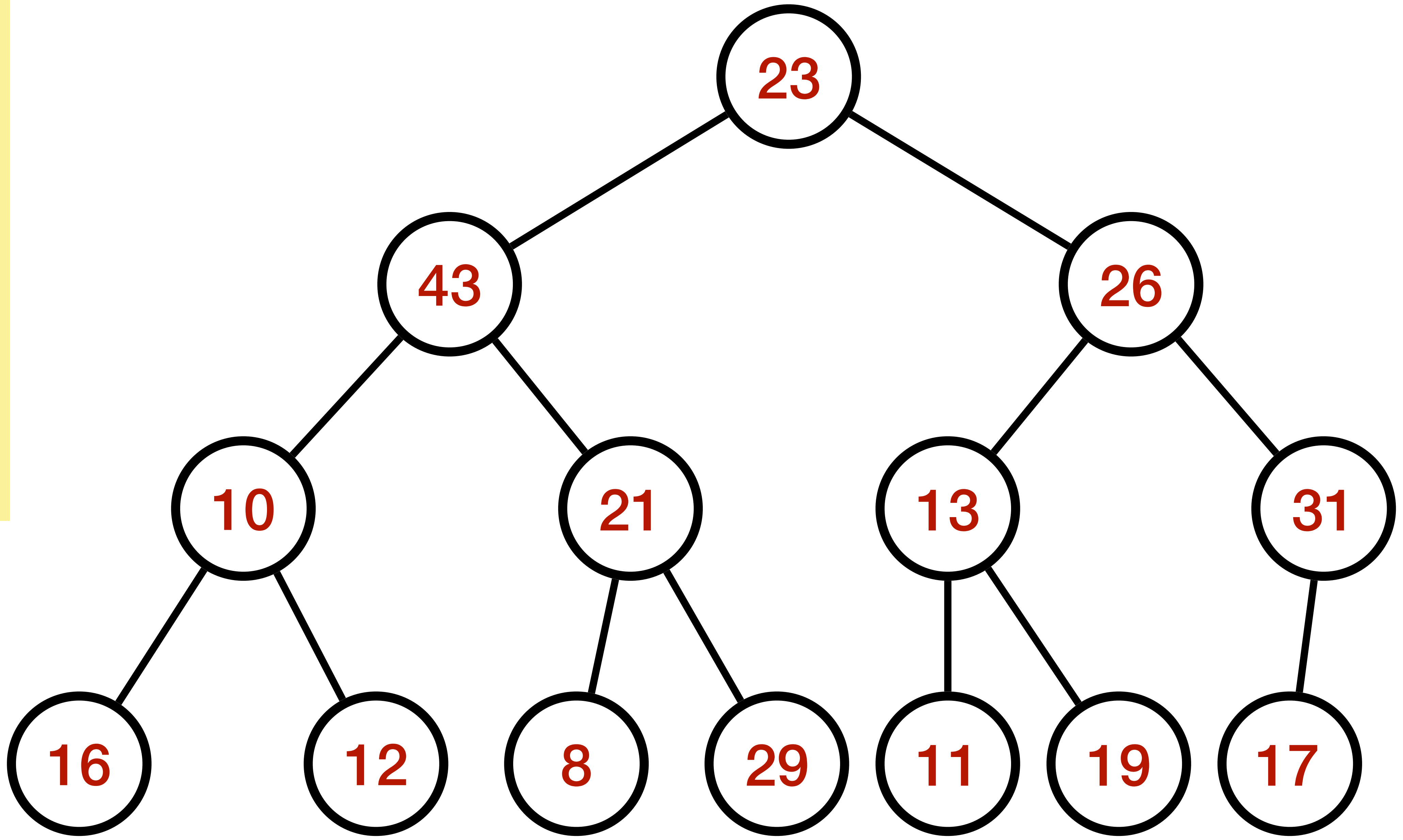

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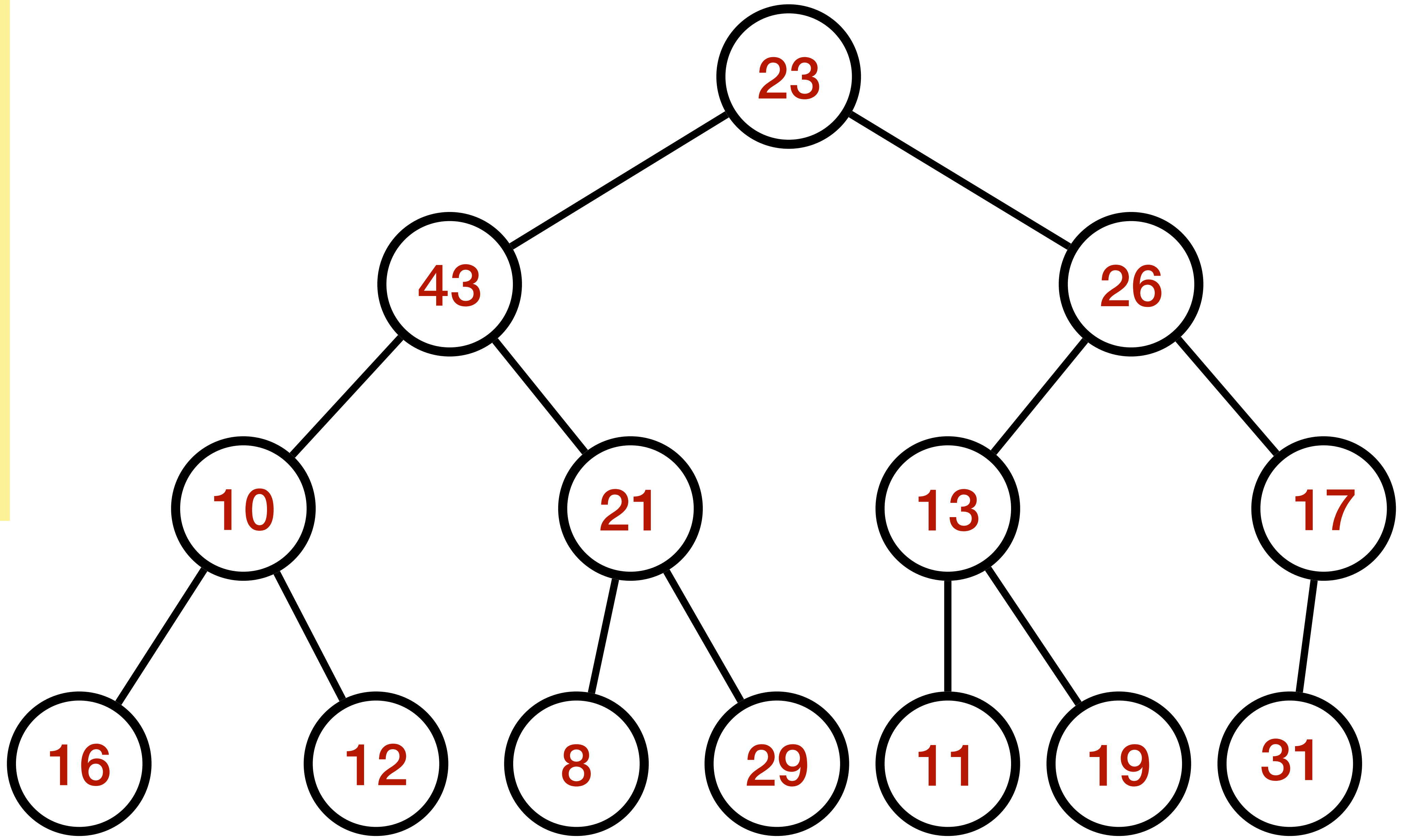

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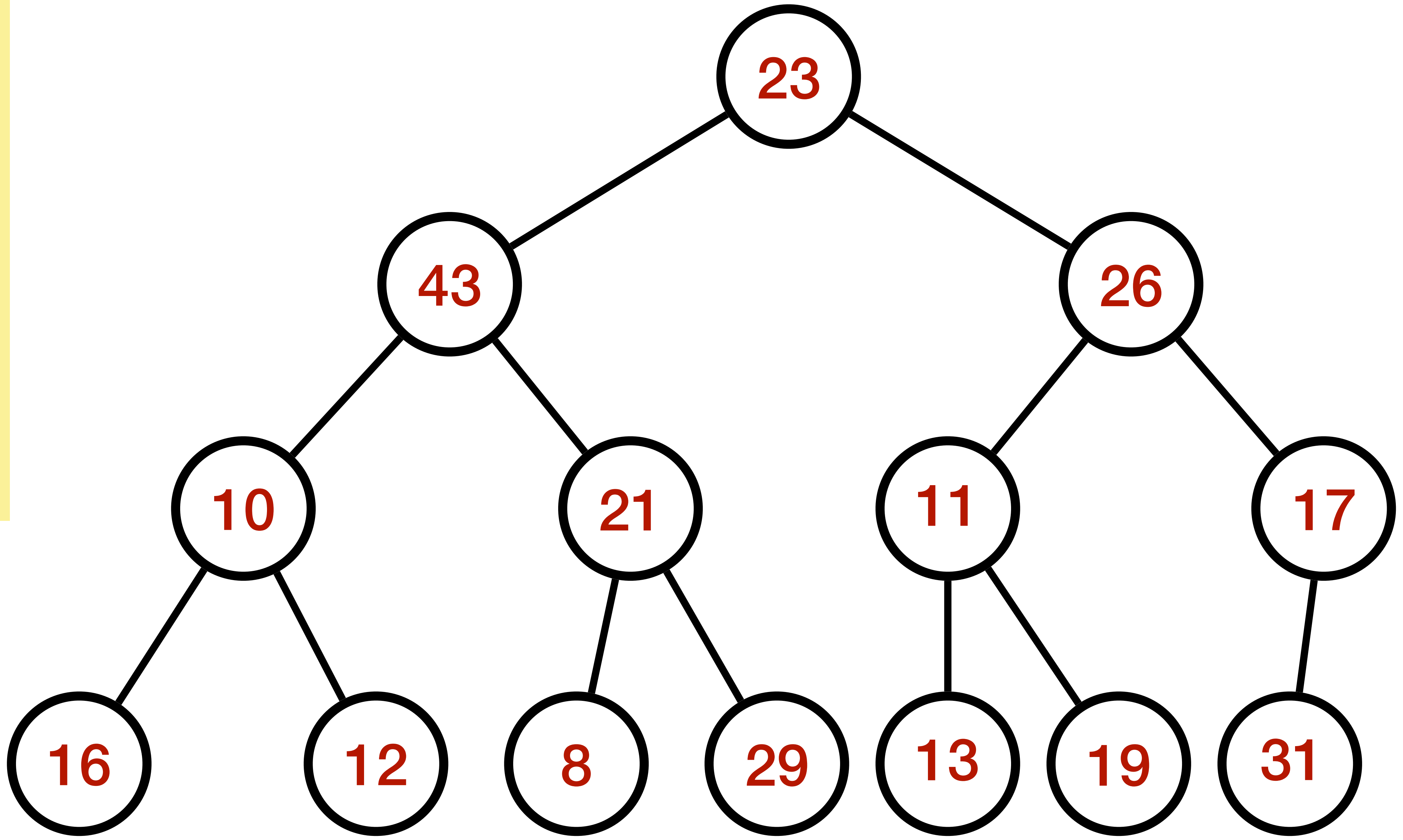

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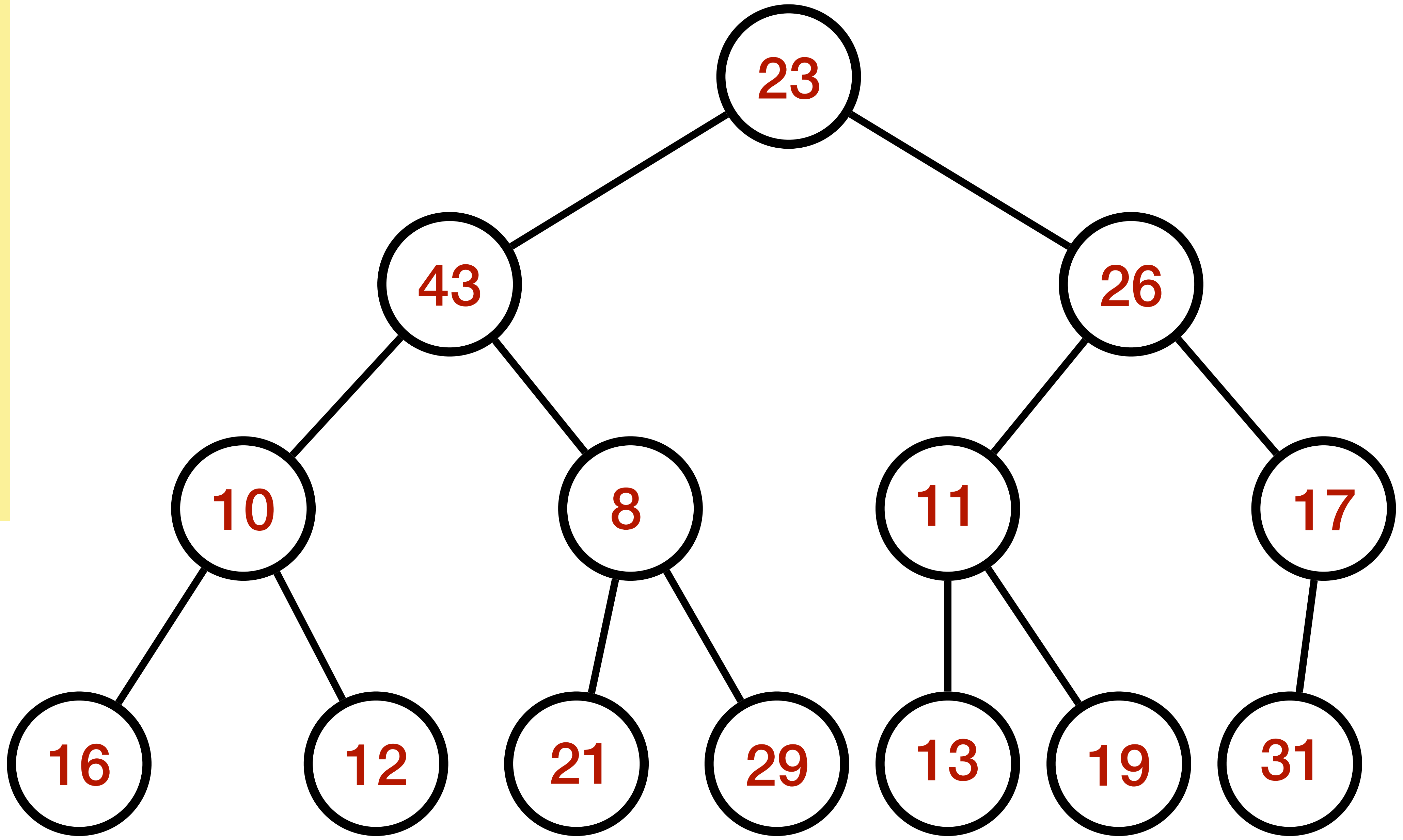

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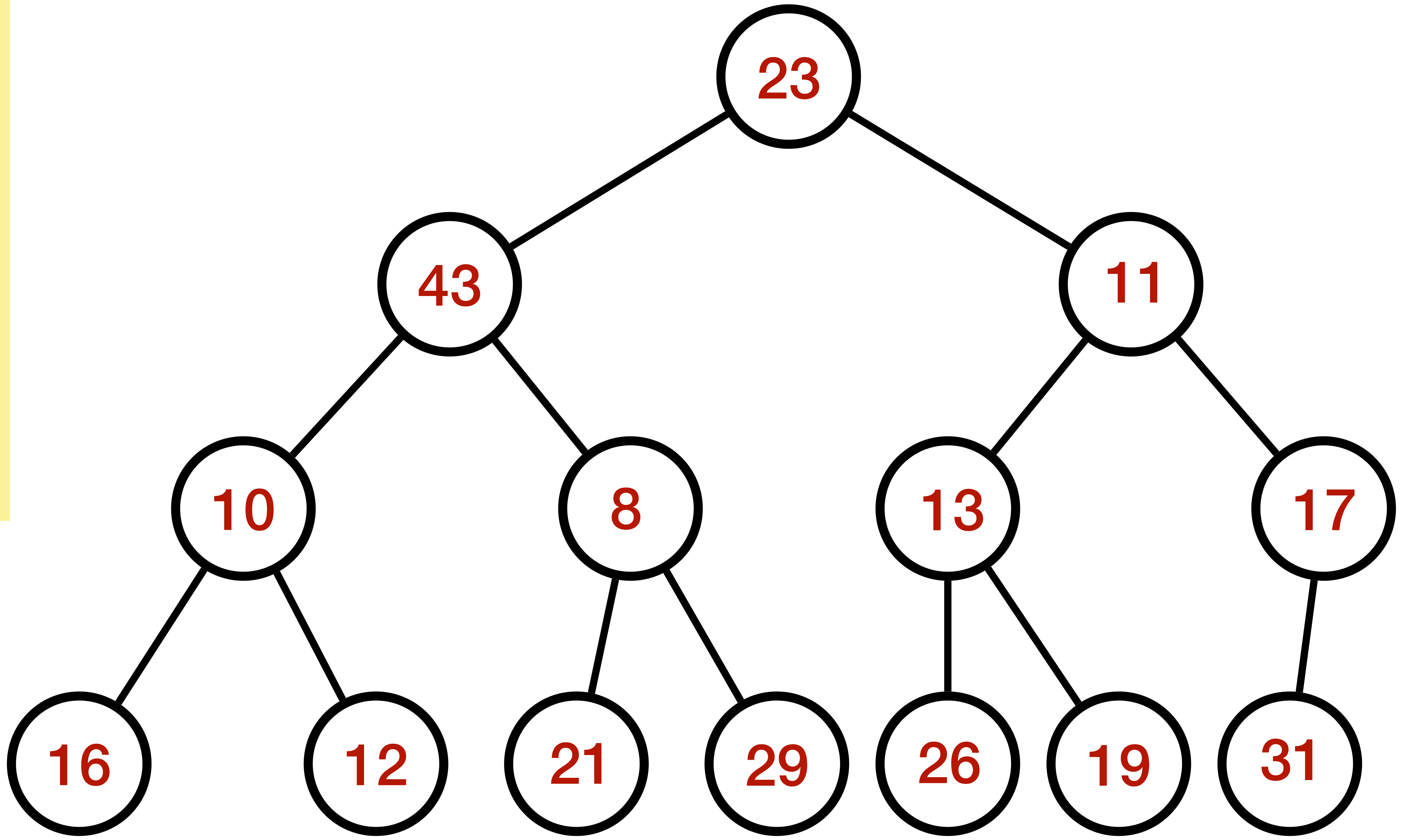

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- Height of node : length of longest path from the node to leaf
- Height of tree: height of root
- Time for HeapifyX(i): O(height of the subtree rooted at i)
- Assume: $n = 2^k - 1$ (a complete binary tree — only help us simplify the analysis)

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- For the n/2 nodes of height 1: Heapify requires at most 1 swap

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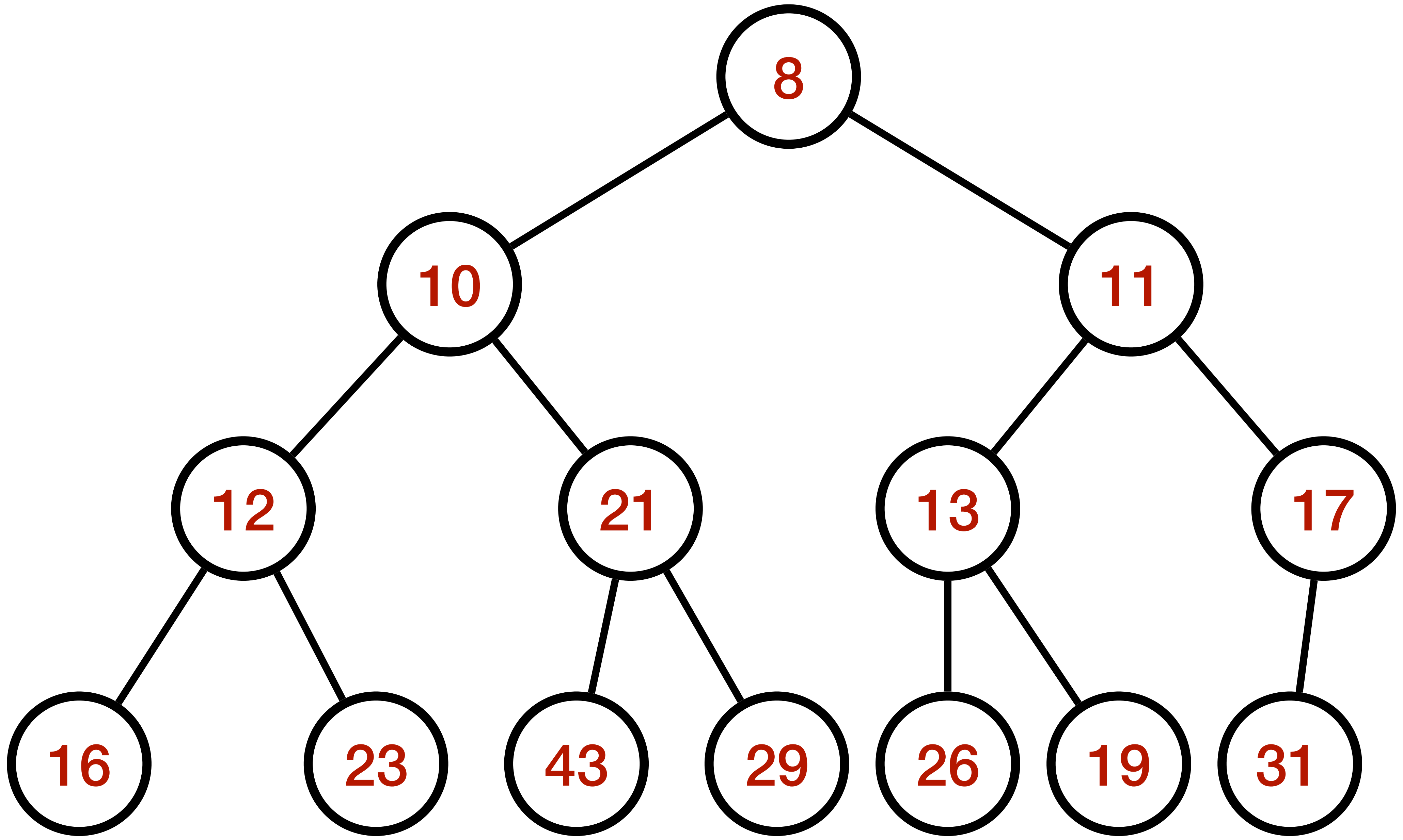
- Total number of swaps: $\Sigma_{i=1}^{log\, n} n\,.\, i/2^i = O(n)$

Heap Sort

- Create a heap: $T(n) = O(n)$
- Do **DeleteMin** repeatedly till the heap becomes empty: $T(n) = O(n \log n)$
- Alternative strategy: No other space constraint, i.e., **in-place sort**
    - Do **DeleteMin** and move the deleted element to the end of the heap
    - Heapify the rest

In-Place Heap Sort

- NOTE: **Heap size is reduced by 1** after each such operation



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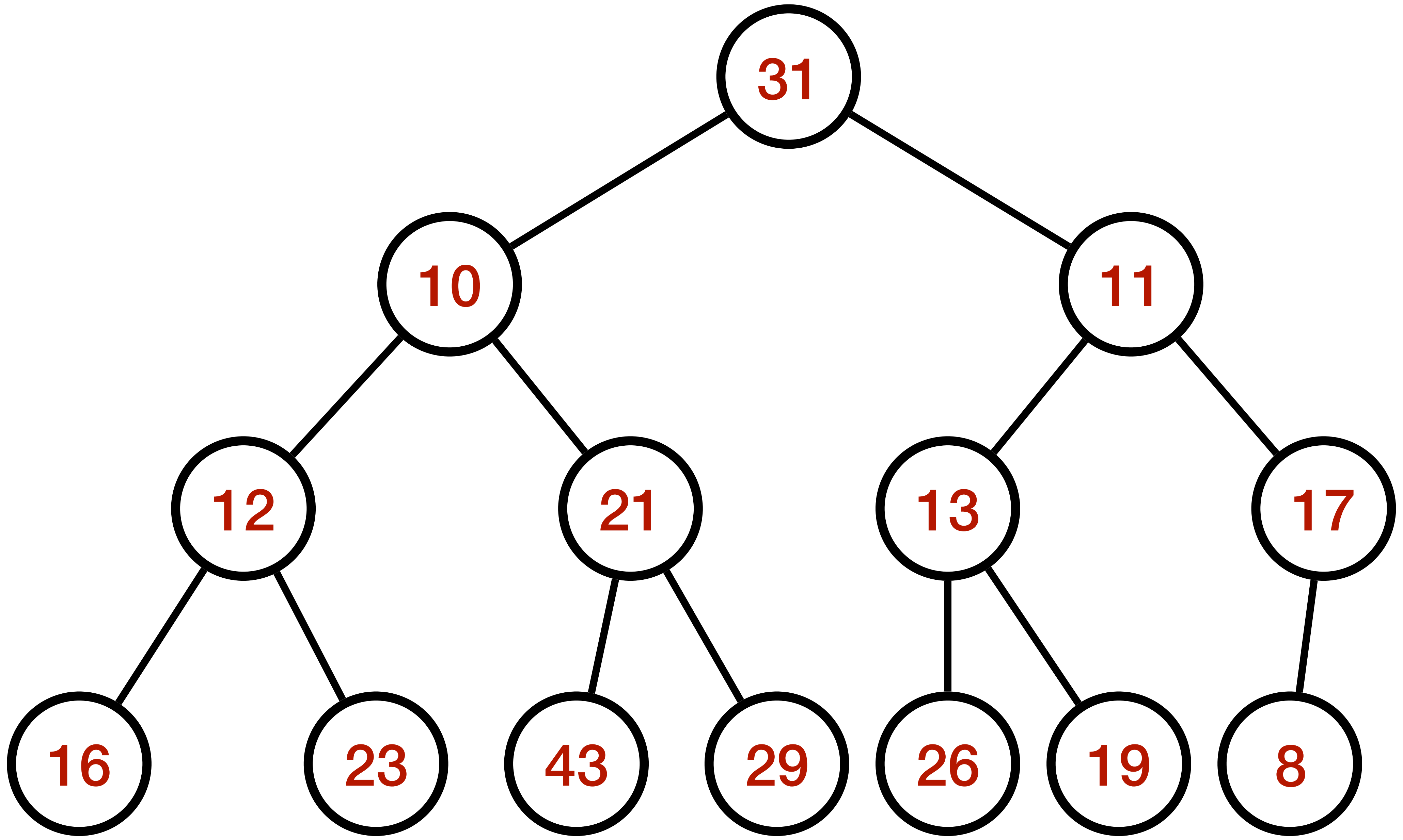

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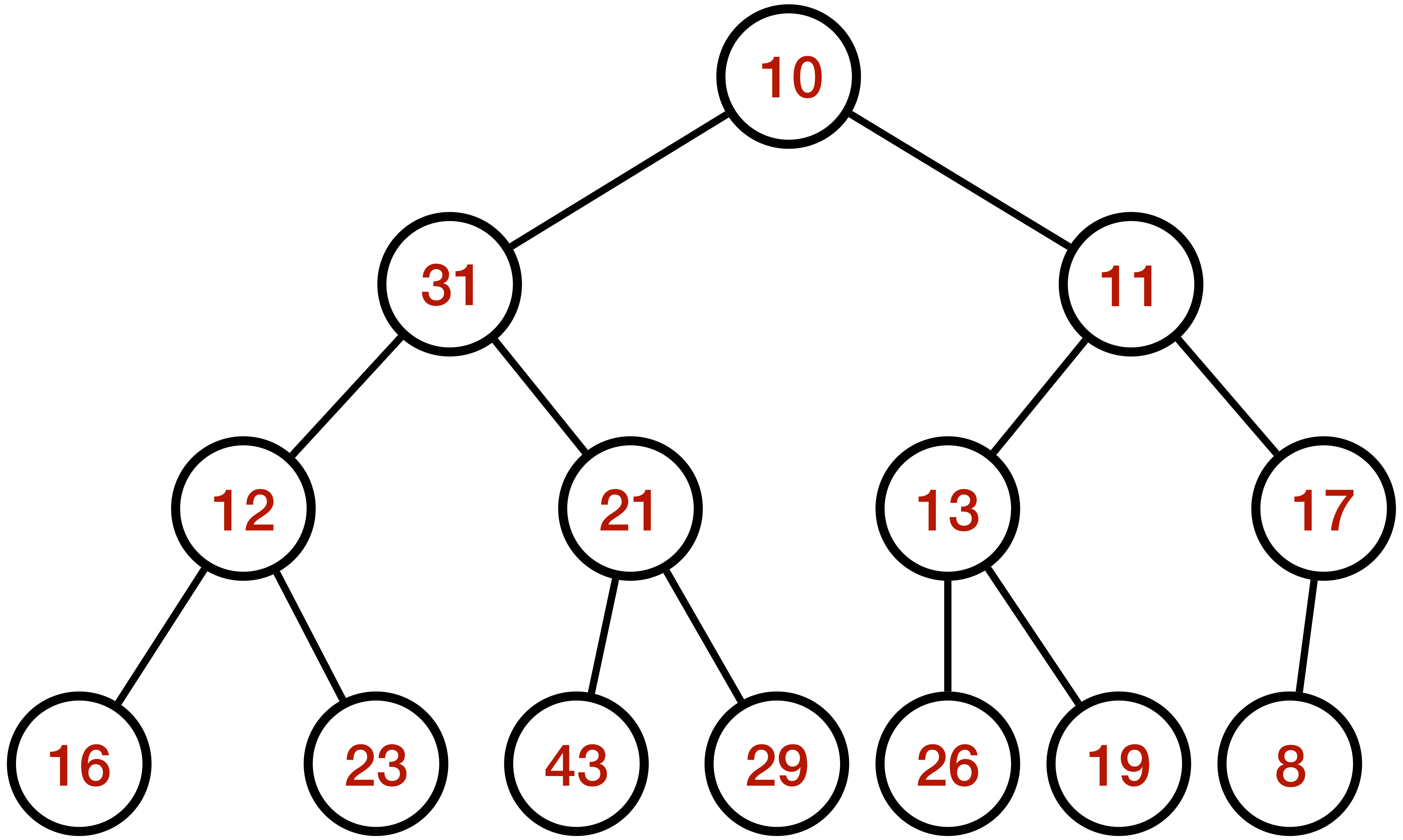

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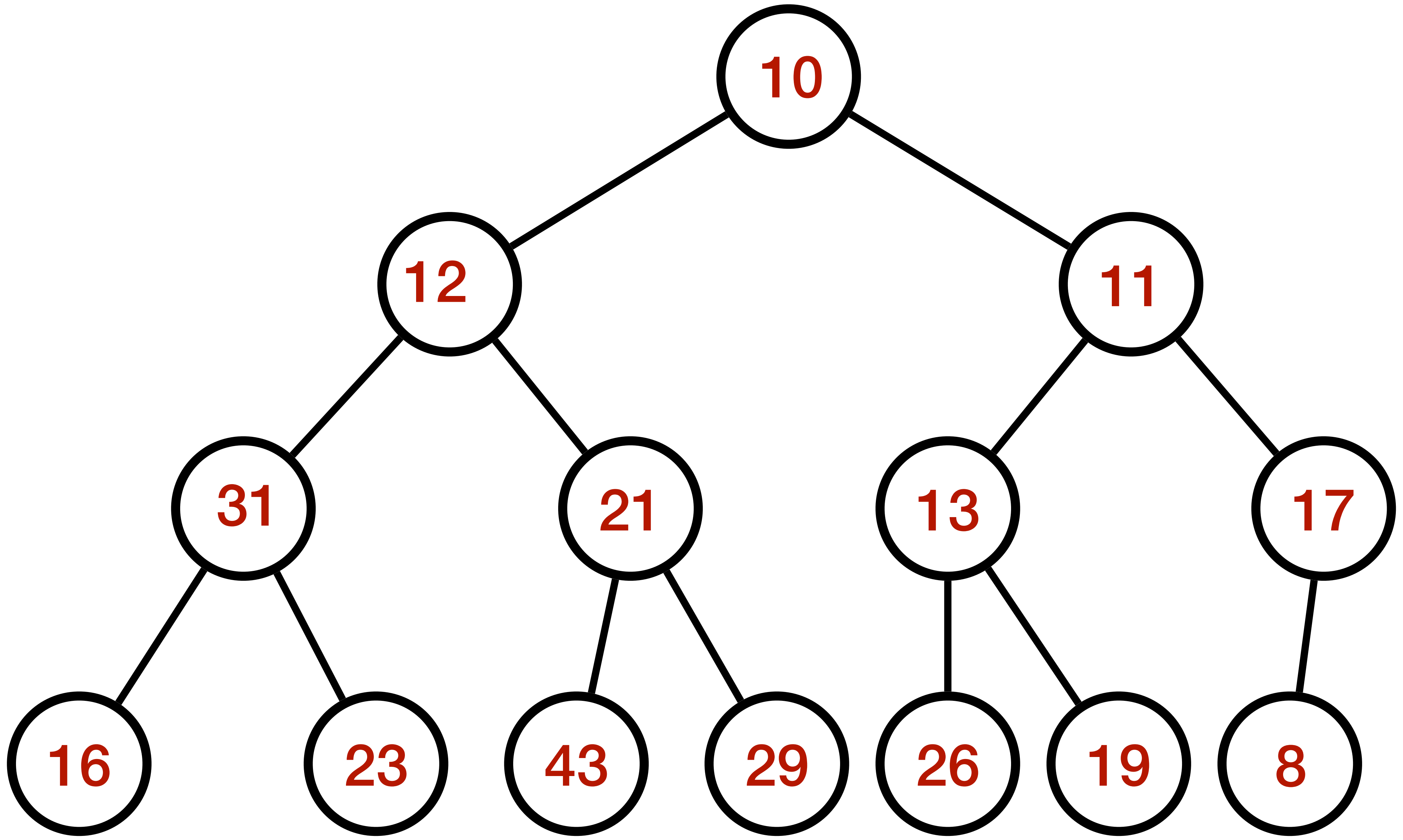

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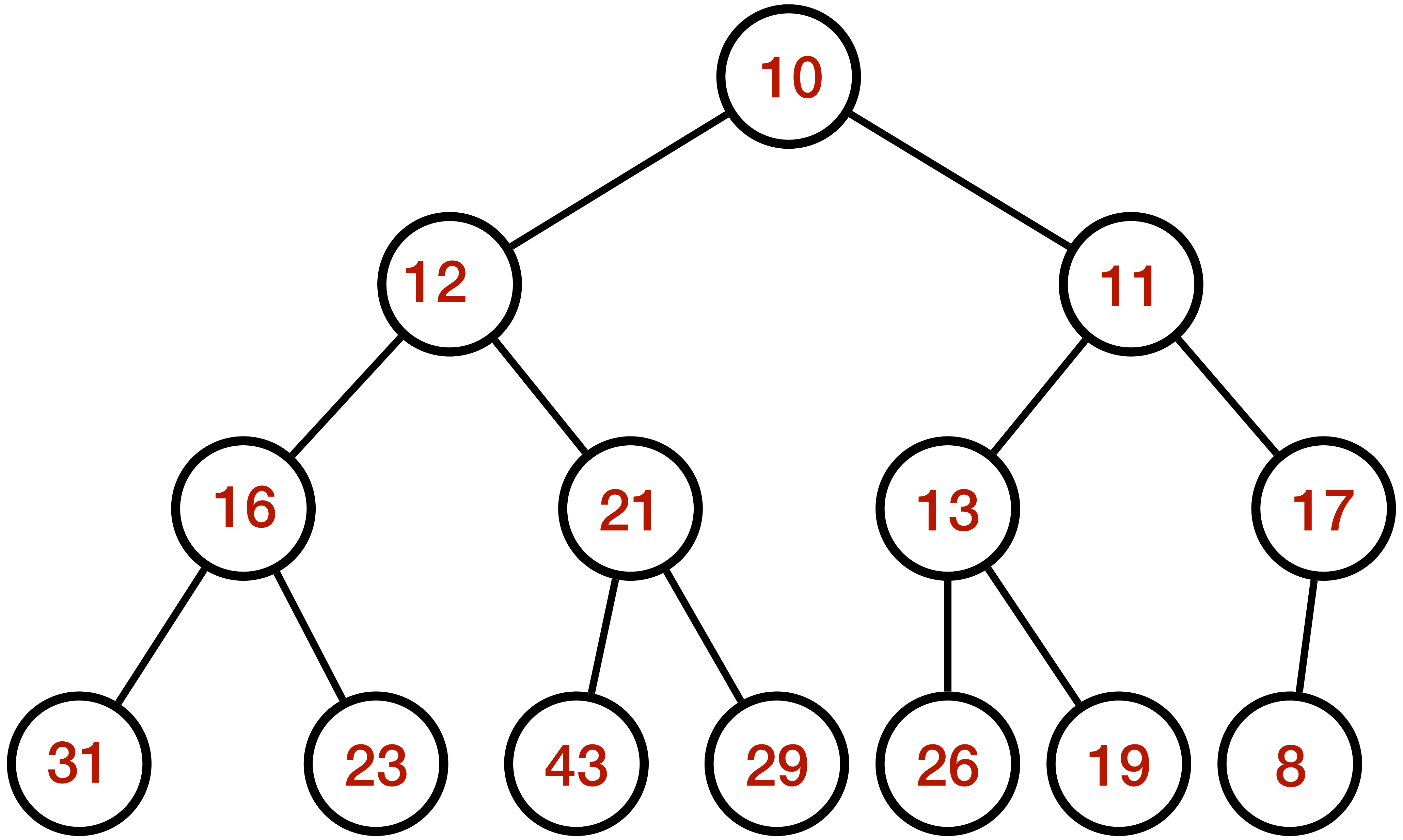

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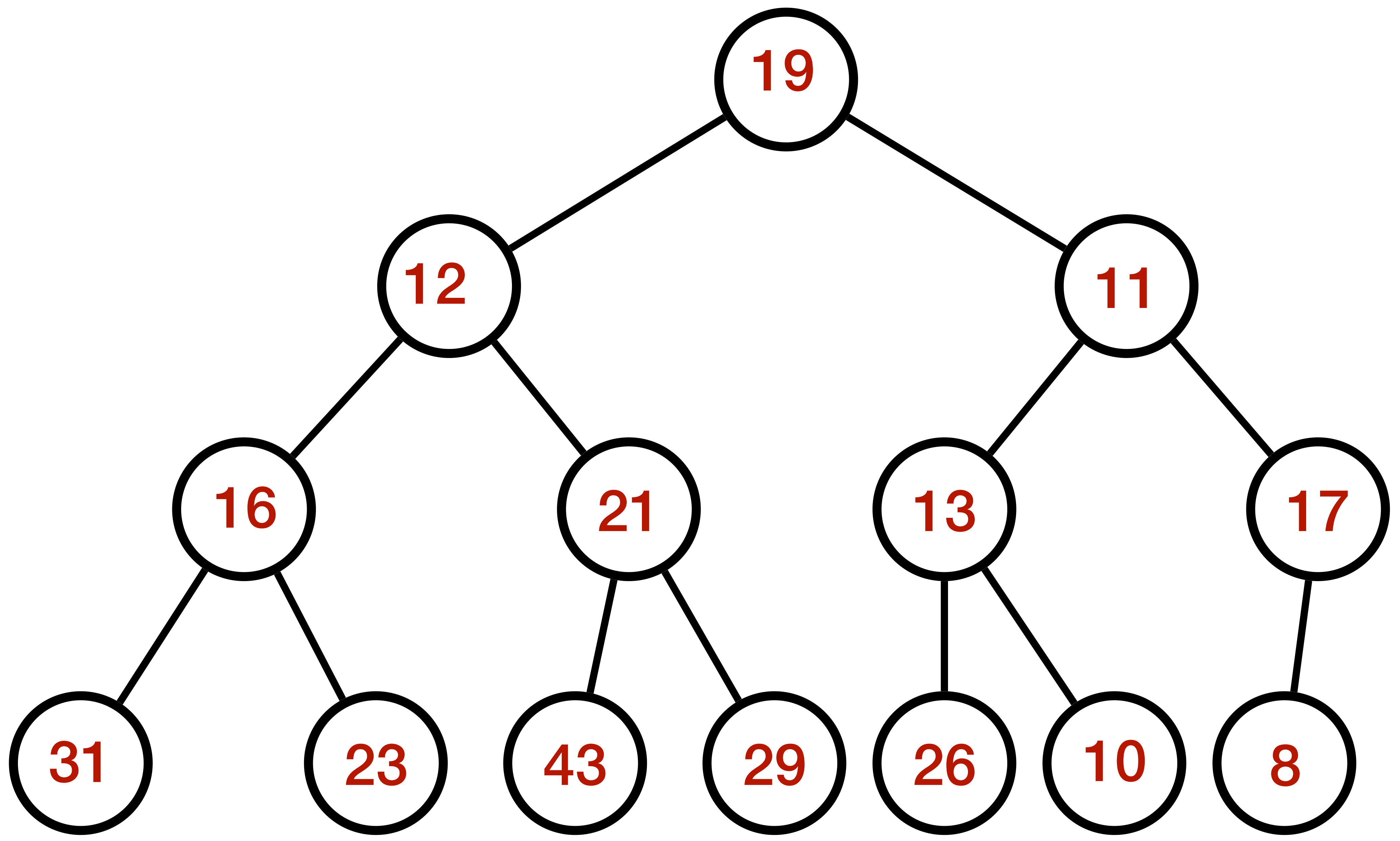

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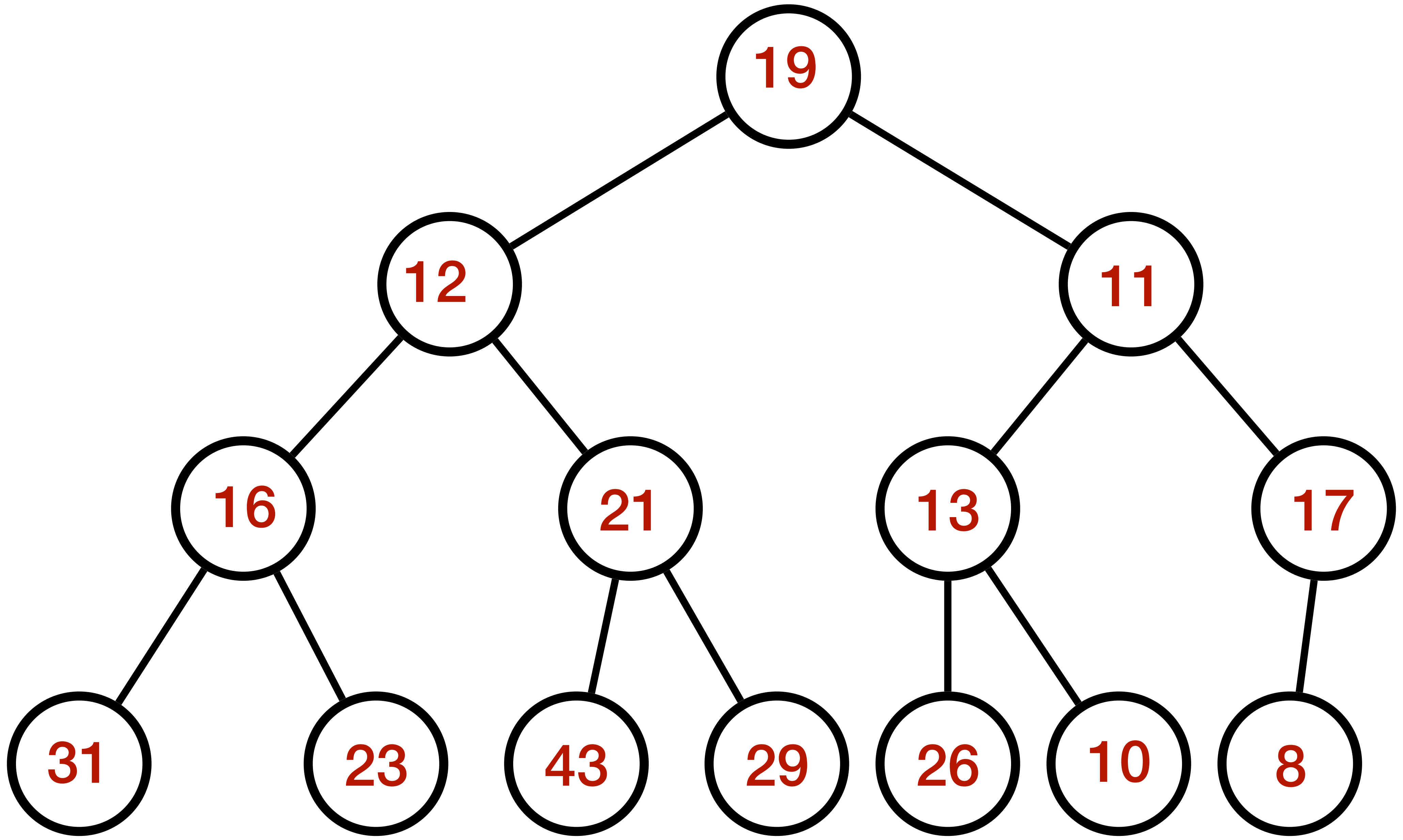
In-Place Heap Sort

- NOTE: **Heap size is reduced by 1** after each such operation

```
void heapSort() {
  int n = heap.size();
  // Extract elements from heap one by one
  for (int i = n - 1; i > 0; i--) {
    // Move the root to the end
    std::swap(heap[0], heap[i]);

    // Heapify the reduced heap
    heapifyDown(i,0);
  }
}
```



- Time complexity: O(n log n)

Runtime Analysis

- A heap of n nodes has height O(log n)
- Insertion (heapifyUp along a path) — at most O(log n) steps
- HeapifyDown — O(log n)
  - An element may be moved all the way to the last level
- DeleteMin — O(log n)
- BuildHeap — O(n)
- HeapSort — O(n log n)

Applications: The Selection Problem

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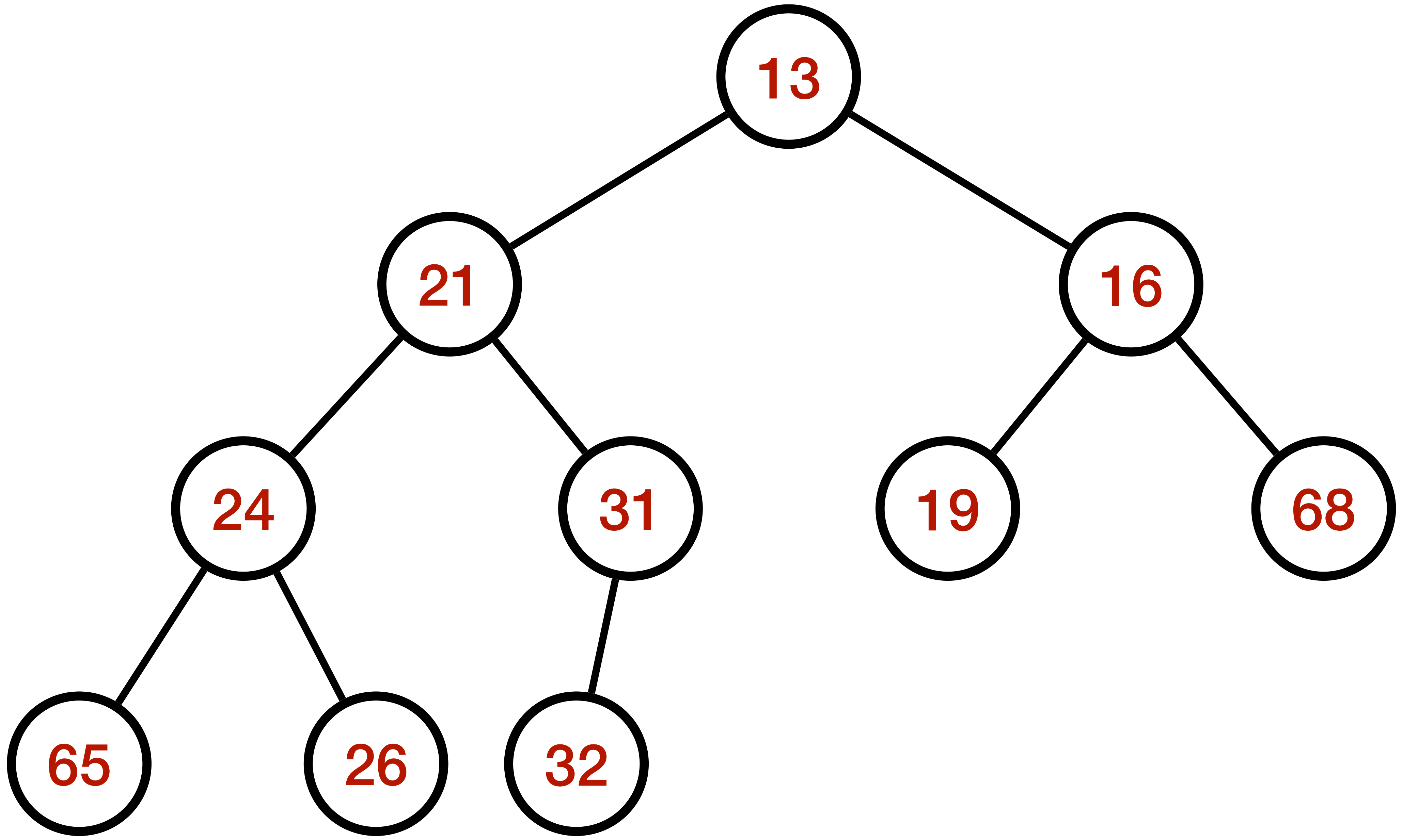
- Problem: Find the k^{th} largest element in a list of n elements
- Algorithm1A:
  - Read the elements in an array
  - Sort the array
  - Return the k^{th} indexed element from the sorted array
  - Time complexity: $O(n^2)$ with simple sorting; $O(n\,log\,n)$ otherwise.

Applications: The Selection Problem



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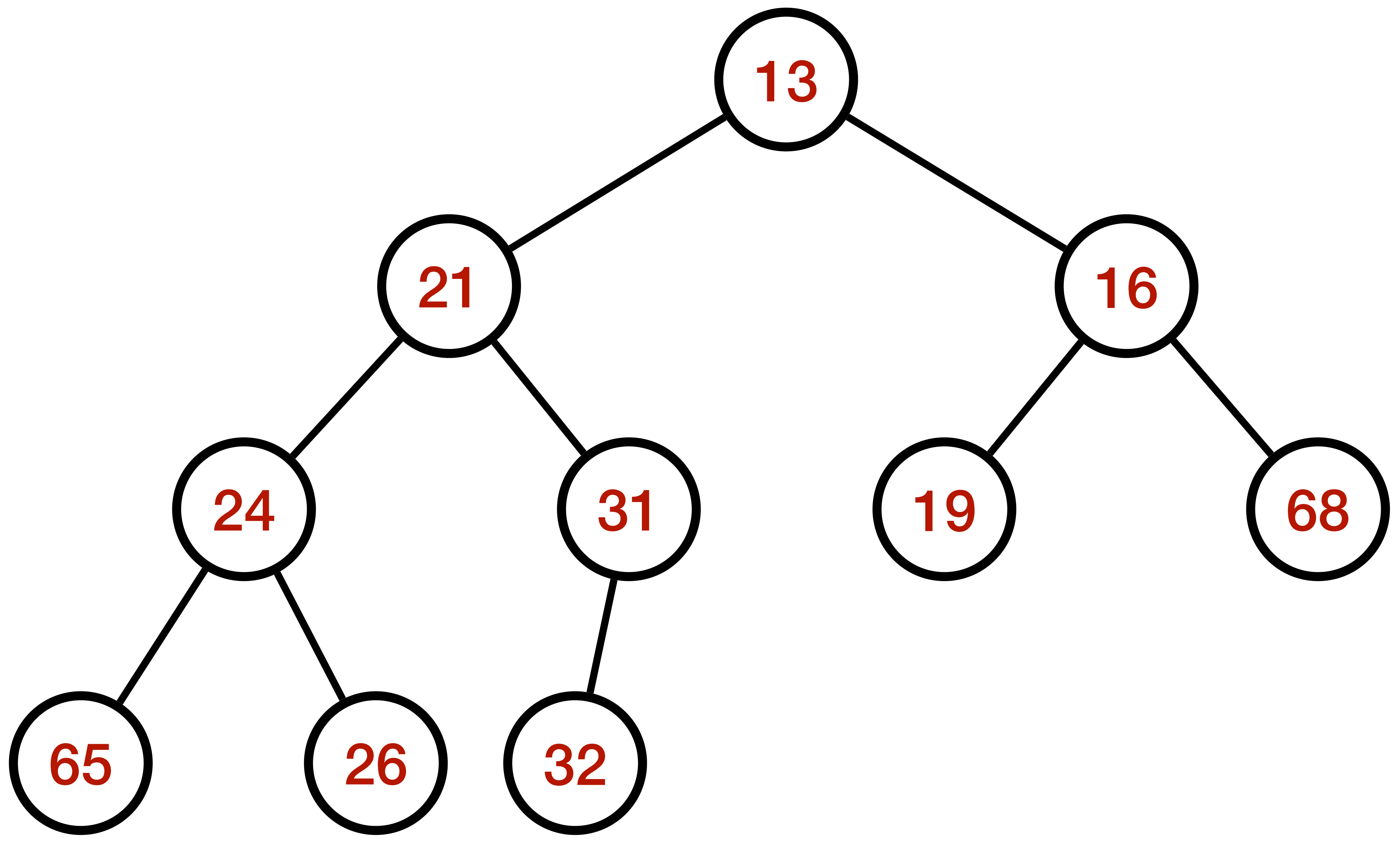

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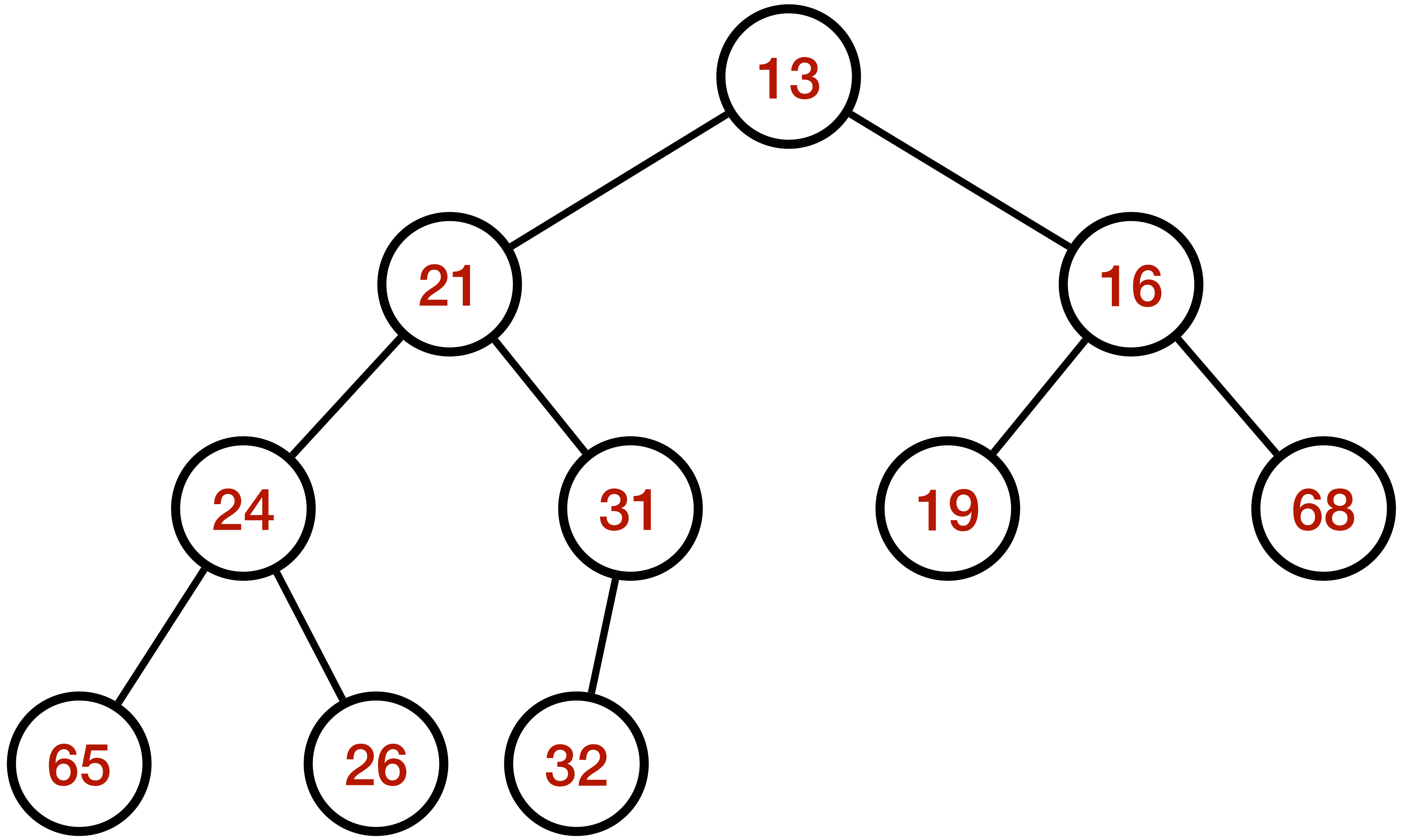

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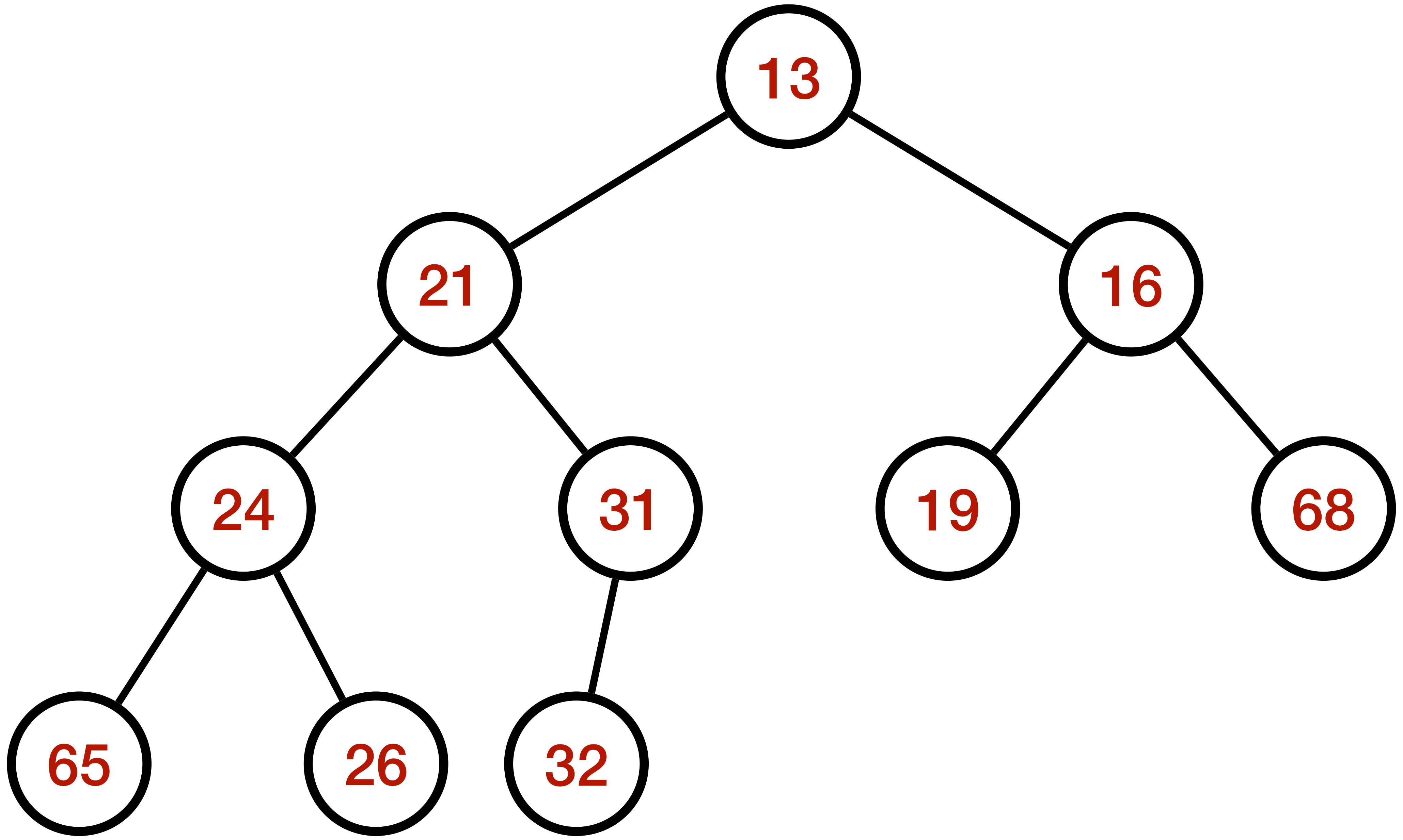

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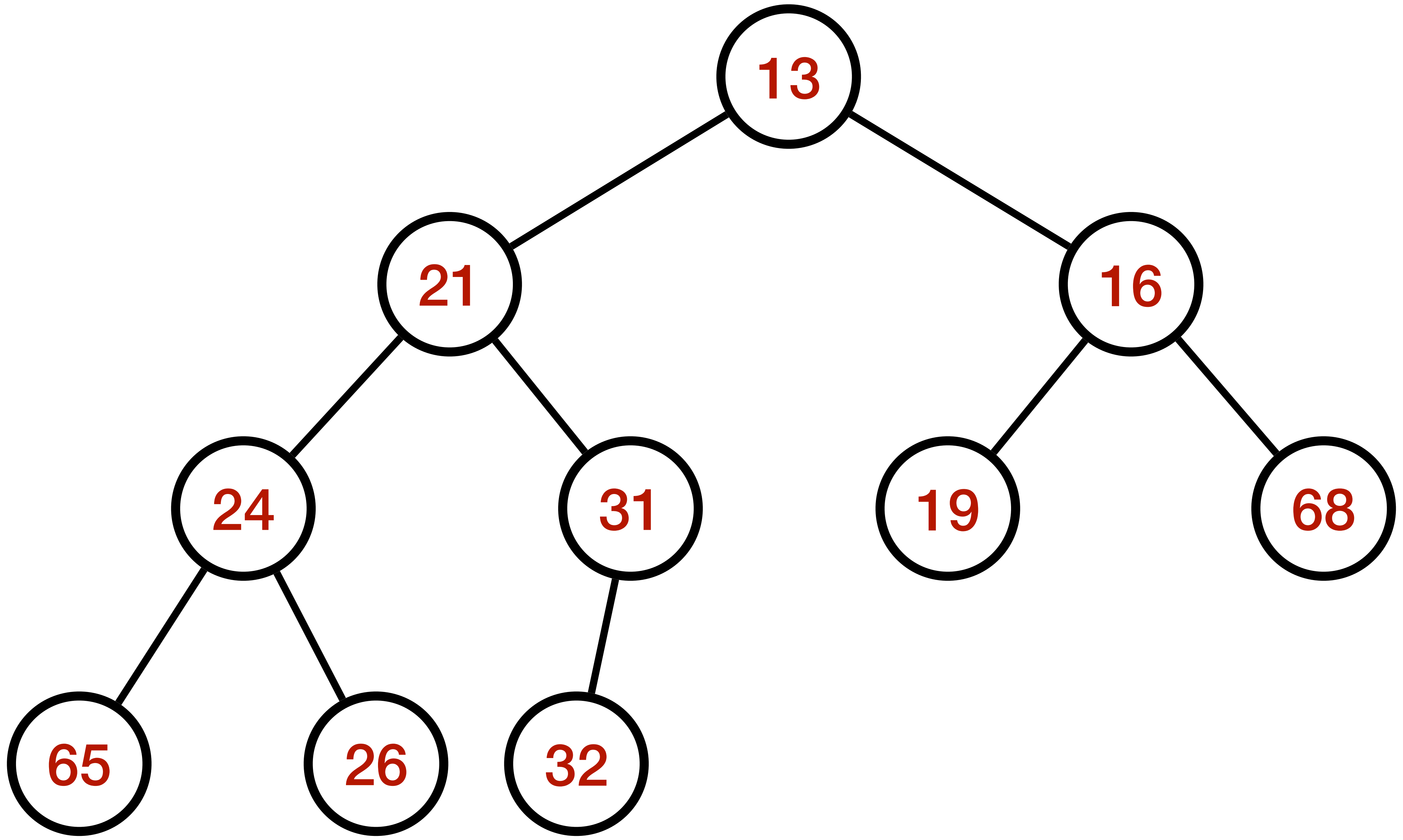

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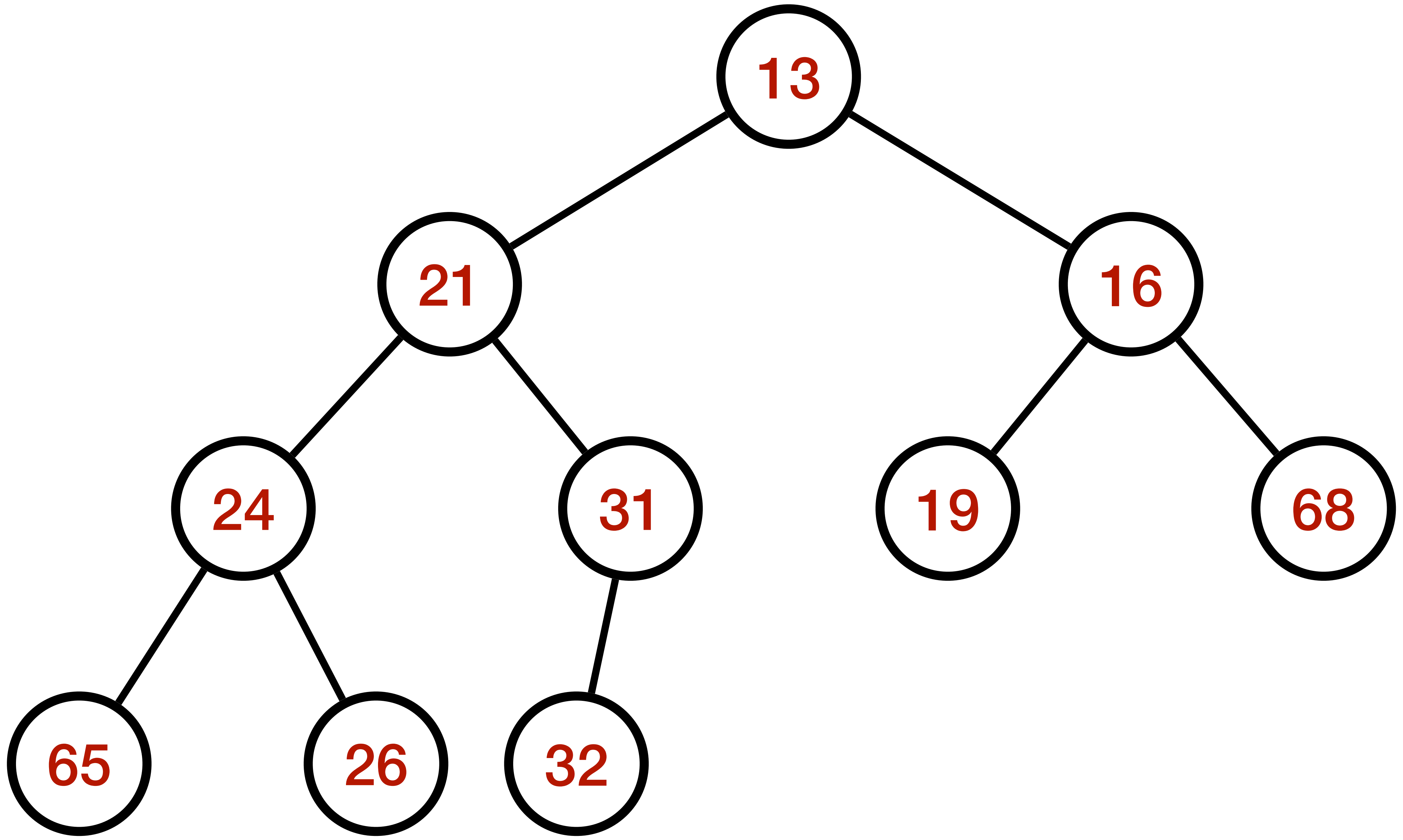
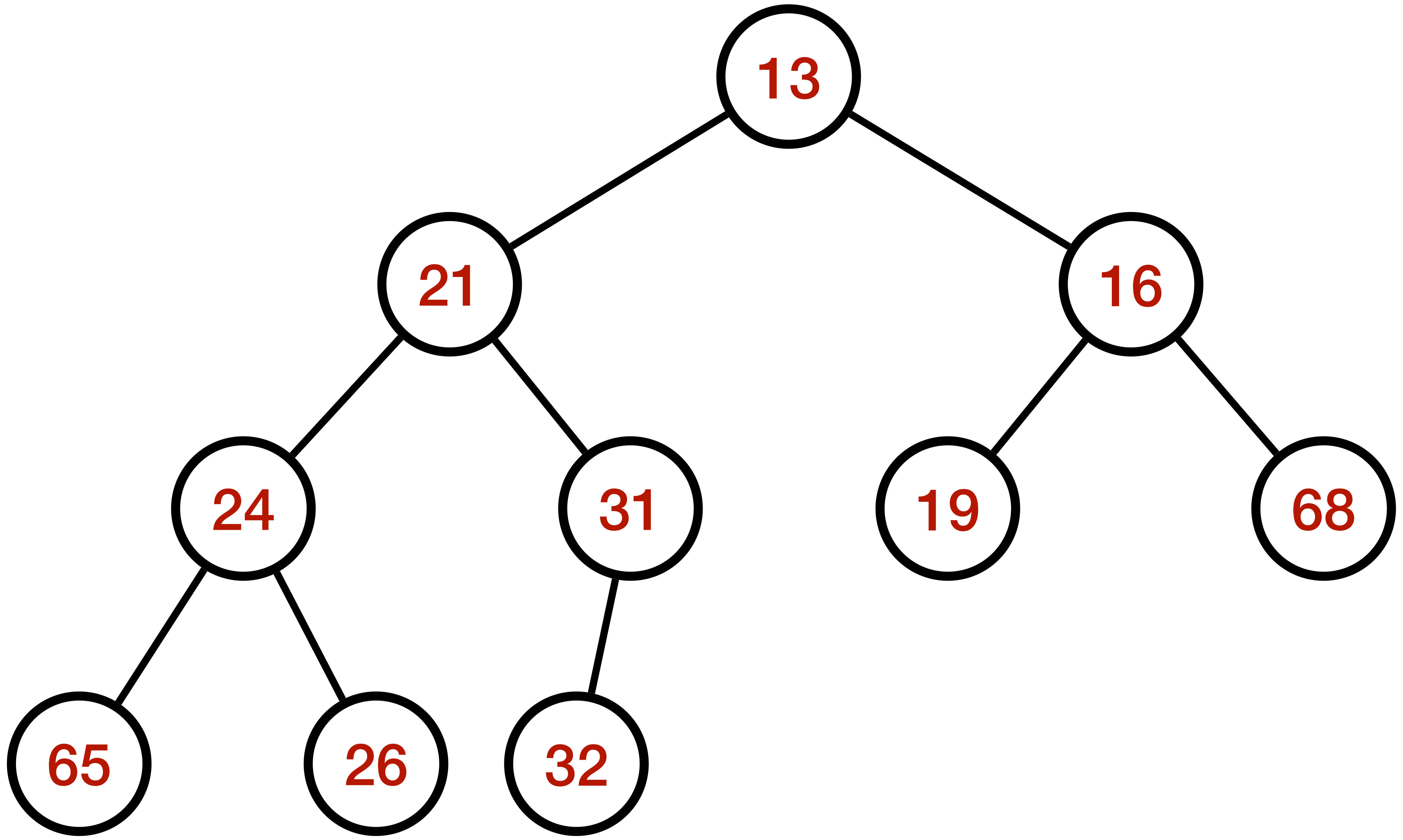
- Algorithm1B:
  - Read **only** k elements in an array
  - Sort the array
  - The smallest is at k^{th} position. For the remaining elements:



Applications: The Selection Problem

- Problem: Find the k^{th} largest element in a list of n elements
- Algorithm1B:
  - Read **only** k elements in an array
  - Sort the array
  - The smallest is at k^{th} position. For the remaining elements:
    - Compare with the k^{th} element —> if the incoming element is larger then replace it with the k^{th} element



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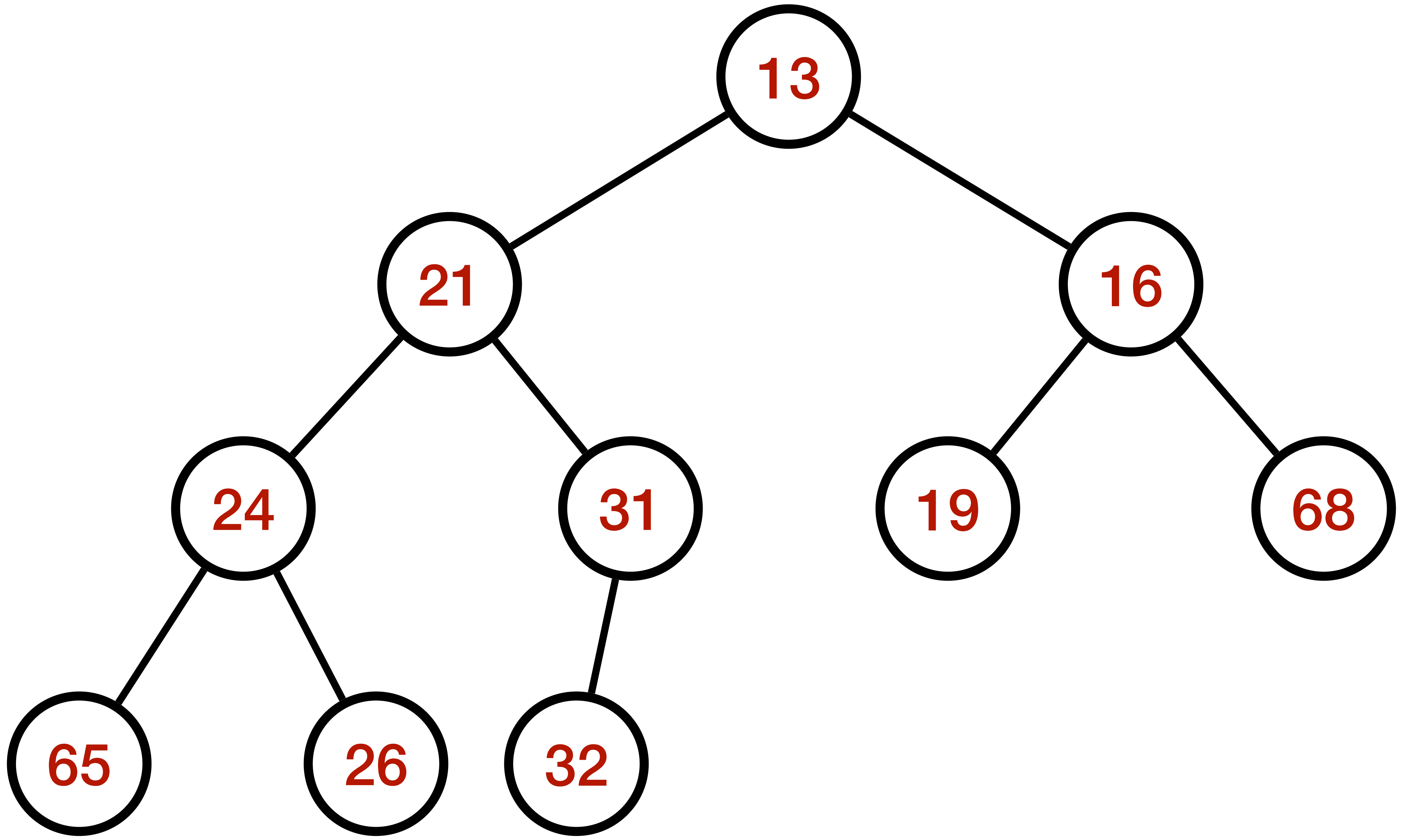
  - Time complexity: ?



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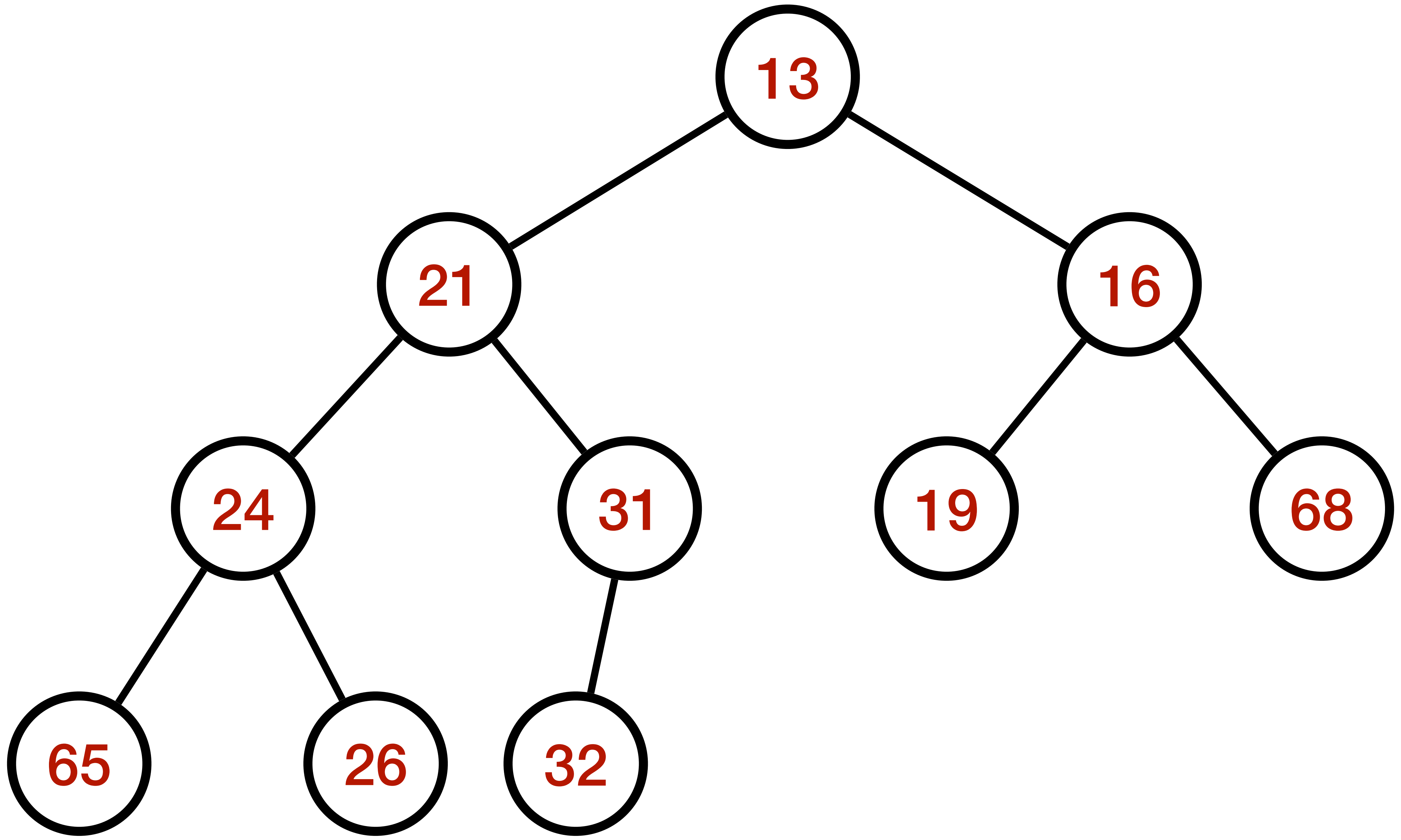
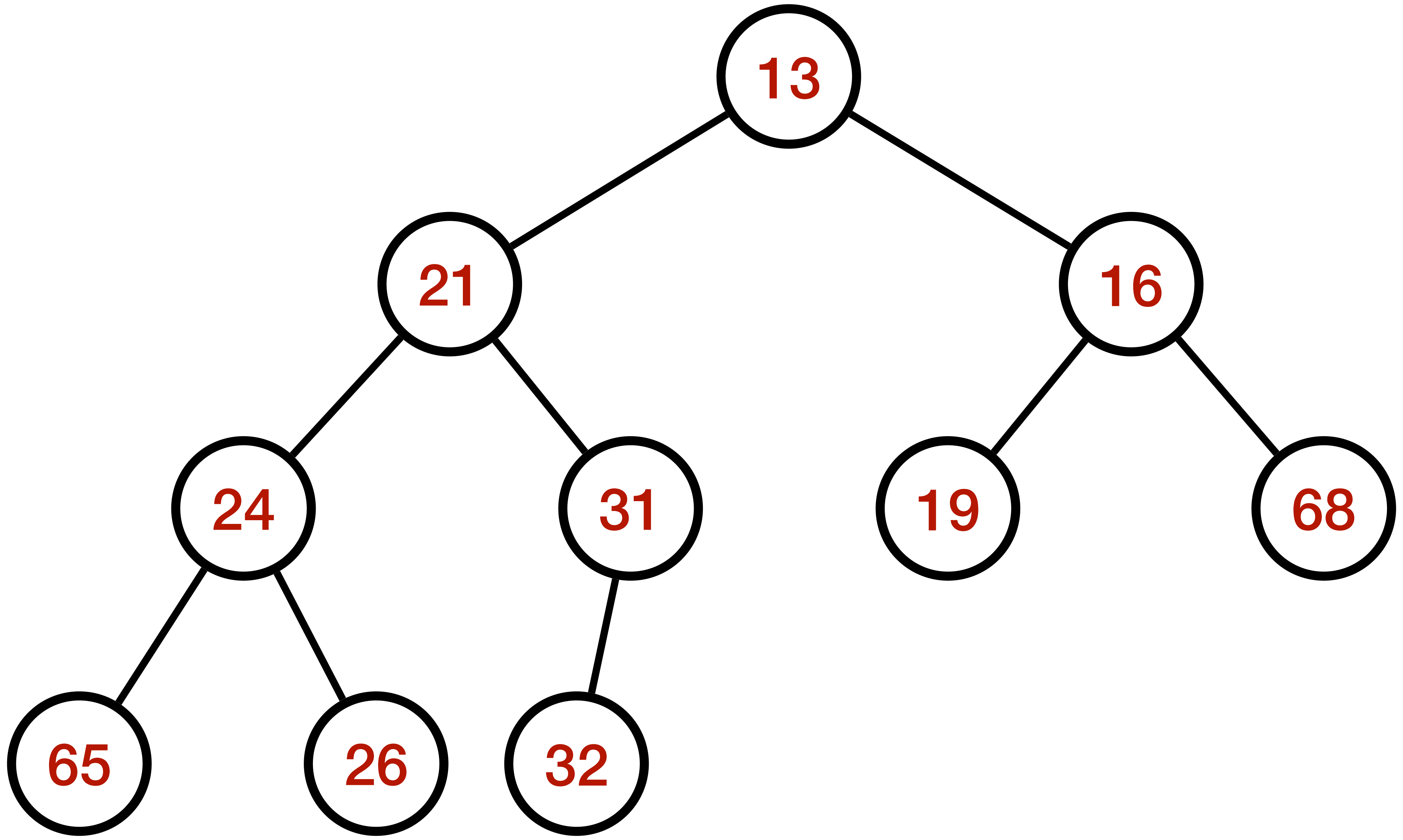


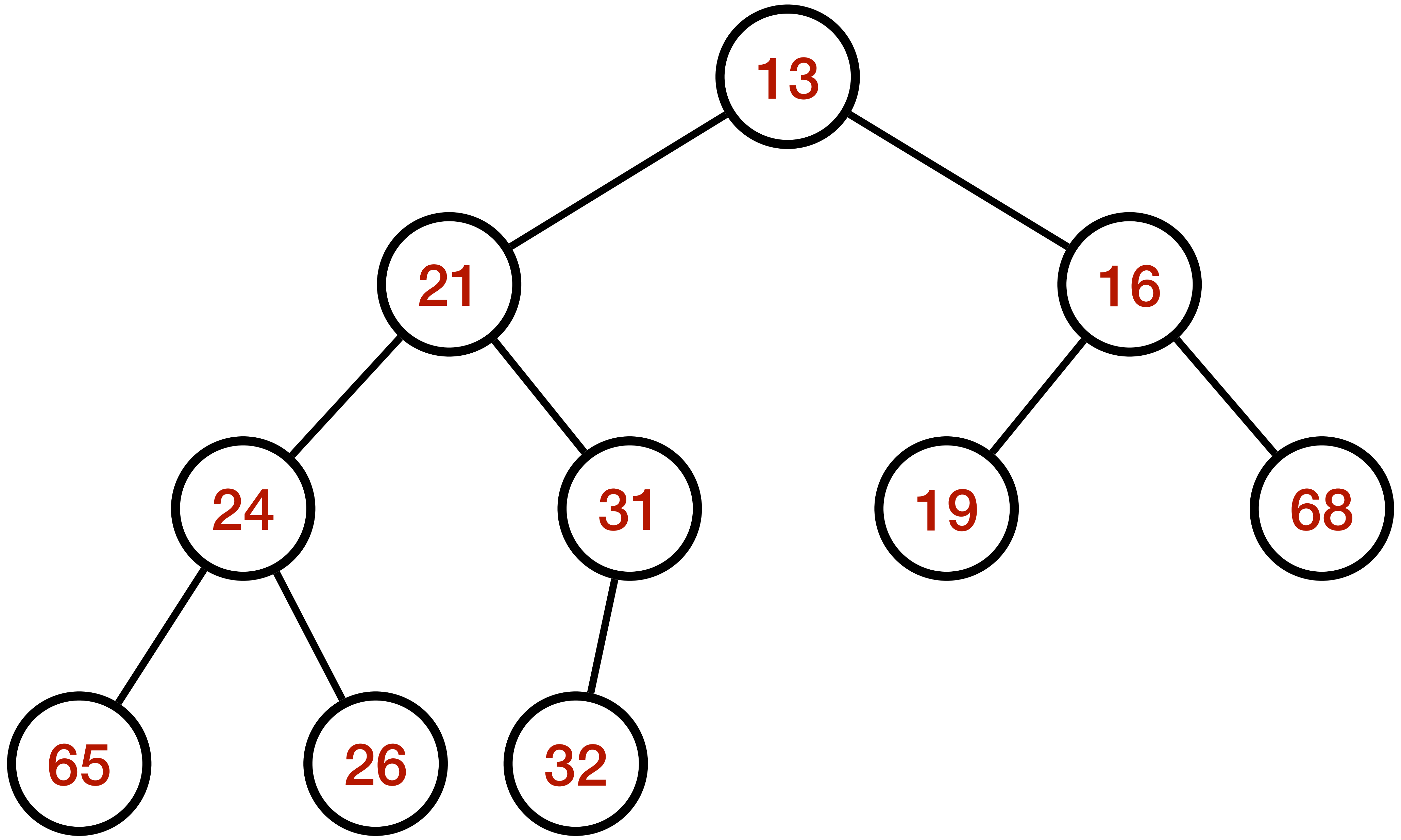
Applications: The Selection Problem

- Changed Problem: Find the k^{th} **smallest** element in a list of n elements



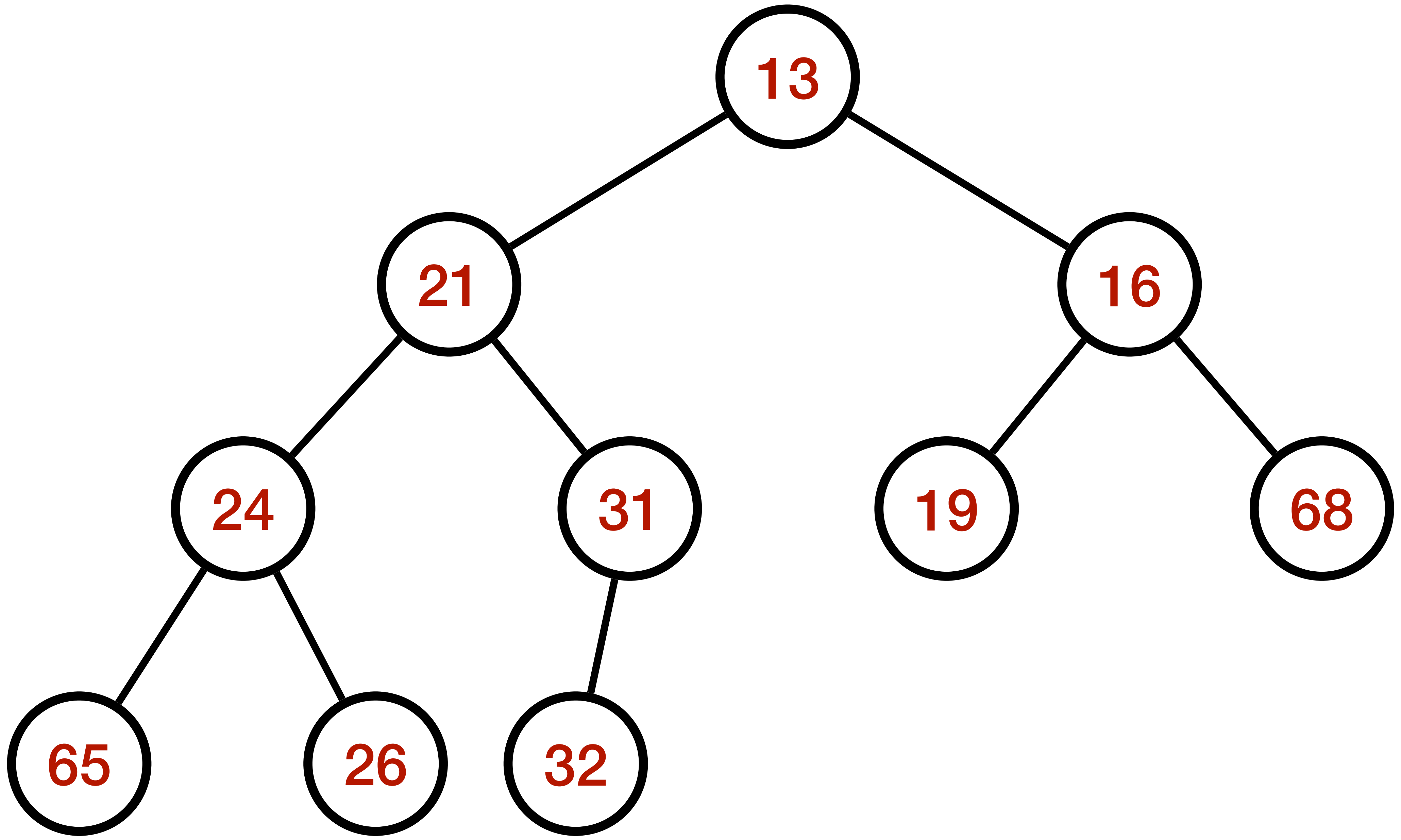
Applications: The Selection Problem

- Changed Problem: Find the k^{th} **smallest** element in a list of n elements
- Algorithm2A:



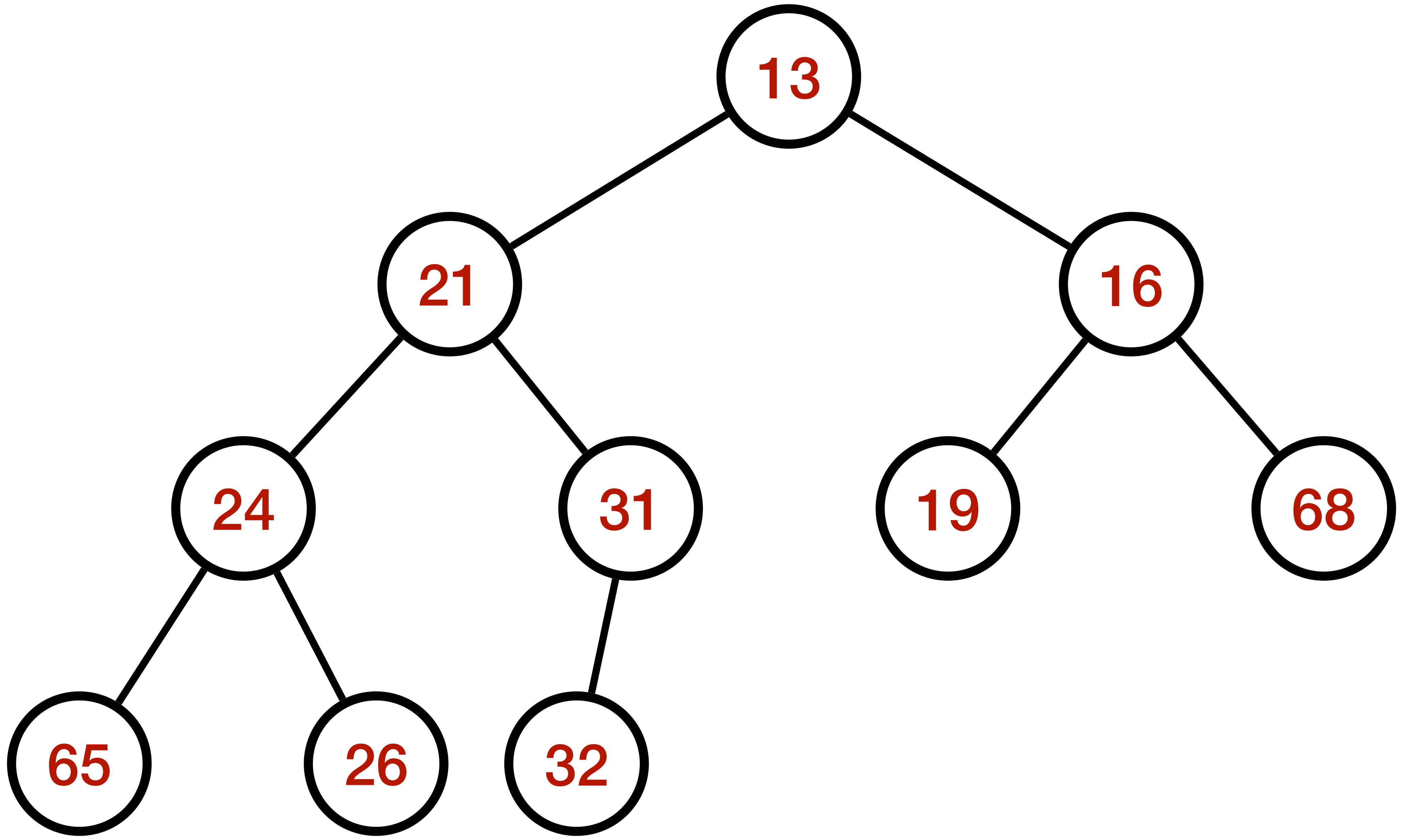
Applications: The Selection Problem

- Changed Problem: Find the k^{th} **smallest** element in a list of n elements
- Algorithm2A:
  - Read elements in an array



Applications: The Selection Problem

- Changed Problem: Find the k^{th} **smallest** element in a list of n elements
- Algorithm2A:
  - Read elements in an array
  - Apply BuildHeap



Applications: The Selection Problem

- Changed Problem: Find the k^{th} **smallest** element in a list of n elements
- Algorithm2A:
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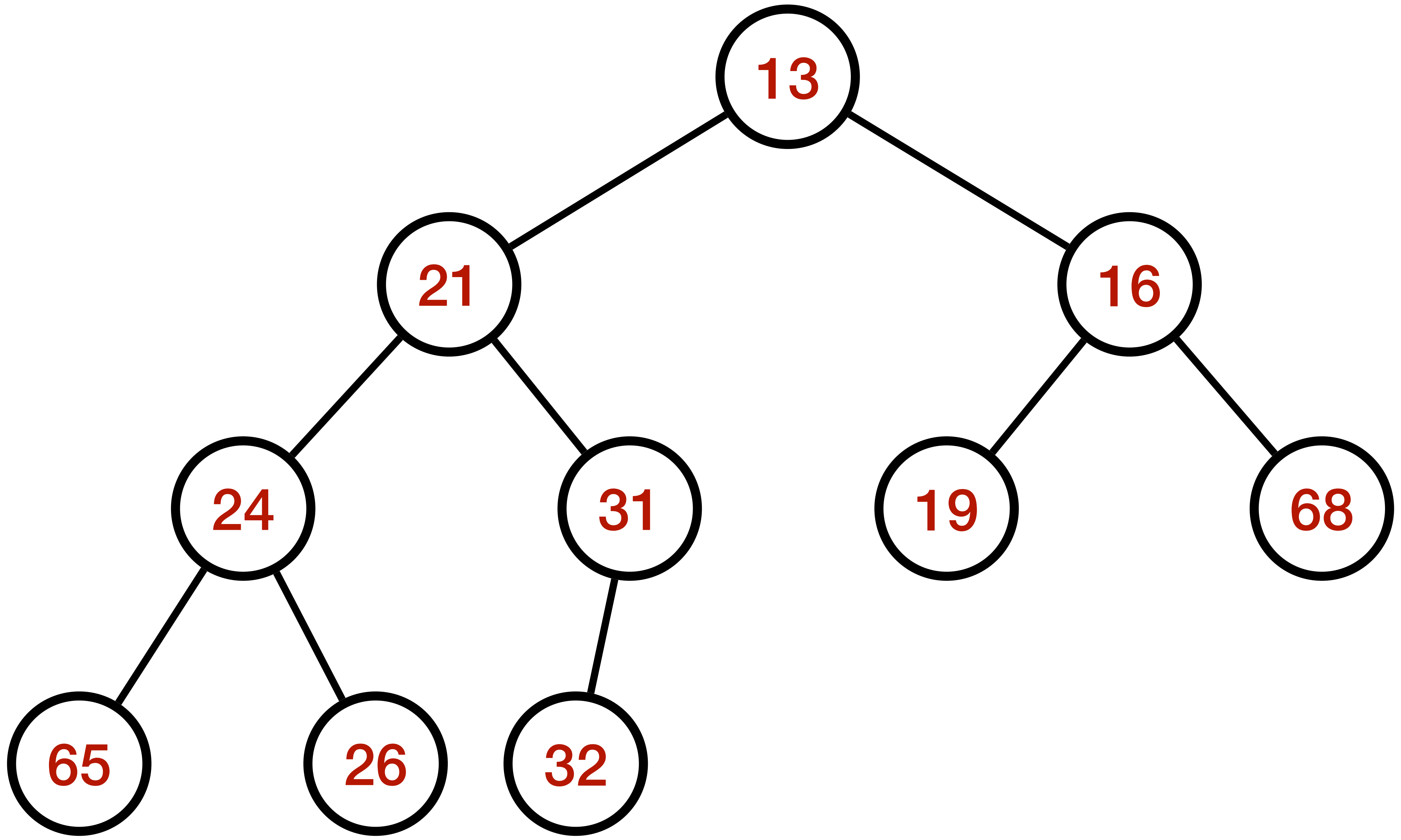
  - Apply k DeleteMin operations



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- Changed Problem: Find the k^{th} **smallest** element in a list of n elements
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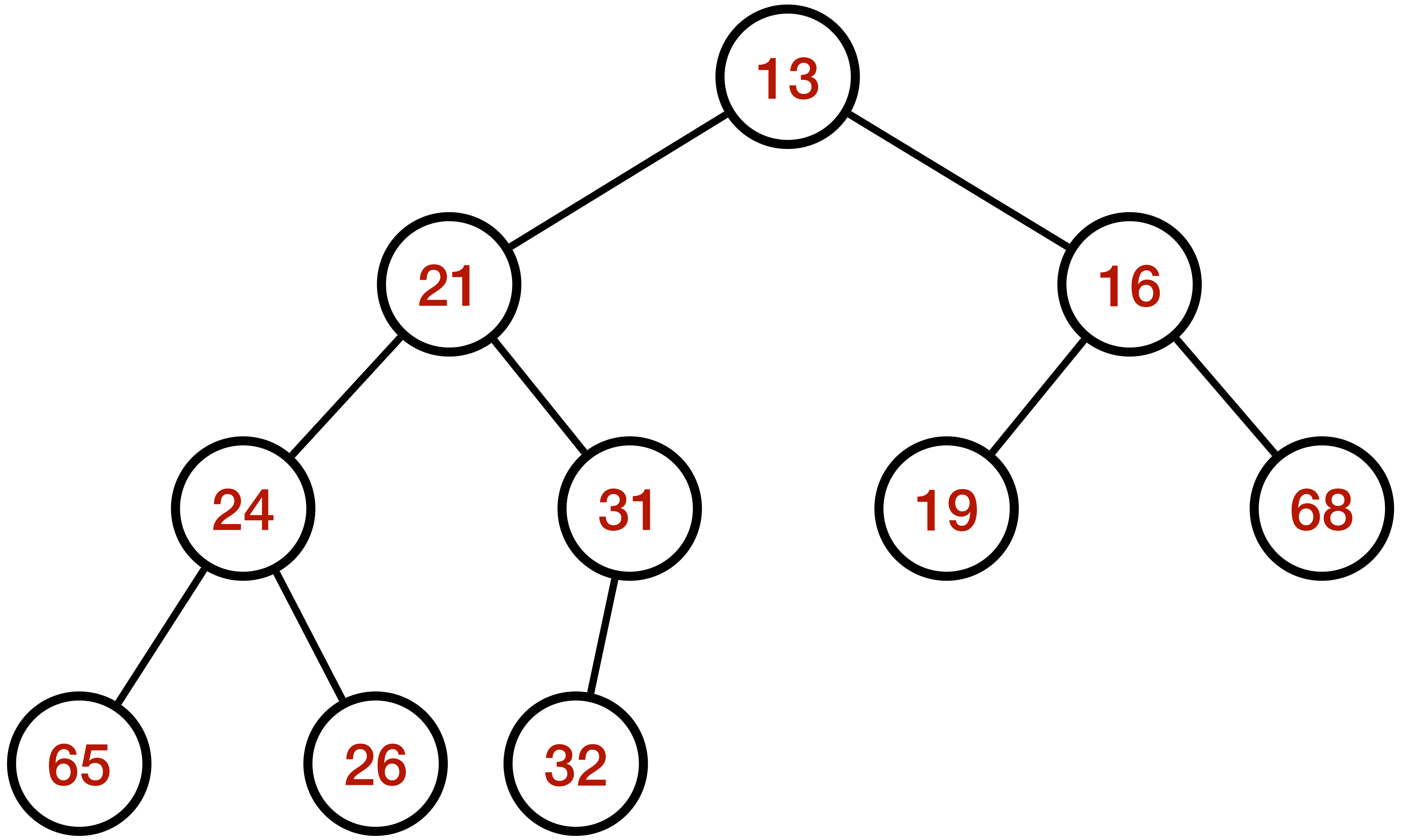

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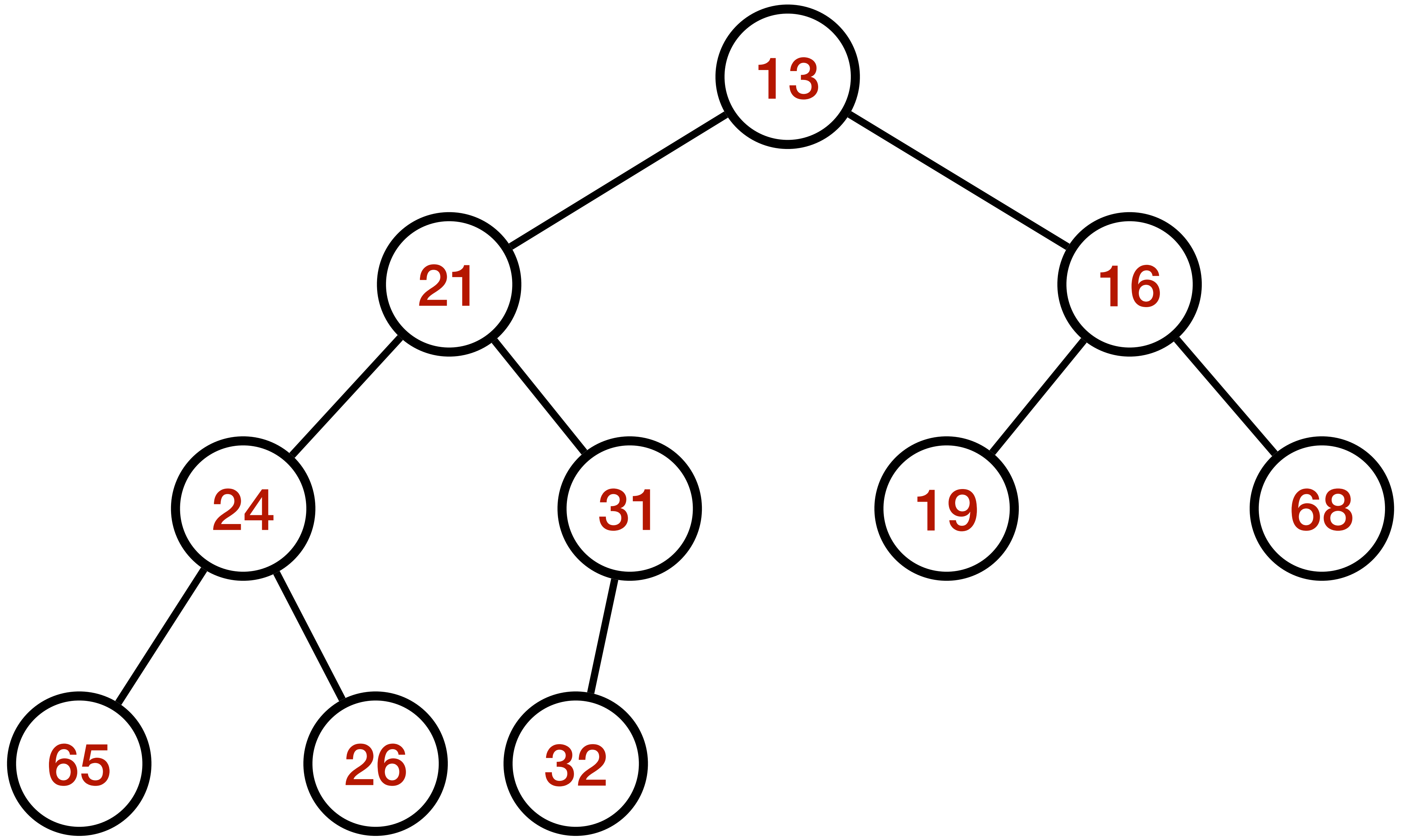

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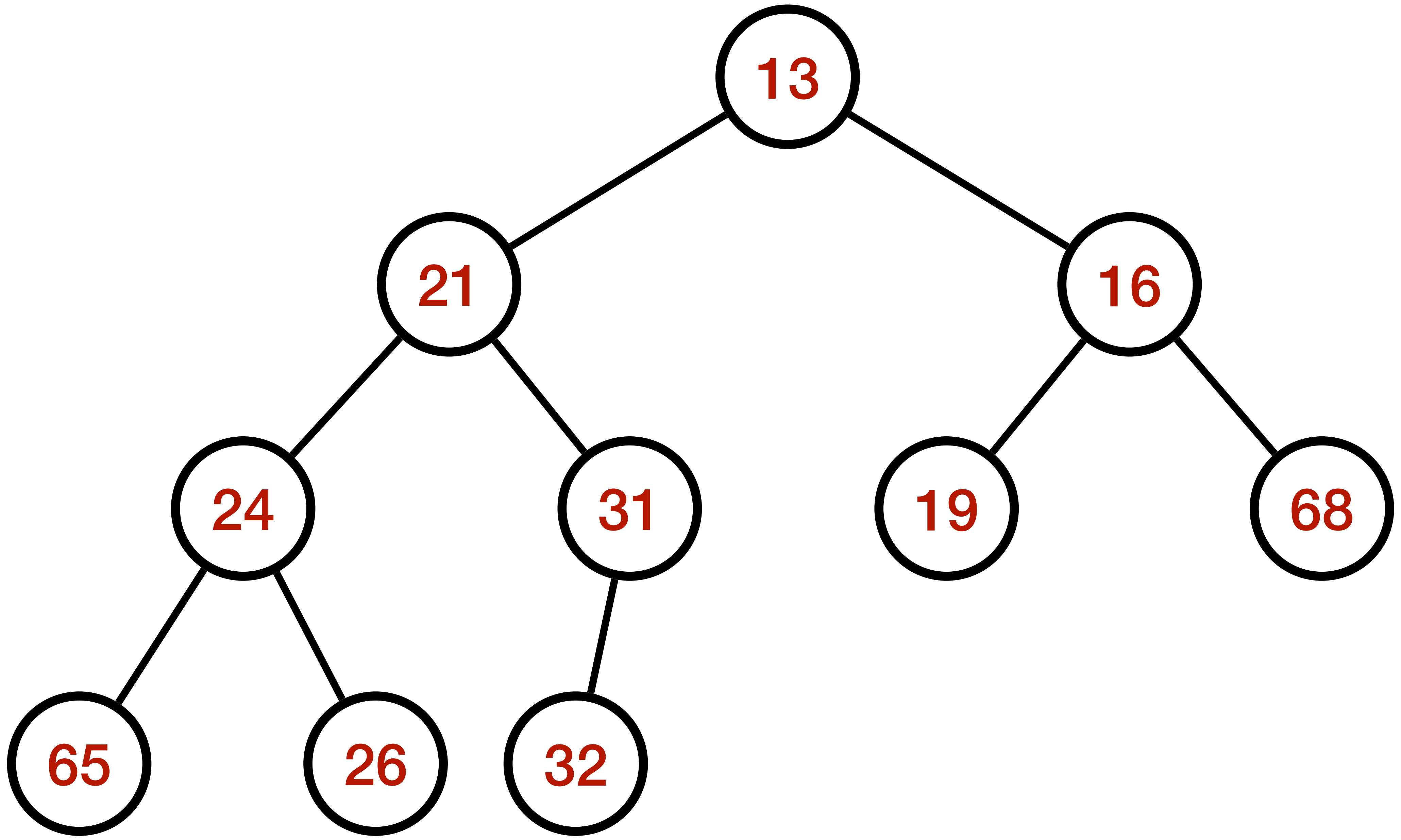
  - Time complexity: O(n + k log n). Why?
    - What happens when $k = \lceil n/2 \rceil$ or when $k = O(n/log\, n)$?



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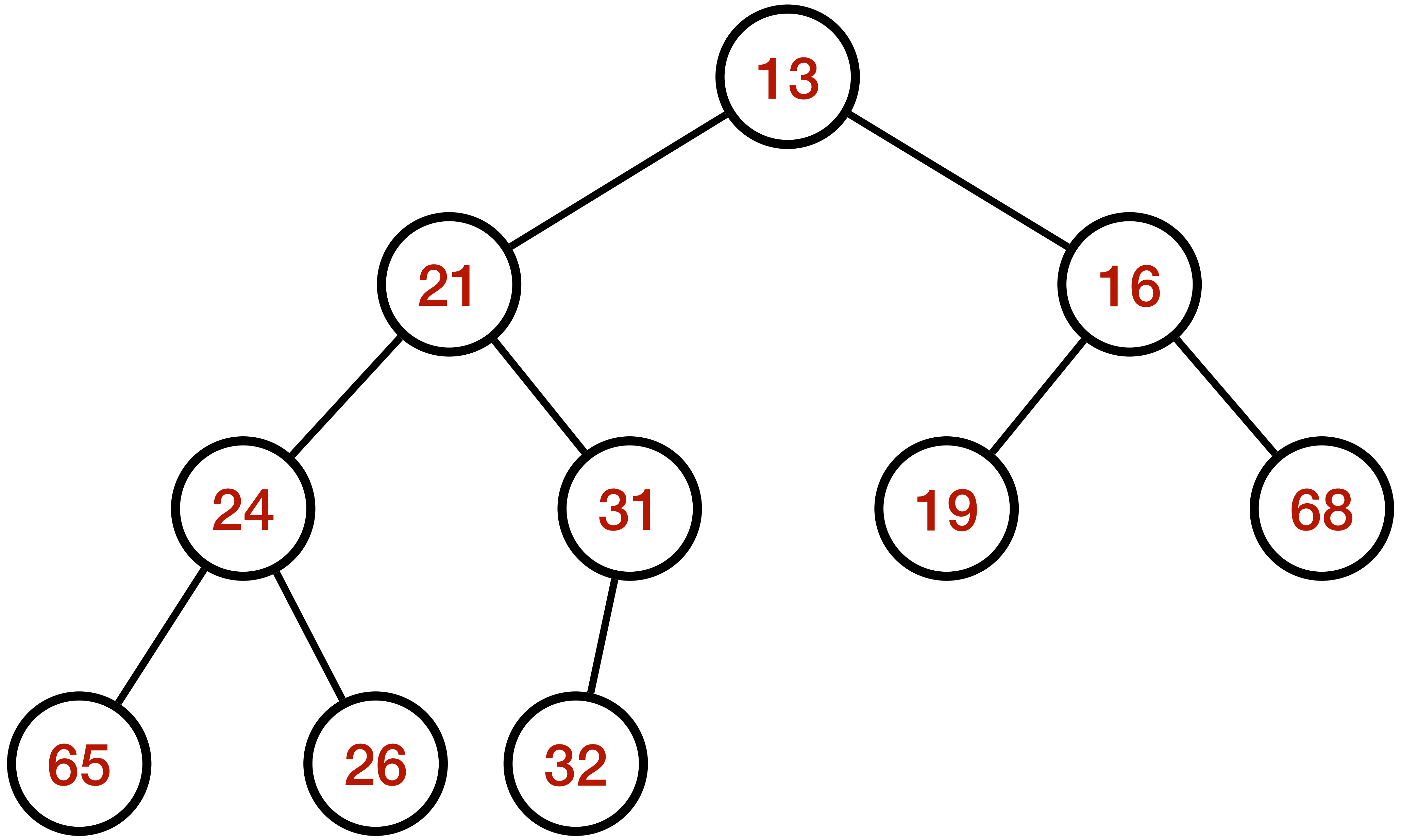
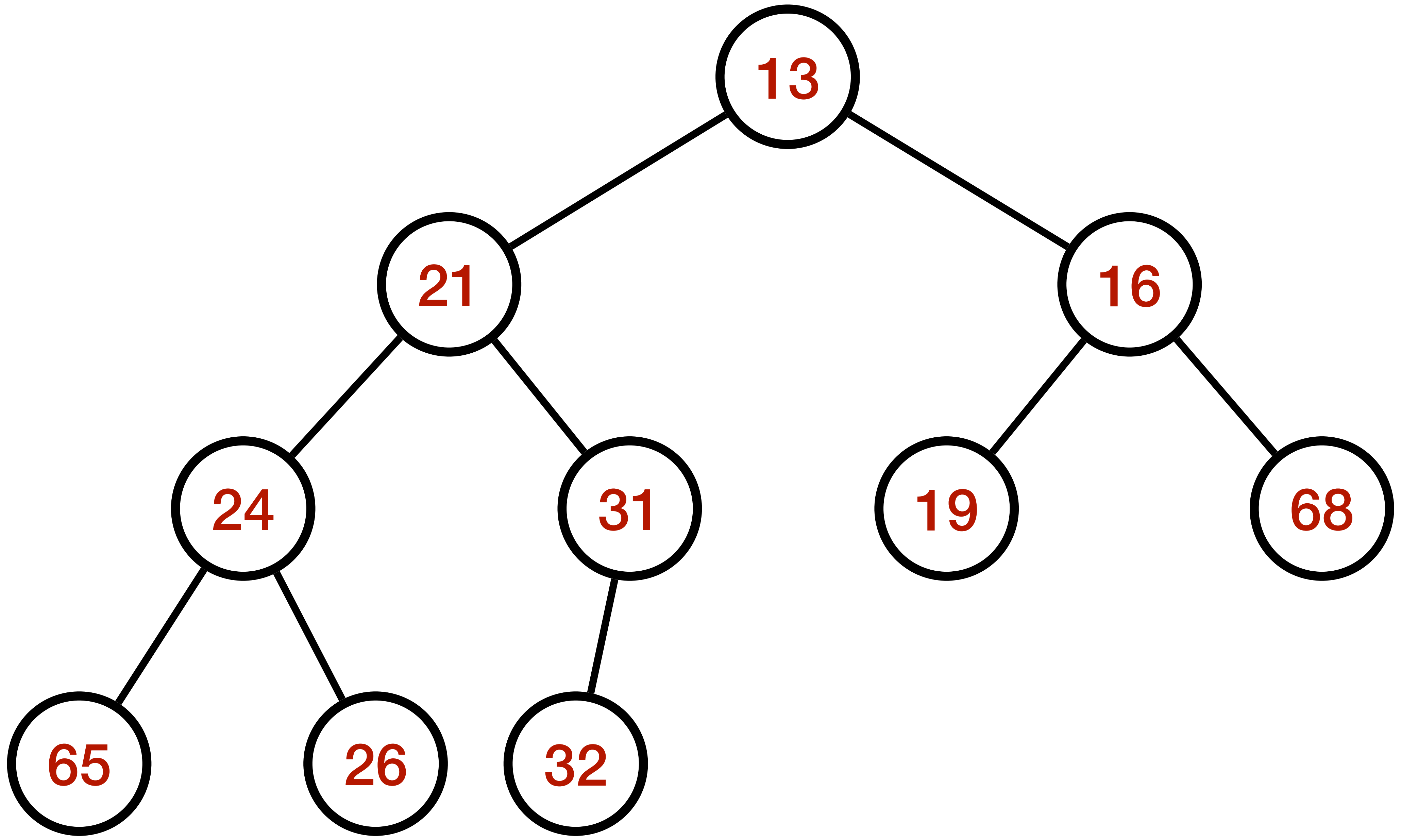


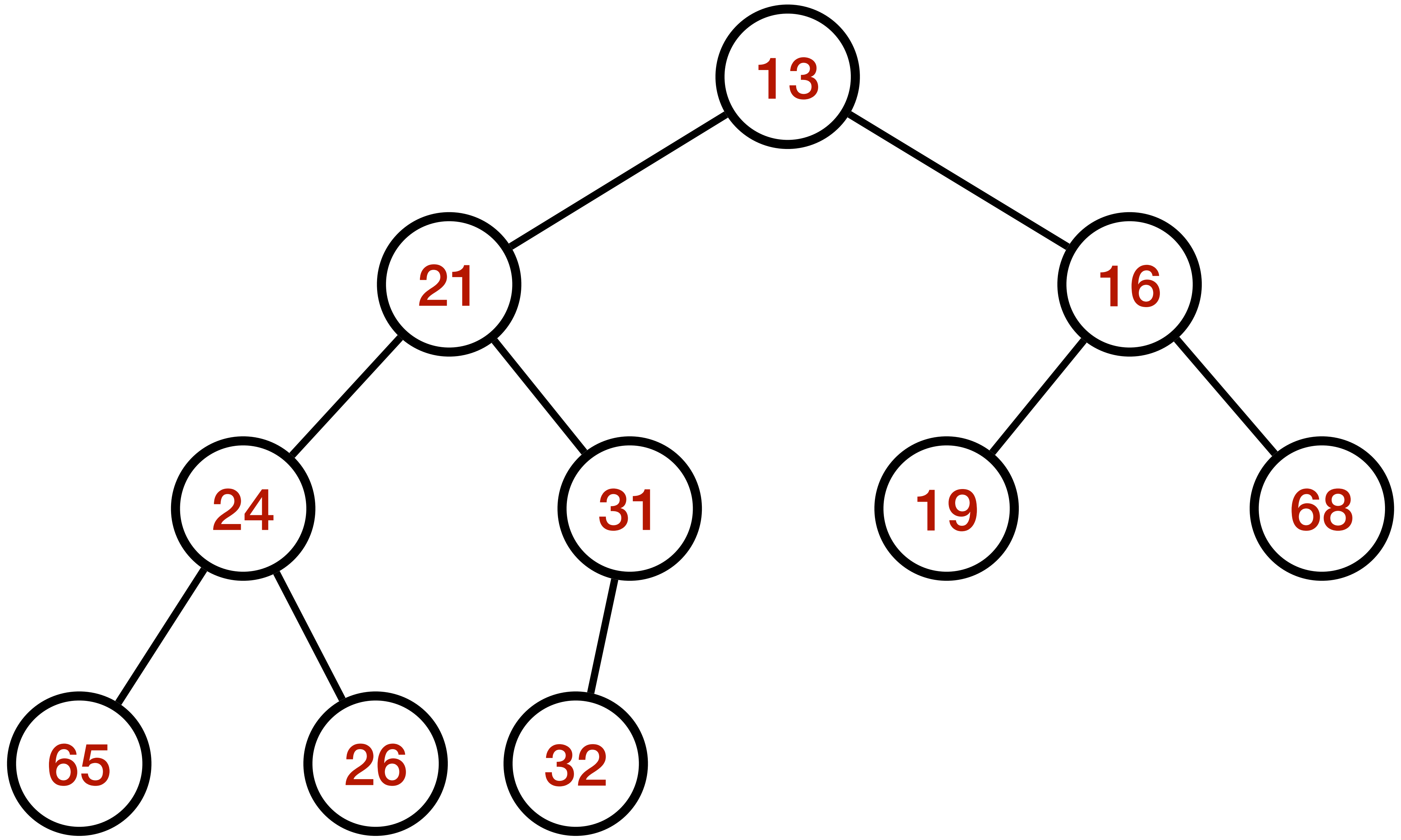
Applications: The Selection Problem

- Problem: Find the k^{th} **largest** element in a list of n elements



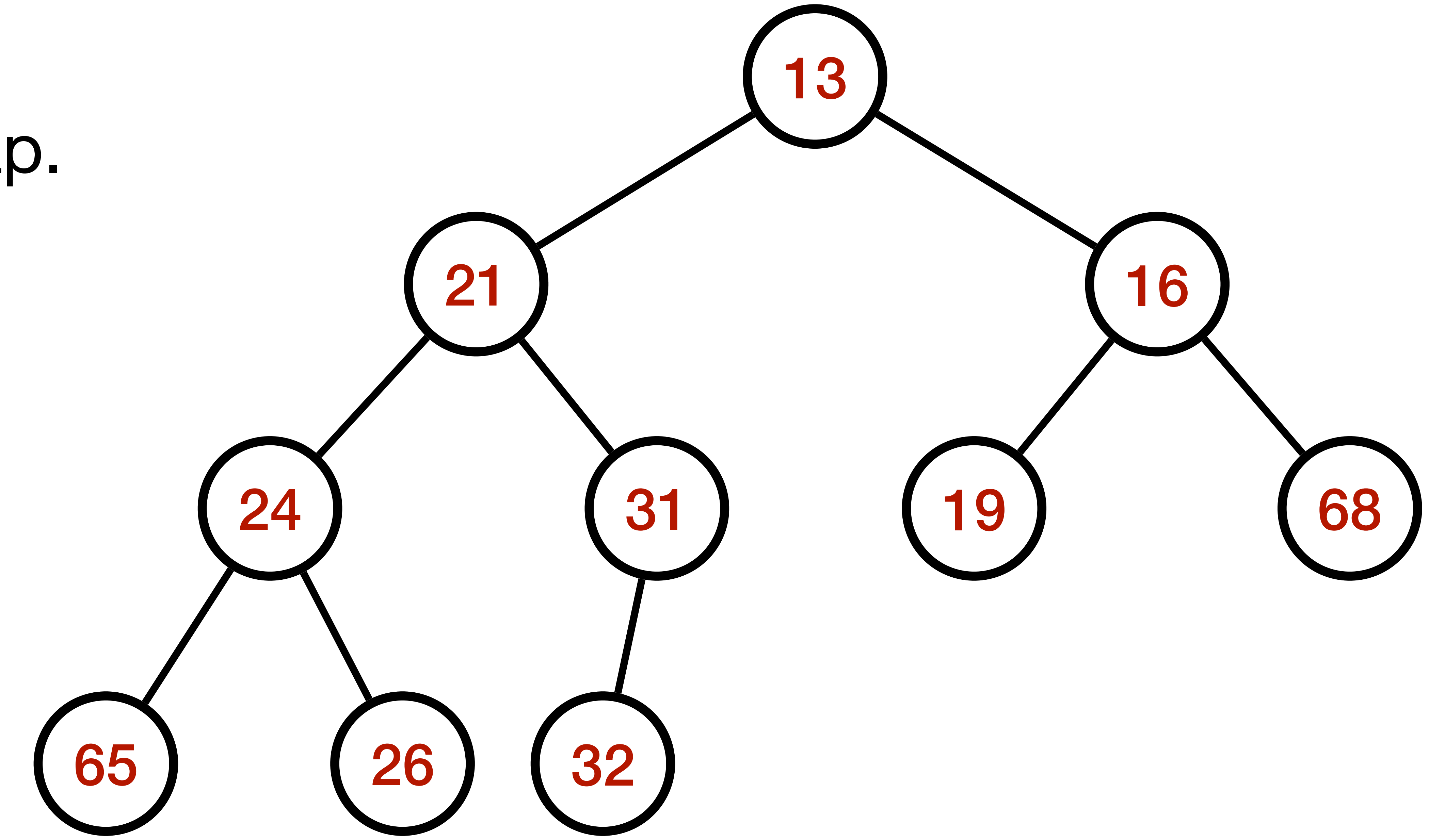
Applications: The Selection Problem

- Problem: Find the k^{th} **largest** element in a list of n elements
- Algorithm2B:



Applications: The Selection Problem

- Problem: Find the k^{th} **largest** element in a list of n elements
- Algorithm2B:
  - Read **only** k elements in an array and build a minheap.



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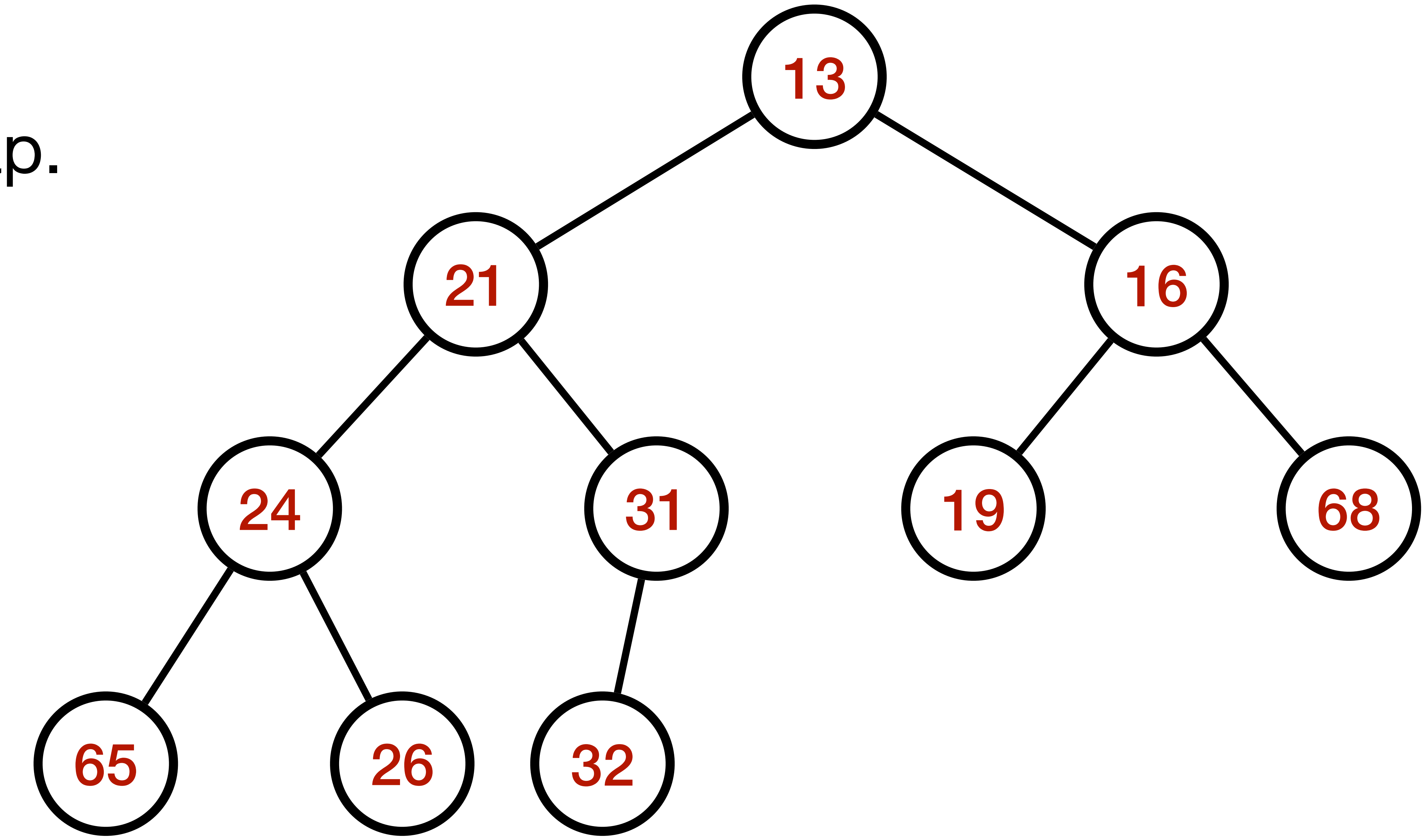

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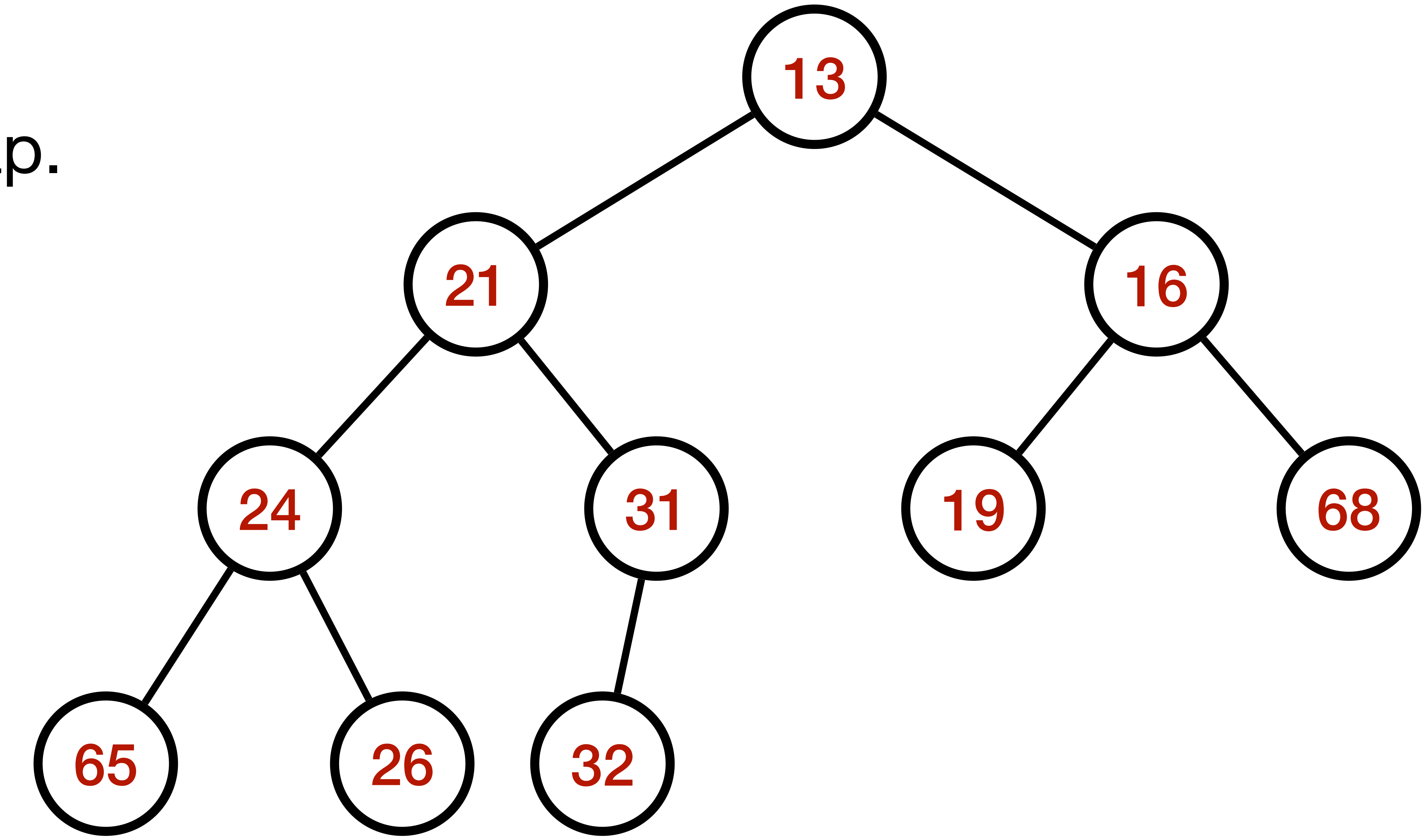

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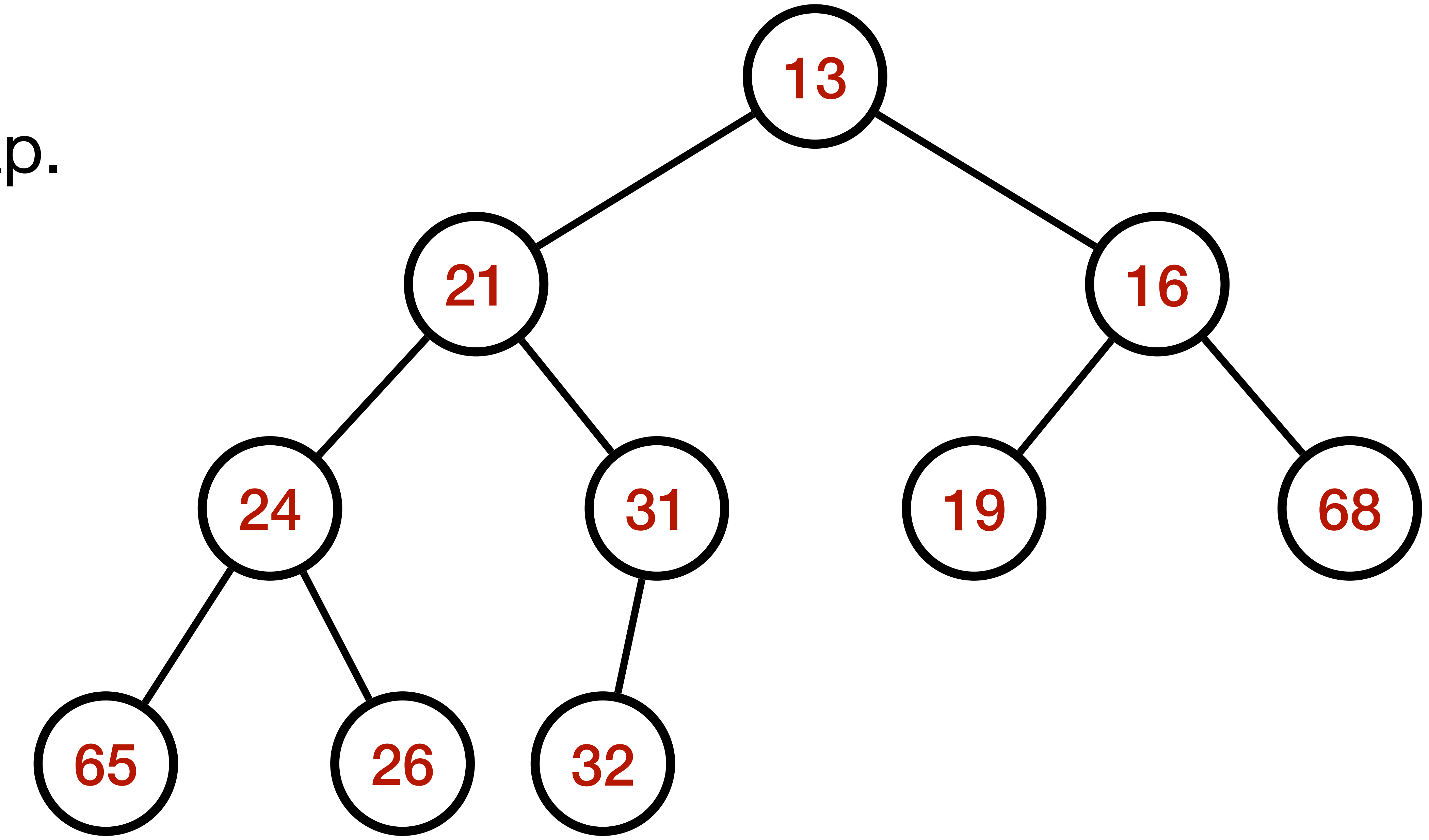

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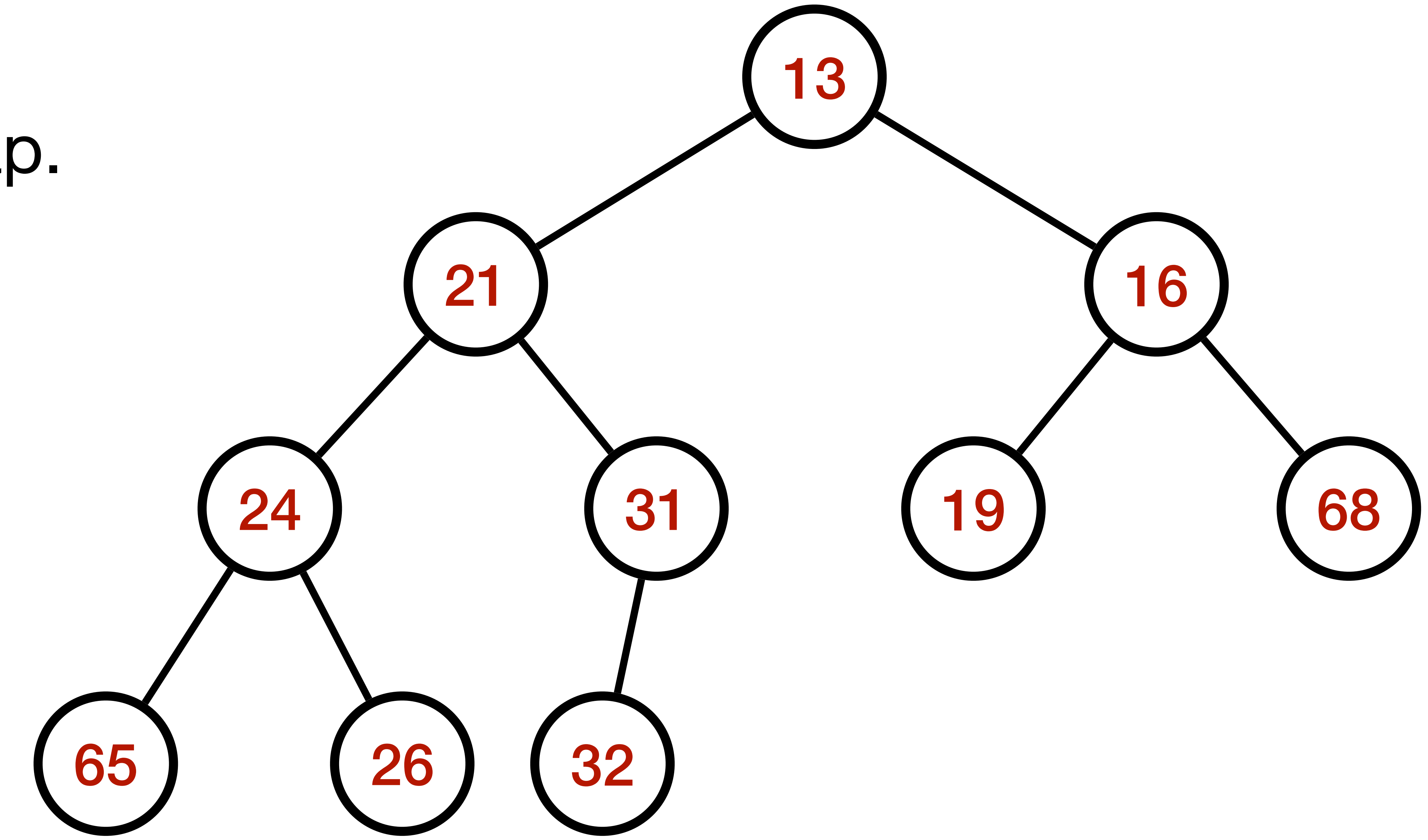

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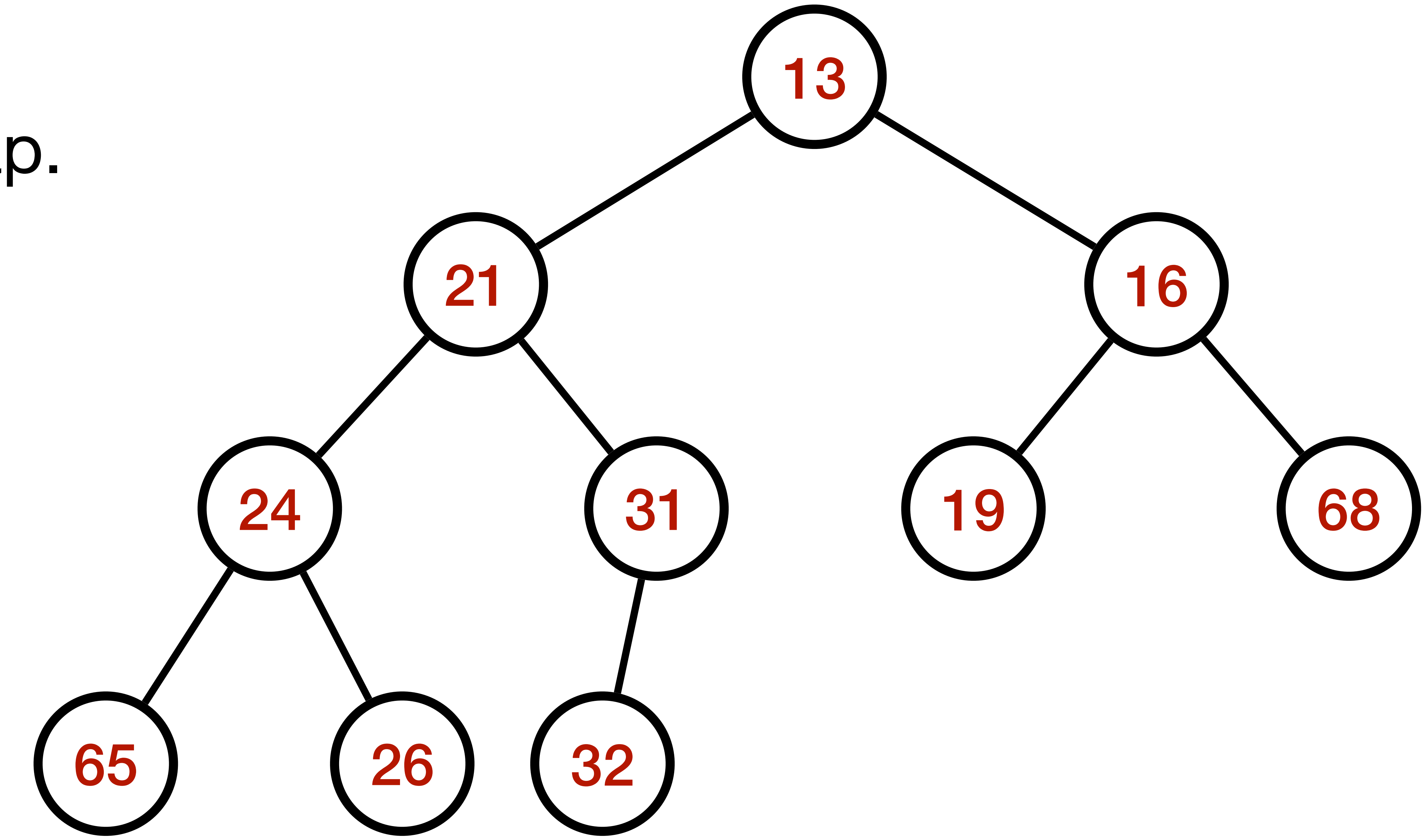
  - Time complexity: O(k + (n-k).log k)
  - Can we do better? Quickselect O(n) average time!



Extra Reading

For those who want to challenge themselves

- Skew Heaps (efficient merge operations)
- Binomial Queues
- Fibonacci Heaps
- Find the median of an array efficiently
- Understand Quick-select