GATE
PSUs

MADE EASY
Publications

POSTAL
STUDY PACKAGE

COMPUTER SCIENCE & IT

2020

THEORY BOOK | **Discrete & Engineering Mathematics**

Well illustrated theory with solved examples

Contents

Discrete and Engineering Mathematics

Discrete & Engineering Mathematics

INTRODUCTION

In this book we tried to keep the syllabus of Discrete & Engineering Mathematics around the GATE syllabus. Each topic required for GATE is crisply covered with illustrative examples and each chapter is provided with Student Assignment at the end of each chapter so that the students get a thorough revision of the topics that he/she had studied. This subject is carefully divided into eight chapters as described below.

Discrete Mathematics:

1. **Propositional Logic**: In this chapter we study logical connectives, well-Formed formulas, rules for inference, predicate calculus with Universal and Existential quantifiers.
2. **Combinatorics**: In this chapter we discuss the basic principles of counting, permutations, combinations, generating functions, binomial coefficients, summations and finally we discuss the recurrence relations.
3. **Set Theory and Algebra**: In this chapter we discuss the basic terms and definitions of set theory, Operations on sets, relations and types of relations , functions and their types and finally group theory, posets, lattices and boolean algebra.
4. **Graph Theory**: In this chapter we discuss the Special Graphs, isomorphism, vertex and edge connectivity, Euler graphs, Hamiltonian and planar graph, trees and enumeration of graphs.

Engineering Mathematics:

5. **Probability:** In this chapter we discuss the basic probability and axioms of probability, Basic concepts of statistics (mean, mode, variance and standard deviation), Discrete and continuous random variables and their distributions.
6. **Linear Algebra:** In this chapter we discuss the Special matrices, Algebra of matrices and their properties, inverse of a matrix, determinant of a matrix, solution of system of linear equations, LU decomposition method, Eigen values and Eigen vectors and finally we discuss the Cayley Hamilton theorem.
7. **Calculus:** In this chapter we discuss about Limits, continuity and differentiability, differentiation, partial derivatives, applications of differentiation (Mean value theorems, increasing and decreasing functions and maxima and minima of functions), methods of integration, and finally definite and indefinite integrals and their properties.

■■■■

CHAPTER 01

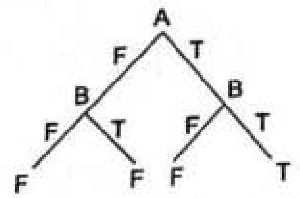
Propositional Logic

1.1 Propositional Logic; First Order Logic

Logic: In general logic is about reasoning. It is about the validity of arguments. Consistency among statements and matters of truth and falsehood. In a formal sense logic is concerned only with the form of arguments and the principle of valid inferencing. It deals with the notion of truth in an abstract sense.

Truth Tables: Logic is mainly concerned with valid deductions. The basic ingredients of logic are logical connectives, and, or, not , if.... then ...., if and only if etc. We are concerned with expressions involving these connectives. We want to know how the truth of a compound sentence like, "$x = 1$ and $y = 2$" is affected by, or determined by, the truth of the separate simple sentences "$x = 1$", "$y = 2$".

Truth tables present an exhaustive enumeration of the truth values of the component propositions of a logical expression, as a function of the truth values of the simple propositions contained in them. An example of a truth table is shown in table 1 below. The information embodied in them can also be usefully presented in tree form.



The branches descending from the node A are labelled with the two possible truth values for A. The branches emerging from the nodes marked B give the two possible values for B for each value of A. The leaf nodes at the bottom of the tree are marked with the values of $A \wedge B$ for each truth combination of A and B.

1.2 Logical Connectives or Operators

The following symbols are used to represent the logical connectives or operators.

And	$\wedge$(Conjuction)
or	$\vee$(Disjunction)
not	$\neg$(Not)
Ex - or	$\oplus$
Nand	$\uparrow$
Nor	$\downarrow$
if....then	$\rightarrow$(Implication)
ifandonlyif	$\leftrightarrow$(Biconditional)

1. **$\wedge$ (And / Conjuction):** We use the letters F and T to stand for false and true respectively.

Table-1

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

It tells us that the conjuctive operation $\wedge$ is being treated as a **binary** logical connective–it operates on two logical statements. The letters A and B are **"Propositional Variables"**.

The table tells us that the compound proposition $A \wedge B$ is true only when both A and B are true separately. The truth table tells us how to do this for the operator. $A \wedge B$ is called a truth function of A and B as its value is dependent on and determined by the truth values of A and B.

A and B can be made to stand for the truth values of propositions as follows:

A : The cat sat on the mat

B : The dog barked

Each of which may be true or false. Then $A \wedge B$ would represent the compound proposition "The cat sat on the mat and the dog barked"

$A \wedge B$ is written as A.B in Boolean Algebra.

2. **$\vee$ (Disjunction):** The truth table for the disjunctive binary operation $\vee$ tells us that the compound proposition $A \vee B$ is false only if A and B are both false, otherwise it is true.

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

This is inclusive use of the operator 'or'.

In Boolean Algebra $A \vee B$ is written as A + B.

3. **$\neg$ (Not):**

The negation operator is a **"unary operator"** rather than a binary operator like $\wedge$ and its truth table is

A	$\neg A$
F	T
T	F

The table presents ¬ in its role i.e the negation of true is false, and the negation of false is true. Notice that ¬ A is sometimes written as ~A or A′.

4. **⊕ (Exclusive OR or Ex - OR):**
   $A \oplus B$ is true only when either A or B is true but not when both are true or when both are false. $A \oplus B$ is also denoted by $A \vee - B$.

A	B	A⊕B
F	F	F
F	T	T
T	F	T
T	T	F

5. **↑ (NAND):**
   $P \uparrow Q \equiv \neg(P \wedge Q)$
6. **↓ (NOR):**
   $P \downarrow Q \equiv \neg(P \vee Q)$

   Note:
   $$P \uparrow P \equiv \neg P$$
   $$P \downarrow P \equiv \neg P$$
   $$(P \downarrow Q) \downarrow (P \downarrow Q) \equiv P \vee Q$$
   $$(P \uparrow Q) \uparrow (P \uparrow Q) \equiv P \wedge Q$$
   $$(P \uparrow P) \uparrow (Q \uparrow Q) \equiv P \vee Q$$
7. **→ (Implication):**

A	B	A→B
F	F	T
F	T	T
T	F	F
T	T	T

Note that $A \to B$ is false only when A is true and B is false. Also, note that $A \to B$ is true, whenever A is false, irrespective of the truth value of B.

8. **↔ (if and only if):** The truth table is

A	B	A↔B
F	F	T
F	T	F
T	F	F
T	T	T

Note that Bi-conditional (if and only if) is true only when both A & B have the same truth values. ($A \leftrightarrow B$ may be written as $A \rightleftharpoons B$)

Equivalences: $B \wedge A$ always takes on the same truth value is as $A \wedge B$.

We say that $B \wedge A$ is logically equivalent to $A \wedge B$ and we can write this as follows $B \wedge A \equiv A \wedge B$

Definition: Two expression are logically equivalent if each one always has the same truth value as the other.

Also,

$$B \wedge A \equiv A \wedge B$$
$$B \vee A \equiv A \vee B$$
$$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$
$$A \vee (B \vee C) \equiv (A \vee B) \vee C$$

These equivalence reveal $\wedge$ and $\vee$ to be commutative and associative operations. But these are not the only important equivalences that hold between logical forms.

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

A	B	$\neg A$	$\neg A \vee B$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

In the last columns in ta bles (Shown Bracketed) we have exactly same sequences of truth values. So, $A \to B \equiv \neg A \vee B$. Thus, we could do without the operation $\to$.

Now, consider the following two truth tables.

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg(\neg A \vee \neg B)$
F	F	T	T	T	F
F	T	T	F	T	F
T	F	F	T	T	F
T	T	F	F	F	T

We see from the bracketed truth values that $A \wedge B$ is logically equivalent to $\neg(\neg A \vee \neg B)$. Thus $\wedge$ could be replaced by a combination of $\neg$ and $\vee$.

Similarly we could show that $\leftrightarrow$ can be replaced by a combination of $\neg$ and $\vee$.

We say therefore that $(\neg, \vee)$ forms a functionally complete set of connectives.

We can also show that $(\neg, \wedge)$ also form a functionally complete set of connectives.

The NAND operator ($\uparrow$) by itself is also a functionally complete set. So is the Nor operator ($\downarrow$). These both are minimal functionally complete set.

Notice that $(\vee, \wedge)$ is not a functionally complete set. Neither is $(\neg)$, $(\vee)$ or $(\wedge)$ by themselves functionally complete.

Example - 1.1 Obtain the truth table for $\alpha = (P \vee Q) \wedge (P \to Q) \wedge (Q \to P)$

Solution:

P	Q	$P \vee Q$	$P \to Q$	$(P \vee Q) \wedge (P \to Q)$	$Q \to P$	α
T	T	T	T	T	T	T
T	F	T	F	F	T	F
F	T	T	T	T	F	F
F	F	F	T	F	T	F

1.2.1 List of Important Equivalences

$A \wedge 0 \equiv 0$	Domination law	$A \cdot 0 \equiv 0$
$A \vee 1 \equiv 1$	Domination law	$A + 1 \equiv 1$
$A \wedge 1 \equiv A$	Identity of $\vee$	$A \cdot 1 \equiv 1$
$A \vee 0 \equiv A$	Identity of $\wedge$	$A + 0 \equiv A$
$A \wedge A \equiv A$	Idempotence	$A \cdot A = A$
$A \vee A \equiv A$	Idempotence	$A + A = A$
$A \wedge \neg A \equiv 0$	Complement law	$A \cdot A = 0$
$A \wedge \neg A \equiv 1$	Complement law	$A + A' = 1$
$\neg\neg A \equiv A$	Law of double negation	$(A')' = A$
$A \wedge B \equiv B \wedge A$	Commutativity	$A \cdot B = B \cdot A$
$A \vee B \equiv B \vee A$	Commutativity	$A + B = B + A$
$A \vee (B \vee C) \equiv (A \vee B) \vee C$	Associativity	$A + (B + C) = (A + B) + C$
$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$	Associativity	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	Distributivity	$A \cdot (B + C) = A \cdot B + A \cdot C$
$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	Distributivity	$A + (B \cdot C) \equiv (A + B) \cdot (A + C)$
$A \wedge (A \vee B) \equiv A$	Absorption law	$A \cdot (A + B) \equiv A$
$A \vee (A \wedge B) \equiv A$	Absorption law	$A + (A \cdot B) \equiv A$
$\neg(A \wedge B) \equiv \neg A \vee \neg B$	De Morgan's law	$(A \cdot B)' \equiv A' + B'$
$\neg(A \vee B) \equiv \neg A \vee \neg B$	De Morgan's law	$(A + B)' \equiv A' \cdot B'$
$A \rightarrow B \equiv \neg A \vee B$		$A \rightarrow B \equiv A' + B$
$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$		$A \leftrightarrow B \equiv (A' \rightarrow B) \cdot (B' \rightarrow A)$

NOTE: $\oplus$ (EX – OR) is commutative and associative, (NAND) and (NOR) are both commutative but not associative. $P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$

Simplication

The various equivalence between logical forms provide us with a means of simplifying logical expressions. For example, we can simply the logical forms.

$(A \vee 0) \wedge (A \vee \neg A)$ as follows:

$$\begin{aligned}(A \vee 0) \wedge (A \vee \neg A) &= A \wedge (A \vee \neg A)\\ &= A \wedge 1 \text{ (since } A \vee \neg A \equiv 1)\\ &= A\end{aligned}$$

Similarly,
$$\begin{aligned}(A \wedge \neg B) \wedge (A \wedge B \wedge C) &\equiv A \wedge (\neg B \vee (B \wedge C))\\ &\equiv A \wedge ((\neg B \vee B) \wedge (\neg B \vee C))\\ &\equiv A \wedge (1 \wedge (\neg B \vee C))\\ &\equiv A \wedge (\neg B \vee C)\end{aligned}$$

Since logic is a boolean algebra, it is always much easier to do simplification of logical expressions by converting them first into its boolean algebra equivalents.

Example: $(A \vee 0) \wedge (A \vee \neg A) \equiv (A + 0) \cdot (A + A') = (A) \cdot (1) = A$

Application to Circuit Design

One of most important application of Boolean algebra is to the design of electronic circuits and especially to the design of computer logic. The basic logic functions 'and', 'or' and 'not' can be realised by electronic devices called gates and these can be combined together to form complicated circuits. The 'and' connective is realised by an 'and-gate' which is symbolised as follows:

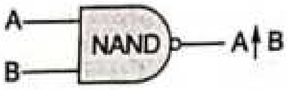
The idea is that if the AND-gate receives input signals on both the A and B input lines, there will be an output signal on the line marked A.B. If there is no input on either A or B or both there will be no output signal. Similarly, an 'OR-gate' is symbolised as follows:

If there is an input signal on either A or B or both, there will be an output signal on A + B.

Finally a NOT gate looks alike

In this case, if there is an input A, there will be no output and If there is no input A, there will be an output. The input is said to be inverted or negated.

Similarly we have the NAND gate and NOR gate represented *y* represented by.



1.3 Well-Formed Formulas (WFFs)

Consider $P \wedge Q$ and $Q \wedge P$, where P and Q are any two propositions (logical statements). The truth table of these two propositions are identical. This happens when we have any proposition in place of P and any propositions in place of Q.

So we can develop the concept of propositional variable (corresponding to propositions) and well formed formulas (corresponding to propositions involving connectives).

Definition: A propositional variable is a symbol representing any proposition. We note that in algebra, a real variable is represented by the symbol x. This means that x is not a real number but can take a real value.

Similarly, a propositional variable is not a proposition but can be replaced by a proposition.

Definition: A **well formed formula** is defined as follows:

(*i*) If P is a propositional variable then it is wff.

(*ii*) If a is a wff, then $\neg$ a is a wff.

(*iii*) If α and β are well formed formula, then $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are well formed formula.

NOTE: A wff is not a proposition, but if we substitute the proposition in place of propositional variable, we get a proposition e.g., $(\neg P \wedge Q) \leftrightarrow Q$ is a wff.

Also note that "$P \neg \wedge Q \leftrightarrow Q$" is not a wff. Similarly "$P \wedge Q \vee$" is not a wff. This is because the above two cannot be derived using rules ***i***, ***ii*** and ***iii*** given above for wffs.

Duality Law

Two formulas A and A' are said to be duals of each other, if either can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\vee \wedge$, 0 by 1 (F by T) and 1 by 0 (T by F). e.g., Dual of $(P \wedge Q) \vee T$ is $(P \vee Q) \wedge F$, Dual of $A + 1 = 1$ is $A . 0 = 0$

Let A and A' be duals consisting of $P_1, P_2, \ldots P_n$ Propositional variables. By repeated application of De Morgan's Law, it can be shown that $\neg A(P_1, P_2, \ldots, P_n) = A'(\neg P_1, \neg P_2, \ldots, \neg P_n)$.

1.3.1 Truth Table for a Well-Formed Formula

If we replace the propositional variables in a formula a by propositions, we get a proposition involving connectives. The table giving the truth value of such proposition obtained by replacing the propositional variables by arbitrary proposition is called the truth table of a.

If a involves n propositional variables, we have 2^n possible combinations of truth values of propositions replacing the variables.

Tautology, Contradiction and Contingency

Definition: A **tautology** is a well formed formula whose truth value is T for all possible assignments of truth values to the propositional variables. Such a wff is also called valid (always true).

NOTE: When it is not clear whether a given formula is tautology, we can construct a truth table and verify that the truth value is T for all possible combinations of truth value of the propositional variables appearing in given formula.

Example - 1.2 Show that $\alpha = (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ is a tautology.

Solution:

Truth table for α:

P	Q	R	$Q\to R$	$P\to(Q\to R)$	$P\to Q$	$P\to R$	$(P\to Q)\to(P\to R)$	α
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Since the truth value of α is T for every possible Combination of truth values of P, Q and R, we can say that α is a tautology.

Definition: A **contradiction** (or absurdity) is a wff whose truth value is F for all possible assignments of truth values to the propositional variables.

e.g., $P \wedge \neg P$ and $(P \wedge Q) \wedge \neg Q$ ← are examples of contradictions

Truth table for $P \wedge \neg P$

P	$\neg P$	$P\wedge\neg P$
T	F	F
F	T	F

Truth table for $(P \wedge Q \wedge \neg Q)$

P	Q	$P\wedge Q$	$\neg Q$	$P\wedge Q\wedge\neg Q$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

NOTE: α is contradiction if and only if $\neg$ a is tautology.

Definition: A **contingency** is a wff which is neither a tautology nor a contradiction. In otherwords, a contingency is a wff which is sometimes true of sometimes false.

Examples of contigency are $P \wedge Q, \neg P \vee Q, P \wedge (P \vee Q)$

Truth table for $P \wedge (P \vee Q)$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Notice that $P \wedge (P \vee Q)$ is sometimes true and sometimes false.

Satisfiable and Unsatisfiable wffs

- A wff which is either a tautology or a contingency is called **satisfiable.**
- A wff which is a contradiction is called **unsatisfiable.**

Equivalence of Well-Formed Formulas

Definition: Two wff a and b in propositional variables $P_1, P_2, ...P_n$ are **equivalent** if the formula $a \leftrightarrow b$ is a tautology.

When α and β are equivalent we write $\alpha \equiv \beta$.

NOTE: α and β are equivalent if and only if truth tables of *a* and *b* are the same.

1.4 Normal forms of Well-Formed Formulas

We know that two formulas are equivalent if and only if they have the same truth table. The number of distinct truth tables for formulas in P and Q is 2^4 (As the possible combination of truth values of P and Q are TT, TF, FT, FF, the truth table of any formula in P and Q has 4 rows. So the number of distinct truth tables is 2^4). Each row may be associated with either T or F value for the function involving P and Q. Thus there are only 16 distinct formulas and any formula in P and Q is equivalent to one of these 16 formulas.

Here there is a method of reducing a given formula to an equivalent form called a 'normal form'. We use 'sum' for disjunctions, 'product' for conjuction, and 'literal' either for P or for $\neg$ P, where P is any propositional variable.

Definition:

1. An elementary product is a product of literals. An elementary sum is a sum of literals.
   e.g. $P \wedge \neg Q, \neg P \wedge Q, P \wedge Q, P$ are elementary products $PV \vee \neg Q, P \vee \neg R, P$ are elementary sums.
2. A formula is in disjunctive normal form (DNF) if it is a sum of elementary products.
   e.g. $(P) \vee (Q \wedge R)$ AND $(P) \vee (\neg Q \wedge R)$ are in disjunctive normal form.

Construction to obtain a disjunctive normal form of a given formula:

Step-1: Eliminate $\rightarrow$ and $\leftrightarrow$ using logical identities.

Step-2: Use De Morgan's law to eliminate $\neg$ before sums or products. The resulting formula has $\neg$ only before propositional variables. i.e. it involve sum, product and literals.

Step-3: Apply distributive laws repeatedly to eliminate product of sums. The resulting formula will be a sum of products of literals i.e. sum of elementary products.

Definition: A min term in n propositional variables $P_1 \ldots P_n$ is $Q_1 \wedge Q_2 \ldots \wedge Q_n$ where each Q_i is either P_i or $\neg P_r$, e.g. The min term in P_1 and P_2 are $P_1 \wedge P_2$. The number of min items in n variables is 2^n.

Definition: A formula α is in principal disjunctive normal form (PDNF), if α is a sum of min terms.

Construction to obtain the Principal Disjunctive Normal form of a given formula:

Step-1: Obtain a disjunctive normal form

Step-2: Drop elementary products which are contradictions (such as $P \wedge \neg P$)

Step-3: P_i and $\neg P_i$ are missing in an elementary product a replace a by $(\alpha \wedge P_i) \vee (\alpha \wedge \neg P_i)$

Step-4: Repeat step 3 until all elementary products are reduced to sum of min terms. Use idempotent laws to avoid repetition of min terms.

Definition:

1. A max term in n propositional variables $P_1, P_2, \ldots P_n$ is $Q_1 \vee Q_2 \ldots \vee Q_n$, where each Q_i is either P_i or $\neg P_r$.
2. A formula α is in **principal conjunctive normal form** if α (PCNF) is a product of max terms.

For obtaining the principal conjunctive normal form of α, we can constant the principal disjunctive normal form of $\neg \alpha$ and apply negation ($\neg$).

NOTE: For a given wff the PDNF form is unique the PCNF form is unique, if PDNF form or PCNF of 2 wffs are same, they are equivalent.

Example-1.3 Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

$$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \quad \text{(By using } p \rightarrow q \equiv \neg p \vee q\text{)}$$
$$\equiv \neg(\neg p) \wedge \neg q \quad \text{(By using Demorgans law)}$$
$$= p \wedge \neg q \quad \text{(By using double negation law)}$$

Example-1.4 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution:

$$\begin{aligned} \text{LHS} &\equiv \neg(p \vee (\neg p \wedge q)) \\ &\equiv (p + (p'q))' \\ &\equiv (p+q)' \\ &\equiv p'q' \\ \text{RHS} &\equiv \neg p \wedge \neg q \\ &\equiv p'q' \end{aligned}$$

Therefore, $\text{LHS} \equiv \text{RHS}$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Example-1.5 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$\begin{aligned} (p \wedge q) \rightarrow (p \vee q) &\equiv pq \rightarrow (p+q) \\ &\equiv (pq)' + (p+q) \\ &\equiv p' + q' + p + q \\ &\equiv p' + p + q' + q \\ &\equiv 1 + q' + q \equiv 1 \end{aligned}$$

Therefore, $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

1.5 Rules of Inferences for Propositional Calculus

In logical reasoning (an argument or proof), a certain number of propositions are assumed to be true and based on that assumption some other propositions are derived. There are some important reasoning or rules of inferences.

The propositions that are assumed to be true are called **hypotheses or premises**. The proposition derived by using the rules of inference is called a **conclusion**.

The process of deriving conclusions based on assumption of premises is called **argument**. An argument is valid iff the conclusion is true whenever the premises are all true.

The rules of inference are commonly known tautologies in the form of implication (i.e. $\alpha \rightarrow \beta$).

e.g. $P \rightarrow (P \vee Q)$ is such a tautology and it is a rule of inference.

We write this in the form of $\dfrac{P}{P \vee Q}$. Here P denotes a premise. The proposition below the line i.e. $P \vee Q$, is the conclusion.

Rules of inference specify which conclusion may be inferred legitimately from known, assumed or previously established premises.

Therefore these are commonly used in mathematical proofs and logical arguments. Infact, most math proofs uses only one or more of the rules of inferences.

1.5.1 Rules of Inference

Rules of Inference	Implication form
1. Addition $\dfrac{P}{\therefore P \vee Q}$	$P \rightarrow (P \vee Q)$
2. Conjunction $\begin{array}{c} P \\ Q \\ \hline \therefore P \wedge Q \end{array}$	$P \wedge Q \rightarrow P \wedge Q$
3. Simplification $\dfrac{P \wedge Q}{\therefore Q}$	$(P \wedge Q) \rightarrow Q$
4. Modus ponens $\begin{array}{c} P \\ P \rightarrow Q \\ \hline \therefore Q \end{array}$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$
5. Modus tollens $\begin{array}{c} \neg Q \\ P \rightarrow Q \\ \hline \therefore \neg P \end{array}$	$(\neg Q \wedge (P \rightarrow Q) \rightarrow \neg P)$
6. Disjunctive syllogism $\begin{array}{c} \neg P \\ P \vee Q \\ \hline \therefore Q \end{array}$	$(\neg P \wedge (P \vee Q)) \rightarrow Q$
7. Hypothetical syllogism $\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore P \Rightarrow R \end{array}$	$((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$

8. Constructive Dilemma $\dfrac{(P \to Q) \wedge (R \to S) \quad P \vee R}{\therefore Q \vee S}$ $\quad (P \to Q) \wedge (R \to S) \wedge (P \vee R) \to (Q \vee S)$

9. Destructive Dilemma $\dfrac{(P \to Q) \wedge (R \to S) \quad \neg Q \vee \neg S}{\therefore \neg P \vee \neg R}$ $\quad (P \to Q) \wedge (R \to S) \wedge (\neg Q \vee \neg S) (\neg P \vee \neg R)$

Arguments: An **argument** is a set of premises followed by a conclusion.

An argument is **valid** if the conjunction of premises → conclusion is a tautology.

An **invalid** argument is also called as **fallacy**.

Example: "If you study hard you will pass the exam. I studied hard. Therefore I will pass the exam", can be translated as

P : I study hard

Q : I will pass the exam $(P \to Q, P) \dashv Q$.

This argument is valid if the wff $(P \to Q) \wedge (P) \to Q$ is a tautology.

It can be verified that, it is indeed a tautology & therefore, the given argument is valid.

Inconsistency and Consistency

A set of wff's $H_1, H_2, H_3 \ldots H_n$ are **inconsistent** if $H_1 \wedge H_2 \wedge H_3 \ldots\ldots \wedge H_n$ is a contradiction (unsatisfiable). The set is **consistent** if $H_1 \wedge H_2 \ldots \wedge H_n$ is satisfiable (i.e. either a tautology or a contingency).

1.6 Predicate Calculus

Let us consider two propositions **"Rita is a student"** and **"Sita is a student"**.

As propositions, there is no relation between them but there is something common between the two statements. Both Rita and Sita share a property of being a student.

We can replace the two proposition by a single statement **"x is a student"**. By replacing x by Rita or Sita (or any other name), we get many propositions. The common feature expressed by **"is a student"** is called a predicate. In predicate calculus we deal with sentences involving **predicates**.

A **predicate** $P(x)$ is a propositional function such as $P(x) : x$ is a student.

Now, P(Rita) has the truth value of the Statement, "Rita is a student".

Another Example: $P(x, y) : x + y = 4$. Here, $P(3, 1)$ is true but $P(3, 2)$ is false.

Statements involving predicates occur in Mathematics and programming languages e.g. **"$2x + 3y = 4z$"**. **"IF (D.GE.O.O)GO TO 20"** are statements in Mathematics and FORTRAN, respectively involving predicates.

Predicates

A part of a declarative sentence describing the properties of an object or relation among objects is in English called a **predicate**. e.g. **"is a student"** is a predicate.

The sentence **"x is the father of y"** also involves a predicate **"is the father of"**. Here the predicate the describes relation between two persons.

We can write this sentence as $P(x, y)$.

Similarly, $2x + 3y = 4z$ can be described by $P(x, y, z)$.

NOTE: Although, $P(x)$, involving a predicate looks like a proposition, it is not a proposition.

As $P(x)$ involves a variable x, we cannot assign a truth value to $P(x)$.

However, if we replace x by a specific object, then we get a proposition.

1.7 Universal and Existential Quantifiers

The phrase 'for all' (denoted by $\forall$) is called the **Universal Quantifier**.
Using this symbol, we can write "for all x, $x^2 = (-x)^2$" as $\forall x\, Q(x)$, where $Q(x)$ is "$x^2 = (-x)^2$".
The phrase 'there exists' (denoted by $\exists$) is called the **Existential Quantifier**.
The sentence, "there exists x such that $x^2 = 5$" can be written as $\exists x\, R(x)$, where $R(x)$ is $x^2 = 5$.
$P(x)$ in $\forall x P(x)$ or in $\exists x\, P(x)$ is called the **'scope of quantifier'** $\forall$ or $\exists$.

NOTE: $\forall$ also be written as $(x)\, P(x)$.

Consider $P(x, y): x + y = y + x$
Now $\forall x \forall y\, P(x, y)$ is true.
Whereas $\forall x \forall y\, Q(x, y)$ is false, where, $Q(x, y) : x = y^2$
Similarly if $P(x) : x^2 \equiv 4$ and $Q(x) : x^2 = -1$, then
$\exists x P(x)$ is true while $\exists x\, Q(x)$ is false.

NOTE: Default domain for numbers is R. Domain may also be specified with quanitifier as follows:
$\exists x \in Z, P(x)$ (here x takes only integer values)

1.7.1 Well-Formed Formulas of Predicate Calculus

A well formed formula of predicate calculus is a string of variables such as $x_1, x_2 .. x_n$, connectives, parenthesis, and quantifiers defined recursively by the following rules:

1. $P(x_1 .. x_n)$ is a wff, when P is a predicate involving n variables $x_1, x_2, .. x_n$.
2. If α is a wff, then $\neg\, \alpha$ is a wff.
3. If α and β are wffs then $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \to \beta$, are also wff.
4. If α is a wff and x is any variable, then $\forall x\, (\alpha)$, $\exists x\, (\alpha)$ are wff.
5. A string is wff if and only if it is obtained by finitely applications rules (1) — (4).

A proposition can be viewed as sentences involving a predicate with O variables. So propositions are wff of predicate calculus by rule (1).

Definition: Let α and β be two predicate formulas in variables $x_1, .. x_n$ and let U be a universe of discourse of α and β. Then, α and β are equivalent to each other over U if, for every possible assignment of values to each variable in α and β, the resulting statements have the same truth values.

We can write $a \equiv b$ over U.

We say that a and b are equivalent to each other $(a \equiv \beta)$ if $a \equiv b$ over U for every universe of discourse U.

Remark: In predicate formulas, the predicate variables may or may not be quantified. We can classify, the predicate variables in a predicate formula, depending on whether they are quantified or not. This gives to the following definitions.

Definition: If a formula of the form $\exists x\, (Px)$ or $\forall x\, P(x)$ occurs as part of a predicate formula α, then such part is called an **x-bound** part of α, and the occurrence of x is called a bound occurrence of x.

An occurence of x is **free** if it is not a bound occurrence.

A predicate variable in α is free if its occurrence is free in any part of α.

In $a = (\exists x_1 P(x_1, x_2)) \wedge (\forall x_2 Q(x_2, x_3))$; for example the occurence of x_1 in $\exists x_1\, P(x_1, x_2)$ is a bound occurence and that of x_2 is free. In $\forall x_2\, Q(x_2, x_3)$, the occurence of x_2 is a bound occurrence the occurence of x_3 in a is free.

NOTE: Quantified parts of predicate formula such as $\forall x\, P(x)$ or $\exists x\, P(x)$ are propositions. We can assign values from the universe of discourse only to free variables in a predicate formula α.

Definition:

1. A predicate formula is **valid** if for all possible assignments of values from any universe of discourse to free variables, the resulting propositions have truth value T.
2. A predicate formula is **satisfiable** if, for some assignments of values to predicate variables the resulting proposition has truth value T.
3. A predicate formula is **unsatisfiable** if, for all possible assignments of values from any universe of discourse to predicate variables, the resulting propositions have truth value F.

Rules of Inference for Predicate Calculus

(*i*) Proposition formula are also predicate formulas.

(*ii*) Predicate formulas where all the variables are quantified are proposition formulas. There fore, all the rules of inference for proposition formulas are also applicable for predicate calculus where ever necessary.

In addition, we have the following 4 laws applicable for predicate calculus.

(US) **Universal Instantiation** (Specification) $\forall x\, A(x) \Rightarrow A(y)$

(ES) **Existential Instantiation** $\exists x\, A(x) \Rightarrow A(y)$

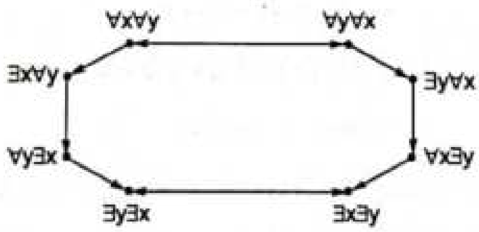
(UG) **Universal Generalisation** $A(x) \Rightarrow \forall y\, A(y)$

(ES) **Existential Generalisation** $A(x) \Rightarrow \exists y\, A(y)$

Equivalence Involving the Two Quantifiers and Valid Implications

1. $\neg(\forall x\, P(x)) \equiv \exists x \neg P(x)$
2. $\neg(\exists x\, Px)) \equiv \forall x \neg P(x)$
3. $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$
4. $\forall x\, P(x) \wedge \forall Q(x) \Rightarrow \forall x(P(x) \vee Q(x))$
5. $\exists x\,(P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$
6. $\exists x\,(P(x) \wedge Q(x)) \Rightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$
7. $\forall x\, P(x) \Rightarrow \exists x\, P(x)$
8. $\forall x\, P \wedge Q(x)) \equiv P \wedge (\forall x\, Q(x))$
9. $\forall x\,(P \vee Q(x)) \equiv P \vee (\forall x\, Q(x))$
10. $\exists x\,(P \wedge Q(x)) \equiv P \wedge (\exists x\, Q(x)$
11. $\exists x\,(P \vee Q(x)) \equiv P \vee (\exists x\, Q(x)$

Graphical Representation of Relation between Sentences Involving Two Quantifiers



Example - 1.6 Let, p: "Maria learns discrete mathematics". q: "Maria will find a good job". Express the statement $p \rightarrow q$ as a statement in English.

Solution:

"If Maria learns discrete mathematics, then she will find a good job" $\{p \rightarrow q\}$.

" Maria will find a good job when she learns discrete mathematics" $\{q$ when $p\}$.

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics" $\{p$ is sufficient for $q\}$.

"Maria will find a good job unless she does not learn discrete mathematics"

$\{q \text{ unless } \neg p \equiv q \vee \neg p \equiv p \rightarrow q\}$.

NOTE: p unless q is same as $p \vee q$.
p nevertheless q is same as $p \wedge q$.

Example - 1.7 Express statement using predicates and quantifiers. "For every person x, if person x is a student in this class then x has studied calculus".

Solution:
We take $C(x)$: "x has studied calculus" consequently if the domain for x consists of the students in the class. We can translate our statement as $\forall x\, C(x)$.
If $S(x)$ represents the statement that person x is in this class, the see that our statement can be expressed as $\forall x\, (S(x) \rightarrow C(x))$.
Note that the statement cannot be expressed as $\forall x\, (S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus.

Example - 1.8 Consider the following formulas:

(*i*) $((p \rightarrow q) \rightarrow (p \wedge q)) \rightarrow p$

(*ii*) $\neg(\forall x (Q(x) \wedge p(x)) \wedge \exists y \neg p(y))$

Which of the above are tautologies?
(a) Only (*i*)
(b) Only (*ii*)
(c) Both (*i*) and (*ii*)
(d) Neither (*i*) nor (*ii*)

Solution:

(*i*)
$$\begin{aligned}((p \rightarrow q) \rightarrow (p \wedge q)) \rightarrow p &= ((\neg p \vee q) \rightarrow (p \wedge q)) \rightarrow p\\ &= ((p \wedge \neg q) \vee (p \wedge q)) \rightarrow p\\ &= \neg[(p \wedge \neg q) \vee (p \wedge q)] \vee p\\ &= ((\neg p \vee q) \wedge (\neg p \vee \neg q)) \vee p\\ &= (\bar{p}+q)(\bar{p}+\bar{q})+p\\ &= \bar{p}+\bar{p}\,\bar{q}+\bar{p}\,q+q\,\bar{q}+p\\ &= \bar{p}+p=1=\text{True}\end{aligned}$$

(*ii*)
$$\begin{aligned}\neg(\forall x(q(x) \wedge p(x)) \wedge \exists x \neg p(x)) &\equiv \exists x(\neg q(x) \vee \neg p(x)) \vee \forall x\, p(x)\\ &\equiv \exists x(\neg q(x)) \vee \exists x(\neg p(x)) \vee \forall x\, p(x)\\ &\equiv \exists x(\neg q(x)) \vee \neg(\forall x p(x)) \vee \forall x\, p(x)\\ &\equiv \exists x(\neg q(x)) \vee 1\\ &\equiv 1\\ &\equiv \text{True}\end{aligned}$$

$\therefore$ Both (*i*) and (*ii*) are tautologies.

Summary

- Two expression are logically equivalent if each one always has the same truth value as the other.
- $\oplus$ (EX – OR) is commutative and associative, (NAND) and (NOR) are both commutative but not associative. $P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$
- A wff is not a proposition, but if we substitute the proposition in place of propositional variable, we get a proposition e.g., $(\neg P \wedge Q) \leftrightarrow Q$ is a wff.
- When it is not clear whether a given formula is tautology, we can construct a truth table and verify that the truth value is T for all possible combinations of truth value of the propositional variables appearing in given formula.
- A contradiction (or absurdity) is a wff whose truth value is F for all possible assignments of truth values to the propositional variables.
- A contingency is a wff which is neither a tautology nor a contradiction. In otherwords, a contingency is a wff which is sometimes true of sometimes false.
- Two wff a and b in propositional variables $P_1, P_2, ...P_n$ are equivalent if the formula $a \leftrightarrow b$ is a tautology.
- For a given wff the PDNF form is unique the PCNF form is unique, if PDNF form or PCNF of 2 wffs are same, they are equivalent.
- Quantified parts of predicate formula such as $\forall x\, P(x)$ or $\exists x\, P(x)$ are propositions. We can assign values from the universe of discourse only to free variables in a predicate formula α.

Student's Assignments

Q.1 The logical expression $((P \wedge Q) \Rightarrow (R' \wedge P)) \Rightarrow P$
(a) a tautology
(b) a contradiction
(c) a contingency
(d) All the above

Q.2 The principal conjunctive normal form is
(a) sum of products
(b) product of sums
(c) sum of max-terms
(d) product of max-terms

Q.3 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I
A. Associative law
B. Absorption law
C. Demorgans law
D. Commutative

List-II
1. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
2. $P \vee Q \equiv Q \vee P$
3. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
4. $P \vee (P \wedge Q) \equiv P$

Codes:

	A	B	C	D
(a)	1	2	3	4
(b)	4	3	1	2
(c)	1	4	3	2
(d)	2	1	4	2

Q.4 Consider the following statements:
$S_1 : R \vee (P \vee Q)$
is a valid conclusion from the premises
$P \vee Q,\ Q \rightarrow R,\ P \rightarrow M$ and $\neg M$
$S_2 : a \rightarrow b,\ \neg(f \vee c) \Rightarrow \neg b$
then
(a) S_1 is true and S_2 is invalid
(b) S_1 is false and S_2 is invalid
(c) Both are true
(d) Both are false

Q.5 The following propositional statement is
$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$
(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) not decidable

Q.6 Identify the correct translation into logical notation of the following assertion
"All connected bipartite graphs are nonplanar"

(a) $\forall x \begin{bmatrix} \sim\text{connected}(x) \vee \sim \text{biparite} \\ (x) \wedge \sim \text{planar}(x) \end{bmatrix}$

(b) $\forall x \begin{bmatrix} \sim\text{connected}(x) \vee \sim \text{biparite} \\ (x) \vee \sim \text{planar}(x) \end{bmatrix}$

(c) $\forall x \begin{bmatrix} \sim\text{connected}(x) \wedge \sim \text{biparite} \\ (x) \wedge \sim \text{planar}(x) \end{bmatrix}$

(d) $\forall x \begin{bmatrix} \sim\text{connected}(x) \wedge \sim \text{biparite} \\ (x) \vee \sim \text{planar}(x) \end{bmatrix}$

Q.7 Which of the following statements are true?
(*i*) It is not possible for the propositions $P \vee Q$ and $\neg P \vee \neg Q$ to be both false, to be both false.
(*ii*) It is possible for the proposition $P \to (\neg P \to Q)$ to be false.
(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

Q.8 Which of the following statements are true?
(*i*) $((P \to Q) \to R) \to ((R \to Q) \to P)$ is a tautology
(*ii*) Let A, B be finite sets, with $|A| = m$ and $|B| = n$. The number of distinct functions $f : A \to B$ is there from A to B is m^n.
(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

Q.9 State whether the following statements are true or false?
(*i*) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ always holds, for all proposition P, Q.
(*ii*) $((P \vee Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow (P \vee Q))$ always holds, for all propositions P.
(a) (*i*) is true, (*ii*) is false
(b) Both (*i*) and (*ii*) are true
(c) (*i*) is false, (*ii*) is true
(d) Both (*i*) and (*ii*) are false

Q.10 Which of the following is tautology?
(a) $x \vee y \to y \wedge z$
(b) $x \wedge y \to y \vee z$
(c) $x \vee y \to y \to z$
(d) $x \to y \to (y \to z)$

Q.11 Suppose
$P(x)$: x is a person.
$F(x, y)$: x is the father of y.
$M(x, y)$: x is mother of y.
What does the following indicates
$(\exists z)\ (P(z) \wedge F(x, z) \wedge M(z, y))$
(a) x is father of mother of y
(b) y is father of mother of x
(c) x is father of y
(d) None of the above

Q.12 Give the converse of "If it is raining then I get wet".
(a) If it is not raining then I get wet
(b) If it is not raining then I do not get wet
(c) If it get wet then it is raining
(d) If I do not get wet then it is not raining

Q.13 Which of the following is true?
(a) $\neg(p \Rightarrow q) \equiv p \wedge \neg q$
(b) $\neg(p \leftrightarrow q) \equiv ((p \vee \neg q) \vee (q \wedge \neg p))$
(c) $\neg(\exists x\ (p(x) \Rightarrow q(x))) \equiv \forall x\ (p(x) \Rightarrow q(x))$
(d) $\exists x\ p(x) \equiv \forall x\ p(x)$

Answer Key:

1. (c)	2. (d)	3. (c)	4. (a)	5. (a)
6. (b)	7. (a)	8. (d)	9. (c)	10. (b)
11. (a)	12. (c)	13. (a)		

Student's Assignments Explanations

1. (c)

The logical expression

$((P \wedge Q) \Rightarrow (R' \wedge P)) \Rightarrow P$

can be converted in Boolean Algebra notation as,

$(pq \Rightarrow r'p) \Rightarrow p$

$\equiv (pq)' + r'p \Rightarrow p$

$\equiv (p' + q' + r'p) \Rightarrow p$

$\equiv ((p' + r'p) + q') \Rightarrow p$

$\equiv ((p' + p)\cdot(p' + r') + q') \Rightarrow p$

$\equiv (p' + r' + q') \Rightarrow p$

$\equiv (p' + r' + q')' + p \equiv prq + p$

$\equiv p$

$\therefore$ The given expression is a contingency.

4. (a)

$S_1 : P \vee Q,\ Q \rightarrow R,\ P \rightarrow M,\ \sim M \Rightarrow R \vee (P \vee Q)$

In boolean algebra notation the above expression is written as

$(p + q)\cdot(q + r')\cdot(p + m')\cdot m' \Rightarrow r + p + q$

$\equiv (q + pr')(m') \Rightarrow r + p + q$

$\equiv qm' + pr'm' \Rightarrow r + p + q$

$\equiv (qm' + pr'm') + r + p + q$

$\equiv (q' + m)(p' + r + m)r + p + q$

$\equiv q'p' + q'r + q'm + mp' + mr + m + r + p + q$

(by absorption law) $\equiv q'p' + r + m + p + q$

$\equiv (p + p')\cdot(p + q') + r + m + q$

$\equiv p + q' + r + m + q$

$\equiv p + r + m + 1 \equiv 1$

$\therefore S_1$ is true

$S_2 : a \rightarrow b, \neg(f \vee c) \Rightarrow \neg b$

In boolean Algebra notation

$S_2 \equiv (a \rightarrow b)\cdot(f \vee c)' \Rightarrow b'$

$\equiv (a' + b)\cdot(f'c') \Rightarrow b'$

$\equiv [(a' + b)\cdot(f'c')] + b'$

$\equiv (a' + b)' + (f'c')^1 + b'$

$\equiv ab' + f + c + b'$

$\equiv f + c + b'$

which is a contingency

$\therefore S_2$ is invalid.

5. (a)

$[(p \rightarrow r) \vee (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$

$\equiv (p' + r)(q' + r) \rightarrow (p + q)' + r$

$\equiv (r + p'q')' \rightarrow (p + q)' + r$

$\equiv (r + p'q')' + (p + q)' + r$

$\equiv r'(p'q')' + p'q' + r$

$\equiv r'(p + q)' + p'q' + r$

$\equiv r'p + r'q + p'q' + r$

$\equiv (r + r')\cdot(r + p) + r'q + p'q'$

$\equiv r + p + r'q + p'q'$

$\equiv (r + r')(r + q) + (p + p')(p + q')$

$\equiv r + q + p + q'$

$\equiv r + p + 1 \equiv 1$

$\therefore$ tautology

6. (b)

The correct translation is

$\forall x[(\text{connected}(x) \wedge \text{bipartite}(x)) \rightarrow \sim \text{planar}(x)]$

however, since $p \rightarrow q \equiv \sim p \vee q$, we can write the above expression also as,

$\forall x[\sim\text{connected}(x) \vee \sim\text{bipartite}(x) \vee \sim \text{planar}(x)]$

7. (a)

If $P \vee Q$ is false, then both P and Q are false.

So, $\neg P \vee \neg Q \equiv \neg F \vee \neg F \equiv T \vee T \equiv T$

$\therefore$ (*i*) is true

Consider (*ii*)

$$\begin{aligned} P \rightarrow (\neg P \rightarrow Q) &\equiv P \rightarrow (P^1 \rightarrow Q) \\ &\equiv P \rightarrow P + Q \\ &\equiv P^1 + P + Q \equiv 1 + Q \equiv 1 \end{aligned}$$

It is a tautology, So (*ii*) is false.

8. (d)

(*i*) Using boolean algebra, we can shown that the given expression reduces to $P + R' + Q'$ which is not a tautology.

(*ii*) For each element $a \in A$, we have n possible choices for value of $f(a)$. Thus there are n^m possible functions.

9. (c)

(*i*) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$

$\equiv (P' + Q) \rightarrow (Q' + P)$

$\equiv (P' + Q)' + Q' + P$

$\equiv PQ' + Q' + P \equiv P + Q'$

Since P + Q′ is a contingency and not a tautology (i) is false

(*ii*) $((P \vee Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow (P \vee Q))$

$\equiv ((P + Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow P + Q)$

$\equiv (P + Q)' + \Rightarrow Q \Rightarrow Q' + P + Q$

$\equiv P'Q' + Q \Rightarrow 1 + P$

$\equiv (Q + P')\cdot(Q + P') \Rightarrow 1$

$\equiv (Q + P') \Rightarrow 1$

$\equiv (Q + P')' + 1 \equiv 1$

$\therefore$ (*ii*) always holds.

10. (b)

(*i*) $x \vee y \rightarrow y \wedge z \equiv x + y \rightarrow yz$

$\equiv (x + y)' + yz$

$\equiv x'y' + yz$

$\not\equiv 1$

$\therefore$ not a tautology

(*ii*) $x \wedge y \rightarrow y \vee z \equiv xy \rightarrow y + z$

$\equiv (xy)' + y + z$

$\equiv x' + y' + y + z$

$\equiv 1 + x' + z$

$\equiv 1$

$\therefore$ It is a tautology

(*iii*) $x \vee y \rightarrow (y \rightarrow z) \equiv x + y \rightarrow (y \rightarrow z)$

$\equiv (x + y)' + (y \rightarrow z)$

$\equiv x'y' + y' + z$

$\equiv y' + z$

$\not\equiv 1$

$\therefore$ not a tautology

(*iv*) $x \rightarrow y \rightarrow (y \rightarrow z) \equiv x' + y \rightarrow (y' + z)$

$\equiv (x' + y)' + y' + z$

$\equiv xy' + y' + z$

$\equiv y' + z$

$\not\equiv 1$

$\therefore$ not a tautology

13. (a)

Option (a) $\neg(p \Rightarrow q) \equiv p \wedge \neg q$ is true, since

$\neg(p \Rightarrow q) \equiv \neg(p' + q) \equiv pq' \equiv p \wedge \neg q$

CHAPTER 02

Combinatorics

2.1 Introduction

Objects (or things) can be arranged in many ways. Suppose there are three objects marked a, b, c on a table from these, two objects can be selected at a line in three different ways as $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. In this way selection of two objects from three objects in three ways is called **Combinations.**

The above selection ab, ac can also be arranged as ab, ba, ac, ca, bc, cb. We can understand that two objects can be selected from three objects and arranged in six ways. These arrangements are called **Permutations.**

Fundamental Concepts

If A is a finite set, then the number of different elements in A is denoted by $n(A)$.e.g.,

If A = $\{2, 5, 7\}$ then $n(A) = 3$

If C = ϕ then $n(C) = 0$

Let us assume that there are three routes say a_1, a_2, a_3 from Delhi to Noida and there are two routes, say b_1, b_2, from Noida to Agra. It may be written as:

$A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$

$n(A) = 3$; $n(B) = 2$

Now we can match the route a_1 from D to N with two routes b_1, b_2 from N, A

i.e., (a_1, b_1), (a_1, b_2)

Similarly the remaining routes can be written as

(a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2).

So to travel from D to A via N, there are 6 different routes

(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_2).

These 6 ways are nothing but the elements of the cartesian product of the two sets A and B.

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$

$n(A \times B) = 6 = 2 \times 3 = n(A) \times n(B)$.

Fundamental Multiplication Principle of Counting

"Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible out comes and if for each outcome of experiment 1 there are possible out comes of experiment 2, then together there are mn possible outcomes of the two experiments".

The Generalized basic Multiplication Principle of Counting

"If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment and if for each of the possible outcomes the first two experiments there are n_3 possible outcomes of the third experiment and so on, then there is a total of $n_1 \times n_2 \dots n_r$ possible outcomes of the r experiments".

Keywords to distinguish permutations from combinations:

Permutations : ordered, arrangement, sequence

Combinations : unordered, selection, set

Some useful properties from Number theory used in Combinatorics:

1. *Method for finding the number of positive divisors of a positive integer n* : If a positive integer n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \dots$ where p_1, p_2 etc. are distinct prime numbers, then the number of positive divisors of n is given by the formula $(n_1 + 1)(n_2 + 1)\dots$

   For example, the number 80 can be broken as $80 = 2^4 \times 5^1$. So the number of positive divisors of 80 is given by $(4 + 1)(1 + 1) = 10$.

2. *Method for finding the number of numbers from 1 to n, which are relatively prime to n* : The number of numbers from 1 to n, which are relatively prime to n i.e., gcd $(m, n) = 1$, is given by the Euler Totient function $\phi(n)$. If n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \dots$ where p_1, p_2 etc. are distinct prime numbers, then $\phi(n) = \phi(p_1^{n_1})\,\phi(p_2^{n_2})\dots$ then by using the property

   $$\phi(p^k) = p^k - p^{k-1}.$$

   we can find each of $\phi(p_1^{n_1}), \phi(p_2^{n_2})\dots$ etc.

   For *example*, the number of numbers from 1 to n, which are relatively prime to 80 can be found as follows: Since $80 = 2^4 \times 5^1$

The number of numbers from 1 to n, which are relatively prime to $80 = \phi(80) = \phi(2^4) \times \phi(5^1)$

Now $\phi(2^4) = 2^4 - 2^3 = 16 - 8 = 8$

Similarly, $\phi(5^1) = 5^1 - 5^0 = 5 - 1 = 4$

So, $\phi(80) = 8 \times 4 = 32$

2.2 Permutations

Permutations with no Repetitions

When we select objects from a set consisting of n distinct objects taking each object exactly once (no repetition), and then arrange them in a straight line, this situation is called permutations with no repetition.

The formula for counting this is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \dots (n-r+1)$$

Example-2.1 How many 2 letter passwords are there using the letters {a, b, c} if no letter is allowed to be used more then once?

Solution:

$$^{3}P_2 = \frac{3!}{(3-2)!} = 3 \times 2 = 6$$

The 6 permutations are ab, ba, ac, ca, bc cb.

Example-2.2 How many ways can four (distinct) dolls be arranged in a straight line?

Solution:

$$^{4}P_4 = \frac{4!}{(4-4)!} = 4! = 4 \times 3 \times 2 \times 1 = 24 \text{ ways.}$$

Alternately more problems can be solved by box method, which is more general and more powerful. For example arranging 4 dolls can be thought of as filling 4 boxes corresponding to position of the dolls.

The first box can be filled in 4 ways. Since repetition is not allowed, the second box can be filled only in 3 ways.

The third box in 2 ways and last box in only 1 way.

$\therefore$ Total arrangements = $4 \times 3 \times 2 \times 1 = 24$ ways.

Permutations with Unlimited Repetition

When we select an object, from a set of distinct objects, taking each object any number of times (unlimited repetition) and arrange them in a straight line, this situation is called permutation with unlimited repetition. The formula for counting this is n^r.

Example-2.3 How many 2 letter passwords can be made from {*a*, *b*, *c*}, if a letter can be used any number of times?

Solution:

$$3^2 = 9 \text{ passwords}$$

The nine permutations are : *aa*, *ab*, *ac*, *ba*, *bb*, *bc*, *ca*, *cb*, *cc*.

NOTE

For objects such as passwords & number, if nothing is mentioned regarding repetition, the default assumption is that unlimited repetition is allowed.

Using box method we make the password by filling 2 boxes. Each can be filled in 3 ways (since repetition allowed) $3 \times 3 = 9$ passwords.

Example-2.4 If there are 10 multiple choice question with four choices for each question, How many answer sheets are possible?

Solution:

5^{10} answer sheets. We can think of each of the question as a box and each of the 10 boxes can be filled is 5 ways (Choice *a*, *b*, *c* or *d* or leave it blank):

$\therefore$ The number of ways of filling up all 10 boxes is $5 \times 5 \times 5 \times \ldots 10 \text{ times} = 5^{10}$

The box method is very powerful for use in all permutation problems (with repetition or without repetition)

Permutations with Limited Repetitions

There are situations where the objects can be used only a limited number of times, due to limited availability. In such cases if n_1 repetition of object 1, n_2 repetition of object 2, ..., n_r repetition of object r is allowed, and total number of objects = n, then the number of permutations is given by

$$P(n; n_1; n_2 ... n_r) = \frac{n!}{n_1! n_2! .... n_r!}$$

Example - 2.5 How many ways are there to arrange 3 indentical blue flags, 2 indentical red flags and 3 indentical yellow flags in a straight line?

Solution:

$$\frac{8!}{3!2!3!} \text{ ways}$$

Example - 2.6 How many ways are there to arrange the letters of the village "VALLIKKAVU". In other words, how many distinct anagrams of VALLIKKAVU are there? There are 2 V's, 2 A's, 2L's, 2K's, 1I, 1U

Solution:

$$\frac{10!}{2!2!2!2!1!1!} = \frac{10!}{(2!)^4}$$

Circular Permutations

The number of ways of arranging n objects in a circle is $(n-1)!$ This is less than $n!$, since in a circle many of the linear permutations become indistinguishable except for rotation.

This formula is derived by putting the first object in any one of the n positions on a circle and then the remaining $(n-1)$ position can be filled by $(n-1)$ objects in $(n-1)!$ ways.

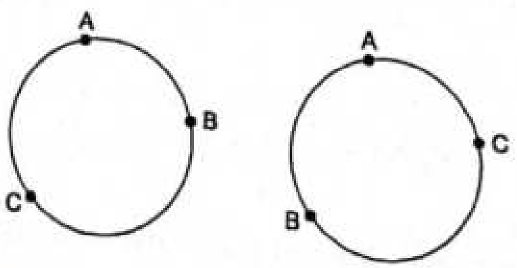
$\therefore 1 \times (n-1)! = (n-1)!$ ways for circular permutation.

Example - 2.7 How many ways are there to arrange 3 children in a circle?

Solution:

$$(3-1)! = 2! = 2 \text{ ways}$$

Let the children A, B, C. The two arrangements on shown below.



If clockwise and anti-clockwise arrangements are also considered to be same, then no of circular permutations (disregarding clockwise and anti-clockwise arrangements) is $\frac{(n-1)!}{2}$

i.e. in the 3 child example; $\therefore$ disregarding clockwise and anticlockwise variations, $\frac{(3-1)!}{2} = 1$ ways only.

2.3 Combinations

Combinations with no Repetitions

Let n, r are integers. If $n \geq r$, for the set of n elements a sub set of r elements is called a "Combination". Therefore a combination is an unordered selection of r elements from a set of n elements.
The number of combination of r elements selected from n elements is denoted by ^nC_r or (n, r).
Here the same element cannot be selected more than once (no repetitions allowed)

Formula: $$\boxed{^nC_r = \frac{n!}{r!(n-r)!}}$$

Results:
1. $^nC_r = {^nC_{n-r}}$
2. $^nC_r = 1$
3. $^nC_0 = 1$

Pascal's formula: $^nC_r = {^{n-1}C_{r-1}} + {^{n-1}C_r}$

Combinations with Unlimited Repetitions

The number of combinations of r elements selected from n elements, when same element can be selected any number of times is given by $^{n-1+r}C_r$ which can also be written as $^{n-1+r}C_{n-1}$.

Example-2.8 10 CD's are available for discount in a music shop. How many ways can a shopper select 3 CD's it if he can choose same CD any number of times? Here; $n = 10$, $r = 3$.

Solution:

$$^{10-1+3}C_3 = {^{12}C_3} = \frac{12 \times 11 \times 10}{1 \times 2 \times 3} = 220 \text{ ways}$$

This same formula can be used in distribution problems involving indentical objects:

Example-2.9 How many ways can we distribute 10 identical balls into 4 (distinct) boxes there:

r = Number of objects to be distributed = 10
n = Number of boxes = 4

Solution:

The same problem can also be used to find the number of non negative integral solutions to the equation

$$x_1 + x_2 + x_3 \ldots + x_n = r$$

Where, $x_1 \geq 0, x_2 \geq 0, \ldots x_n \geq 0$

Solution using $^{n-1+r}C_r$;

Notice that question can also be reduced to non negative integral solution problem as follows.
Let x_1, x_2, x_3, x_4 be the number of balls put into boxes 1, 2, 3 and 4 respectively. Since we have totally 10 balls.

$\therefore \quad x_1 + x_2 + x_3 + x_4 = 10$

Also we can only put 0 or more balls in each box.

$\therefore x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

$$^{n-1+r}C_r = {^{4-1+10}C_{10}} = {^{13}C_{10}} = {^{13}C_3} \text{ ways}$$

Distribution of Distinct Objects

When the problem involved distributing distinct objects to distinct people the formula to be used is $\frac{n!}{n_1!n_2!...n_r!}$. This is the same formula used in permutation with limited repetition, but the distribution problem here in different, although we are using the same formula.

Example-2.10 How many ways can we deal a deck of 52 cards to 4 people giving equal no of card to each?

Solution:

$$\frac{52!}{13!13!13!13!} = \frac{52!}{(13!)^4}$$

Example-2.11 How many ways can we divide a group of 10 boys into 3 teams such that team 1 has 2 boys, team 2 has 5 boys and team 3 has 3 boys?

Solution:

$$\frac{10!}{2!5!3!} \text{ ways}$$

Notice that in both the above problem, within the teams there is no particulars order of the boys, although the teams themselves than order (as in team 1, 2 and 3). Similarly also the card distribution problem. Although the 4 people are distinct, the 13 cards given to a person is in no particulars order.

∴ These problems are called ordered partitions.

The general formula for ordered partition of n objects of the type $(n_1, n_2, n_3, ..., n_r)$ is given by

$$P(n, n_1, n_2, n_3, ... n_r) = \frac{n!}{n_1!n_2!n_3!...n_r!}$$

(Which is really same as the permutation with limited repetition formula).

Now if the type of partition is unspecified we must take every possible type of partitioning into account.

Now the number of ways of distributing n distinct objects to r distinct people is

$$= \Sigma P(n, n_1, n_2, n_3 ... n_r) = n_r$$
$$n_1 + n_2 + ... n_r = n$$

(which is really the same as formula for permutations with unlimited repetitions).

Example-2.12 How many ways can 4 (distinct) dolls be distributed amongst 3 (distinct) children? Here $n = 4$, $r = 3$

Solution:

Alternatively we can think of each 4 dolls as boxes each of which can be issued 3 ways (3 children)

$$3 \times 3 \times 3 \times 3 = 3^4 \text{ ways} = 81 \text{ ways}$$

Notice that a child may receives 0 or more dolls.

If the above problem is changed follows: How many ways can 4 dolls be given to 3 children such that first child gets 2 dolls, second 1 doll and third 1 doll, then the answer would be $P(4; 2, 1, 1) = \frac{4!}{2!1!1!}$

(Since the method of distributing the dolls is specified in this case).

Unordered Partition

In the above problem the children were distinct.
But consider the problem of dividing 6 dolls into 3 groups of 2 dolls each.
Now this is a problem of unordered partition.

The number of ways of dividing 6 dolls into 3 groups of 2 dolls each are $\frac{6!}{(2!)^3 3!}$.

The general formula for dividing n objects into t groups of r objects is given by $\frac{n!}{(r!)^t t!}$.

We have to divide by t! the ordered the ordered partition formula since t!, permutations are indistinguishable as the groups or (cells as they are called) are indistinguishable in order.

2.4 Binomial Identities

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots \binom{n}{k}x^k + \ldots \binom{n}{n}x^n \qquad \text{...(2.1)}$$

There are some basic properties of binomial coefficients:

1. **Symmetry identity:**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \qquad \text{...(2.2)}$$

This identity says that the number of ways to select a subset of k objects out of a set of n objects is equal to the number of ways to select a subset of $(n-k)$ of the objects to set aside, for rejection.

2. **Pascal's identity:**

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{...(2.3)}$$

The left hand side counts the number of ways of selecting to k things from n things.
The r. h. s. counts the same as a sum of two types of selections.
One selection ${}^{n-1}C_k$ is the number of ways of selecting the k things so that a particular element say "x" is excluded and the other selection ${}^{n-1}C_{k-1}$ is the number of ways of selecting the k things so that same particular element "x" is always included.
In addition to these identities there are a few more important identities to remember:

3. **Newton's identity:**

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \qquad \text{...(2.4)}$$

The left hand side counts the ways to select a group of k people choosen from a set of n people and then to select a subset of m leaders from the group of k people.
Equivalently, as counted on the right side, we could first select the subset of m leaders from the set of n people and then select the remaining $k-m$ non leading members of the group from $(n-m)$ people.

4. Row summation:

$$\sum_{k=0}^{n} {}^nC_k = {}^nC_0 + {}^nC_1 + {}^nC_2 \dots + {}^nC_n = C(n, 0) + C(n, 1) + C(n, 2) \dots C(n, n) = 2^n$$

5. Alternating sign row summation:

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 \dots (-1)^n\, {}^nC_n = 0$ or

$({}^nC_0 + {}^nC_2 + {}^nC_4 \dots +) = ({}^nC_1 + {}^nC_3 + {}^nC_5 \dots) = 2^{n-1}$

i.e. even summation = odd summation = 2^{n-1}

Example: Row summation: ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 = 16$

Even summation and Odd summation:

${}^4C_0 + {}^4C_2 + {}^4C_4 = {}^4C_1 + {}^4C_3 = 2^{4-1} = 8$

The row summation identity can be understood as RHS = 2^n = no of subsets of a set of n elements. Now, LHS is the same counted as no of 0 element subsets (= ${}^nC_o$) + No of 1 element subsets (= ${}^nC_1$) + no of 2 element subsets (= ${}^nC_2$) ... and so m + no of n element subsets (= ${}^nC_r$). Obviously these two counts should be equal. i.e. LHS = RHS.

6. Column summation:

$$\sum_{k=0}^{n} {}^kC_r = \sum_{k=r}^{n} {}^kC_r = {}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r \dots + {}^nC_r = {}^{n+1}C_{r+1}$$

Example: ${}^2C_2 + {}^3C_2 + {}^4C_2 = {}^5C_3$

7. Vandermondes indentity:

${}^{n+m}C_r = {}^nC_0{}^mC_r + {}^nC_1{}^mC_{r-1} + {}^nC_2{}^mC_{r-2} \dots {}^nC_r{}^mC_0$

This indentity can be understood as selecting r objects from a group of $n + m$ objects as written in LHS.

RHS counts the same as, selecting 0 objects from group of n objects and r objects from group of m objects or selecting 1 object from group of n objects and $r - 1$ objects from group of m objects and so on ...

$\therefore$ LHS = RHS

8. Row square summation:

Special case of vandermodes indentity is row square summation obtained by putting $n = m$ and $r = n$ in vandermondes identity.

$$\begin{aligned} {}^{2n}C_n &= {}^nC_0{}^nC_n + {}^nC_1{}^nC_{n-1} + \dots {}^nC_n{}^nC_0 \\ &= {}^nC_0{}^nC_0 + {}^nC_1{}^nC_1 + \dots {}^nC_n{}^nC_n \\ &= {}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 \dots + {}^nC_n^2 \end{aligned}$$

$$\therefore \quad \sum_{k=0}^{n} {}^nC_k^2 = {}^{2n}C_n$$

9. Another special case of Vandermonde's identity is obtained by putting $r = n$

$${}^{n+m}C_n = {}^nC_0\,{}^mC_n + {}^nC_1\,{}^mC_{n-1} + \dots {}^nC_n\,{}^mC_0$$

10. Yet another binomial indentity is written as follows:

$$\therefore \quad \sum_{r=1}^{n} r\,{}^nC_r = 1\,{}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 \dots + n\,{}^nC_n = n\,2^{n-1}$$

2.4.1 Multinomial Coefficients

$$(x_1 + x_2 + ... + x_t)^n = \Sigma P(n, q_1, q_2, ... q_t) = x_1^{q_1} x_2^{q_2} x_3^{q_3} ... x_t^{q_t} \; q_1 + q_2 ... + q_t = n$$

The coefficient of the term $x_1^3 x_2^4 x_3^2$ in the expansion of $(x_1+x_2+x_3)^9$, would be exactly $P(9; 3, 4, 2) = \frac{9!}{3!4!2!}$

$$\Sigma P(9, q_1, q_2, ... q_t) = 3^9 \; q_1 + q_2 ... q_t = n$$

e.g. $\Sigma P(q, q_1, q_2, ... q_3) = 3^9 \; q_1 + q_2 + q_3 + = 9$ (summation of all multinomial coefficients)

2.5 Generating Functions

Consider a sequence $(a_0, a_1, a_2 ..., a_r) = \{a_r\} = \{a_r\}_r^\infty = 0$ of real numbers.

We may write a power series of the form $A(X) = a_0X^0 + a_1X^1 + a_2X^2 + a_3X^3 ... + a_rX^r = \sum_{r=0}^{\infty} a_rX^r$

Now, A(X) is called as the generating function corresponding to the sequence a_r.

Generating functions can be successfully used to solve counting problems involving constraints such as in finding the number of ways to fill r balls into n boxes with upper constraints on the no of balls that can go into each box.

The following identities are useful:

$$\sum_{r=0}^{n} X^r = 1 + X + X^2 + ... X^n = \frac{1-X^{n+1}}{1-X} \quad ...(2.5)$$

$$\sum_{r=0}^{\infty} X^r = 1 + X + X^2 + ... = \frac{1}{1-X} \quad ...(2.6)$$

$$\sum_{r=0}^{n} {}^nC_rX^r = 1+\binom{n}{1}X+\binom{n}{2}X^2+...\binom{n}{n}X^n = (1+X)^n \quad ...(2.7)$$

$$\sum_{r=0}^{\infty} {}^{n-1+r}C_rX^r = \frac{1}{(1-X)^n} \quad ...(2.8)$$

In the equations (2.6) and (2.7) if we replace x by –x we can get.

$$\sum_{r=0}^{\infty} (-1)^r X^r = 1 - X + X^2 - X^3 ... = \frac{1}{1+X}$$

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_rX^r = 1-\binom{n}{1}X+\binom{n}{2}X^2-\binom{n}{3}X^3 ... (-1)^n\binom{n}{n}X^n = (1-X)^n$$

Also : if in equation (2.6), we replace X by aX and $-aX$ we can get

$$\sum_{r=0}^{\infty} a^rX^r = 1 + aX + a^2X^2 + a^3X^3 = \frac{1}{1-aX}$$

$$\sum_{r=0}^{\infty} (-1)^r a^rX^r = 1 - aX + a^2X^2 - a^3X^3 ... = \frac{1}{1+aX}$$

NOTE

- When generating function is written as a series $1 + X + X^2 + \ldots$ it said to be in open form and when it written instead as $\frac{1}{1-X}$, it is said to be in closed form.
- Notice that the sequence corresponding to (2.1) $\sum_{r=0}^{n} X^r$ is $[a_r]_{r=0}^{n} = 1$.

The sequence corresponding to (2.6) is $[a_r]_{r=0}^{\infty} = 1$.

The sequence corresponding to (2.7) is $\left[{}^nC_r\right]_{r=0}^{n}$ and the sequence corresponding (2.8) is $\left\{\left({}^{n-1+r}C_r\right)\right\}_{r=0}^{\infty}$.

Example - 2.13 Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \ldots)^5$. What is the coefficient of x^r?

Solution:

To simplify the expression, we extract x^2 from each polynomial factor and then apply identity (2.6).

$(x^2 + x^3 + x^4 + x^5)^5 = [x^2(1 + x + x^2 + \ldots)]^5 = x^{10}(1 + x + x^2 + \ldots)^5 = x^{10}\,\frac{1}{(1-x)^5}$.

Thus the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \ldots)^5$ is the coefficient of x^{16} in $x^{10}(1-x)^{-5}$. But the coefficient of x^{16} in this latter expression will be the coefficient of x^6 in $(1-x)^{-5}$.

Now, we need to find coefficient of X^6 in $\frac{1}{(1-X)^5}$.

We expand $\frac{1}{(1-X)^5}$ using identity (4)

$$\frac{1}{(1-X)^n} = \sum_{r=0}^{\infty} {}^{n-1+r}C_r X^r$$

$$\therefore \quad \frac{1}{(1-X)^5} = \sum_{r=0}^{\infty} {}^{5-1+r}C_r X^r = \sum_{r=0}^{\infty} {}^{r+4}C_r X^r$$

Now the coefficient of X^r in this expansion is ${}^{r+4}C_4$. Therefore, coefficient of X^6 in this expansion would be ${}^{6+4}C_4 = {}^{10}C_4$. Which is the required answer for this problem.

NOTE: Ball in the box problems, when balls are indistinguishable, along with upper unstraints on the no of balls that can be put into a box, can be effectively converted into a problem of finding coefficient of some power of X in the expansion of a corresponding generating function.

2.6 Summation

Basic Results

(i) $1 + 2 + 3 \ldots + n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(*iii*) $1^3+2^3+3^3+ \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

In general sum to n terms is denoted as S_n where the summation is of the sequence u_n.

i.e. $S_n = t_1 + t_2 + \dots + t_n$

Arithmetic Progression

$a, a + d, a + 2d, \dots a + (n-1)d + \dots$ If the sequence is an A. P. (arithmetic progression) then, $t_n = a + (n-1)d$ (n^{th} term of sequence).

$$S_n = a + (a + d) \dots = \frac{n}{2}(2a + (n-1)d) \text{ (sum up to } n \text{ terms)}$$

$$= \frac{n}{2}(a + l) \qquad \text{(where } l \text{ is the last term)}$$

Geometric Progression

$a, ar, ar^2, \dots ar^{n-1} + \dots t_n = ar^{n-1}$

$$S_n = a + ar + ar^2 \dots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}$$

$$S_\infty = \frac{a}{1-r} \text{ [if } |r| < 1]$$

Example: A. P : 1, 3, 5, 7 ...

Here,

$$a = 1$$
$$d = 3 - 1 = 5 - 3 = 2$$
$$u_{20} = 1 + (20 - 1)\,2 = 39$$
$$S_{20} = \frac{20}{2}(2\times1+(20-1)\times2) = 10\,(40) = 400$$

Example: G. P. : 1, 2, 4, 8, 16, ...

Here,

$$a = 1, r = \frac{2}{1} = \frac{4}{2} = 2$$
$$u_{10} = 1 \times 2^{(10-1)} = 2^9 = 51$$
$$S_{10} = \frac{a(r^n - 1)}{r-1} = \frac{1\times(2^{10}-1)}{2-1} = 2^{10} - 1 = 1023$$

Example: Infinite G. P. : $1, \frac{1}{2}, \frac{1}{4}, \dots$

Here,

$$a = 1, r = \frac{1}{2}, \text{ since } |r| < 1,$$

$\therefore$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

Method of Summation when u_n is a product of r successive terms of an A. P.

Example: 1.4 + 4.7 + 7.10 + ...

The first factor is in A.P. So is the second factor.

$$n^{th} \text{ term of } 1^{st} \text{ factor} = 1 + (n-1)3 = 3n-2$$

$$n^{th} \text{ term of } 2^{nd} \text{ factor} = 4 + (n-1)3 = 3n+1$$

$$\therefore \quad u_n = (3n-2)(3n+1) = 9n^2 - 3n - 2$$

$$\sum u_n = 9\sum n^2 - 3\sum n - 2\sum 1 = \frac{9n(n+1)(2n+1)}{6} - 3\frac{(n)(n+1)}{2} - 2n$$

$$= n(3n^2 + 3n - 2)$$

Method of Summation when u_n is the reciprocal of the product of r successive terms of an A.P.

Example - 2.14 Find the sum to n terms and if possible to infinity the series $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots$

Solution:

Here, $$u_n = \frac{1}{(3n-2)(3n+1)}$$

Let $$V_n = \frac{1}{3n+1}, \text{ then } V_{n-1} = \frac{1}{3n-2}$$

$$V_n - V_{n-1} = \frac{1}{3n+1} - \frac{1}{3n-2} = \frac{-3}{(3n+1)(3n-2)} = -3u_n$$

$$\therefore \quad u_n = -\frac{1}{3}(v_n - v_n - 1)$$

Now, $$u_1 = -\frac{1}{3}(v_1 - v_0)$$

$$u_2 = -\frac{1}{3}(v_2 - v_1)$$

$$u_3 = -\frac{1}{3}(v_3 - v_2)$$

$$u_n = -\frac{1}{3}(v_n - v_{n-1})$$

Adding, we get $$S_n = \sum_{n=1}^{n} u_n = -\frac{1}{3}(v_n - v_0) = -\frac{1}{3}\left(\frac{1}{3n+1} - \frac{1}{1}\right)$$

$$\therefore \quad S_n = \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$$

$$\lim_{n\to\infty} S_n = \frac{1}{3}\left(1 - \frac{1}{\infty}\right) = \frac{1}{3}$$

Arithmetic-Geometric Series

When numerator of u_n is in arithmetic series and denominator is in geometric series, we say, it is an Arithmetic- Geometric series.

Sum to infinity, the series below:

Example - 2.15 $1+\frac{5}{3}+\frac{9}{3^2}+\frac{13}{3^3}+\frac{17}{3^4}+...\infty$, find sum.

Solution:

Let,

$$S_n = 1+\frac{5}{3}+\frac{9}{3^2}+\frac{13}{3^3}+\frac{17}{3^4}+...\infty$$

$$\frac{S_n}{3} = \frac{1}{3}+\frac{5}{3^2}+\frac{9}{3^3}+\frac{13}{3^4}+\frac{17}{3^5}+...\infty$$

$$S_n-\frac{S_n}{3} = 1+\left(\frac{5}{3}-\frac{1}{3}\right)+\left(\frac{9}{3^2}-\frac{5}{3^2}\right)+\left(\frac{13}{3^3}-\frac{9}{3^3}\right)...\infty$$

$$= 1+\left(\frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}+...\infty\right) = 1+4\left(\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+...\infty\right)$$

$$\frac{2S_n}{3} = 1+\frac{4\left(\frac{1}{3}\right)}{1-\frac{1}{3}} = 1+2=3$$

$$\therefore \quad S_n = \frac{3\times 3}{2} = 4.5$$

2.7 Recurrence Relations

Sometimes it may be difficult to define the n^{th} term of a sequence explicitly in terms of n. However, it may be fairly easy to define the n^{th} term in terms of either the previous term, or in terms of a collection of previous terms. e.g.,

(i) The sequence of powers of 2 is given by $a_n = 2a_{n-1}$ and $a_0 = 1$ for $n \geq 1$. Here $a_n = 2a_{n-1}$ is called a recurrence relation and $a_0 = 1$ is called initial condition.

(ii) The recursive definition of binomial coefficients $\binom{n}{k}$ or ${}^nC_k$ or $C(n, k)$ where $n \geq 0$, $k \geq 0$ and $n \geq k$ is given by $C(n, k) = C(n-1, k) + C(n-1, k-1)$ with initial conditions, ${}^1C_0 = 1$, ${}^1C_1 = 1$.

Recurrence Relations

Let the number of bacteria in a colony double every hour. If a colony begins with five bacteria, how many will be present in n hours?

To solve this problem let a_n be the number of bacteria at the end of n hours. Since the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This relationship, together with initial condition $a_0 = 5$, uniquely determines a_n for all non negative integer n.

Definition: A recurrence relation for the sequence $\{a_n\}$ is a formula that expresses a_n in terms of one or more of the previous term of sequence, namely a_0, a_1 ...a_{n-1}, for all integers n with $n \geq n_0$ where n_0 is a non-negative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example-2.16 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3 \ldots$ and suppose that $a_0 = 3$, $a_1 = 5$. What are a_2 and a_3?

Solution:

Given the recurrence relation $a_n = a_{n-1} - a_{n-2}$.

Recurrence relation for $n = 2$ is $a_2 = a_1 - a_0 = 5 - 3 = 2$ and for $n = 3$ is $a_3 = a_2 - a_1 = -3$.

Example: The recurrence relation for the Fibonacci sequence is:

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ together with initial conditions $f_1 = 1$ and $f_2 = 1$.

It is possible to find any term of a recurrence relation by starting from initial condition and repeatedly applying the recurrence formula.

For example: Find the 5[th] term of the Fibonacci sequence

$f_n = f_{n-1} + f_{n-2} \quad n \geq 3, \quad f_1 = 1, \quad f_2 = 1$

Now,
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$

However this brute force process is tedious and we wish to explicitly find the solution for n^{th} term, as a function of n, so that the n^{th} term may be obtained easily for even large values of n.

Recursively Defined Functions

To define a function with the set of non negative integers as its domain.

(*i*) Specify the value of the function at zero.

(*ii*) Give a rule for finding its value as an integer from its values at smaller integers such a definition is called a recursive or inductive definition.

Example-2.17 Give an inductive definition of factorial function $f(n) = n!$

Solution:

The initial value of the function is defined as F(0) = 1, then giving a rule for $F(n+1)$ from $F(n)$.

$$F(n+1) = (n+1)\,F(n)$$

For
$$n = 5, F(6)$$
$$= 6F(5) = 6.5F(4) = 6.5.4F(3)$$
$$= 6.5.4.3.2.F(0)$$

Now, since
$$F(0) = 1$$
$$F(6) = 6.5.4.3.2.1 = 720$$
$$F(6) = 6! = 720$$

2.8 Solving Recurrence Relations

Types of Recurrence Relations

1. Linear recurrence relations with constant coefficient
2. Non-linear recurrence relations
3. Indeterminate order recurrence relations

1. Linear recurrence relations with constant coefficients:

A k^{th} order linear recurrence relation on S with constant coefficients can be written of the form,

$S(n) + C_1S(n-1) + C_2S(n-2) \ldots + C_kS(n-k) = f(n)$ $[k \geq n]$

Where, $C_1, C_2, \dots C_k$ are constant and f is a numeric function defined for $k \le n$

When $f(n) = 0$, such a recurrence is called homogenous.

Examples:

(a) $f_n = f_{n-1} + f_{n-2}$ (Fibonacci sequence)

(b) $f_n - f_{n-1} - f_{n-2} = 0$

(c) $T_n - 3T_{n-1} + 4T_{n-2} = 2^n$

(d) $S_n = 2S_{n-1} + 5 + n^2$ i.e. $S_n - 2S_{n-1} = 5 + n^2$

(*i*) is a 2nd order homogenous linear recurrence

(*ii*) is a 2nd order inhomogenous linear recurrence

(*iii*) is a 1st order inhomogenous linear recurrence

All linear homogenous equations can be solved by standard methods. Linear inhomogenous equations also can be solved by standard methods, provided the right hand side function $f(n)$ is in some specific allowed forms.

2. **Non-linear recurrence relations:**

If the powers of the recurrence variable is more than 1 or if product of recurrence variables is found in the equation, then the recurrence becomes non-linear.

Example:

$S_n^2 - 2S_{n-1} = n + 2$ is a non linear 1st order relation

and $S_n S_{n-1} + S_{n-2} = 5n$ is a non linear 2nd order relation

Some non-linear recurrences can be solved after converting them to linear homogenous or inhomogenous type, by appropriate substitutions.

Example: $S_n^2 - 2S_{n-1}^2 = n$

Let, $T_n = S_n^2$ than $T_{n-1} = S_{n-1}^2$

∴ The above non linear recurrence can be converted to a linear recurrence.

$T_n - T_{n-1} = n$ and solved.

3. **Indeterminate order recurrence relations:**

If the difference between the largest subscript of the recurrence and smallest subscript in not an integer but a function of n, we say that such a recurrence relation, is of indeterminate order.

Example: $S_n = 2S_{n/2} + 1$

Now $n - n/2 = n/2$ is not an integer, but a function of n.

∴ This is an indeterminate order recurrence relation.

These can be solved in some cases by converting into linear recurrence relation, by appropriate substitutions.

In some cases, the Master's theorem may be applied, to find the complexity order of the solution as a function of n.

Example: $S_n = 2S_{n/2} + 1 \quad S(1) = 2$

$S_n - 2S_{n/2} = 1$ Let $n = 2^k$

$S_n = S_{2^k} = B(k)$ and $S_{n/2} = S_{2^{k-1}} = B(k-1)$

∴ The recurrence becomes converted to the inhomogenous linear recurrence, $B(k) - 2B(k-1) = 1$ and can be solved by standard methods.

Note: The initial condition S(1) = 2 must also be changed, using the same transformation $n = 2^k$
Here, $S(1) = S(2^0) = B(0) = 2$
$\therefore$ The initial condition, S(1) = 2 is changed to B (0) = 2

Solving (Linear Homogenous Recurrence Relations with Constant Coefficients) by Characteristic Roots Method

Example-2.18 Solve the following recurrence relation by using characteristic root method:
$a_n - 7a_{n-1} + 12a_{n-2} = 0, \quad n \geq 2, a_0 = 1, a_1 = 2$

Solution:
Characteristic equation:
$$C(t) = t^2 - 7t + 12 = 0$$
i.e. $$(t-3)(t-4) = 0$$
Root are 3, 4
$\therefore$ The general solution is $a_n = C_1 3^n + C_2 4^n$
C_1, C_2 can be found from initial conditions.
$$a_0 = C_1 3^0 + C_2 4^0 = 1$$
$$\Rightarrow \quad C_1 + C_2 = 1 \quad \ldots(1)$$
$$a_1 = C_1 3^1 + C_2 4^1 = 2$$
$$\Rightarrow \quad 3C_1 + 4C_2 = 2 \quad \ldots(2)$$
From equation (1) and (2), $C_2 = -1, C_1 = 2$
$$\therefore \quad a_n = 2.3^n - 4^n$$
If the roots of characteristic equations are not distinct, then one of the terms of the solution has to be multiplied by n.

Example: $a_n - 4a_{n-1} + 4a_{n-2} = 0$
$$C(t) = t^2 - 4t + 4 = 0 = (t-2)(t-2) = 0 \text{ root are } 2, 2$$
Now solution to recurrence is $a_n = C_1 2^n + C_2.n.2^n$
C_1, C_2 can now be solved from initial conditions.
If the number of repeated roots is 3, then we multiply one term by n, another term by n^2, to keep all terms distinct in form.

Solving Linear Inhomogenous Equations by Characteristic Roots Method

To solve linear inhomogenous equation, we must

Step-1: first put the R.H.S. $f(n)$ to Zero and solve the homogenous case. Let us call the homogenous solution as a_n^h. Do not solve for constants C_1, C_2 etc at this stage.

Step-2: Then we choose a trial particular solution a_n^p based on the form of the RHS function according to the table:

RHS	Form of trial particular solution (a_n^p)	
Constant	C	d
Linear	$C_0 + C_1 n$	$d_0 + d_1 n$
Quadratic	$C_0 + C_1 n + C_2 n^2$	$d_0 + d_1 n + d_2 n^2$
Power fn	$C.a^n$	da^n
Power fn * Poly	$a^n (C_0 + C_1 n \ldots)$	$a^n (d_0 + d_1 n \ldots)$

If any of the terms of the trial solution is same as a term of the homogenous solution (in form), then the entire particular solution is multiplied by n.

If this does not make the particulars solution terms to be distinct from homogenous solution, then multiply by n^2 instead and so on until all terms of particulars solution are distinct in form from those of the homogenous solution.

Step-3: Now, substitute the trial solution into the recurrence and solve for constants $d_0, d_1, \ldots$ etc.

Step-4: Now get general solution $a_n = a_n^h + a_n^p$

Step-5: Lastly, solve for constants C_1, C_2 etc in homogenous solution by substituting the initial conditions.

Example-2.19 Solve the following linear inhomogenous recurrence relation by using characteristic root method, $S(k) + 5S(k-1) = 9$, $S(0) = 6$.

Solution:

Step-1:

$$S(k) + 5\,S(k-1) = 0$$

$$C(t) = t + 5 = 0 \Rightarrow t = -5$$

$$S_k^h = C_1(-5)^k$$

Step-2:

Let,

$$S_k^p = d$$

Step-3:

$$d + 5d = 9 \Rightarrow d = 9/6 = 1.5$$

Step-4:

$\therefore$

$$S_k = S_k^h + S_k^p = C_1(-5)^k = 1.5$$

Step-5:

Substituting the initial condition,

$$S(0) = C_1(-5)^0 + 1.5 = 6 \Rightarrow C_1 = 4.5$$

$\therefore$

$$S_k = 4.5(-5)^k + 1.5$$

Example-2.20 Solve the following linear inhomogenous recurrence relation by using characteristic root method, $T(k) - 7T(k-1) + 10T(k-2) = 6 + 8k$.

$$T(0) = 1 \text{ and } T(1) = 2$$

Solution:

Step-1:

Here, the characteristic equation,

$$t^2 - 7t + 10 = 0 \Rightarrow t = 2, 5$$

$\therefore$

$$T^h(k) = C_1 2^k + C_2 5^k$$

Step-2:

$$T^p(k) = d_0 + d_1 k$$

Step-3:

Substituting in recurrence we get,

$$(d_0 + d_1 k) - 7(d_0 + d_1(k-1)) + 10(d_0 + d_1(k-2)) = b + 8k$$

$\Rightarrow$

$$(4d_0 - 13d_1) + 4d_1 k = 6 + 8k$$

Equating ooefficients, $4d_0 - 13d_1 = 6$ and $4d_1 = 8$

$$d_0 = 8,\ d_1 = 2$$

$\Rightarrow$

$$T^p(k) = 8 + 2k$$

Step-4:

$\therefore$

$$T(k) = T^h(k) + T^p(k) = C_1 2^k + C_2 5^k + 8 + 2k$$

Step-5:

Now, C_1, C_2 can be solved using initial conditions:

$$T(0) = 1 \Rightarrow C_1 + C_2 = -7$$

$$T(1) = 2 \Rightarrow 2C_1 + 5C_2 = -8$$

$$\therefore \quad C_1 = -9,\ C_2 = 2$$

$$T(k) = -9.2^k + 2.5^K + 8 + 2K$$

Summary

- The number of ways of arranging n objects in a circle is $(n-1)!$ This is less than $n!$, since in a circle many of the linear permutations become indistinguishable except for rotation.
- Combinations with no Repetitions $\boxed{{}^nC_r = \dfrac{n!}{r!(n-r)!}}$
- Combinations with Unlimited Repetitions
  ${}^{n-1+r}C_r$ which can also be written as ${}^{n-1+r}C_{n-1}$.
- Distribution of Distinct Objects: $\dfrac{n!}{n_1!n_2!...n_r!}$
- **Vandermondes indentity:** ${}^{n+m}C_r = {}^nC_0{}^mC_r + {}^nC_1{}^mC_{r-1} + {}^nC_2{}^mC_{r-2} \cdots {}^nC_r{}^mC_0$
- Special case of Vandermonde's identity is obtained by putting $r = n$
  $${}^{n+m}C_n = {}^nC_0{}^mC_n + {}^nC_1{}^mC_{n-1} + \ldots {}^nC_n{}^mC_0$$
- When generating function is written as a series $1 + X + X^2 + \ldots$ it said to be in open form and when it written instead as $\dfrac{1}{1-X}$, it is said to be in closed form.
- Notice that the sequence corresponding to (4.1) $\sum_{r=0}^{n} X^r$ is $[a_r]_{r=0}^{n} = 1$.
- Ball in the box problems, when balls are indistinguishable, along with upper unstraints on the no of balls that can be put into a box, can be effectively converted into a problem of finding coefficient of some power of X in the expansion of a corresponding generating function.
- Solving Linear Inhomogenous Equations by Characteristic Roots Method

RHS	Form of trial particular solution (a_n^p)	
Constant	*C*	*d*
Linear	$C_0 + C_1n$	$d_0 + d_1n$
Quadratic	$C_0 + C_1n + C_2n^2$	$d_0 + d_1n + d_2n^2$
Power fn	$C.a^n$	da^n
Power fn * Poly	$a^n(C_0 + C_1n...)$	$a^n(d_0 + d_1n...)$

Student's Assignment

Q.1 How many strings are there of lowercase letters of length four or less?

Q.2 A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
(a) How many socks must he take out to be sure that he has at least two socks of the same color?
(b) How many socks must he take out to be sure that he has at least two black socks?

Q.3 A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
(a) How many balls must she select to be sure of having at least three balls of the same color?
(b) How many balls must she select to be sure of having at least three blue balls?

Q.4 What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Q.5 Atleast how many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

Q.6 How many subsets with an odd number of elements does a set with 10 elements have?

Q.7 A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
(a) are there in total?
(b) contain exactly three heads?
(c) contain at least three heads?
(d) contain the same number of heads and tails?

Q.8 What is the coefficient of $x^{12} y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Q.9 Find the coefficient of $x^5 y^8$ in $(x + y)^{13}$.

Q.10 How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?

Q.11 What is the coefficient of $x^{101} y^{99}$ in the expansion of $(2x - 2y)^{200}$?

Q.12 In how many ways can 5 numbers be selected and arranged in ascending order from the set {1,2,3...10}?

Q.13 In how many different ways can 5 ones and 20 twos be permuted so that each one is followed by atleast 2 twos?

Q.14 How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

Q.15 A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose
(a) six bagels?
(b) a dozen bagels?
(c) a dozen bagels with at least one of each kind?
(d) a dozen bagels without least three egg bagels and no more than two salty bagels?

Q.16 How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Q.17 How many solutions are there to the equation
$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$
where x_i, i=1, 2, 3, 4, 5, is a non-negative integer such that
(a) $x_1 \geq 1$
(b) $x_i \geq 2$ for i = 1, 2, 3, 4, 5
(c) $0 \leq x_1 \leq 10$

Q.18 How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of twelve 0 bits, and must have at least two 0 bits following each 1 bit?

Q.19 An agency has 10 available foster families $F_1 \ldots F_{20}$ and 6 children $C_1 \ldots C_6$ to place. In how many ways can they do this if

(a) No family can get more than one child.

(b) A family can get more than one child.

Q.20 How many distinguishable permutation can be generated from word "BANANA"?

(a) 720 (b) 60
(c) 240 (d) 120

Q.21 How many integers in S = {1, 2, 3, ..., 1000} are divisible by 3 or 5?

(a) 599 (b) 467
(c) 333 (d) 66

Q.22 How many numbers from a set {1, 2, 3, ..., 20} should be chosen in order to be sure to have one number multiple of another.

(a) 10 (b) 11
(c) 3 (d) 9

Q.23 How many numbers must be chosen from set {1, 2, 3, ..., 8}, such that atleast 2 of them must have Sum = 9

(a) 28 (b) 9
(c) 5 (d) 10

Q.24 In a string of length n, with distinct letters, how many substrings can be generated (other than null string)?

(a) 2^n (b) n^2
(c) $n(n+1)/2$ (d) $n(n-1)/2$

Answer Key:

20. (b) **21.** (b) **22.** (b) **23.** (c) **24.** (c)

Student's Assignments Explanations

1. *Here repetition is allowed by default (nothing is mentioned about repetition).*

Number of possible strings of 0 length (empty string) : 1

Number of possible strings of length 1 : 26

Number of possible strings of length 2 : 26 × 26

Number of possible strings of length 3 : 26 × 26 × 26

Number of possible strings of length 3 : 26 × 26 × 26 × 26

Therefore the total number of strings of length 4 or less = $1+26+26^2+26^3+26^4$

2. **(a)**

He must take out three socks to make sure that he gets a pair. Because the first socks could be of one colour and second socks could be of another colour.

Hence the third socks that he draws from the drawer is of one of the colour of the two socks that he had drawn earlier, which is sufficient to make a pair.

(b)

He must take out 14 socks to be sure that he has at least 2 black socks. Because there are 12 black and 12 brown socks in the drawer. So when he draws first 12 socks they could be all brown and hence 2 more socks need to be drawn to make sure a pair of black socks.

3. **(a)**

She must select five balls to be sure of having at least 3 balls of the same colour. This is because first 4 attempts have the following possibilities which does not ensure three balls of same colour.

Red, Blue, Red, Blue or

Blue, Red, Red, Blue ...and so on

Hence we need fifth ball to ensure three balls of same colour.

(b)

Thirteen balls must be selected to be sure of having atleast three blue balls.

The first ten balls that she draws could be all ten red balls. Hence she need to draw three more balls to ensure three blue balls.

4. Here the number of students are similar to number of pigeons and number of states are similar to pigeon holes.

Number of pigeons = n

Number of pigeons hole = 50

Therefore $\left\lfloor \frac{n-1}{50} \right\rfloor + 1 = 100$

The minimum value of n is obtained after removing the floor function and solving for n.
$\Rightarrow n = 4951$ pigeons (students).

5. The number of pairs that add upto 7 are the pigeon holes.
{1, 6}, {2, 5}, {3, 4} are the holes.
Let 'N' be the number of such numbers. If two numbers sit in any one of these pigeon holes, we have a pair whose total is 7.

Therefore $\left\lfloor \frac{N-1}{3} \right\rfloor + 1 = 2$

$\Rightarrow$ Atleast N = 4 numbers must be selected.

6. The number of subsets with 1 element = ${}^{10}C_1$
The number of subsets with 3 elements = ${}^{10}C_3$
The number of subsets with 5 elements = ${}^{10}C_5$
The number of subsets with 7 elements = ${}^{10}C_7$
The number of subsets with 9 elements = ${}^{10}C_9$
Hence the total number of subsets with an odd number of elements are:
$${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9 = 2^{10-1} = 2^9 = 512$$

7. When a coin is tossed there are only 2 possibilities (H and T).
(a) Total number of possibilities are:
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$
(b) The total number of possibilities with exactly three heads are: ${}^8C_3$ (any of the 3 coins can turn up heads).
(c) The total number of possibilities with atleast three heads are: $2^8 - \{{}^8C_0 + {}^8C_1 + {}^8C_2\}$.
(d) The total number of possibilities with same number of heads and tails are: ${}^8C_4$ (any of the 4 coins could be head and remaining 4 coins could be tails).

8. First, note that this expression equals $(2x+(-3y))^{25}$. By the Binomial Theorem, we have
$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j$$
Consequently, the coefficient of $x^{12}y^{13}$ in the expression is obtained when $j = 13$, namely,
$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25}{13!\,12!} 2^{12}3^{13}$$

9. The coefficient of $x^{n_1}y^{n_2}$ is $\frac{n}{n_1!\,n_2!}$. Therefore the coefficient is $\frac{13!}{5!8!}$ which is $= 13 \times 11 \times 9 =$ 1287.

10. We will get like terms when we expand $(x+y)^{100}$ as $(x+y)(x+y)(x+y)(x+y)$ ...100 times.
The binomial expansion collects all like terms. In the expansion the terms are in the form $x^{n_1}y^{n_2}$ with the condition that $n_1 + n_2 = 100$ always. Thus, the number of solution of this equation gives the number of terms in the binomial expansion.
$${}^{2-1+100}C_{100} = 101$$
$\therefore$ The number of solutions of this equation gives the number of terms are 101.

11. $\frac{200!}{100!\,99!}(2x)^{101}(-2y)^{99}$
$$= -\frac{200!}{100!\,99!} \times 2^{200} \times x^{101} \times y^{99}$$

12. Since, after selection there is only one way to put 5 numbers in ascending order, the answer is ${}^{10}C_5 \times 1 = {}^{10}C_5 = 252$.

13. First itself, attach 2 twos to each 1 to form the compound symbols 122,122,122,122,122 and permute these 5 symbols along with the remaining 10 twos in $\frac{15!}{5!\,10!}$ ways.

14. First job can be given to all the 5 employees.
Second job can be given to all the 5 employees.
Third job can be given to all the 5 employees.
$\therefore$ The total number of ways of distributing three jobs among five employees are $5 \times 5 \times 5 = 125$.

15. Onion bagels (x_1), Poppy seed bagels (x_2), Egg bagels (x_3), Salty bagels (x_4), Pumpernickel bagels (x_5), Sesame seed bagels (x_6), Raisin bagels (x_7), and Plain bagels (x_8).

(a) This problem is same as distributing six chocolates among eight children.

i.e. $x_1 + x_2 + x_3 + ... + x_8 = 6$

$(n = 8,\ r = 6)$

${}^{8-1+6}C_6 = 1716$

(b) $r = 12, n = 8$

${}^{8-1+12}C_{12} = 50388$

(c) The problem is same as

$$x_1 + x_2 + x_3 + ... + x_8 = 12$$

with the constraints $x_1 \geq 1, x_2 \geq 1, ... x_8 \geq 1$.

Now select one bagel of each type to satisfy the constraints.

Now the number of ways to choose $12 - 8 = 4$ bagels from among the 8 type of bagels is same as the number of solutions to the equation

$$x_1 + x_2 + x_3 + ... + x_8 = 4$$

which is ${}^{8-1+4}C_4 = {}^{11}C_4$

(d) The problem is same as

$$x_1 + x_2 + x_3 + ... + x_8 = 12$$

with the constraints $x_3 \geq 3$ and $x_4 \leq 2$.

Now to meet the constraint on the egg bagel, choose 3 egg bagels. We now have choice in choosing the remaining 9 bagels.

Now we are left with the problem of finding the number of solutions to the equation

$$x_1 + x_2 + x_3 + ... + x_8 = 9$$

with the constraint $x_4 \leq 2$

We can solve this problem by complementary counting method.

Solve the complementary problem

$$x_1 + x_2 + x_3 + ... + x_8 = 9$$

with the constraint $x_4 \geq 3$

which has a solution same as

$$x_1 + x_2 + x_3 + ... + x_8 = 6$$

with the constraint $x_4 \geq 0$

The solution of this is ${}^{8-1+6}C_6 = {}^{13}C_6$

Now subtract from the universal set which is ${}^{8-1+9}C_9 = {}^{16}C_9$ and the final answer is ${}^{16}C_9 - {}^{13}C_6$.

16. There are only two varieties hence $n = 2$.

We need to choose 8 coins.

This problem is same as selecting 8 chocolates from 2 types of chocolates.

$$x_1 + x_2 = 8$$

$${}^{2-1+8}C_8 = {}^9C_8 = {}^9C_1 = 9 \text{ ways}$$

17. **(a)** Treat the right side as 21 balls. Put one ball in First box, to satisfy the constraint $x_1 \geq 1$.

The remaining 20 balls can be distributed into 5 boxes in

$${}^{5-1+20}C_{20} = {}^{24}C_{20} = {}^{24}C_4 \text{ ways}$$

(b) Treat the right side as 21 balls. Put 2 balls in each of the 5 boxes, to satisfy the constraints $x_i \geq 2$.

The remaining 11 balls can be distributed into 5 boxes in ${}^{5-1+11}C_{11} = {}^{15}C_{11} = {}^{15}C_4$ ways.

(c) This problem can be solved by complementary counting method.

The given problem is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $0 \leq x_1 \leq 10$.

The universal set for this problem is the unconstrained solution to this problem which is

$${}^{5-1+21}C_{21} = {}^{25}C_{21} = {}^{25}C_4$$

The complementary problem is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $x_1 \geq 11$.

The solution to this is to put 11 balls into first box and distribute the remaining 10 balls into 5 boxes in

$${}^{5-1+10}C_{10} = {}^{14}C_{10} = {}^{14}C_4 \text{ ways.}$$

So the final answer to the given problem is obtained by subtracting ${}^{14}C_4$ from the universal set which is ${}^{25}C_4$ to given the final answer of ${}^{25}C_4 - {}^{14}C_4$.

18. 100 □ 100 □ 100 □ 100 □

The remaining four 0's can be placed in four boxes as shown above.

This is same as distributing four identical balls to four boxes.

$$x_1 + x_2 + x_3 + x_4 = 4$$

$$\therefore \quad {}^{4-1+4}C_4 = {}^7C_4 = 35 \text{ ways}$$

19. (a) Since all children must be placed, consider the children as boxes and assign one family to each box (permutation with no repetition).
$10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151200$ ways.

(b) Since each family can be assigned more than one child, each of the boxes (children) can be assigned to any one of the ten families (permutation with unlimited repetition).
$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$ ways.

20. (b)
Total letters = 6
Repetition frequency per word = N : 2, A : 3, B : 1

Number of permutation $\frac{6!}{3!\,2!\,1!} = 60$

21. (b)

$$D_3 = \{n \mid n\%3 = 0\}$$

$$\Rightarrow \quad |D_3| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$$

$$D_5 = \{n \mid n\%5 = 0\}$$

$$\Rightarrow \quad |D_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$D_3 \cap D_5 = \{n \mid n\,\%3 = 0 \text{ and } n\%5 = 0\}$
Since 15 is LCM of 3 and 5, $D_3 \cap D_5$ certains exactly those numbers which are divisible by 15.

$$|D_3 \cap D_5| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$$

Therefore,

$$|D_3 \cup D_5| = |D_3| + |D_5| - |D_3 \cap D_5|$$
$$= 333 + 200 - 66 = 467$$

22. (b)
{11, 12, 13, ... 20} is the largest set with no multiples. Adding one more number will make it sure that there is at least one number which is multiple of another in this set.

23. (c)
{1, 8}, {2, 7}, {3, 6} and {4, 5}
are the 4 pairs which give sum = 9. Maximum 4 numbers can belong to different sets. The 5[Th] number will pair up and form 9 sum, for sure.

24. (c)
suppose, S = abcd, |S| = 4. The substrings are a, b, c, d, ab, bc, cd, abc, bcd, abcd. Total substrings are 10. Here there is a trend, with strings of length n, we have

substring of length 1 = n
substring of length 2 = $n-1$
substring of length 3 = $n-3$
⋮
substring of length n = 1

Total substring = $1 + 2 + \ldots + (n-1) + n$
$= n(n+1)/2$

CHAPTER 03

Set Theory and Algebra

3.1 Introduction

Definition: The term set is an undefined primitive of mathematical system under our study. According to George canter. "The set is a collection of definite well defined objects of perception or thought".

Objects: Objects constituting a set are called its elements and we express the fact that a is an element of a set S, symbollically as $a \in S$. (Pronounced as a belongs to S).

Since "object" is very general in scope, an element of a set may be letters, numbers or any symbols, or ordered pairs, or even a set itself.

Sets are denoted by capital letters A, B, C etc. and its elements by small letters *a*, *b*, *c*, ... Sets are either be written by enumerating all its elements (listing method) or by a rule (rule method or set builder method) or by statement method.

(a) There are no particular order in a set, since it is only a collection of elements. i.e. A = {1, 2, 3} and B = {2, 3, 1} are both the same set. We say A = B.

(b) Repetition of element is meaningless in a set, since an element is taken only once i.e. A = {1, 2, 2, 3} and B = {1, 2, 3} are both the same sets.

3.2 Sets

Finite and Infinite Sets

Sets which have a finite number of elements are called **finite sets** and those having infinite number of elements are called **infinite sets.** If S is a set, then $n(S)$ or $|S|$ denotes the number of elements in S. $n(S)$ is also called as cardinal number of S or cardinality of S.

Examples of finite set:

(*i*) Set of days in a week

(*ii*) Set of dates in a month

(*iii*) Set of chairs in a classroom

Examples of infinite set:

(*i*) Set of natural number

(*ii*) Set of points on a plane

(*iii*) Set of lines passing through one point

Equivalent Sets

Two finite sets are equivalent if their cardinal numbers are same. Notice that two equivalent sets need not be equal.

Example: Let $A = \{1, 2, 3\}$
$B = \{x, y, z\}$

then, A and B are equivalent sets.

Equal Sets

Two sets are equal if they have exactly same elements

Example: If $A = \{1, 2, 3\}$ $B = \{2, 3, 1\}$ then $A = B$.

Empty or Null Sets

A set which does not possesses any element is called empty or null or void set and is denoted by ϕ or $\{\ \}$.

Example: If $A = \{x : x \in N \text{ and } 2 < x < 3\}$ then $A = \phi$.

3.2.1 Subset

A set 'A' is **subset** of B if each element of A is also an element of B. A is called **proper subset** of B if B has at least one element more than that of A and all elements of A are contained in B.

For subset we use $\subseteq$ and for **proper subset** we use $\subset$.

Subset Properties:

1. If a set A has n elements, then total number of subsets of A is 2^n.
   Example: If a set A is $\{1, 2\}$, then subsets of A are $\{\ \}$, $\{1\}$, $\{2\}$ and $\{1, 2\}$
   Here total number of subsets are 2^2 i.e., 4.
2. If $X \in A \Rightarrow X \in B$ (where x is any arbitrary element)
   Then we can say that $A \subseteq B$
   This is the strategy that is used to check or to prove that $A \subseteq B$.
3. $A \subseteq B$ and $B \subseteq A$ then $A = B$
   i.e. $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$
   This is the strategy that is used to prove that some two arbitrary sets are equal.
4. **Power Set:** Let A be a set, then power set of A is P(A) given by $P(A) = \{S : S \subseteq A\}$.
   If A is the set of n elements, then the number of elements in P(A) is 2^n.
   $\Rightarrow n[P(A)] = 2^n$.
   Example: If $S = \{a, b, c\}$, then
   $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
   Here S has 3 elements, so P(S) has $2^3 = 8$ elements.
5. If A is the subsect of B, then B is the superset of A Superset is denoted by the symbol "$\supseteq$".
   i.e. $A \subseteq B \Rightarrow B \supseteq A$
6. Every set is a subset of itself and null set is a subset of every set.
   i.e. $A \subseteq A$ (for all A)
   and $\phi \subseteq A$ (for all A)

3.2.2 Universal Set

Universal set U is the superset of all the sets under consideration.

It is a set which contains all the sets in the domain of discussion. For example if we are discussing numbers such as integers, rational nos etc, the real no set is a convenient universal set since it contains all the numbers discussed.

3.2.3 Complement of a Set

Let U be an universal set and A be any element of it, then A^C or A' is the complement of A given by $A^C = \{x : x \notin A \text{ and } x \in U\}$

Let U = {1, 2, 3, 4, 5}

A = {1, 4,}, A^C = {2, 3, 5}

Properties of Complements:

1. $(A^C)^C = A$ or $(A')' = A$
   (law of double complementation)
2. $A \cup A^C = U$ or $A \cap A^C = \phi$
   (That is to say that A and A^C together contain everything and A and A^C have nothing in common between themselves).
   ∴ We can write $A^C = U - A$ (where "–" represent difference of two sets as will be discussed later).

3.2.4 Union of Sets

Let A and B be two sets. A set consisting of the elements of both A and B is called union of set A and B and is denoted by $A \cup B$.

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

i.e. $A \cup B$ contains elements belonging to A or B or both A and B. The "or" is being used in inclusive sense. (includes elements belonging to both A & B also).

So, "∪" is the inclusive or.

Example:

Let A = {*a*, *b*, *c*, *d*}, B = {*a*, *e*, *f*},

$A \cup B$ = {*a*, *b*, *c*, *d*, *e*, *f*}

3.2.5 Intersection of Sets

Let A and B be two sets, then the set which consists the common elements of A and B is called intersection of A and B and it is denoted by $A \cap B$.

⇒ $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example: It A = {*a*, *b*, *c*, *d*}, B = {*a*, *e*, *f*}, then $A \cap B$ = {*a*}. Properties of "∪" & "∩".

1. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
2. $A \cap B \subseteq A$ and $A \cap B \cup \subseteq B$

3.2.6 Disjoint Sets

Two sets A and B are said to be disjoint sets, if there is no common element in A and B. If A and B are disjoint sets, then $A \cap B = \phi$.

Example: Let A = {*a*, *b*, *c*}, B = {*x*, *y*, *z*} then A and B are disjoint sets, because $A \cap B = \phi$.

3.2.7 Difference of Sets

Let A and B be two sets. Then the set of all those elements of A which are not in B is called difference set of A and B and denoted by $A - B = \{x : x \in A, x \notin B\}$

Also $B - A = \{x : x \in B, x \notin A\}$

Example: If A = {1, 2, 3, 4}, B = {2, 3, 5} then A – B = {1, 4} and B – A = {5}.

A – B includes all elements which belong to A only & (not B) and B – A includes all elements which belong to B only (& not A) A – B is also called as relative complement of B in A.

Properties of Set Difference:

1. In general $A - B \neq B - A$
2. $A^C = \cup - A$
3. $A - B = A - (A \cap B)$
   [i.e. A – B can be obtained by removing from A, the elements common to both A & B]
4. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ = (A only) or (B only) or (both A & B)

3.2.8 Symmetric Difference of Sets

Let A and B be two sets. Then the set of all those elements which are in A but not in B or in B but not in A is called symmetric difference of A & B, denoted by $A \oplus B$.

i.e. $A \oplus B$ contains all elements belonging to either A only or B only but not both.

This is also called as the "XOR" operation. (Exclusive - or).

Example:

If A = { 1, 2, 3, 4 } and B = {3, 4, 5, 6}

Then $A \oplus B$ = {1, 2, 5, 6}

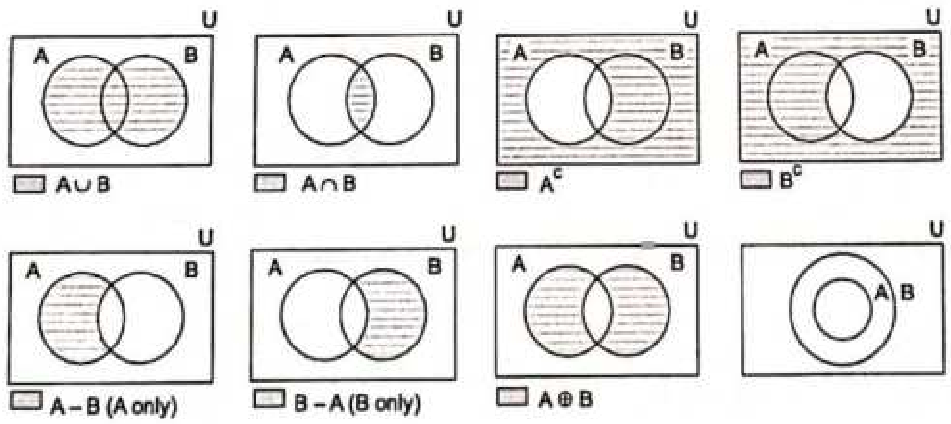
Properties of Symmetric Difference:

1. $A \oplus B = B \oplus A$ (Commutative)
2. $A \oplus B = (A - B) \cup (B - A)$ = A only or B only
3. $A \oplus B = (A \cup B) - (B \cap A)$
4. $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ (Associative)

3.2.9 Venn Diagrams

Most of relationship between the sets can be represented by diagrams known as venn diagrams. A universal set U is represented by points in the interior of a rectangle and any of its non empty subsets by points in interior of closed curves (usually circles).

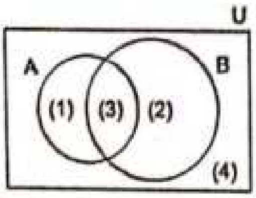
The venn diagrams for common set operations is shown below.



NOTE: Venn diagrams can be effectively used for proving equality of set expressions or for answering question regarding counting of elements of sets.

3.2.10 Fundamental Products

Fundamental products are the disjoint partitions (regions) of a venn diagram with two or more sets. For example consider a venn diagram with two arbitrtory sets A and B. The four fundamental products are shown below as (1), (2), (3) and (4).

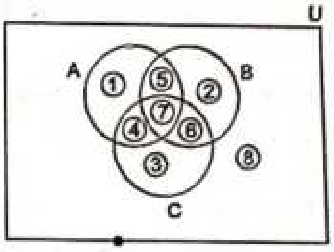


The number of fundamental products is always = 2^n, where n is the number of sets under consideration.

1. $A \cap B^C$ [A only] [A and not B]
2. $A^C \cap B$ [B only] [B and not A]
3. $A \cap B$ [A & B]
4. $A^C \cap B^C$ [neither A nor B]

Similarly for a 3 set venn diagram, there are $2^3 = 8$ fundamental products as shown below:

1. $A \cap B^C \cap C^C$ [A only]
2. $A^C \cap B \cap C^C$ [B only]
3. $A^C \cap B^C \cap C$ [C only]
4. $A \cap B^C \cap C$ [A & C but not B]
5. $A \cap B \cap C^C$ [A & B but not C]
6. $A^C \cap B \cap C$ [B & C but not A]
7. $A \cap B \cap C$ [all three]
8. $A^C \cap B^C \cap C^C$ [none of them]



Fundamental products are useful in counting since they are disjoint in nature and therefore provides no chance for double-counting.

3.2.11 Law of Set Theory

1. $\left.\begin{array}{l} A \cup \phi = A \\ A \cap U = A \end{array}\right\}$ Identity Laws
2. $\left.\begin{array}{l} A \cup \phi = \phi \\ A \cup U = U \end{array}\right\}$ Domination Laws
3. $\left.\begin{array}{l} A \cup A = A \\ A \cap A = A \end{array}\right\}$ Idempotent property
4. $\left.\begin{array}{l} A \cup B = B \cup A \\ A \cap B = B \cap A \end{array}\right\}$ Commutative Property
5. $\left.\begin{array}{l} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{array}\right\}$ Associative property
6. $\left.\begin{array}{l} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{array}\right\}$ Distributive property

7. $\left.\begin{array}{l} A \cup A^C = \cup \\ A \cap A^C = \phi \end{array}\right\}$ Complement laws
8. $(A^C)^C = A$ Law of double complement
9. $\left.\begin{array}{l} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{array}\right\}$ Demorgan's Laws

Set Theory: More results

1. $A - B = A \cap B^C = A\,(A \cap B)$
2. $A - B = B^C - A^C$
3. $A \subseteq B \Leftrightarrow B^C \subseteq A^C$
4. $A \subset B$ and $C \subset D \Rightarrow A \times C \subset B \times D$
5. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
6. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
7. $n(A^C) = n(U) - n(A)$
8. $n(A - B) = n(A \cap B^C) = n(A) - n(A \cap B)$ 10, 11, 12 and 13 are used in counting problem involving sets.

3.2.12 Cartesian Product of Sets

Let A and B be two sets, then $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ $A \times B$ is called Cartesian product of sets.
The elements of $A \times B$ are of the form (a, b) called ordered pairs.
If A has m elements and B has n elements then $A \times B$ has mn elements.

Example: Let $A = \{a, b\}$, $B = \{c, d, e\}$, then $A \times B = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$
$B \times A = \{(c, a), (c, b), (d, a), (d, b), (e, a), (e, b)\}$

Here, $A \times B$ has $2 \times 3 = 6$ elements
$B \times A$ also has $3 \times 2 = 6$ elements

Properties of Cartesian Product:

1. $A \times B \neq B \times A$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
4. $A \times (B - C) = (A \times B) - (A \times C)$
5. $(A \times B) \cap (C \times D) = (A \cap C) \times (C \cap D)$
6. $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

3.3 Relations

Definition:

Let A and B be two non empty sets, then a relation R from A to B is a subset of $A \times B$.

1. Let $R \subseteq A \times B$. If $(x, y) \in R$ then we say "x is related to y", denote it by $x\,Ry$. In this notes, whenever you see $x\,Ry$, read it as x related to y (by relation R).
2. Let $R \subseteq A \times B$, given by $R = \{(x, y) : x \in A, y \in B\}$, then Domain $(R) = \{x : (x, y) \in R\}$, and Range $(R) = \{y : (x, y) \in R\}$
3. A relation R on set A is a subset of $A \times A$ and is called a binary relation on A.
   Example: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$
   Then a relation R defined on $A \times B$ is any subset of $A \times B$. For instance,
   $R = \{(1, a), (2, c), (2, b)\}$
   Now since $(1, a) \in R$, we say 1Ra (1 is related to a by R)
   Domain $(R) = \{1, 2\}$

[The set of all first elements of the ordered pairs of R]

Range (R) = {a, b, c}

[The set of all second elements of the ordered pairs of R]

Note: Since ϕ is also a subset of A × B it is also a relation.

R= ϕ ={ } is called the null relation or void relation.

It is the smallest possible relation on A and B.

Since A × B ⊆ A × B. A × B itself is a relation. It is the biggest possible relation on A and B, since it contains all possible ordered pair combinations from element of A and B. Similarly, the largest relation from a set A to itself is A × A, which is the universal relation in A.

3.3.1 R-relative Sets

For any element $x \in$ R, we can define a set called R-relative set of x as $R(x) = \{y \mid xRy\}$ i.e. R relative set of x is all the elements which are related to the element x by relation R.

Example: R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 3)}

R (1) = {1, 2, 3}

R (2) = { 1, 2} and R (3) = {3}

Now for some B ⊆ A, we can define also R(B) = $\{y \mid xRy,\ \forall x \in B\}$. In example above, if B = {1, 3}, then R(B) = all elements related to 1 or 3 = {1, 2, 3}.

3.3.2 Representation of Relations

Since relations are also sets (of ordered pairs), They can be represented by listing, set builder or statement methods, used for representing sets.

However, relations can be represented by other methods such as matrix method, arrow diagram method, graphical method or digraph method.

Consider a relation R on

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 3, 4\}$$

Set Builder: R = $\{(x, y) \mid x < y\}$ is a relation expressed in set builder method.

Listing: R = {(1, 2),(1, 3),(1, 4) (2, 3), (2, 4) (3, 4)} is the same relation expressed in listing method.

$$\textbf{Matrix: } M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$ is a matrix representation of the same relation.

If $(x, y) \in$ R, then there will be a 1 in the position corresponding to row representing element x and column representing element y. All other entries in M_R are made zero.

Notice, that the row and column labels are shown for reference only and can be omitted as follows, if order of elements listed in A and B is fixed.

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

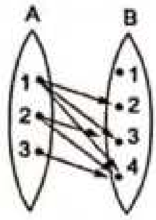
The elements of M_R may therefore be defined as to follows:

q_{ij} = 1 {if there is a relation between element i of A and Element j of B}

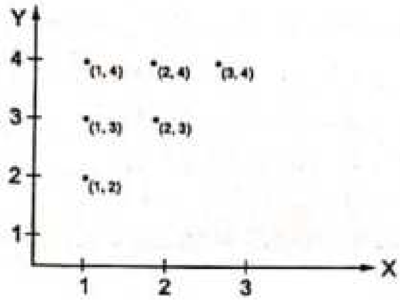
= 0 otherwise

Arrow Diagram

The arrow diagram representation of same relation would be



The graphical representation of R would be as follows:



Digraph

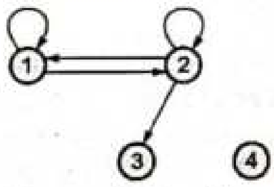
A Digraph (Directed graph) representation is suitable only if the relation is between a set A and itself, i.e. on A × A.

Example: A = {1, 2, 3, 4,}

Consider a relation on A × A given as follows.

R = {(1,1), (2,2), (1,2), (2,1) (2,3)}

The Digraph for the above relation is shown below.



The lines representing (1, 1) and (2, 2) are called self loops.

NOTE: The representation of a relation in set builder form is complete only when the sets A and B are clearly specified.

For Example R = {$(x, y) / x \le y$} has a different meaning if specified on Z × Z, than when specified on R × R. (Note: the default set for numbers is the set of real nos).

So, if you wish to allow only integer values of x and y the correct representation will be, let R = {$(x, y) / x \le y$} on Z × Z.

Identity Relation

Let A be a non empty set. Then the relation {$(x, y) : x, y \in$ A and $x = y$} is called identity relation.

Example: Let A = {a, b, c, d} then the identity relation I_A = {$(a, a), (b, b), (c, c), (d, d)$}

Identity relation is also known as diagonal relation, since in matrix representation of I_A, the diagonal elements are all 1 (Identity matrix).

3.3.3 Operations on Relations

Since relations are sets, all set operations can be performed on relations also. i.e. if R and S are two relations, than the following are defined.

$R \cup S$, $R \cap S$, R^C, S^C, $R - S$, $R \cup \oplus S$

Example: R = {(1, 1), (1, 2), (2, 3)}

S = {(1,2), (2, 3), (3,3)}

on $A \times A$ where A = {1, 2, 3}

$$R \cup S = \{1, 1\}, (1, 2), (2, 3), (3, 3)\}$$
$$R \cap S = \{(1, 2), (2, 3)\}$$
$$\bar{R} = R^C = U - R = (A \times A) - R = \{1, 3) (2, 1), (2, 2\}, (3, 1) (3, 2) (3, 3)\}$$
$$\bar{S} = S^C = U - S = (A \times A) - S = \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2)\}$$
$$R - S = R - (R \cap S) = \{(1, 1)\}$$
$$S - R = S - (S \cap R) = \{(3, 3)\}$$
$$R \oplus S = (R-S) \cup (S-R) = (R \cup S) - (R \cap S) = \{(1, 1), (3, 3)\}$$

In addition to the above operations, the following are also defined on R and S. i.e. R^{-1}, S^{-1}, RoS, SoR.

Definition:

$$R^{-1} = \{(y, x) / (x, y) \in R\}$$

In this example: $R^{-1} = \{(1, 1), (2, 1), (3, 2)\}$

Note that if R relates x to y, then R^{-1} relates y back to x.

$$S^{-1} = \{(2, 1), (3, 2), (3, 3)\}$$

Using matrics M_{R-1} can be obtaines by taking transpose of M_R, i.e. $M_{R-1} = (M_R)^T$

Composition of Relations

$$RoS = \{(x, y)/(x, z) \in S \text{ and } (z, y) \in R\}$$

RoS is called **composition** of S with R.

Similarly, SoR is composition of R with S.

To find elements of RoS., start with S and for each $(x, z) \in S$ identify elements of the type $(z, y) \in R$ and write $(x, z) \in$ RoS. This must be done for each of the ordered pair of S.

Example: In above relation, with R = {(1, 1), (1, 2), (2, 3)} and S = {(1, 2), (2, 3), (3, 3)}

NOTE: Here, $(1, 2) \in S$ and $(2, 3) \in R \Rightarrow (1, 3) \in RoS$

There is no composition for ordered pairs (2, 3) and (3, 3) of S, since no element in R starts with 3. We write

RoS = {(1, 3)}

$(1, 1) \in R$ & $(1, 2) \in S$. $\therefore (1, 2) \in SoR$

$(1, 2) \in R$ & $(2, 3) \in S$. $\therefore (1, 3) \in SoR$

$(2, 3) \in R$ & $(3, 3) \in S$. $\therefore (2, 3) \in SoR$

SoR = {(1, 2), (1, 3), (2, 3)}

Note: RoS ≠ SoR (Composition is not commutative)

But, Ro (SoT) = (RoS) oT (Composition is Associative)

Using matrices for R & S, RoS can also be obtained as follows:

$M_{RoS} = M_S \odot M_R$ where $\odot$ is boolean multiplication of matrices.

3.3.4 Types of Relations

1. Reflexive Relation

A relation R on A is called reflexive, if $\forall x \in A\ (x, x) \in R$

i.e. $\forall x \in A,\ x\,R\,x$

Example: Let S be a set of all straight line. The relation R in S defined by "x is parallel to y", is reflexive because every straight line is parallel to itself.

(a) The matrix of a reflexive relation will contain 1's in all the diagonal position.

Example:

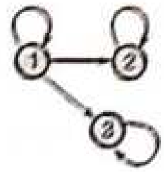
Let, R = {(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)} be defined in A × A where A = {1, 2, 3}

Now, $$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that diagonal elements are all 1s.

∴ This is a reflexive relation.

(b) The digraph of a reflexive relation will have self loops on every node. For example for above relation R, the digraph is



(c) R is reflexive iff R^{-1} is reflexive.

(d) Note that when checking for reflexive property, check that every element is related to itself.

(e) To check a set builder relation for reflexivity let $x\,Rx$ for an arbitrary x and see if it is true. Then it is reflexive,

Example: R = {(x, y) | x divides y} Let $x\,Rx \Rightarrow x$ divides x which is true $\forall x$.

∴ R is reflexive.

2. Symmetric Relation:

A relation R in A is called symmetric relation iff $(x, y) \in R \Rightarrow (y, x) \in R$

i.e., $x\,Ry \Rightarrow y\,Rx\ \forall x, y \in A$

(a) The matrix of a symmetric relation will be such as that $M_R = M_R^T$ i.e. the matrix will be symmetric matrix.

Since M_R^T represents the inverse relation R^{-1}, ∴ a necessary & sufficient condition for a relation to be symmetric is $R = R^{-1}$

Example: R{(1, 1), (1, 2), (2, 1), (3, 2), (2, 3)} is a symmetric relation

Here, $$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Notice that M_R is a symmetric matrix.

Also, $R^{-1} = R$ & $(M_R)^T = M_R$

(b) The digraph of a symmetric relation will be such that all arrows (which are not self loops) will be bidirectional i.e.
If an arrow goes from (a) to (b), there will be an arrow from (b) to (a). Of course, since self loops are always bidirectional and can be excluded while checking a digraph for symmetric property.

(c) The check a set builder relation for symmetry: Let xRy be true & see if this $\Rightarrow yRx$. If this is so, then R is symmetric.
Example: R $\{(x, y) / x + y = 10\}$
Now, Let xRy be true $\Rightarrow x + y = 10$
$\Rightarrow y + x = 10 \Rightarrow yRx$
$\therefore$ R is symmetric.

(d) Since an implication is true whenever LHS is false, then if xRy itself false, then by default R is symmetric.
$\therefore$ The empty relation is always symmetric.

3. Anti Symmetric Relation:

A relation R on A is called **anti symmetric** iff $xRy \Rightarrow y \not{R} x$, unless $x = y$
However the following definition is easier to use in practice. A relation R is antisymmetric iff $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$
i.e., xRy and $yRx \Rightarrow x = y \; \forall x, y \in A$.

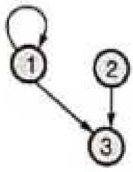
(a) Antisymmetric property basically means that all relations are one way, (Except for self loops, which are always two-ways).

(b) The matrix of an antisymmetric relation will have a "0" in the mirror image position (using diagonal as mirror) for every "1" in the off diagonal.
To check for antisymmetry, check the 1s in off diagonal and see if a "0" is there in corresponding mirror image position. Ignore diagonal 1s in this check.

(c) To check a digraph for antisymmetry, ignore self loops and check that for every arrow going form a to b (a, b distinct), there is no arrow from b to a, i.e. All arrows (Except self loops) are unidirectional.
Example: R = {(1, 1), (2, 3), (1, 3)}is antisymmetric

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The digraph is unidirectional except for self-loops.



(d) To check a set builder relation for antisymmetry, Let xRy and yRx and solve. If the only solution is $x = y$, then R is antisymmetric.
Example: R = $\{(x, y) / x$ divides $y\}$ $x, y \in N$
Now Let xRy and $yRx \Rightarrow x$ divides y and y divides $x \Rightarrow x = y$.
$\therefore$ R is antisymmetric.

(e) If xRy and yRx cannot be satisfied by any elements i.e. LHS is false, by default the implication becomes true, i.e. R is antisymmetric.

Scanned with CamScanner

Example: R = {(x, y) / x is father of y}

Now Let xRy and yRx

$\Rightarrow x$ is father of y and y is father of x.

Now, this is not at all possible. Always false.

$\therefore$ by default the implication is true and the relation is antisymmetric.

4. Transitive Relation:

A relation R on A is called transitive iff $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$

i.e., xRy and $yRz \Rightarrow xRz \quad \forall x, y, z \in A$

Example: A relation "greater than" defined on the set of natural numbers N is transitive because $x, y, z \in N$ if $x > y, y > z$ then $x > z$.

(a) Transition property is difficult to check with a matrix.

(b) Transitive property can be checked on a digraph by scanning each node systematically all possible $(x, y), (y, z)$ arrow and seeing if (x, z) arrow also exists. Self loops can be ignored in this analysis since they always are transitive.
This procedure although tedious, can be used for checking a small digraph for transitivity.

(c) To check a set builder relation for transitivity, Let xRy and yRz and solve these two equations. If there is no solution, or if the solution results in xRz, then relation is transitive.

Example: R = {(x, y) / x + y is even}

Now Let xRy and yRz

$\Rightarrow \quad x + y = 2k_1, \quad \text{... (1)}$

and $\quad y + z = 2k_2 \quad \text{... (2)}$

Adding (1) and (2) we get

$$x + 2y + z = 2(k_1 + k_2)$$

$$\Rightarrow \quad x + z = 2(k_1 + k_2 - y)$$

$\therefore \Rightarrow x + z$ is also even $\Rightarrow xRz$

$\therefore$ R is transitive

(d) Notice that if xRy and yRz is always false (i.e. no solution to xRy and yRz), then by default R is transitive.

5. Irreflexive Relation:

A relation R on A is called **irreflexive** iff $\forall x \in A, (x, x) \notin R$.

i.e. $\forall x \in A, \; x \not R x$

Example: Let S be the set of all straight lines, the relation R on S defined by "x is perpendicular to y". is irreflexive, since no line is perpendicular to itself.

(a) Irreflexive property means strictly no self loops in digraphs. Strictly no 1s in the diagonal of the matrix representation. (i.e. all 0's in diagonal of M_R).

(b) In the builder form this can be checked by putting xRx and seeing if this is always false.
Example: R = {(x, y) / x is one inch from y} defined on set of pts in a plane.
Let $xRx \Rightarrow$ pt x is one inch from itself. Which is always false. Hence R is irreflexive.

(c) An irreflexive relation is surely not reflexive, but a not reflexive relation may or may not be irreflexive.
i.e. irreflexive $\Rightarrow$ not reflexive
not reflexive $\not\Rightarrow$ irreflexive
Example: R = {(1, 1), (2, 3), (3, 1)} on A = {1, 2, 3} is neither reflexive, nor irreflexive.

6. **Asymmetric Relation:**

A relation R on A is an **asymmetric** relation iff $(x, y) \in R \Rightarrow (y, x) \notin R \quad xRy \Rightarrow y \not R x$.

This is similar to antisymmetric property in that all relations are unidirectional, except that in antisymmetric the self loops are allowed, but here in asymmetry even self loops are not allowed (i.e. strictly unidirectional).

Example: $R = \{(x, y) | x$ is father of $y\}$

Let $xRy \Rightarrow x$ is father of $y \Rightarrow y$ is not father of x

i.e. $xRy \Rightarrow y \not R x$. $\therefore$ R is asymmetric.

Notice that there are no self loops here, i.e. x cannot be father of x.

(a) The matrix of an asymmetric relation must have 0'S in all diagonal positions (no self loops). Also wherever a "1" is in off diagonal, a "0" must be there in corresponding mirror image position.

(b) The digraph of a relation can be easily checked for asymmetry, as follows. Check that there are no self-loops. Also check that every arrow is unidirectional.

(c) To check a relation in set builder for asymmetry.

Let xRy. This must imply that $y \not R x$.

(d) If xRy is always false, then by default the relation is asymmetric i.e. ϕ is an asymmetric relation.

7. **Equivalence Relation:** A relation R on a non empty set A is called equivalence relation iff

(a) R is reflexive i.e $xRx \ \forall x \in A$

(b) R is symmetric i.e $xRy = yRx$

(c) R is transitive i.e xRy and $yRz \Rightarrow xRz \quad \forall x, y, z \in A$

Example: $R = \{(x, y) | x \parallel y\}$ on straight lines on a plane. Here, $\parallel$ means "Parallel to".

Let xRx.

$xRx \Rightarrow x \parallel x$ is always true since every line is parallel to itself.

$\therefore$ R is reflexive.

Set $xRy \Rightarrow x \parallel y \Rightarrow y \parallel x \Rightarrow yRx$

$\therefore$ R is symmetric

Now, Let xRy and $yRe \Rightarrow x \parallel y$ and $y \parallel z \Rightarrow x \parallel z \Rightarrow xRz$.

$\therefore$ R is transitive.

Now we say that, R is an equivalence relation since it is reflexive, symmetric and transitive.

8. **Partial Order Relation:** A relation R on a non empty set A is called a partial order relation iff.

(a) R is reflexive $\forall x \in A, xRx$

(b) R is antisymmetric xRy and $yRx \Rightarrow x = y$

(c) R is transitive xRy and $yRz \Rightarrow xRz$

Example: $R = \{(A, B) | A \subseteq B\}$ on sets

Now, $\forall A \ A \subseteq A$ is true. $\qquad \therefore$ R is reflexive

Let $A \subseteq B$ and $B \subseteq A$

Now, this $\Rightarrow A = B$ $\qquad \therefore$ R is antisymmetric

Let $A \subseteq B$ and $B \subseteq C$

Now, this $\Rightarrow A = C$ $\qquad \therefore$ R is transitive

We now say that R is a partial order relation, since it is reflexive, antisymmetric and transitive.

Equivalence Relation, Equivalence Classes and Quotient Set

Let R be an equivalence relation on A × A.

Now equivalence class of $x \in$ A can be written as $[x]$.

We define $[x] = \{y | xRy\}$ i.e. for every element of A we can define its equivalence class as the set of all elements related to it.

Example: R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)} is an equivalence relation on A = {1, 2, 3}

Now,
$$[1] = \{1, 2\}$$
$$[2] = \{2, 1\}$$
$$[3] = \{3\}$$

Notice that [1] = [2] i.e. There are only 2 distinct equivalence classes. Now the set of all equivalence classes is called the quotient set of A induced by R, denoted as A/R

Here A/R = {[1], [2], [3]} = {{1, 2}, {3}}

NOTE

- **Theorem:** Every quotient set A/R is also a partition of A.
  Here, the converse is also true.
- **Theorem:** Corresponding to every partition P of A, there exists an unique equivalence relation whose quotient set is exactly P.

To find the equivalence relation corresponding to a given partition, Simply take the union of the cross product of the blocks of the partition with themselves.

Example: Let A = {1, 2, 3, 4}

Find the Equivalence relation corresponding to the partition $P_1 = \{\{1, 2\}, \{3, 4\}\}$

Now there are two blocks $A_1 = \{1, 2\}$ & $A_2 = \{3, 4\}$ in P_1

The equivalence relation R corresponding to partition P_1 is simply,
$$R = A_1 \times A_1 \cup A_2 \times A_2$$
$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$$

Similarly the equivalence relation corresponding to partition $P_2 = \{\{1, 2, 3\}, \{4\}\}$ is
$$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$$

Theorem: The relation congruence modulo m is defined as R = $\{(x, y)\} \mid x = y \bmod m\}$ (where m is a fixed integer). This relation partitions the set of integers Z into exactly m distinct equivalence classes.

It can be shown that the m distinct equivalence classes are
$$[0] = \{y | y = km, k \in z\}$$
$$[1] = \{y | y = km + 1, k \in z\}$$
$$[m-1] = \{y | y = km + (m-1), k \in z\}$$

These are also called residue classes since 0, 1, 2, ... m – 1 are the residues obtained upon dividing any integer by m.

Example - 3.1 Set A has 'm' elements and Set B has 'n' elements. What is the total number of relations possible on $_A R_B$ (from A to B)?

Solution:

The cross product A × B has $m * n$ ordered pairs.

And every relation R is either a subset or proper subset of A × B. This problem reduces to number of subsets of ordered pairs in A × B. There total number of relations are 2^{mn}.

Example - 3.2 What is the total number of reflexive relations from Set A to itself having 'K' elements in the set.

Solution:

$$\begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix}_{K \times K}$$

We can represent relation as matrix of size $K \times K$.
There are 'K' reflexive pairs and hence diagonal elements are fixed as 1.
There are $(K^2 - K)$ non-diagonal pair.
Therefore number of reflexive relations are equal to number of subsets of non-diagonal elements.

$\therefore \quad 2^{K^2-K}$

Example - 3.3 What is the total number of symmetric, relations from set A to itself that has 'n' elements?

Solution:
Let us consider relation as $n \times n$ matrix. There are 'n' diagonal elements representing reflexive pairs or self loops. Therefore 2^n subsets of reflexive pairs. In symmetric relation, the ordered pairs above the diagonal elements are mirror image of the ordered pairs below the diagonal.

Therefore, $\dfrac{n^2-n}{2}$ pairs are to be taken.

$\therefore$ Total $\qquad 2^n \times 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$ symmetric relations

Example - 3.4 What is the total number of antisymmetric relations from Set A to itself which has 'n' elements?

Solution:
Self loops are always allowed. Therefore 2^n subsets of reflexive pairs. In antisymmetric relations symmetric ordered pairs are not allowed.
i.e. If (a, b) exists then (b, a) should not be present.

The image pairs can take {10, 01, 00} but not {11}. Therefore, 3 possibilities for all $\left(\dfrac{n^2-n}{2}\right)$ pairs.

$\therefore$ Total antisymmetric relations are $2^n \times 3^{\frac{n^2-n}{2}}$

Example - 3.5 Let R = {(a, b), (b, c), (c, d)} be a relation on set {a, b, c, d}. Which of the following is transitive closure of R?

(a) {(a, b), (b, c), (c, d), (a, d)}
(b) {(a, b), (b, c), (c, d), (a, c), (b, d), (a, d)}
(c) {(a, b), (b, c), (c, d), (b, a), (c, b), (d, c)}
(d) None of these

Solution: (b)

$$R = \{(a, b), (b, c), (c, d)\}$$

$$\text{Transitive closure of } R = \{(a, b), (b, c), (c, d), (a, c), (b, d), (a, d)\}$$

Missing elements in R:

(i) (a, b) and (b, c) $\Rightarrow$ (a, c)

(ii) (b, c) and (c, d) $\Rightarrow$ (b, d)

(iii) (a, c) and (c, d) $\Rightarrow$ (a, d)

3.4 Functions

Definition: A **function or mapping** is a relation between the elements of A and those of B having no ordered pairs with the same first component.

In other words, a function is a unique valued relation. i.e. every element of A is mapped to only one element of B. However, elements of B may be related to more than one element of A.

Note that every function is a relation, but a relation may or may not be a function.

If the first element may be thought of as input and the second element as output, then, in a function, every input has a unique output.

Example: $f = \{(1, 1), (2, 3), (3, 3)\}$ is a function on A × A, where A = {1, 2, 3}

Here;
$$f(1) = \{1\} = 1$$
$$f(2) = \{3\} = 3$$
$$f(3) = \{3\} = 3$$

Whereas,
$$R = \{(1, 1), (2, 3), (2, 4), (3, 3)\} \text{ is not function, since}$$
$$R(1) = \{1\},\ R(2) = \{3, 4\},\ R(3) = \{3\}$$

Here R(2) has 2 values 3 and 4 and hence R is not a function.

There are two ways to write a function, one as a formula and other as a relation.

Example: $f(x) = x^2$ and $f = \{(x, y) \mid y = x^2\}$ are both one and the same function.

If a function is written as $f : A \to B$, it means that f is a mapping that takes all elements of A and maps each, to a unique element of B. It must be noted here

(a) that there may be some elements of the set B which are not associated to any element of set A.

(b) that each element of set A must be associated with one and only one element of set B.

Then A is the domain of f and B is the Co-domain of "f".

If $(x, y) \in f$, it is customary to write $y = f(x)$. y is called the image of x and x is called the preimage of y.

y is also called value of f at x. The set consisting of all images of elements of A is called the range of f. It is denoted by $f(A)$.

Range of $f = f(A) = \{f(x) \mid x \in A\}$

To check if a given relation is a function $f : A \to B$ check the following:

(i) $\forall x \in A$, is $f(x)$ defined and belongs to B?

(ii) $f(x)$ is unique, and single valued.

Example: $S = \{(x, y) \mid y = 3x + 1\}$ on R × R is a function, since,

(i) $\forall x \in R$, $S(x) = y\ 3x + 1 \in R$

(ii) $3x + 1$ has a single value for any real value x,

$\therefore$ S is a function, We say $S : R \to R$ or $S_{R \to R}$.

3.4.1 One-One Mapping

A function $f : A \to B$ is said to be one-one if different elements of A have different f-images in B i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or equivalently $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

One-one mapping are also called **injection.**

To check if a function is one to one, Let $f(x_1) = f(x_2)$ and see if this leads to a single solution i.e. $x_1 = x_2$. If so, f is one-to-one. Else, it is many to one. (Assuming, it is a function.)

Example: $S = \{(x, y) \mid y = 3x + 1\}$ on $R \times R$

We already have checked that indeed S is a function.

Now to check 1 – 1, we set $S(x_1) = S(x_2)$

$\Rightarrow y_1 = y_2 \Rightarrow 3x_1 + 1 = 3x_2 + 1 \Rightarrow x_1 = x_2$

$\therefore$ S is one-to-one function.

3.4.2 Many-One Mapping

A function $f : A \to B$ is said to be many one iff two or more different elements in A have the same f-image in B.

A function which is not one-to-one is many to one.

Example: $T = \{(x, y) \mid y = x^2\}$ on $R \times R$

Not it can easily be checked that T is indeed a function.

Now let $T(x_1) = T(x_2) \Rightarrow y_1 = y_2 \Rightarrow x_1^2 = x_2^2$

Now, $x_1^2 = x_2^2$ has two solutions $x_1 = x_2$ or $x_1 = -x_2$

$\therefore$ We say, $x_1^2 = x_2^2 \not\Rightarrow x_1 = x_2$

$\therefore \qquad T(x_1) = T(x_2)$

$\therefore \qquad T(x_1) = T(x_2) \not\Rightarrow x_1 = x_2$

This means T is not one-to-one, i.e., It is many-to-one function.

Let $f : A \to B$ (here $f : A \to A$ since $A = B$)

3.4.3 Into-Mapping

The mapping f is said to be into iff there is at least one element in B which is not the f-image of any element in A.

In this case $f(A) \subset B$.

i.e., range of A is a proper subset of B.

Example: $f = \{(1, 1), (2, 3), (3, 4), (4, 3)\}$

Where, $\qquad A = B = \{1, 2, 3, 4\}$

Now, $\qquad f(A) = \{1, 3, 4\} \subset \{1, 2, 3, 4\}$

$\therefore$ f is an into function.

3.4.4 Onto Mapping

The mapping f is said to be onto iff every elements in B, is the f image of at least one element in A (i.e. every element of B has atleast one pre-image in A).

In this case $f(A) = B$

i.e., the range of f = Co-domain

Onto Mapping is also called **surjection.**

Example: Let $f: A \to B$ be

where, $f = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$

$A = B = \{1, 2, 3, 4\}$

Now, $f(A) = \{1, 2, 3, 4\} = B$

$\therefore$ f is an onto function.

We say, f is A onto B.

To check if a given function given in formula or set builder notation is onto or not, see if every element $y \in B$ has a preimage in A.

Example: Check if $f: R \to R$

$f(x) = 3x + 1$ is onto or not

Let, $f(x) = y = 3x + 1$

Now solve x in terms of y.

i.e. $x = \dfrac{(y-1)}{3}$

$\forall y \in R$, $\dfrac{y-1}{3}$ is also real

i.e. $\forall y \in R, x \in R$

$\therefore$ Every element y in second set has a primage in the first set. i.e. f is onto function.

NOTE: If a function $f: A \to B$ is both one-one and onto, then it is called a bijection function or a bijection. A bijection is also called a one-one correspondence between A and B.

If two sets A and B are in 1-1 correspondence, then $|A| = |B|$, that is they have exactly same number of elements.

3.4.5 Composition of Function

Definition:

Let $f: A \to B$ and $g: B \to C$

The composition of f and g denoted by gof, read as gof results in a new function from $A \to C$ and is given by (gof). $(x) = g(f(x)) \quad \forall y \in A$

Example: Let $A = \{1, 2, 3\}$, $B = \{a, b\}$ and $C = \{r, s\}$ and $f: A \to B$ is defined by $f(1) = a$ and $f(2) = a$ and $f(3) = b$ and $g: B \to C$ be defined by $g(a) = s$, $g(b) = r$.

Then gof : $A \to C$ is defined by,

(gof) $(1) = g(f(1)) = g(a) = s$

(gof) $(2) = g(f(2)) = g(a) = s$

(gof) $(3) = g(f(3)) = g(b) = r$

Example:

Let $f: R \to R$ be $f(x) = x + 2$

Let $g: R \to R$ be $g(x) = x^2$

Now (gof) $(x) = g(f(x)) = g(x + 2) = (x + 2)^2$

(fog)$(x) = f(g(x)) = f(x^2) = x^2 + 2$

Note that fog $\neq$ gof (composition is not commutative)

However, fo(goh) = (fog)oh (composition of functions is associative)

Theorems

(a) If f and g are one-one then gof is one-one

(b) If f and g are onto, then gof is onto

(c) If f and g are bijections, then gof is also a bijection.

Identity Mapping

If A is a non- empty set then $f: A \rightarrow B$ such that $f(x) = x$, $\forall x \in A$ is called identity mapping. It is denoted by I_A.

Inverse Mapping

If f is one-one and onto (bijective), from $f: A \rightarrow B$, then f^{-1} exists and it carries elements of B back to A.
Example: Let $f = \{(1, 2), (2, 3), (3, 1)\}$ is a function on $f: A \rightarrow A$ where $A = \{1, 2, 3\}$
Now $f^{-1} = \{(2, 1), (3, 2), (1, 3)\}$ is the inverse function.
$f(1) = 2$ and $f^{-1}(2) = 1$
To find inverse of set builder functions, the following procedure is given:
Example: Find inverse of $f(x) = 3x + 1$, $f: R \rightarrow R$
$f(x) = y = 3x + 1$

1. Write x in terms of y

$$x = \frac{y-1}{3}$$

2. Now, if $f(x) = y$, $x = f^{-1}(y)$

$$\therefore f^{-1}(y) = x = \frac{y-1}{3}$$

i.e. $f^{-1}(y) = \dfrac{y-1}{3}$

3. Since y is a dummy variable, we can replace it with x also.

i.e. $f^{-1}(x) = \dfrac{x-1}{3}$

$\therefore$ if $f(x) = 3x + 1$, then $f^{-1}(x) = \dfrac{x-1}{3}$

Here the inverse exists because $f(x) = 3x + 1$ is a bijection from R to R.

Example - 3.6 How many onto functions from a set with six elements to a set with three elements?

Solution:

Let P_1, P_2 and P_3 be the properties that b_1, b_2 and b_3 are not in the range of the function, respectively. Note that a function is ONTO if and only if it has none of properties P_1, P_2 or P_3.
Hence by using inclusion-exclusion principle the number of ONTO functions from a set with six elements to a set with three elements are:

$$n(\bar{P}_1 \bar{P}_2 \bar{P}_3) = \text{Total \# of functions possible} - [n(P_1) + n(P_2) + n(P_3)] + [n(P_1P_2) + n(P_1P_3) + n(P_2P_3)] - n(P_1P_2P_3)$$

$n(P_i) \rightarrow$ is the number of functions that do not have b_i in their range.
$n(P_iP_j) \rightarrow$ is the number of functions that do not have b_i and b_j in their range.
$n(P_iP_jP_k) \rightarrow$ is the number of functions that do not have b_i, b_j and b_k in their range.
The total number of functions are 3^6.
$n(P_1) = 2^6$ (since b_1 is not in range every element in domain have two choices (b_2 and b_3).
Similarly,

$$n(P_2) = n(P_3) = n(P_1) = 2^6$$

$\Rightarrow$ $$n(P_1P_2) = n(P_2P_3) = n(P_1P_3) = 1^6 = 1$$
(Every element in the domain will have only one choice)

$\Rightarrow$ $$n(P_1P_2P_3) = 0$$
Because this term is the number of functions that have none of b_1, b_2 and b_3 in their range. Clearly, there are no such functions.

$\therefore$ Number of ONTO functions $= 3^6 - 3 \bullet 2^6 + 3 \times 1^6 = 729 - 192 + 3 = 540$

Note: Let m and n be positive integers with $m \geq n$, then there are

$n^m - {}^nC_1\,(n-1)^m + {}^nC_2\,(n-2)^m - {}^nC_3\,(n-3)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}\,1^m$

ONTO functions from a set with m elements to a set with n elements.

Example - 3.7 **How many ways are there to assign five different jobs to four different employees if every employee is assigned to atleast one job?**

Solution:

Consider the assignment of jobs as a function from the set of five jobs to the set of four employees. An assignment where every employee gets atleast one job is same as an onto function from the set of jobs to the set of employees.

Hence number of onto functions are:

$n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots + (-1)^{n-1}.\,{}^nC_{n-1}.1^m$

where $n = 4$, $n = 5$

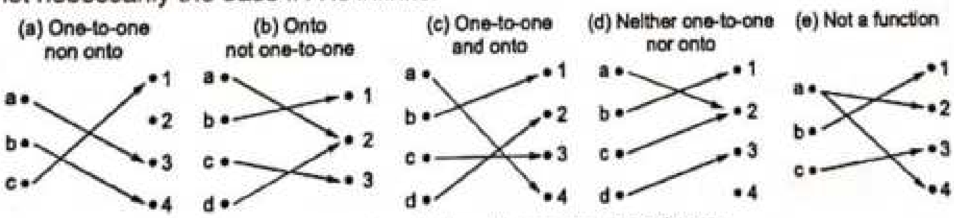
$\therefore$ $$4^5 - {}^4C_1(3)^5 + {}^4C_2\,(2)^5 - {}^4C_3(1)^5 = 1024 - 4 \times 243 + 6 \times 32 - 4 = 240$$

Example - 3.8 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is f a bijection.

Solution:

The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the co-domain are images of elements in the domain. Hence f is a bijection.

Figure displays four functions where the first is one-to-one but not onto, the second is onto but not one-to-one, the third is both one-to-one and onto, and the fourth is neither one-to-one nor onto. The fifth correspondence in figure is not a function, because it sends and element to two different elements. Suppose that f is a function from a set A to itself. If A is finite, then f is one-to-one if and only if it is onto. This is not necessarily the case if A is infinite.



Examples of different types of correspondences

Example - 3.9 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if it is what is its inverse?

Solution:

The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverse the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Example - 3.10 Let $f: Z \to Z$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution:
The function f has an inverse because it is a one-to-one correspondence, as we have shown. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of Z that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

Example - 3.11 Let f be the function from R to R with $f(x) = x^2$. Is f invertible?

Solution:
Because $f(-2) = f(2) = 4$. f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence f is not invertible. Sometimes we can restrict the domain or the co-domain of a function, or both, to obtain an invertible function.

Example - 3.12 Show that if we restrict the function $f(x) = x^2$ in example previous to a function from the set of all non-negative real numbers to the set of all non-negative real numbers, then f is invertible.

Solution:
The function $f(x) = x^2$ from the set of non-negative real numbers to the set of non-negative real numbers is one-to-one. To see this note that if $f(x) = f(x)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both x and y are non-negative, we must have $x = y$. So, this function is one-to-one. Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all non-negative real numbers, because each none-negative real number has a square root. That is, if y is a non-negative real number, there exists a non-negative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of non-negative real numbers to the set of non-negative real numbers is one-to-one and onto, it invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$.

Example - 3.13 Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution:
The composition f o g is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$. Note that g o f is not defined, because the range of f is not a subset of the domain of g.

Example - 3.14 Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution:

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3)$$
$$= 3(2x + 3) + 2 = 6x + 11$$

3.5 Equal Functions

Two functions f and g on same domain A are equal if $f(x) = g(x), \forall x \in A$.

Symmetric Function

If f and f^{-1} are equal then f is said to be **symmetric function** for example

Let $f = \{(2, 7), (3, 8), (7, 2), (8, 3)\}$ then $f^{-1} = \{(7, 2), (8, 3), (2, 7), (3, 8)\}$

Here $f = f^{-1}$

Hence f is symmetric function.

Binary Operation as a Function

Consider a set 'A' and an operation denoted by '$*$' which when placed between two elements a and b produces a unique result denoted by $a * b$ which may or may not belong to A.

If '$*$' is **binary operation** on A and $a * b \in A\ \forall\ a, b \in A$, then '$*$' is said to be closed and we say 'A' is closed with respect to binary operation '$*$'.

Since $a * b$ is unique (single valued) and also if "$*$" is closed on A, then we may look at a binary operation as a function from A × A to A.

In such a case instead of $a * b$, we may even write it in functional notation as $*(a, b)$.

For example: $a + b = +(a, b)$ defined on Z × Z maps pairs of integers to a value which is their sum. i.e. $+(1, 2) = 1 + 2 = 3$

$\therefore$ (1, 2) is mapped to 3.

$\therefore$ In this case this is a function $+ : Z \times Z \to Z$

A binary operation may be defined as a formula or as a binary operation table.

Example: $a * b = a + b - ab$ is a binary operation defined by formula method

Say $*$ is defined in Z

Then $2 * 3 = 2 + 3 - 2 \times 3 = -1$

$1 * 2 = 1 + 2 - (1 \times 2) = 1$ and so on.

Example: Let a binary operation be defined as in the following binary operation table.

$*$	a	b	c
a	c	a	b
b	b	c	c
c	a	b	b

from table we can see that $a * b = a$ & $b * b = c$ & $a * (b * c) = a * c = b$

3.6 Groups

Algebraic Structure: A non empty set S along with one or more binary operations is called an algebraic structure. Suppose $*$ is binary operation on S. Then $(S, *)$ is an algebraic structure. $(N, +), (Z, +), (Z, -), (R, +, \times)$ are all examples of algebraic structures.

3.6.1 Semi Group

An algebraic structure $(G, *)$ is called a semi group if the binary operation $*$ is closed on G ($a * b \in G\ \forall a, b \in G$) and is associative in G ($(a * b) * c = a * (b * c)\ \forall a, b, c \in G$).

Example: (Z , +) is a semi group since

1. $\forall a, b \in Z, a + b \in Z$ (closure)
2. $\forall a, b, c \in Z, a + (b + c) = (a + b) + c$

However, (Z, –) is not semigroup since although closure property holds, associative property does not hold for "–", $a-(b-c)^{1}(a-b)-c$.

3.6.2 Monoid

An algebraic structure (M, $*$) is called a monoid, if.

(*i*) " $*$ " is closed on G

(*ii*) " $*$ " is associative &

(*iii*) $e \in$ G such that $\forall a \in M$, $e * a = a = a * e$

Such an element "e" is unique and is called the identity element for the monoid.

NOTE: Every monoid is a semigroup but the converse in not true.

Example: (Z, +) is not only a semigroup, but is also a monoid since

(*i*) and (*ii*) holds and $\forall x \in Z, x + 0 = 0 + x = x$

$\therefore$ 0 is the identity element for the binary operation "+" on z. Notice that (N,+) is also a monoid since $0 \in N$.

However, (Z^+, +) is semigroup, but not a monoid since $0 \notin Z^+$.

NOTE: N = {0, 1, 2, 3, 4....} ← (set of +ve integers)

Set of non - negative integer

→ $Z^+ = \{1, 2, 3, 4....\}$ & $Z^- = \{-1, -2, -3, ...\}$ & $Z = \{0, \pm 1, \pm 2, ...\}$ ← set of integers

3.6.3 Group

An algebraic structure (G, $*$) is called a group, if the binary operation satisfies the following postulates.

1. **Closure property:** $a * b \in G \ \forall a, b \in G$
2. **Associativity:** $(a * b) * c = a * (b * c) \ \forall a, b, c \in G$
3. **Existence of identity:** There exists an element $e \in G$ such that $e * a = a = a * e \ \forall a \in G$. The element e is called identity for '$*$' in G.'
4. **Existence of inverse:** Each element of G possesses an inverse. In other words for each $a \in G$, there exists an element $b \in G$. Such that $a * b = b * a = e$. The element b is called inverse of a and we write $b = a^{-1}$. Thus a^{-1} is an element of G, such that $a * a^{-1} = a^{-1} * a = e$.

Example: (Z, +) is not only a semigroup and a monoid, but it is also a group.

(Z, +) has already has been shown to satisfy (*i*) closure (*ii*) association property and (*iii*) existence of identity. Now we shall show that condition (*iv*) for group, also holds for (Z, +).

$\forall x \in Z$, If inverse exists it must satisfy., $x * x^{-1} = x^{-1} * x = e$

$\Rightarrow \quad x + x^{-1} = x^{-1} + x = 0$ (since 0 is the identity element for +)

$\Rightarrow \quad x^{-1} = -x$ since $-x \in z \qquad \therefore \forall x \in Z, x^{-1} \in Z$ exists

Just like identity element, the inverse is also unique for a given element. Notice, however that there is only one identity element for the entire group, whereas there is a unique inverse for each element of G.

In (Z, +), the identity element is 0 for the entire group, while inverse of 1 is –1, inverse of 2 is –2, and so on.

Note however that (Z, ×) is not a group since although it is closed, associative, identity exists (= 1), inverse does not exist for all elements.

$$a * a^{-1} = a^{-1} * a = 1 \Rightarrow a^{-1} = \frac{1}{a}$$

but $0 \in Z$ and does not have an inverse, since $\frac{1}{0} = \infty \notin Z$

$\therefore$ (Z, x) is not a group.

3.6.4 Abelian Group or Commutative Group

A group G is said to be **abelian or commutative**, if in addition to the above four postulates, the following postulates are also satisfied.

Commutative: i.e., $a * b = b * a\ \forall a, b \in G$.

Example: (Z, +) is an abelian group, since it has already been shown to be a group, and it has the commutation property also.

i.e. $\forall x, y \in Z\ x + y = y + x$.

Notice that the set of (2 × 2 non singular matrices, $*$) when "$*$" is matrix multiplication is a group but not an abelian group, since matrix multiplication is not commutative.

i.e $A * B \neq B * A, \forall A, B \in$ (2 × 2 non singular matrices)

3.6.5 Finite or Infinite Groups

If in a group G, the underlying set G consists of a finite number of elements, then the group is called finite group, otherwise as infinite group.

Order of the Group

The number of elements in a finite group is called the order of a group. An infinite group is said to be of infinite order.

Some General Properties of Groups:

Suppose our group consists of a non-empty set G equipped with a binary operation denoted by $*$. Then,

1. The identity element in a group is unique.
2. The inverse of each element of a group is unique.
3. If the inverse of a is a^{-1}, then the inverse of a^{-1} is a i.e $(a^{-1})^{-1} = a$.
4. The inverse of the product of two elements of a group G is the product of the inverse taken in reverse order $(ab)^{-1} = b^{-1}a^{-1}\ \forall a, b \in G$.
5. If a, b, c are any elements of G, then $ab = ac \Rightarrow b = c$ (Left Cancellation Law) $ba = ca \Rightarrow b = c$. (Right Cancellation Law).
6. If a, b are any two elements of a group G, then the equation $ax = b$ and $ya = b$ have unique solution in G, given by $x = a^{-1}b$ and $y = ba^{-1}$, respectively.
7. The left inverse of an element is also its right inverse i.e if a^{-1} is left inverse of a (i.e. $a^{-1}a = e$), then $a a^{-1} = e$, which means that a^{-1} is also the right inverse of a.

3.6.6 Cayley Table

The binary operation table for a finite group are called cayley tables.

A cayley table for a group with only 4 elements is presented below:

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

It can be verified that ({e, a, b, c}, $*$) is indeed a group. It is conventional to put the identity element in the front of both row and column in a cayley table.

Properties of Cayley Tables:

1. The row and column containing e will be a copy of the column headers & row headers respectively. This is because by definition $x * e = e * x = x \quad \forall x \in G$
2. Since $ax = b$ and $ya = b$ have a unique solution, Every element in every row (or column) of the table must be different.
   That it to say that no element can be repeated in any row (or column) i.e. Each row or column is only a permutation of the element of G.
   In the example given each row and column can be seen to be a permutation of *e, a, b, c*.
3. The cayley table tells us if the group is abelian or not. If the cayley table is symmetric about the diagonal (as is the case in the example given), then the group is abelian.
4. In the cayley table, if e appears anywhere in off diagonal, then another e has to appear in its mirror image location (using diagonal as the mirror).
   This is because, an off diagonal e means $a \neq b$ and $a * b = e$, but, this implies that $b * a = e$, Thereby the mirror image location must also have an *e*.
   e appears in the diagonal, then the corresponding element is its own inverse.

Theorem: If every element of a group is its own inverse, then the group is abelian. (The converse is not necessarily true).

In other words, if G is a group and $\forall x \in G$ if $x^2 = e$, then the group is abelian.

Multiplication Modulo p

We shall now define a new type of multiplication known as 'multiplication modulo *p*' and written as $a \times_p b$ where *a* and *b* are any integers and *p* is a fixed positive integer. By definition. We have $a \times_p b = r, 0 \leq r < p$ where *r* is the least nonnegative remainder when ab is divided by *p*. For Ex.

$$8 \times_5 3 = 4 \text{ (since } 24 = 4(5) + 4)$$

Also, $$4 \times_7 2 = 1 \text{ (since } 4 \times 2 = 8 = 1(7) + 1)$$

Some Properties of Integers:

Let, $Z = \{......-3, -2, -1, 0, 1, 2, 3......\}$ be the set of integers

Division Algorithm

Let $a \in Z$ and $d \in Z^+$. Then we can divide *a* by *b* to get nonnegative remainder *r* which is smaller in size than b. In other words if $a \in Z$ and $d \in Z^+$, then there exist unique integers *q* and *r* such that

$a = dq + r$ where $0 \leq r < d$

Example: $a = 23$, $d = 3$, then $23 = 3 \times 7 + 2$.

(*q* is called quotient and *r* is called as reminder).

Divisibility in Set of Integers

Let $a, b \neq 0 \in |$. We say that a is divisible by *b* if $a = bm$ where *m* is some integer. i.e. if *b* divides *a*, then *a* is a multiple of *b*.

Greatest Common Divisor

Let *a* and *b* be any two integers. Then the positive integer *c* is said to be greatest common divisor of *a* and *b* if

(*i*) $c|a$ and $c|b$

(*ii*) Whenever $d|a$ and $d|b$, then $d|c$

The greatest common divisor of integer *a* and *b* will be symbollically denoted by GCD (*a*, *b*).

3.6.7 Some Classic Examples of Group's

1. Let B = {0, 1} & operation + is defined as:

+	0	1
0	0	1
1	1	0

Then (B, +) is an abelian group with 0 as identity and each element is its own inverse. Infact this + as defined in table is nothing but the XOR operation

2. $(Z_m, +_m)$ is an abelian group for every $m \in Z^+$, where Z_m is the set of equivalence classes for the relation congruence modulo m & $+_m$ is the modulo m addition. The operation table for $(Z_5, +_5)$ is

$+_5$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

Here, [1] = set of all integers which leave a remainder of 1 when divided by 5 = {±6, ±11, ±16, ±...., ± $(5m+1)$} of course, a simpler form of this is simply ({0, 1, 2, 3, 4}, $+_5$).

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. When p is prime, ({1, 2, 3, ... p–1}, $\times_p$) is always on abelian group.
Example: ({1, 2, 3, 4}, $\times_5$)

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Here the inverse of 2 is 3, 3 is 2, 4 is 4 & 1 is 1. A more general version of this is $\{Z_p - \{0\}, \times_p\}$, is also an abelian group.

$\times_5$	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]
[2]	[2]	[4]	[1]	[3]
[3]	[3]	[1]	[4]	[2]
[4]	[4]	[3]	[2]	[1]

4. **Symmetric group of permutations:**

Let S = {1, 2, 3} Let S_n be the set of all permutations on S. There are 3! = 6 permutations. Each permutations is a one-one, onto map from S to S. The S_n form a group under the operation composition of mappings. This group S_n is called the symmetric group of permutations of order *n*.

Consider S = {1, 2, 3}. Then $S_3 = \{p_1, p_2, p_3, p_4, p_5, p_6\}_7$. Where

$$p_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, p_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, p_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$p_4 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, p_5 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, p_6 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

The compositions 0 are given in table below:

0	p_1	p_2	p_3	p_4	p_5	p_6
p_1	p_1	p_2	p_3	p_4	p_5	p_6
p_2	p_2	p_1	p_6	p_5	p_4	p_3
p_3	p_3	p_5	p_1	p_6	p_2	p_4
p_4	p_4	p_6	p_5	p_1	p_3	p_2
p_5	p_5	p_3	p_4	p_2	p_6	p_1
p_6	p_6	p_4	p_2	p_3	p_1	p_5

p_1, is the identity element. S_3 is called the group of symmetric of a triangle.

This group is not abelian.

Power of an Element

Let (G, $*$) be a group and let $a \in$ G, for any position integer *m*, we define, $a^m = a * a * a * \ldots\ldots * a$ (*m* times) and $a^{-m} = (a^{-1}) * (a^{-1}) * (a^{-1}) \ldots\ldots * (a^{-1})$ (*m* times) $a^0 = e$ and if *m* & *n* are position integers, then $a^{m+n} = a^m * a^n$

Example: On (*z*, +) which is a group

$$1^3 = 1 + 1 + 1 = 3$$
$$2^3 = 2 + 2 + 2 = 6$$
$$2^{-3} = 2^{-1} + 2^{-1} + 2^{-1} = (-2) + (-2) + (-2) = -6$$

Order of an Element of a Group

Suppose G is group. By the order of an element $a \in$ G, is meant the least positive integer n, if one exists, such that $a^n = e$ (the identity of G).

If there exists no positive integer n such that $a^n = e$, then we say that a is of infinite order. We shall use the symbol O(*a*) to denote the order of *a*.

Example: Consider the group given below

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

The order of the group = |G| = 4

The order of element *e* is 1 since $e^1 = e$

The order of element *a* is 4 since $a^4 = a * a * a * a = b * a * a = c * a = e$.

The order of element *b* is 2 since $b^2 = b * b = e$ and the order of element *c* is 4 since $c^4 = c * c * c * c = b * c * c = a * c = e$ in (Z, +), the order of each element other than 0 is ∞ and the order of element 0 is 1.

Some Results Regarding Order of an Element

1. The order of every element of a finite group is finite and is less than or equal to order of group.
2. The order of an element a of a group is the same as that of its inverse a^{-1}.
3. The order of any integral power of an element a, cannot exceed the order of a.
4. If the element a of G is of order m, then $a^n = e$ iff m is a divisor of n.
5. The order of the elements a and $x^{-1}ax$ are the same where a, x are any two elements of a group.
6. Order of ab is same as that of ba where a and b are any elements of a group.
7. If a is an element of order n and p is prime to n, then a^p is also of order n.

3.6.8 Cyclic Group

A group (a, $*$) is called a cyclic group if there exists an element a $\in$ G such that every element of G can be written as a^n for some integer n. That is G = $\{a^n | n \in z\}$. We say that G is generated by a. a is the generator of G, we may then write G – {a} or G (a). Naturally, a cyclic group is abelian.

The order of a cyclic group is same as that of its generator.

1. (Z, +) is a cyclic group generated by 1.
2. (Z_m, t_m) is generated by [1].

Note that in example 1, order of G = order of 1 = ∞ and in example 2, order of G = $|Z_m|$ = order of [1] = m.

Properties of Cyclic Group:

1. Every cyclic group is an abelian group.
2. If a is generator of a cyclic group G, then a^{-1} is also a generator of G.
3. A cyclic group G with generator a of finite order n, is isomorphic to multiplicative group of n, n^{th} roots of unity.
4. A cyclic group G with a generator of finite order n is isomorphic to the additive group of residue classes modulo n.
5. If a finite group of order n contains element of order n, the group must be cyclic.
6. Every group of prime order is cyclic.
7. Every subgroup of a cyclic group is cyclic.

Method for Finding the number of generators of a cyclic group of order n: The number of generators of a cyclic group of order n is same as the number of numbers from 1 to n, which are relatively prime to n.

Method for finding the number of numbers from 1 to n, which are relatively prime to n: The number of numbers from 1 to n, which are relatively prime to n i.e., gcd (m, n) = 1, is given by the Euler Totient function $\phi(n)$. If n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2}$.... where p_1, p_2 etc. are distinct prime numbers, then

$\phi(n) = \phi(p_1^{n_1})\,\phi(p_2^{n_2})$... then by using the property

$$\phi(p^k) = p^k - p^{k-1}$$

we can find each of $\phi(p_1^{n_1})$, $\phi(p_2^{n_2})$... etc.

For *example*, let us find the number of generators of a cyclic group of order 80:

The number of generators of a cyclic group of order 80 = The number of numbers from 1 to n, which are relatively prime to 80.

Since $80 = 2^4 \times 5^1$.

The number of numbers from 1 to n, which are relatively prime to 80 = $\phi(80) = \phi(2^4) \times \phi(5^1)$
Now $\phi(2^4) = 2^4 - 2^3 = 16 - 8 = 8$.
Similarly, $\phi(5^1) = 5^1 - 5^0 = 5 - 1 = 4$.
So, $\phi(80) = 8 \times 4 = 32$.
So, the number of generators of a cyclic group of order 80 is exactly 32.

3.6.9 Subgroup

Let (G, $*$) be a group. A non empty subset H of G is called a subgroup of G if the following conditions are satisfied.

1. $a \in H, b \in H \Rightarrow a * b \in H$ (closure)
2. The identity $e \in H$ also (existence of identity)
3. $a \in H \Rightarrow a^{-1} \in H$ (existence of inverse)

In other words, (H,$*$) is a subgroup of (G,$*$), if $H \subseteq G$ and (H, $*$) is itself a group (since associative law holds in H also.)

Example: (E, +) where E is the set of even integers is a subgroup. In fact, $(kz, +)$ when $k \in z^+$, is a subgroup of $(z, +)$.

Properties of Subgroup:

1. For any group (G, $*$), ({e}, $*$) and (G, $*$) are called trivial subgroups. Other subgroups (if any) of (G, $*$) are called proper subgroups.
2. The identity of a sub group is same as that of the group (as seen in definition).
3. The inverse of any element of a subgroup is same as the inverse of that element when regarded as part of the group.
4. The order of any element of a subgroup is same as the order of that element when regarded as member of the group.

Important Results

1. A necessary and sufficient condition for a non-empty subset H of a group to be a subgroup is that $a \in H, b \in H, \Rightarrow ab^{-1} \in H$ where b^{-1} is the inverse of b in G.
2. A necessary and sufficient condition for a non empty finite subset H of a group G, to be a subgroup is that H must be closed with respect to multiplication i.e $a \in H, b \in H \Rightarrow ab \in H$.
3. If H, K are two subgroups of a group G, then HK is a subgroup of G iff HK = KH.
4. If H, K are subgroups of an abelian group G, then HK is subgroup of G.
5. If H_1, H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G.
6. Arbitrary intersection of subgroups i.e the intersection of any family of subgroups of a group is a subgroup.
7. The union of two subgroups is not necessarily a subgroup.

Cayley's Theorem

Every finite group G is isomorphic to a permutation group.

Cosets

Let (G, $*$) be a group and (H, $*$) be a sub group of G for any $a \in G$, the set
$aH = \{a * h \mid h \in H\}$ is called the left coset of H, determined by a.
$Ha = \{h * a \mid h \in H\}$ is called the right coset of H, determined by a.

Example: Consider the group $(\{0, 1, 2, 3\}, +_4)$, whose table is given below:

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Now consider $H = \{0, 2\}$

H since $H \subseteq G$,

$a \in H$, $b \in H$, $a * b \in H$, (it is closed), identity $e = 0 \in H$,

$0^{-1} = 0 \in H$ and $2^{-1} = 2 \in H$

$\therefore$ H is clearly a subgroup of G.

Now the left coset determined by 0 is $\{0, 2\}$

Now the left coset determined by 1 is $\{1, 3\}$

Now the left coset determined by 2 is $\{0, 2\}$

Now the left coset determined by 3 is $\{1, 3\}$

$\therefore$ There are only two distinct left cosets of H in G.

Note: If $a \in H$, $aH = H$

Similarly the right coset determined by 0 is $\{0, 2\}$

Similarly the right coset determined by 1 is $\{1, 3\}$

Similarly the right coset determined by 2 is $\{0, 2\}$

Similarly the right coset determined by 3 is $\{1, 3\}$

Since in this subgroup the set of left cosets & right cosets of H in G are same, H is a normal subgroup of G.

Notice that although $H = \{0, 2\}$ is a subgroup of G, $T = \{1, 3\}$ is not a subgroup of H (closure property does not hold).

3.6.10 Normal Subgroup

A subgroup H of a group G is said to be a **normal subgroup** of G iff $aH = Ha \ \forall a \in G$ (Where aH and Ha are the left and right cosets of H in G).

Alternatively, if for every $x \in G$, and for every $h \in H$, $x h x^{-1} \in H$, then H is a normal subgroup of G.

A group having no proper normal subgroups is called a simple group.

Some important results on normal subgroups:

1. A subgroup H of a group G is normal iff $x Hx^{-1} = H \ \forall x \in G$.
2. A subgroup H of a group G is a normal subgroup of G iff each left coset of H in G is a right coset of H in G i.e $aH = Ha \ \forall a \in G$.
3. The intersection of any two normal subgroups of a group is a normal subgroup.
4. The intersection of any collection of normal subgroups is itself a normal subgroup.

Lagrange's Theorem

The order of each subgroup of a finite group is a divisor of the order of the group.

NOTE: Converse of Lagrange's theorem is not true.

Corrollary: Order of the product of two subgroups of a group G.

Let H and K be finite subgroups of a group G.

$$O(HK) = \frac{O(H) \circ O(K)}{O(H \cap K)}$$

Example for lagranges theorem:

We have seen that in the group ({0, 1, 2, 3} $+_4$)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

H = {0, 2} is a sub group.

The order of this group is $|G| = 4$

The order of this subgroup is $|H| = 2$

Clearly $|H|$ divides $|G|$, which verifies the lagranges theorem.

In fact from lagranges theorem, we can conclude that in this case a subgroup of order 3 is not possible since 3 does not divide 4.

3.7 Lattice

Posets: A non empty set P, together with a binary relation R is said to form a partially ordered set or a poset if following conditions are satisfied

1. **Reflexivity:** aRa for all $a \in P$
2. **Anti symmetry:** If $a\,Rb$ and $b\,Ra$ then $a = b (\forall a, b \in P)$
3. **Transitivity:** If $a\,Rc$, $b\,Rc$ then $a\,Rc$ $(\forall a, b, c \in P)$

In other words, a non empty set P, together with a partial order relation is called as a poset (or partially ordered set.)

For convenience, we generally use the symbol ≤ in place of R. We read ≤ as "less than or equal to" (although it may have nothing to do with the usual "less than or equal to" that we are so familiar with).

If $a \leq b$ or $b \leq a$ in a poset, we say that a and b are comparable. Two elements of a poset may or may not be comparable. If $a \leq b$ and $a \neq b$, we will write $a < b$ (and read as "a is less than b").

Example: (S, ⊆) is a poset where S is the set of all sets. So if (P(A), ⊆) where P(A) is the power set of a given set A. The set (Z, ≤) is also a poset where "≤" is the usual numerical ≤.

The set (Z^+, divides) denoted also as (Z^+, |) is also a poset where "|" symbol means aRb iff $a \mid b$ (a divides b).

3.7.1 TOSET

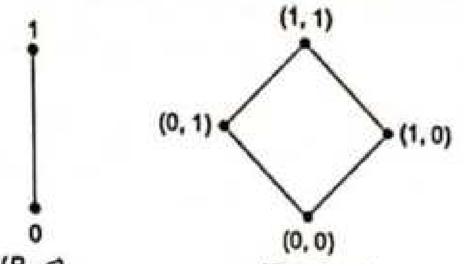
A poset (P, ≤) in which every pair of element $a, b \in P$ are comparable (i.e. either $a \leq b$ or $b \leq a$) is called a toset (totally ordered set) or a chain. Example (Z, ≤) is a toset.

Product Partial Order

If $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are two partial order. Then we define a new partial order called product partial order which is $(P_1 \times P_2, \leq)$ in this way.

$(a_1, b_1) \le (a_2, b_2)$ iff $a_1 \le_1 a_2$ and $b_1 \le_2 b_2$ where $(a_1, b_1), (a_2, b_2) \in P_1 \times P_2$

Example:



$(P_1, \le)$ $(P_1 \times P_2, \le)$

Here, $P_1\{0, 1\} = P_2(P_1 \times P_2, \le)$

If the posets P_1 and P_2 are lattices, then $(P_1 \times P_2, \le)$ is called the product lattice.

3.7.2 Hasse Diagram

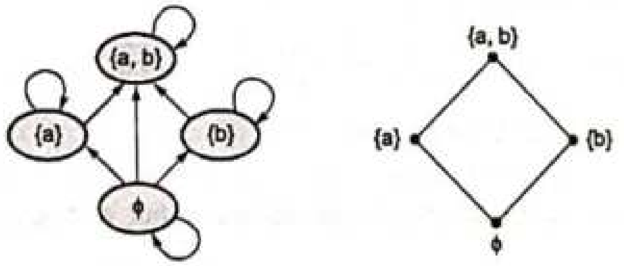
The digraph of a poset may be very complicated. To simplify this diagram while retaining the essential features of the poset, a diagram called Hasse diagram is drawn. The Hasse diagram contains all information regarding poset; but is much more simplified and easier to interpret and use than a digraph. The procedure for constructing a Hasse diagram of a poset is as follows:

1. Draw the digraph of the poset, so that all arrows are pointing upwards.
2. Reduce the circle of the nodes to points with labels adjacent to the points.
3. Remove all self loops (since it is understood that a partial order relation is always reflexive)
4. Remove all arrows which can be inferred by transitive property (i.e. aRb, bRc and aRc, then remove arrow corresponding to aRc.)
5. Remove all the arrow heads (since it is understood that the arrows are pointing upwards)

The result of the above 5 steps is a Hasse diagram of the poset.

NOTE: The Hasse diagram of a Toset (or a chain) will have no branches. (i.e. it will be a line of nodes in a chain).

Example: The digraph and the corresponding hasse diagram for the Poset (P(A),$\subseteq$), where A = {a, b} is shown next page:



Digraph Hasse Diagram

3.7.3 Supremum and Infimum of Poset

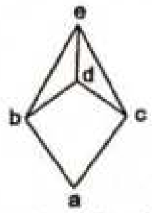
Let S be a non empty subset of a poset P. An element $a \in$ P is called an upper bound of S if $x \le a \, \forall x \in$ S. Further if a is an upper bound of S such that $a \le b$ for all upper bounds b of s, then a is called least upper bound ($\ell.u.b$) or supremum of S. We write sup S for supremum S.

It is important to note that there can be more than one upper bound of a set. But sup, if it exists, will be unique. Again comparing with the definition of greatest element we notice whereas the greatest element belonged to the set itself, an upper bound or sup can lie outside the set.

An element $a \in$ P will be called a lower bound of S if $a \leq x$, $\forall x \in$ S and a will be called greatest lower bound ($g.\ell.b$) or infimum S or Inf S, if $b \leq a$ for all lower bounds b of S.

In the hasse diagram example above, if we take say S = {{a}, {b}}, then LUB(S) =Sup (S) = {a, b} GLB(S) = Inf (S) = ϕ

In the hasse diagram given below:



Let, S = {b, c, d}, Now UB (S) = upper bounds of S = {d, e}

LUB (S) = Sup (S) = d

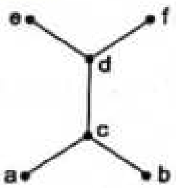
GLB (S) = Inf (S) = a

NOTE: Although, there may be many upper and lower bounds for a given subset S of a poset, there can be only one LUB and one GLB of S. i.e. LUB (or Sup (S)) and GLB (or Inf (S)) of S in unique.

3.7.4 Maximal and Minimal Elements of Poset

An element $a \in$ P is called maximal if there exist no element x such that $a \leq x$. (i.e. no element is above a in hasse diagram) An element $a \in$ P is called minimal if there exist no element x such that $x \leq a$. (i.e. no element is below **a** in the hasse diagram).

Example: In the hasse diagram below:



e and f are maximal elements of the poset.

a and b are the minimal elements of the poset.

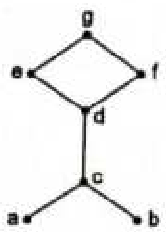
NOTE: There can be more than one maximal or minimal elements in a poset.

3.7.5 Greatest and Least Elements of a Poset

An element $a \in$ P is called the greatest element if $\forall x \in$ P, $x \leq a$. (Usually the greatest element s sometimes devoted by I in the hasse diagram) (i.e. In hasse diagram, element a is above all elements of the poset).

An element $a \in$ P is called the least element if. $\forall x \in$ P, $a \leq x$ (i.e. in hasse diagram, element a is below every element of the poset). Usually the least element is sometimes denoted by 0 in the Hasse diagram.

Example:



In the hasse diagram shown above, g is the greatest element. But there is no least element in this poset. Notice, however, that there are two minimal elements in this poset, none of which is the least element (since they are not comparable).

NOTE: There can be only one greatest element and only one least element, if they exist for any poset. A chain is a set of all comparable elements and an antichain is a set of all incomparable elements in a poset.

Theorem

If the longest chain in a partial order is of length n, then the partial order can be written as a partition of n antichains.

Dual of the Above Theorem

If the longest antichain has size t, then the set can be partitioned into t-chains.

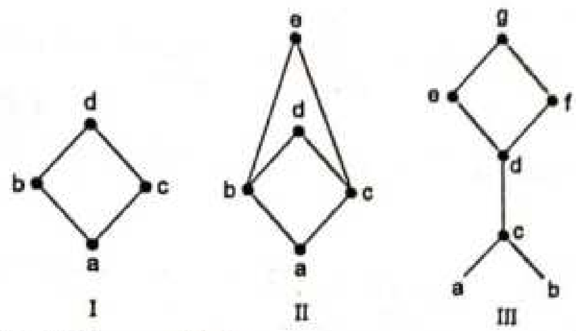
Lattices

A poset (L, ≤) is said to form a lattice if for every pair of elements $a, b \in L$, Sup{a, b} and Inf {a, b} exist in L.

In that case, we write

Sup{a, b} = $a \vee b$ (read 'a join b') = LUB (a, b)

Inf{a, b} = $a \wedge b$ (read 'a meet b') = GLB (a, b)



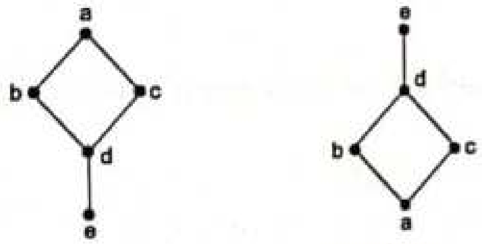
(*i*) is a lattice, while II and III are not lattices.

(*ii*) is not a lattice, since $b \vee c$ does not exist (There are two upper bounds to {b, c}, which are d and e. But d and e are not comparable, hence no LUB.

(*iii*) is not an lattice since, GLB (a, b) does not exist.

Dual Lattice

For a lattice (P, ≤), the dual is (P, ≥). The duals are shown in figure below. The diagram of (P, ≥) is obtained from that of (P, ≤) by simply turning it upside down.



$a \le b \le d \le e$ $\qquad$ $e \ge d \ge b \ge a$

$a \le c \le d \le e$ $\qquad$ $e \ge d \ge c \ge a$

Example: Let A be a non empty set then the poset $(P(A), \subseteq)$ of all subset of A is a lattice. Here for A, B $\in$ P(A) $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.

Example: The poset {2, 3, 4, 6} under divisibility is not a lattice as $4 \vee 6$ = LCM (4, 6) does not exist.

NOTE: A poset $(L, \leq)$ is a lattice iff every non empty finite subset of L has sup and Inf.

Some Lattice Results

If L is any lattice, then for any $a, b, c \in L$, the following results hold

1. $a \wedge b \leq a, b \leq a \vee b$
2. $a \leq b \Leftrightarrow a \wedge b = a$ (Consistency)
   $\Leftrightarrow a \vee b = b$
3. $a \wedge a = a, a \vee a = a$ (Idempotency)
4. $a \wedge b = b \wedge a, a \vee b = b \vee a$ (Commutativity)
5. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (Associativity)
   $a \vee (b \vee c) = (a \vee b) \vee c$
6. Domination Laws:
   If $0, I \in L$, then
   $0 \wedge a = 0, 0 \vee a = a$
   $I \wedge a = a, I \vee a = I$
7. $a \wedge (a \vee b) = a$
   $a \vee (a \wedge b) = a$ (Absorption Laws)
8. $a \leq b, c \leq d \Rightarrow a \wedge c \leq b \wedge d$
   $a \vee c \leq b \vee d$
   In particular
   $a \leq b \Rightarrow a \wedge x \leq b \wedge x$ and $a \vee x \leq b \vee x \ \forall x \in L$
9. In any lattice the distributive inequalities hold.
   $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
   $a \wedge (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
10. In any lattice L, the modular inequality holds.
    $a \wedge (b \vee c) \geq b \vee a \wedge c$
    holds for all $a, b, c \in L, a \geq b$.
11. In any lattice L
    $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Another Definition of a Lattice

A non empty set L together with two binary operations $\wedge$ and $\vee$ is said to form a lattice if $\forall a, b, c \in L$, the following conditions hold.

(i) Idempotency: $a \wedge a = a, a \vee a = a$
(ii) Commutativity: $a \wedge b = b \wedge a, a \vee b = b \wedge a$
(iii) Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c$
(iv) Absorption: $a \wedge (a \vee b) = a, a \vee (a \wedge b) = a$

NOTE

- Dual of lattice is a lattice.
- Product of two lattice is a lattice.
- A finite lattice surely has least and greatest elements i.e. a finite lattice is always bounded.
- Every chain is a lattice.
- The set $(D_n, |)$ is always a lattice, where D_n is the set of divisors of integer n. Here LUB (a, b) = LCM (a, b) and GLB (a, b) = GCD (a, b).
- The set $(Z^+, |)$ is also a lattice.
- The set $(P(A), \subseteq)$ is a lattice, (In fact, it is also a Boolean Algebra). Here for any two sets A and B.
  LUB $(A, B) = A \cup B$ and GLB$(A, B) = A \cap B$.

3.8 Types of Lattices

Bounded Lattice: A Lattice $(L, \leq)$ is called bounded, if the lattice has a greatest and least element, usually denoted by I and O respectively. (or sometimes 1 and 0).

3.8.1 Bounded Lattice Properties

1. $\forall a \in L$ $\qquad 0 \leq a \leq 1,$
2. $0 \wedge a = 0,$ $\qquad 0 \vee a = a$
3. $I \wedge a = a,$ $\qquad I \vee a = I$

Example: $(P(A), \subseteq)$ is bounded where $A = \{a, b\}$ with $I = \{a, b\}$ & $0 = \phi$ while $(Z, \leq)$ is an unbounded lattice.

3.8.2 Complemented Lattice

The complement a of any element in a lattice $(L, \leq)$ is an element which satisfies both the properties give below:

$a \wedge a' = 0$ & $a \wedge a' = I$

Obviously, complement is defined only for a bounded lattice. If in a lattice $(L, \leq)$, if at least one complement exists for every element $a \in L$, then such a lattice is called a complemented lattice.

3.8.3 Distributive Lattice

A Lattice $(L, \leq)$ is called distributive, if it satisfies both the distributive laws.

i.e. $\forall a, b, c \in L$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

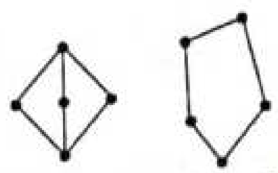
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Checking if a given hasse diagram corresponds to a distributive lattice or not, is tedious.

However the following result is useful for checking if a lattice is non distributive or not.

Theorem

A lattice is non-distributive iff it contains a sublattice, isomorphic to one of the non distributive lattices given below.



Another result that is useful for establishing if a lattice is distributive or not is the following.

Theorem

In a distributive bounded lattice, if a complement exists, then it is unique.

Another way to understand this is that if in a lattice, there is more than one compliment for some element of the lattice, then such a lattice cannot be distributive. This theorem cannot be used to prove that the lattice is distributive. It can only be used to show that a lattice is non-distributive.

NOTE: A bounded, complemented and distributive lattice is also called a Boolean Algebra.

3.8.4 Semi Lattices

1. A poset $(P, \leq)$ is called a meet semi lattice if for all $a, b \in P$, Inf $\{a, b\}$ exists.
2. A non empty set P together with a binary composition $\wedge$ is called a meet semi lattice if $a, b, c \in P$.
   (i) $a \wedge a = a$
   (ii) $a \wedge b = b \wedge a$
   (iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

3.8.5 Complete Lattices

A lattice L is called a complete lattice if every non empty subset of L has its Sup and Inf in L.

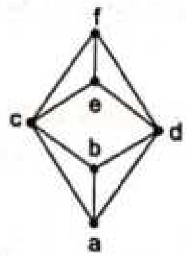
Results

1. Dual of a complete lattice is complete.
2. If $(P, \leq)$ is a poset with greatest element I such that every non empty subset S of P has Inf, then P is a complete lattice.
3. If $(P, \leq)$ is a poset with least element 0 such that every non empty subset S of P has Sup then, P is a complete lattice.

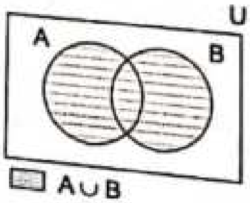
3.8.6 Sub-lattices

A non empty subset s of a lattice L is called a **sublattice** if, $a, b \in S \Rightarrow a \wedge b, a \vee b \in S$ (it is understood that $\wedge$ and $\vee$ are taken in L).

Example: Consider the lattice given below:



Now the lattice given below is a sublattice of the lattice given above.



A sublattice S of a lattice L is called a **"convex sublattice"** if for all $a, b \in S$ $[a \wedge b, a \vee b] \subseteq S$.

Results Regarding Sub-lattices

1. ϕ is a subset of every lattice (as it vacuously satisfies definition).
2. Every lattice is a sublattice of itself.

3. If L is any lattice and $a \in L$ be any element then {a} is a sublattice of L.
4. Every non empty subset of a chain is a sublattice (called a subchain)
5. The union of two sublattice may not be a sublattice.
6. A lattice is a chain iff every non-empty subset of it is a sublattice.

3.9 Boolean Algebra

Definition: A Lattice is called a **boolean algebra** if it is bounded, complemented and distributive.

A non empty set alongwith two binary operations "$\vee$" and "$\wedge$" (i.e. Sup and Inf), is called a Boolean Algebra if it is satisfies the following 6 axioms.

We may substitute + 4 for $\vee$ and . for $\wedge$ in a Boolean Algebra.

Axioms

1. **Closure:** $\forall a, b \in S, a + b \in S\ a.b \in S$
2. **Commutativity:** $\forall a, b \in S, a + b = b + a\ a.b = b.a$
3. **Associativity:** $\forall a, b, c \in S, a + (b + c) = (a + b) + c\ a.(b.c) = (a.b).c$
4. **Distributivity:** $\forall a, b, c \in S, a + (b.c) = (a + b).(a + c)\ a.(b + c) = (a.b) + (a.c)$
5. **Existence of Identity:** $\forall a \in S, \exists e$ (unique) such that $a + e = e + a = a$
6. **Existence of Compliment:** $\forall a \in S, \exists e' \in$ such that $a + a' = a' + a = 1$ and $a.a' = a'.a = 0$

Other Derived Laws of Boolean Algebra

1. $\left.\begin{matrix} a+a=a \\ a.a=a \end{matrix}\right]$ idempotent laws

2. $(a')' = a$ - double complement law

3. $\left.\begin{matrix} a+a.b=a \\ a.(a+b)=a \end{matrix}\right]$ absorption laws

4. $\left.\begin{matrix} (a+b)'=a'.b' \\ (a.b)'=a'+b' \end{matrix}\right]$ Demorgan's Laws

5. $\left.\begin{matrix} a+0=a, & a.0=0 \\ a+1=1, & a.1=a \end{matrix}\right]$ Domination Laws

Operator Precedence in Boolean Expressions

1. Expressions are scanned from left to right.
2. Expressions are evaluated with following precedence, ( ), complement, ., +

   Examples: A + B. C will be Evaluated as A + (B. C)

   In $\overline{(A+B)}$, (A+B) is evaluated first and then complemented.

Simplification of Boolean Expressions

Boolean algebraic expressions may be simplified by using axioms and derived laws of Boolean Algebra.

Example:

$$\begin{aligned} (a + ab)'(a + b) &= (a)'(a + b) && \text{(Absorption law)} \\ &= a'a + a'b && \text{(Distributive law)} \\ &= 0 + a'b && \text{(Complement law)} \\ &= a'b && \text{(Domination law)} \end{aligned}$$

Summary

- Sets which have a finite number of elements are called **finite sets** and those having infinite number of elements are called **infinite sets**.
- Most of relationship between the sets can be represented by diagrams known as venn diagrams.
- Venn diagrams can be effectively used for proving equality of set expressions or for answering question regarding counting of elements of sets.
- **Properties of Cartesian Product:**
  1. $A \times B \neq B \times A$
  2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
  3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
  4. $A \times (B - C) = (A \times B) - (A \times C)$
  5. $(A \times B) \cap (C \times D) = (A \cap C) \times (C \cap D)$
  6. $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$
- The representation of a relation in set builder form is complete only when the sets A and B are clearly specified.
- A relation R on A is called **reflexive**, if $\forall x \in A\ (x, x) \in R$ i.e. $\forall x \in A, xRx$
- A relation R in A is called symmetric relation iff $(x, y) \in R \Rightarrow (y, x) \in R$
  i.e., $xRy \Rightarrow yRx\ \forall x, y \in A$
- A relation R on A is called **anti symmetric** iff $xRy \Rightarrow y\not{R}x$, unless $x = y$
- A relation R on A is called transitive iff $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$
  i.e., xRy and $yRz \Rightarrow xRz \quad \forall x, y, z \in A$
- A relation R on A is called **irreflexive** iff $\forall x \in A, (x, x) \notin R$. i.e. $\forall x \in A, x\not{R}x$
- A relation R on A is an **asymmetric** relation iff $(x, y) \in R \Rightarrow (y, x) \notin R \quad xRy \Rightarrow y\not{R}x$.
- A relation R on a non empty set A is called equivalence relation iff
  (a) R is reflexive i.e $xRx\ \forall x \in A$
  (b) R is symmetric i.e $xRy = yRx$
  (c) R is transitive i.e xRy and $yRz \Rightarrow xRz \quad \forall x, y, z \in A$
- A relation R on a non empty set A is called a partial order relation iff.
  (a) R is reflexive $\forall x \in A, xRx$
  (b) R is antisymmetric xRy and $yRx \Rightarrow x = y$
  (c) R is transitive xRy and $yRz \Rightarrow xRz$
- Every quotient set A/R is also a partition of A. Here, the converse is also true.
- Corresponding to every partition P of A, there exists an unique equivalence relation whose quotient set is exactly P.
- A **function or mapping** is a relation between the elements of A and those of B having no ordered pairs with the same first component.
- If in a group G, the underlying set G consists of a finite number of elements, then the group is called finite group, otherwise as infinite group.
- **Properties of Cyclic Group:**
  1. Every cyclic group is an abelian group.
  2. If a is generator of a cyclic group G, then a^{-1} is also a generator of G.
  3. A cyclic group G with generator a of finite order n, is isomorphic to multiplicative group of n, n^{th} roots of unity.

4. A cyclic group G with a generator of finite order n is isomorphic to the additive group of residue classes modulo n.
5. If a finite group of order n contains element of order n, the group must be cyclic.
6. Every group of prime order is cyclic.
7. Every subgroup of a cyclic group is cyclic.

- Every finite group G is isomorphic to a permutation group.
- A subgroup H of a group G is said to be a **normal subgroup** of G iff $aH = Ha \ \forall a \in G$ (Where aH and Ha are the left and right cosets of H in G).
- A poset $(P, \leq)$ in which every pair of element $a, b \in P$ are comparable (i.e. either $a \leq b$ or $b \leq a$) is called a toset (totally ordered set) or a chain. Example $(Z, \leq)$ is a toset.
- Although, there may be many upper & lower bounds for a given subset S of a poset, there can be only one LUB and one GLB of S. i.e. LUB (or Sup (S)) & GLB (or Inf (S)) of S in unique.

Student's Assignment

Q.1 An equivalence relation is a relation which is
(a) Reflexive and symmetric
(b) Symmetric and Transitive
(c) Reflexive, Symmetric and transitive
(d) None of the above

Q.2 A function $f: N \to N$ defined by $f(n) = 2n + 3$ is
(a) Surjective (b) Injective
(c) Both (d) Non injective

Q.3 Let L be the set of lines in the Euclidean plane. Let R be the relation on L defined by "is parallel to". Then which of the following is true?
(a) R is an equivalence relation
(b) R is POSET
(c) R is reflexive symmetric
(d) None of the above

Q.4 If a lattice is distributive, then
(a) $a * (b \cdot c) = (a \cdot b) * (b \cdot c)$
(b) $a \cdot (b * c) = (a \cdot b) * (a \cdot c)$
(c) $a * (b * c) = (a * b) \cdot (a * c)$
(d) None of these

Q.5 Let A and B be sets with cardinalities 2 and 4 respectively. The number of one-one mapping from A to B is
(a) 4^2 (b) ${}^4P_2$
(c) 4! (d) 1

Q.6 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I	List-II
A. Identity	1. Monoids
B. Associativity	2. Abelian groups
C. Commutative	3. Semi groups
D. Left inverse	4. Groups

Codes:

	A	B	C	D
(a)	1	3	2	4
(b)	1	2	3	4
(c)	4	3	1	2
(d)	2	3	4	1

Q.7 Given set S = {1, 5, 7, 11}. Then S is a group w.r.t.
(a) multiplication modulo 12
(b) addition modulo 6
(c) summation modulo 8
(d) S is not a group

Q.8 Let P and Q be any two equivalence relations on a non-empty set S, then choose the correct one
(a) $P \cup Q$, $P \cap Q$, are both equivalence relations
(b) $P \cup Q$ is an equivalence relation
(c) $P \cap Q$ is an equivalence relation
(d) Neither $P \cup Q$ nor $P \cap Q$ is an equivalence relation

Q.9 Consider the following binary relation

$S = \{(x, y) \mid y = x + 1$ and
$x, y \in \{0, 1, 2, ...\}\}$

The symmetric closure of S is

(a) $\{(x, y) \mid x = y + 1$ and $x, y \in \{0, 1, 2, ...\}\}$
(b) $\{(x, y) \mid y = x + 1$ and $x, y \in \{0, 1, 2, ...\}\}$
(c) $\{(x, y) \mid y = x \pm 1$ and $x, y \in \{0, 1, 2, ...\}\}$
(d) None of the above

Q.10 Which of the following statements is not true?

(a) If z is the set of integers and $\leq$ is the usual ordering on z, then $[z; \leq]$ is partially ordered and totally ordered.
(b) If z is the set of integers and $\leq$ is the usual ordering on z, then $[z; \leq]$ is partially ordered but not totally ordered
(c) U be an arbitrary set and A = P (U) be the collection of all subsets of U. Then $[P(U); \subseteq]$ is a poset.
(d) If U contains more than one element then it is not totally ordered.

Q.11 Consider the following relation:

$\{(a, a), (a, b), (a, c)\} \subseteq \{a, b, c\} \times \{a, b, c\}$

Which of the following statement is true about the above relation

(a) It is not a function
(b) It is a function which is not one-to-one or onto
(c) It is a function which is one-to-one but not onto
(d) It is a function which is both one-to-one and onto

Q.12 N denotes the set of natural numbers, {0, 1, 2, ...3). Z denotes the integers, {..., –2, –1, 0, 1, 2, ...} which of the following statements are true?

(*i*) For all $p \in Z$, $p > 5 \rightarrow$ There exists $x \in N$, $x^2 \equiv 1(\text{mod}\, p)$
(*ii*) If m is any natural number satisfying $m \equiv 1(\text{mod}2)$, then the equation $2048\, x \equiv 1(\text{mod}m)$ is guaranteed to have a solution for x.

(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

Q.13 If $|A| = k$ and $|B| = m$, how many relation are between A and B? If in addition $|C| = n$ how many relations are there between there in $A \times B \times C$?

(a) $k + m$ and $k + m + n$
(b) $k \times m$ and $k \times m \times n$
(c) 2^{k+m} and 2^{k+m+n}
(d) 2^{km} and 2^{kmn}

Q.14 (G, $*$) is an abelian group. Then

(a) $X = X^{-1}$, for any X belonging to G
(b) $X = X^2$, for any X belonging to G
(c) $(X * Y)^2 = X^2 * Y^2$, for any X, Y belonging to G
(d) G is of finite order

Q.15 Enumerate each of the following sets

(*i*) $\phi \times \{3, 5, 9\}$
(*ii*) 2^{ϕ}
(*iii*) $2^{\{3, 5, 9\}}$

(a) ϕ, $\{\phi\}$, $\{\phi, \{3\}, \{5\}, \{9\}, \{3,5\}, \{5,9\}, \{3,9\}, \{3,5,9\}\}$
(b) $\{\phi\}$, ϕ, $\{\phi, \{3\}, \{5\}, \{9\}, \{3,5\}, \{5,9\}, \{3,9\}, \{3,5,9\}\}$
(c) $\{\phi\}$, ϕ, $\{\phi, \{3\}, \{5\}, \{9\}, \{5,9\}, \{3,9\}, \{3,5,9\}\}$
(d) None of these

Q.16 Let $R \subseteq A \times A$ and $S \subseteq A \times A$ be a binary relations as defined below:

Let A be the set of positive integers. And
$R = \{(a, b) \mid b$ is divisible by $a\}$.
Let $A = N \times N$ and $S = \{((a, b), (c, d)) \mid a \leq c$ or $b \leq d\}$.

Which of the following statements are true?

(a) R is partial order but not total order and S is partial order but not a total order
(b) R is both partial order and total order and S is neither partial order nor a total order
(c) R is partial order but not total order and S is neither partial order nor a total order
(d) R is neither partial order nor a total order and S is neither partial order nor a total order

Q.17 N denotes the set of natural numbers, {0, 1, 2, ...}. Z denotes the integers, {....., -2, -1, 0, 1, 2, ...?

which of the following statements are true?

(*i*) $\forall w \in z, \exists x \in z, \forall y \in z, \exists z \in z$, such that $w + x = y + z$
(*ii*) $\exists x \in N, \forall p \in z, p > 5 \rightarrow x^2 \equiv 1(\text{mod}\, p)$

(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

Q.18 Let $R \subset A \times A$ and $S \subset B \times B$ be binary relations as defined below:
Let $A = N$ and $R = \{(a, b) \mid b = a \text{ or } b = a + 1\}$
Let B be the set of English words, and let $(a, b) \in S$ when a is not longer than b.
(a) R is partial order but not total order and S is partial order but not a total order.
(b) R is both partial order and total order and S is neither partial order nor a total order.
(c) R is partial order but not total order and S is neither partial order nor a total order.
(d) R is neither partial order nor a total order and S is neither partial order nor a total order.

Q.19 Which of the following statements are true?
(*i*) Let $\Sigma_1 = \{a, b\}$ and $\Sigma_2 = \{0, 1, 2\}$ be disjoint alphabets.
Let Σ_1^* be the set of (finite-length) strings over S_1 and let Σ_2^* be the set of (finite-length) strings over S_2.
We can show that card (Σ_1^*) = card (Σ_2^*)
(*ii*) Let $S = \{2^i \mid i \in N\}$ be the set of integers that are powers of two. We can show that S. is uncountable.
(a) Only (*i*) is true
(b) Only (ii) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

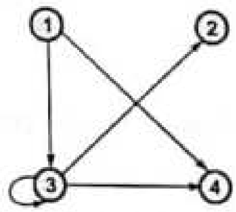
Q.20 Each of the following defines a relation on the set N of positive integers. Determine which of the following relations are reflexive.
(a) R : x is greater than y
(b) S : $x + y = 10$
(c) T : $x + 4y = 10$
(d) None of the above

Q.21 Which of the following are symmetric
(a) R : x is greater than y
(b) S : $x + y = 10$
(c) T : $x + 4y = 10$
(d) None of the above

Q.22 Let P(X) be the collection of all subsets of a set X with atleast three elements. Each of the following defines a relation on P(X):
R : $A \subseteq B$
S : A is disjoint from B
T : $A \cup B = X$
Determine which of the following relation is antisymmetric.
(a) R : $A \subseteq B$
(b) S : A is disjoint from B
(c) T : $A \cup B = X$
(d) None of the above

Q.23 Determine which of the following relation is transitive
(a) R : $A \subseteq B$
(b) S : A is disjoint from B
(c) T : $A \cup B = X$
(d) None of the above

Q.24 Find the transitive closure R* of the relation R on A = {1, 2, 3, 4} defined by the directed graph



(a) R* = {(1,2), (2,3), (1,3), (1,4), (3,2) (3,3), (3,4)}
(b) R* = {(1,2), (1,3), (1,4), (3,2), (3,3), (3,4)}
(c) R* = {(1,1), (2,2), (3,3), (4,4)}
(d) None of the above

Q.25 Let S = {1, 2, 3, 4, 5, 6}. Determine which of the following is a partition of S:
(a) P_1 = [{1,2,3}, {1,4,5,6}]
(b) P_2 = [{1,2}, {3,5,6}]
(c) P_3 = [{1,3,5}, {2,4}, {6}]
(d) P_4 = [{1,3,5}, {2,4,6,7}]

Q.26 Let X = {1, 2, ..., 8, 9}
Determine whether each of the following is a partition of X
(*i*) [{1,3,6}, {2,8}, {5,7,9}]
(*ii*) [{1,5,7}, {2,4,8,9}, {3,5,6}]
(*iii*) [{2,4,5,8}, {1,9}, {3,6,7}]
(*iv*) [{1,2,7}, {3,5}, {4,6,8,9}, {3,5}]

(a) (i) and (ii)
(b) (ii) and (iii)
(c) (iii) and (iv)
(d) (iv) and (i)

Q.27 Let A be set of integers and let ~ be the relation on A × A defined by

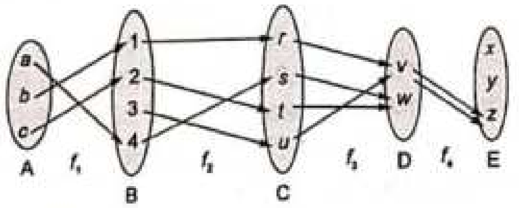
$(a, b) \sim (c, d)$ iff $a + d = b + c$

This relation satisfies
(a) Reflexive, symmetric
(b) Symmetric, transitive
(c) Reflexive, symmetric and transitive
(d) None of above

Q.28 The relation R = {(1,1), (1,2), (2,1), (2,2), (3,3)} is an equivalence relation of the set S = {1, 2, 3}. Find the quotient S/R
(a) {[2]}
(b) {[1], [2]}
(c) {[1]}
(d) {[1], [3]}

Q.29 Determine which of the following is a partition of the set R of real numbers.
(a) $[\{x : x > 4\}, \{x : x < 5\}]$
(b) $[\{x : x > 0\}, \{0\}, \{x : x < 0\}]$
(c) $[\{x : x^2 > 11\}, \{x : x^2 < 11\}]$
(d) None of the above

Common Data Questions (30 and 31):



Functions $f_1 : A \to B$, $f_2 : B \to C$, $f_3 : C \to D$ and $f_4 : D \to E$,

Q.30 Which of the functions are one-to-one
(a) f_1 and f_2
(b) f_2 and f_3
(c) f_3 and f_4
(d) f_4 and f_1

Q.31 Which of the following functions are onto functions
(a) f_1 and f_2
(b) f_2 and f_3
(c) f_3 and f_4
(d) f_4 and f_1

Q.32 Which of the following functions are invertible
(a) f_1
(b) f_2
(c) f_3
(d) f_4

Q.33 Let R be a binary relation on the set of all positive integers such that
R = {(a, b) | a − b is an odd positive integer}
Thus R is
(a) anti-symmetric relation
(b) reflexive and symmetric relation
(c) equivalence relation
(d) partial ordering relation

Q.34 $A \cup B = A \cap B$ if and only if
(a) A is empty set
(b) B is empty set
(c) A and B are non-empty sets
(d) A = B

Q.35 Let A and B be sets with cardinalities m and n. The number of one-one mappings from A to B, when $m < n$ is
(a) m^n
(b) ${}^nP_m$
(c) ${}^mC_n$
(d) ${}^nC_m$

Q.36 Which additional properties are true if a partial order "≤" must become a linear order
(i) for any a and b is S, atleast one of $a \le b$ (or) $b \le a$ is true.
(ii) for all a, b and c in S, if $a \le b$ and $b \le c$, then $a \le c$.
(iii) for any a and b in S, exactly one of $a \le b$, (or) $b \le a$ is true.
(a) Only (i)
(b) Both (ii) and (iii)
(c) Only (iii)
(d) All of the above

Q.37 Suppose A = { }, B = {1, 2, 3}. What does the set B × A contain?
(a) { }
(b) {1, 2, 3}
(c) {(1), (2), (3)}
(d) None of these

Q.38 Consider a binary relation R shown in the following matrix on set

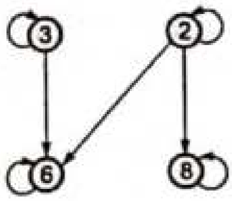
S = {1, 2, 3, 4}.

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The relation R is

(a) Equivalence relation
(b) Irreflexive and antisymmetric
(c) Irreflexive, symmetric and transitive
(d) Transitive but neither reflexive nor irreflexive

Q.39 Following figure shows relation on set $S = \{2, 3, 6, 8\}$.



The relation is
(a) Equivalence relation
(b) Poset (partial order relation)
(c) Symmetric and reflexive relation
(d) None of the above

Common Data Questions (40 and 41):

Let $X = \{1, 2, 3, 4\}$ if

$R = \{<x, y> \mid x \in X;\ y \in X;\ |x-y| > 0;\ |x-y|\%2 = 0\}$

$S = \{<x, y> \mid x \in X;\ y \in X;\ |x-y| > 0;\ |x-y|\%3 = 0\}$

Q.40 Find $|R \cup S|$ and $|R \cap S|$
(a) $|R \cup S| = 6,\ |R \cap S| = 0$
(b) $|R \cup S| = 3,\ |R \cap S| = 6$
(c) $|R \cup S| = 2,\ |R \cap S| = 2$
(d) $|R \cup S| = 5,\ |R \cap S| = 3$

Q.41 If $X = \{1, 2, 3 \ldots\}$, what is $R \cap S$?

(a) $R = \left\{ \begin{matrix} <x,y> \mid x \in X; y \in X; (x-y) > 0; \\ (x-y)\%2 = 0 \text{ or } (x-y)\%3 = 0 \end{matrix} \right\}$

(b) $R = \{<x, y> \mid x \in X;\ y \in X;\ (x-y) > 0;\ (x-y)\%6 = 0\}$

(c) $R = \{<x, y> \mid x \in X;\ y \in X;\ (x-y) > 0;\ (x-y)\%5 = 0\}$

(d) None of the above

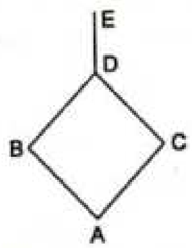
Q.42 An empty relation f is
(a) symmetric but reflexive
(b) equivalence relation
(c) partial order
(d) None of the above

Q.43 Let A be the set of non-zero integers and let # be the relation on A × A defined as (a, b) # (c, d) iff ad = bc. The relation A is
(a) Equivalence relation
(b) Poset
(c) Antisymmetric
(d) Reflexive and symmetric but not transitive

Q.44 Which of the following statements is true about $B = \{D, \{A\}\}$
(a) $A \in B$
(b) $\{A\} \in B$
(c) $\{A\} \subseteq B$
(d) $\{D, A\} \in \text{pow}(B)$

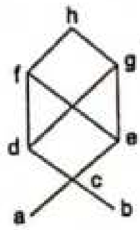
Q.45 $R : A \rightarrow B$.
A_1 is subset of A and A_2 is also a subset of A. Which of the following statements is not correct?
(a) $R(A_1 \cup A_2) \subseteq R(A_1) \cup R(A_2)$
(b) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$
(c) $R(A_1) \cup R(A_2) \subseteq R(A_1 \cup A_2)$
(d) $R(A_1) \cap R(A_2) \subseteq R(A_1 \cap A_2)$

Q.46 Consider the following figure which of the following is true?



(a) There exists a Euler path but not Euler circuit
(b) There exists a Euler circuit
(c) Euler path is not possible
(d) None of the above

Q.47 Consider the poset A = {a, b, c, d, e, f, g, h}. The Hasse diagram is given below.



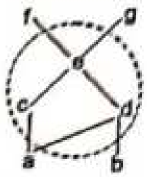
Find the lower and upper bound for B = {a, b} respectively.

(a) {a, b} and {c}
(b) {a, b} and {f}
(c) {} and {c, d, e, f, g, h}
(d) {} and {c}

Q.48 With respect to previous question, the lower and upper bound for $B_1 = \{c, d, e\}$ respectively are
(a) {a, b} and {h}　(b) {c} and {h}
(c) {c, a, b} and {h}　(d) {c, a, b} and {f, g, h}

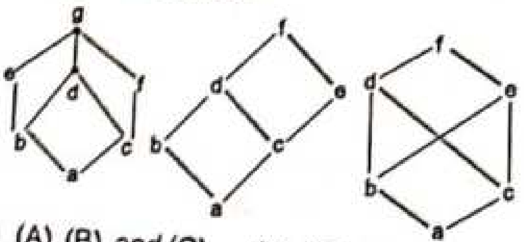
Q.49 What is the cardinality of a multiset having letters "MI SSI SSI PPI"?
(a) 4　(b) 11
(c) 3　(d) 6

Q.50 Let V = {a, b, c, d, e, f, g} be a partially ordered set as shown in figure and let X = {c, d, e}. Find the upper and lower bounds of *x*.



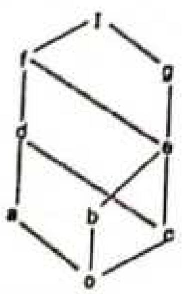
(a) Upper bounds-e,f, and g, lower bound-a
(b) Upper bounds-d, e, and f, lower bound b
(c) Upper bounds-c, d, and e, lower bound-a
(d) None of the above

Q.51 Identify which of the partially ordered sets shown in the figure are lattices



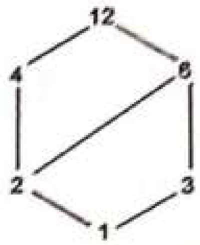
(a) (A), (B), and (C)　(b) (B), (C)
(c) (C), (A)　(d) (A), (B)

Q.52 Find the join- irreducible elements of the lattice *k* shown in figure.



(a) a, b, c, d　(b) b, c, d, g
(c) a, b, c, g　(d) b, c, g, e

Q.53 Consider the lattice $D_{12} = \{1, 2, 3, 4, 6, 12\}$, the divisors of 12 ordered by divisibility as shown in figure. Find
1. Lower bound and upper bound of D_{12}
2. The complements of 4 and 6
3. Is D_{12} a complemented lattice?



(a) 1. low bound is 2 and upper bound is 12
　2. complement of 4 is 3, 6 has no complement
　3. NO
(b) 1. L. B. is 1 and U. B. is 12
　2. complement of 4 is 3, 6 has no complement
　3. NO
(c) 1. L. B. is 1 and U. B. is 12
　2. complement 4 is 3. complement of 6 is 3
　3. YES
(d) 1. L. B. is 1 and U. B. is 12
　2. 4 has no complement, complement of 6 is 3
　3. YES

Q.54 Which of the following statements are true?
(*i*) If *x* is positive and irrational, then $\sqrt{x}$ is also irrational.
(*ii*) Let $\{0, 1\}^*$ denote the set of all binary strings $y.z$ denotes the concatenation of two strings y and z.
Every string $X \in \{0, 1\}^*$ can be written in the from $x = y.z$ where the number of 0' s in y is the same as the number of 1's in z. (Empty strings are allowed).
(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are true

Q.55 Set A has 4 elements and Set B has 2 elements. What is the total number of relations from B to A?

Q.56 Set S has '9' elements. Suppose you where asked to find the number of Irreflexive relations possible from set S to itself, what would be your answer.

Q.57 Set S has '6' elements, what is the total number of reflexive relations possible from set S to itself?

Q.58 What is the total number of asymmetric relations from Set A to itself which has 'n' elements?

Answer Key:

1. (c)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (b)
6. (a)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (b)
11. (a)	**12.** (c)	**13.** (d)	**14.** (c)	**15.** (a)
16. (c)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (d)
21. (b)	**22.** (a)	**23.** (a)	**24.** (b)	**25.** (c)
26. (c)	**27.** (c)	**28.** (d)	**29.** (b)	**30.** (a)
31. (b)	**32.** (b)	**33.** (a)	**34.** (d)	**35.** (b)
36. (a)	**37.** (a)	**38.** (d)	**39.** (b)	**40.** (a)
41. (b)	**42.** (d)	**43.** (a)	**44.** (b)	**45.** (d)
46. (a)	**47.** (c)	**48.** (d)	**49.** (b)	**50.** (a)
51. (d)	**52.** (c)	**53.** (b)	**54.** (c)	**55.** 256
56. 2^{72}	**57.** 2^{30}	**58.** $3^{\frac{n^2-n}{2}}$		

Student's Assignments Explanations

1. (c)

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

2. (b)

$$f(n) = 2n+3$$

Now, $f(x_1) = f(x_2)$

$\Rightarrow$ $2x_1 + 3 = 2x_2 + 3$

$\Rightarrow$ $x_1 = x_2$

$\therefore$ $f(n)$ is one-to-one, i.e. injective.

To check for onto, write the function as

$$y = 2x+3$$

$\Rightarrow$ $x = \dfrac{y-3}{2}$

Here, $y = 4 \in N$ but $x = \dfrac{4-3}{2} = \dfrac{1}{2} \notin N$

$\therefore$ f is not onto, i.e. not surjective

7. (a)

The composition table (Cayley table) of S w.r.t. multiplication module 12 is

X_{12}	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

8. (c)

PUQ may not be transitive.

9. (c)

Let us take some samples which satisfy the given binary relation. They are {(0,1), (1, 2), (2, 3), ...}

The symmetric closure of this is

{(0,1), (1, 0), (1,2), (2,1), (2,3), (3,2), ...}

The above samples satisfy the equation

$\{(x, y) \mid y = x \pm 1$ and $x, y \in \{0, 1, 2, —\}\}$

completely

11. (a)

It is not a function, since (a, b) and (a, c) are in the relation.

12. (c)

Consider A. Since, $I^2 \equiv 1 \bmod (p)$ always,

$\therefore$ $\exists x \in N$ such that $x^2 \equiv 1 \bmod (p)$, A is true.

Consider B. Since $m \equiv 1 \pmod 2$, this means m is a odd number.

This means 2048 and m are relatively prime.

The equation, $ax \equiv b \pmod m$ has a solution whenever a ad m are relatively prime.

$\therefore$ The equation $2048x \equiv 1 \pmod m$ is guaranted to have a solution for x, since 2048 and m are relatively prime.

So B is also true.

13. (d)
The number of binary relations between A and B is the number of subsets of $A \times B$.
Similarly, the number of 3-ary relations between A and B and C is the number of subsets of $A \times B \times C$.
The answers are therefore $2^{k \times m}$ and $2^{k \times m \times n}$

14. (c)

$$\begin{aligned}(x * y)^2 = (x * y)*(x * y) &= (x * (y * x) * y) \\ &= (x * (x * y) * y) \\ \text{(since (G, *) is abelian)} &= ((x * x) * (y * y)) \\ &= (x^2 * y^2) \\ &= x^2 * y^2\end{aligned}$$

16. (c)
R : $\{(a, b) \mid b$ is divisible by A$\}$ on $A \times A$
S : $\{((a, b), (c, d)) \mid a \le c$ or $b \le d$ on $A \times A$ where
$A = z^+$
R is reflexive, antisymmetric and transitive and hence is a partial order.
R is not a total order, as can be seen by a counter example such as $3 \in A$, $5 \in A$. Here, 3 does not divide 5 ad 5 does not divide 3.
i.e. 3 R 5 and 5/3.
∴ 3 and 5 are not comparable.
R is therefore not a total order.
Consider the relation S.
S is neither a partial order nor total order since S is not antisymmetric and it is not transitive.
S is not antisymmetric since (1, 2) S(4, 1) and (4, 1) S(1, 2) but $(1, 2) \neq (4, 1)$
S is not transitive since (4, 8) S (8, 4) and (8, 4) S(3, 6) but (4, 8) S (3, 6).

18. (d)
R is neither partial order nor a total order, because R is not transitive. This can be seen from the following counter example

$$\begin{aligned} &(1,2) \in R \\ &(2,3) \in R \\ \text{but} \quad &(1,3) \notin R\end{aligned}$$

S is neither partial order nor a total order, because *s* is not antisymmetric, this can be seen from the two distinct english words fox and cat. Both (fox, cat) ∈ S and (cat, fox) ∈ S, since both words have the same length, but fox ≠ cat.

19. (d)
Consider statement (*i*)
Since we cannot set up a one-to-one correspondence from Σ^* to Σ_2^*, we cannot show that $|\Sigma_1^*| = |\Sigma_2^*|$. Therefore, (*i*) is false.
Consider (*ii*).
Since $f(i) = 2^i$ is a one-to-one correspondence from the set N to the set S, and since N is countable so is S.
∴ Statement (*ii*) is also false.

20. (d)
None of these are reflexive, since (1, 1) does not belong R, S or T.

21. (b)
R is not symmetric since $x > y \not\Rightarrow y > x$
S is symmetric since $x + y = 10 \Rightarrow y + x = 10$
T is not symmetric since $x + 4y = 10 \not\Rightarrow y + 4x = 10$

22. (a)
Since $A \subseteq B$ and $B \supseteq A \Rightarrow A = B$
∴ R is antisymmetric.
Since
A is disjoint from B and B is disjoint from A
$\not\Rightarrow A = B$
∴ S is not antisymmetric
Since $A \cup B = X$ and $B \cup A = X \not\Rightarrow A = B$
∴ T is not antisymmetric.

23. (a)
Since $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$
∴ R is transitive.
Consider the velation S : A is disjoint from B let
$A = \{1, 2, 3\}$ $B = \{a, b\}$ $C = \{2, 3, 5\}$
Here $A \cap B = \phi$, $B \cap C = \phi$ but $A \cap C \neq \phi$
∴ A is disjoint from B and B disjoint from C
$\not\Rightarrow$ A disjoint from C
S is therefore, not transitive.
Consider the velation T : $A \cup B = X$
Let $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{1, 2, 3\}$ and $X = \{1, 2, 3, 4\}$
clearly, $A \cap B = X$ and $B \cap C = X$ but $A \cap C \neq X$
∴ T is not transitive.

24. (b)
$R = \{(1, 3) (1, 4), (3, 3), (3, 2), (3, 4)\}$
$(1, 3) \in R$ and $(3, 2) \in R \Rightarrow (1, 2) \in R*$
$\therefore R* = \{(1, 2), (1, 3) (1, 4), (3, 2), (3, 3), (3, 4)\}$
All the elements are now transitive in R*.

25. (c)
P_1 is not partition since $1 \in S$ elong to two cells.
P_2 is not partition since $4 \in S$ does not belong to any cell.
P_3 is a partition of S.
P_4 is not partition sice $\{2, 4, 6, 7\}$ is not a subset of S.

26. (c)
Part (c) because each element of X belongs to exactly one cell. In other words the cells are disjoint and their union is X.

27. (c)
$(a, b) \sim (c, d)$ iff $a + d = b + c$
Reflexive Property
Since $a + b = b + a$, $(a, b) \sim (a, b)$
$\therefore \sim$ is reflexive.
Symmetry Property
Let $(a, b) \sim (c, d)$
$\Rightarrow a + d = b + c \Rightarrow c + b = d + a$
$\Rightarrow (c, d) \sim (a, b)$
$\therefore (a, b) \sim (c, d) \Rightarrow (c, d) \sim (a, b)$
$\sim$ is symmetric.
Transitive property
Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$
$\Rightarrow a + d = b + c$ and $c + f = d + e$
Adding then two equation we get
$a + d + c + f = b + c + d + e$
$\Rightarrow a + f = b + e \Rightarrow (a, b) \sim (e, f)$
$\therefore (a, b) \sim (c, d)$ & $(c, d) \sim (e, f) \Rightarrow (a, b) \sim (e, f)$
$\sim$ is transitive.

28. (d)
Under the relation R, $[1] = \{1, 2\}$, $[2] = \{1,2\}$ and $[3] = \{3\}$. Noting that $[1] = [2]$. We have $SIR = \{[1], [3]\}$

29. (b)
(a) No, since the two cells are not disjoint e.g. 4.5 belongs to both cells.
(b) Yes, since the three cells are mutually disjoint and their union is R.
(c) No, since $\sqrt{11}$ in R does not belong to either cell.

30. (a)
The function f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u) = v$ and $f_4(v) = f_4(w) = z$.

31. (b)
The functions f_2 and f_3 are both onto function since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C. i.e. $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto, since $3 \in S$ is not the image under f_1 of any element of A, and f_4 is not onto since $x \in S$ is not the image under f_4 of any element of D.

32. (b)
The function f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However f_2 is both one-to-one and onto, i.e., f_2 is bijective function between A and B. Hence f_2 is invertible anf f_2^{-1} is a function from C to B.

33. (a)
R is equivalence relation if it is reflexive symmetric and transitive.
R is partial ordering relation if it is reflexive, antisymmetric and transitive.
as a-b odd positive integer, b-a is not odd positive hence antisymmetric a-b is odd positive, b-c is odd positive but (a-c) is even positive, hence not transitive.
$\therefore$ R is antisymmetric.

34. (d)
(*i*) If A is empty set then $A \cup B = B$ and $A \cap B = \phi$.
$\therefore A \cup B \neq A \cap B$
(*ii*) Same for if B is empty set.

(*iii*) Consider A = {1} B = {2} $A \cup B = \{1, 2\}$ and

$A \cap B = \phi$

$\therefore A \cup B \neq A \cap B$

(*iv*) is correct since if A = B,

$A \cup B = A \cup A = A$

and $A \cap B = A \cap A = A$

$\therefore A \cup B = A \cap B$

and conversely if $A \cup B = A \cap B$

$A = A \cap (A \cup B) = A \cap (A \cap B) = A \cap B$

$B = B \cap (A \cup B) = B \cap (A \cap B) = A \cap B$

$\therefore A = B$

$\therefore A \cup B = A \cap B$ iff A = B

35. (b)

The first element of A can be mapped in n different ways. The second element of A can be mapped only in $(n-1)$ ways since function is one-to-one. and so on.

$\therefore$ Total number of one-to-one mappings from A to B is $n(n-1)(n-2) \ldots (n-m+1) = {}^{n}P_{m}$

37. (a)

Suppose A = {a, b} and B = {1,2}, the set B × A will contain {(1, a), (1, b), (2, 1), (2, b)}.

However if either of set in relation, for instance, A or B in B×A is { }, the relation is also an empty set i.e. { }. Thus, B × A = { } = ϕ

38. (d)

One could see from the matrix for R that

- All entires in the diagonal are not 1, hence the matrix is not reflexive.
- All entires in the diagonal are not 0, hence the relation is also not irreflexive.
- The relation A is transitive since

  $(2, 3), (3, 1) \Rightarrow (2, 1)$

  $(3, 2), (2, 3) \Rightarrow (3, 3)$

  $(2, 3), (3, 2) \Rightarrow (2, 2)$

  $(3, 2), (2, 4) \Rightarrow (3, 4)$

$\therefore$ The answer is that the relation is transitive but neither reflexive nor irreflexive.

39. (b)

The relation is {<2, 2>, <3, 3>, <6, 6>, <8, 8>, <2, 8>, <2, 6>, <3, 6>}

We could see that

- The relation is reflexive as <2, 2>, <3, 3>, <6, 6>, and <8, 8> are present.
- The relation is not symmetric, since <2, 6> ∈ R but <6, 2> ∉ R.
- The relation is antisymmetric, That means, the pair (x, y), (y, x) is present if and only if $x = y$. (i.e. all arrows are unidirectional except self loops).
- The relation is transitive.

  The relation is reflexive, antisymmetric and transitive that means, its is a partially ordered set or poset.

40. (a)

R = {<1, 3>, <3, 1>, <2, 4>, <4, 2>}

S = {<1, 4>, <4, 1>}

$R \cup S$ {<1, 3>, <3, 1>, <2, 4>, <4, 2>, <1, 4>, <4, 1>}

$R \cap S = \{\}$, $|R \cup S| = 6$, $|R \cup S| = 0$

42. (d)

f is not reflexive

f is symmetric, antisymmetric and transitive

43. (a)

Given (a, b) # (c, d) iff ad = bc

1. Since ab = ba, (a, b) # (a,b)

   $\therefore$ Relation # is reflexive

2. Let (a, b) # (c, d) ⇒ ad = bc ⇒ cb = da

   ⇒ (c, d) # (a, b)

   $\therefore$ Relation # is symmetric

3. Let (a, b) # (c, d) & (c, d) # (e, f)

   ⇒ ad = bc and cf = de

   ⇒ adcf = bcde ⇒ af = be (cancelling cd from both sides)

   Now since af = be ⇒ (a, b) # (e, f)

   $\therefore$ We have how shown that

   (a, b) # (c, d) and (c, d) # (e, f) ⇒ (a, b) # (e, f)

   $\therefore$ The relation # is transitive.

   $\therefore$ # is an equivalence relation.

45. (d)

1. Since $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$, statement (a) and (c) are correct

2. $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$ is true. So, statement (b) is correct.
Only statement (d) is false.

46. (a)
Since there are exactly 2 vertices (E and D) in this graph with odd degree, this graph has an euler path but not an euler circuit.

49. (b)
The multi-set
$$S = \{M * 1, I * 4, S * 4, P * 2\}$$
$$|S| = 11$$

50. (a)
The elements e, f and g succeed every element of x; hence e, f and g are the upper bounds of x. The element a precedes every elements of x; hence it is the lower bound of x. Note that b is not a lower bound since b does not precede c; b and c are not comparable.

51. (d)
An ordered set S is a lattice if and only if sup (x, y) and inf (x, y) exist for each pair (x, y) in S. Posets (A) and (B) of given figures are lattices. Poset (C) is not a lattice since (b, c) has three upper bounds d, e and f and no one of them precedes the other two (d, e being incomparable) hence sup (b, c) does not exist.

52. (c)
The join-irreducible elements of the lattice are those with a unique predecessor. Therefore join-irreducible elements are a, b, c and g.

53. (b)
1. Lower bound is 1 and upper bound is 12.
2. The complement of 4 is 3 since
   g.c.d (4, 3) = 1 (which is the least element) and l.c.m. (4, 3) = 12 (which is the greatest element).
   6 has no complement since there is no element x satisfying both gcd $(6, x) = 1$ and lcm$(6, x) = 12$.
3. D_{12} is not a complemented lattice, since 6 has no complement.
   (In a complementes lattice, every element must have at least one complement).

56. Solution:
2^{72} Irreflexive relations [Hint: Diagonal elements are fixed as 0].

57. Solution:
2^{30} reflexive relations [Hint: Diagonal elements are fixed as 1].

CHAPTER 04

Graph Theory

4.1 Fundamental Concepts

Graph: A graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge two vertices called its end points. Vertices are sometimes called as nodes.

A loop is an edge whose end points are equal. Multiple edges are edges having the same pair of end points.

4.1.1 Types of Graphs

Simple Graph

A Simple graph is an undirected graph having no loops or multiple edges. We specify a simple graph by its vertex set and edge set treating the edge set as a set of unordered pairs of vertices (undirected graph) and writing $e = \{u, v\}$ for an edge e with endpoints u and v.

When u and v are the endpoints of an edge, they are adjacent and are neighbors. We write $u \leftrightarrow v$ for u is adjacent to v and we say that the edge

$e = \{u, v\}$ is incident on u and v.

Multigraph

A Multigraph is an undirected graph in which multiple edges between pairs of vertices is allowed. However, self loops are not allowed.

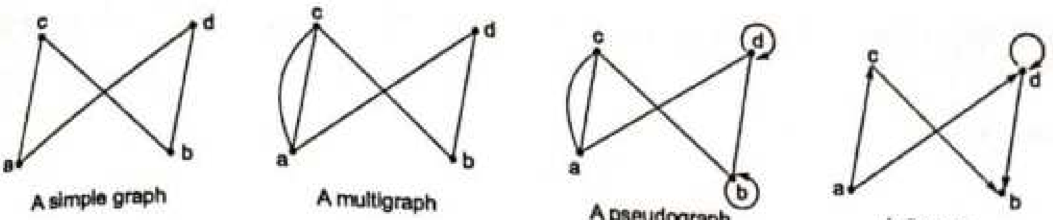
Psuedograph

A psuedograph is an undirected graph in which multiple edges as well as self loops are allowed.

4.1.2 Directed Graph

A graph in which the edges have direction. i.e. A graph (V, E) such that E is a set of directed edges that are ordered pairs of vertices of V.

NOTE: In an undirected graph, E is a set of undirected edges that are unordered pairs of vertices of V.



A simple graph A multigraph A pseudograph A digraph

Vertex Degrees & Counting

The **degree** of a vertex v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges, incident to v, except that each loop at v counts twice. The maximum degree is $\Delta(G)$, the minimum degree is $\delta(G)$, and G is regular if $\Delta(G) = \delta(G)$. i.e. all vertices have the same degree. It is **k-regular** if the common degree is k. The neighbourhood of v written $N(v)$, is the set of vertices adjacent to v.

The **order** of a graph G, written $n(G)$, is the number of vertices. An "n-vertex graph" is a graph of order n. The **size** of a graph G written $e(G)$, is the number of edges in G.

Degree-sum formula for undirected graphs (Handshaking theorem): If G is an undirected graph, then

$$\sum_{v \in V} d(v) = 2e.$$

Corollary

In a graph G, the average vertex degree is $\frac{2e}{n}$ and $\delta(G) \le \frac{2e}{n} < \Delta(G)$.

NOTE: A vertex with zero degree is called a **lone vertex** or isolated vertex and a vertex with exactly one degree is called a **pendent** vertex or end vertex.

Some Results:

1. Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree.
2. A k-regular graph with n vertices has $\frac{nk}{2}$ edges.
3. The minimum number of edges in a connected graph with n vertices is $(n-1)$.
4. If G is a simple n-vertex graph with $\delta(G) \ge \frac{(n-1)}{2}$ then G is connected.

4.1.3 Directed Graphs

In a directed graph, if (u, v) is an edge in G, then u is **adjacent** to v and v is adjacent to u.
u is called **initial vertex** and v is called **terminal vertex** of the edge (u, v).

Indegree and outdegree of a vertex (v) in a directed graph: The indegree of a vertex v denoted by $d^+(v)$ is the number of edges with v as their terminal vertex. The **out degree** of a vertex denoted by $d^-(v)$ is the no of edges with v is their initial vertex.

Loops contribute 1 to the indegree and 1 to out degree of a vertex on which they are present.

Degree-sum Formula for Directed Graphs

$$\Sigma d^-(v) = \Sigma d^+(v) = e$$

i.e. $$\Sigma d^-(v) + \Sigma d^+(v) = 2e$$

Example - 4.1 Given a graph with 10 vertices and 12 edges. What is highest value of minimum degree?

Solution:

$$\delta_{max} = \left\lfloor \frac{2e}{n} \right\rfloor = 2$$

is the highest value of minimum degree that is possible.

Also the least value of maximum degree is

$$\left\lceil \frac{2e}{n} \right\rceil \leq \Delta$$

i.e.

$$\left\lceil \frac{2 \times 12}{10} \right\rceil \leq \Delta$$

$$3 \leq \Delta$$

Hence the least value of maximum degree that is possible is 3.

Example - 4.2 Given a tree with n_2 vertices with degree 2, n_3 vertices with degree 3, n_4 vertices with degree 4, and so on such that n_k vertices with degree k. How many leaf nodes are there?

Solution:

$$\text{Total degree} = x * 1 + 2 * n_2 + 3 * n_3 + \ldots + k * n_k$$

In tree the number of edges ($|E|$) = $|V| - 1$

i.e. $x + 2n_2 + 3n_3 + \ldots + kn_k = 2(x + n_1 + n_2 + \ldots + n_k - 1)$

Solving for x, we get

$$x = n_3 + 2n_4 + 3n_5 + \ldots + (k-2)\, n_k + 2$$

'x' is the number of vertices with degree 1 and so are the leaf nodes.

Example - 4.3 What is the order and size of $\bar{G}$, given that order and size of G are 5 and 7 respectively?

Solution:

$$O(\bar{G}) = 5$$

Because the number of vertices do not change in the compliment of a graph.

Now, Size = Number of edges

= Number of edges in k_5 – Number of edges in G

$$= \frac{5(5-1)}{2} - 7 = 3$$

4.1.4 Havell-Hakimi Theorem

Let G be a graph with vertex set

$$V(G) = \{V_1, V_2, V_3 \ldots V_n\}$$

and let deg V_1, deg V_2, ..., deg V_n be the degree sequence.

NOTE: A sequence $d_1, d_2, \ldots, d_n$ of non-negative integer is graphical if it is a degree sequence of some graph. We study a powerful tool to determine whether a particular sequence is graphical.

Theorem

Let D be the sequence $d_1, d_2, \ldots d_n$ with $d_1 \geq d_2 \geq \ldots \geq d_n$ and $n \geq 2$.
Let D′ be the sequence obtained from D by
- → discarding d_1 and
- → subtracting 1 from each of the next d_1 entries of D

i.e. D′ is the sequence.

$d_2 - 1, d_3 - 1, d_4 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2} - 1, \ldots, d_n$ when sequence has only 1's and 0's and the number of 1's are even then we say that the sequence is graphical.

Example - 4.4 Given a graph with 14 edges. What is the missing value 'k' in the following degree sequence, 1, 2, 3, 3, 4, k, k, k.

Solution:

By using hand shaking theorem,

$$2 \times 14 = 1 + 2 + 3 + 3 + 4 + 3k$$
$$28 = 13 + 3k$$
$$15 = 3k$$
$$k = 5$$

Hence the actual degree sequence is 1, 2, 3, 3, 4, 5, 5, 5.

Example - 4.5 Check whether the following sequences are valid degree sequences or not

(a) 5, 3, 3, 3, 2, 2, 1, 1 (b) 6, 5, 5, 5, 4, 4, 2, 1 (c) 8, 7, 6, 6, 5, 3, 2, 2, 2, 1

Solution:

(a) 5, 3, 3, 3, 2, 2, 1, 1

Apply Havel-Hakimi theorem,

5, 3, 3, 3, 2, 2, 1, 1 is graphical iff

2, 2, 2, 1, 1, 1, 1 is graphical iff (obtained by subtracting 1 from next 5 degree in descending order)

1, 1, 1, 1, 1, 1 is graphical (obtained by subtracting 1 from next 2 degree in descending order)

The last sequence is graphical hence all the above sequence would be graphical.

Note: Here the last sequence 1, 1, 1, 1, 1, 1 is graphical only because number of 1's are even.

(b) 6, 5, 5, 5, 4, 4, 2, 1

4, 4, 4, 3, 3, 1, 1

3, 3, 2, 2, 1, 1

2, 1, 1, 1, 1

0, 0, 1, 1

Here number of 1's are even graphical sequence.

(c) 8, 7, 6, 6, 5, 3, 2, 2, 2, 1

6, 5, 5, 4, 2, 1, 1, 1, 1

4, 4, 3, 1, 0, 0, 1, 1

Rearrange in decreasing order

4, 4, 3, 1, 1, 1, 0, 0

3, 2, 0, 0, 1, 0, 0

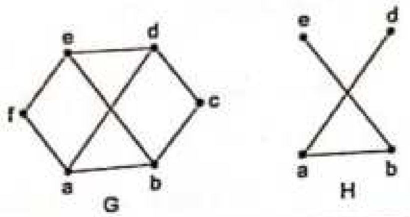
1, 0, −1, 0, 0, 0, 0

Hence not a graphical sequence

4.1.5 Subgraphs

A subgraph of a graph G is a graph H such that V(H) ⊆ V(G) and E (H) ⊆ E(G) and the assignment of end points to edges in H is the same as in G. We then write H ⊆ G and say that "G" contains H"

Example: Here H ⊆ G



NOTE: Every graph is its own subgraph. G ⊆ G. Null graph is a subgraph of every graph.

4.2 Special Graphs

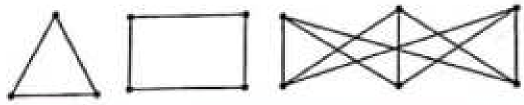
(a) Finite Graph: A graphs is finite if its vertex set and edge set are both finite. If a graph is not finite, it is infinite.

(b) Null Graph: A graph with vertices but no edges. i.e. A graph in which the edge set is empty, but vertex set is not empty is called a null graph.

NOTE: The vertex set cannot be empty, i.e. a graph must have at least one vertex.

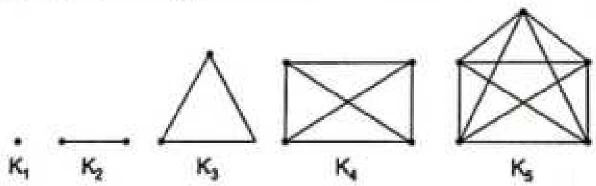
(c) Regular Graph: A graph is which all vertices have the same degree is called a regular graph. In a regular graph $\delta(G) = \Delta F$

Examples:



NOTE: A regular graph with $\delta(G) = \Delta G = k$ is called k-regular.

(d) Complete Graph: The complete graph of n vertices denoted by K_n is a simple graph that contains exactly one edge between every pair of distinct vertices.

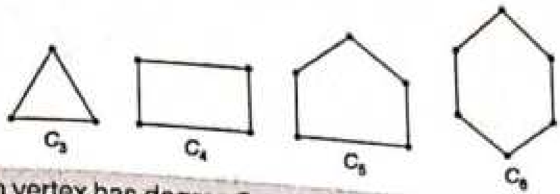


NOTE

- A complete graph is a simple graph with maximum number of possible edges.
- The number of edges in $K_n = \dfrac{n(n-1)}{2} = nC_2$

  Example: K_5 has 5 vertices and $\dfrac{5(5-1)}{2} = 10$ edges
- If a simple graph has nC_2 edges it is complete.
- Maximum edges possible in an n-vertex simple graph is $\dfrac{n(n-1)}{2}$

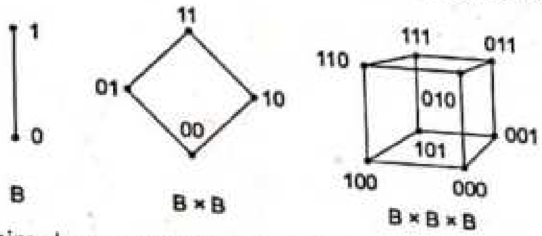
(e) Cycle Graph: The cycle C_n, $n \geq 3$ consists of n vertices $v_1, v_2, \ldots v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}\ldots$ and so on until $\{v_n, v_1\}$.



NOTE: In a cycle graph each vertex has degree 2 and all cycle graphs are 2-regular.

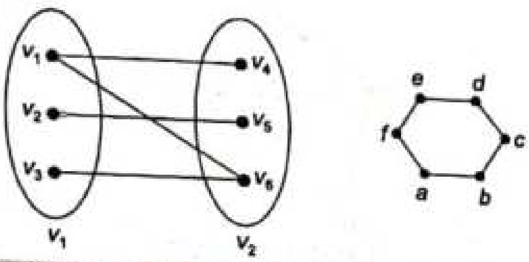
(f) Wheel Graph: A cycle graph C_n with a vertex v_{n+1} added in centre and edges $\{v_1, v_{n+1}\}, \{v_2, v_{n+1}\}, \ldots \{v_n, v_{n+1}\}$ also added, makes a wheel graph W_n.

(g) n-Cubes: Consist of exactly 2^n vertices (labelled by bit strings) and connected by the relation 2 bitstrings are adjacent iff Hamming distance between them is 1.



(h) Bipartitc Graphs: A simple graph G is called bipartite if its vertex set V can be partitioned into two distinct non empty sets, V_1, V_2 such that any edge in G connects a vertex in V_1 with a vertex in V_2 (so that no edge connects either 2 vertices in V_1 or 2 vertices in V_2).

Example:

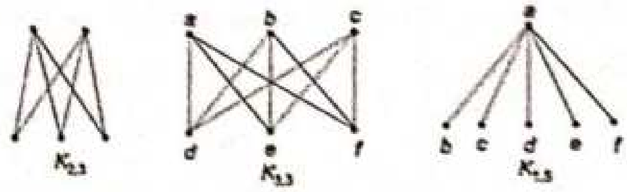


NOTE: Cycle graph C_6 above is bipartite $V_1 = \{a, c, e\}$ and $V_2 = \{b, d, f\}$. K_4, K_3 are not bipartite, but K_2 is bipartite.

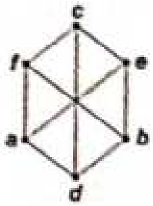
Lemma: A graph is bipartite iff it has no odd cycle. We can show a graph to be not bipartite, if we detect an odd cycle (Biclique).

(i) Complete Bi-partite Graphs: $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 and V_2 of m and n vertices, respectively. There is an edge between 2 vertices iff one vertex is in first subset and another vertex is in second subset.

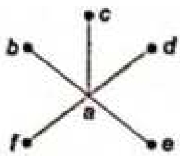
In other words, in a complete bipartite graph every vertex in v_1 is connected to every vertex v_2 by an edge.



$K_{3,3}$ can also be drawn as:



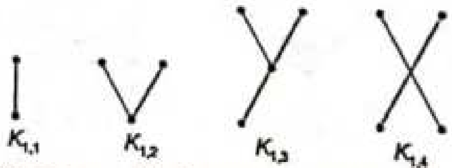
$K_{1,5}$ can also drawn as:



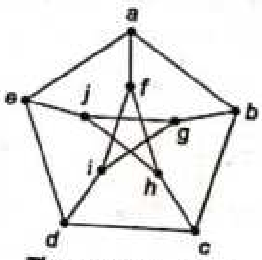
Result:

(*i*) The no of vertices in $K_{m,n}$ is exactly $m + n$ and the no of edges in $K_{m,n}$ is exactly mn. i.e. $K_{3,4}$ has exactly $3 + 4 = 7$ vertices and $3 \times 4 = 12$ edges.

(*ii*) Any graph that is $K_{1,n}$ is also called a star graph.



(j) **Petersen Graph:** The petersen graph is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets. If two vertices are non adjacent in the petersen graph, then they have exactly one common neighbor.



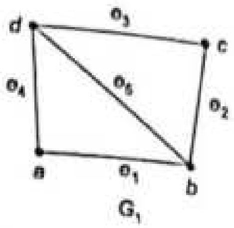
The petersen graph

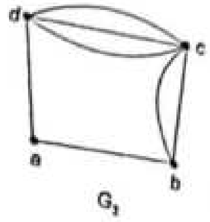
The **girth** of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. The Petersen graph has girth 5.

4.3 Graph Representations

Definition: Let G be a loopless graph with vertex set $V(G) = \{v_1, \dots v_n\}$ and edge set $E(G) = \{e_1, \dots e_n\}$. The **adjacency matrix** of G, written A(G), is the n-by-n matrix in which entry $a_{i,j}$ is the number edges in G with end points $\{v_i, v_j\}$. The **incidence matrix** M(G) is the n-by-n matrix in which entry $m_{i,j}$ is 1 if v_i is an end point of e_i. If vertex *v* is an endpoint of edge *e*, the *v* and e are **incident**. The **degree** of vertex *v* is the number of edges, incident upon it.

Example:





Matrix representations:

Adjacency matrix of G_1 is given below:

$$X_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix of G_2

$$X_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

Incidence matrix of G_1

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Adjacency list representation:

The Adjacency list for G_1 is

Vertex	Adjacent Vertices
a	b, d
b	a, c, d
c	b, d
d	a, b, c

4.4 Isomorphism

An **isomorphism** from a simple graph g to a simple graph H is a bijection $f: V(G) \to V(H)$ such that $uv \in (G)$ iff $f(u)\, f(v) \in E(H)$.

We say "G is isomorphic to H" written as $G \cong H$, if there is an isomorphsion from G to H.

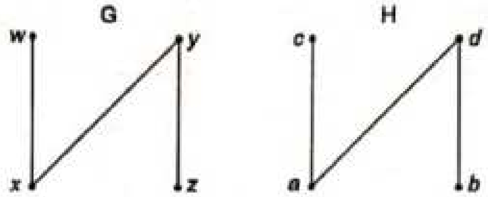
Example: The graph G and H drawn below are 4-vertex paths.

Define the function $f: V(G) \to V(H)$ by $f(w) = a$, $f(x) = d$, $f(y) = b$, $f(z) = c$.

To show that f is an isomorphism, we check that f preserves edges and non-edges.

Note that rewriting A(G) by placing the rows in the order w, y, z, x and the column also in that order yields A(H), this verifies that f is an isomorphism.

Another isomorphism maps w, x, y, z to c, b, d, a respectively.



$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{matrix} w & x & y & z \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ w \\ y \\ z \\ x \end{array}\begin{array}{c} \begin{matrix} w & y & z & x \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Finding Isomorphisms

Presenting the adjacency matrices, with vertices ordered so that the matrices are identical, is one way to show an isomorphism.

One can also verify preservation of adjacency relation without writing out the matrices.

In order for an explicit bijection to be an isomorphism from G to H, the image in H of a vertex v in G must behave in H as v does in G.

The isomorphism relation is an equivalence relation on the set of (simple) graphs.

An "isomorphism class" of graphs is an equivalence class of graphs under the isomorphism relation.

NOTE: An automorphism of G is an isomorphism from G to G. A graph G is vertex transitive if for every pair $u, v \in V(G)$ there is an automorphism that maps u to v.

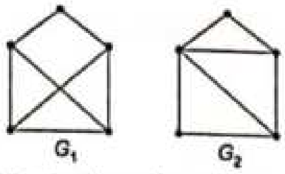
4.5 Invariants of Isomorphic Graphs

Isomorphic simple graphs must have

1. Same number of vertices
2. Same number of edges.
3. Same degree sequence. i.e. degree of corresponding vertices must be same.
4. Number of simple circuits of a given length must be same in both graphs.

All the above are called invariants in an isomorphism. If any of the above are violated, then the graphs are not isomorphic. Note that the above invariants are necessary, but not sufficient to prove isomorphism but violation of these invariants can be used to prove that the graphs are not isomorphic.

Example:



Although G_1 and G_2 have same number of vertices as well as same number of edges, they are not isomorphic since there degree sequences are not the same. The degree sequence of G_1 is (2,3,3,3,3) and the degree sequence of G_2 is (2,2,3,3,4).

4.6 Operations on Graphs

Complement of a Graph

If H is a simple graph with n vertices, the complement $\overline{H}$ of H is the complement of H in K_n i.e. $\overline{H}$ is the graph of $K_n - E(H)$.

From this definition, we can say that,

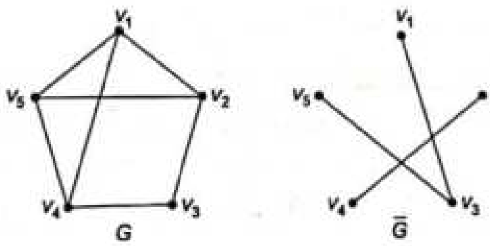
(i) $V(\overline{H}) = V(H)$ and $e(\overline{H}) = \dfrac{n(n-1)}{2} - e(H)$ (where $n = V(H) = \#$ of vertices of H)

(ii) Any two vertices are adjacent in $\overline{H}$ iff they are not adjacent in H.

(iii) The degree of a vertex in $\overline{H}$ plus its degree in H is $n-1$, when $n = |V(H)|$.

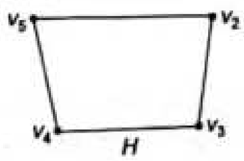
NOTE: If G is a simple graphs of n vertices, then $G \cup \overline{G}$ is K_n, the complete graph on n vertices.

Example:

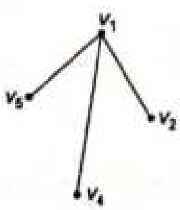


Relative complement of a Graph

Let H be a graph as shown below. $\overline{H}(G) = G - H$, is the relative complement of H in G. It is obtained by starting with G and deleting the edges common to H and G.



Now, $\overline{H}(G) = G - H$ is shown below:



NOTE: Two simple graphs are isomorphic iff their complements are isomorphic.

Decompositions and Special Graphs

A graph is **self-complementary** if it is isomorphic to its complement.

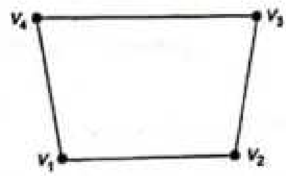
A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

An n-vertex graph H is self-complementary if and only if K_n has a decomposition consisting of two copies of H.

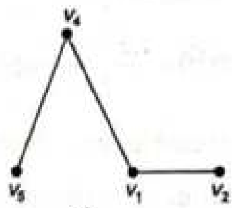
Union and Intersection of Graphs

Let G and G′ be two graphs. The **intersection** of G and G′ written as G ∩ G′, is the graph where vertex set is V(G) ∩ V(G′) and where edge set is E (G) ∩ E(G′). Similarly the **union** of G and G′ is the graph with vertex set V(G) ∪ V(G′) and edge set E(G) ∪ (G′).

Example: Let G and G′ be two graphs shown below.

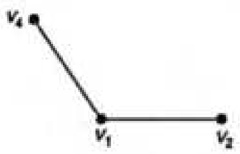


G

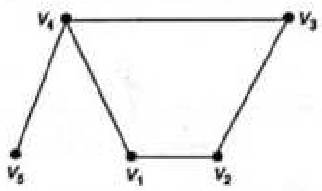


G

Then G ∩ G′ and G ∪ G′ are shown below:



G∩G′



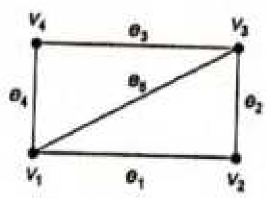
G∪G′

Definition: The "union" of graph G_1, G_2 ...G_k, written $G_1 \cup G_2 \cup \ldots \cup G_k$ is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. The complete graph K_n can be expressed as the union of k bipartite graphs if and only if $n \le 2^k$.

4.7 Walks, Paths and Cycles

A **walk** is defined as a finite alternating sequences of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more then once in a walk. A vertex may however appear more than once.

Example:



The sequence $v_1e_1v_2e_2v_3e_5v_1e_4v_4$ is a walk. The set vertices and edges constituting a walk is clearly a subgraph of G.

The vertices with which a walk begins and ends are called terminal vertices. If a walk starts and ends at the same vertex, it is called a closed walk. Else it is a open walk. A open walk, is which no vertex appears more than once is called a **path** or a simple path or elementary path. A path is basically a non-self intersecting open walk.

The no of edges is a path is called the length of the path. A self loop can be included in a walk but not in a path. A closed walk in which no vertex (except the initial and final vertex) appears more than once, is called a **circuit or cycle**. A circuit is therefore a closed, non intersecting walk.

In the graph above, $v_1e_1v_2e_2v_3e_5v_1$ is a circuit or a cycle of length 3.

NOTE: A self loop is a cycle of length 1. A walk from u to v is called a u-v walk. Every u-v walk contains a u-v path also.

4.8 Connected Graphs, Disconnected Graphs and Components

A graph G is said to be **connected** if there is at least one path between every pair of vertices in G. Otherwise, G is **disconnected.**

A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a **component.**

$\therefore$ A connected graph is a graph with only one component. A disconnected graph has at least 2 components. A null graph with more than one vertex is disconnected.

Results:

1. If a graph (connected or disconnected) has exactly 2 vertices of odd degree, there must be a path joining these two vertices.
2. A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

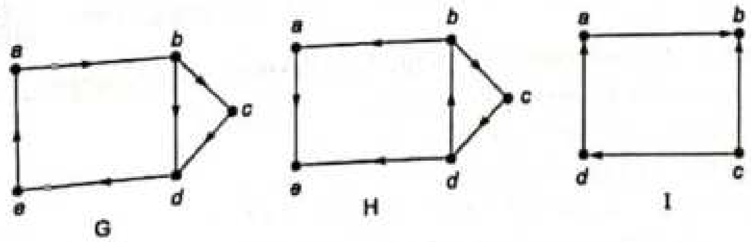
NOTE: From above, when $k=1$, a simple graph with n vertices can have atmost $(n-1)\,n/2$ edges which is the maximum possible edges in a connected simple graph.

3. Every graph with n vertices and k edges has at least $n-k$ components (where $n \geq k$)

Connectedness in Directed Graphs

1. A directed graph is strongly connected, if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
2. A directed graph is weakly connected, if there is a path between every two vertices in the underlying undirected graph.
3. A directed graph is unilaterally connected, if for any pair of vertices a, b, either there is a path from a to b or from b to a.

Example:



G is strongly connected.
H is weakly connected, and I is unilaterally connected, but not strongly connected.

NOTE

- Strongly connected ⇒ Unilateral as well as weakly connected but converse is not true
- Unilaterally connected ⇒ Weakly connected but converse is not true
- If a diagraph is not even weakly connected, then such a graph will be a disconnected graph.

4.8.1 Cut vertex, Cut set and Bridge

A **cut vertex** of a graph G is a vertex whose removal increases the number of components. Clearly, if v is a cut vertex of a connected graph G, then its removal must disconnect the graph. A cut vertex is also called a cut point, a cut node or an articulation point.

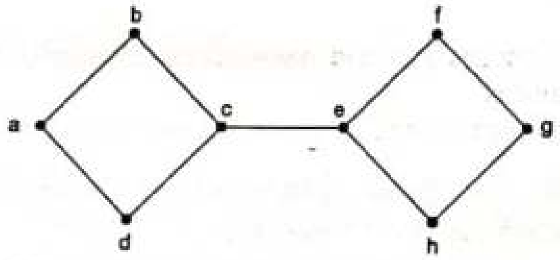
Similarly, an edge whose removal produces a graph with more components than the original graph is called a **cut edge** or a bridge.

Clearly, if e is a cutedge of a connected graph, its removal would surely disconnect the graph.

The set of all minimum number of edges of G whose removal disconnects a graph G is called a **cut set** of G. Thus a cut set S satisfies the following conditions.

1. S is a subset of the edge set E of G.
2. Removal of edges in S from G, disconnects G.
3. No proper subset of S satisfies this condition.

Example:



In above graph, vertex "c" is a cut vertex. So is vertex "e".
Edge {c, e} is a cut edge.
{{c, e}} is a cut set.
{{e, f}, {f, g}} is another cut set.
{{b, c}, {c, d}} is another cut set.

NOTE: A set consisting of only a cut edge or bridge is always a cut set.

Theorem: A vertex v in a connected graph G is a cutvertex if and only if there exists two vertices x and y in G such that every path between x and y passes through v.

4.8.2 Connectivity

Edge connectivity: Let G be connected graph. Then the edge connectivity of G is the minimum number of edges whose removal disconnects the graph.

The edge connectivity is denoted by $\lambda(G)$

1. If G is disconnected graph, $\lambda(G) = 0$.
2. If G is a connected graph with a bridge, then $\lambda(G) = 1$.
3. $k(K_n) = n - 1$, where K_n is the complete graph of n vertices.
4. The vertex connectivity of a graph is one if and only if it has a cut vertex.

5. $k(C_n) = 2$, where C_n is the cycle graph of n vertices.
6. A connected graph is said to be separable if its vertex connectivity is one. All other connected graphs are called non separable.
7. In a separable graph, the vertex whose removal disconnects the graph is a cut vertex.
8. In a tree, every vertex with degree greater than one is a cut-vertex.

Theorem: The edge connectivity of a graph cannot exceed the degree of the vertex with smallest degree in G.

Theorem: The vertex connectivity of a graph G is always less than or equal to its edge connectivity. i.e. $k(G) \le \lambda(G)$.

Definition: A graph is said to be ***n*-connected** if $k(G) = n$ and n-line connected if $\lambda(G) = n$. A 1-connected graph is the same as separable graph.

Theorem: $k(G) \le \lambda(G) \le 2e/n$ and max. vertex connectivity possible $= \left\lfloor 2e/n \right\rfloor$.

Example: for a graph with 8 vertices and 18 edges, we can achieve a vertex connectivity as high as four $\left(= \left\lfloor \frac{2 \times 18}{8} \right\rfloor = 4\right)$.

Example - 4.6 What is the number of edges in the graph Q_7?

Solution:

We know that, Q_n is hasse diagram of boolean algebra. Q_n is always a n-regular graph. There are 2^n vertices in Q_n.

Hence, $\quad n \times 2^n = 2e$

i.e. $\quad e = n \times 2^{n-1}$

Hence is Q_7 we have $7 \times 2^6 \rightarrow 448$ edges.

Example - 4.7 15 vertices and 4 components in the graph. What is the minimum number of edges?

Solution:

We know that, $$n - k \le e \le \frac{(n-k+1)(n-k)}{2}$$

$$n - k \le e$$

$$\therefore \quad 15 - 4 \le e$$

$$11 \le e$$

Therefore minimum number of edges for the graph with 15 vertices and 4 components are 11.

Example - 4.8 What is the maximum number of edges with 17 vertices and 6 components?

Solution:

$$e \le \frac{(n-k+1)(n-k)}{2}$$

$$e \le \frac{(17-6+1)(17-6)}{2} \le 6 \times 11 \le 66$$

$\therefore$ Maximum edges are 66.

Scanned with CamScanner

Example-4.9 How many cut vertices in tree with '20' nodes and '9' leaf nodes?

Solution:

In tree all the internal nodes are cut vertices

$$n = i + l$$

$$20 = i + 9 \Rightarrow i = 11$$

$\therefore$ 11 cut vertices are tree.

Example-4.10 In a computer lab, 50 computer and 75 cables are there. How many ways can we connect to have strongest connectivity?

Solution:

We know that complete graph (k_n) gives the strongest connectivity.

But we do not have $\frac{50(50-1)}{2}$ cables.

Hence we can make as strong as possible from given cables.

$$\text{Vertex connectivity of } (K_G) \le \left\lfloor \frac{2e}{n} \right\rfloor \le \left\lfloor \frac{2\times 75}{50} \right\rfloor \le 3$$

Hence, given 50 computer and 75 cables, there are 3 computers which if removed will disconnect the network.

Example-4.11 The order of the graph is 20 and it's size in 35. Then what is the maximum vertex connectivity possible?

Solution:

$$\left\lfloor \frac{2e}{n} \right\rfloor = \left\lfloor \frac{2\times 35}{20} \right\rfloor = 3 \text{ is the strongest connectivity.}$$

Example-4.12 The number of vertices and edges in a graph are 10 and 15 respectively. What is the maximum vertex connectivity and edge connectivity, given that it's minimum degree is 2?

Solution:

We know two relations,

$$\text{Vertex connectivity} \le \left\lfloor \frac{2e}{n} \right\rfloor$$

This given vertex connectivity $\le \left\lfloor \frac{2\times 15}{10} \right\rfloor \le 3$. But we have more strong answer

Vertex connectivity (K_G) $\le$ Edge connectivity (λ_G) $\le \delta$ (minimum degree) $\le \left\lfloor \frac{2e}{n} \right\rfloor$

$\delta = 2$ is given which gives a stronger upper bound for vertex connectivity and edge connectivity.

$$\therefore \quad K_G \le \lambda_G \le 2 \le 3$$

So in this problem the maximum possible vertex and edge connectivity is 2.

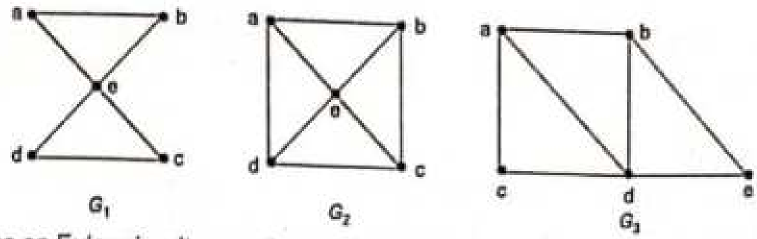
If minimum degree (δ) is not given then we take the average degree $\left(\left\lfloor \frac{2e}{n} \right\rfloor\right)$ as the upper bound for K_G as well as λ_G.

4.9 Euler Graphs

Euler Path: An open path in a graph G is called **Euler path** if it includes every edge exactly one. It is also called Euler Trail.

Euler Circuit: A circuit in a graph G that includes every edge exactly once is called an Euler circuit or Euler Cycle. A graph with an Euler cycle is called an **Euler Graph.**

Example:



Graph G_1 has an Euler circuit a e c d e b a. Graph G_2 does not have an Euler path or a circuit.

Graph G_3 has an Euler path (a c d e b d a b) but not an Euler circuit.

Theorem: A connected multigraph has an Euler circuit if and only if each of vertices has even degrees.

All vertices in G_1 are even degrees, hence it is an Euler graph.

Theorem: A connected multigraph has an Euler path but not an Euler circuit, if and only if it has exactly two vertices of odd degree. Such a path will begin at one of these odd vertices and end at the other.

G_3 has exactly two vertices of odd degree. These are vertices a and b respectively. Hence there is an Euler path but not circuit.

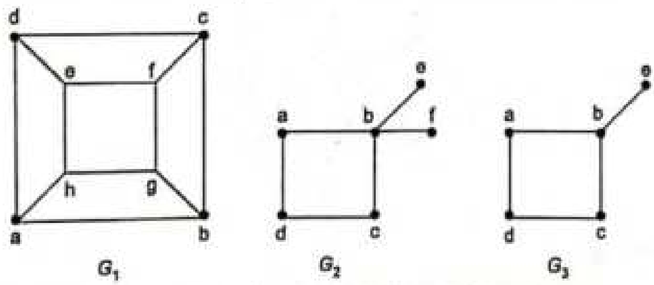
Theorem: A directed multigraph has an Euler circuit if and only if it is unilaterally connected and the indegree of every vertex in G is equal to its outdegree.

4.10 Hamiltonian Graphs

A **hamiltonian path** is a simple open path that visits all vertices of G, exactly once.

A hamiltonian circuit is a circuit that visits all vertices of G, exactly once.

A graph G is called a Hamiltonian Graph, If it contains a hamiltonian circuit.



Graph G_1 is a hamiltonian graph, since it contains a hamiltonian circuit, d c b a h g f e d.

A graph which contains a hamiltonian circuit will surely contain a hamiltonian path (by leaving out the last vertex of hamiltonian cycle). For example, G_1 contains the hamiltonian path d c b a h g f e. G_2 has neither a hamiltonian circuit, not a hamiltonian path. G_3 has a hamiltonian path, but not a hamiltonian circuit.

Hence G_3 is not a hamiltonian graph.

NOTE: K_n has a hamiltonian circuit whenever $n \geq 3$.

i.e. K_n, $n \geq 3$ is always a hamiltonian graph.

Theorem: In K_n, there are $(n-1)/2$ edge-disjoint hamiltonian circuits, if n is an odd number ≥ 3.

Theorem: A graph with n vertices and with no loops or parallel edges (i.e. a simple graph) which has at least $1/2(n-1)(n-2)+2$ edges is hamiltonian. (The converse may or may not be true).

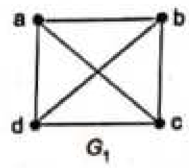
Dirac's Theorem: If G is a simple graph with n vertices, $n \geq 3$, such that the degree of every vertex in G is atleast $n/2$, then G has a hamiltonian circuit.

Ore's Theorem: If G is a simple graph with n vertices, $n \geq 3$, such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a hamiltonian circuit.

NOTE: Unlike for Euler graphs, there as yet no necessary and sufficient condition developed for a graph to be a hamiltonian graph.

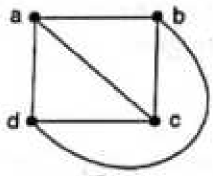
4.11 Planar Graphs

A graph G is said to be **planar** if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges cross each other. The points of crossings are called crossovers. A representation of a graph drawn in a plane without edges crossing is called its planer representation or plane embedding.



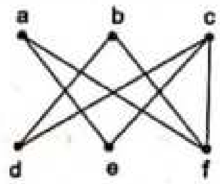
G_1

G_1 is a planer graph, since it has a planer representation, given below as G_2.



G_2

However the following graph G_3 is non-planar. i.e. it has no possible planar representation or embedding.

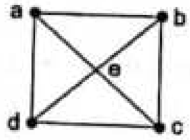


G_3

Euler's Formula for Planar Graphs

Theorem: If a connected planar graph G has n vertices, e edges and r regions, then $r = e - n + 2$.

Example:



Here, $n = 5$, $e = 8$ and $r = 5$

Since, $e - n + 2 = 8 - 5 + 2 = 5 = r$

$\therefore$ Euler's formula for planar graphs is verified.

Corollary: If a planar graph has k components, then $r = e - n + (k+1)$

Corollary: If G is a connected, planar, simple graph with $n\,(\geq 3)$ vertices and e edges, then $e \leq 3n-6$.

Note: This corollary can be used to prove that a given graph is non-planar.

i.e. if in a simple and connected graph $e \not\leq 3n-6 \Rightarrow$ graph is not planar.

Corollary: If G is a connected, planar, simple graph with $n\,(\geq 3)$ vertices and e edges and no circuit of length 3, then $e \leq 2n-4$. This corollary also can be used to prove that some graphs are non-planar.

Example-4.13 A connected planar graph having 15 vertices and 12 regions. Find the number of edges connecting the vertices in this graph.

(a) 15
(b) 20
(c) 25
(d) 30

Solution: (c)

Euler's formula: $V = E - R + 2$

$\Rightarrow$ $E = V + R - 2$

Given, V = 15 and R = 12

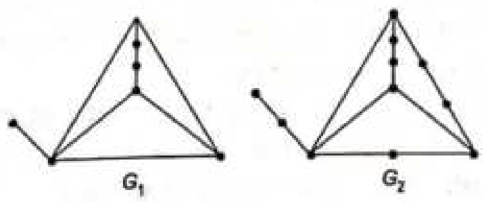
$$\text{Number of edges (E)} = 15 + 12 - 2 = 25$$

Kuratowski's Theorem (Necessary and sufficient condition for planarity) A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

NOTE: K_5 and $K_{3,3}$ are known as Kuratowski's two graphs. These two graphs are special since, K_5 is the non-planar graph with minimum number of vertices and $K_{3,3}$ is the non-planar graph with minimum number of edges.

Definition: Two graphs are said to be homeomorphic if both can be obtained from the same graph by elementary sub-divisions (insertion of new vertices of degree 2 into its edges or by merger of edges in series).

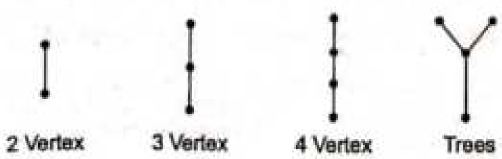
Example: G_1 and G_2 below are homeomorphic.



4.12 Trees

A connected, acyclic graph is a tree. i.e. A connected graph with no circuits is a tree. Its edges are called branches.

A tree with only one vertex is a trivial tree, otherwise it is a non trival tree.

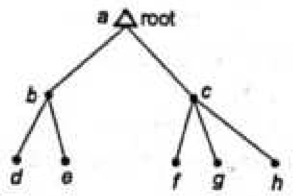


Properties of Trees:

1. A graph is a tree if and only if there is one and only one path between every pair of vertices.
2. A tree with n vertices has exactly $n-1$ edges.
3. If a connected graph G, has n vertices and $n-1$ edges, it is a tree.
   ∴ A graph with n vertices is called a tree if
   (a) It is connected and acyclic or
   (b) It is connected and has $n-1$ edges or
   (c) If it is acyclic and has $n-1$ edges or
   (d) If there is exactly one path between every pair of vertices in G.
4. Every edge of a tree is a cut set
5. Adding one edge to a tree forms exactly one cycle.
6. A tree is a minimally connected graph.

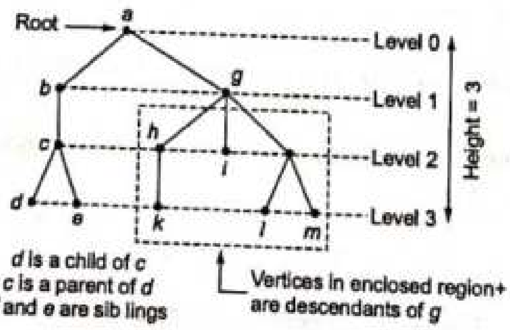
4.12.1 Rooted Trees

A tree in which a particulars vertex (called root) is distinguished from others is called a rooted tree.



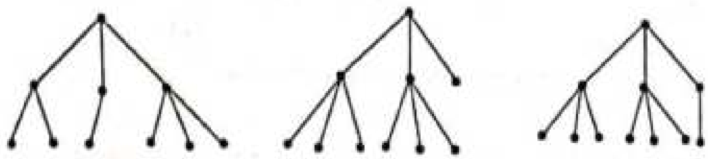
A Rooted Tree

1. The **level** of a vertex is the number of edges along the unique path between it and the root. The level of the root is defined as 0. The vertices immediately under the root are said to be in level 1 and so on.
2. The **height** of a rooted tree is the maximum level to any vertex of the tree. * The **depth** of a vertex v in a tree is the length of the path from the root to v.
3. Given any internal vertex v of a rooted tree, the **children** of v are all those vertices that are adjacent to v and are one level further away from the root than v. If w is a child v, the v is called the **parent** of w, and two vertices that are both children of the same parent are called **siblings**.
4. If the vertex u has no children, then u is called a **leaf** (or a **terminal vertex**). If u has either one or two children, then u is called **an internal vertex**.
5. The **descendants** of the vertex u is the set consisting of all the children of u together with the descents of those children. Given vertices v and w, if v lies on the unique path between w and the root, then v is an **ancestor** of w and w is a descendant of v.
   These terms are illustrated in given figure.



Definition: A rooted tree is an **m-ary tree** if every internal vertex has at most m children. A m-ary tree is a full m-ary tree if every internal vertex has exactly m children. In particular, the 2-ary tree is called **binary tree.** A full **binary tree** is a binary tree in which each internal vertex has exactly two children.

A **complete m-ary tree** is an m-ary tree in which all its levels, except possibly the last have maximum number of nodes and if all nodes of the last level appear as far left as possible.



A 3-ary tree — A full 3-ary tree — A complete 3-ary tree

The relationship between i, the number of internal vertices and l, the number of leaves of a full m-ary can be proved by using the following theorem.

Theorem: A full m-ary tree with i internal vertex has $n = mi + 1$ vertices.

Proof: Since the tree is full m-ary, each internal vertex has m children and the number of internal vertex is i, the total number of vertex except the root is mi.

Therefore, the tree has $n = mi + 1$ vertices.

Since l is the number of leaves, we have $n = l + i$, Using the two equalities $n = mi + 1$ and $n = l + i$, the following results can easily be deduced.

A full m-ary tree with

(*i*) n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)\,n + 1]/m$ leaves.

(*ii*) i internal vertices has $n = mi + 1$ vertices and $l = (m-1)\,i + 1$ leaves.

(*iii*) l leaves has $n = (ml-1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

Theorem:

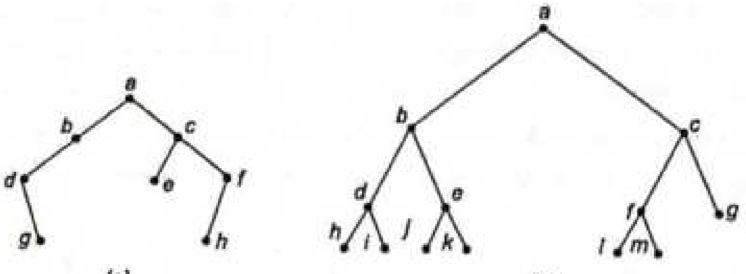
1. The maximum number of leaves in an m-ary tree of height h is m^h i.e., $l \le m^h$.
2. The maximum number of nodes in an m-ary tree of height h is $\dfrac{m^{h+1}-1}{m-1}$ i.e., $n \le \dfrac{m^{h+1}-1}{m-1}$.

4.12.2 Binary Tree

A binary tree is a rooted tree in which each vertex has atmost two children. Each child in a binary tree is designated either a **left child** or a **right child** (not both), and an internal vertex has at most one left and one right child. A **full binary tree** is a tree in which each internal vertex has exactly two children.

Given an internal vertex v of a binary tree, T, the **left subtree** of v is the binary tree whose root is the left child of v, whose vertex consist of the left child of v and all its descendants, and whose edges consist of all those edges of T that connect the vertices of the left subtree together. The **right subtree** of v is defined analogously.

Figure (a) below, is a binary tree and figure (b) is a full binary tree, since each of its internal vertices has exactly two children.



(a) (b)

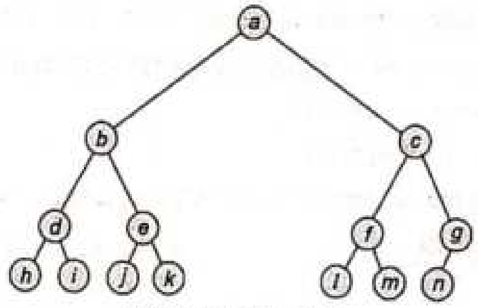
Theorem: If T is full binary tree with i internal vertices, then T has $i + 1$ terminal vertices (leaves) and $2i + 1$ total vertices.

Theorem: The maximum number of vertices on level n of a binary tree is 2^n where $n \geq 0$.

Theorem: The maximum number of vertices in a binary tree of depth d is $2^{d+1} - 1$ where $d \geq 1$.

Complete Binary Tree

If all the leaves of a full binary tree are at level d, then we call such a tree as a complete binary tree of depth d. A complete binary tree of depth of 3 is shown below.

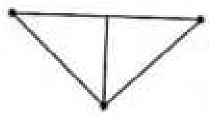


A complete binary tree

4.12.3 Spanning Trees

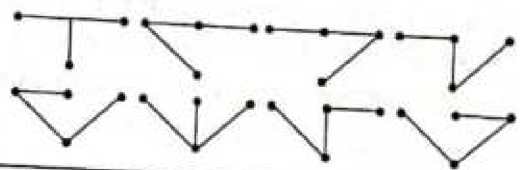
A subgraph T of G is called a spanning tree if T is a tree and if T includes every vertex of G i.e. V(T) = V(G).

Example - 4.14 Find all spanning trees of the graph G shown below.



Solution:

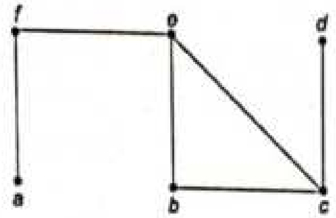
The graph G has four vertices and hence each spanning tree must have 4 – 1 = 3 edges. Thus each tree can be obtained by deleting two of the five edges of G. This can be done in 10 ways, except that two of the ways lead to disconnected graphs. Thus there are eight spanning trees as shown in figure below.



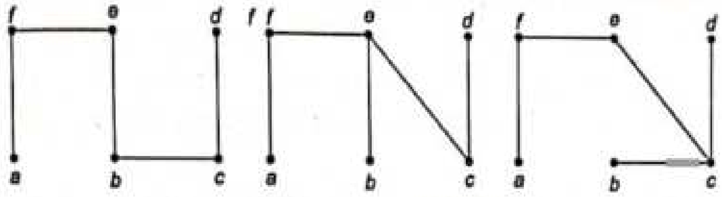
Theorem: A simple graph G has a spanning tree if and only if G is connected.

Example-4.15 Find all spanning trees for the graph G shown in figure. By removing the edges in simple circuits.

Solution:



The graph G has one cycle cbec and removal of any edge of the cycle gives a tree. There are three edges in the cycle and hence there are 3 spanning trees possible as shown below.



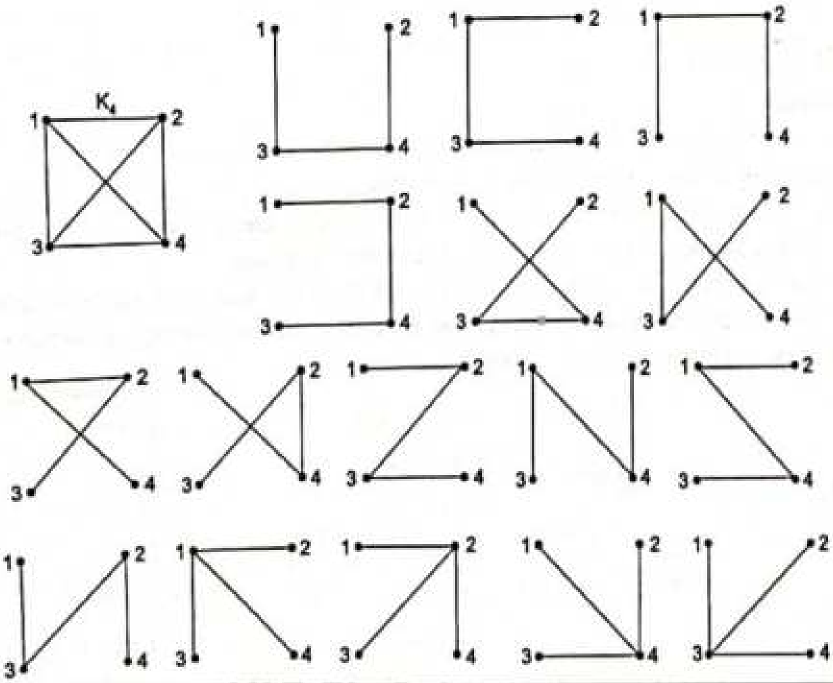
The number of different spanning trees on the complete graph K_n can be found from Cayley's theorem which is given below without any proof.

Cayley's Theorem: The complete graph K_n has n^{n-2} different spanning trees.

Example-4.16 Give all the spanning trees of K_4.

Solution:

Here $n = 4$, so there will be $4^{4-2} = 16$ different spanning trees. All the spanning trees of K_4 are shown in figure.



Weighted Graph

A **weighted graph** is a graph G in which each edge e has been assigned a non-negative number $w(e)$, called the weight (or length) of e.

Figure below shows a weighted graph. The weight (or length) of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path. Many optimisation problems amount to finding, in a suitable weighted graph, a certain type of subgraph with minimum (or maximum) weight.

4.12.4 Minimal Spanning Trees

Let G be a weighted graph. A minimal spanning tree of G is a spanning tree of G with minimum weight.

Algorithm for Minimal Spanning Trees

There are several methods available for actually finding a minimal spanning tree in a given graph.

Two algorithms due to Kruskal and Prim for finding a minimal spanning tree of a connected weighted graph where no weight is negative are available. These algorithms are example of greedy algorithms. A greedy algorithm is a procedure that makes an optimal choice at each of its steps without regard to previous choices.

4.13 Enumeration of Graphs

1. The number of simple, labeled graphs with n vertices and e edges is $\begin{pmatrix} \frac{n(n-1)}{2} \\ e \end{pmatrix} = {}^{\frac{n(n-1)}{2}}C_e$.
2. The number of simple, labeled graphs of n vertices is $= 2^{n(n-1)/2}$.
3. The number of labeled trees with n vertices $(n \geq 2) = n^{n-2}$.
4. The number of different rooted, labeled trees with n vertices is n^{n-1}
5. Enumerating unlabeled trees is more complicated due to isomorphism and involves the use of generating functions and partitions.

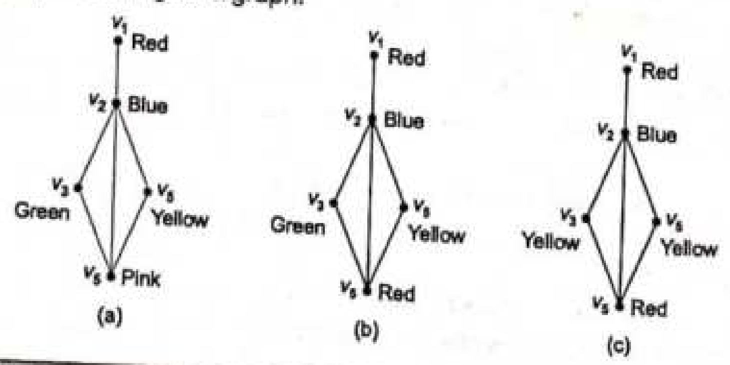
4.13.1 Graph Coloring

Chromatic Number

The chromatic number of a graph G written K(G) is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors.

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the **proper coloring** (or sometimes simply coloring) of a graph.

A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph. Usually a given graph can be properly colored in many different ways. Figure above shows three different proper colorings of a graph.



The proper coloring which is of interest to us is one that requires the minimum number of colors. A graph G that requires k different colors for its proper coloring, and no less, is called a **k-chromatic graph**, and the number k is called the **chromatic number** of G. You can verify that the graph in above figure is 3-chromatic.

$$K(G) = k \text{ means}$$

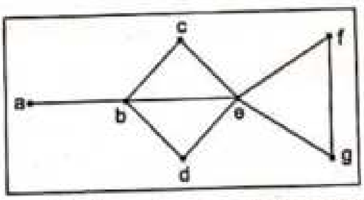
1. The graph can be colored with k colors.
2. The graph cannot be colored with fewer than k colors.

In coloring graphs there is no point in considering disconnected graphs. How we color vertices in one component of a disconnected graph has no effect on the coloring of the other components. Therefore, it is usual to investigate coloring of connected graphs only. All parallel edges between two vertices can be replaced by a single edge without affecting adjacency of vertices. Self-loops must be disregarded. Thus for coloring problems we need to consider only simple, connected graphs.

Some observations that follow directly from the definitions just introduced are:

1. A graph consisting of only isolated vertices is 1-chromatic.
2. A graph with one or more edges (not a self-loop, of course) is at least 2-chromatic (also called bichromatic).
3. A complete graph of n vertices is **n-chromatic**, as all its vertices are adjacent. Hence a graph containing a complete graph of r vertices is at least r-chromatic. For instance, every graph having a triangle is at least 3-chromatic.
4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2 chromatic if n is even and 3-chromatic if n is odd. (This can be seen by numbering vertices 1, 2, ..., n in sequence and assigning one color to odd vertices and another to even, no adjacent vertices will have the same color. If n is odd, the nth and first vertex will be adjacent and will have the same color, thus requiring a third color for proper coloring.)
   This means $K(C_n) = 2$ if n is even and $K(C_n) = 3$ if n is odd
   Where, C_n is the cycle graph with n vertices.
5. **Theorem:** Every tree with 2 or more vertices is 2-chromatic.
6. **Theorem:** A graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length. i.e. bipartite graphs (which have no circuits of odd length) are 2-chromatic. Also, $K(K_{m,n}) = 2$.
7. **Theorem:** If d_{max} is the maximum degree of the vertices in a graph G, Chromatic number of G $\leq 1 + d_{max}$.
8. Apple and Haken proved the **4 color theorem** which says that every planar graph can be colored by at most 4 colors. That is chromatic number of every planar graph is 4 or less.
   Proper coloring of a given graph is simple enough, but a proper coloring with the minimum number of colors is, in general, a difficult task. In fact, there has not yet been found a simple way of characterizing a k-chromatic graph. (The brute-force method of using all possible combinations can, of course, always be applied, as in any combinatorial problem. But brute force is highly unsatisfactory, because it gets out of hand as soon as the size of the graph increases beyond a few vertices).

4.13.2 Independent Sets



A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in figure (graph coloring: previous page) produces the partitioning $\{v_1, v_4\}$, $\{v_2\}$ and $\{v_3, v_4\}$.

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set; more formally:

A set of vertices in a graph is said to be an **independent set** of vertices or simply an independent set (or an internally stable set) if no two vertices in the set are adjacent. For example, in figure below, {a, c, d} is an independent set. A single vertex in any graph constitutes an independent set.

A **maximal independent set** (or maximal internally stable set) is an independent set to which no other vertex can be added without destroying its independence property. The set {a, c, d, f} in figure above, is a maximal independent set. The set {b, f} is another maximal independent set. The set {b, g} is a third one. From the preceding example, it is clear that a graph, in general, has many maximal independent sets, and they may be of different sizes. Among all maximal independent sets, one with the largest number of vertices is often of particular interest.

The number of vertices in the largest independent set of a graph G is called, the **independence number** (or coefficient of internal stability), β(G).

Consider a *k*-chromatic graph G of *n* vertices properly colored with *k* different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number β(G), we have the inequality,

$$\beta(G) \geq \frac{n}{k}$$

NOTE: A graph G is *k*-partite if V(G) can be expressed as union of *k* independent sets.

4.13.3 Dominating Sets

A **dominating set** (or an externally stable set) in a graph G is a set of vertices that dominates every vertex *v* in G in the following sense: Either *v* is included in the dominating set or is adjacent to one or more vertices included in the dominating set.

For instance, the vertex set {b, g} is a dominating set in Figure above. So is the set {a, b, c, d, f} a dominating set.

A dominating set need not be independent. For example, the set of all its vertices is trivially a dominating set in every graph.

In many applications one is interested in finding minimal dominating sets defined as follows:

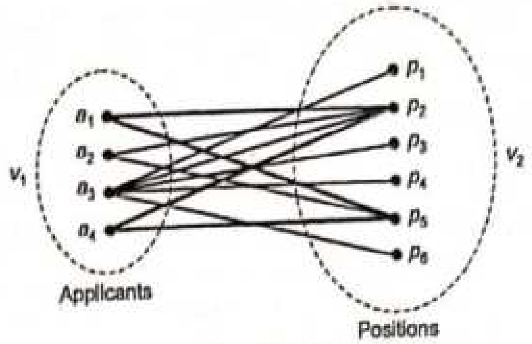
A **minimal dominating set** is a dominating set from which no vertex can be removed without destroying its dominance property. For example, in above figure {b, e,} is a minimal dominating set. And so is {a, c, d, f}.

Observations that follows from these definitions are:

1. Any one vertex in a complete graph constitutes a minimal dominating set.
2. Every dominating set contains at least one minimal dominating set.
3. A graph may have many minimal dominating sets, of different sizes. [The number of vertices in the smallest minimal dominating set of a graph G is called domination number, α (G).
4. A minimal dominating set may or may not be independent.
5. Every maximal independent set is a dominating set. For if an independent set does not dominate the graph, there is at least one vertex that is neither in the set nor adjacent to any vertex in the set. Such a vertex can be added to the independent set without set without destroying its independence. But then the independent set could not have been maximal.
6. An independent set has the dominance property only if it is a maximal independent set. Thus an independent dominating set is the same as a maximal independent set.
7. In any graph G, a(G) ≤ β (G).

4.13.4 Matchings

Suppose that four applicants a_1, a_2, a_3 and a_4 are available to fill six vacant positions P_1, P_2, P_3, P_4, P_5 and P_6. Applicant a_1 is qualified to fill position P_2 or P_5. Applicant a_2 is qualified to fill position P_2 or P_5. Applicant a_3 is qualified for P_1, P_2, P_3, P_4 or P_6. Applicant a_4 can fill jobs P_2 or P_5. This situation is represented by the graph in figure below. The vacant positions and applicants are represented by vertices. The edges represent the qualifications of each applicant for filling different positions. The graph clearly is bipartite, the vertices falling into two sets $V_1 = \{a_1, a_2, a_3, a_4\}$ and $V_2 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$.



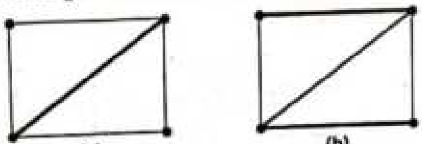
Bipartite graph

The questions one is most likely to ask in this situation are : is it possible to hire all applicants and assign each a position for which he is suitable. If the answer is no, what is the maximum number of positions that can be filled from the given set of applicants?

This is a problem of matching (or assignment) of one set of vertices into another. More formally, a **matching** in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is obviously a matching.

A **maximal matching** is a matching to which no edge in the graph can be added. For example, in a complete graph of three vertices (i.e., a triangle) any single edge is a maximal matching.

The edges shown by heavy lines in above figure are two maximal matchings. Clearly, a graph may have many different maximal matchings, and of different sizes. Among these, the maximal matching with the largest number of edges are called the **largest maximal matching**. In figure (b), a largest maximal matching is shown in heavy lines. The number of edges in a largest maximal matching is called the **matching number** of the graph.



(a) (b)

Although matching is defined for any graph, it is mostly studied in the context of bipartite graphs, as suggested by the introduction to this section.

In a bipartite graph having a vertex partition V_1 and V_2, a **complete matching** of vertices in set V_1 into those in V_2 is a matching in which there is one edge incident with every vertex in V_1. In other words, every vertex in V_1 is matched against some vertex in V_2. Clearly, a complete matching (if it exists) is a largest maximal matching, whereas the converse is not necessarily true.

For the existence of a complete matching of set V_1 into set V_2, first we must have at least as many vertices in V_2 as there are in V_1. In other words there must be at least as many vacant positions as the number of applicants if are to be hired. This condition, however, is not sufficient. For example, the above figure, although there are six positions and four applicants, a complete matching does not exist. Of the three applicants a_1, a_2 and a_4 each qualifies for the same two positions P_2 and P_5 and therefore one of the three applicants cannot be matched.

This leads us to another necessary condition for a complete matching: Every subset of r verties in V_1 must collectively be adjacent to at least r vertices in V_2, for all values or $r = 1, 2 ..., |V_1|$. This condition is not satisfied in above figure. The subset $\{a_1, a_2, a_4\}$ of three vertices has only two vertices P_2 and P_5 adjacent to them. That this condition is also sufficient for existence of a complete matching is indeed surprising. Theorem below is a formal statement and proof of this result.

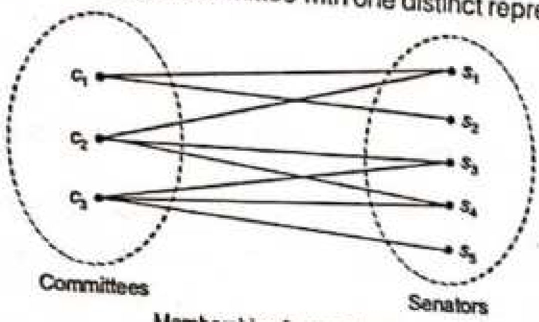
NOTE: ***Theorem:*** A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.

Let us illustrate this important theorem with an example:

Problem of Distinct Representatives: Five senators s_1, s_2, s_3, s_4 and s_5 are members of three committees c_1, c_2 and c_3. The membership is shown in figure below. One member from each committee is to be represented in a super committee. Is it possible to send one distinct representative from each of the committees?

This problem is one of finding a complete matching of a set V_1 into set V_2 in a bipartite graph. Let us use theorem above and check if r vertices from V_1 are collectively adjacent to at least r vertices from V_2, for all values of r. The result is shown in the Table below (ignore the last column for the time being).

Thus for this example the condition for the existence of a complete matching is satisfied as stated in Theorem. Hence it is possible to form the supercommittee with one distinct representative from each committee.



Membership of committees

	V_1	V_2	$r-q$
$r=1$	$\{c_1\}$	$\{s_1, s_2\}$	-1
	$\{c_2\}$	$\{s_1, s_3, s_4\}$	-2
	$\{c_3\}$	$\{s_3, s_4, s_5\}$	-2
$r=2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$	-2
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$	-2
	$\{c_3, c_1\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-3
$r=2$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-2

In above table, $r = |V_1|$ and $q = |V_2|$

The problem of distinct representatives just solved was a small one. A larger problem would have become unwieldy. If there are M vertices in V_1. Theorem requires that we take all $2^M - 1$ non-empty subsets of V_1 and find the number of vertices of V_2 adjacent collectively to each of these. In most cases, however, the following simplified version of the theorem will suffice for detection of a complete matching in any large graph.

NOTE: ***Theorem:*** In a bipartite graph a complete matching of V_1 into V_2 exists if (but not only if) there is a positive interger m for which the following condition is satisfied:
Degree of every vetex in V_1 ³ m ³ degree of every vertex in V_2.

In the bipartite graph of above figure. Degree of every vertex in $V_1 \geq 2 \geq$ degree of every vertex in V_2.

Therefore, there exists a complete matching.

In the bipartite graph of figure in previous page (applicants and positions), no such number is found, because the degree of $p_2 = 4 >$ degree of a_1.

It must be emphasized that the condition of Theorem above, is a sufficient condition and not necessary for the existence of a complete matching. It will be instructive for the reader to sketch a bipartite graph that does not satisfy the theorem and yet has a complete matching.

If one fails to find a complete matching, he is most likely to be interested in finding a maximal matching, that is, to pair off as many vertices of V_1 with those in V_2 as possible. For this purpose, let us define a new term called deficiency, δ (G), of a bipartite graph G.

A set of r vertices in V_1 is collectively incident on, say, q vertices of V_2. Then the maximum value of the number $r - q$ taken over all values of $r = 1, 2, \ldots$ and all subsets of V_1 is called the deficiency δ(G) of the bipartite graph G.

Theorem above, expressed in terms of the deficiency, states that a complete matching in a bipartite graph G exists if and only if $\delta(G) \le 0$. For example, the deficiency of the bipartite graph in above figure is -1 (the largest number in the last column of above table). It is suggested that you prepare a table for the graph in previous page (applicants and positions) similar to above table and verify that the deficiency is $+1$ for this graph.

The next theorem gives the size of the maximal matching for a bipartite graph with a positive deficiency.

Theorem: The maximal number of vertices in set V_1, that can be matched into V_2 is equal to number of vertices in $V_1 - \delta(G)$.

The size of a maximal matching in the problem of applicants and positions, using above theorem, is obtained as follows:

Number of vertices in $V_1 - \delta(G) = 4 - 1 = 3$

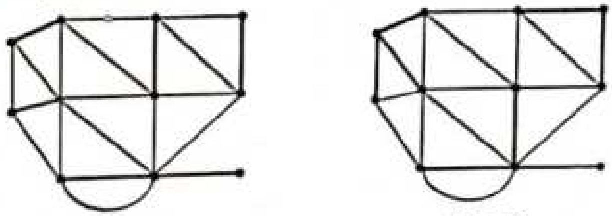
4.13.5 Coverings

In a graph G, a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g.

A set of edges that covers a graph G is said to be an **edge covering**, a **covering subgraph**, or simply a **covering** of G.

For example a graph G is trivially its own covering. A spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering.

A Hamiltonian circuit (if it exists) in a graph is also a covering. Just any covering is too general to be of much interest. We have already dealt with some coverings with specific properties, such as spanning trees and Hamiltonian circuits. In this section we shall investigate the **minimal covering** - a covering from which no edge can be removed without destroying its ability to cover the graph.



(a) e = 6 (b) e = 8

A Graph and two of its minimal coverings

In above figure a graph and two of its minimal coverings are shown in heavy lines. The following observations should be made:

1. A covering exists for a graph if and only if the graph has no isolated vertex.
2. A covering of an n-vertex graph will have at least $\lceil n/2 \rceil$ edges. ($\lceil x \rceil$ denotes the smallest integer not less than x).
3. Every pendant edge in a graph is included in every covering of the graph.
4. Every covering contains a minimal covering.
5. No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than $n - 1$ edges.

6. A graph, in general, has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges). The number of edges in a minimal covering of the smallest size is called the covering number of the graph.

Theorem: A covering g of a graph is minimal if and only if g contains no paths of length three or more.

Suppose that the graph in above figure represented the street map of a part of a city. Each of the vertices is a potential trouble dot and must be kept under the surveillance of a patrol car. How will you assign a minimum number of patrol cars to keep every vertex covered? The answer is a smallest minimal covering. The covering (a) shown in above figure is an answer, and it requires six patrol cars. Clearly, since there are 11 vertices and no edge can cover more than two, less than six edges cannot cover the graph.

Summary

- A vertex with zero degree is called a **lone vertex** or isolated vertex and a vertex with exactly one degree is called a **pendent** vertex or end vertex.
- A complete graph is a simple graph with maximum number of possible edges.
- The number of edges in $K_n = \frac{n(n-1)}{2} = nC_2$

  Example: K_5 has 5 vertices and $\frac{5(5-1)}{2} = 10$ edges
- If a simple graph has nC_2 edges it is complete.
- Maximum edges possible in an n-vertex simple graph is $\frac{n(n-1)}{2}$.
- Isomorphic simple graphs must have:
  - Same no of vertices
  - Same no of edges
  - Degrees of corresponding vertices must be same
  - Number of simple circuits of a certain length must be same in both graphs. All the above are called invariants in an isomorphism.
- If G is a simple graphs of n vertices, then $G \cup \overline{G}$ is K_n, the complete graph on n vertices.
- When $k=1$, a simple graph with n vertices can have atmost $(n-1)n/2$ edges which is the maximum possible edges in a connected simple graph.
- Strongly connected ⇒ Unilateral as well as weakly connected but converse is not true
  Unilaterally connected ⇒ Weakly connected but converse is not true
  If a diagraph is not even weakly connected, then such a graph will be a disconnected graph.
- K_5 and $K_{3,3}$ are known as Kuratowski's two graphs. These two graphs are special since, K_5 is the non planar graph with minimum number of vertices and $K_{3,3}$ is the non planar graph with minimum number of edges.
- Every tree with 2 or more vertices is 2-chromatic.
- A graph with atleast one edge is 2-chromatic if and only if it has no circuits of odd length. i.e. bipartite graphs (which have no circuits of odd length) are 2-chromatic. Also, $K(K_{m,n}) = 2$.

- If d_{max} is the maximum degree of the vertices in a graph G, Chromatic number of $G \leq 1 + d_{max}$.
- A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.
- In a bipartite graph a complete matching of V_1 into V_2 exists if (but not only if) there is a positive interger m for which the following condition is satisfied: Degree of every vetex in $V_1^3\ m^3$ degree of every vertex in V_2.

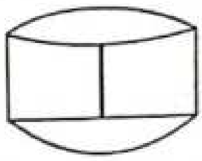
Student's Assignment

Q.1 The minimum number of edges in a connected cyclic graph on 2 vertices is
(a) 1 (b) 2
(c) 3 (d) None of these

Q.2 If the following degree vertex (5, 3, 2, 4, 3), can't form a simple graph then which of the following degree vertex should be added to make it form a simple graph?
(a) 1 (b) 5
(c) 2 (d) Both (a) and (b)

Q.3 How many edges are possible in a forest with n vertices and k trees?
(a) $n - k + 1$ (b) $n - k - 1$
(c) $n - k$ (d) Insufficient data

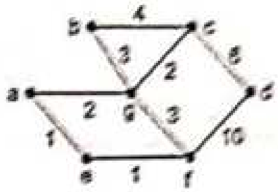
Q.4 The minimum number of colours required to colour the following graph is



(a) 1 (b) 2
(c) 3 (d) 4

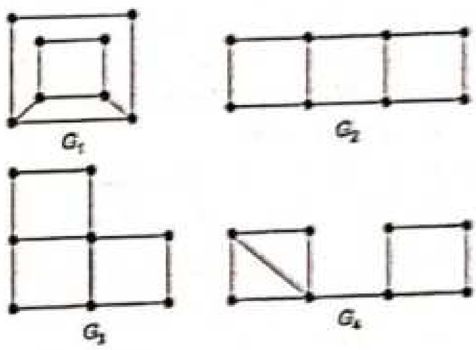
Q.5 What is the maximum length of cycles in a diagraph of partial order on A having n elements?
(a) 2 (b) 3
(c) 1 (d) n

Q.6 Consider the following graph.



While constructing the minimum spanning tree of the above graph using Kruskal's algorithm, which of the following is a possible order in which the edges are added to the minimum spanning tree?
(a) (a,e), (a,g), (g,f), (b,c), (c,d), (d,f)
(b) (a,e), (e,f), (a,g), (g,c), (g,f), (b,g)
(c) (a,e), (e,f), (a,g), (g,c), (b,g), (b,c)
(d) (e,f), (a,e), (g,c), (a,g), (b,g), (c,d)

Q.7 Determine which pairs G_i, G_j of the graphs below are isomorphic



(a) $(G_1; G_2)$
(b) $(G_1; G_2)$ and $(G_1; G_3)$
(c) $(G_1; G_2)$ and $(G_2; G_3)$
(d) $(G_1; G_2)$ and $(G_3; G_4)$

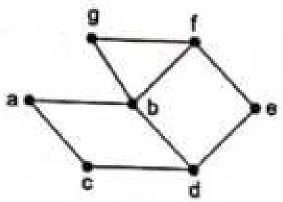
Q.8 Which of the following statements are true?
(i) Euler's problem can be solved in polynomial time

(*ii*) Hamiltan's problem is believed to lie in class NP (Non-deterministic polynomial)

(*iii*) Traveling saleman problem can also be proved to be in NP

(a) Only (*i*) and (*iii*) are true
(b) Only (*i*) and (*ii*) are true
(c) Only (*ii*) and (*iii*) are true
(d) All (*i*), (*ii*) and (*iii*) are true

Q.9 Consider the following graph:



Which of the following statements is true about the graph?

(a) It has an Eulerian path, but not Eulerian cycle
(b) It has an Eulerian cycle, but not Eulerian path
(c) It has both Eulerian path and Eulerian cycle
(d) It doesn't have an eulerian path or Eulerian cycle

Q.10 Which of the following statements is/are false?

(*i*) An edge e of a graph G is cut edge if e is not part of any cycle in g.

(*ii*) A connected graph G is Euler if and only if the degree of every vertex is odd.

(*iii*) A relation on a set P is said to be partial if it is reflexive, symmetric and transitive.

(*iv*) All of the above

(a) (*ii*) only
(b) (*i*) and (*ii*)
(c) (*ii*) and (*iii*)
(d) (*iv*)

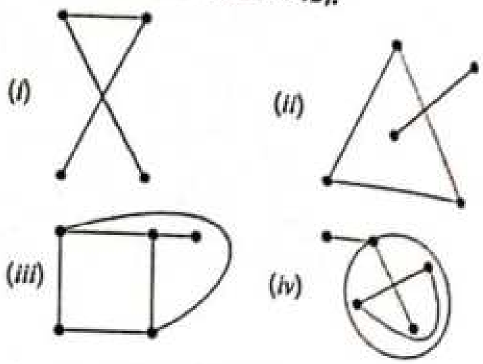
Q.11 Suppose G = G (V, E) has five vertices. Find the maximum number *m* of edges in E if G is a simple graph.

(a) 10
(b) 15
(c) 20
(d) None of these

Q.12 Suppose G = G (V, E) has five vertices, find the maximum number *m* of edges in E if G is a multigraph

(a) Finite
(b) Infinite
(c) ${}^{m}C_2$
(d) None of the above

Common Data Questions (13 to 15):



Q.13 From the above graphs, which of them are connected.

(a) (*i*) and (*iii*)
(b) (*ii*) and (*iii*)
(c) (*iii*) and (*iv*)
(d) (*iv*) and (*i*)

Q.14 Which of the graphs are multigraphs (i.e., have no loops, multiple edges allowed)

(a) (*i*), (*ii*), (*iii*)
(b) (*ii*), (*iii*), (*iv*)
(c) (*iii*), (*iv*), (*i*)
(d) (*iv*), (*i*), (*ii*)

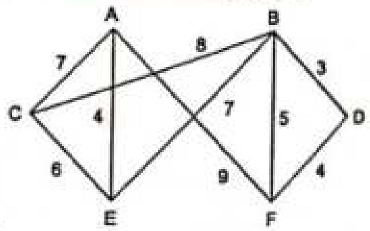
Q.15 Which of the above are simple graphs?

(a) (*i*), (*ii*) and (*iii*)
(b) (*ii*) and (*iii*)
(c) (*iii*) and (*iv*)
(d) None of these

Q.16 Number of edges of a complete binary tree with 16 leaf nodes is

(a) 14
(b) 30
(c) 32
(d) 28

Q.17 Find the length of minimal spanning tree for graph represented in the following figure.

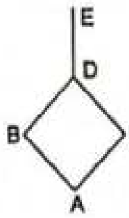


(a) 28
(b) 55
(c) 24
(d) None of these

Q.18 If the degree of every non-pendent vertex in a tree is 3, then the number of vertex of the tree is.

(a) odd
(b) even
(c) odd or even
(d) such a tree is not possible

Q.19 Consider the following figure which of the following is true?



(a) There exists a Euler path but not Euler circuit
(b) There exists a Euler circuit
(c) Euler path is not possible
(d) None of the above

Q.20 How many edges are there in a complete graph having 12 nodes?

(a) 12 (b) 144
(c) 66 (d) None of these

Answer Key:

1. (b)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (c)
6. (d)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (c)
11. (a)	**12.** (b)	**13.** (a)	**14.** (a)	**15.** (a)
16. (b)	**17.** (c)	**18.** (b)	**19.** (a)	**20.** (c)

Student's Assignments Explanations

2. (d)

By using Hevell-Hakimi theorem:

$$(\cancel{5}43321)$$
$$(\cancel{3}2210)$$
$$(1100)$$

Since number of 1 are even. Hence 1 is possible.

Similarly by using Hevell-Hakimi theorem:

$$(\cancel{5}54332)$$
$$(\cancel{4}3221)$$
$$(\cancel{2}110)$$
$$(000)$$

Since number of 1 are even. Hence 5 is possible.

So both 1 and 5 are possible.

3. (c)

As the problem definition says 'n' vertices are divided in to 'k' trees. Let these trees are having $x_1, x_2, x_3, ..., x_k$, vertices respectively. The sum of $x_1, x_2, ..., x_k$ should be 'n' as we have only 'n' vertices in total. The definition of tree says that any tree says that any tree with 'n' vertices will have '$n - 1$' edges. So all the 'k' trees will have '$x_1 - 1 + x_2 - 1 + ... + x_k - 1$' edges

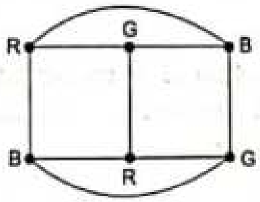
Result $= x_1 - 1 + x_2 - 1 + ... + x_k - 1$

$= x_1 + x_2 + ... + x_k - k$

$= n - k$

4. (c)

Let's say we have 3 colors namely R, G and B.

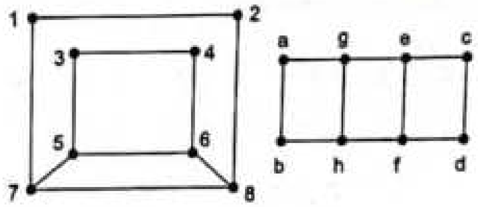


7. (a)

The degree sequence of the above graphs is:

G_1: (2, 2, 2, 2, 3, 3, 3, 3);
G_2: (2, 2, 2, 2, 3, 3, 3, 3);
G_3: (2, 2, 2, 2, 2, 3, 3, 4);
G_4: (2, 2, 2, 2, 2, 3, 3, 4).

G_1 is isomorphic to G_2 is shown by the following labeling:



1 corresponding to a, 2 to b, etc.

The differing degree sequences show neither G_3 nor G_4 can be isomorphic to either G_1 or G_2. Also G_3 is not isomorphic G_4 because G_4 has triangles (circuits of length 3) and G_3 does not.

9. (a)

The graph has no Euler cycle since f and d have odd degree.

However, since all vertices have even degrees except exactly 2 vertices namely f and d, therefore the graph does have an Eulerian path. One such eulerian path is f, b, g, f, e, d, b, a, c, d.

Other variants are possible, but in each variant the first and last vertex has to be f and d or d and f since these are the two vertices with odd degrees.

11. (a)

There are C(5, 2) = 10 ways of choosing two vertices from V to make an edge; hence $m = 10$.

12. (b)

Since multiple edges are permitted (in a multi-graph), G can have any number of edges between 2 vertices. Hence $m = \infty$.

13. (a)

Only (*i*) and (*iii*) are connected.

14. (a)

Only (*iv*) has a loop, (*i*), (*ii*) and (*iii*) are loop free.

15. (a)

Only (*i*) and (*ii*) are simple graphs. The multigraph (*iii*) has multiple edges and (*iv*) has multiple edges and a loop.

16. (b)

$$n = 2i + 1$$

Also $\quad n = \ell + i$

Since $\quad \ell = 16$

$$n = 16 + i = 2i + 1$$

$\Rightarrow \quad i = 15$

$$n = \ell + i = 16 + 15 = 31$$

$$e = n - 1 = 31 - 1 = 30$$

17. (c)

Use Kruskal's algorithm as given below:

- Arrange the edges in ascending order of weight.
- Add selectively the edges, such that addition of edge doesn't form a cycle.

The edges added are in the following order BD, DF, AE, CE, BE.

Total length is 24.

18. (b)

Let n be the number of vertices of the tree and let p be the number of pendent vertices.

Now, $\quad \Sigma deg = 2e$

In a tree $e = n - 1$

$\therefore \quad \Sigma deg = 2(n - 1)$

Since in this tree every non-pendent vertex has degree 3, and since pendent vertices have degree 1 always, we can say that

$$p + 3(n - p) = 2(n - 1)$$

Solving which we get

$$n = 2p - 2 = 2(p - 1)$$

which is even.

19. (a)

Since there are exactly 2 vertices (E and D) in this graph with odd degree, this graph has an euler path but not an euler circuit.

20. (c)

The number of edges in K_n is

$$^nC_2 = \frac{n(n-1)}{2}$$

K_{12} has $\dfrac{12(12-1)}{2} = 6.11 = 66$ edges

CHAPTER 05

Probability

5.1 Some Fundamental Concepts

Sample Space and Event: Consider an experiment whose outcome is not predictable with certainty. Such an experiment is called a **random experiment**. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the **sample space** of experiment and is denoted by S.

Some examples follows:

(*i*) If the outcome of an experiment consist in the determination of the sex of a newborn child, then S = {*g*, *b*} where the outcome *g* means that the child is a girl and *b* is the boy.

(*ii*) If the outcome of an experiment consist of what comes up on a single dice, then S = {1, 2, 3, 4, 5, 6}

(*iii*) If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7; then S = {all 7! permutations of the (1, 2, 3, 4, 5, 6, 7)}

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instances, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

Any subset E of the sample space is known as **Event**. That is, an event is a set consisting of some or all of the possible outcomes of the experiment.

If the outcome of the experiment is contained in E, then we say that E has occurred. Always $E \subseteq S$.

Since E and S are sets, theorems of set theory may be effectively used to represent and solve probability problems which are more complicated.

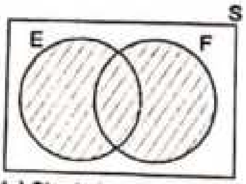
Examples: In the preceding example – (i) If $E_1 = \{g\}$, then E_1 is the event that the child is a girl.

Similarly, if $E_2 = \{b\}$.

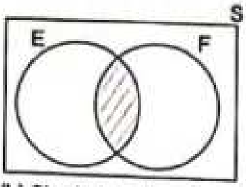
Then E_2 is the event that the child is a boy. These are examples of simple events. Compounded events may consist of more than one outcome. Such as E = {1, 3, 5} for an experiment of throwing a dice. We say event E has happened if the dice comes up 1 or 3 or 5.

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consists of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F or both occurs. For instances, in dice example (i) if event $E_1 = \{1, 2\}$ and $E_2 = \{3, 4\}$, then $E \cup F = \{1, 2, 3, 4\}$.

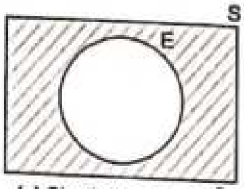
That is $E \cup F$ would be another event consisting of 1 or 2 or 3 or 4. The event $E \cup F$ is called union of event E and the event F. Similarly, for any two events E and F we may also define the new event $E \cap F$, called intersection of E and F, to consists of all outcomes that are common to both E and F.



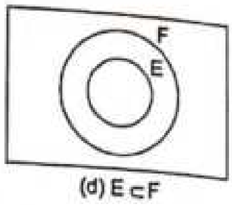
(a) Shaded region : $E \cup F$



(b) Shaded region : $E \cap F$



(c) Shaded region : E^C



(d) $E \subset F$

Mutually Exclusive Events

Two events E and F are mutually exclusive, if $E \cap F = \phi$ i.e. $P(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).

Collectively Exhaustive Events

Two events E and F are collectively exhaustive, if $E \cup F = S$ i.e. together E and F include all possible outcomes, $P(E \cup F) = P(S) = 1$

DeMorgan's Law

(i) $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$ (ii) $\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$

Example: $(E_1 \cup E_2)^C = E_1^C \cap E_2^C$ $(E_1 \cup E_2)^C = E_1^C \cup E_2^C$

Note that $E_1^c \cap E_2^c$ is called neither E_1 nor E_2. $E_1 \cup F_2$ is called either E_1 or E_2 (or both).

5.1.1 Approaches to Probability

There are 2 approaches to quantifying probability of an Event E.

1. **Classical Approach:** $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{|E|}{|S|}$

   i.e. the ratio of number of ways an event can happen to the number of ways sample space can happen, is the probability of the event. Classical approach assumes that all outcomes are equally likely.

2. **Frequency Approach:** Since sometimes all outcomes may not be equally likely, a more general approaches is the frequency approach, where probability is defined as the relative frequency of occurrence of E.

   $P(E) = \lim_{N \to \infty} \dfrac{n(E)}{N}$ where N is the number of times exp is performed & n(E) is the no of times the event E occurs.

5.1.2 Axioms of Probability

Consider an experiment whose sample space is S. For each event E of the sample space S we assume that a number P(E) is defined and satisfies the following three axioms.

Axiom-1: $0 \le P(E) \le 1$
Axiom-2: $P(S) = 1$
Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, ...$(that is, events for which $E_i \cap E_j = \phi$ when $i \ne j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ (E_1, E_2 are mutually exclusive)

Some Simple Propositions

It is to be noted that E and E^c are always mutually exclusive and since $E \cup E^c = S$. We have by Axiom-(2) and (3) that : $P(E \cup E^c) = P(E) + P(E^c) = P(S) = 1$

Proposition-1: $P(E^c) = 1 - P(E)$
Proposition-2: If $E \subseteq F$, then $P(E) \le P(F)$
Proposition-3: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
Prop - 3 is more general then axiom 3, since here E & F need not be mutually exclusive
Prop - 3 reduces to axiom - 3 when E, F mutually exclusive ($E \cap F = \phi$)
Prop - 3 may be extended for union of more sets as follows:
$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) + P(E \cap F \cap G)$

5.1.3 Conditional Probability

$$E/F = \frac{P(E \cap F)}{P(F)}$$

E/F is called the **conditional probability** of E given F.

Example - 5.1 A coin is flipped twice. What is the conditional probability that both flips result in heads, given that the first flip does?

Solution:

$$E/F = \frac{P(E \cap F)}{P(F)}$$

i.e. P (both are heads I first is heads)

$$= \frac{P(\text{both heads \& first is head})}{P(\text{first is head})}$$

$$= \frac{P(\text{both heads})}{P(\text{first head})} = \frac{1/4}{1/2} = \frac{1}{2}$$

5.1.4 The Multiplication Rule

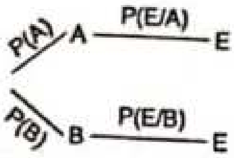
$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1) \quad ...(5.1)$$
$$= P(E_2) \cdot P(E_1/E_2) \quad ...(5.2)$$

Notice that (1) and (2) can be obtained from the following conditional probability formulas after cross multiplication.

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad \text{and} \quad P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

5.1.5 Rule of Total Probability and Bayes Theorem

Consider an event E which occurs via two different events A and B. Further more, Let A and B be mutually exclusive and collectively exhaustive events. This situation may be represented by following tree diagram



Now, the probability of E is given by value of total probability as:

$$P(E) = P(A \cap E) + P(B \cap E)$$
$$= P(A) * P(E/A) + P(B) *(E/B)$$

Sometimes we wish to know that, given that the event E has already occurred, what is the probability that it occurred with A?

i.e.
$$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A \cap E)}{P(A \cap E) + P(B \cap E)}$$

$$= \frac{P(A) * P(E|A)}{P(A) * P(E|A) + P(B) * P(E|B)}$$

Notice that the denominator of Bayes theorem formula is obtained by using rule of total probability.

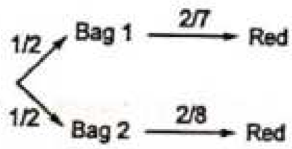
Example - 5.2 Suppose we have 2 bags. Bag 1 contains 2 red and 5 green marbles. Bag 2 contains 2 red and 6 green marbles. A person tosses a coin and if it is heads goes to bag 1 and draws a marble. If it is fails he goes to bag 2 and draws a marble. In this situation.

1. What is the probability that the marble drawn this is Red?
2. Given that the marble draw is red, what is probability that it came from bag 1.

Solution:

The tree diagram for above problem,

1. $\therefore$ $P(\text{Red}) = 1/2 \times 2/7 + 1/2 + 2/8$



2. $$P(\text{bag1}|\text{Red}) = \frac{P(\text{bag1} \cap \text{Red})}{P(\text{Red})}$$

$$= \frac{\frac{1}{2} \times \frac{2}{7}}{\frac{1}{2} \times \frac{2}{7} \times \frac{1}{2} \times \frac{2}{8}} = \frac{\frac{1}{7}}{\frac{15}{56}} = 8/15$$

5.1.6 Independent Events

Two events are said to be **independent** if equation (A) holds.

$$P(E \cap F) = P(E) * P(F) \quad \text{...(A)}$$

Two events are said to be dependent if they are not independent.

Also it E and F are independent

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \times P(F)}{P(F)} = P(E)$$

Similarly, $P(E|F) = P(F)$

P(E|F) is called **conditional probability** of E given F and P (E) in called **marginal probability** of E to distinguish it from P(E|F).

P(F) is the marginal probability of F.

Example: A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then

$$P(E \cap F) = P(\text{Ace and Spade}) = \frac{1}{52}$$

$$P(E) = P(\text{Ace}) = \frac{4}{52} \text{ and } P(F) = P(\text{Spade}) = \frac{13}{52}$$

Here,

$$P(E \cap F) = P(F) * P(F)$$

$\therefore$ E and F independent.

Proposition: If E and F are independent, then so are E and F^C, E^C & F, E^C & F^C.

Condition for three Events to be Independent: The events E, F ad G are said to be independent if

$P(EFG) = P(E)\,P(F)\,P(G)$

$$\left.\begin{array}{l} \text{and } P(EF) = P(E)\,P(F) \\ \text{and } P(EG) = P(E)\,P(G) \\ \text{and } P(FG) = P(F)\,P(G) \end{array}\right. \begin{bmatrix} \text{E, F, G} \\ \text{pairwise} \\ \text{independent} \end{bmatrix}$$

It should be noted that if E, F and G are independent, then E will be independent of any event formed from F and G. For instance, E is independent of $F \cup G$.

5.2 Mean

Arithmetic Mean

The formula for calculating the arithmetic mean is: $\bar{x} = \frac{\Sigma x}{n}$

$\bar{x}$-arithmetic mean

x-refers to the value of an observation

n-number of observations.

Example - 5.3 The number of visits made by ten mothers to a clinic were 8 6 5 5 7 4 5 9 7 4. Calculate the average number of visits.

Solution:

Σx = total of all these numbers of visits, that is the total number of visits made by all mothers.

$$8 + 6 + 5 + 5 + 7 + 4 + 5 + 9 + 7 + 4 = 60$$

$$\text{Number of mothers } n = 10$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{60}{10} = 6$$

The Arithmetic Mean of a Frequency Distribution

The formula for the arithmetic mean calculated from a frequency distribution has to be amended to include the frequency. It becomes:

$$\bar{x} = \frac{\Sigma(fx)}{\Sigma f}$$

Arithmetic Mean of a Grouped Data

To show how we can calculate the arithmetic mean of a grouped frequency distribution, there is a example of weights of 75 pigs. The classes and frequencies are given in following table:

Weight (kg)	Midpoint of class x	Number of pigs (frequency)	fx
Under 20	$\simeq 15$	1	15
20 & under 30	25	7	175
30 & under 40	35	8	280
40 & under 40	45	11	495
50 & under 60	55	19	1045
60 & under 70	65	10	650
70 & under 80	75	7	525
80 & under 90	85	5	425
90 & under 100	95	4	380
Over 100	$\simeq 105$	3	215
Total		75	4305

With such a frequency distribution we have a range of values of the variable comprising each group. As our values for x in the formula for the arithmetic mean we use the midpoints of the classes.

$$\text{In this case } \bar{x} = \frac{\Sigma(fx)}{\Sigma f} = \frac{4305}{75} = 57.4 \text{ kg}$$

5.3 Median

Arithmetic mean is the central value of the distribution in the sense that positive and negative deviations from the arithmetic mean balance each other. On the other hand, **median** is the central value of the distribution in the sense that the number of values less than the median is equal to the number of values greater than the median.

Median is the central value in a sense different from the arithmetic mean. In case of the arithmetic mean it is the "**numerical magnitude**" of the deviations that balances. But, for the median it is the 'number of values greater than the median which balances against the number of values of less than the median. In general, if we have n values of x, they can be arranged in ascending order as:

$x_1 < x_2 < \ldots\ldots < x_n$

Suppose n is odd, then

$$\text{Median} = \text{the } \frac{(n+1)}{2}\text{-th value}$$

However, if n is even, we have two middle points

$$\text{Median} = \frac{\left(\frac{n}{2}\right)\text{-th value} + \left(\frac{n}{2}+1\right)\text{-th value}}{2}$$

Example - 5.4 The heights (in cm) of six students in class are 160, 157, 156, 161, 159, 162. What is median height?

Solution:

Arranging the heights in ascending order

156, 157, 159, 160, 161, 162

Two middle most values are the 3rd and 4th.

$$\text{Median} = \frac{1}{2}(159 + 160) = 159.5$$

Median for Grouped Data

1. Identify the median class which contains the middle observation ($\simeq (n+1/2)^{th}$ observation). This can be done by observing the first class in which the cumulation frequency is equal to or more than $n+1/2$.
2. Calculate Median as follows:

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] \times h$$

Where,

L = Lower limit of median class

N = Total number of data items = ΣF

F = Cumulative frequency of the class immediately preceding the median class

f_m = Frequency of median class

h = width of median class

Median for Grouped Data

Consider the following table giving the marks obtained by students in an exam

Mark Range	f No. of Students	Cumulative Frequency
0 – 20	2	2
20 – 40	3	5
40 – 60	10	15
60 – 80	15	30
80 – 100	20	50

Here $\frac{N+1}{2} = 25.5$

The class 60-80 is the median class since cum-freq is 30 > 25.5

$$\text{Median} = \frac{60 + [25.5 - (15+1)]}{15} \times 20 = 69.66$$

∴ Median marks of the class is approximately 69.7 (at most).

i.e. (at least) half the students got less than 69.7 and (almost) half got more than 69.7 marks.

5.4 Mode and Standard Deviation

Mode: Mode is defined as the value of the variable which occurs most frequently.

Calculation of Mode: Mode is that value of x for which the frequency is maximum. If the values of x are grouped into the classes (such that they are uniformly distributed within any class) and we have a frequency distribution then:

(*i*) Identify the class which has the largest frequency (modal class)

(*ii*) Calculate the mode as:

$$\text{Mode} = L + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

Where,

L = Lower limit of the modal class

f_0 = Largest frequency (frequency of Modal Class)

f_1 = Frequency in the class preceding the modal class

f_2 = Frequency in the class next to the modal class

h = Width of the modal class

Example - 5.5 Data relating to the height of 352 school students are given in the following frequency distribution. Calculate the modal height.

Heigh (in feet)	Number of students
3.0 – 3.5	12
3.5 – 4.0	37
4.0 – 4.5	79
4.5 – 5.0	152
5.0 – 5.5	65
5.5 – 6.0	7
Total	352

Solution:

Since 152 is the largest frequency, the modal class is (4.5 – 5.0).

Thus $L = 4.5$, $f_0 = 152$, $f_1 = 79$, $f_2 = 65$, $h = 0.5$.

$$\text{Mode} = 4.5 + \frac{152 - 79}{2(152) - 79 - 65} \times 0.5 = 4.73 \text{ (approx.)}$$

While mean, median and mode are measures of central tendency.

5.5 Standard Deviation

Standard Deviation is a measure of dispersion or variation amongst data.

Instead of taking absolute deviation from the arithmetic mean, we may square each deviation and obtain the arithmetic mean of squared deviations. This gives us the 'variance' of the values.

The positive square root of the variance is called the **'Standard Deviation'** of the given values.

Standard Deviation from Raw Data

Suppose $x_1, x_2, \dots x_n$ are n values of the x, their arithmetic mean is:

$\bar{x} = \frac{1}{N}\Sigma x_i$ and $x_1 - \bar{x}, x_2 - \bar{x} \dots x_n - \bar{x}$ are the deviations of the values of x from $\bar{x}$. Then

$\sigma^2 = \frac{1}{n^2}\Sigma(x_i - \bar{x})^2$ is the variance of x. It can be shown that

$$\sigma^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = \frac{1}{n}\Sigma x_i^2 - \bar{x}^2 = \frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}$$

It is conventional to represents the variance by the symbol σ^2. Infact, σ is small sigma and Σ is capital sigma.

Square root of the variance is the standard deviation

$$\sigma = +\sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2} = \sqrt{1/n\,\Sigma x_i^2 - \bar{x}^2} = \sqrt{\frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{n^2}}$$

Calculation of Standard Deviation from Grouped Data

Calculation for standard deviation for grouped data can be shown by this example:

The frequency distribution for heights of 150 young ladies in a beauty contest is given below for which we have to calculate standard deviation.

Height (in inches)	Mid values x	Frequency f	$f_i \times x_i$	$f_i \times x_i^2$
62.0 – 63.5	62.75	12	753.00	47250.75
63.5 – 65.0	64.25	20	1285.00	82561.25
65.0 – 66.5	65.75	28	1841.00	121045.75
66.5 – 68.0	67.25	18	1210.50	81406.125
68.0 – 69.5	68.75	19	1306.25	89806.125
69.5 – 71.0	70.25	20	1405.00	89804.6875
71.0 – 72.5	71.75	30	2152.50	98701.25
72.5 – 74.0	73.25	3	219.75	154441.875
				16096.6875
Total		150	10173.00	691308.375

Thus, $$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{10173}{150} = 67.82$$

and $$\frac{\Sigma f_i x_i^2}{\Sigma f_i} = 4608.7225$$

where, $$N = \Sigma f_i = 150$$

Therefore, the variance of x is

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - \bar{x}^2 = \frac{N\Sigma f_i x_i^2 - (\Sigma f_i x_i)^2}{N^2} = 9.1701$$

$\sigma_x = 3.03$ (inches) and standard deviation is 3.03 (inches).

5.6 Random Variables

It is frequently the case when an experiment is performed that we are mainly interested in some function of the outcome as opposed to the actual outcome itself.

For instances, in tossing dice we are often interested in the sum of two dice and are not really concerned about the separate value of each die. That is, we may be interested in knowing that the sum is 7 and not be concerned over whether the actual outcome was (1, 6) or (2, 5) or (3, 4) or (4, 3) or (5, 2) or (6, 1).

Also, in coin flipping we may be interested in the total number of heads that occur and not care at all about the actual head tail sequence that results. These quantities of interest, or more formally, these real valued functions defined on the sample space, are known as random variables.

Because the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

Types of random variable: Random variable may be discrete or continuous.

Discrete random variable: A variable that can take one value from a discrete set of values.

Example: Let x denotes sum of 2 dice, Now x is a discrete random variable as it can take one value from the set {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, since the sum of 2 dice can only be one of these values.

Continuous random variable: A variable that can take one value form a continuous range of values.

Example: x denotes the volume of Pepsi in a 500 ml cup. Now x may be a number from 0 to 500, any of which value, x may take.

5.7 Distributions

Based on this we can divide distributions also into **discrete distribution** (based on a disc random variable) or **continuous distribution** (based on a continuous random variable).

Examples of discrete distribution are binomial, poisson and hypergeometric distributions.

Examples of continuous distribution are uniform, normal and exponential distribution.

Properties of Discrete Distribution

- $SP(x) = 1$
- $E(x) = \Sigma x\, P(x)$
- $V(x) = E(x^2) - (E(x))^2$
- $V(x) = \Sigma x^2 P(x) - [\Sigma x\, P(x)]^2$

Properties of Continuous Distribution

- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $F(x) = \int_{-\infty}^{x} f(x)dx$ (cumulative distribution function)
- $E(x) = \int_{-\infty}^{\infty} x f(x)dx$
- $V(x) = \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} x f(x)dx\right]^2$
- $P(a < x < b) = P(a \le x \le b) = \int_{a}^{b} f(x)dx$

5.7.1 Types of Discrete Distributions

1. Binomial Distribution

Suppose that a trial or an experiment, whose outcome can be classified as either a success or a failure is performed.

Suppose now that n independent trials, each of which results in a successes with probability p and in a failure with probability $1 - p$, are to be performed.

If X represents the number of successes that occur in the n trials, then X is said to be binomial random variable with parameters (n, p). The Binomial distribution occurs when experiment performed satisfies the three assumptions of bernolli trials:

1. Only 2 outcomes are possible, success and failure
2. Probability of success (p) and failure ($1 - p$) remains same from trial to trial.
3. The trials are statistically independent. i.e The outcome of one trial does not influence subsequent trials.

The probability of x success from n trials is given by $P(X = x) = {}^nC_x\, p^x(1-p)^{n-x}$.

Where p is the probability of success in any trial and $(1 - p) = q$ is the probability of failure.

Example-5.6 10 dice are thrown. What is the probability of getting exactly 2 sixes?

Solution:

$$P(X=2) = {}^{10}C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8 = 0.2907$$

Example-5.7 It is known that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution:

If X is the number of defective screws in a packages, then X is a binominal variable with parameters (10, 0.01). Hence, the probability that a package will have to be replaced is:

$$P(X \geq 2) = 1 - [P(X \leq 1)] = 1 - [P\{X = 0\} + P\{X = 1\}]$$

$$= 1 - \left[\binom{10}{0}(0.01)^0(0.99)^{10} + \binom{10}{1}(0.01)^1(0.99)^9\right] \approx 0.004$$

Hence only 0.4% of packages will have to be replaced.

For Binomial distribution:

$$\text{Mean} = E[X] = np$$

$$\text{Variance} = V[X] = np(1-p)$$

Example-5.8 100 dice are thrown. How many are expected to fall 6. What is the variance in the number of 6's?

Solution:

$$E(x) = np = 100 \times 1/6 = 16.7$$

$$V(x) = np(1-p) = 100 \times 1/6 \times (1 - 1/6) = 13.9$$

2. Poisson Distribution

A random variable X, taking on one of the values 0, 1, 2 .... is said to be a Poisson random variable with parameter λ if for some $\lambda > 0$,

$$P(x = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

For Poisson distribution:

$$\text{Mean} = E(x) = \lambda$$
$$\text{Variance} = V(x) = \lambda$$

Therefore, expected value and variance of a Poisson random variable are both equal to its parameter λ.

Example - 5.9 A certain airport receives on an average of 4 aircrafts per hour. What is the probability that no aircraft lands in a particular 2 hr period?

Solution:

α = rate of occurrence of event = 4/hr

λ = average no of occurrences of event in specified observation period $\Delta t = \alpha\Delta t$

In this case α = 4/hr and $\Delta t = 2h$

$\therefore$ $\lambda = 4 \times 1 = 8$

Now we wish that no aircraft should land for 2 hrs. i.e. $x = 0$

$$P(x = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-8}8^0}{0!} = e^{-8}$$

Frequently, poisson distribution is used to approximate binominal distribution when n is very large and p is very small. Notice that direct computation of $nC_x p^x (1-p)^{n-x}$ may be erroneous or impossible when n is very large and p is very small. Hence, we resort to a poisson approximation with $\lambda = np$.

Example - 5.10 A certain company sells tractors which fail at a rate of 1 out of 1000. If 500 tractors are purchased from this company what is the probability of 2 of them failing within first year.

Solution:

$$\lambda = np\ 500 \times \frac{1}{1000} = \frac{1}{2}$$

$$P(x = 2) = \frac{e^{-1/2}\left(\frac{1}{2}\right)^2}{2!} = 0.1011$$

5.7.2 Types of Continuous Distributions

1. Uniform distribution
2. Exponential distribution
3. Normal distribution
4. Standard normal distribution

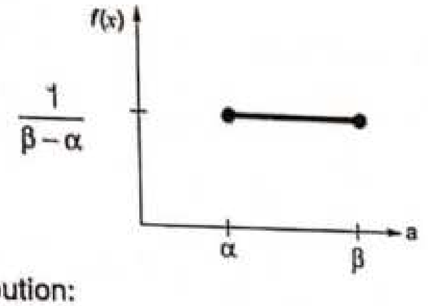
1. Uniform Distribution

In general we say that X is a uniform random variable on the interval (α, β) if its probability density function is given by:

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Since $f(x)$ is a constant, all values of x between α and β are equally likely (uniform).

Graphical representation:



For discrete uniform distribution:

$$\text{Mean} = E[X] = \frac{\beta+\alpha}{2}$$

$$\text{Variance} = V(X) = \frac{(\beta-\alpha)^2}{12}$$

Example - 5.11 If X is uniformly distributed over (0, 10), calculate the probability that:

(a) $X < 3$ (b) $X > 6$ (c) $3 < X < 8$

Solution:

$$P\{X < 3\} = \int_0^3 \frac{1}{10}dx = \frac{3}{10}$$

$$P\{X < 6\} = \int_6^{10} \frac{1}{10}dx = \frac{4}{10}$$

$$P\{3 < X < 8\} = \int_3^8 \frac{1}{10}dx = \frac{1}{2}$$

2. Exponential Distribution

A continuous random variable whose probability density function is given for some $\lambda > 0$ by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is said to be exponential random variable with parameter λ. The cumulative distributive function $F(a)$ of an exponential random variable is given by:

$$F(a) = P(X \leq a) = \int_0^a \lambda e^{-\lambda x}dx = \left(-e^{-\lambda x}\right)_0^a = 1 - e^{-\lambda a},\ a \geq 0$$

For Exponential distribution:

$$\text{Mean} = E[X] = \frac{1}{\lambda}$$

$$\text{Variance} = v(x) = 1/\lambda^2$$

Example-5.12 Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait,

(a) More than 10 minutes (b) Between 10 and 20 minutes

Solution:

Letting X denote the length of the call made by the person in the booth, we have that the desired probabilities are:

(a)
$$\begin{aligned} P\{X > 10\} &= 1 - P(x < 10) \\ &= 1 - F(10) \\ &= 1 - (1 - e^{-\lambda \times 10}) \\ &= e^{-10}\lambda = e^{-1} = 0.368 \end{aligned}$$

(b)
$$\begin{aligned} P\{10 < X < 20\} &= F(20) - F(10) \\ &= (1 - e^{-20\lambda}) - (1 - e^{-10\lambda}) \\ &= e^{-1} - e^{-2} = 0.233 \end{aligned}$$

3. Normal Distribution

We say that X is a normal random variable, or simply that X is normally distributed, with parameters μ and σ^2 if the probability density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

The density function is a bell-shaped curve that is symmetric about μ.

For normal distribution:

$$\text{Mean} = E(X) = \mu$$
$$\text{Variance} = V(X) = \sigma^2$$

4. Standard Normal Distribution

Since the for N (μ, σ^2) varies with μ & σ^2 & the integral can only be evaluated numerically, it is more reasonable to reduce this distribution to another distribution called Standard normal distribution N (0, 1) for which, the shape and hence the integral values remain constant.

Since all N (μ, σ^2) problems can be reduced to N(0, 1) problems, we need only to consult a standard table giving calculations of area under N (0, 1) from 0 to any value of z.

The conversion from N (μ, σ^2) to N (0, 1) is effected by the following transformation,

$$Z = \frac{X - \mu}{\sigma}$$

Where Z is called standard normal variate.

For Standard Normal distribution:

$$\text{Mean} = E(X) = 0$$
$$\text{Variance} = V(X) = 1$$

Hence the standard normal distribution is also referred to as the N(0, 1) distribution.

Summary

- Two events E and F are mutually exclusive, if $E \cap F = f$ i.e. $P(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).
- **Axioms of Probability:**

  Axiom-1: $0 \le P(E) \le 1$

  Axiom-2: $P(S) = 1$

  Axiom-3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$(that is, events for which $E_i \cap E_j = \phi$ when $i \ne j$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- **Median for Ungrouped Data:**

$$\text{Median} = \text{the } \frac{(n+1)}{2}\text{-th value}$$

  However, if n is even, we have two middle points

$$\text{Median} = \frac{\left(\frac{n}{2}\right)\text{th value} + \left(\frac{n}{2}+1\right)\text{th value}}{2}$$

- **Median for Grouped Data:**

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] \times h$$

  Where,

  L = Lower limit of median class

  N = Total number of data items = ΣF

  F = Cumulative frequency of the class immediately preceding the median class

  f_m = Frequency of median class

  h = width of median class
- **Standard Deviation** is a measure of dispersion or variation amongst data. The positive square root of the variance is called the **'Standard Deviation'** of the given values.
- The probability of x success from n trials is given by $P(X = x) = nC_x\, p^x\, (1-p)^{n-x}$. Where p is the probability of success in any trial and $(1-p) = q$ is the probability of failure.
- **Uniform Distribution:**

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = E[X] = \frac{\beta+\alpha}{2}$$

$$\text{Variance} = V(X) = \frac{(\beta-\alpha)^2}{12}$$

Scanned with CamScanner

- **Exponential Distribution:**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{If } x \geq 0 \\ 0 & \text{If } x < 0 \end{cases}$$

$$\text{Mean} = E[X] = \frac{1}{\lambda}$$

$$\text{Variance} = v(x) = 1/\lambda^2$$

- **Normal Distribution:**

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

$$\text{Mean} = E(X) = \mu$$

$$\text{Variance} = V(X) = \sigma^2$$

- **Standard Normal distribution:**

$$\text{Mean} = E(X) = 0$$

$$\text{Variance} = V(X) = 1$$

Hence the standard normal distribution is also referred to as the N(0, 1) distribution.

Student's Assignments

Q.1 Let P (E) denotes the probability of the event E. Given P(A) = 1, P(B) = $\frac{1}{2}$, then if A and B are independent, then the values of $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$ respectively are

(a) $\frac{1}{4}, \frac{1}{2}$ (b) $\frac{1}{2}, \frac{1}{4}$

(c) $\frac{1}{2}, 1$ (d) $1, \frac{1}{2}$

Q.2 If P(A) = $\frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$ then $P\left(\frac{B}{A}\right) =$

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{3}{4}$ (d) 0

Q.3 A bag contains 5 black, 2 red, and 3 white marbles. Three marbles are drawn simultaneously. The probability that the drawn marbles are of the different color is

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$

(c) $\frac{5}{6}$ (d) None of these

Q.4 A and B are equally likely and independent events. $p(A \cup B) = 0.1$. Then what is the value of p(A)?

(a) 0.032 (b) 0.046

(c) 0.513 (d) 0.05

Q.5 The probability of occurrence of an event. A is 0.7, the probability of non-occurrence of an event B is 0.45 and the probability of at least one of A and B not occurring is 0.6. The probability that at least one of A and B occurs is

(a) 0.4 (b) 0.6

(c) 1 (d) 0.85

Q.6 The probability density function of a variable

X	0	1	2	3	4	5	6
P(X)	K	3K	5K	7K	9K	11K	13K

Find the value of K

(a) $\frac{1}{49}$ (b) $\frac{1}{50}$

(c) $\frac{1}{51}$ (d) $\frac{1}{52}$

Q.7 $p(3 < x \le 6) = ?$ (From previous question)

(a) $\frac{33}{49}$ (b) $\frac{17}{25}$

(c) $\frac{34}{51}$ (d) $\frac{1}{4}$

Q.8 Given $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and

$P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$, $P(B \cap C) = ?$

(a) $\frac{1}{12}$ (b) $\frac{1}{9}$

(c) $\frac{1}{15}$ (d) $\frac{1}{18}$

Q.9 In a lottery, 2 tickets are drawn at a time out of 6 tickets numbered from 1 to 6. The expected value of the sum of the numbers on the tickets drawn is

(a) 7 (b) 6

(c) 5 (d) 4

Q.10 Two dice are thrown simultaneously. The probability that atleast one of them will have 6 facing up is

(a) $\frac{1}{36}$ (b) $\frac{1}{3}$

(c) $\frac{25}{36}$ (d) $\frac{11}{36}$

Q.11 Let X be a contiuous random variable with following distribution

$$f(x) = kx \quad 0 \le x \le 2$$
$$= 0 \quad \text{elsewhere}$$

The value of k and $P(1 \le x \le 2)$ are respectively,

(a) $\frac{1}{2}, \frac{1}{4}$ (b) $\frac{1}{2}, \frac{3}{4}$

(c) $\frac{1}{4}, \frac{2}{3}$ (d) $\frac{1}{2}, \frac{1}{2}$

Q.12 A gambler has 4 coins in her pocket. Two are double-headed, one is double-tailed, and one is normal. The coins can not be distinguished unless one looks at them. The gambler takes a coin at random, opens her eyes and sees that the upper face of the coin is a head. What is the probability that the lower face is a head?

(a) $\frac{5}{8}$ (b) $\frac{4}{5}$

(c) $\frac{2}{3}$ (d) $\frac{1}{2}$

Q.13 Let X be uniformly distributed on {0, 1, ..., 32} what is $Pr[3x + 12 \equiv 0 (\text{mod } 33)]$?

(a) $\frac{1}{34}$ (b) $\frac{2}{34}$

(c) $\frac{1}{22}$ (d) $\frac{1}{11}$

Q.14 Suppose you are given a bag containing n unbaised coins you are told that $n - 1$ of these are normal coins, with heads on one side and tails on the other; however the remaining coin has heads on both its sides.

Suppose you reach in to the bag, pickout a coin uniformly at random, flip it and get a head. What is the (conditional) probability that this coin you choose is the fake (i.e., double-headed) coin?

(a) $\frac{1}{n+1}$ (b) $\frac{2}{n+1}$

(c) $\frac{1}{n-1}$ (d) $\frac{2}{n-1}$

Q.15 Two random variables X and Y are independent if the pair of events X_i and Y_j are independent no matter how you choose the values i and j. Which of the following most accurately expresses the proposition that X and Y are not independent?

(a) for all i, j Pr[X_i and Y_j] ≠ Pr[X_i] Pr[Y_j]
(b) for all i, some j, Pr[X_i AND Y_j] ≠ Pr [X_i] Pr[Y_j]
(c) for some j, all i, Pr[X_i AND Y_j] ≠ Pr[X_i] Pr[Y_j]
(d) for some i, j, Pr[X_i AND Y_j] ≠ Pr[X_i] Pr[Y_j]

Q.16 Suppose you are given a bag containing n unbiased coins. You are total that $n-1$ of these are normal coins, with heads on one side and tails on the other; however, the remaining coin has heads on both its sides.
Suppose you reach into the bag, pick out a coin uniformly at random.
Suppose you flip the coin k times after picking it (instead of just once) and see k heads. What is now the conditional probability that you picked the fake coin?

(a) $\dfrac{2^k}{(n+1+2^k)}$ (b) $\dfrac{2^k}{(n+2^k)}$

(c) $\dfrac{2^k}{(n-1)+2^k}$ (d) None of these

Q.17 The expectation and variance of a random variable $z = X_1 + X_2$ where X_1 and X_2 are independent random variables with expectation μ and variance σ^2.
(a) μ, σ (b) $\mu, 2\sigma$
(c) $2\mu, \sigma^2$ (d) $2\mu, 2\sigma$

Q.18 For each square of an 8 × 8 checker board, flip a fair coin, and color that square black or red according to whether you get heads or tails. Assume that all coin flips are independent. A same-color row in a row on the board were all squares in the row have the same color (i.e., all red, or all black). Let the random variable X denote the number of same colour rows. What is the Pr(X = 0)

(a) $\dfrac{1}{(2)^7}$ (b) $1-\left(\dfrac{1}{(2^7)}\right)$

(c) $\left(\dfrac{1}{(2)^7}\right)^8$ (d) $\left(1-\left(\dfrac{1}{(2^7)}\right)\right)^8$

Q.19 Suppose you are given a bag containing n unbiased coins and you are told that $n-1$ of these are normal coins, with heads on one side and tails and the other; however, the remaining coin has heads on both its sides.
Suppose you reach in to the bag, pick out a coin uniformly at random.
Suppose you wanted to decide whether the chosen coin was fake by flipping it k times; The decision procedure returns FAKE if all k-flips come up heads, otherwise it returns NORMAL. What is the (unconditional) probability that this procedure makes an error?
(a) $(1/2)^k\,(n)/(n+1)$
(b) $(1/2)^k\,(n-1)/n$
(c) $(1/2)^{k+1}\,(n-1)/n$
(d) None of the above

Q.20 In a multi-user operating system, 20 requests are made to use a particular resource per hour, on an average. The probability that no requests are made in 45 minutes is
(a) e^{-15} (b) e^{-5}
(c) $1-e^{-5}$ (d) $1-e^{-10}$

Common Data Questions (21 and 22):
A random variable x has PDF

$$P(x) = \frac{1}{2}a \text{ for } -a < x < a \text{ and } P(x) = 0, \text{ else}$$

where

Q.21 Find the central moments
(a) All even central moments are zero and odd central moment are $\frac{1}{3}a^2, \frac{1}{5}a^4, \frac{1}{7}a^6$
(b) All odd central moments are zero and even central moments are $\frac{1}{3}a^2, \frac{1}{5}a^4, \frac{1}{7}a^6$
(c) All the odd and even central moments are equal to zero
(d) All the odd and even central moments are not equal to zero.

Q.22 For the above distribution value of is

$$P\left(|x| \geq \frac{\sqrt{3}}{2}a\right)$$

(a) $\geq \frac{9}{4}$ (b) $\leq \frac{9}{5}$

(c) $\leq \frac{4}{9}$ (d) $\geq \frac{5}{9}$

Common Data Questions (23 and 24):

Analysis of the daily registration at an Examination on a certain day indicated that the source of registration from North India are 15%, South India are 35% and from western part of India are 50%. Further suppose that the probabilities that a registration being a free registration from these parts are 0.01, 0.05, and 0.02, respectively.

Q.23 Find the probability that a registration chosen at random is a free registration

(a) 0.603 (b) 0.029

(c) 0.009 (d) None of these

Q.24 Find the probability that a randomly chosen registration comes from south India, given that it is a free registration.

(a) 60% (b) 3%

(c) 17 % (d) None of these

Q.25 A manufacturer produces IC chips, 1% of which are defective. Find the probability that in a box containing 100 chips, no defective are found. Use Poisson distribution approximation to binomial distribution?

(a) 0.366 (b) 0.368

(c) 0.1 (d) None of these

Answer Key:

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (d)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (d)
16. (c)	**17.** (d)	**18.** (d)	**19.** (b)	**20.** (a)
21. (b)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (b)

Student's Assignments Explanations

1. (d)

Since A and B independent events

$p(A|B) = p(A) = 1$ and $p(B|A) = p(B) = \frac{1}{2}$.

2. (b)

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

5. (d)

Given,

$$p(A) = 0.7$$
$$p(\bar{B}) = 0.45$$
$$p(\bar{A} \cup \bar{B}) = 0.6$$
$$p(A \cup B) = ?$$
$$p(B) = 1 - p(\bar{B})$$
$$= 1 - 4.5 = 0.55$$
$$p(A \cap B) = 1 - p(\overline{A \cap B})$$
$$= 1 - p(\bar{A} \cup \bar{B})$$
$$= 1 - 0.6 = 0.4$$

Now, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$$= 0.7 + 0.55 - 0.4 = 0.85$$

$\therefore$ $p(A \cup B) = 0.85$

6. (a)

If X is a randon variable, then

$$\Sigma p(X) = 1$$

$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

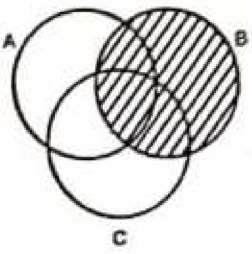
$\Rightarrow$ $k = \frac{1}{49}$

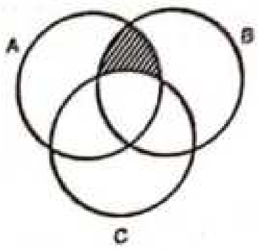
7. (a)

$P(3 < x \leq 6) = 9k + 11k + 13k = 33k$

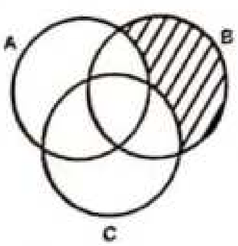
$\therefore$ $P(3 < x \leq 6) = \frac{33}{49}$

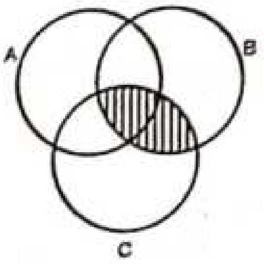
8. (a)





$P(B)\ P(A \cap B \cap \bar{C}) = \frac{1}{3}$





$P(\bar{A} \cap B \cap \bar{C})$ $P(B \cap C)$

From the above Venn diagram

$P(B \cap C)\ P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$

$$= \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{12}$$

9. (a)

Let X be the vandon variable that represents the sum of 2 tickets.

The probability distribution table of X is

X	3	4	5	6	7	8	9	10	11
$p(X)$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$

$$E(X) = \Sigma X p(X)$$

$$= 3 \times \frac{1}{15} + 4 \times \frac{1}{15} + 5 \times \frac{2}{15} + \ldots$$

$$= \frac{105}{15} = 7$$

10. (d)

The possible combinations for at least one dice being 6 is given by 11 ordered pairs below:

(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)

$\therefore$ Probability that at least one dice is 6 = $\frac{11}{36}$

Alternatively we can solve this problem by another method:

p(6 on I dice or 6 on II dice)

$= 1 - p$(not 6 on I dice and not 6 on II dice)

$$= 1 - \frac{5}{6} \times \frac{5}{6} = 1 - \frac{25}{36} = \frac{11}{36}$$

11. (b)

For $f(x)$ to be a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_0^2 kx\,dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^2}{2}\right]_0^2 = 1 \Rightarrow 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2}$$

$$p(1 \le x \le 2) = \int_1^2 f(x)dx$$

$$= \int_1^2 \frac{1}{2}x\,dx = \frac{3}{4}$$

12. (b)

There are 5 faces that are heads out of a total of 8, so the probability is $\frac{5}{8}$. Let A be the event that the upper face is a head, and B be the event that the lower face is heads.

$$\Pr[A] = \Pr[B] = \frac{5}{8}$$

$$\Pr[A \cap B] = \frac{2}{4} = \frac{1}{2}$$

So, $$\Pr[B|A] = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5}$$

13. (d)

$$3X + 12 \equiv 0 \pmod{33}$$

$$\Rightarrow \quad 3X \equiv -12 \pmod{33}$$

$$3X \equiv 21 \pmod{33}$$
$\Rightarrow$ $X \equiv 7 \bmod 11$
$\Rightarrow$ $X = 7 + 11k$
$\Rightarrow$
Since, $X = 0, 1, 2, ..., 32$

Only solutions are 7, 18 and 29

Now,

pr[$3X + 12 \equiv 0 \pmod{33}$]

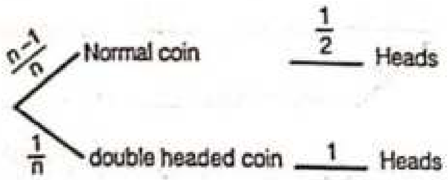
$= pr[(x = 7) \text{ or } (x = 18) \text{ or } (x = 29)]$

$= pr[x = 7] + p[x = 18] + p[x = 29]$

$$= \frac{1}{33} + \frac{1}{33} + \frac{1}{33} = \frac{3}{33} = \frac{1}{11}$$

14. (b)

The tree diagram with probabilities for the given problem is shown below:

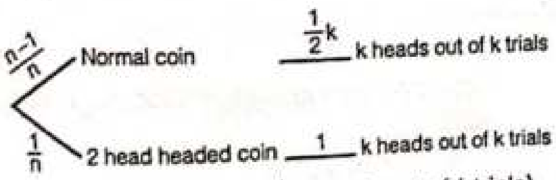


$$p(\text{double head coin} \mid \text{Heads}) = \frac{\frac{1}{n} \times 1}{\frac{n-1}{n} \times \frac{1}{2} + \frac{1}{n} \times 1}$$

$$= \frac{2}{n+1}$$

16. (c)

The tree diagram with probabilities for this problem is shown below:



$\therefore$ p(2 headed coin | k heads out of k trials)

$$= \frac{\frac{1}{n} \times 1}{\frac{n-1}{n} \cdot \frac{1}{2^k} + \frac{1}{n} \times 1} = \frac{2^k}{(n-1) + 2^k}$$

17. (d)

$E(X_1 + X_2) = E(X_1) + E(X_2)$ always

$= \mu + \mu = 2\mu$

$V(aX_1 + bX_2) = a^2 V(X_1) + b^2 V(X_2)$

(If X_1 and X_2 are independent)

Putting, $a = b = 1$

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$
$$= \sigma^2 + \sigma^2 = 2\sigma^2$$

18. (d)

p(a row being same color)

$= p$(a row being all black) + p(a row being all red)

$= p$(all heads) + p(all tails)

$$= 8C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0 + 8C_8\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^8$$

$$= \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8$$

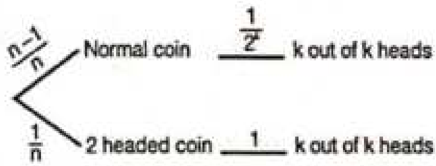
$$= \frac{3}{4}$$

pr(X = 0) = pr(0 out of 8 rows being of same colour)

$$= 8C_0\left(\frac{1}{2^7}\right)^0\left(1-\frac{1}{2^7}\right)^8 = \left(1-\frac{1}{2^7}\right)^8$$

19. (b)

The tree diagram with probabilities is



p(procedure is in error) = p(normal coin and k out of k heads) + p(fake coin and not k out of k heads)

$$= \frac{n-1}{n} \times \left(\frac{1}{2^k}\right) + \frac{1}{n} \times 0$$

$$= \frac{n-1}{n}\left(\frac{1}{2^k}\right) = \left(\frac{1}{2}\right)^k \frac{(n-1)}{n}$$

20. (a)

The arrival pattern follows poisson distribution.

$$p(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Here $\lambda = \alpha \Delta t$

where, α = number of events/unit time = 20/hr

$$\Delta t = 45 \text{ min} = \frac{3}{4} \text{hr}$$

$$\therefore \qquad \lambda = \alpha \Delta t = 20 \times \frac{3}{4} = 15$$

$$p(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} = \frac{15^0}{0!} e^{-15} = e^{-15}$$

21. (b)

$$\mu = \int_{-\infty}^{\infty} x p(x) dx = \int_{-a}^{a} \frac{x}{2a} dx$$

$$\mu = \int_{-\infty}^{\infty} (x-\mu)^r p(x) dx = \int_{-a}^{a} \frac{x^r}{2a} dx$$

$$= \frac{1}{2a}\left\{\frac{x^{r+1}}{r+1}\right\}_{-a}^{a} = \begin{cases} 0 \text{ if } r \text{ is odd} \\ \dfrac{a^r}{r+1} \text{ if } r \text{ is even} \end{cases}$$

$\therefore$ odd central moments are zero, even central moments are

$$\mu_2 = \frac{1}{3}a^2, \mu_2 = \frac{1}{5}a^4, \mu_2 = \frac{1}{7}a^5 \ldots$$

22. (d)

Here, $\mu = 0$ and $\sigma = \frac{a}{\sqrt{3}}$

Using Bienayme-Chebyshev rule,

$$p(\mu - k\sigma \le x \le \mu + k\sigma) \ge 1 - \frac{1}{k^2}$$

$$p(-k\sigma \le x \le +k\sigma) \ge 1 - \frac{1}{k^2}$$

$$k\sigma = \frac{\sqrt{3}}{2}a$$

$$\Rightarrow \qquad k \cdot \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{2}a$$

$$\Rightarrow \qquad k = \frac{3}{2}$$

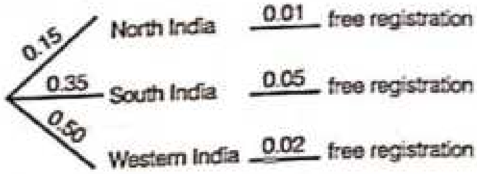
$$\therefore \; p\left(\frac{-\sqrt{3}}{2}a \le x \le \frac{\sqrt{3}}{2}a\right) \ge 1 - \frac{1}{\left(\frac{3}{2}\right)^2}$$

$$p\left(|x| \ge \frac{\sqrt{3}}{2}a\right) \ge 1 - \left(\frac{2}{3}\right)^3$$

$$p\left(|x| \ge \frac{\sqrt{3}}{2}a\right) \ge \frac{5}{9}$$

23. (b)

The tree diagram is shown below:



$p(\text{free registration}) = 0.15 \times 0.01 + 0.35 \times 0.05 + 0.50 \times 0.02 = 0.029$

24. (a)

$p(\text{South India I free registration})$

$$= \frac{p(\text{South India and free registration})}{p(\text{free registration})}$$

$$= \frac{0.35 \times 0.05}{0.029} = 0.6034 \simeq 60\%$$

25. (b)

$p = 0.01$, $n = 100$

Using poisson approximation to binomial distribution,

$\lambda = np = 100 \times 0.01 = 1$

$$p(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-1} = 0.368$$

CHAPTER 06

Linear Algebra

6.1 Introduction

Definition of Matrix

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m \times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

6.2 Special Types of Matrices

1. **Square Matrix:** An m × n matrix for which $m = n$ (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. The elements a_{11}, a_{22}.... are called **DIAGONAL ELEMENTS** and the line along which they lie is called **PRINCIPLE DIAGONAL** of matrix.

   Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3}$ is an square Matrix

NOTE: A square sub-matrix of a square matrix A is called a "**principle sub-matrix**" if its diagonal elements are also the diagonal elements of the matrix A.

2. **Diagonal Matrix:** A square matrix in which all of non-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero.

Example: $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ is a diagonal matrix

The above matrix can also be written as A = diag [3, 5, 9]

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal.

Example: $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix.

5. **Null Matrix:** The m × n matrix whose elements are all zero is called null matrix.

Null matrix is denoted by O.

Example: $O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

6. **Upper Triangular Matrix:** A upper triangular matrix is a square matrix whose lower off diagonal elements are zero, i.e. $q_{ij} = 0$ whenever $i < j$

It is denoted by U.

Example: $U = \begin{pmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{pmatrix}$

7. **Lower Triangular Matrix:** A lower triangular matrix is a square matrix whose triangular elements are zero, ilel $q_{ij} = 0$ whenever $i > j$. It is denoted by L,

Example: $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{pmatrix}$

6.3 Algebra of Matrices

Equality of Two Matrices

Two matrices $A= [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if,

(a) They are of same size.

(b) The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Addition of Matrices

Let A and B be two matrices of the same type $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$; $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$

Scalar Properties of Matrix Addition:

(a) Matrix addition is commutative $A + B = B + A$.
(b) Matrix addition is associative $(A + B) + C = A + (B + C)$
(c) Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every $m \times n$ matrix A.
(d) Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.
Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.
$\Rightarrow$ Matrix $-A$ is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order $m \times n$.
(e) Cancellation laws holds good in case of addition of matrices.
$A + X = B + X \Rightarrow A = B$
$X + A = X + B \Rightarrow A = B$
(f) The equation $A + X = 0$ has a unique solution in the set of all $m \times n$ matrices.

Substraction of Two Matrices

If A and B are two m × n matrices, then we define, $A - B = A + (-B)$.

Thus the difference $A - B$ is obtained by substracting from each element of A corresponding elements of B.

NOTE: Subtraction of matrices is neither commutative nor associative.

Multiplication of a Matrix by a Scalar

Let A be any $m \times n$ matrix and k be any real number called scalar. The $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

$\Rightarrow$ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

If $A = \begin{pmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{pmatrix}$ then, $3A = \begin{pmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{pmatrix}$

Properties of Multiplication of a Matrix by a Scalar:

(a) Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.
(b) If p and q are two scalars and A is any $m \times n$ matrix then, $(p + q)A = pA + qA$.
(c) If p and q are two matrices and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.
(d) If $A = [a_{ij}]_{m \times n}$ matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.

Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is called the product of matrices A and B in that order and we write C = AB.

Properties of Matrix Multiplication:

(a) Multiplication of matrices is not commutative. In fact if the product of AB exists, then it is not necessary that the product of BA will also exist.

(b) Matrix multiplication is associative if conformability is assured. i.e., A (BC) = (AB)C where A, B, C are $m \times n$, $n \times p$, $p \times q$ matrices respectively.

(c) Multiplication of matrices is distributive with respect to addition of matrices. i.e., A (B+C) = AB + AC.

(d) The equation AB = O does not necessarily imply that at least one of matrices A and B must be a zero matrix.

(e) In the case of matrix multiplication if AB = O then it is not necessarily imply that BA = O.

6.4 Properties of Matrices

Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by tr A.

Thus if $A = [a_{ij}]_{n \times n}$ then, $\text{Tr}(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \dots a_{nn}$.

Let, $$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{pmatrix}$$

Then, trace (A) = tr(A) = 1 + (–3) + 5 = 3

Properties: Let A and B be two square matrices of order n and λ be a scalar. Then,

(a) tr (λA) = λ tr A

(b) tr (A + B) = tr A + tr B

(c) tr (AB) = tr (BA)

Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

Let, $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{pmatrix}$ then, $A^T = A = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{pmatrix}$

Properties: If A' and B' be transposes of A and B respectively then,

(a) $(A')' = A$

(b) $(A + B)' = A' + B'$

(c) $(kA)' = kA'$, k being any complex number

(d) $(AB)' = B'A'$

Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$; $\bar{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$

If $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively. Then,

(a) $\overline{(\bar{A})} = A$

(b) $\overline{(A+B)} = \bar{A}+\bar{B}$

(c) $\overline{(kA)} = \bar{k}\bar{A}$, k being any complex number

(d) $\overline{(AB)} = \bar{A}\bar{B}$, A and B being conformable to multiplication.

Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ or A^* or $(\bar{A})^T$. It is also called conjugate transpose of A.

Some properties: If A^θ and B^θ be the transposed conjugates of A and B respectively then,

(a) $(A^\theta)^\theta = A$

(b) $(A + B)^\theta = A^\theta + B^\theta$

(c) $(kA)^\theta = \bar{k}A^\theta$, $k \to$ complex number

(d) $(AB)^\theta = B^\theta A^\theta$

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i and j.

In a symmetric matrix, $A^T = A$

Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

Skew Symmetric Matrix:

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}\ \forall\ i, j$.

In a skew symmetric matrix $A^T = -A$.

A skew symmetric matrix must have all 0's in the diagonal.

Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Orthogonal Matrix

A square matrix A is said be orthogonal if:

$A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if

$AA^T = I = A^TA$.

Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i, j)^{th}$ element of A is equal to conjugate complex of $(j, i)^{th}$ element of A. i.e., if $a_{ij} = \bar{a}_{ji}\ \forall\ i\ \&\ j$

Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$

A necessary and sufficient condition for a matrix A to be Hermitian is that $A^{\theta} = A$.

Skew Hermition Matrix

A square matrix $A = [a_{ij}]$ is said to be skew Hermitian if $(i, j)^{th}$ element of A is equal to negative of conjugate complex of $(j, i)^{th}$ element of A i.e.,

if $a_{ij} = a_{ij} = -\bar{a}_{ji}\ \forall\ i.j.$

Example: $A = \begin{bmatrix} 0 & -2+i \\ 2+i & 0 \end{bmatrix}_{2\times 2}$

A necessary and sufficient condition for a matrix to be skew Hermitian if $A^{\theta} = -A$.

Unitary Matrix

A square matrix A is said to be unitary if:

$A^{\theta} = A^{-1} \Rightarrow AA^{\theta} = AA^{-1} = I$

Thus A will be unitary matrix if

$AA^{\theta} = I = A^{\theta} A$

6.5 Determinants

Definition: Let $a_{11}, a_{12}, a_{21}, a_{22}$ be any four number. The symbol $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}a_{22} - a_{12}a_{21}$ and is called determinants of order 2. The number $a_{11}, a_{12}, a_{21}, a_{22}$ are called elements of the determinant and the number $a_{11}a_{22} - a_{21}a_{12}$ is called the value of determinant.

6.5.1 Minors and Cofactors

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor

Leaving the row and column passing through the elements a_{ij}, then the second order determinant thus obtained is called the minor of element a_{ij} and we will be denoted by M_{ij}.

Example: The Minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

Similarly Minor of element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$

Cofactors

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij}. We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$.

$$\text{Cofactor of element } a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{by cofactor of element } a_{32} = A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Therefore, "in a determinant the sum of the products of the elements of any row or column with corresponding cofactors is equal to value of the determinant.

6.5.2 Determinant of Order n

A determinant of order n has n-row and n-columns. It has $n \times n$ elements.

A determinant of order n is a square array of $n \times n$ quantities enclosed between vertical bars.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & \cdots\cdots\cdots & a_{2n} \\ a_{n1} & a_{n2} \cdots\cdots & a_{nn} \end{vmatrix}$$

Cofactor of A_{ij} of elements a_{ij} in D is equal to $(-1)^{i+j}$ times the determinants of order $(n-1)$ obtained from D by leaving the row and column passing through element a_{ij}.

Example - 6.1 Compute the determinant of each of the following matrices and determine which is singular.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & 5 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{bmatrix}$

Solution:

(a) The determinant is simply the product of the diagonal running left ro right minus the product of the diagonal running from right to left. So, here is the determinant for this matrix. The only thing we need to worry about is paying attention to minus signs. It is easy to make a mistake with minus signs in these computations if you are not paying attention.

$$\det(A) = (3)(5) - (2)(-9) = 33$$

(b)
$$\det(B) = \begin{vmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{vmatrix} = 3\times\begin{vmatrix} -1 & 8 \\ 1 & 7 \end{vmatrix} - 5\times\begin{vmatrix} -2 & 8 \\ -11 & 7 \end{vmatrix} + 4\times\begin{vmatrix} -2 & -1 \\ -11 & 1 \end{vmatrix} = -467$$

(c)
$$\det(C) = \begin{vmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{vmatrix} = 2\times\begin{vmatrix} -8 & 3 \\ 1 & 1 \end{vmatrix} + 6\times\begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} + 2\times\begin{vmatrix} 2 & -8 \\ -3 & 1 \end{vmatrix} = 0$$

Since only det(C) is zero, only C is singular.

Example-6.2 For the following matrix compute the co-factors C_{12}, C_{24} and C_{32}.

$$A = \begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix}$$

Solution:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

In order to compute the co-factors we will first need the minor associated with each co-factor. Remember that in order to compute the minor we will remove the i^{th} row and j^{th} column of A. So, to compute M_{12} (which we will need for C_{12}) we will need to compute the determinate of the matrix we get by removing the 1st row and 2nd column of A. Here is that work.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{12} = \begin{vmatrix} -1 & 3 & 9 \\ 5 & -1 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 160$$

We have marked out the row and column that we eliminated. Now we can get the co-factor,

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (160) = -160$$

Let's now move onto the second co-factor,

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{24} = \begin{vmatrix} 4 & 0 & 10 \\ 5 & -5 & -1 \\ 3 & 7 & 1 \end{vmatrix} = 508$$

The co-factor in this case is, $C_{24} = (-1)^{2+4} M_{24} = (-1)^6 (508) = 508$

For the final co-factor,

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{32} = \begin{vmatrix} 4 & 10 & 4 \\ -1 & 3 & 9 \\ 3 & 1 & -2 \end{vmatrix} = 150$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (150) = -150$$

6.5.3 Properties of Determinants

1. The value of a determinant does not change when rows and columns are interchanged. i.e. $|A^T| = |A|$
2. If any row (or column) of a matrix A is completely zero, then $|A| = 0$.
   Such a row (or column) is called a zero row (or column).
   Also if any two rows (or columns) of a matrix A are indential, then $|A| = 0$.
3. If any two rows or two columns of a determinant are interchanged the value of determinant is multiplied by -1.
4. If all elements of the one row (or one column) of a determinant are multiplied by same number k the value of determinant is k times the value of given determinant.
5. If A be n-rowed square matrix, and k be any scalar, then $|kA| = k^n|A|$

6. (a) In a determinant the sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of any row or column is equal to the determinant value.
   (b) In determinant the sum of the products of the elements of any row (or column) with the cofactors of some other row or column is zero.

Example: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then,
$$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$
$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$
$$a_1A_3 + b_1B_3 + c_1C_3 = 0$$
$$a_2A_2 + b_2B_2 + c_2C_2 = \Delta$$
$$a_2A_1 + b_2B_1 + c_2C_1 = 0 \text{ etc}$$

where A_1, B_1, C_1 etc., be cofactors of the elements a_1, b_1, c_1 in D.

7. If to the elements of a row (or column) of a determinant are added m times the corresponding elements of another row (or column) the value of determinant thus obtained is equal to the value of original determinant.
8. $|AB| = |A| \cdot |B|$ and $|A^n| = (|A|)^n$
9. Let A be a square matrix of order n then,
   (a) $|\bar{A}| = |\bar{A}|$
   (b) $|A^\theta| = |\bar{A}|$

6.6 Inverse of Matrix

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be any $n \times n$ matrix. The transpose B of the matrix $B = [A_{ij}]_{n \times n}$ where A_{ij} denotes the cofactor of element a_{ij} in the determinant $|A|$ is called the adjoint of matrix A and is denoted by symbol Adj A.

Results:

1. If A be any n-rowed square matrix, then (Adj A) A = A (Adj A) = $|A| I_n$.
2. Every invertible matrix possesses a unique inverse.
3. The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

Non-Singular Matrix

A square matrix is said to be non-singular or singular as $|A| \neq 0$ or $|A| = 0$.

NOTE: If A and B be two n-rowed non-singular matrices, then AB is also non-singular then, $(AB)^{-1} = B^{-1}A^{-1}$ i.e., the inverse of a product is product of the inverse taken in the reverse order.

Results:

1. If A be an $n \times n$ non-singular matrix, then $(A')^{-1} = (A^{-1})'$.
2. If A be an $n \times n$ non-singular matrix then $(A^{-1})^\theta = (A^\theta)^{-1}$.
3. Formula to determine the inverse of matrix A is $A^{-1} = \frac{1}{|A|} \text{Adj A}$.

Rank of Matrix

Some Important Concepts:

1. **Submatrix of a Matrix:** Suppose A is any matrix of the type $m \times n$. Then a matrix obtained by leaving some rows and some columns from A is called sub-matrix of A.
2. **Rank of a Matrix:** A number r is said to be the rank of a matrix A if it possesses the following properties:
   (a) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.
   (b) If the matrix A contains any square sub-matrix of order $(r + 1)$, then the determinant of every square sub-matrix of A of order $(r + 1)$ should zero.

Important Points:

(a) The rank of a matrix is $\leq r$, if all $(r + 1)$ - rowed minors of the matrix vanish.
(b) The rank of a matrix is $\geq r$, if there is at least one r-rowed minor of the matrix which is not equal to zero.

NOTE: The rank of transpose of a matrix is same as that of original matrix. i.e. $r(A^T) = r(A)$

Elementry Matrices

A matrix obtained from unit matrix by a single elementry transformation is called an elementry matrix.

Results:

1. Elementry transformations do not change the rank of a matrix.
2. The rank of a product of two matrices cannot exceed the rank of either matrix. i.e. $r(AB) \leq r(A)$ and $r(AB) \leq r(B)$.
3. Rank of sum of two matrices cannot exceed the sum of their ranks $r(A + B) \leq r(A) + r(B)$.
4. If A, B are two n-rowed square matrices then Rank (AB) $\geq$ (Rank A) + (Rank B) $- n$.

6.7 System of Linear Equations

6.7.1 Homogenous Linear Equations

Suppose,

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = 0 \end{array}\right\} \quad \dots(6.1)$$

is a system of m homogenous equations in n unknowns $x_1, x_2, \dots x_n$.

Let,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix}_{m \times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

where A, X, O are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively. Then obviously we can write the system of equations in the form of a single matrix equation A X = O ...(6.2)

The matrix A is called coefficient matrix of the system of equation (1).

The set $S = \{x_1 = 0, x_2 = 0, \dots x_n = 0\}$ i.e., X = 0 is a solution of equation (1).

Again suppose X_1 and X_2 are two solutions of (2). Then their linear combination, $R_1X_1 + R_2X_2$ when R_1 and R_2 are any arbitrary numbers, is also solution of (2).

Important Results:

The number of linearly independent solutions of m homogenous linear equations in n variables, $AX = O$, is $(n - r)$, where r is rank of matrix A.

Some Important results regarding nature of solutions of equation $AX = O$:

Suppose there are m equations in n unknowns. Then the coefficient matrix A will be of the type $m \times n$. Let r be rank of matrix A. Obviously r cannot greater than n. Therefore we have either $r = n$ or $r < n$.

Case-1: If $r = n$; the equation $AX = O$ will have $n - n$ i.e., no linearly independent solution.

Case-2: If $r < n$ we shall have $n - r$ linearly independent solutions. Any linear combination of these $(n - r)$ solutions will also be a solution of $AX = O$. Thus in this case the equation $AX = O$ will have infinite solutions.

NOTE: That $r < n \Rightarrow |A| = 0$ i.e. A is a singular matrix.

Case-3: Suppose $m < n$ i.e., the number of equations is less than the number of unknowns. Since $r \le m$ therefore r is less than n. Hence in this case the given system of equation must possess a non zero solution.

$\Rightarrow$ In this case the number of solutions of the equation $AX = O$ will be infinite.

6.7.2 System of Linear Non-Homogeneous Equations

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m \end{array}\right\} \quad \dots(6.3)$$

be a system of m non-homogenous equations in n unknown, $x_1, x_2 \dots x_n$

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} \dots\dots a_{1n} \\ a_{21} & \dots\dots\dots a_{2n} \\ \dots\dots & \dots\dots\dots\dots \\ a_{m1} & a_{m2} \dots\dots a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m \times 1}$$

where A, X, B are $m \times n$, $n \times 1$, and $m \times 1$ matrices respectively. The above equations can be written in the form of a single matrix equation $AX = B$.

"Any set of values of $x_1, x_2 \dots\dots x_n$ which simultaneously satisfy all these equation is called a solutions of the system. When the system of equations has one or more solutions, the equation are said to be consistent otherwise they are said to be inconsistent".

$$\text{The matrix } [A \;\; B] = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} & b_1 \\ a_{21} & a_{22} \dots a_{2n} & b_2 \\ \dots\dots & \dots\dots\dots & \\ a_{m1} & a_{m2} \dots a_{mn} & b_m \end{bmatrix}$$

is called augmented matrix of the given system of equations.

Condition for Consistency

The system of equations AX = B is consistent i.e., possess a solution iff the coefficient matrix A and the augmented matrix [A B] are of the same rank, i.e. ρ (A) = ρ (A, B).

If $\rho(a) \neq \rho(A, B)$ the system A x = B, has no solution. We say that such a system is inconsistent.

Now, when $\rho(A) = \rho(A|B) = r$.

We say, that the rank of the system is r. Now two cases arise.

Case-1: If $\rho(A) = \rho(A|B) = r = n$ (where n is the number of unknown variables of the system), then the system is not only consistent but also has a unique solution.

Case-2: If $\rho(A) = \rho(A|B) = r < n$, then the system is consistent, but has infinite number of solutions.

In summary we can say the following:

1. If $\rho(A) = \rho(A|B) = r = n$ (consistent and unique solution)
2. If $\rho(A) = \rho(A|B) = r < n$ (consistent and infinite solution)
3. If $\rho(A) \neq \rho(A|B)$ (Inconsistent and hence, no solution)

Result:

If A be an n-rowed non-singular matrix, X be an $n \times 1$ matrix, B be an $n \times 1$ matrix, the system of equations A X = B has a unique solution. i.e. if $|A| \neq 0$, then the system AX = B has a unique solution.

The rank of a system of equations as well as its solution (if it exists) can be obtained by a procedure called Gauss-Elimination method.

6.8 Solution of System of Linear Equation by LU Decomposition Method (Factorisation or Triangularisation Method)

This method is based on the fact that a square matrix A can be factorised into the form LU where L is unit lower triangular and U is a upper triangular, if all the principal minors of A are non singular i.e., it is a standard result of linear algebra that such a factorisation, when it exists, is unique.

We consider, for definiteness, the linear system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the form

$$AX = B \qquad \text{...(6.1)}$$

Let, $$A = LU \qquad \text{...(6.2)}$$

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad \text{...(6.3)}$$

and $$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \qquad \text{...(6.4)}$$

Equation (6.1) becomes, LUX = B ...(6.5)

If we set UX = Y ...(6.6)

then (6.5) way be written as LY = B ...(6.7) which is equivalent to the system $y_1 = b_1$

$$\ell_{21}y_1 + y_2 = b_2,\ \ell_{31}y_1 + \ell_{32}y_2 + y_3 = b_3$$

and can be solved for y_1, y_2, y_3 by the forward substitution. When Y is known, the system (6.6) become

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

which can be solved by backward substitution we shall now describe a scheme for computing the matrices L and U, and illustrate the procedure with a matrix of order 3. From the relation (2), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the watrices on the left and equating the corresponding claments of both sides we get

$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$

$$\ell_{21}u_{11} = a_{21} \text{ or } \ell_{21} = \frac{a_{21}}{u_{11}}$$
$$\ell_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - \ell_{21}u_{12}$$
$$\ell_{31}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - \ell_{21}u_{13}$$
$$\ell_{31}u_{11} = a_{31} \Rightarrow \ell_{31} = \frac{a_{31}}{u_{11}}$$
$$\ell_{31}u_{12} = \ell_{32}u_{22} = a_{32} \Rightarrow \ell_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

Lastly, $\ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} = a_{33}$

$\Rightarrow$
$$u_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23}$$

$\therefore$ the variables are solved in the following

order u_{11}, u_{12}, u_{13}

then $\ell_{21}, u_{22}, u_{23}$

lastly, $\ell_{31}, \ell_{32}, u_{33}$

Example-6.3 Solve the equations by the factorisation method.

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Clearly, $u_{11} = 2,\ u_{12} = 3,\ u_{13} = 1$

Also, $\ell_{21}u_{11} = 1$, so that $\ell_{21} = \frac{1}{2}$

$$\ell_{21}u_{12} + u_{22} = 2$$

$\Rightarrow$
$$u_{22} = 2 - \ell_{21}u_{12} = \frac{1}{2}$$

$$\ell_{21}u_{13} + u_{23} = 3$$

from which we obtain $u_{23} = \frac{5}{2}$

$$\ell_{31}u_{11} = 3 \Rightarrow \ell_{31} = \frac{3}{2}$$

$$\ell_{31}u_{12} + \ell_{32}u_{22} = 1 \Rightarrow \ell_{32} = -7$$

$$\ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 18$$

It follows that.
$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

and hence the given system of equations can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

or as
$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

solving this system by forward substitution, we get

$$y_1 = 9, \ \frac{y_1}{2} + y_2 = 6 \Rightarrow y_2 = \frac{3}{2}$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \text{ or } y_3 = 5$$

Hence the solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

which when solved by back substitution process.

$$x = \frac{35}{18}; \ y = \frac{29}{18}; \ z = \frac{5}{18}$$

6.9 Eigenvalues and Eigenvectors

Definitions: Let $A = [a_{ij}]_{n \times n}$ be any n-rowed square matrix and λ is a scalar. The matrix $A - \lambda I$ is called characteristic matrix of A, where I is the unit matrix of order n. Also the determinant

$$|A - \lambda I| = \begin{bmatrix} a_{11-\lambda} & a_{12} \dots\dots\dots a_{1n} \\ a_{21} & a_{22-\lambda} \dots\dots\dots a_{2n} \\ \dots\dots & \dots\dots\dots\dots\dots \\ a_{n1} & a_{n2} \dots\dots\dots a_{nn-\lambda} \end{bmatrix}$$

which is ordinary polynomial in λ of degree n is called "characteristic polynomial of A". The equation $|A - \lambda I| = 0$ is called **"characteristic equation of A"** and the roots of this equation is called **"characteristic roots or characteristic values or latent roots or proper values"** of the matrix A. The set of eigenvalues of A is called the **"spectrum of A"**.

If I is a characteristic root of the matrix A, then $|A - \lambda I| = 0$ and the matrix $A - \lambda I$ is singular. Therefore there exist a non-zero vector X such that $(A - \lambda I)X = 0$ or $AX = \lambda X$

Characteristic Vectors

If I is a characteristic root of an $n \times n$ matrix A, then a non-zero vector X such that $AX = \lambda X$ is called characteristic vector or eigenvector of A corresponding to characteristic root λ.

Some Results:

1. λ is a characteristic root of a matrix A iff there exist a non-zero vector X such that $AX = \lambda X$.
2. If X is a characteristic vector of matrix A corresponding to characteristic value λ, then kX is also a characteristic vector of A corresponding to the same characteristic value λ where k is non-zero vector.
3. If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.
4. The characteristic roots of a hermitian matrix are real.
5. The characteristic roots of a real symmetric matrix are all real.
6. Characteristic roots of a skew Hermitian matrix are either pure imaginary or zero.
7. The characteristic roots of a real skew symmetric matrix are either pure imaginary or zero, for every such matrix is skew Hermitian.
8. The characteristic roots of a unitary matrix are of unit modulus. i.e., $|\lambda| = 1$.
9. **Theorem:** The maximum value of $x^T Ax$ where the maximum is taken over all x that are the unit eigen-vectors of A is the maximum eigen value of A.
10. The value of the dot product of the eigenvectors corresponding to any pair of different eigen values of any symmetric positive definite matrix is 0.

6.9.1 Process of Finding the Eigenvalues and Eigenvectors of a Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n first we should write the characteristic equation of the matrix A. i.e., the equation $|A - \lambda I| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will be the n eigenvalues of the matrix A.

If I_1 is an eigenvalue of A, the corresponding eigenvectors of A will be given by the non-zero vectors $X = [x_1, x_2, \dots\dots\dots\dots x_n]'$ satisfying the equations $AX = \lambda_1 X$ or $(A - \lambda_1 X)X = 0$.

NOTE

- If $l_1, l_2 \ldots l_n$ are the eigenvalues of A, then $kl_1, kl_2 \ldots kl_n$ are eigenvalues of kA.
- the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

  i.e. if $\lambda_1, \lambda_2, \ldots \lambda_n$ are two eigenvalue of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen value of A^{-1}.
- If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ are the eigen values of A^k.
- Eigen values of A = Eigen values of A^T.
- Number of eigen values = size of A.
- Sum of eigen values = Trace of A = Sum of diagonal elements.
- Product of eigen values = $|A|$.
- If a is a characteristic root of a non-singular matrix A, the $\frac{|A|}{\alpha}$ is characteristic root of AdjA.

Example-6.4 Find all the eigenvalues the for the given matrices.

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ (b) $A = \begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}$ (c) $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Solution:

(a) We will do this one with a little more detail than we will do the other two. First we will need the matrix $\lambda I - A$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} \lambda-6 & -16 \\ 1 & \lambda+4 \end{bmatrix}$$

Next we need the determinant of this matrix, which gives us the characteristic polynomial.

$$\det(\lambda I - A) = (\lambda-6)(\lambda+4) - (-16) = \lambda^2 - 2\lambda - 8$$

Now, set this equal to zero and solve for the eigenvalues.

$$\lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 4$$

So, we have to eigenvalues and since they occur only once in the list they are both simple eigenvalues.

(b) Here is the matrix $\lambda I - A$ and its characteristic polynomial.

$$\lambda I - A = \begin{bmatrix} \lambda+4 & -2 \\ -3 & \lambda+5 \end{bmatrix} \quad \det(\lambda I - A) = \lambda^2 + 9\lambda + 14$$

We will leave it to you to verify both of these. Now, set the characteristic polynomial equal to zero and solve for the eigenvalues.

$$\lambda^2 + 9\lambda + 14 = (\lambda+7)(\lambda+2) = 0 \Rightarrow \lambda_1 = -7, \lambda_2 = -2$$

Again, we get two simple eigenvalues.

(c) Here is the matrix $\lambda I - A$ and its characteristic polynomial.

$$\lambda I - A = \begin{bmatrix} \lambda-7 & 1 \\ -4 & \lambda-3 \end{bmatrix} \quad \det(\lambda I - A) = \lambda^2 - 10\lambda + 25$$

Now, set the characteristic polynomial equal to zero and solve for the eigenvalues,

$$\lambda^2 - 10\lambda + 25 = (\lambda-5)^2 = 0 \Rightarrow \lambda_{1,2} = 5$$

In this case we have an eigenvalue of multiplicity two. Sometimes we call this kind of eigenvalue a double eigenvalue. Notice as well that we used the notation $\lambda_{1,2}$ to denote the fact that this was a double eigenvalue.

Example - 6.5 Find all the eigenvalues for the given matrices.

(a) $A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (d) $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution:

(a) As with the previous example we will do this one in a little more detail than the remaining two parts. First, we will need $\lambda_{1,2}$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda-4 & 0 & -1 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{bmatrix}$$

Now, let's take the determinant of this matrix and get the characteristic polynomial for A.

$$\det(\lambda I - A) = \left| \begin{matrix} \lambda-4 & 0 & -1 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{matrix} \right| \begin{matrix} \lambda-4 & 0 \\ 1 & \lambda+6 \\ -5 & 0 \end{matrix}$$

$$= \lambda(\lambda-4)(\lambda+6) - 5(\lambda+6) = \lambda^3 + 2\lambda^2 - 29\lambda - 30$$

Next, set this equal to zero.

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = 0$$

Suppose we are trying to find the roots of an equation of the form,

$$\lambda_n + c_{n-1}\lambda_{n-1} + \ldots + c_1\lambda + c_0 = 0$$

where the c_1 are integer solutions to this (and there may NOT be) then we know that they may be divisors of c_0. This won't give us any integer solutions, but it will allow us to write down a list of possible integer solution. The list will be all possible divisors of c_0.

In this case the list of possible integer solutions is all possible divisors of –30.

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$$

Start with the smaller possible solutions and plug them in until you find one (i.e. until the polynomial is zero for one of them) and then stop. In this case the smallest one in the list that works is –1. This means that,

$$\lambda - (-1) = \lambda + 1$$

must be a factor in the characteristic polynomial. In other words, we can write the characteristic polynomial as,

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + I)\, q(\lambda)$$

where $q(\lambda)$ is a quadratic polynomial. We find $q(\lambda)$ by performing long division on the characteristic polynomial. Doing this in this case gives,

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + 1)(\lambda^2 + \lambda - 30)$$

At this point all we need to do is find the solutions to the quadratic and nicely enough for us that factors in this case. So, putting all this together gives,

$$(\lambda + 1)(\lambda + 6)(\lambda - 5) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -6, \lambda_3 = 5$$

So, this matrix has three simple eigenvalues.

(b) Here $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - \lambda^2 - 16\lambda - 20$$

Now, in this case the list of possible integer solutions to the characteristic polynomial are,

$$\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$$

Again, if we start with the smallest integers in the list we will find that -2 is the first integer solution. Therefore, $\lambda - (-2) = \lambda + 2$ must be a factor of the characteristic polynomial. Factoring this out of the characteristic polynomial gives,

$$\lambda^3 - \lambda^2 - 16\lambda - 20 = (\lambda + 2)(\lambda^2 - 3\lambda - 10)$$

Finally, factoring the quadratic and setting equal to zero gives us,

$$(\lambda + 2)^2 (\lambda - 5) = 0 \Rightarrow \lambda_{1,2} = -2, \lambda_3 = 5$$

So, we have one double eigenvalue ($\lambda_{1,2} = -2$) and one simple eigenvalue ($\lambda_3 = 5$).

(c) Here is $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - 3\lambda - 2$$

We have a very small list of possible integer solutions for this characteristic polynomial.

$$\pm 1, \pm 2$$

The smallest integer that works in this case is -1 and the complete factored form is characteristic polynomial is, $\lambda^3 - 3\lambda - 2 = (\lambda + 1)^2 (\lambda - 2)$

and so we can see that we have got two eigenvalues $\lambda_{1,2} = -1$ (a multiplicity 2 eigenvalue) and $\lambda_3 = 2$ (a simple eigenvalue)

(d) Here is $\lambda I - A$ and the characteristic polynomial for this matrix,

$$\lambda I - A = \begin{bmatrix} \lambda-4 & 0 & 1 \\ 0 & \lambda-3 & 0 \\ -1 & 0 & \lambda-2 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - 9\lambda^2 + 27\lambda - 27$$

In this case the list of possible integer solutions is, $\pm 1, \pm 3, \pm 9, \pm 27$

The smallest integer that will work in this case is 3. The factored form of the characteristic polynomial is,

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = (\lambda - 3)^3$$

and so we can see that if we set this equal to zero and solve we will have on eigenvalue of muliplicity 3 (sometimes called a triple eigenvalue),

$$\lambda_{1,2,3} = 3$$

Example-6.6 For each of the following matrices determine the eigenvectors corresponding to each eigenvalue and determine a basis of the eigenspace of the matrix corresponding to each eigenvalue.

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Solution:

We determined the eigenvalues for each of these in example above so refer to that example for the details in finding them. For each eigenvalue we will need to solve the system,

$$(\lambda I - A)x = 0$$

to determine the general form of the eigenvector. Once we have that we can use the general form of the eigenvector to find a basis for the eigenspace.

(a) We know that the eigenvalues of this matrix are $\lambda_1 = -2$ and $\lambda_2 = 4$.

Let's first find the eigenvector(s) and eigenspace for $\lambda_1 = -2$. Referring to example 2 for the formula for $\lambda I - A$ and plugging $\lambda_1 = -2$ into this we can see that the system we need to solve is,

$$\begin{bmatrix} -8 & -16 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We will leave it to you to verify that the solution to this system is,

$$x_1 = -2t \qquad x_2 = t$$

Therefore, the general eigenvector corresponding to $\lambda_1 = -2$ is of the form,

$$x = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ t \end{bmatrix}$$

The eigenspaces is all vectors of this form and so we can see that a basis for the eigenspace corresponding to $\lambda_1 = -2$ is,

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Now, let's find the eigenvector(s) and eigenspace for $\lambda_2 = 4$. Plugging $\lambda_2 = 4$ into the formula for $\lambda I - A$ from previous example gives the following system we need to solve,

$$\begin{bmatrix} -2 & -16 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution to this system is (you should verify this),

$$x_1 = -8t \qquad x_2 = t$$

The general eigenvector and a basis for the eivenspace corresponding to $\lambda_2 = 4$ is then,

$$x = \begin{bmatrix} -8t \\ t \end{bmatrix} = t\begin{bmatrix} -8 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

Note that if we wanted our hands on specific eigenvalues for each eigenvector the basis vector for each eigenspace would work. So, if we do that we could use the following eigenvectors (and their corresponding eigenvalues) if we'd like,

$$\lambda_1 = -2 \quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = 4 \quad v_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

Note as well that these eigenvectors are linearly independent vectors.

(b) From previous example we know that $\lambda_{1,2} = 5$ is a double eigenvalue and so there will be a single eigenspace to compute for this matrix. Using the formula for $\lambda I - A$ from example 2 and plugging $\lambda_{1,2}$ into this gives the following system that we will need to solve for the eigenvector and eigenspace.

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution to this system is,

$$x_1 = \frac{1}{2}t, \quad x_2 = t$$

The general eigenvector and a basis for the eigenspace corresponding $\lambda_{1,2} = 5$ is then,

$$x = \begin{bmatrix} 1/2t \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ t \end{bmatrix} \text{ and } v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

In this case we get only a single eigenvector and so a good eigenvalue/eigenvector pair is,

$$\lambda_{1,2} = 5 \qquad v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

6.9.2 The Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic equation i.e., if for a square matrix A of order n.

Important Result:

If A and B are two square matrices of the same order then AB and BA are two square matrices of the same order then AB and BA have the same characteristic roots.

Any matrix polynomial in A of size $n \times n$ can be expressed as a polynomial of degree $n-1$ in A by using Cayley-Hamilton theorem.

Example: Process to express a polynomial of a 2×2 Matrix as a linear polynomial in A:

Example-6.7 Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then we have to express the $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Solution:

Step 1: First of all write the characteristic equation of A.

In this case, $$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$
$$= (3-\lambda)(2-\lambda) + 1$$
$$= \lambda^2 - 5\lambda + 7$$

Thus the characteristic equation of A is $|A - \lambda I| = 0$

i.e., is $\lambda^2 - 5\lambda + 7 = 0$ ...(1)

Step 2: By Cayley Hamilton theorem, matrix A satisfies the equation (1).

Therefore, putting $A = I$ in (1) we get

$$A^2 - 5A + 7 = 0$$
$$\Rightarrow \quad A^2 = 5A - 7I \qquad ...(2)$$

Step 3: Find the A^5, A^4, A^3 with the help of (2). In this case,

$$A^3 = 5A^2 - 7A$$
$$\Rightarrow \quad A^4 = 5A^3 - 7A^2$$
$$\Rightarrow \quad A^4 = 5A^4 - 7A^3$$
$$2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$$
$$= 7A^4 - 14A^3 + A^2 - 4I$$

$$= 7[5A^3 - 7A^2) - 14A^3 + A^2 - 4I$$
$$= 21A^3 - 48A^2 - 4I$$
$$= 21(5A^2 - 7A) - 48A^2 - 4I$$
$$= 57A^2 - 147A - 4I$$
$$= 57[5A - 7I] - 147A - 4I$$
$$= 138A - 403I$$

which is a linear polynomial in A.

Summary

- A square sub-matrix of a square matrix A is called a "**principle sub-matrix**" if its diagonal elements are also the diagonal elements of the matrix A.
- Subtraction of matrices is neither commutative nor associative.
- Properties of multiplication of a matrix by a scalar:
  (a) Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.
  (b) If p and q are two scalars and A is any $m \times n$ matrix then, $(p + q)A = pA + qA$.
  (c) If p and q are two matrices and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.
  (d) If $A = [a_{ij}]_{m \times n}$ matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.
- If A and B be two n-rowed non-singular matrices, then AB is also non-singular then, $(AB)^{-1} = B^{-1}A^{-1}$ i.e., the inverse of a product is product of the inverse taken in the reverse order.
- The rank of transpose of a matrix is same as that of original matrix. i.e. $r(A^T) = r(A)$.

Student's Assignment

Q.1 The number of linearly independent solutions of the system of equations

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ is equal to}$$

(a) 1 (b) 2
(c) 3 (d) 4

Q.2 The eigen values of the matrix $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ are

(a) 2, 4, 6 (b) –2, –4, –6
(c) 1, 0, 7 (d) –2 , 4, 6

Q.3 The number of linearly independent eigen vector(s) of the above matrix is

(a) 1 (b) 2
(c) 3 (d) None of these

Q.4 The nullity of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 3 & 4 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) 4

Q.5 If matrix A is of order 3 × 4 and matrix B is 4 × 5. The number of multiplication operation and addition operations needed to calculate the matrix product AB.

(a) 240, 60 (b) 60, 45
(c) 60, 60 (d) 240, 32

Q.6 If $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$Q = PAP^{-1}$ and $X = P^{-1}Q^{2005}P$, then X is equal to

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(d) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Q.7 The rank of matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

Q.8 What could we say about the following system of linear equations?

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

(a) the system is consistent with a unique solution
(b) the system is consistent
(c) the system is inconsistent
(d) none of the above

Q.9 Find eigen values of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

(a) 2 and –6 (b) 3 and 2
(c) 4 and –6 (d) 3 and –7

Q.10 $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 8 & 9 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $|A| = ?$

(a) 0 (b) 120
(c) 252 (d) 40

Q.11 $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ find A^{-1}

(a) $A = \begin{bmatrix} 3/2 & -2 \\ 5/2 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} -3 & -4 \\ -5 & -6 \end{bmatrix}$

(c) $A = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$ (d) A is not invertible

Q.12 $A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$ Compute $|A|$.

(a) 90 (b) 20
(c) 0 (d) 4

Q.13 Find the eigen values of the following matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

(a) 4, 0, –3 (b) 4, 1
(c) 1, –3 (d) 2, –1.5

Q.14 Consider $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -3 & 4 & -2 \end{bmatrix}$ which of the following statements is not true?

(a) A is invertible (b) A is non-invertible
(c) $|A| \neq 0$ (d) None of above

Q.15 Consider the following system of linear equations

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α, does this system of equations have infinitely many solutions?

(a) 0
(b) 1
(c) 2
(d) Infinitely many

Q.16 What is the index to prove matrix

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$ a nilpotent matrix?

(a) 2
(b) 1
(c) 3
(d) None of above

Common Data Questions (17 and 18):

Two eigen values of 3 × 3 matrix A are 2, 1 and $|A| = 12$.

Q.17 Find the third eigen value of matrix A.

(a) 2
(b) 6
(c) 12
(d) None of these

Q.18 The product of eigen values of A^3 and A^5 are

(a) 1728, 248832
(b) 248832, 1728
(c) 243288, 1827
(d) None of these

Q.19 For a system of m linear equations in n unknowns, the cramer's rule is applicable when

(a) $m = n$
(b) $m \neq n$
(c) $m = n$ and coefficient matrix is non singular
(d) $m = n$ and the coefficient matrix is singular

Q.20 If $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$ and $a, b, c \neq 0$, then

(a) $c = (a + b)/2$
(b) $a = b$ or $a = c$ or $b = c$
(c) $c = (a - b)/2$
(d) $a = 2b = 2c$

Q.21 Choose the incorrect statements:

1. The determinant of a matrix equals the sum of its eigen values.
2. A square matrix satisfies its characteristic equation.
3. The sum of principal diagonal elements of a matrix equals the sum of its eigen values.
4. If a row of a matrix is same as one of its columns, its determinant value is 0.

(a) 1 and 2
(b) 1 and 3
(c) 1 and 4
(d) 3 and 4

Q.22 The given equations are solved by using LU-decomposition method.

$$x_1 + 3x_2 - 8x_3 = 4$$
$$x_1 + 4x_2 + 3x_3 = -2$$
$$x_1 + 3x_2 + 4x_3 = 1$$

Find lower triangular matrix value.

(a) $\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 12 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$
(d) None of above

Q.23 Find the U matrix from the previous question.

(a) $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 11 \\ 0 & 0 & -8 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 3 & -8 \\ 0 & 1 & 11 \\ 0 & 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 3 & -8 \\ 0 & 11 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -8 & 11 & 1 \end{bmatrix}$

Answer Key:

1. (a)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (b)
6. (a)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (c)
11. (c)	**12.** (b)	**13.** (a)	**14.** (b)	**15.** (a)
16. (a)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (b)
21. (c)	**22.** (b)	**23.** (b)		

Student's Assignments Explanations

1. (a)

$$\begin{bmatrix}1 & 0 & 2\\1 & -1 & 0\\2 & -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix};$$

$$x_1 + 2x_3 = 0$$
$$x_1 - x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

$$x_1 = x_2 = t \text{ and } x_3 = \frac{-x_1}{2} = \frac{-t}{2}$$

The solution is $\begin{bmatrix}t\\t\\-t/2\end{bmatrix}$ for all values of t.

$\therefore$ The number of linearly independent solution is 1.

3. (c)

Characteristic equation is

$$\begin{vmatrix}5-\lambda & 0 & 1\\0 & -2-\lambda & 0\\1 & 0 & 5-\lambda\end{vmatrix} = 0$$

$$\lambda = -2, 4, 6$$

$[A-\lambda I]X = 0$ is the eigen value problem

$$\begin{bmatrix}5-\lambda & 0 & 1\\0 & -2-\lambda & 0\\1 & 0 & 5-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ...(i)$$

Putting $\lambda = -2$ in eqn (i)

$$\begin{bmatrix}7 & 0 & 1\\0 & 0 & 0\\1 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$7x_1 + x_3 = 0$$
$$x_1 + 7x_3 = 0$$

Solving which we get $x_1 = 0, x_3 = 0$

Putting $x_2 = t$

We get one eigen vector as $\begin{bmatrix}0\\t\\0\end{bmatrix}$

Putting, $\lambda = 4$ in equation ...(i)

$$\begin{bmatrix}1 & 0 & 1\\0 & -6 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving which we get $x_2 = 0$, $x_1 = -x_3$ putting $x_1 = t$, we get $x_3 = -t$

$\therefore$ Another eigen vector is $\begin{bmatrix}-t\\0\\t\end{bmatrix}$

Putting, $\lambda = 6$ in equation (i),

$$\begin{bmatrix}-1 & 0 & 1\\0 & -8 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving we get $x_1 = x_3$ and $x_2 = 0$

Putting $x_1 = t$ we get, $x_3 = t$

$\therefore$ Another eigen vector is $\begin{bmatrix}t\\0\\t\end{bmatrix}$

All three are linearly independent

$\therefore$ No of linearly independent eigen vectors = 3

4. (a)

$|A| = 0$. So rank is not 3.

The minor $\begin{bmatrix}5 & 0\\7 & 3\end{bmatrix} = 15 \neq 0$

So, Rank = 2

Nullity = number of columns − rank = 3 − 2 = 1

5. (b)

Let matrix A is of order $m \times n$

Let matrix B is of order $n \times P$

for matrix product AB,

The number of multiplication operations = mpn

The number of addition operations = $mp(n-1)$

Here $m = 3$, $n = 4$, $P = 5$

No. of multiplication operations = $3 \times 4 \times 5 = 60$

No. of addition operations = $3 \times 5\,(4-1) = 45$

6. (a)

$P^{-1}P = I$

$Q = PAP^{-1}$, so that

$X = P^{-1}Q^{2005}P = P^{-1}(PAP^{-1})^{2005}P$

Now, $(PAP^{-1})^2 = (PAP^{-1})(PAP^{-1})$

$= PA(P^{-1}P)AP^{-1} = PAIAP^{-1}$

$= PA^2P^{-1}$

Similarly, $(PAP^{-1})^3 = PA^3P^{-1}$ and so on

$\therefore \quad (PAP^{-1})^n = PA^nP^{-1}$

$\therefore \quad X = P^{-1}(PAP^{-1})^{2005}P$

$= P^{-1}(PA^{2005}P^{-1})P$

$= (P^{-1}P).A^{2005}.(P^{-1}P) = A^{2005}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

and so on

$$\therefore \quad A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

7. (c)

$$\begin{vmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{vmatrix} = 0$$

$\therefore \quad$ rank $\neq 3$

and $\quad \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$

Since 2×2 minor is not zero, rank = 2

8. (c)

Use gauss elimination on augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right] \xrightarrow{R(1,2)} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 5/2R_1} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right] \xrightarrow{R_3 + 1/2R_2}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right]$$

rank (A) = 2

rank (A | B) = 3

rank (A) $\neq$ rank (A | B)

This system is therefore inconsistent and has no solution.

9. (d)

The characteristic equation of matrix is

$|A - tI| = 0$

$$|A - \tau I| = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$(2-\tau)(-6-\tau) - (3)(3) = 0$

$\tau^2 + 4\tau + 21 = 0$

$(\tau - 3)(\tau + 7) = 0$

which has roots 3 and –7, which are the eigen values.

10. (c)

In case of triangular matrix, the value of determinant is equal to multiplication of diagonal elements = 2*6*7*3 = 252

11. (c)

Suppose, $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is invertible if $ad - bc \neq 0$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

substituting the values, $a = 3$, $b = 4$, $c = 5$ and $d = 6$,

we get, $\quad A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$

12. (b)

Taking advantage of 0s in third column start cofactor expansion down the third column to obtain 3 × 3 matrix.

$$= (-1)^{(1+3)} \cdot 2 \cdot \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

again taking advantage of 0s in the first column, we expand using cofactors of column 1 elements.

$$= 2 \cdot (-1)^{2+1} \cdot -5 \cdot \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

13. (a)

Eigen values of triangular matrix are entries in the diagonal: 4, 0 and –3.

14. (b)

The quick test for invertibility is the value of determinant of the matrix. If the determinant of a matrixis non-zero, then it is invertible. Since $|A| = -30 \neq 0$. The matrix is invertible.

15. (a)

Gauss elimination on argumented matrix gives rank (A) = rank (A|B) = 3 for all values of α.

$\therefore$ for all values of α, we get unique solution.

i.e. there is no value of α, which gives infinite solution for this system.

16. (a)

To prove a matrix A to be nilpotent, A should be a square matrix, and $A^n = 0$.

$n \rightarrow$ index (least positive integer that satisfies $A^n) = 0$

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$A^2 = A \,.\, A = A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^2 = 0$$

Therefore, index is 2.

17. (b)

Product of eigen values = |A|

$\therefore \quad 2 \times 1 \times \tau = 12$

$\tau = 6$

Third eigen values = 6

18. (a)

Eigen values of $A^3 \Rightarrow 1^3, 2^3, 6^3$

Eigen values of $A^5 \Rightarrow 1^5, 2^5, 6^5$

Product of eigen values of

$A^3 \Rightarrow 1 \times 2^3 \times 6^3 = 1728$

Product of eigen values of

$A^5 \Rightarrow 1 \times 2^5 \times 6^5 = 248832$

20. (b)

$$\text{Given} \quad \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} \rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$\rightarrow abc[(b-a)(c^2-a^2)-(c-a)(b^2-a^2)]$

$\rightarrow abc[(b-a)(c-a)(c+a)-(c-a)(b-a)(b+a)]$

$\rightarrow abc\{(b-a)(c-a)[(c+a)-(b+a)]\}$

$\rightarrow abc[(b-a)(c-a)(c-b)]$

Since the determinant is zero

$abc\,(b-a)(c-a)(c-b) = 0$

since $a, b, c \neq 0$ (given)

$b-a=0$ or $c-a=0$ or $c-b=0$

$\Rightarrow a = b$ or $a = c$ or $b = c$

21. (c)
A is false since, determinant of a matrix is equal to product of eigen values.
B is true
C is true
D is false since determinate is zero, if a row is same as another row or if a column is same as another column.

22. (b)
The system may be written in matrix form as

$$\begin{bmatrix}1 & 3 & -8\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\-2\\1\end{bmatrix}$$

$$A = \begin{bmatrix}1 & 3 & -8\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$$

$$LU = A$$

$$= \begin{bmatrix}1 & 3 & -8\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$$

$$\ell_{11} = 1, \ell_{21} = \ell, \ell_{31} = \ell$$
$$\ell_{11} U_{12} = 3 \Rightarrow U_{12} = 3,$$
$$\ell_{21} U_{12} + \ell_{22} = 4 \Rightarrow \ell_{22} = 4 - 1.3 = 1$$
$$\ell_{31} U_{12} + \ell_{32} = 3 \Rightarrow \ell_{32} = 3 - 1 \times 3 = 0$$

$$\ell_{11} U_{13} = -8 \Rightarrow U_{13} = \frac{-8}{1} = -8$$

$$\ell_{21} U_{13} + \ell_{22} U_{23} = 3 \Rightarrow U_{23} = 11$$
$$\ell_{31} U_{13} + \ell_{32} U_{23} + \ell_{33} = 4 \Rightarrow \ell_{33} = 12$$

$$\therefore L = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 0 & 12\end{bmatrix} \text{ and } U = \begin{bmatrix}1 & 3 & -8\\0 & 1 & 11\\0 & 0 & 1\end{bmatrix}$$

23. (b)
See explanation of previous problem.

CHAPTER 07

Calculus

7.1 Limit

Definition: A number A is said to be limit of f at $x = a$ iff for any arbitrarily choosen positive integer ϵ, however small but not zero there exist a corresponding number d greater than zero such that: $|f(x) - A| < \epsilon$ for all values of x for which $0 < |x - a| < \partial$ where $|x - a|$ means the absolute value of $(x - a)$ without any regard to sign.

Right and Left Hand Limits

If x approaches a from the right, that is, from larger value of x than a, the limit of f as defined before is called the right hand limit of $f(x)$ and is written as: $\underset{x \to a+0}{\text{Lt}} f(x)$ or $f(a+0)$

Working rule for finding right hand limit is, put $a + h$ for x in $f(x)$ and make h approach zero.

In short, we have, $f(a+0) = \underset{h \to 0}{\text{Limit}}\, f(a+h)$

Similarly if x approaches a from left, that is from smaller values of x than a, the limit of f is called the left hand limit and is written as: $\underset{x \to a-0}{\text{Lt}} f(x)$ or $f(a-0)$

In this case, we have $f(a-0) = \underset{h \to 0}{\text{Limit}}\, f(a-h)$

If both right hand and left hand limit of f, as $x \to a$ exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

Indeterminate Form

A fraction whose numerator and denominator both tend to zero as $x \to a$ is called indeterminate form 0/0. It has not definite values. Other indeterminate forms are: ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0, ∞^0.

L' Hospital Rule

If $f(x)$ and $\phi(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = \phi(a) = 0$ then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}$ provided, the latter limit exists, finite or infinite.

Working Rule

If the limit of $\frac{f(x)}{\phi(x)}$ as $x \to a$ takes the form 0/0, differentiate the numerator and denominator separately with respect to x and obtain a new function $\frac{f'(x)}{\phi'(x)}$. Now as $x \to a$ if it again takes the form 0/0, differentiate the numerator and denominator again with respect to x and repeat the above process, till indeterminate form persists.

Caution

Before applying Hospital's rule at any stage be sure that the form is 0/0. Do not go on applying this rule even if the form is not 0/0.

Various Formulae

$$(1+x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(x\log a)^2 + \frac{x^3}{3!}(x\log a)^3 + \ldots..$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots...$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots...$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.. |x| < l$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots..\right) |x| < l$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots..$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots..$$

$$\sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

$$\cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$

REMEMBER: $\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

Some useful results:

(i) $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$

(ii) $\underset{x \to 0}{\text{Lt}} \cos x = 1$

(iii) $\underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1$

(iv) $\underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$

(v) $\underset{x \to 0}{\text{Lt}} (1+nx)^{\frac{1}{x}} = e^n$

(vi) $\underset{x \to \infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$

(vii) $\underset{x \to \infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^x = e^a$

Form-II

∞/∞: If $f(x)$ and $\phi(x)$ be two functions such that, $\underset{x \to a}{\text{Limit}} f(x) = \infty$ and $\underset{x \to a}{\text{Limit}} \phi(x) = \infty$, then

$\underset{x \to a}{\text{Limit}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Limit}} \frac{f'(x)}{\phi'(x)}$, provided the limit exists.

Form-III

$0 \times \infty$: This form can be easily reduced to the form 0/0 or to the form ∞/∞.

Let $\underset{x \to a}{\text{Limit}} f(x) = 0$ and $\underset{x \to a}{\text{Limit}} \phi(x) = \infty$.

Then we can write

$\underset{x \to a}{\text{Limit}} f(x) \cdot \phi(x) = \underset{x \to a}{\text{Limit}} \frac{f(x)}{1/\phi(x)}$ [form 0/0] or $\underset{x \to a}{\text{Limit}} \frac{\phi(x)}{1/f(x)}$ [form ∞/∞]

Thus $\underset{x \to a}{\text{Limit}} f(x) \cdot \phi(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L' Hospital rule or otherwise.

Form-IV

$0^0, 1^\infty, \infty^0$: Suppose $\underset{x \to a}{\text{Limit}} [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then let $y = \underset{x \to a}{\text{Limit}} [f(x)]^{\phi(x)}$

Taking log on both sides, we get

$\log y = \underset{x \to a}{\text{Limit}} \phi(x) \cdot \log f(x)$.

Now in any of these above cases $\log y$ takes the form $0 \times \infty$ which is changed to the form 0/0 or ∞/∞ then it can be evaluated by previous methods.

Example-7.1 Find the value of 'a'?

$$\text{If } \lim_{n \to 0} \frac{\sin 2x + a\sin x}{x^3} = \text{finite.}$$

Solution:

$$\lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3} = \frac{0}{0} \quad \text{(Indeterminate form)}$$

Apply L-Hospital's rule

$$\Rightarrow \quad \lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^3} = \frac{2+a}{0} = \text{finite (given)}$$

$$\therefore \quad 2 + a = 0 \Rightarrow a = -2$$

Example-7.2 Consider the following function given below:

$$f(x) = \begin{cases} \dfrac{\sin[x]}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [0] = 0 \end{cases}$$

The reason for $f(x)$ be discontinuous at $x = 0$ is

(a) $f(0)$ is not defined.

(b) $f(0)$ is defined but $\lim_{x \to 0} f(x)$ does not exist.

(c) $\lim_{x \to 0} f(x)$ exists, $f(0)$ is defined but $\lim_{x \to 0} f(0) \neq f(0)$

(d) $f(x)$ is continuous at $x = 0$

Solution:

$$f(x) = \begin{cases} \dfrac{\sin(-1)}{(-1)} & \text{for } -1 \le x < 0 \; [x] = 1 \\ 0 & \text{for } 0 \le x < 1 \; [x] = 0 \end{cases}$$

$$f(0) = 0$$

Left limit is sin1. Right limit is 0.

$\therefore$ Limit does not exists.

Example-7.3 Find the limits of the following function:

(a) $\lim_{x \to 0} \dfrac{1 - \cos 3x}{x \sin x}$

(b) $\lim_{x \to 0} \dfrac{\cos\left(\frac{\pi}{2} x\right)}{1 - \sqrt{x}}$

Solution:

(a)

$$\lim_{x \to 0} \left[\frac{1 - \cos 3x}{x \sin x}\right] = \frac{0}{0} \quad \text{(Indeterminate form)}$$

to overcome the indeterminate form we use L Hospital's rule.

$$\lim_{x \to 0} \left[\frac{3\sin 3x}{\sin 2x + 2x\cos x}\right] = \frac{0}{0} \quad \text{(This still in indeterminate form)}$$

Apply L Hospital's rule again

$$\lim_{x\to 0}\left[\frac{9\cos 3x}{2\cos 2x + 2\cos 2x - 4x\sin 2x}\right] = \frac{9(1)}{2(1)+2(1)-4(0)} = \frac{9}{4}$$

(b)
$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}} = \left(\frac{0}{0}\right) \quad \text{(Indeterminate form)}$$

$$= \lim_{x\to 1}\frac{-\frac{\pi}{2}\sin\frac{\pi x}{2}}{-\frac{1}{2}\sqrt{x}} = \frac{\frac{\pi}{2}\sin\frac{\pi}{2}}{\frac{1}{2}} = \pi$$

7.2 Continuity

Definition: A function $f(x)$ is defined for $x = a$ is said to be continuous at $x = a$ if:

(*i*) $f(a)$ i.e., the value of $f(x)$ at $x = a$ is a definite number

(*ii*) the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.

Otherwise the function is discontinuous at $x = a$.

A function is said to be continuous in an interval (a, b) if it is continuous at every point of interval (a, b).

Arithmetical Definition of Continuity: A function $f(x)$ is said to be continuous at $x = a$, if for any arbitrarily choosen positive number ϵ, however small (but not zero) $\exists$ a corresponding number δ such that, $|f(x) - f(a)| < \epsilon$ for all values of x for which $|x - a| < \delta$.

NOTE: On comparing the definitions of limit and continuity we find that a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$

Thus $f(x)$ is continuous at $x = a$ if we have $f(a+0) = f(a-0) = f(a)$, otherwise it is discontinuous at $x = a$.

Continuity from Left and Continuity from Right

Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a if $\lim_{x\to a+0} f(x)$ exists and is equal to $f(a)$.

Continuity in an Open Interval

A function f is said to be continuous in open interval $[a, b]$ if it is continuous at each point of interval.

Continuity in a Closed Interval

Let f be a function defined on the closed interval $[a, b]$. We say that f is continuous at a if it is continuous from the right a and also that f is continuous at b if it is continuous from the left at b. Further f is said to be continuous on the closed interval $[a, b]$ if (i) it is continuous from the right and (ii) continuous from the lift at b and (iii) continuous on the open interval $[a, b]$.

7.3 Differentiability

Derivative at a point: Let-I denote the open interval [a, b] in R and let $x_0 \in I$. Then a function $f: I \to R$ is said to be differentiable at x_0, iff:

$$\underset{x\to 0}{\text{Limit}} \frac{f(x_0+h)-f(x_0)}{h} \text{ or } \underset{x\to x_0}{\text{Limit}} \frac{f(x)-f(x_0)}{x-x_0}$$ exist (finitely) and is denoted by $f'(x_0)$.

Progressive and Regressive Derivatives

The progressive derivative of f at $x = x_0$ is given by

$$\underset{h\to 0}{\text{Limit}} \frac{f(x_0+h)-f(x_0)}{h}, h>0$$ and is denoted by $Rf'(x_0)$ or by $f'(x_0+0)$

The regressive derivative of f at $x = x_0$ is given by

$$\underset{h\to 0}{\text{Limit}} \frac{f(x_0-h)-f(x_0)}{-h}, h>0$$ and is denoted by $Lf'(x_0)$ or by $f'(x_0-0)$

Differentiability in [a, b]

A function f: [a, b] → R is said to be differentiable at a iff $Rf'(a)$ exists, differentiable at b iff $Lf'(b)$ exists, f is said to be differentiable at every point [a, b].

Result:

Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.
i.e. differentiability ⇒ continuity

But continuity $\not\Rightarrow$ differentiability

Example-7.4 What can be said about the continuity and differentiability of $f(x)$, where $f(x) = \frac{1}{1+|x|}$; at $x = 0$?

Solution:

$$f(x) = \begin{cases} \frac{1}{1+x} & \text{for} \quad x>0 \\ \frac{1}{1-x} & \text{for} \quad x<0 \\ 1 & \text{for} \quad x=0 \end{cases}$$

$$\left.\begin{aligned} f(0) = 1, \text{ left limit} &= \frac{1}{1+0} = 1 \\ \text{similarly right limit} &= 1 \end{aligned}\right\} \text{continuous at } x = 0$$

$$\text{Left hand derivative} = \lim_{h\to 0}\left[\frac{\frac{1}{1-(-h)}-1}{-h}\right] = \lim_{h\to 0}\frac{-h}{-h(1+h)} = 1$$

$$\text{Right hand derivative} = \lim_{h\to 0}\left[\frac{\frac{1}{1+h}-1}{h}\right] = \lim_{h\to 0}\frac{-h}{h(1+h)} = -1$$

$\therefore$ Left hand derivative = Right hand derivative

$\therefore$ It is continuous but not differentiable at $x = 0$

Example-7.5 Let $f(x) = x|x|$ where $x \in R$, then $f(x)$ at $x = 0$ is

(a) continuous and differentiable
(b) continuous but not differentiable
(c) differentiable but not continuous
(d) neither differentiable not continuous

Solution:

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$f(0) = 0$, Left limit = 0, Right limit = 0

$$\text{L.H.D.} = \lim_{h\to 0}\left[\frac{-(-h)^2 - 0}{-h}\right] = 0$$

$$\text{R.H.D.} = \lim_{h\to 0}\left[\frac{(+h)^2 - 0}{h}\right] = \lim_{h\to 0} h = 0$$

$\therefore$ It is both continuous and differentiable.

7.4 Mean Value Theorems

Rolle's Theorem

If a function $f(x)$ is such that:

(i) $f(x)$ is continuous in the closed interval $a \le x \le b$.

(ii) $f'(x)$exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$, then there exists at least one value of x, say c where $a < c < b$ such that $f'(c) = 0$

NOTE

Role's theorem will not hold good.

- If $f(x)$ is discontinuous at some point in the interval $a < x < b$
- If $f'(x)$ does not exist at some point in the interval $a \le x \le b$ or
- If $f(a) \ne f(b)$

Example-7.6 The Mean value C for the below function:

$$f(x) = e^x[\sin x - \cos x] \text{ in } \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \text{ is \_\_\_\_\_}.$$

Solution:

$$f'(x) = e^x[\sin x - \cos x] + e^x[\cos x + \sin x] = 2e^x \sin x$$

$$f\left(\frac{\pi}{4}\right) = 0, \quad f\left(\frac{5\pi}{4}\right) = 0$$

By Rolle's theorem there exist $C \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Such that $f'(c) = 0$, $\quad 2e^C \sin C = 0 \Rightarrow \sin C = 0$

$$C = 0, \pm\pi, \pm 2\pi, \ldots$$

$\Rightarrow$
$$C = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Lagrange's Mean Value Theorem or First Mean Value Theorem

If a function $f(x)$ is

(i) continuous in closed interval $a \le x \le b$ and

(ii) differentiable in open interval (a, b) i.e., $a < x < b$, then there exist at least one value c of x lying in the open interval $a < x < b$ such that,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Some important deductions from mean value theorem:

(i) If a function $f(x)$ be such that $f'(x)$ is zero throughout the interval, then $f(x)$ must be constant throughout the interval.

(ii) If $f(x)$ and $\phi(x)$ be two functions such that $f'(x) = f'(x)$ throughout the interval (a, b), then $f(x)$ and $\phi(x)$ differ only by a constant.

(iii) If $f(x)$ is:

(a) continuous in closed interval $[a, b]$

(b) differentiable in open interval (a, b)

(c) $f(x)$ is –ve in $a < x < b$, then $f(x)$ is monotonically decreasing function in the closed interval $[a, b]$

Example-7.7 Verify Lagrange's mean mean value theorem for the following functions in the given interval and find 'c' of this theorem.

(a) $f(x) = x^2 + 2x + 3$ in $[4, 6]$

(b) $f(x) = px^2 + qx + r,\ p \ne 0$, in $[a, b]$

Solution:

(a) Given $f(x) = x^2 + 2x + 3$ ... (i)

(i) $f(x)$ being a polynomial function is continuous in $[4, 6]$.

(ii) $f(x)$ being a polynomial function is derivable in $(4, 6)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(4, 6)$ such that

$$f'(c) = \frac{f(6) - f(4)}{6 - 4}$$

$$f(6) = 6^2 + 2.6 + 3 = 51,\ f(4) = 4^2 + 2.4 + 3 = 27$$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2x + 2 \Rightarrow f'(c) = 2c + 2$$

$$f'(c) = \frac{f(6) - f(4)}{6 - 4} \quad 2c + 2 = \frac{51 - 27}{2} \Rightarrow 2c + 2 = 12$$

$\therefore$
$$2c = 10 \Rightarrow c = 5$$

$\Rightarrow$ Thus, there exists $c = 5$ in $(4, 6)$ such that $f'(5) = \frac{f(6) - f(4)}{6 - 4}$

Hence, Lagrange's mean value theorem is verified and $c = 5$.

(b) Given $f(x) = px^2 + qx + r, p \neq 0$

(*i*) f being a polynomial function is continuous in $[a, b]$

(*ii*) f being a polynomial function is derivable in (a, b).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in (a, b) such that $f'(x) = \dfrac{f(b)-f(a)}{b-a}$.

$$f(b) = pb^2 + qb + r, f(a) = pa^2 + qa + r$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 2px + q \Rightarrow f'(c) = 2pc + q$$

$$\therefore \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2pc + q = \frac{(pb^2+qb+r)-(pa^2+qa+r)}{b-a}$$

$$\Rightarrow \quad 2pc + q = \frac{p(b^2+a^2)+q(b-a)}{b-a} \Rightarrow 2pc = p(a+b)$$

$$\Rightarrow \quad c = \frac{a+b}{2} \text{ and } \frac{a+b}{2} \in (a, b)$$

Thus, there exist $c = \dfrac{a+b}{2}$ in (a, b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Hence Lagrange's mean value theorem is verified and $c = \dfrac{a+b}{2}$

Example-7.8 Find a point on the graph of $y = x^3$ where the tangent is parallel to the chord joining (1, 1) and (3, 27).

Solution:

$$f(x) = x^3 \text{ in the interval } [1, 3]$$

(a) $f(x)$ being a polynomial is continuous in [1, 3].

(b) $f(x)$ being a polynomial is derivable in (1, 3).

Thus, both the conditions of Lagrange's mean value theorem are satisfied by the function $f(x)$ in [1, 3], therefore, there exists atleast one real number c in (1, 3) such that

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = 1^3 = 1$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$$

Now
$$f'(c) = \frac{f(3)-f(1)}{3-1} \Rightarrow 3c^2 = \frac{27-1}{3-1} \Rightarrow 3c^2 = 13$$

$$\Rightarrow \quad c^2 = \frac{13}{3} = \frac{39}{9} \Rightarrow c = \pm\frac{\sqrt{39}}{3}$$

But, $\qquad c \in (1, 3) \Rightarrow c = \dfrac{\sqrt{39}}{3}$

When, $\qquad x = \dfrac{\sqrt{39}}{3}$, from (1) $y = \dfrac{\sqrt{39}}{3}$

Hence, there exists a point $\left(\dfrac{\sqrt{39}}{3}, \dfrac{13\sqrt{39}}{9}\right)$ on the given curve $y = x^3$ where the tangent is parallel to the chord joining the points (1, 1) and (3, 27).

7.5 Theorems of Integral Calculus

1. The integral of the product of a constant and a function is equal to be product of the constant and the integral of function.

   Thus if λ is constant, then $\int \lambda f(x)dx = \lambda \int f(x)dx$.

2. The integral of a sum of or difference of a finite number of functions is equal to sum or difference of integrals. Symbolically

$$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \dots \pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \dots \pm \int f_n(x)dx$$

Fundamental Formulae

(i) $\int x^n\, dx = \dfrac{x^{n+1}}{n+1}$

(ii) $\int \dfrac{1}{x}\, dx = \log x$

(iii) $\int \sin x\, dx = \cos x$

(iv) $\int \cos x\, dx = \sin x$

(v) $\int \sec^2 x\, dx = \tan x$

(vi) $\int \text{cosec}^2 x\, dx = -\cot x$

(vii) $\int \text{cosec}\, x \cot x = -\text{cosec}\, x$

(viii) $\int \dfrac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x$

(ix) $\int \dfrac{1}{1+x^2}\, dx = \tan^{-1} x$

(x) $\int \dfrac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x$

(xi) $\int \cos hx = \sin hx$

(xii) $\int \sin hx\, dx = \cos hx$

(xiii) $\int \text{cosec}\, h^2 x\, dx = -\cot hx$

(xiv) $\int \sec hx \tan hx\, dx = -\sec hx$

(xv) $\int \text{cosec}\, hx \cot hx = -\text{cosec}\, hx$

7.6 Methods of Integration

There are various methods of integration by which we can reduce the given integral to one of fundamental known integral. There are four principal methods of integration.

7.6.1 Integration by Substitution

A change in the variable of integration often reduces an integral to one of fundamental integrals.

Let $I = \int f(x)dx$, then by differentiation w.r.to x we have $\frac{dI}{dx} = f(x)$. Now put,

$$x = \phi(t), \text{ so that } \frac{dx}{dt} = \phi'(t)$$

Then, $$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = f(x)\cdot\phi'(t) = f\{\phi(t)\cdot\phi'(t)\} \text{ for } x = \phi(t)$$

This gives $$I = \int f\{\phi(t)\cdot\phi'(t)\}\,dt$$

Rule to Remember

To evaluate $\int f\{\phi(x)\cdot\phi'(x)\}\,dx$

Put $\phi(x) = t$ and $\phi'(x)dx = dt$

where $\phi'(x)$ is the differential coefficient of $\phi(x)$ with respect to x.

Three forms of Integrals

1. $$\int \frac{f'(x)}{f(x)}dx = \log f(x)$$

Put $f(x) = t$ differentiating we get $f'(x)\cdot dx = dt$

$\Rightarrow$ $$\int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \log t = \log f(x)$$

Thus the integral of a fraction whose numerator is the exact derivative of its denominator is equal to the logarithmic of its denominator.

Example: $\int \frac{4x^3}{1+x^4}dx = \log(1+x^4)$ ...(7.1)

Because, if we put $(1+x^4) = t$

$\Rightarrow$ $4x^3\,dx = dt$

Equation (7.1) reduces to $\Rightarrow \int \frac{dt}{t} \Rightarrow \log t \Rightarrow \log(1+x^4)$.

Some important formulae based on the above form

(i) $$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{(-\sin x)}{\cos x}dx$$
$$= -\log\cos x = \log(\cos x)^{-1}$$
$$= \log\sec x$$

(ii) $$\int \cot x dx = \log\sin x$$

(iii) $$\int \text{cosec}\, x dx = \log\tan\frac{x}{2}$$

2. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{(n+1)}$ when $n \neq 1$: If the integrand consists of the product of a constant power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$, to obtain the integral we increase the index by unity and then divide by increased index. This is known as power formula.

Formulae:

(i) $\int f'(ax+b)dx = \frac{f(ax+b)}{a}$

(ii) $\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right) = \log\left[x+\sqrt{x^2+a^2}\right]$

(iii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left[x+\sqrt{x^2-a^2}\right] = \cosh^{-1}\left(\frac{x}{a}\right)$

(v) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$ or $\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\{x+\sqrt{x^2+a^2}\right\}$

(vi) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

7.6.2 Integral of the Product of Two Functions

Integration by parts: Let u and v be two functions of x. Then we have from differential calculus.

$$\frac{d}{dx}(uv) = u \times \frac{dv}{dx} + v \times \frac{du}{dx} \quad ...(7.2)$$

Integrating both sides of (7.2) with respect to x, we have

$$uv = \int u \cdot \frac{dv}{dx}dx + \int v \cdot \frac{du}{dx}dx$$

$$\Rightarrow \quad \int u\frac{dv}{dx}dx = uv - \int v \cdot \frac{du}{dx} \cdot dx \quad ...(7.3)$$

i.e. $\int udv = uv - \int v \cdot du$

This can also be written as $\int uv dx = u\int v dx - \int [du \int v dx]dx$

Formulae based Upon Above Method

1. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

2. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

7.6.3 Integration by Partial Fractions

1.
$$I = \int \frac{1}{x^2 - a^2}, (x > a)$$

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\}$$

$$= \frac{1}{2a}\{\log(x-a) - \log(x+a)\} = \frac{1}{2a}\log\frac{x-a}{x+a}$$

Thus, $\displaystyle\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\frac{x-a}{x+a}, x > a$

2.
$$I = \int \frac{1}{a^2 - x^2} dx \ (x < a)$$

In this case $\displaystyle\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\frac{a+x}{a-x}, x < a$

7.7 Definite Integrals

If $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of $f(x)$ between the limit of a and b. $b \rightarrow$ upper limit; $a \rightarrow$ lower limit.

Fundamental Properties of Definite Integrals

1. We have $\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e., the value of a definite integral does not change with the change of variable of integration provided the limits of integration remain the same.

   Let, $\int f(x)dx = F(x)$ and $\int f(t)dt = F(t)$

   Now, $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

   $\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$

2. $\int_a^b f(x)dt = -\int_a^b f(x)dt$. Interchanging the limits of a definite integral does not change in the absolute value but change the sign of integrals.

3. We have $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

NOTE: 1. This property also holds true even if the point c is exterior to the interval (a, b). 2. In place of one additional point c, we can take several points. Thus several points.

Thus, $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \dots + \int_{c_n}^{b} f(x)dx$

4. We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: Let $I = \int_0^a f(x)dx$

Put $x = a - t \Rightarrow dx = -dt$ where $x = 0$, $t = a$ and when $x = a$, $t = 0$

$\Rightarrow \quad I = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)dt = \int_0^a f(a-x)dx$

5. $\int_{-a}^{+a} f(x)dx = 0$ or $2\int_0^a f(x)dx$ according as $f(x)$ is an odd or even function of x.

Odd and Even function:

(*i*) An odd function of x if $f(-x) = -f(x)$

(*ii*) An even function of x if $f(-x) = f(x)$

6. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$ and $\int_0^{2a} f(x)dx = 0$ if $f(2a-x) = -f(x)$

Corollary: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Example-7.9 Evaluate the following definite integrals:

(a) $\int_{-5}^{5} |x+2|dx$ (b) $\int_{1}^{4} (|x|+|x-3|)dx$

Solution:

(a) Since for $-5 \le x \le -2$, $x+2 \le 0$

$\Rightarrow \quad |x+2| = -(x+2)$

and for $-2 \le x \le 5$, $x + 2 \ge 0$

$\Rightarrow \quad |x+2| = x+2,$

$$\therefore \int_{-5}^{5} |x+2|\,dx = \int_{-5}^{-2} |x+2|dx + \int_{-2}^{5} |x+2|dx \qquad \text{(Property 3)}$$

$$= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx = \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= (-2+4) - \left(-\frac{25}{2} + 10\right) + \left(\frac{25}{2} + 10\right) - (2-4) = 29$$

(b) Since for $1 \le x \le 3$, $x \ge 0$, $x - 3 \le 0 \Rightarrow |x| = x$, $|x-3| = -(x-3)$

Also for $3 \le x \le 4$, $x \le 0$, $x - 3 \ge 0 \Rightarrow |x| = x$, $|x-3| = x-3$

$$\therefore \int_1^4 (|x|+|x-3|)dx = \int_1^3 (|x|+|x-3|)dx + \int_3^4 (|x|+|x-3|)dx \qquad \text{(Property 3)}$$

$$= \int_1^3 (x-(x-3))dx + \int_2^4 (x+x-3)dx = \int_1^3 3dx + \int_3^4 (2x-3)dx$$

$$= 3[x]_1^3 + \left[2.\frac{x^2}{2} - 3x\right]_3^4$$

$$= 3(3-1) + (16-12) - (9-9) = 16 + 4 - 0 = 10$$

Example-7.10 Evaluate the following definite integrals:

(a) $\int_{-1}^{2} f(x)dx$ where $f(x) = \begin{cases} 2x+1, & x \le 1 \\ x-5, & x > 1 \end{cases}$ (b) $\int_{-1}^{1} \frac{|x|}{x} dx$

(c) $\int_0^1 [3x]dx$

Solution:

(a) First note that the given function is discontinuous at $x = 1$.

$$\therefore \int_{-1}^{2} f(x)dx = \int_{-1}^{1} f(x)dx + \int_1^2 f(x)dx$$

$$= \int_{-1}^{1} (2x+1)dx + \int_1^2 (x-5)dx = \left[x^2+x\right]_{-1}^{1} + \left[\frac{x^2}{2} - 5x\right]_1^{-2}$$

$$= (1+1)-(1-1)+(2-10)-\left(\frac{1}{2}-5\right) = 2-0-8+\frac{9}{2} = -\frac{3}{2}$$

(b) First note that $\frac{|x|}{x}$ is discontinuous at $x = 0$.

$$\therefore \int_{-1}^{1} \frac{|x|}{x}dx = \int_{-1}^{0} \frac{|x|}{x}dx + \int_0^1 \frac{|x|}{x}dx = \int_{-1}^{0} \frac{-x}{x}dx + \int_0^1 \frac{x}{x}dx$$

$$(\because -1 \le x \le 0 \Rightarrow |x| = -x \text{ and } 0 \le x \le 1 \Rightarrow |x| = x)$$

$$= \int_{-1}^{0} -1dx + \int_0^1 1dx = [-x]_{-1}^{0} + [x]_0^1$$

$$= -(0-(-1)) + (1-0) = -1 + 1 = 0$$

(c) First note that $[3x]$ is discontinuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$,

$$\therefore \int_0^1 [3x]\,dx = \int_0^{1/3} [3x]\,dx + \int_{1/3}^{2/3} [3x]dx + \int_{2/3}^{1} [3x]dx$$

$$= \int_0^{1/3} 0\,dx + \int_{1/3}^{2/3} 1\,dx + \int_{2/3}^{1} 2\,dx = 0 + [x]_{1/3}^{2/3} + 2[x]_{2/3}^{1}$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right) + 2\left(1-\frac{2}{3}\right) = \frac{1}{3}+\frac{2}{3} = 1$$

Example-7.11 Evaluate the following definite integrals:

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

Solution:

Let,
$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \text{... (i)}$$

Then, by using property 4*b*, we get

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Example-7.12 Evaluate the following definite integrals:

(a) $\int_0^1 \log\left(\frac{1}{x}-1\right) dx$ (b) $\int_0^{\pi/2} \sin 2x \log(\tan x) dx$

Solution:

(a) Let,
$$I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{...(i)}$$

Then, by using property 4*b*, we get

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx = \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right)^{-1} dx = \int_0^1 -1.\log\left(\frac{1-x}{x}\right) dx = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx = -I$$

$\Rightarrow$ $2I = 0$

$\Rightarrow$ $I = 0$

(b) Let
$$I = \int_0^{\pi/2} \sin 2x \log(\tan x) dx \quad \text{...(i)}$$

Then, by using property 4*b*, we get

Let,
$$I = \int_0^{\pi/2} \sin\left(2\left(\frac{\pi}{2}-x\right)\right) \log\left(\tan\left(\frac{\pi}{2}-x\right)\right) dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log(\cot x) dx = \int_0^{\pi/2} \sin 2x \log\left((\tan x)^{-1}\right) dx$$

$$= \int_0^{\pi/2} \sin 2x(-1)\log(\tan x)dx = \int_0^{\pi/2} \sin 2x \log(\tan x)dx$$

$$= -I \qquad \text{[using (}i\text{)]}$$

$$\Rightarrow \qquad 2I = 0$$

$$\Rightarrow \qquad I = 0$$

Example-7.13 Evaluate the following definite integrals:

$$\int_0^{\pi} \log(1+\cos x)\, dx$$

Solution:

$$I = \int_0^{\pi} \log(1+\cos x)dx \qquad ...(i)$$

Then, by using property 4*b*, we get

$$I = \int_0^{\pi} \log(1+\cos(\pi-x))dx = \int_0^{\pi} \log(1-\cos x)dx \qquad ...(ii)$$

On adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} (\log(1+\cos x)+\log(1-\cos x))dx = \int_0^{\pi} \log\left(1-\cos^2 x\right)dx$$

$$= \int_0^{\pi} \log(\sin^2 x)dx = 2\int_0^{\pi} \log \sin x\, dx$$

$$\Rightarrow \qquad I = \int_0^{\pi} \log \sin x\, dx$$

Let $f(x) = \log \sin x \Rightarrow f(\pi - x) = \log(\sin(\pi - x)) = \log \sin x = f(x)$, therefore, by using property 6, we get

$$I = 2\int_0^{\pi/2} \log \sin x\, dx = 2.\left(-\frac{\pi}{2}\log 2\right) = -\pi \log 2.$$

7.8 Partial Derivatives

Definition of Partial Derivative

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a partial derivative. The operation of finding the partial derivative of a function of more than one independent variables is called **Partial Differentiation**.

The symbols $\partial/\partial x$, $\partial/\partial y$ etc., are used to denote such differentiations and the expressions $\partial u/\partial x$, $\partial u/\partial y$ etc., are respectively called partial differential coefficients of u with respect to x and y.

If $u = f(x, y, z)$ the partial differential coefficient of u with respect to x i.e., $\partial u/\partial x$ is obtained by differentiating u with respect to x keeping y and z as constants.

Second Order Partial Differential Coefficients

If $u = f(x, y)$ then $\partial u/\partial x$ or f_x and $\partial u/\partial y$ or f_y are themselves function of x and y can be again differentiated partially.

We call $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$ as second order partial derivatives of u and these are respectively denoted by $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$.

NOTE

If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial i.e., $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Example-7.14 Let $f = y^x$.

What is $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2$, $y = 1$?

(a) 0
(b) ln 2
(c) 1
(d) $\frac{1}{\ln 2}$

Solution: (c)

$$f = y^x$$

Treating x as constant, we get

$$\frac{\partial t}{\partial y} = xy^{x-1}$$

Now we treat y as a constant and get,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(y^{x-1}x\right) = y^{x-1} + xy^{x-1}\ln y$$

whose value at $x = 2$ and $y = 1$ is $= 1^{(2-1)}(1 + 2.\ \ln 1) = 1$

Example-7.15 If $z = xy\ln(xy)$, then

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$
(b) $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$
(c) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$
(d) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

Solution: (c)

$$\frac{\partial z}{\partial x} = y\ln(xy) + \frac{xy}{xy}xy$$

$$\frac{\partial z}{\partial x} = y[\ln(xy) + 1] \qquad ...(i)$$

$$\frac{\partial z}{\partial x} = x\ln(xy) + \frac{xy}{xy} \times x$$

$$\frac{\partial z}{\partial x} = x[\ln(xy)+1] \quad ...(ii)$$

Here, $$\boxed{x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}}$$

Homogenous Functions

An expression in which every term is of the same degree is called homogenous function. Thus, $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n$ is a homogenous function of x and y of degree n. This can also be written as,

$$x^n\left\{a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ... + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right\}$$

or $x^n f\left(\frac{y}{x}\right)$, where $f\left(\frac{y}{x}\right)$ is some function of $\frac{y}{x}$.

NOTE

- To test whether a given function $f(x, y)$ is homogenous or not we put tx for x and ty for y in it.
  If we get $f(tx, ty) = t^n f(x, y)$ the function $f(x, y)$ is homogenous of degree n otherwise $f(x, y)$ is not a homogenous function.
- If u is a homogenous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogenous function of x and y each being of degree $(n-1)$.

Euler's Theorem on Homogenous Functions

If u is a homogenous function of x and y of degree n, then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Euler's theorem can be extended to a homogenous function of any number of variables. Thus if $f(x_1, x_2, ... x_n)$ be a homogenous function of $x_1, x_2, ... x_n$ of degree n then, $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + ... + x_n\frac{\partial f}{\partial x_n} = nf$

7.9 Total Derivatives

If $u = f(x, y)$, where $x = \phi_1(t)$ and $y = \phi_2(t)$,

then, $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Here $\frac{du}{dt}$ is called the total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u.

In the same way if $u = f(x, y, z)$ where x, y, z are all functions of some variable t, when

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

This result can be extended to any number of variables.

Corollary-1: If u be a function of x and y, where y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

Corollary-2: If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

and

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

Corollary-3: If x and y are connected by an equation of the form $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{\partial f/\partial x}{\partial f/\partial y}$$

7.10 Maxima and Minima (of function of a single independent variable)

Definitions: A function $f(x)$ is said to be "maximum" at $x = a$, if there exist a positive number δ such that $f(a + h) < f(a)$ for all values of h other than zero, in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to be minimum at $x = a$, if there exists a positive number δ such that $f(a + h) > f(a)$ for all values of h, other than zero, in the interval $(-\delta, \delta)$.

Maximum and Minimum values of a function are also called extreme values or turning values and the points at which they are attained are called points of maxima and minima.

The points at which a function has extreme values are called Turning Points.

7.10.1 Properties of Maxima and Minima

1. At leastone maximum or one minimum must lie between two equal values of a function.
2. Maximum and minimum values must occur alternatively.
3. There may be several maximum or minimum values of same function.
4. A function $y = f(x)$ is maximum at $x = a$, if dy/dx changes sign from +ve to –ve as x passes through a.
5. A function $y = f(x)$ is minimum at $x = a$, if dy/dx changes sign from –ve and +ve as x passes through a.
6. If the sign of dy/dx does not change while x passes through a_1 then y is neither maximum nor minimum at $x = a$.

Conditions for Maximum or Minimum Values

The necessary condition that $f(x)$ should have a maximum or a minimum at $x = a$ is that $f'(a) = 0$.

Definition of Stationary Values

A function $f(x)$ is said to be stationary at $x = a$ if $f'(a) = 0$.

Thus for a function $f(x)$ to be a maximum or minimum at $x = a$ it must stationary at $x = a$.

Sufficient Conditions of Maximum or Minimum Values

There is a maximum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ is negative.

Similarly there is a minimum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ positive.

NOTE: If $f''(a)$ is also equal to zero, then we can show that for a maximum or a minimum of $f(x)$ at $x = a$, we must have $f'''(a) = 0$. Again, if $f^{iv}(a)$ is negative, there will be a maximum at $x = a$ and if $f''(a)$ is positive there will be minimum at $x = a$.

In general if, $f'(a) = f''(a) = f'''(a) = \dots f^{n-1}(a) = 0$ and $f^n(a) \neq 0$ then n must an even integer for maximum or minimum. Also for a maximum $f^n(a)$ must be negative and for a minimum $f^n(a)$ must be positive.

7.10.2 Working rule for Maxima and Minima of $f(x)$

1. Find $f'(x)$ and equate to zero.
2. Solve the resulting equation for x. Let its roots be $a_1, a_2, \dots\dots$ Then $f(x)$ is stationary at $x = a_1, a_2, \dots\dots\dots\dots$ Thus $x = a_1, a_2, \dots\dots\dots$ are the only points at which $f(x)$ can be maximum or a minimum.
3. Find $f''(x)$ and substitute in it by terms $x = a_1, a_2, \dots\dots\dots\dots$
4. If $f''(a_1) = 0$, find $f'''(x)$ put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is –ve, we have maximum at $x = a_1$, if it is positive there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on. Repeat the above process for each root of the equation $f'(x) = 0$.

Example-7.16 Prove that the function $f(x) = ax + b$ is strictly increasing iff $a > 0$.

Solution:

Given: $f(x) = ax + b$, $D_f = R$

Note that f is continuous and differentiable for all $x \in R$.

Differentiating the given function w.r.t. x, we get $f'(x) = a$.

Now the given function is strictly increasing iff $f'(x) > 0$ i.e. iff $a > 0$.

Hence, the given function is strictly increasing for all $x \in R$ iff $a > 0$.

Example-7.17 Prove that the function e^{2x} is strictly increasing on R.

Solution:

Let, $$f(x) = e^{2x},\ D_f = R.$$

Differentiating w.r.t. x, we get

$$f'(x) = e^{2x}.\,2 > 0 \text{ for all } x \in R.$$

$\Rightarrow$ $f(x)$ is strictly increasing on R.

Example-7.18 Prove that $(2/x) + 5$ is a strictly decreasing function.

Solution:

Let, $$f(x) = \frac{2}{x} + 5,\ D_f = R - [0].$$

Differentiate it w.r.t. x, we get $f'(x) = 2.(-1.\,x^{-2}) + 0 = -\dfrac{2}{x^2}$

Since, $x^2 > 0$ for all $x \in R$, $x \neq 0$, therefore,

$f'(x) < 0$ for all $x \in R$, $x \neq 0$, i.e., for all $x \in D_f$

$\Rightarrow$ the given function is strictly decreasing.

Example-7.19 Prove that the function $f(x) = x^3 - 6x^2 + 15x - 18$ is strictly increasing on R.

Solution:

Given, $f(x) = x^3 - 6x^2 + 15x - 18,\ D_f = R$

Differentiate it w.r.t. x we get, $f'(x) = 3x^2 - 6.2x + 15.1 = 3(x^2 - 4x + 5)$

$= 3[(x-2)^2 + 1] \geq 3 \quad (\because\ (x-2)^2 \geq 0 \text{ for all } x \in R)$

$\Rightarrow$ $f'(x) > 0$ for all $x \in R$

$\Rightarrow f(x)$ is strictly increasing function for all $x \in R$.

Example-7.20 Find the intervals in which the following functions are strictly increasing or strictly decreasing

(a) $f(x) = 10 - 6x - 2x^2$ (b) $f(x) = x^2 - 12x^2 + 36x + 17$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

Solution:

(a) Given, $f(x) = 10 - 6x - 2x^2,\ D_f = R$

Differentiating it w.r.t. x, we get

$$f'(x) = 0 - 6.1 - 2.\ 2x = -6 - 4x = -4\left(x + \frac{3}{2}\right)$$

Putting, $f'(x) = 0$,

we get, $\dfrac{20 \pm \sqrt{400 - 156}}{2} = 0$

$\Rightarrow$ $x + \dfrac{3}{2} = 0 \Rightarrow x = -\dfrac{3}{2}$

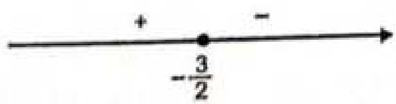
So there is only one critical point which is $x = -\dfrac{3}{2}$

Plotting this critical point on the number line we get the following picture

So the critical point divides the real number line into two regions which are $x \in \left(-\infty, -\dfrac{3}{2}\right)$ and $x \in \left(-\dfrac{3}{2}, \infty\right)$.

Now we find $f'(0) = -6$ which is negative and so the region $x \in \left(-\dfrac{3}{2}, \infty\right)$ (which contains $x = 0$) is the region where the function is strictly decreasing.

Therefore in the other region i.e. $x \in \left(-\infty, -\dfrac{3}{2}\right)$ is the region in which the function is strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



(b) Given, $f(x) = x^3 - 12x^2 + 36x + 17,\ D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 24x + 36$$
$$= 3(x^2 - 8x + 12) = 3(x-2)(x-6)$$

Putting, $f'(x) = 0$ i.e. $3(x-2)(x-6) = 0$

$\Rightarrow$ $(x-2)(x-6) = 0$

$\Rightarrow$ $x = 2$ or $x = 6$ are the two critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, 2)$ and $x \in (2, 6)$ and $x \in (6, \infty)$.

Now we find $f'(0) = 3(0-2)(0-6) = +36$ which is positive and so in the region $x \in (-\infty, 2)$ (which contains $x = 0$), the function is strictly increasing.

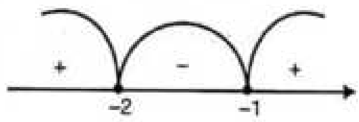
Therefore in the next region i.e. $x \in (2, 6)$, the function is strictly decreasing and in the next region $x \in (6, \infty)$, the function is again strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

So the final region in which the function strictly increasing is $x \in (-\infty, 2) \cup (6, \infty)$ and the region in which the function is strictly decreasing is $x \in (2, 6)$.

(c) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1,\ D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = -6x^2 - 18x - 12$$
$$= -6(x^2 + 3x + 2)$$
$$= -6(x+2)(x+1)$$



Putting, $f'(x) = 0$ i.e. $-6(x+2)(x+1) = 0$

$\Rightarrow$ $(x+2)(x+1) = 0$

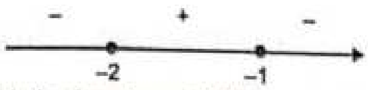
$\Rightarrow$ $x = -2$ and $x = -1$ are the critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, -2)$ and $x \in (-2, -1)$ and $x \in (-1, \infty)$.

Now we find $f'(0) = -6(0+2)(0+1) = -12$ which is negative and so in the region $x \in (-1, \infty)$. (which contains $x = 0$), the function is strictly decreasing.

Therefore in the next adjacent region on the left i.e. $x \in (-2, -1)$, the function is strictly increasing and in the next adjacent region on the left $x \in (-\infty, -2)$, the function is again strictly decreasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



So the final region in which the function strictly increasing is $x \in (-2, -1)$ and the region in which the function is strictly decreasing is $x \in (-\infty, -2) \cup (-1, \infty)$.

Example-7.21 The distance between the origin and the point nearest to it on the surface $z^2 = 1 + xy$ is

(a) 1
(b) $\frac{\sqrt{3}}{2}$
(c) $\sqrt{3}$
(d) 2

Solution: (a)

Let the point be (x, y, z) on surface $z^2 = 1 + xy$

$$\text{Distance from origin} = l = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

$$l = \sqrt{x^2 + y^2 + 1 + xy} \quad [\text{since } z^2 = 1 + xy \text{ is given}]$$

This distance is shortest when l is minimum we need to find minima of $x^2 + y^2 + 1 + xy$

Let, $$u = x^2 + y^2 + 1 + xy$$

$$\frac{\partial u}{\partial x} = 2x + y$$

$$\frac{\partial u}{\partial y} = 2y + z$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$\Rightarrow$ $$2x + y = 0 \quad \text{and} \quad 2y + x = 0$$

Solving simultaneously, we get

$$x = 0 \quad \text{and} \quad y = 0$$

is the only solution and so (0, 0) is the only stationary point.

Now, $$r = \frac{\partial^2 u}{\partial x^2} = 2$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2$$

Since, $$rt = 2 \times 2 = 4 > s^2 = 1$$

We have case 1, i.e. either a maximum or minimum exists at (0, 0)

Now, since $r = 2 > 0$, so it is a minima at (0, 0).

Now at $x = 0, \quad y = 0, \quad z = \sqrt{1+xy} = \sqrt{1+0} = 1$

So, the point nearest to the origin on surface $z^2 = 1 + xy$ is (0, 0, 1)

The distance, $$l = \sqrt{0^2 + 0^2 + 1^2} = 1$$

So, correct answer is choice (a).

7.10.3 Maxima and Minima (of function of a two independent variable)

Definitions: Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively.

The $f(a, b)$ is said to be maximum and a minimum value of $f(x, y)$ according as $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k. Positive negative, provided both of than are not equal to zero.

Necessary Conditions

The necessary conditions that $f(x, y)$ should have a maximum or minimum at $x = a$, $y = b$ is that

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} = 0 \text{ and } \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} = 0$$

Sufficient condition for Maxima or Minima

Let,
$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}};\ s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}};\ t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$$

Case-1: $f(x, y)$ will have a maximum or a minimum at $x = a$, $y = b$, if $rt > s^2$. Further, $f(x, y)$ is maximum or minimum according as r in negative or positive.

Case-2: $f(x, y)$ will have neither maximum or minimum at $x = a$, $y = b$ if $rt < s^2$. i.e. $x = a$, $y = b$ is a saddle point.

Case 3: If $rt = s^2$ this case is doubtful case and further investigation is needed to determine whetherr $f(x, y)$ is a maximum or minimum at $x = a$, $y = b$ or not.

Example-7.22 Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$.

The optimal value of $f(x, y)$

(a) is a minimum equal to 10/3
(b) is a maximum equal to 10/3
(c) is a minimum equal to 8/3
(d) is a maximum equal to 8/3

Solution: (a)

$$f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$$

$$\frac{\partial f}{\partial x} = 8x - 8$$

$$\frac{\partial f}{\partial y} = 12y - 4y$$

Putting,
$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$8x - 8 = 0 \text{ and } 12y - 4y = 0$$

Given,
$$x = 1 \text{ and } y = \frac{1}{3}$$

$\left(1, \frac{1}{3}\right)$ is the only stationary point.

$$r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8$$

$$s = \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{\left(1, \frac{1}{3}\right)} = 0$$

$$t = \left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1, \frac{1}{3}\right)} = 12$$

Since, $rt = 8 \times 12 = 96$

$s^2 = 0$

Since, $rt > s^2$

we have either a maxima or minima at $\left(1, \dfrac{1}{3}\right)$

also since, $r = \left[\dfrac{\partial^2 f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8 > 0$, the point $\left(1, \dfrac{1}{3}\right)$ is a point of minima.

The minimum value is

$$f\left(1, \frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{3^2} - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$$

So the optimal value of $f(x, y)$ is a minimum equal to $\dfrac{10}{3}$.

Summary

- If $f(x)$ and $f(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = f(a) = 0$ then $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \dfrac{f'(x)}{\phi'(x)}$ provided, the latter limit exists, finite or infinite.
- Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a if $\underset{x \to a}{\text{Limit}} f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a if $\underset{x \to a+0}{\text{Limit}} f(x)$ exists and is equal to $f(a)$.
- Continuity is a necessary but not a sufficient condition for the existence of a finite derivative. i.e. differentiability $\Rightarrow$ continuity. But continuity $\not\Rightarrow$ differentiability
- Role's theorem will not hold good:
  - (*i*) If $f(x)$ is discontinuous at some point in the interval $a < x < b$
  - (*ii*) If $f'(x)$ does not exist at some point in the interval $a \le x \le b$ or
  - (*iii*) If $f(a) \ne f(b)$
- If a function $f(x)$ is (*i*) continuous in closed interval $a \le x \le b$ and (*ii*) differentiable in open interval (a, b) i.e., $a < x < b$, then there exist at least one value c of x lying in the open interval $a < x < b$ such that, $\dfrac{f(b) - f(a)}{b - a} = f'(c)$

- If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial i.e., $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.
- To test whether a given function $f(x, y)$ is homogenous or not we put tx for x and ty for y in it. If we get $f(tx, ty) = t^n f(x, y)$ the function $f(x, y)$ is homogenous of degree n otherwise $f(x, y)$ is not a homogenous function.
- If u is a homogenous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogenous function of x and y each being of degree $(n-1)$.
- If $f''(a)$ is also equal to zero, then we can show that for a maximum or a minimum of $f(x)$ at $x = a$, we must have $f'''(a) = 0$. Again, if $f^{iv}(a)$ is negative, there will be a maximum at $x = a$ and if $f'(a)$ is positive there will be minimum at $x = a$.

Student's Assignment

Q.1 Evaluate $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi a}{2x}}$

(a) $e^{2/\pi}$ (b) $2/e^{\pi}$
(c) $e^{\pi/2}$ (d) $e^{-2/\pi}$

Q.2 Check whether the following integrals and its values are true or false?

1. $\int_0^{\pi} \cos^{2n+1} x dx = 0$
2. $\int_0^{\pi} \cos^{2n} x dx = 2\int_0^{\pi/2} \cos^{2n} x dx$
3. $\int_0^{\pi} \sin^{2n} x dx = \int_0^{\pi/2} \sin^{2n} x dx$

For all values of n

(a) 1 and 3 are true (b) 1 and 2 are true
(c) 3 only true (d) 2 and 3 are true

Q.3 If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$ and $I_n + n(n-1) I_{n-2} = n(\pi/2)^{n-1}$ then find I_2

(a) $2-\pi$ (b) $\pi-2$
(c) $\pi/2$ (d) $(\pi/2)^2$

Q.4 Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(a) 0 (b) 1
(c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x^2}{\sqrt{x^2-1}}$

Q.5 $\int \log x \, dx =$

(a) $(x \log x - 1)$ (b) $\log x - x$
(c) $x(\log x - 1)$ (d) None of these

Q.6 If $0 < x < 1$ then

(a) $\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1} x} < 1$

(b) $\frac{\sqrt{1-x}}{\sqrt{1+x}} > \frac{\log(1+x)}{\sin^{-1} x} > 1$

(c) $\frac{\sqrt{1-x}}{\sqrt{1+x}} > \frac{\log(1+x)}{\sin^{-1} x} < 1$

(d) $\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1} x} > 1$

Q.7 If $u = f(y/x)$ then

(a) $\frac{x\,du}{dx} - \frac{y\,du}{dy} = 0$

(b) $\frac{x\,du}{dx} + \frac{y\,du}{dy} = 0$

(c) $\frac{x\,du}{dx} + \frac{y\,du}{dy} = 2u$

(d) $\frac{x\,du}{dx} + \frac{y\,du}{dy} = 1$

Q.8 The function $f(x, y) = xy(a - x - y)$, $a > 0$ attains extreme values at $\left(\frac{a}{3}, \frac{a}{3}\right)$ and its value is

(a) minimum value, $\frac{a^2}{3}$

(b) minimum value, $\frac{a^3}{27}$

(c) maximum value, $\frac{a^3}{3}$

(d) maximum value, $\frac{a^3}{27}$

Q.9 Match List-I (Sequence) with List-II (Generating function) and select the correct answer using the codes given below the lists:

List-I	List-II
A. 1	1. $x(1 + x)(1 - x)^{-3}$
B. $(-1)^n$	2. $\frac{1}{1-x}$
C. n^2	3. $x(1 - x)^{-2}$
D. n	4. $\frac{1}{1+x}$

Codes:

	A	B	C	D
(a)	1	2	4	3
(b)	2	1	3	4
(c)	4	3	2	1
(d)	2	4	1	3

Q.10 The Domain of the function $\frac{1}{\sqrt{|x| - x}}$ is

(a) $(-\infty, 0)$ (b) $(0, \infty)$

(c) $(0, x)$ (d) $(0, 1)$

Q.11 The function is continuous in [0, 1], such that $f(0) = -1$, $f(1/2) = 1$ and $f(1) = -1$.
We can conclude that

(a) f attains the value zero at least twice in [0, 1]

(b) f attains the value zero exactly once in [0, 1]

(c) f is non-zero in [0, 1]

(d) f attains the value zero exactly twice in [0, 1]

Q.12 The function $f(x) = |x| - |x + 1|$

(a) is less than 1, for all x

(b) equal $f(-x)$

(c) equals $1 - f(1/x)$

(d) none of the above

Q.13 $f(x)$ and $g(x)$ are two functions differentiable in [0, 1] such that $f(0) = 2$; $g(0) = 0$; $f(1) = 6$; and $g(1) = 2$. Then there must exist a constant C in

(a) (0, 1), such that $f'(c) = 2g'(c)$

(b) [0, 1], such that $f'(c) = g'(c)$

(c) (0, 1), such that $2f'(c) = g'(c)$

(d) [0, 1], such that $2f'(c) = g'(c)$

Answer Key:

1. (a)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)
6. (a)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (a)
11. (a)	**12.** (d)	**13.** (a)		

Student's Assignments Explanations

1. (a)

$$\lim_{x \to a}\left(2 - \frac{a}{x}\right)^{\tan\left(\frac{\pi a}{2x}\right)}$$

Let, $f(x) = 2 - \frac{a}{x}$ and $g(x) = \tan\frac{\pi x}{2a}$

Since $\lim_{x \to a} f(x) = 1$

and $\lim_{x \to a} g(x) = \tan\frac{\pi}{2} = \infty$

$$\lim_{x \to a}[f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)\{f(x) - 1\}}$$

$$= e^{\lim_{x \to a} \tan\frac{\pi a}{2a}\left(1 - \frac{a}{x}\right)}$$

Now, $\lim_{x \to a} \tan\frac{\pi a}{2x}\left(1 - \frac{a}{x}\right) = \lim_{x \to a}\frac{1 - \frac{a}{x}}{\cot\frac{\pi a}{2x}}$

$$= \lim_{x\to a} \frac{1-\frac{a}{x}}{\tan\left(\frac{\pi}{2}-\frac{\pi a}{2x}\right)}$$

$$= \lim_{x\to a} \frac{1-\frac{a}{x}}{\tan\left[\frac{\pi}{2}\left(1-\frac{a}{x}\right)\right]}$$

$$= \frac{2}{\pi}\lim_{x\to a} \frac{\frac{\pi}{2}\left[1-\frac{a}{x}\right]}{\tan\frac{\pi}{2}\left[1-\frac{a}{x}\right]}$$

$$= \frac{2}{\pi}\lim_{x\to a} \frac{t}{\tan t}$$

Where, $t = \frac{\pi}{2}\left(1-\frac{a}{x}\right)$ as $x \to a$, $t \to 0$

$$= \frac{2}{\pi}\cdot 1 = \frac{2}{\pi}$$

Now the required limit is

$$e^{\lim_{x\to a} \tan\frac{\pi a}{2x}\left(1-\frac{a}{x}\right)} = e^{2/\pi}$$

2. (b)

1. $\int_0^{\pi}\cos^{2n+1}x\,dx = 0$

$$f(x) = \cos^{2n+1}x$$

Since, $f(\pi-x) = \cos^{2n+1}(\pi-x)$

$$= (-\cos x)^{2n+1} = -f(x)$$

$$\therefore \int_0^{\pi}\cos^{2n+1}x\,dx = 0$$

1 is true.

2. $\int_0^{\pi}\cos^{2n}x\,dx = 2\int_0^{\pi/2}\cos^{2n}x\,dx$

$$f(x) = \cos^{2n}(x)$$
$$f(\pi-x) = \cos^{2n}(\pi-x)$$
$$= (-\cos x)^{2n}$$
$$= \cos^{2n}x = f(x)$$

$$\therefore \int_0^{\pi}\cos^{2n}x\,dx = 2\int_0^{\pi/2}\cos^{2n}x\,dx$$

2 is true.

3. $\int_0^{\pi}\sin^{2n}x\,dx = \int_0^{\pi/2}\sin^{2n}x\,dx$

$$f(x) = \sin^{2n}(x)$$
$$f(\pi-x) = (\sin(\pi-x))^{2n}$$
$$= (\sin x)^{2n} = f(x)$$

$$\Rightarrow \int_0^{\pi}\sin^{2n}x\,dx = 2\int_0^{\pi/2}\sin^{2n}x\,dx$$

$\therefore$ 3 is false.

3. (b)

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Put, $n = 2$

$$I_2 + 2(2-1)I_0 = 2\left(\frac{\pi}{2}\right)^{2-1}$$

$$\therefore \quad I_2 = \pi - 2I_0$$

$$I_0 = \int_0^{\pi/2}\dot{x}\sin x dx$$

$$= \int_0^{\pi/2}\sin x\,dx$$

$$= [-\cos x]_0^{\pi/2}$$

$$= -0 + 1 = 1$$

$$\therefore \quad I_2 = \pi - 2(1) = \pi - 2$$

4. (b)

Let, $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

putting $x = \tan\theta$

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta$$

$$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{2}{2} = 1$$

6. (c)
Use integration by parts.

7. (a)
Consider the functions
$f(x) = \log(1+x)$ and $g(x) = \sin^{-1} x$
Both of these are continuous and differentiable for $0 \le x \le 1$.
$\therefore$ These satisfy the conditions of the Cauchy's Mean value theorem in interval $[0, x]$ when $x < 1$

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(C)}{g'(C)}$$

$$\frac{\log(1+x)-\log 1}{\sin^{-1}x-\sin^{-1}0} = \frac{1}{1+C}\sqrt{1-C^2}$$

$$\frac{\log(1+x)}{\sin^{-1}x} = \frac{\sqrt{1-C}}{\sqrt{1+C}}$$

Since $0 < C < x < 1$ we have

$$\frac{\sqrt{1-C}}{\sqrt{1+C}} < 1 \text{ and } \frac{\sqrt{1-C}}{\sqrt{1+C}} > \frac{\sqrt{1-x}}{\sqrt{1+x}}$$

$$\text{Hence } \frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\sqrt{1-C}}{\sqrt{1+C}} < 1$$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$$

8. (b)

$$u = f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\frac{-y}{x^2}$$

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x \cdot f'\left(\frac{y}{x}\right)\cdot\frac{-y}{x^2} + y \cdot f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$
$$= 0$$

9. (d)

$$f(x, y) = xy(a-x-y)$$

$$\frac{\partial f}{\partial x} = xy(-1) + (a-x-y)y$$
$$= ay - 2xy - y^2$$

Similarly, $\frac{\partial f}{\partial y} = ax - 2xy - x^2$

Putting $\frac{\partial f}{\partial x} = 0$, we get

$$ay - 2xy - y^2 = 0$$
$$\Rightarrow y = 0 \text{ or } 2x + y = a$$

and putting $\frac{\partial f}{\partial y} = 0$, we get

$$ax - 2xy - x^2 = 0$$
$$\Rightarrow x = 0 \text{ or } 2y + x = a$$

Solving $2x + y = a$
and $2y + x = a$

we get $x = \frac{a}{3}, y = \frac{a}{3}$

$\therefore$ The function attains extreme values at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$\Rightarrow r \text{ value at } \left(\frac{a}{3}, \frac{a}{3}\right) = \frac{-2a}{3}$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y$$

$$\Rightarrow s \text{ value at } \left(\frac{a}{3}, \frac{a}{3}\right) = -\frac{a}{3}$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

$$\Rightarrow t \text{ value at } \left(\frac{a}{3}, \frac{a}{3}\right) = \frac{-2a}{3}$$

$$r \times t = \frac{-2a}{3} \times \frac{-2a}{3} = \frac{4a^2}{9}$$

$$s^2 = \frac{a^2}{9}$$

$\Rightarrow \quad rt > s^2$

This means that at $\left(\frac{a}{3}, \frac{a}{3}\right)$, this function reaches a maximum or minimum.

However, since $r = \frac{-2a}{3} < 0$, it is maximum

$$f_{max} = \frac{a}{3} \times \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}$$

10. (a)

f is defined if $|x| - x > 0$ i.e., $|x| > x$

If $x \geq 0$, $|x| = x$ so, $x > x$, which has no feasible solution

If $x < 0$, $|x| = -x$, so $-x > x$, which has solution $x < 0$.

$\therefore$ Domain $= (-\infty, 0)$

11. (a)

Whenever $f(a)$ and $f(b)$ are of different signs, f has an odd number of roots (at least one root) between a and b.

$\therefore$ There exists at least one root between 0 and $\frac{1}{2}$ and at least 1 more root between $\frac{1}{2}$ and 1

i.e. $f(x)$ has at least 2 zeroes between 0 and 1.

12. (d)

$$f(x) = |x| - |x+1|$$
$$f(-2) = |-2| - |-2+1| = 1$$

$\therefore$ (a) is false.

$$f(2) = |2| - |2+1| = -1$$
$$f(2) \neq f(-2).$$

So (b) is false.

$$f\left(\frac{1}{2}\right) = -1$$

$$1 - f\left(\frac{1}{2}\right) = 1 - (-1) = 2$$

$$f(2) = -1$$

So, $\quad f(2) \neq 1 - f\left(\frac{1}{2}\right)$

(c) is false.

So "none of the above" is the answer.

13. (a)

Consider the function $\phi(x) = f(x) - 2g(x)$

$$\phi(0) = \phi(1) = 2,$$

So, $f(x)$ satisfies the conditions of Roll's theorem in [0, 1] so,

$$\phi'(x) = f'(x) - 2g'(x)$$

has atleast one 0 at (in (0, 1)

i.e., $\quad \phi'(C) = 0$

$\Rightarrow \quad f'(C) - 2g'(C) = 0$

$\Rightarrow \quad f'(C) = 2g'(C)$