Thoroughly Revised and Updated

Engineering Mathematics

GATE 2018
and ESE 2018 Prelims

Comprehensive Theory with Solved Examples
including **Previous Solved Questions of**
GATE (2003-2017) and ESE-Prelims 2017

Note: Syllabus of ESE Mains [illegible] also covered

MADE EASY
[illegible] Publications

MADE EASY Publications

[illegible]

[illegible]

[illegible]

[illegible]

Engineering Mathematics for GATE 2018 and ESE 2018 Prelims

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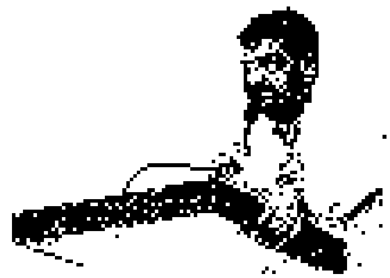
Preface



B. Singh (Ex. IES)

Over the period of time the GATE and ESE examination have become more challenging due to increasing number of candidates. Though every candidate has ability to succeed but competitive environment, in-depth knowledge, quality guidance and good source of study is required to achieve high level goals.

The new edition of **Engineering Mathematics for GATE 2018 and ESE 2018 Prelims** has been fully revised, updated and edited. The whole book has been divided into topicwise sections.

I have true desire to serve student community by way of providing good source of study and quality guidance. I hope this book will be proved an important tool to succeed in GATE and ESE examination. Any suggestions from the readers for the improvement of this book are most welcome.

B. Singh (Ex. IES)
Chairman and Managing Director
MADE EASY Group

SYLLABUS

GATE and ESE Prelims: Civil Engineering

GATE and ESE Prelims: Mechanical Engineering

GATE and ESE Prelims: Electrical Engineering

Structural Engineering ESE Mains

GATE and ESE Prelims: Electronics Engineering

GATE: Instrumentation Engineering

GATE: Computer Science & IT Engineering

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1.1 Introduction

[illegible]

1.2 Algebra of Matrices

1.2.1 Definition of Matrix

[illegible]

[illegible]

[illegible]

1.2.2 Special Types of Matrices

1. **Square Matrix** [illegible] DIAGONAL ELEMENTS [illegible] PRINCIPLE DIAGONAL [illegible]

Example [illegible]

NOTE

A square sub-matrix of a square matrix A is called a "principle sub-matrix" if its diagonal elements are also the diagonal elements of the matrix A. So $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$ is a principle sub-matrix of the matrix A given above, but $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$ is not.

2. **Diagonal Matrix:** A square matrix in which all off diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} & \text{if } i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix

The above matrix can also be written as $A = \text{diag}\,[3, 5, 9]$

Properties of Diagonal Matrix:

$\text{diag}\,[x, y, z] + \text{diag}\,[p, q, r] = \text{diag}\,[x + p, y + q, z + r]$

$\text{diag}\,[x, y, z] \times \text{diag}\,[p, q, r] = \text{diag}\,[xp, yq, zr]$

$(\text{diag}\,[x, y, z])^{-1} = \text{diag}\,[1/x, 1/y, 1/z]$

$(\text{diag}\,[x, y, z])^T = \text{diag}\,[x, y, z]$

$(\text{diag}\,[x, y, z])^n = \text{diag}\,[x^n, y^n, z^n]$

Eigen value of $\text{diag}\,[x, y, z] = x, y$ and z.

Determinant of $\text{diag}\,[x, y, z] = |\text{diag}\,[x, y, z]| = xyz$

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} = k & \text{if } i = j \end{cases}$

Example: $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} = 1 & \text{if } i = j \end{cases}$

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Properties of Identity Matrix:

(i) I is identity element for multiplication, so it is called multiplicative identity.

(ii) $AI = IA = A$

(iii) $I^n = I$

(iv) $I^{-1} = I$

(v) $|I| = 1$

5. **Null Matrix:** The $m \times n$ matrix whose elements are all zero is called null matrix.
Null matrix is denoted by O. Null matrix need not be square. i.e. $a_{ij} = 0\ \forall\ i, j$

Example: $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Properties of Null Matrix

(i) $A + O = O + A = A$
So, O is additive identity.

(ii) $A + (-A) = O$

6. **Upper Triangular Matrix:** An square matrix whose all elements below the principal diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$.
It is denoted by U.

The diagonal and upper off diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i > j \\ a_{ij} & \text{if } i \le j \end{cases}$

Example: $U = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 7 & 8 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Lower Triangular Matrix:** A square matrix is said to be lower triangular if all the upper off diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off diagonal elements may or may not be zero.

$\begin{cases} a_{ij} = 0 & \text{if } i < j \\ a_{ij} & \text{if } i \ge j \end{cases}$

It is denoted by L.

Example: $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 5 \end{bmatrix}$

8. **Idempotent Matrix:** A matrix A is called idempotent iff $A^2 = A$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are examples of idempotent matrices.

9. **Involuntary Matrix:** A matrix A is called involutory iff $A^2 = I$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is involutory. Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is involutory since $A^2 = I$.

10. **Nilpotent Matrix:** A matrix A is said to be nilpotent of class x or index x iff $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$.

Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$, $A^2 \neq 0$, but $A^3 = 0$

11. **Singular matrix:** If the determinant of a square matrix is 0, then matrix is called as singular matrix.

[illegible]

If determinant is not zero, then matrix is known as non-singular matrix.

If matrix is singular then its inverse doesn't exist.

1.2.3 Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if:

1. They are of same size.
2. The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example: Let [illegible]

Then [illegible]

$\Rightarrow$ [illegible]

1.2.4 Addition of Matrices

Two matrices A and B are compatible for addition only if they both have exactly the same size say $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$,

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Properties of Matrix Addition:

1. Matrix addition is commutative $A + B = B + A$
2. Matrix addition is associative $(A + B) + C = A + (B + C)$
3. Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every $m \times n$ matrix A.
4. Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.

   Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.

   — Matrix $-A$ is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order $m \times n$.
5. Cancellation laws holds good in case of addition of matrices, which is $X = -A$.

   $A + X = B \Rightarrow X = B - A$

   $X + A = X + B \Rightarrow A = B$
6. The equation $A + X = O$ has a unique solution in the set of all $m \times n$ matrices.

1.2.5 Substraction of Two Matrices

[illegible]

[illegible]

NOTE: [illegible]

1.2.6 Multiplication of a Matrix by a Scalar

[illegible]

[illegible]

Properties of Multiplication of a Matrix by a Scalar:

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]

1.2.7 Multiplication of Two Matrices

[illegible] of rows in B.

[illegible] and we write $C = AB$.

Properties of Matrix Multiplication:

1. [illegible]
2. [illegible]
3. [illegible] $= AB + AC$
4. [illegible] matrix. For example, [illegible]
5. [illegible]
6. [illegible]

   $AB = AC \Rightarrow B = C$ [illegible]

   $BA = CA \Rightarrow B = C$ [illegible]

1.2.8 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by Tr(A).

Thus if $A = [a_{ij}]_{n \times n}$, then $Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$

Let $A = \begin{bmatrix} [illegible] & 2 & 3 \\ 8 & [illegible] & [illegible] \\ -1 & [illegible] & [illegible] \end{bmatrix}$

Then Trace (A) = Tr (A) = [illegible]

Properties of Trace of a Matrix:

Let A and B be two square matrices of order n and λ be a scalar. Then,

1. $tr(\lambda A) = \lambda\, tr\, A$
2. $tr(A + B) = tr\, A + tr\, B$
3. $tr(AB) = tr(BA)$ [illegible]

1.2.9 Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then [illegible] rows into columns and columns into rows is called the transpose of A and is denoted by A′ or A^T.

[illegible] $A = \begin{bmatrix} [illegible] & 3 \\ 2 & [illegible] \\ [illegible] & [illegible] \end{bmatrix}$ then $A' = A^T =$ [illegible]

[illegible] $B =$ [illegible]

Then $B^T =$ [illegible]

Properties of Transpose of a Matrix:

Let A^T and B^T be transposes of A and B respectively, then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$, k being any complex number
4. $(AB)^T = B^T A^T$
5. $(ABC)^T = C^T B^T A^T$

1.2.10 Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

Example: If A = [illegible]

$\bar{A} =$ [illegible]

Properties of Conjugate of a Matrix:

If $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively. Then,

1. $\overline{(\bar{A})} = A$
2. $\overline{(A+B)} = \bar{A}+\bar{B}$
3. $\overline{(kA)} = \bar{k}\bar{A}$, k is being any complex number
4. $\overline{(AB)} = \bar{A}\bar{B}$, A and B being conformable for multiplication
5. $\bar{A} = A$ iff A is real matrix
   $\bar{A} = -A$ iff A is purely imaginary matrix

1.2.11 Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ} or A^{*}.

$A^{\theta} = (\bar{A})^T$. It is also called conjugate transpose of A.

Example: If $A = \begin{bmatrix} 2-i & 3-i \\ -4 & 1-i \end{bmatrix}$

To find A^{θ} we first find $\bar{A} = \begin{bmatrix} 2+i & 3+i \\ -4 & 1+i \end{bmatrix}$

Then $A^{\theta} = (\bar{A})^T = \begin{bmatrix} 2+i & -4 \\ 3+i & 1+i \end{bmatrix}$

Some properties: If A^{θ} & B^{θ} be the transposed conjugates of A and B respectively then

1. $(A^{\theta})^{\theta} = A$
2. $(A+B)^{\theta} = A^{\theta}+B^{\theta}$
3. $(kA)^{\theta} = \bar{k}A^{\theta}$, $k \rightarrow$ complex number
4. $(AB)^{\theta} = B^{\theta}A^{\theta}$

1.2.12 Classification of Real Matrices

Real matrices can be classified into the following three types based on the relationship between A^T and A.

1. [illegible]
1. Symmetric Matrices ($A^T = A$)
2. Skew Symmetric Matrices ($A^T = -A$)
3. Orthogonal Matrices ($A^T = A^{-1}$ or $AA^T = I$)

1. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ element is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i & j.

   In a symmetric matrix $A^T = A$.

   Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix, since $A^T = A$.

   Note: For any matrix A,

   (a) AA^T is always a symmetric matrix.

(iii) $\frac{A+A^T}{2}$ is always symmetric matrix.

Note: If A and B are symmetric, then

(a) $A + B$ and $A - B$ are also symmetric

(b) AB, BA need not be symmetric

2. **Skew Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A i.e. $a_{ij} = -a_{ji}$

In a skew symmetric matrix $A^T = -A$

A skew symmetric matrix must have all 0's in the diagonal.

Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Note: For any matrix A, the matrix $\frac{A-A^T}{2}$ is always skew symmetric.

3. **Orthogonal Matrix:** A square matrix A is said to be orthogonal if

$A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if $AA^T = I = A^TA$

Example: The identity matrix is orthogonal since $I^T = I^{-1} = I$.

Note: Since A is orthogonal matrix:

$$AA^T = I$$

$$\Rightarrow \quad |AA^T| = |I| = 1$$

$$\Rightarrow \quad |A||A^T| = 1$$

$$\Rightarrow \quad |A|^2 = 1$$

$$|A| = \pm 1$$

So the determinant of an orthogonal matrix always has a modulus of 1.

1.2.13 Classification of Complex Matrices

Complex matrices can be similarly classified into the following three types based on A and its conjugate transpose (A^θ):

1. Hermitian Matrix ($A^\theta = A$)
2. Skew-Hermitian Matrix ($A^\theta = -A$)
3. Unitary Matrix ($A^\theta = A^{-1}$ or $AA^\theta = I$)

1. **Hermitian Matrix:** A necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

Example: $A = \begin{bmatrix} a & b-ic \\ b+ic & d \end{bmatrix}$ is a Hermitian matrix.

2. **Skew-Hermitian Matrix:** A necessary and sufficient condition for a matrix to be skew-Hermitian is $A^\theta = -A$.

Example: $A = \begin{bmatrix} 0 & -2+i \\ 2+i & i \end{bmatrix}$ is skew-Hermitian.

3. **Unitary Matrix:** A square matrix A is said to be unitary if:

$$A^\theta = A^{-1}$$

[illegible]

[illegible]

$$[illegible]$$

Example: $A = [illegible]$ [illegible]

1.3 Determinants

1.3.1 Definition

[illegible] $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ [illegible]

[illegible]

1.3.2 Minors, Cofactors and Adjoint

[illegible] $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

[illegible] a_{ij} [illegible] M_{ij}

Example: [illegible] $= \begin{vmatrix} [illegible] \end{vmatrix} = M_{[illegible]}$

[illegible] $= \begin{vmatrix} [illegible] \end{vmatrix} = M_{[illegible]}$

1.3.3 Cofactors

[illegible] M_{ij} [illegible] a_{ij} [illegible]

Example: [illegible]

[illegible] $= (-1)^{[illegible]} M_{[illegible]} = \begin{vmatrix} [illegible] \end{vmatrix}$

[illegible] $= \begin{vmatrix} [illegible] \end{vmatrix}$

[illegible]

Example: II $A = \begin{bmatrix} 1 & 2 & [illegible] \\ [illegible] & 6 & 1 \\ [illegible] & [illegible] & 2 \end{bmatrix}$

[illegible] $\text{adj}(A) = \begin{bmatrix} 2 & [illegible] & 12 \\ 4 & [illegible] & [illegible] \\ 2 & -1 & [illegible] \end{bmatrix}$

$|A| = (1 \times [illegible]) + (2 \times 4) - ([illegible] \times 12)$

$= [illegible]$

$= (2 \times 2) - (0 - 4) + (2 \times 5) = [illegible]$

1.3.4 Adjoint

[illegible] known as adjoint of matrix 'A'.

$$A_{ij} = C_{ij}$$

$$\text{Adj}(A) = [C_{ij}]^T$$

Properties of adjoint matrix [illegible]

1. [illegible] $|A| \times I$

2. $A^{-1} = \frac{1}{|A|} \, \text{adj}\, A$

1.3.5 Determinant of order n

[illegible]

[illegible]

$$A = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & \ldots\ldots & a_{2n} \\ a_{n1} & a_{n2} \;\; \ldots & a_{nn} \end{vmatrix}$$

[illegible]

[illegible] $\sum A_{ij} [illegible] = \sum [illegible] = \sum [illegible] = \sum [illegible]$

[illegible]

1.3.6 Properties of Determinants

1. [illegible] rows and columns are interchanged [illegible]

2. [illegible] $|A| = 0$

   [illegible]

   [illegible] $|A| = 0$

3. [illegible] by -1

4. If all elements of one row (or one column) of a determinant are multiplied by same number k, the value of new determinant is k times the value of given determinant.

5. If A is n-rowed square matrix and k be any scalar, then $|kA| = k^n |A|$

6. (a) In a determinant the sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of the same row (or column) is equal to the determinant value.

   (b) In a determinant the sum of the products of the elements of any row (or column) with the cofactors of the corresponding elements of any other row or column is zero.

Example:

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $a_1A_1 + b_1B_1 + c_1C_1 = |A|$

$a_1A_2 + b_1B_2 + c_1C_2 = 0$

$a_1A_3 + b_1B_3 + c_1C_3 = 0$

$a_2A_2 + b_2B_2 + c_2C_2 = |A|$

$a_3A_1 + b_3B_1 + c_3C_1 = 0$ etc.

where A_1, B_1, C_1, ... are cofactors of elements a_1, b_1, c_1 in A.

7. If to the elements of a row (or column) of a determinant are added k times the corresponding elements of another row (or column) the value of the determinant thus obtained is equal to the value of original determinant.

$$\text{e.g. } A \xrightarrow{R_2 \to R_2 + kR_1} B \text{ then } |A| = |B|$$

$$\text{e.g. } A \xrightarrow{C_1 \to C_1 + kC_2} B \text{ then } |A| = |B|$$

8. $|AB| = |A| \times |B|$ and based on this we can prove the following:

(a) $|A^n| = (|A|)^n$

(b) $|A^{-1}| = \dfrac{1}{|A|}$

Proof of a: $|A^n| = |A \times A \times A \ldots n \text{ times}|$

$= |A| \times |A| \times |A| \ldots n \text{ times}$

$= (|A|)^n$

Proof of b: $AA^{-1} = I$

$|AA^{-1}| = |I| = 1$

Now since, $|AA^{-1}| = |A|\,|A^{-1}|$

$\therefore \quad |A|\,|A^{-1}| = 1$

$$|A^{-1}| = \frac{1}{|A|}$$

9. Using the fact that $A(\text{Adj } A) = |A|\,I$, the following results can be proved for $A_{n \times n}$

(a) $|\text{Adj } A| = |A|^{n-1}$

(b) $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2}$

1.4 Inverse of Matrix

The inverse of a matrix A exists if A is non-singular (i.e., $|A| \neq 0$) and is given by the formula

$$A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

Inverse of a matrix, if exists, is unique.

1.4.1 Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be any $n \times n$ matrix. The transpose B^T of the matrix $B = [A_{ij}]_{n \times n}$, where A_{ij} denotes the cofactor of element a_{ij}, is called the adjoint of matrix A and is denoted by symbol Adj A.

$$\therefore \quad \text{Adj}(A) = [\text{cof}(A)]^T$$

Properties of Adjoint:

If A be any n-rowed square matrix, then $(\text{Adj } A)\, A = A\,(\text{Adj } A) = |A|\, I_n$

where I_n is the $n \times n$ identity matrix.

1.4.2 Properties of Inverse

1. $AA^{-1} = A^{-1}A = I$
2. A and B are inverse of each other iff $AB = BA = I$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
5. If A be an $n \times n$ non-singular matrix, then $(A')^{-1} = (A^{-1})'$.
6. If A be an $n \times n$ non-singular matrix, then $(A^{-1})^{\theta} = (A^{\theta})^{-1}$.
7. For a 2×2 matrix there is a short cut formula for inverse as given below

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1.5 Rank of A Matrix

Rank is defined for any matrix $A_{m \times n}$ (need not be square).

Some Important concepts:

1. **Submatrix of a Matrix:** Suppose A is any matrix of the type $m \times n$. Then a matrix obtained by leaving some rows and some columns from A is called a submatrix of A.
2. **Rank of a Matrix:** A number r is said to be the rank of a matrix A if it possesses the following properties:
   (a) There is at least one square submatrix of A of order r whose determinant is not equal to zero.
   (b) If the matrix A contains any square submatrix of order $(r + 1)$ and above, then the determinant of every such matrix should be zero.
   Thus together property (a) and (b) give the definition of the rank of a matrix as the order of the largest non-zero minor.

Note:

(a) The rank of a null matrix is zero, i.e., $r(A) = 0$ if every element of the matrix is zero.

(b) The rank of a matrix is ≥ 1 if there is at least one element in the matrix which is not equal to zero.

(c) The rank of transpose of a matrix is same as that of original matrix i.e. $\rho(A^T) = \rho(A)$.

(e) [illegible]

(f) [illegible]

(g) [illegible]

(g) [illegible]

(h) [illegible]

Echelon form: [illegible]

1. [illegible]

2. [illegible]

[illegible]

(i) [illegible]

(j) [illegible]

1.5.1 Elementary Matrices

[illegible]

1.5.2 Results

1. [illegible]
2. [illegible]
3. [illegible]
4. Rank [illegible]
5. [illegible]

1.6 Sub-Spaces : Basis and Dimension

1.6.1 Introduction

[illegible]

1.6.2 Vector

Definition: [illegible]

column matrix. [illegible]

The n-vector space: [illegible]

1.6.3 Linearly dependent and Linearly Independent Sets of Vectors

1.6.3.1 Linear dependence and independence of vector

[illegible]

1. [illegible]
2. [illegible] independent.

1.6.3.2 Dependence / Independency of vector by matrix method

1. [illegible] independent.
2. [illegible] dependent.

1.6.3.3 A vector as a Linear Combination of a Set of Vectors

Definition: [illegible]

Example: [illegible]

Example:

1. [illegible]
2. [illegible]

Solution:

1. Consider the relation [illegible]

   [illegible]

   [illegible]

2. [illegible]

   [illegible]

1.6.4 Some properties of linearly Independent and Dependent Sets of Vectors

[illegible]

1. [illegible]

[illegible]

[illegible]

$$\alpha_1 \xi_1 = \bar{0}$$

2. If a set $\{\xi_1, \ldots, \xi_r\}$ is linearly independent and $\{\xi_1, \ldots, \xi_r, \eta\}$ is linearly dependent, then η is a linear combination of the set $\{\xi_1, \ldots, \xi_r\}$.
3. Every superset of a linearly dependent set is linearly dependent.
4. It may also be easily shown that every subset of a linearly independent set is linearly independent.

1.6.5 Subspaces of an N-vector space V_n

Definition: Any non-empty set, S, of vectors of $V_n(F)$ is called a subspace of $V_n(F)$, if when

1. ξ_1, ξ_2 are any two members of S, then $\xi_1 + \xi_2$ is also a member of S and
2. ξ is a member of S and k is a scalar, then $k\xi$ is also a member of S.

Briefly we may say that a set S of vectors of $V_n(F)$ is a subspace of $V_n(F)$ if it is closed with the compositions of addition and multiplication with scalars.

Every subspace of V_n contains the zero vector, being the product of any vector with the scalar zero.

Example: $S = \{[a_1, a_2, 0]\}$ [illegible] V_3 [illegible] a_1, a_2 being variable.

1.6.5.1 Construction of Subspaces

Theorem 1: The set S of all linear combinations of any given set of vectors of $V_n(F)$ is a subspace of V_n.

Def. 1 A Subspace Spanned by a Set of Vectors. A subspace which consists of all linear combinations of any given set of vectors is said to be spanned by the given set of vectors.

Def. 2. Basis of a Subspace. A set of vectors is said to be a basis of a subspace if

1. the subspace is spanned by the set, and
2. the set is linearly independent.

It is important to notice that the set of vectors

$e_1 = [1, 0, 0, \ldots, 0], e_2 = [0, 1, 0, \ldots, 0], \ldots, e_n = [0, 0, \ldots, 0, 1]$

is a basis of the vector space V_n, for if

$x_1e_1 + x_2e_2 + \ldots + x_ne_n = 0$

then $x_1 = x_2 = \ldots = x_n = 0$ so that the set is linearly independent and any vector

$$\xi = [a_1, a_2, \ldots, a_n]$$

of V_n is expressible as

$$\xi = a_1e_1 + a_2e_2 + \ldots + a_ne_n$$

Theorem 2: A basis of a subspace S can always be selected from a set of vectors which span S.

Let $\{\xi_1, \ldots, \xi_r\}$

be a set of vectors which span a subspace S.

If this set is linearly independent, then it is already a basis. In case it is linearly dependent, then some member of the set is a linear combination of the preceding members. Deleting this member, we obtain another set which also spans S.

Continuing in this manner, we shall ultimately arrive at [illegible] basis of S.

NOTE: It may also be shown that every subspace S of V_n possesses a basis and that the number of vectors in every basis of S is the same.

1.6.6 Row and column spaces of a matrix. Row and column ranks of a Matrix

[illegible]

[illegible]

[illegible] Row space of [illegible] matrix A.

[illegible]

[illegible] Column space of [illegible] matrix A.

[illegible] Row rank [illegible] Column rank of the matrix.

Theorem 1: [illegible]

[illegible] column rank of a matrix.

1.6.6.1 Equality of row rank, column rank and rank

Theorem 2: [illegible]

Theorem 3: [illegible]

Corollary 1: [illegible]

Corollary 2: [illegible]

1.6.6.2 Connection between Rank and Span

[illegible] $X_1, X_2, X_3, \ldots X_n$ [illegible]

Example: Check [illegible] $[1\ 2\ -1]$, [illegible] span [illegible]

Solution:

Step 1: [illegible] $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & ? & 5 \end{bmatrix}$

Step 2: Find its rank

Since $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & ? & 5 \end{vmatrix}$ [illegible]

$= 0$

So rank $= 2$

$\therefore$ The vectors are [illegible]

Example: Check if the vectors $[1\ 2\ 3]$, $[4\ 5\ 6]$ and $[7\ 8\ 9]$ span [illegible]

Solution:

Since $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= 0$$

So, $\text{rank} \neq 3$

Since $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

$\therefore$ rank $= 2$

So the vectors $[1\ 2\ 3]$, $[4\ 5\ 6]$ and $[7\ 8\ 9]$ span a subspace of R^3 but do not span R^3.

1.6.7 Orthogonality of Vectors

1. Two vectors X_1 and X_2 are orthogonal if they are non-zero and the dot product $X_1^T X_2 = 0$.

   Example: The vectors [illegible] are orthogonal since

   $$[\text{illegible}] \times [\text{illegible}] = 0$$

   or [illegible] $= 0$

   Example: The vectors $[1\ 2]$ and $[-2\ 1]$ are orthogonal since

   $$[1\ 2] \times [-2\ 1] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T [-2\ 1]$$

   $$= (1 \times -2) + (2 \times 1)$$

   $$= 0$$

   Example: The vectors $[1\ 2\ 3]$ and [illegible] are not orthogonal since

   $$(1 \times [\text{illegible}]) + (2 \times [\text{illegible}]) + (3 \times [\text{illegible}]) = [\text{illegible}] \neq 0$$

2. Three vectors X_1, X_2 and X_3 are orthogonal if each is non-zero and they are pairwise orthogonal.

   i.e. $X_1^T X_2 = 0$

   and $X_2^T X_3 = 0$

   and $X_3^T X_1 = 0$

   Example: The vectors $[1\ 0\ 0]$, $[0\ 1\ 0]$ and $[0\ 0\ 1]$ are orthogonal since

   $$[1\ 0\ 0]^T [0\ 1\ 0] = [0\ 0\ 0]$$

   and $[0\ 1\ 0]^T [0\ 0\ 1] = [0\ 0\ 0]$

   and $[1\ 0\ 0]^T [0\ 0\ 1] = [0\ 0\ 0]$

3. If n vectors $X_1, X_2, \ldots, X_n$, each of which is in R^n, are orthogonal, then they are surely linearly independent and hence span R^n and they also form a basis for R^n.

   Example: The vectors $[1\ 0\ 0]$, $[0\ 1\ 0]$ and $[0\ 0\ 1]$ are orthogonal and hence are linearly independent and hence span R^3. They form a basis for R^3.

   The vectors $[1\ -2]$, [illegible] are orthogonal and hence are linearly independent and span R^2 and form a basis of R^2.

4. The set of vectors $X_1, X_2, X_3, \ldots X_n$ are called orthonormal if they are
(a) orthogonal and
(b) each vector has unit length.
The two conditions may be written as

$$X_i^T X_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Any set of orthogonal vectors X can be converted to a set of orthonormal vectors by dividing each vector by its length (i.e. normalized by its length). (Find each norm $\|X_i\|$)

Example: The set {(1, 2, 1), (2, 1, −4) and (3, −2, 1)} are three linearly independent vectors for R^3, since they are pairwise orthogonal and hence are linearly independent for a set of basis of R^3.
To convert this set to an orthonormal basis of R^3 we need to divide each vector by its length.

$$\|X_1\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\|X_2\| = \sqrt{4+1+16} = \sqrt{21}$$

$$\|X_3\| = \sqrt{9+4+1} = \sqrt{14}$$

So an orthonormal basis of R^3 is $\left\{\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right) \text{ and } \left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)\right\}$

1.7 System of Equations

1.7.1 Homogenous Linear Equations

Suppose,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots \ldots + a_{2n}x_n &= 0 \\ \ldots \quad \ldots\ldots \quad \ldots\ldots & \\ \ldots \quad \ldots\ldots \quad \ldots\ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n &= 0 \end{aligned} \qquad \ldots (i)$$

is a system of m homogeneous equations in n unknowns $x_1, x_2, \ldots x_n$.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & & & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

where $A = [a_{ij}]_{m \times n}$, $X = [x_1, x_2, \ldots, x_n]'$ and O is a null matrix. Then obviously we can write the system of equations in the form of a single matrix equation

$$AX = O \qquad \ldots(ii)$$

The matrix A is called coefficient matrix of the system of equations (ii).

The set $X = [x_1 = 0, x_2 = 0, \ldots, x_n = 0]$ i.e. $X = O$ is always a solution of equation (ii).

But in general there may be infinite number of solutions of equation (ii).

Again suppose X_1 and X_2 are two solutions of (ii). Then their linear combination, $k_1X_1 + k_2X_2$ where k_1 and k_2 are any arbitrary numbers, is also a solution of (ii).

1.7.1.1 Important Results

The number of linearly independent solutions of the homogeneous linear system of m equations in n variables, $AX = O$, is $(n - r)$, where r is rank of matrix A.

$n - r$ is also called the number of parameters in the $AX = O$ solution.

1.7.1.2 Some important results regarding nature of solutions of equation AX = O

Suppose there are m equations in n unknowns. Then the coefficient matrix A will be of the type $m \times n$. Let r be rank of matrix A. Obviously r cannot be greater than n. Therefore we have either $r = n$ or $r < n$.

Case 1: Inconsistency: This case is not possible in homogeneous systems since such a system is always consistent. Since the trivial solution $X = [0, 0, \ldots, 0]'$ always exists for a homogeneous system.

Case 2: Consistent Unique Solution: If $r = n$, the equation $AX = O$ will have only the trivial, unique solution $X = [0, 0, \ldots, 0]'$.

Note: That $r = n \Rightarrow |A| \neq 0$ i.e. A is non-singular.

Case 3: Consistent Infinite Solutions: If $r < n$, we shall have $n - r$ linearly independent solutions. Any linear combination of these $n - r$ solutions will also be a solution of $AX = O$. Thus in this case, the equation $AX = O$ will have infinite solutions.

Note: That $r < n \Rightarrow |A| = 0$ i.e. A is a singular matrix.

1.7.2 System of Linear Non-Homogeneous Equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots\ldots\ldots \quad \ldots\ldots \\ \ldots\ldots\ldots \quad \ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \qquad \ldots(i)$$

For a system of m non-homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$

If we write
$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

are $m \times n$, $n \times 1$ and $m \times 1$ matrices respectively, the above equations can be written in the form of a single matrix equation $AX = B$.

"Any set of values of $x_1, x_2, \ldots, x_n$ which simultaneously satisfy all these equation is called a solution of the system. When the system of equations has one or more solutions, the equations are said to be consistent otherwise they are said to be inconsistent."

$$\text{The matrix } [A|B] = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} & & b_1 \\ a_{21} & a_{22} & a_{2n} & b_2 \\ \ldots & \ldots & & \\ a_{m1} & a_{m2} & a_{mn} & b_m \end{bmatrix}$$

is called augmented matrix of the given system of equations.

Condition for Consistency: The system of equations $AX = B$ is consistent i.e. it possesses a solution, if the coefficient matrix A and the augmented matrix $[A|B]$ are of the same rank i.e. $\rho(A) = \rho(A|B)$.

Case 1: Inconsistency: If $\rho(A) \neq \rho(A|B)$ the system $AX = B$ has no solution. We say that such a system is inconsistent.

Case 2 and 3 consistent systems: Now when $\rho(A) = \rho(A|B) = r$. The system is solvable (has at least one solution). We say that the rank of the system is r. Now two cases arise.

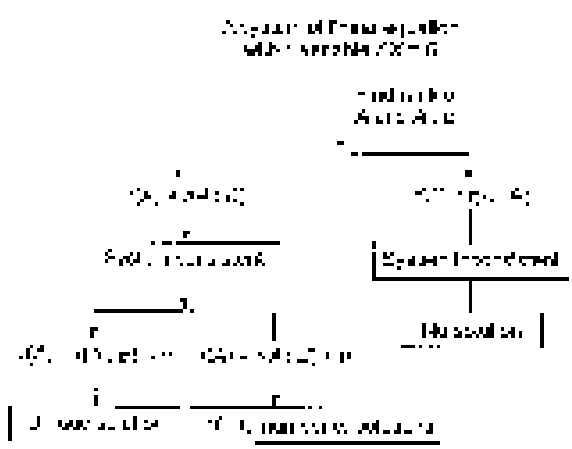
Case 2: Consistent Unique Solution. If $\rho(A) = \rho(A|B) = n$ (where n is the number of unknowns/variables in the system), then the system is said to be consistent and also has a unique solution.

Case 3: Consistent Infinite solution: If $\rho(A) = \rho(A|B) < n$, then the system is consistent and has infinite number of solutions.

In summary we can say the following:

1. If $\rho(A) \neq \rho(A|B)$, the system is inconsistent (no solution).
2. If $\rho(A) = \rho(A|B) = n$, the system is consistent (unique solution).
3. If $\rho(A) = \rho(A|B) < n$, the system is consistent (infinite solutions).

The method of solving a system of equations by reducing [illegible] to Echelon form and then solving is called Gauss-Elimination method. In this process the matrix is reduced to Echelon form and then by counting the number of non-zero rows in [illegible] we get the rank of A.



1.7.3 Homogenous Polynomial

The quadratic forms are defined as a homogenous polynomial of degree [illegible] in any number of variables.

Two variables: $ax^2 + 2hxy + by^2$ [illegible]

Three variables: $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ [illegible]

[illegible]

Quadratic form can be expressed as a product of matrices.

$$Q(x) = X^T A X$$

$$= [x_1\ x_2\ x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{23} + a_{32})x_2x_3 + (a_{31} + a_{13})x_3x_1$$

From here, coefficient of $x_1x_2 = a_{12}$ and a_{21}

coefficient of $x_2x_3 = a_{23}$ and a_{32}

coefficient of $x_1x_3 = a_{13}$ and a_{31}

In general, the coefficient of $x_ix_j = a_{ij}$ and a_{ji}

Let $c_{ij} = c_{ji} = \frac{1}{2}(a_{ij} + a_{ji})$ = new coefficient of x_ix_j

$$C = \frac{1}{2}(A + A^T) = \text{symmetric matrix}$$

In the coefficient matrix C in quadratic form is always symmetric [illegible] without loss of generality [illegible]

$$X^TAX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1$$

Matrix A is [illegible]

1.8 Eigenvalues and Eigenvectors

Let $A = [a_{ij}]_{n \times n}$ be any n-rowed square matrix and λ is a scalar. The equation $AX = \lambda X$ is called eigenvalue problem. We want to find the values of λ satisfying the eigenvalue problem, and those non-zero solutions of X are called as the **eigen vectors** of A. The corresponding λ values are called **eigen values** of A.

1.8.1 Definitions

The matrix $A - \lambda I$ is called **characteristic matrix** of A, where I is the unit matrix of order n. Also the determinant

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

which is a ordinary polynomial in λ of degree n is called "**characteristic polynomial of A**". The equation $|A - \lambda I| = 0$ is called "**characteristic equation of A**".

Characteristic Roots: The roots of the characteristic equation are called characteristic roots or [illegible] values or latent roots or proper values or eigen values of the matrix A. The set of all eigenvalues of A is called the "**spectrum of A**".

[illegible]

Characteristic Vectors: [illegible]

1.8.2 Some Results Regarding Characteristic Roots and Characteristic Vectors

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]
5. [illegible]
6. [illegible]
7. [illegible]
8. [illegible]
9. [illegible]
10. [illegible]

1.8.3 Process of Finding the Eigenvalues and Eigenvectors of a Matrix

[illegible]

[illegible]

1.8.4 Properties of Eigen Values

1. [illegible]
2. [illegible]

   [illegible]
3. [illegible]
4. [illegible]
5. [illegible]
6. [illegible]

7. Sum of eigen values = Trace of A = Sum of diagonal elements.
8. Product of eigen values = $|A|$ (i.e. if at least one eigen value is zero then A is singular).
9. In a triangular and diagonal matrix, eigen values are diagonal elements themselves.
10. Similar matrices have same eigen values. Two matrices A and B are said to be similar if there exists a non singular matrix P such that $B = P^{-1}AP$.
11. If A and B are two matrices of same order then the matrix AB and BA will have same characteristic roots.

1.8.5 The Cayley-Hamilton Theorem

This theorem is an interesting and useful theorem and alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree. We give below the statement of the theorem without proof.

Statement of the Theorem: Every square matrix satisfies its own characteristic equation.

This means that, if $c_0\lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$c_0A^n + c_1A^{n-1} + \dots + c_{n-1}A + c_nI = 0$ ...(i)

NOTE: When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by c_nI to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n.

Also 0 in the R.H.S. of (i) is a null matrix of order n.

1.8.5.1 Finding Inverse of a Matrix by using Cayley-Hamilton Theorem

Example: Find A^{-1} by Cayley-Hamilton theorem, if

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda - 10 = 0$$

By Cayley-Hamilton Theorem

$$A^2 - 3A - 10I = 0$$

$$\Rightarrow \quad I = \frac{1}{10}[A^2 - 3A]$$

Premultiplying by A^{-1} we get

$$A^{-1} = \frac{1}{10}[A - 3I] = \frac{1}{10}\left(\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{10}\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{10} \\ \frac{2}{5} & -\frac{1}{10} \end{bmatrix}$$

1.8.5.2 Finding Higher Powers of a Matrix in Terms of its Lower Powers

Example: If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, express A^5 as a linear polynomial in A.

Characteristic equation is

$$\lambda^2 - 3\lambda - 10 = 0$$

By Cayley Hamilton theorem,

$$A^2 - 3A - 10I = 0$$

$$\Rightarrow \quad A^2 = 3A + 10I$$

If A is $n \times n$ matrix, any power of A can be written as linear combination of maximum degree $n - 1$.

Here since A is 2×2 we can write any power of A as a polynomial of degree 1, i.e. as linear polynomial of A as shown below

$$A^3 = 3A^2 + 10A \quad \ldots(1)$$

$$A^4 = 3A^3 + 10A^2 \quad \ldots(2)$$

Substituting (1) again in (1) we get

$$A^3 = 3(3A + 10I) + 10A = 9A + 30I + 10A$$

Now $\quad A^3 = 19A + 30I \quad \ldots(3)$

Again we substitute equation (3) in equation (2) to get

$$A^4 = 3(19A + 30I) + 10(3A + 10I) = 57A + 90I + 30A + 100I$$

Now $\quad A^4 = 87A + 190I \quad \ldots(4)$

Again substituting equation (4) in equation (3) we get

$$A^5 = 3(87A + 190I) + 10(19A + 30I) = 451A + 870I$$

Which is the desired result.

1.8.5.3 Expressing Any Matrix Polynomial in A of size n × n as a Polynomial of Degree n − 1 in A by using Cayley Hamilton Theorem

Example: We want to express a polynomial of a 2 × 2 Matrix as a linear polynomial in A.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Step 1: Find $|A - \lambda I|$ which is the characteristic equation of A.

In this case

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}$$

$$= (2 - \lambda)(3 - \lambda) + 1$$

$$= \lambda^2 - 5\lambda + 7$$

So, the characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\quad \lambda^2 - 5\lambda + 7 = 0 \quad \ldots(1)$

Step 2: By Cayley Hamilton theorem, matrix A satisfies the equation (1). Therefore substituting $\lambda = A$ in (1) we get

$$A^2 - 5A + 7I = 0$$

$$\Rightarrow \quad A^2 = 5A - 7I \quad \ldots(2)$$

Step 3: [illegible]

[illegible]

1.8.6 Similar Matrices

Two matrices A and B are said to be similar if there exists a non-singular matrix P such that $B = P^{-1}AP$.

1.8.6.1 Properties of Similar Matrices

1. A is always similar to A.
   Proof: [illegible]
2. If A is similar to B then B is similar to A.
   Proof: [illegible]
   [illegible]
   So B is similar to A.
3. If A is similar to B and B is similar to C then A is similar to C.
   Proof: [illegible] ...(i)
   [illegible] ...(ii)
   [illegible]
   [illegible]
4. [illegible]
5. Similar matrices have the same eigenvalues.

1.8.7 Diagonalisation of a Matrix

[illegible]

$$[illegible]$$

where D is a diagonal matrix.

Condition for a Matrix to be Diagonalisable:

1. [illegible]
2. [illegible]
   [illegible]

[illegible]

Practical application of Diagonalization:

[illegible]

[illegible]

[illegible]

■■■■

Previous GATE and ESE Questions

Q.1 [illegible]

[CE, GATE-2000, 1 mark]

Q.2 [illegible]

[ME, GATE-2003, 2 marks]

Q.3 [illegible]

CS, GATE-2005, 2 marks]

Q.4 [illegible]

[ME, GATE-2006, 1 mark]

Q.5 [illegible]

[ME, GATE-2004, 2 marks]

Q.6 [illegible]

[CS, GATE-2004, 1 mark]

Q.7 [illegible]

[CS, GATE-2004, 2 marks]

Q.8 The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

(a) are 1 and 4 (b) are –1 and 2
(c) are 0 and 5 (d) cannot be determined

[CE, GATE-2004, 2 marks]

Q.9 The sum of the eigen values of the matrix given below is

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

(a) 5 (b) 7
(c) 9 (d) 18

[ME, GATE-2004, 1 mark]

Q.10 Consider the matrices $X_{(4\times3)}$, $Y_{(4\times3)}$ and $P_{(2\times3)}$. The order of $[P(X^TY)^{-1}P^T]^T$ will be

(a) (2×2) (b) (3×3)
(c) (4×3) (d) (3×4)

[CE, GATE-2005, 1 mark]

Q.11 Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \ [AA^T]^{-1} \text{ is}$$

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.12 If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5 6 4] (b) [5 –3 1]
(c) [2 0 –1] (d) [2 –1 1/2]

[EE, GATE-2005, 2 marks]

Q.13 Let, $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$

Then $(a + b) =$

(a) $\frac{7}{20}$ (b) $\frac{3}{20}$
(c) $\frac{19}{60}$ (d) $\frac{11}{20}$

[EC, GATE-2005, 2 marks]

Q.14 Consider a non-homogeneous system of linear equations representing mathematically an over-determined system. Such a system will be

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

[CE, GATE-2005, 1 mark]

Q.15 A is a 3 × 4 real matrix and $Ax = b$ is an inconsistent system of equations. The highest possible rank of A is

(a) 1 (b) 2
(c) 3 (d) 4

[ME, GATE-2005, 1 mark]

Q.16 In the matrix equation $Px = q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector x?

(a) Augmented matrix $[P|q]$ must have the same rank as matrix P
(b) Vector q must have only non-zero elements
(c) Matrix P must be singular
(d) Matrix P must be square

[EC, GATE-2005, 1 mark]

Q.17 Consider the following system of equations in three real variables x_1, x_2 and x_3

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 - 2x_2 + 5x_3 = 2$$
$$-x_1 - 4x_2 + x_3 = 3$$

This system of equations has
(a) no solution
(b) a unique solution
(c) more than one but a finite number of solutions
(d) an infinite number of solutions

[CS, GATE-2005, 2 marks]

Q.18 Which one of the following is an eigenvector of the matrix

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

[ME, GATE-2005, 2 marks]

Q.19 For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen values is equal to -2. Which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

[EE, GATE-2005, 2 marks]

Q.20 Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigen vector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.21 What are the eigen values of the following 2×2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(a) -1 and 1 (b) 1 and 6
(c) 2 and 5 (d) 4 and -1

[CS, GATE-2005, 2 marks]

Q.22 Consider the system of equations $A_{(n \times n)} x_{(n \times 1)} = \lambda_{(n \times 1)}$, where λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigenvalue and its corresponding eigen vector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statements is NOT correct?
(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ having a nontrivial solution, the rank of $(A - \lambda I)$ is less than n.
(b) For matrix A^m, m being a positive integer, (λ_i^m, x_i^m) will be the eigen-pair for all i.
(c) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all i.
(d) If $A^T = A$, then λ_i is real for all i.

[CS, GATE-2005, 2 marks]

Q.23 Multiplication of matrices E and F is G. Matrices E and G are

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[ME, GATE-2005, 2 marks]

Q.24 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I
A. Singular matrix
B. Non-square matrix
C. Real symmetric
D. Orthogonal matrix

List-II
1. Determinant is not defined
2. Determinant is always one
3. Determinant is zero
4. Eigenvalues are always real
5. Eigenvalues are not defined

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	[illegible]

[ME, GATE-2006, 2 marks]

Q.25 The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[EC, GATE-2006, 1 mark]

Q.26 $P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}$, $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}$ are three vectors. An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Q.27 The following vector is linearly dependent upon the solution to the previous problem

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Q.28 Solution for the system defined by the set of equations $4y + 3z = 8$; $2x - z = 2$ and $3x + 2y = 5$ is

(a) $x = 0$; $y = 1$; $z = 4/3$
(b) $x = 0$; $y = 1/2$; $z = 2$
(c) $x = 1$; $y = 1/2$; $z = 2$
(d) non-existent

[CE, GATE-2006, 1 mark]

Q.29 For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to the eigenvector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(a) 2 (b) 4
(c) 6 (d) 8

[EC, GATE-2006, 2 marks]

Q.30 For a given matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$, one of the eigenvalues is 3. The other two eigenvalues are

(a) 2, −5 (b) 3, −5
(c) 2, 5 (d) 3, 5

[CE, GATE-2006, 2 marks]

Q.31 Eigen values of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1. What are the eigen values of the matrix $S^2 = SS$?

(a) 1 and 25 (b) 6 and 4
(c) 5 and 1 (d) 2 and 10

[ME, GATE-2006, 2 marks]

Q.32 The eigen values and the corresponding eigen vectors of a 2 × 2 matrix are given by

Eigen value	Eigen vector
$\lambda_1 = 8$	$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

[EC, GATE-2006, 2 marks]

Q.33 [A] is square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which of the following statements is TRUE?

(a) Both [S] and [D] are symmetric

(b) Both [S] and [D] are skew-symmetric

(c) [S] is skew-symmetric and [D] is symmetric

(d) [S] is symmetric and [D] is skew-symmetric

[CE, GATE-2007, 1 mark]

Q.34 The inverse of the 2 × 2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is

(a) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

[CE, GATE-2007, 2 marks]

Q.35 $X = [x_1, x_2, \ldots, x_n]^T$ is an n-tuple nonzero vector. The $n \times n$ matrix $V = XX^T$

(a) has rank zero (b) has rank 1

(c) is orthogonal (d) has rank n

[EE, GATE-2007, 1 mark]

Q.36 It is given that $X_1, X_2, \ldots, X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the 2M vectors $X_1, X_2, \ldots, X_M$, $-X_1, -X_2, \ldots, -X_M$ is

(a) 2M

(b) M + 1

(c) M

(d) dependent on the choice of $X_1, X_2, \ldots, X_M$

[EE, GATE-2007, 2 marks]

Q.37 Consider the set of (column) vectors defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T = [x_1, x_2, x_3]^T\}$. Which of the following is TRUE?

(a) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a basis for the subspace X.

(b) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X.

(c) X is not a subspace for R^3

(d) None of the above

[CS, GATE-2007, 2 marks]

Q.38 For what values of α and β, the following simultaneous equations have an infinite number of solutions?

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

(a) 2, 7 (b) 3, 8

(c) 8, 3 (d) 7, 2

[CE, GATE-2007, 2 marks]

Q.39 The number of linearly independent eigenvectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) infinite

[ME, GATE-2007, 2 marks]

Q.40 The linear operation L(x) is defined by the cross product $L(x) = b \times x$, where $b = [0\ 1\ 0]^T$ and $x = [x_1\ x_2\ x_3]^T$ are three dimensional vectors. The 3 × 3 matrix M of this operation satisfies

$$L(x) = M\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then the eigen values of M are

(a) 0, +1, −1 (b) 1, −1, 1

(c) i, −i, 1 (d) i, −i, 0

[EC, GATE-2007, 2 marks]

Q.41 The minimum and the maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are −2 and 6, respectively. What is the other eigen value?

(a) 5 (b) 3

(c) 1 (d) −1

[PI, GATE-2007, 1 mark]

Q.42 If a square matrix A is real and symmetric, then the eigenvalues
(a) are always real
(b) are always real and positive
(c) are always real and non-negative
(d) occur in complex conjugate pairs
[ME, GATE-2007, 1 mark]

Statement for Linked Answer Questions 43 and 44

Cayley-Hamilton Theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

Q.43 A satisfies the relation
(a) $A + 3I + 2A^{-1} = 0$
(b) $A^2 + 2A + 2I = 0$
(c) $(A + I)(A + 2I) = 0$
(d) $\exp(A) = 0$
[EE, GATE-2007, 2 marks]

Q.44 A^9 equals
(a) $511A + 510I$
(b) $309A + 104I$
(c) $154A + 155I$
(d) $\exp(9A)$
[EE, GATE-2007, 2 marks]

Q.45 The product of matrices $(PQ)^{-1}P$ is
(a) P^{-1}
(b) Q^{-1}
(c) $P^{-1}Q^{-1}P$
(d) PQP^{-1}
[CE, GATE-2008, 1 mark]

Q.46 A is a $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A^+ = (A^TA)^{-1}A^T$. Then, which one of the following statement is FALSE?
(a) $AA^+A = A$
(b) $(AA^+)^2 = AA^+$
(c) $A^+A = I$
(d) $AA^+A = A^+$
[EC, GATE-2008, 2 marks]

Q.47 If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?
(a) Q will have four linearly independent rows and four linearly independent columns
(b) Q will have four linearly independent rows and five linearly independent columns
(c) QQ^T will be invertible
(d) Q^TQ will be invertible
[EE, GATE-2008, 1 mark]

Q.48 The following simultaneous equations
$x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + kz = 6$
will NOT have a unique solution for k equal to
(a) 0
(b) 5
(c) 6
(d) 7
[CE, GATE-2008, 2 marks]

Q.49 For what value of a, if any, will the following system of equations in x, y and z have a solution?
$2x + 3y = 4$, $x + y + z = 4$, $x + 2y - z = a$
(a) Any real number
(b) 0
(c) 1
(d) There is no such value
[ME, GATE-2008, 2 marks]

Q.50 The system of linear equations
$4x + 2y = 7$
$2x + y = 6$
has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) exactly two distinct solutions
[EC, GATE-2008, 1 mark]

Q.51 The following system of equations
$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 2$$
$$x_1 + 4x_2 + ax_3 = 4$$
has a unique solution. The only possible value(s) for a is/are
(a) 0
(b) either 0 or 1
(c) one of 0, 1 or −1
(d) any real number other than 5
[CS, GATE-2008, 1 mark]

Q.52 The eigenvalues of the matrix $[P] = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$ are
(a) −7 and 8
(b) −6 and 5
(c) 3 and 4
(d) 1 and 2
[CE, GATE-2009, 2 marks]

Q.53 The eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?
(a) 0
(b) 1/2
(c) 1
(d) 2
[ME, GATE-2008, 2 marks]

Q.54 How many of the following matrices have an eigenvalue 1?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(a) one (b) two
(c) three (d) four

[CS, GATE-2008, 2 marks]

Q.55 The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigenvalue equal to 3. The sum of the other two eigenvalues is

(a) p (b) $p-1$
(c) $p-2$ (d) $p-3$

[ME, GATE-2008, 1 mark]

Q.56 All the four entries of the 2×2 matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are nonzero, and one of its eigenvalues is zero. Which of the following statements is true?

(a) $p_{11}p_{22} - p_{12}p_{21} = 1$
(b) $p_{11}p_{22} - p_{12}p_{21} = -1$
(c) $p_{11}p_{22} - p_{12}p_{21} = 0$
(d) $p_{11}p_{22} + p_{12}p_{21} = 0$

[EC, GATE-2008, 1 mark]

Q.57 The characteristic equation of a (3 × 3) matrix P is defined as

$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$

If I denotes identity matrix, then the inverse of matrix P will be

(a) $(P^2 + P + 2I)$ (b) $(P^2 + P + I)$
(c) $-(P^2 + P + I)$ (d) $-(P^2 + P + 2I)$

[EE, GATE-2008, 1 mark]

Q.58 A square matrix B is skew-symmetric if

(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

[CE, GATE-2009, 1 mark]

Q.59 For a matrix $[M] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$. The value of x is given by

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$
(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

[ME, GATE-2009, 1 mark]

Q.60 The trace and determinant of a 2 × 2 matrix are known to be −2 and −35 respectively. Its eigenvalues are

(a) −30 and −5 (b) −37 and −1
(c) −7 and 5 (d) 17.5 and −2

[EE, GATE-2009, 1 mark]

Q.61 The eigenvalues of the following matrix are

$$\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) $3, 3+5j, 6-j$ (b) $-6+5j, 3+j, 3-j$
(c) $3+j, 3-j, 5+j$ (d) $3, -1+3j, -1-3j$

[EC, GATE-2009, 2 marks]

Q.62 The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is

(a) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

[CE, GATE-2010, 2 marks]

Q.63 For the set of equations

$x_1 + 2x_2 + x_3 + 4x_4 = 2$
$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$

The following statement is true

(a) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exists
(b) There are no solutions
(c) A unique non-trivial solution exists
(d) Multiple non-trivial solutions exist

[EE, GATE-2010, 2 marks]

Q.64 One of the eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is

(a) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (b) $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$

(c) $\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

[ME, GATE-2010, 2 marks]

Q.65 An eigenvector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $[-1 \quad 1 \quad 1]^T$ (b) $[1 \quad 2 \quad 1]^T$

(c) $[1 \quad -1 \quad 2]^T$ (d) $[2 \quad 1 \quad -1]^T$

[EE, GATE-2010, 2 marks]

Q.66 The eigenvalues of a skew-symmetric matrix are

(a) always zero

(b) always pure imaginary

(c) either zero or pure imaginary

(d) always real [EC, GATE-2010, 1 mark]

Q.67 Consider the following matrix

$$A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$$

If the eigenvalues of A are 4 and 8, then

(a) $x = 4, y = 10$ (b) $x = 5, y = 8$

(c) $x = -3, y = 9$ (d) $x = -4, y = 10$

[CS, GATE-2010, 2 marks]

Q.68 Consider the following system of equations

$2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$

This system has

(a) a unique solution

(b) no solution

(c) infinite number of solutions

(d) five solutions [ME, GATE-2011, 2 marks]

Q.69 The system of equations

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$

has NO solution for values of λ and μ given by

(a) $\lambda = 6, \mu = 20$ (b) $\lambda = 6, \mu \neq 20$

(c) $\lambda \neq 6, \mu = 20$ (d) $\lambda \neq 6, \mu \neq 20$

[EC, GATE-2011, 2 marks]

Q.70 Eigen values of a real symmetric matrix are always

(a) positive (b) negative

(c) real (d) complex

[ME, GATE-2011, 1 mark]

Q.71 Consider the matrix as given below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

Which one of the following options provides the CORRECT values of the eigen values of the matrix?

(a) 1, 4, 3 (b) 3, 7, 3

(c) 7, 3, 2 (d) 1, 2, 3

[CS, GATE-2011, 2 marks]

Q.72 The eigen values of the matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are

(a) -2.42 and 6.86 (b) 3.48 and 13.53

(c) 4.70 and 6.86 (d) 6.86 and 9.50

[EE, GATE-2012, 2 marks]

Q.73 $x + 2y + z = 4$

$2x + y + 2z = 5$

$x - y + z = 1$

The system of algebraic equations given above has

(a) A unique solution of $x = 1, y = 1$ and $z = 1$

(b) only the two solutions of ($x = 1, y = 1, z = 1$) and ($x = 2, y = 1, z = 0$)

(c) infinite number of solutions

(d) no feasible solution

[ME, GATE-2012, 2 marks]

Q.74 Let A be the 2×2 matrix with elements

$$a_{11} = a_{12} = a_{21} = +1 \text{ and } a_{22} = -1$$

Then the eigenvalues of the matrix A^{19} are

(a) 1024 and -1024

(b) $1024\sqrt{2}$ and $-1024\sqrt{2}$

(c) $4\sqrt{2}$ and $-4\sqrt{2}$

(d) $512\sqrt{2}$ and $-512\sqrt{2}$

[CS, GATE-2012, 1 mark]

Q.75 For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, ONE of the normalized eigen vectors is given as

(a) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(c) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

[ME, GATE-2012, 2 marks]

Q.76 Given that

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the value of A^3 is

(a) $15A + 12I$ (b) $19A + 30I$
(c) $17A + 15I$ (d) $17A + 21I$

[EC, EE, IN GATE 2012, 2 marks]

Q.77 There are three matrices $P(4 \times 2)$, $Q(2 \times 4)$ and $R(4 \times 1)$. The minimum of multiplication required to compute the matrix PQR is

[CE, GATE-2013, 1 Mark]

Q.78 Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that determinant $(I_m + AB)$ = determinant $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

(a) 2 (b) 5
(c) 8 (d) 16

[EC, GATE-2013, 2 Marks]

Q.79 Which one of the following does NOT equal $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$?

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

[CS, GATE-2013, 1 Mark]

Q.80 The dimension of the null space of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ is

(a) 0 (b) 1
(c) 2 (d) 3

[IN, GATE-2013, 1 mark]

Q.81 Choose the CORRECT set of functions, which are linearly dependent.

(a) $\sin x$, $\sin^2 x$ and $\cos^2 x$
(b) $\cos x$, $\sin x$ and $\tan x$
(c) $\cos 2x$, $\sin^2 x$ and $\cos^2 x$
(d) $\cos 2x$, $\sin x$ and $\cos x$

[ME, GATE-2013, 1 Mark]

Q.82 The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) no solution
(b) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
(c) non-zero unique solution
(d) multiple solutions

[EE, GATE-2013, 1 Mark]

Q.83 One pair of eigenvectors corresponding to the two eigen values of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$

[IN, GATE-2013, 2 marks]

Q.84 The eigen values of a symmetric matrix are all

(a) complex with non-zero positive imaginary part
(b) complex with non-zero negative imaginary part
(c) real
(d) pure imaginary

[ME, GATE-2013, 1 Mark]

Q.85 A matrix has eigen values −1 and −2. The corresponding eigen vectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\-2\end{bmatrix}$ respectively. The matrix is

(a) $\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\-2 & -4\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}$ (d) $\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}$

[EE, GATE-2013, 2 Marks]

Q.86 The minimum eigenvalue of the following matrix is

$$\begin{bmatrix}3 & 5 & 2\\5 & 12 & 7\\2 & 7 & 5\end{bmatrix}$$

(a) 0 (b) 1

(c) 2 (d) 3

[EC, GATE-2013, 1 Mark]

Q.87 Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$, $[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$ and $[F]_{5\times1}$ are given. Matrices [B] and [E] are symmetric.

Following statements are made with respect to these matrices.

1. Matrix product $[F]^T[C]^T[B][C][F]$ is a scalar.
2. Matrix product $[D]^T[F][D]$ is always symmetric.

With reference to above statements, which of the following applies?

(a) Statement 1 is true but 2 is false
(b) Statement 1 is false but 2 is true
(c) Both the statements are true
(d) Both the statements are false

[CE, GATE-2004, 1 Mark]

Q.88 Given the matrices $J = \begin{bmatrix}3 & 2 & 1\\2 & 4 & 2\\1 & 2 & 6\end{bmatrix}$ and $K = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, the product K^TJK is ______.

[CE, GATE-2014 : 1 Mark]

Q.89 With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates: $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; and $(x_3, y_3) = (4, 3)$. The area of the triangle is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$

(c) $\frac{4}{5}$ (d) $\frac{5}{2}$

[CE, GATE-2014 : 1 Mark]

Q.90 Which one of the following equations is a correct identity for arbitrary 3 × 3 real matrices P, Q and R?

(a) $P(Q + R) = PQ + RP$
(b) $(P - Q)^2 = P^2 - 2PQ + Q^2$
(c) $\det(P + Q) = \det P + \det Q$
(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

[ME, GATE-2014 : 1 Mark]

Q.91 Which one of the following statements is true for all real symmetric matrices?

(a) All the eigenvalues are real.
(b) All the eigenvalues are positive.
(c) All the eigenvalues are distinct.
(d) Sum of all the eigenvalues is zero.

[CE, GATE-2014 : 1 Mark]

Q.92 For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$
(b) $(cM)^T = c(M)^T$
(c) $(M + N)^T = M^T + N^T$
(d) $MN = NM$

[EE, GATE-2014 : 1 Mark]

Q.93 Which one of the following statements is NOT true for a square matrix A?

(a) If A is upper triangular, the eigenvalues of A are the diagonal elements of it
(b) If A is real symmetric, the eigenvalues of A are always real and positive
(c) If A is real, the eigenvalues of A and A^T are always the same
(d) If all the principal minors of A are positive, all the eigenvalues of A are also positive

[EC, GATE-2014 : 2 Marks]

Q.94 The determinant of matrix $\begin{bmatrix}0 & 1 & 2 & 3\\1 & 0 & 3 & 0\\2 & 3 & 0 & 1\\3 & 0 & 1 & 2\end{bmatrix}$ is ______.

[CE, GATE-2014 : 1 Mark]

Q.95 Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is -12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is

(a) -96 (b) -24

(c) 24 (d) 96

[ME, GATE-2014 : 1 Mark]

Q.96 The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is ______.

[EC, GATE-2014 : 1 Mark]

Q.97 The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ______.

[EC, GATE-2014 : 2 Marks]

Q.98 If the matrix A is such that

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 \end{bmatrix}$$

then the determinant of A is equal to ______.

[CS, GATE-2014 : 1 Mark]

Q.99 The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ______.

[CE, GATE-2014 : 2 Marks]

Q.100 Two matrices A and B are given below:

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix};\ B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$$

If the rank of matrix A is N, then the rank of matrix B is

(a) $\frac{N}{2}$ (b) $N-1$

(c) N (d) $2N$

[EE, GATE-2014 : 1 Mark]

Q.101 Given a system of equations:

$x + 2y + 2z = b_1$

$5x + y + 3z = b_2$

Which of the following is true regarding its solutions

(a) The system has a unique solution for any given b_1 and b_2

(b) The system will have infinitely many solutions for any given b_1 and b_2

(c) Whether or not a solution exists depends on the given b_1 and b_2

(d) The system would have no solution for any values of b_1 and b_2

[EE, GATE-2014 : 1 Mark]

Q.102 The system of linear equations

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 14 \end{bmatrix} \text{ has}$$

(a) a unique solution

(b) infinitely many solutions

(c) no solution

(d) exactly two solutions

[EC, GATE-2014 : 2 Marks]

Q.103 Consider the following system of equations:

$3x + 2y = 1$

$4x + 7z = 1$

$x + y + z = 3$

$x - 2y + 7z = 0$

The number of solutions for this system is ______

[CS, GATE-2014 : 1 Mark]

Q.104 The sum of Eigen values of matrix, $[M]$ is

$$\text{where } [M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$$

(a) 915 (b) 1355

(c) 1640 (d) 2180

[CE, GATE-2014 : 1 Mark]

Q.105 Consider a 3×3 real symmetric matrix S such that two of its eigenvalues are $a \neq 0$, $b \neq 0$ with

respective eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $a \neq b$

then $x_1y_1 + x_2y_2 + x_3y_3$ equals

(a) a (b) b

(c) ab (d) 0

[ME, GATE-2014 : 1 Mark]

Q.106 One of the eigenvectors of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is

(a) $\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ (b) $\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

(c) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

[ME, GATE-2014 : 1 Mark]

Q.107 A system matrix is given as follows.

$$\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The absolute value of the ratio of the maximum eigenvalue to the minimum eigenvalue is ___

[EE, GATE-2014 : 2 Marks]

Q.108 A real (4 × 4) matrix A satisfies the equation $A^2 = I$, where I is the (4 × 4) identity matrix. The positive eigenvalue of A is ______

[EC, GATE-2014 : 1 Mark]

Q.109 The value of the dot product of the eigenvectors corresponding to any pair of different eigenvalues of a 4-by-4 symmetric positive definite matrix is ______.

[CS, GATE-2014 : 1 Mark]

Q.110 The product of the non-zero eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is ________. [CS, GATE-2014 : 2 Marks]

Q.111 Which one of the following statements is TRUE about every $n \times n$ matrix with only real eigenvalues?

(a) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its eigenvalues is negative.

(b) If the trace of the matrix is positive, all its eigenvalues are positive.

(c) If the determinant of the matrix is positive, all its eigenvalues are positive.

(d) If the product of the trace and determinant of the matrix is positive, all its eigenvalues are positive.

[CS, GATE-2014 : 1 Mark]

Q.112 If any two columns of a determinant $P = \begin{vmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{vmatrix}$ are interchanged, which one of the following statements regarding the value of the determinant is CORRECT?

(a) Absolute value remains unchanged but sign will change

(b) Both absolute value and sign will change

(c) Absolute value will change but sign will not change

(d) Both absolute value and sign will remain unchanged

[ME, GATE-2015 : 1 Mark]

Q.113 Perform the following operations on the matrix

$$\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$$

1. Add the third row to the second row.
2. Subtract the third column from the first column.

The determinant of the resultant matrix is ______

[CS, GATE-2015 : 2 Marks]

Q.114 For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, the determinant of $A^T A^{-1}$ is

(a) $\sec^2 x$ (b) $\cos 4x$

(c) 1 (d) 0

[EC, GATE-2015 : 1 Mark]

Q.115 For a given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$, where $i = \sqrt{-1}$, the inverse of matrix P is

(a) $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$ (b) $\frac{1}{25}\begin{bmatrix} i & 4-3i \\ 4+3i & -i \end{bmatrix}$

(c) $\frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$ (d) $\frac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

[ME, GATE-2015 : 2 Marks]

Q.116 Let $A = [a_{ij}]$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i \cdot j$. The rank of A is
(a) 0 (b) 1
(c) $n - 1$ (d) n
[CS, GATE-2015 : 1 Mark]

Q.117 For what value of p the following set of equations will have no solution?

$$2x + 3y = 5$$
$$3x + py = 10$$

[CE, GATE-2015 : 1 Mark]

Q.118 We have a set of 3 linear equations in 3 unknowns. '$X \equiv Y$' means X and Y are equivalent statements and '$X \not\equiv Y$' means X and Y are not equivalent statements.
P: There is a unique solution.
Q: The equations are linearly independent.
R: All eigenvalues of the coefficient matrix are nonzero.
S: The determinant of the coefficient matrix is nonzero.
Which one of the following is TRUE?
(a) $P \equiv Q \equiv R \equiv S$ (b) $P \equiv R \not\equiv Q \equiv S$
(c) $P \equiv Q \not\equiv R \equiv S$ (d) $P \not\equiv Q \not\equiv R \not\equiv S$
[EE, GATE-2015 : 1 Mark]

Q.119 Consider the system of linear equations:

$$x - 2y + 3z = -1$$
$$x - 3y + 4z = 1, \text{ and}$$
$$-2x + 4y - 6z = k$$

The value of k for which the system has infinitely many solutions is ________.
[EC, GATE-2015 : 1 Mark]

Q.120 If the following system has non-trivial solution,
$px + qy + rz = 0$
$qx + ry + pz = 0$
$rx + py + qz = 0$
then which one of the following options is TRUE?
(a) $p - q + r = 0$ or $p = q = -r$
(b) $p + q - r = 0$ or $p = -q = r$
(c) $p + q + r = 0$ or $p = q = r$
(d) $p - q + r = 0$ or $p = -q = -r$
[CS, GATE-2015 : 2 Marks]

Q.121 Let A be an $n \times n$ matrix with rank r $(0 < r < n)$. Then $AX = 0$ has p independent solutions, where p is
(a) r (b) n
(c) $n - r$ (d) $n + r$
[IN, GATE-2015 : 1 Mark]

Q.122 The smallest and largest Eigen values of the following matrix are:

$$\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$$

(a) 1.5 and 2.5 (b) 0.5 and 2.5
(c) 1.0 and 3.0 (d) 1.0 and 2.0
[CE, GATE-2015 : 2 Marks]

Q.123 The lowest eigenvalue of the 2×2 matrix $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is ________.
[ME, GATE-2015 : 1 Mark]

Q.124 The value of p such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an Eigenvector of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$ is ________.
[EC, GATE-2015 : 1 Mark]

Q.125 The larger of the two Eigenvalues of the matrix $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ is ________.
[CS, GATE-2015 : 1 Mark]

Q.126 In the given matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, one of the Eigen values is 1. The Eigen vectors corresponding to the Eigen value 1 are
(a) $\{\alpha(4, 2, 1) \mid \alpha \ne 0, \alpha \in R\}$
(b) $\{\alpha(-4, 2, 1) \mid \alpha \ne 0, \alpha \in R\}$
(c) $\{\alpha(\sqrt{2}, 0, 1) \mid \alpha \ne 0, \alpha \in R\}$
(d) $\{\alpha(-\sqrt{2}, 0, 1) \mid \alpha \ne 0, \alpha \in R\}$
[CS, GATE-2015 : 1 Mark]

Q.127 The two eigen values of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & p \end{bmatrix}$ have a ratio of 3 : 1 for p = 2. What is another value of p for which the Eigen values have the same ratio of 3 : 1?

(a) −2
(b) 1
(c) 7/3
(d) 14/3

[CE, GATE-2015 : 2 Marks]

Q.128 At least one eigen value of a singular matrix is
(a) positive
(b) zero
(c) negative
(d) imaginary

[ME, GATE-2015 : 1 Mark]

Q.129 The maximum value of 'a' such that the matrix $\begin{bmatrix} -3 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & a & -2 \end{bmatrix}$ has three linearly independent real eigenvectors is

(a) $\frac{2}{3\sqrt{3}}$
(b) $\frac{1}{3\sqrt{3}}$
(c) $\frac{1+2\sqrt{3}}{3\sqrt{3}}$
(d) $\frac{1+\sqrt{3}}{3\sqrt{3}}$

[EE, GATE-2015 : 2 Marks]

Q.130 The value of x for which all the eigen values of the matrix given below are real is

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(a) 5 + j
(b) 5 − j
(c) 1 − 5j
(d) 1 + 5j

[EC, GATE-2015 : 1 Mark]

Q.131 Consider the following 2 × 2 matrix A where two elements are unknown and are marked by a and b. The eigen values of this matrix are −1 and 7. What are the values of a and b?

$$A = \begin{pmatrix} 1 & 4 \\ b & a \end{pmatrix}$$

(a) a = 6, b = 4
(b) a = 4, b = 6
(c) a = 3, b = 5
(d) a = 5, b = 3

[CS, GATE-2015 : 2 Marks]

Q.132 A real square matrix A is called skew-symmetric if
(a) $A^T = A$
(b) $A^T = A^{-1}$
(c) $A^T = -A$
(d) $A^T = A + A^{-1}$

[ME, GATE-2016 : 1 Mark]

Q.133 Let $M^4 = I$, (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:
(a) M^{4k+1}
(b) M^{4k+2}
(c) M^{4k+3}
(d) M^{4k}

[EC, GATE-2016 : 1 Mark]

Q.134 The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det(A) = 100 and trace(A) = 14. The value of |a − b| is ______.

[EC, GATE-2016 : 2 Marks]

Q.135 Let A be a 4 × 3 real matrix with rank 2. Which one of the following statement is TRUE?
(a) Rank of A^TA is less than 2.
(b) Rank of A^TA is equal to 2.
(c) Rank of A^TA is greater than 2.
(d) Rank of A^TA can be any number between 1 and 3.

[EE, GATE-2016 : 2 Marks]

Q.136 Let $P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Consider the set S of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $a^2 + b^2 = 1$ where $\begin{pmatrix} a \\ b \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$. Then S is
(a) a circle of radius $\sqrt{10}$
(b) a circle of radius $\frac{1}{\sqrt{10}}$
(c) an ellipse with major axis along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
(d) an ellipse with minor axis along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

[EE, GATE-2016 : 2 Marks]

Q.137 If the vectors $e_1 = (1, 0, 2)$, $e_2 = (0, 1, 0)$ and $e_3 = (-2, 0, 1)$ form an orthogonal basis of the three dimensional real space R^3, then the vector $u = (4, 3, -3) \in R^3$ can be expressed as

(a) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(b) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(c) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(d) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

[EC, GATE-2016 : 2 Marks]

Q.138 Consider the following linear system
$x + 2y - 3z = a$
$2x + 3y + 3z = b$
$5x + 9y - 6z = c$
This system is consistent if a, b and c satisfy the equation
(a) $7a - b - c = 0$ (b) $3a + b - c = 0$
(c) $3a - b + c = 0$ (d) $7a - b + c = 0$

[CE, GATE-2016 : 2 Marks]

Q.139 The solution to the system of equations is

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

(a) 6, 2 (b) -6, 2
(c) -6, -2 (d) 6, -2

[ME, GATE-2016 : 1 Mark]

Q.140 Consider the systems, each consisting of m linear equations in n variables.
I. If $m < n$, then all such systems have a solution
II. If $m > n$, then none of these systems has a solution
III. If $m = n$, then there exists a system which has a solution
Which one of the following is CORRECT?
(a) I, II and III are true
(b) Only II and III are true
(c) Only III is true
(d) None of them is true

[CS, GATE-2016 : 1 Mark]

Q.141 [illegible] matrices [illegible] column [illegible] equal [illegible] up to 1, the real eigen value of M is
(a) 4 (b) 3
(c) 2 (d) 1

[CE, GATE-2016 : 1 Mark]

Q.142 Consider a [illegible] eigen [illegible] equal to 1 [illegible]

[CE, GATE-2016 : 1 Mark]

Q.143 The [illegible] eigen values of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ [illegible]

(a) [illegible] $\frac{1}{2}$ (b) [illegible] 2

(c) [illegible] 0 (d) [illegible] $\frac{1}{2}$

[ME, GATE-2016 : 1 Mark]

Q.144 Consider a 2×2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where x is unknown. If the eigen values of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to
(a) $+j\omega$ (b) $-j\omega$
(c) $+\omega$ (d) $-\omega$

[EC, GATE-2016 : 1 Mark]

Q.145 Two eigen values of a 3×3 real matrix P are $(2 + \sqrt{-1})$ and 3. The determinant of P is ______.

[EE, GATE-2016 : 1 Mark]

Q.146 Consider a linear time-invariant system $\dot{x} = Ax$, with initial condition $x(0)$ at $t = 0$. Suppose α and β are eigen vectors of (2×2) matrix A corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Then the response $x(t)$ of the system due to initial condition $x(0) = \alpha$ is

(a) $e^{\lambda_1 t}$ (b) $e^{\lambda_2 t}\beta$

(c) $e^{\lambda_1 t}\alpha$ (d) $e^{\lambda_1 t}\alpha + e^{\lambda_2 t}\beta$

[EE, GATE-2016 : 2 Marks]

Q.147 Let the eigenvalues of a 2 × 2 matrix A be 1, −2 with eigenvectors x_1 and x_2 respectively. Then the eigenvalues and eigenvectors of the matrix $A^2 - 3A + 4I$ would respectively, be

(a) 2, 14; x_1, x_2
(b) 2, 14; $x_1 + x_2$, $x_1 - x_2$
(c) 2, 0; x_1, x_2
(d) 2, 0; $x_1 + x_2$, $x_1 - x_2$

[EE, GATE-2016 : 2 Marks]

Q.148 The number of linearly independent eigenvectors of matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ is}$$

[ME, GATE-2016 : 2 Marks]

Q.149 The value of x for which the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$$

has zero as an eigenvalue is ____

[EC, GATE-2016 : 1 Mark]

Q.150 Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose eigenvalues are 1, −1 and 3. Then Trace of $(A^3 - 3A^2)$ is

[IN, GATE-2016 : 2 Marks]

Q.151 Suppose that the eigenvalues of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is

[CS, GATE-2016 : 1 Mark]

Q.152 A 3 × 3 matrix P is such that, $P^3 = P$. Then the eigenvalues of P are

(a) 1, 1, −1
(b) 1, 0.5 + j0.866, 0.5 − j0.866
(c) 1, −0.5 + j0.866, −0.5 − j0.866
(d) 0, 1, −1

[EE, GATE-2016 : 1 Mark]

Q.153 A sequence $x[n]$ is specified as

$$\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ for } n \geq 2.$$

The initial conditions are $x[0] = 1$, $x[1] = 1$, and $x[n] = 0$ for $n < 0$. The value of $x[12]$ is ____

[EC, GATE-2016 : 2 Marks]

Q.154 Let A be $n \times n$ real valued square symmetric matrix of rank 2 with $\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 = 50$. Consider the following statements.

I. One eigenvalue must be in [−5, 5]
II. The eigenvalue with the largest magnitude must be strictly greater than 5.

Which of the above statements about eigenvalues of A is/are necessarily CORRECT?

(a) Both I and II
(b) I only
(c) II only
(d) Neither I nor II

[CS, GATE-2017 : 2 Marks]

Q.155 The determinant of a 2 × 2 matrix is 50. If one eigenvalue of the matrix is 10, the other eigenvalue is ____________.

[ME, GATE-2017 : 1 Mark]

Q.156 Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigenvectors corresponding to eigenvalues λ_1 and λ_2 are $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$, respectively. The value of $x_1^T x_2$ is

[ME, GATE-2017 : 2 Marks]

Q.157 The product of eigenvalues of the matrix P is

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

(a) −6
(b) 2
(c) 6
(d) −2

[ME, GATE-2017 : 1 Mark]

Q.158 Consider the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

Which one of the following statements about P is INCORRECT?

(a) Determinant of P is equal to 1.
(b) P is orthogonal.
(c) Inverse of P is equal to its transpose.
(d) All eigenvalues of P are real numbers.

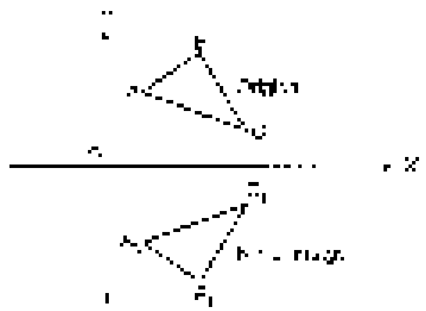
[ME, GATE-2017 : 2 Marks]

Q.159 The eigenvalues of the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$ are

(a) $-1, 5, 6$ (b) $1, -5 \pm j6$
(c) $1, 5 \pm j6$ (d) $1, 5, 5$

[IN, GATE-2017 : 1 Mark]

Q.160 The figure shows a shape ABC and its mirror image $A_1B_1C_1$ across the horizontal axis (X-axis). The coordinate transformation matrix that maps ABC to $A_1B_1C_1$ is



(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

[IN, GATE-2017 : 1 Mark]

Q.161 The eigenvalues of the matrix given below are

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

(a) $(0, -1, -3)$ (b) $(0, -2, -3)$
(c) $(0, 2, 3)$ (d) $(0, 1, 3)$

[EE, GATE-2017 : 2 Marks]

Q.162 The matrix $A = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$ has three distinct eigenvalues and one of its eigenvectors is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Which one of the following can be another eigenvector of A?

(a) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$
(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

[EE, GATE-2017 : 1 Mark]

Q.163 The rank of the matrix,

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

is ______.

[EE, GATE-2017 : 1 Mark]

Q.164 Consider the 5×5 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

It is given that A has only one real eigenvalue. Then the real eigenvalue of A is

(a) -2.5 (b) 0
(c) 15 (d) 25

[EC, GATE-2017 : 1 Mark]

Q.165 The rank of the matrix $M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[EC, GATE-2017 : 1 Mark]

Q.166 Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices. Then the rank of $P + Q$ is ______.

[CS, GATE-2017 : 1 Mark]

Q.167 The characteristic polynomial of a 3 × 3 matrix M over $\mathbb{R}$ (the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$, $a \in \mathbb{R}$, and one eigen value of M is 2. Then the largest among the absolute values of the eigen values of M is ______

[CS, GATE-2017 : 2 Marks]

Q.168 Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i a_i = 0$ where a_i are column vectors in R^n.

Consider the set of linear equations
$Ax = b$
where $A = [a_1, \ldots, a_n]$ and $b = \sum_{i=1}^{n} a_i$. The set of equations has

(a) a unique solution at $x = J_n$ where J_n denotes a n-dimensional vector of all 1
(b) no solution
(c) infinitely many solutions
(d) finitely many solutions

[CS, GATE-2017 : 1 Mark]

Q.169 Consider the following simultaneous equations (with c_1 and c_2 being constants):
$3x_1 + 2x_2 = c_1$
$4x_1 + x_2 = c_2$
The characteristic equation for these simultaneous equations is

(a) $\lambda^2 - 4\lambda - 5 = 0$
(b) $\lambda^2 - 4\lambda + 5 = 0$
(c) $\lambda^2 + 4\lambda - 5 = 0$
(d) $\lambda^2 + 4\lambda + 5 = 0$

[CE, GATE-2017 : 1 Mark]

Q.170 If $A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$, AB^T is equal to

(a) $\begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$
(b) $\begin{bmatrix} 3 & 40 \\ 42 & 8 \end{bmatrix}$
(c) $\begin{bmatrix} 43 & 27 \\ 34 & 50 \end{bmatrix}$
(d) $\begin{bmatrix} 38 & 32 \\ 28 & 56 \end{bmatrix}$

[CE, GATE-2017 : 2 Marks]

Q.171 The matrix P is the inverse of a matrix Q. If I denotes the identity matrix, which one of the following options is correct?

(a) PQ = I but QP ≠ I
(b) QP = I but PQ ≠ I
(c) PQ = I and QP = I
(d) PQ − QP = I

[EE, GATE-2017 : 1 Mark]

Q.172 Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. Which one of the following statements is TRUE for the eigenvalues and eigenvectors of this matrix?

(a) Eigenvalue 3 has a multiplicity of 2, and only one independent eigenvector exists.
(b) Eigenvalue 3 has a multiplicity of 2, and two independent eigenvectors exist.
(c) Eigenvalue 3 has a multiplicity of 2, and no independent eigenvector exists.
(d) Eigenvalues are 3 and −3, and two independent eigenvectors exist.

[CE, GATE-2017 : 2 Marks]

Q.173 The solution of the system of equations
$x + y + z = 4$, $x - y + z = 0$, $2x + y + z = 5$ is

(a) $x = 2, y = 2, z = 0$
(b) $x = 1, y = 2, z = 1$
(c) $x = 1, y = 1, z = 2$
(d) $x = 1, y = 2, z = 1$

[ESE Prelims-2017]

■■■■

Answers Linear Algebra

1.	(c)	2.	[illegible]	3.	(b)	4.	(a)	5.	(d)	6.	(a)	7.	(c)	8.	[illegible]	9.	(b)
10.	(c)	11.	(c)	12.	(b)	13.	(a)	14.	(c) (a) + (b)	15.	(b)	16.	(a)	17.	(b)		
18.	(c)	19.	(d)	20.	(a)	21.	(b)	22.	[illegible]	23.	(c)	24.	(c)	25.	(a)	26.	(a)
27.	(b)	28.	(d)	29.	(a)	30.	(b)	31.	(d)	32.	(c)	33.	(c)	34.	(a)	35.	(b)
36.	(c)	37.	(c)	38.	(d)	39.	(b)	40.	(c)	41.	(c)	42.	(a)	43.	(d)	44.	(d)
45.	(b)	46.	(c)	47.	(d)	48.	(c)	49.	[illegible]	50.	(c)	51.	(c)	52.	(b)	53.	(b)
54.	(b)	55.	(c)	56.	(a)	57.	(c)	58.	(d)	59.	(c)	60.	(c)	61.	(c)	62.	(b)
63.	(c)	64.	(b)	65.	(c)	66.	(d)	67.	(c)	68.	(c)	69.	(b)	70.	(a)	71.	(a)
72.	(b)	73.	(c)	74.	(c)	75.	(c)	76.	(c)	78.	(c)	79.	(a)	80.	(b)	81.	(a)
82.	(c)	83.	(c)	84.	(c)	85.	(c)	86.	(d)	87.	(c)	89.	(a)	90.	(c)	91.	(a)
92.	(c)	93.	(b)	96.	(c)	100.	(d)	101.	(c)	102.	(d)	104.	(a)	105.	(c)	108.	(c)
111.	(d)	112.	(a)	114.	(c)	115.	(a)	116.	(c)	118.	(c)	120.	(c)	121.	(d)	122.	(c)
126.	(c)	127.	(c)	128.	(c)	129.	(c)	130.	(c)	131.	(d)	132.	(a)	133.	(a)	135.	(b)
136.	(c)	137.	(c)	138.	(a)	139.	(c)	140.	(c)	141.	(d)	143.	(a)	144.	(c)	146.	(a)
147.	(c)	152.	(c)	154.	(d)	157.	(c)	158.	(d)	159.	(c)	160.	(c)	161.	(a)	162.	(c)
164.	(b)	166.	(a)	168.	(c)	169.	(c)	170.	(d)	171.	(c)	172.	(a)	173.	(c)		

Explanations Linear Algebra

1. (a)

[illegible]

2. (c)

[illegible]

Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$

By gauss elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rho(A) = 2$$
$$\rho(A \mid B) = 3$$

Since the rank of coefficient matrix is 2 and the rank of augmented matrix is 3, which are not equal. Hence system has no solution i.e., system is inconsistent.

3. (b)

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right]$$

Performing Gauss elimination on above matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right] \xrightarrow{[illegible]} [illegible]$$

$$\xrightarrow{[illegible]} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5 - 2\alpha \\ 0 & 0 & 0 & \frac{5\alpha - 1}{2} \end{array}\right]$$

Now for infinite solutions it is necessary that at least one row must be completely zero

$$\therefore \frac{5\alpha - 1}{2} = 0$$
$$\alpha = 1/5 \text{ is the solution}$$

There is only one value of α for which infinite solution exists.

4. (c)

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Now, $|A - \lambda I| = 0$

Where λ = eigenvalue

$$\therefore \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$
$$(4 - \lambda)^2 - 1 = 0$$
$$\Rightarrow (4 - \lambda - 1)(4 - \lambda + 1) = 0$$
$$\Rightarrow (3 - \lambda)(5 - \lambda) = 0$$
$$\therefore \lambda = 3, \lambda = 5$$

5. (a)

For singular matrix, $\begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$

$\Rightarrow 8(0 - 12) - x(0 - 24) = 0$

$\therefore x = 4$

6. (b)

A, B, C, D are $n \times n$ matrix.

Given $ABCD = I$

$\Rightarrow ABCD(D^{-1}C^{-1}) = D^{-1}C^{-1}$

$\Rightarrow AB = D^{-1}C^{-1}$

$\Rightarrow A^{-1}AB = A^{-1}D^{-1}C^{-1}$

$\Rightarrow B = A^{-1}D^{-1}C^{-1}$

$B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$

$= (C^{-1})^{-1}(D^{-1})^{-1}(A^{-1})^{-1}$

$= CDA$

7. (c)

[illegible]

The augmented matrix is [illegible]

Using gauss elimination on above matrix, we get

[illegible]

Rank $[A \mid B] = 2$ (number of non zero rows in $[A \mid B]$)

Rank $[A] = 2$ (number of non zero rows in $[A]$)

Rank $[A \mid B]$ = Rank $[A]$

= 2 = number of variables

∴ Unique solution exists. Correct choice is (c)

8. (c)

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} [illegible] - \lambda & 2 \\ 2 & [illegible] - \lambda \end{vmatrix} = 0$$

$(5 - \lambda)(2 - \lambda) - [(-2) \times (-2)] = 0$ [illegible]

$\lambda^2 - [illegible]\lambda = 0$

$\Rightarrow \lambda(\lambda - [illegible]) = 0$

Hence, $\lambda = 0$, [illegible] are the eigen values.

9. (b)

Sum of eigen values of given matrix = sum of diagonal elements of given matrix = 1 + [illegible] = 7

10. (a)

With the given order we can say that order of matrices are as follows:

$X^T \rightarrow 3 \times 4$

$Y \rightarrow 4 \times 3$

$X^T Y \rightarrow 3 \times 3$

$(X^T Y)^{-1} \rightarrow 3 \times 3$

$P \rightarrow 2 \times 3$

$P^T \rightarrow 3 \times 2$

$P(X^T Y)^{-1} P^T \rightarrow (2 \times 3)(3 \times 3)(3 \times 2) \rightarrow 2 \times 2$

$\therefore (P(X^T Y)^{-1} P^T)^T \rightarrow 2 \times 2$

11. (a)

For orthogonal matrix

$AA^T = I$ = identity matrix

$\therefore (AA^T)^{-1} = I^{-1} = I$

12. (b)

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 2 & 2 & [illegible] \end{bmatrix}$$

$$B^{-1} = \frac{\text{Adj}(B)}{|B|} = \frac{\text{cofactor}(B)^T}{|B|}$$

$|B| = $ [illegible]

$= 5 - 4 = 1$

Since we need only the first row of B^{-1}, we need to find only first column of cofactor (B) which after transpose will become first row of adj (B).

cof (1, 1) = $+\begin{vmatrix} 1 & 1 \\ [illegible] & [illegible] \end{vmatrix}$ = [illegible] = 5

cof (2, 1) = $-\begin{vmatrix} 0 & -1 \\ 2 & [illegible] \end{vmatrix}$ = [illegible]

cof (3, 1) = $+\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}$ = +1

$\therefore$ cof (B) = [illegible]

Adj (A) = [cof (A)]^T = [illegible]

Dividing by $|B| = 1$ gives

$$B^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ [illegible] & & \\ & & \end{bmatrix}$$

Top row of $B^{-1} = [5 \ -3 \ 1]$

13. (a)

$[AA^{-1}] = I$

$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & [illegible] \end{bmatrix} \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2a + 0.5 \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 2a + 0.5 = 0 \Rightarrow a = \frac{-0.5}{2}$ ... (i)

$3b = 1 \Rightarrow b = \frac{1}{3}$

Now substitute in equation (i) we get

$a = \frac{[illegible]}{[illegible]}$

So, $a + b = $ [illegible]

$= \frac{[illegible]}{[illegible]} = \frac{[illegible]}{[illegible]}$

14. (a), (b) and (c) all possible.

In an over determined system having more equations than variables, all three possibilities can exist (a) consistent unique (b) consistent infinite and (c) inconsistent with no solution.

15. (b)

$\rho(A_{[illegible]}) \le \min(4, [illegible])$

So, Highest possible rank = Least of 4 [illegible] and [illegible] is highest possible rank (based on size of A) = 3. However, if the rank of A = 3, the rank of [A | B] also would be 3 which means the system would be consistent. But it is given that the system is inconsistent. So the maximum rank of A could only be 2.

16. (a)

Rank (A) = Rank $[A|B]$ is necessary for existence of at least one solution to $AX = B$.

17. (b)

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & -4 & 1 & 0 \end{array}\right]$$

Using gauss elimination method on above, we get,

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & -4 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + \frac{1}{2}R_1]{R_2 - \frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 7/2 & 1/2 & 1/2 \\ 0 & -9/2 & 5/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_3 + \frac{9}{7}R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 7/2 & 1/2 & 1/2 \\ 0 & 0 & 22/7 & 8/7 \end{array}\right]$$

Rank $([A|B]) = 3$

Rank $([A]) = 3$

Since Rank $([A|B])$ = Rank $([A])$ = number of variables. The system has unique solution.

18. (a)

First solve for eigen values by solving characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 & 0 & 0 \\ 0 & 5-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$= (5-\lambda)[(5-\lambda)\{(2-\lambda)(1-\lambda) - 3\}]$

$= 0$

$= (5-\lambda)(5-\lambda)(\lambda^2 - 3\lambda - 1) = 0$

$$\lambda = 5, 5, \frac{3 \pm \sqrt{13}}{2}$$

At $\lambda = 5$, $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-5 & 1 & 0 & 0 \\ 0 & 5-5 & 0 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_2 = 0$

$-3x_3 + x_4 = 0$

$3x_3 - 4x_4 = 0$

Solving all three we get $x_3 = 0$, $x_4 = 0$ x_1 and x_2 may be anything.

The eigen vector corresponding to $\lambda = 5$, may be written as

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1, k_2 may be any real numbers. Since choice (a) is the only one, which is of this form with both x_3 and $x_4 = 0$, so this is the correct answer.

Since we already got a correct answer, there is no need to derive the eigen vector corresponding to $\lambda = \dfrac{3 \pm \sqrt{13}}{2}$.

19. (d)

Since matrix is triangular, the eigen values are the diagonal elements themselves namely $\lambda = 3, -2$ and 1. Corresponding to eigen value $\lambda = -2$ let us find the eigen vector.

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$ in above equation we get,

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$5x_1 - 2x_2 + 2x_3 = 0$ ...(i)

$x_3 = 0$ ...(ii)

$3x_3 = 0$ ...(iii)

Since eqn (ii) and (iii) are same we have

$5x_1 - 2x_2 + 2x_3 = 0$ ...(i)

$x_3 = 0$ ...(ii)

Putting $x_2 = k$ in eqn (i) we get

$5x_1 - 2k + 2 \times 0 = 0$

$\Rightarrow \quad x_1 = 2/5\, k$

$\therefore$ Eigen vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k/5 \\ k \\ 0 \end{bmatrix}$$

[illegible]

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [illegible] \text{ is an eigen vector of matrix } A$$

20. (a)

First, find the eigen values of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-4-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 20 = 0$$

$$\Rightarrow (\lambda + 5)(\lambda - 4) = 0$$

$$\Rightarrow \lambda_1 = -5 \text{ and } \lambda_2 = 4$$

Corresponding to $\lambda_1 = -5$ we need to find eigen vector.

The eigen value problem is $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

Putting $\lambda = -5$

we get $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 = 0 \quad \text{(i)}$$

$$4x_1 + 8x_2 = 0 \quad \text{(ii)}$$

Since (i) and (ii) are the same equation we take

$$x_1 = -2x_2$$

$$x_1 + 2x_2 = 0$$

$$\Rightarrow \frac{x_1}{x_2} = -2$$

Now from the answers given, we look for any vector in this form and we find option (a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is

in the ratio $\frac{x_1}{x_2} = \frac{2}{-1} = -2$

So vector (a) is an eigen vector corresponding to $\lambda = -5$

Since we already got an answer, there is no need to find the second eigen vector corresponding to $\lambda = 4$.

21. (b)

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

The characteristic equation of the above matrix is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda = 1, 6$$

∴ The eigen values of A are 1 and 6.

22. (a)

[illegible] the corresponding eigen values [illegible] are the corresponding eigen vectors.

23. (c)

Method 1:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

According to problem

$$A \times B = I$$

$$\text{or } \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we see B is pre-multiplied to (3 × 3) unit matrix so we need to find the inverse of A.

$$B = A^{-1} = \frac{\text{adj}(A)}{|A|}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:

A lesser method for finding B is by multiplying A with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix. Again the answer is (c).

24. (a)
A. Singular matrix → Determinant is zero
B. Non-square matrix → Determinant is not defined
C. Real symmetric → Eigenvalues are always real
D. Orthogonal matrix → Determinant is always one

25. (c)
Perform Gauss elimination

[illegible matrix reduction]

This is now Echelon form.
So rank = the number of non zero rows in this form.
i.e., rank = 2

26. (a)
We are looking for a diagonal vector lying in span that contain [illegible]

[illegible] and [illegible]

Find the [illegible] by finding their determinant
= [illegible] = 0
The space spanned by these two vectors is

[illegible] ...(i)

The vector [illegible] lies in span [illegible]

We can show this by [illegible] solving equation (i) [illegible] and [illegible] for λ and μ [illegible].
[illegible] choice (c) and (d) are wrong [illegible] by taking [illegible].

27. (b)
The vector $[\ldots\ 30]^T$ is linearly [illegible] upon the solution obtained in previous question namely $[-6\ -3\ 6]^T$ and $[4\ \ 2\ \ 3]^T$.
This can be easily checked by finding determinant

[illegible]

= [illegible] = 0
Hence it is linearly dependent.

28. (d)
The augmented matrix for given system is

[illegible]

then by Gauss elimination procedure

[illegible]

For last row we see 0 = 2 which is inconsistent.
Also we can see rank(A) = 2, while rank(A|B) = 3, rank(A) ≠ rank(A|B), inconsistent.
∴ Solution does not exist for above system.

29. (c)

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix},\ |A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

Given eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)(101) + 2(101) = 0$$

$$\Rightarrow \lambda = 6$$

30. (c)

$$\Sigma\lambda_i = \text{Trace}(A)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A) = [illegible] = 1$$

Now $\lambda_1 = 3$

$$3 + \lambda_2 + \lambda_3 = 1$$

$$\Rightarrow \lambda_2 + \lambda_3 = -2$$

Only choice (c) satisfies this condition.

31. (a)
[illegible]

32. (a)
[illegible] Trace [illegible]

33. (d)
[illegible]
∴ [illegible] is symmetric
[illegible]
[illegible] is Skew-Symmetric

34. (a)
[illegible]

35. (b)
[illegible]
Rank [illegible]

36. (a)
[illegible]

37. (a)
[illegible]
1. [illegible] linearly independent
2. [illegible]
Step 1: Now, [illegible] linearly independent [illegible]
Step 2: Now we need to check if the set spans X.
[illegible]
The general [illegible] solution of [illegible]
[illegible] linearly independent solutions [illegible]
[illegible]

38. (a)
The augmented matrix for this system is
[illegible]

Using Gauss elimination method, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{R_2}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{array}\right]$$

Now, for infinite solutions last row must be completely zero

i.e., $\alpha - 2$ and $\beta - 7 = 0$

$\Rightarrow \qquad \alpha = 2$ and $\beta = 7$

39. (c)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \qquad (2-\lambda)^2 = 0$

$\Rightarrow \qquad \lambda = 2$

Now consider the eigenvector problem

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 2$, we get

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0 \qquad \text{...(i)}$$

$$0 = 0 \qquad \text{...(ii)}$$

The solution is therefore $x_2 = 0$, $x_1 = k$ where k

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

40. (d)

The cross product of $a = [0 \ 1 \ 0]^T$

and $X = [x_1 \ x_2 \ x_3]^T$ can be written as

$$a \times X = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

$$= x_3\hat{i} - 0\hat{j} - x_1\hat{k}$$

$$= [x_3 \ 0 \ -x_1]^T$$

Now $\qquad a \times X = [x_3 \ 0 \ -x_1]^T = M\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

where M is a 3×3 matrix.

Let $\qquad M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Now $M\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a \times X$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

By matching LHS and RHS, we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

So $\qquad M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Now we have to find the eigenvalues of M

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad -\lambda(\lambda^2) - 0 + 1(0 - \lambda) = 0$

$\Rightarrow \qquad \lambda^3 + \lambda = 0$

$\Rightarrow \quad \lambda(\lambda^2 + 1) = 0$

$\Rightarrow \qquad \lambda = 0, \lambda = \pm i$

So, the eigenvalues of M are 0, $+i$ and $-i$

One real and two complex.

41. (b)

$$\Sigma\lambda = \text{Trace}(A)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 1 + 5 = 7$$

Now $\qquad \lambda_1 = 2, \lambda_2 = 0$

$\therefore \qquad 2 + 0 + \lambda_3 = 7$

$$\lambda_3 = 5$$

42. (a)

The eigenvalues of any symmetric matrix are always real.

43. (a)

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(-\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

A will satisfy this equation according to Cayley Hamilton theorem

i.e. $A^2 + 3A + 2I = 0$

multiplying by A^{-1} on both sides we get,

$A^{-1}A^2 + 3A^{-1}A + 2A^{-1}I = 0$

$A + 3I + 2A^{-1} = 0$

44. (a)

The characteristic

We know $A^2 + 3A + 2I = 0$ which has been derived above

$\Rightarrow \quad A^2 = -3A - 2I$

$A^4 = A^2 \times A^2 = (-3A - 2I)(-3A - 2I)$

$= 9A^2 + 12A + 4I$

$= 9(-3A - 2I) + 12A + 4I$

$= -15A - 14I$

$A^8 = A^4 \times A^4$

$= (-15A - 14I)(-15A - 14I)$

$= 225A^2 + 420A + 196I$

$= 225(-3A - 2I) + 420A + 196I$

$= -255A - 254I$

$A^9 = A^8 \times A$

$= (-255A - 254I)A$

$= -255A^2 - 254A$

$= -255(-3A - 2I) - 254A$

$= 511A + 510I$

45. (b)

[illegible]

46. (a)

Choice (a): $AA^+A = A$ is correct

Since, $AA^+A = A[(A^TA)^{-1}A^T]A$

$= A[(A^TA)^{-1}(A^TA)]$

But $(A^TA)^{-1}(A^TA) = I$

Then $= A[I] = A$

47. (c)

If rank of a 5×6 matrix is 4 then surely it must have exactly 4 linearly independent rows as well as 4 linearly independent columns, since rank = row rank = column rank.

48. (d)

The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right]$$

Using Gauss elimination we reduce this to an upper triangular form to investigate its rank.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & k-1 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-7 & 0 \end{array}\right]$$

If $k \neq 7$

rank (A) = rank (A | B) = 3

∴ Unique solution

If $k = 7$, rank (A) = rank (A | B) = 2

which is less than number of variables

∴ when $k = 7$, unique solution is not possible and only infinite solution is possible.

49. (a)

Augmented matrix is $\left[\begin{array}{cccc} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$

Performing gauss elimination on this matrix, we get.

$$\left[\begin{array}{cccc} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right] \xrightarrow[R_3 - \frac{1}{2}R_1]{R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 0 & -1/2 & 1 & 2 \\ 0 & 1/2 & -1 & a-2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{cccc} 2 & 3 & 0 & 4 \\ 0 & -1/2 & 1 & 2 \\ 0 & 0 & 0 & a \end{array}\right]$$

If $a \neq 0$, $r(A) = 2$ and $r(A | B) = 3$, hence the system will have no solutions.

If $a = 0$, $r(A) = r(A | B) = 2$ then the system will be consistent and will have infinitely many solutions.

50. (b)

The system of simultaneous equations

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

The Augmented matrix $[A \mid B]$ is given by

$$\begin{bmatrix} 4 & 2 & 7 \\ 2 & 1 & 6 \end{bmatrix}$$

Performing Gauss elimination on this $[A|b]$ as follows:

$$\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 1 & 6 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 4 & 2 & 7 \\ 0 & 0 & 2.5 \end{array}\right]$$

Now, Rank $[A|b] = 2$

(The number of non zero rows in $[A \mid B]$)

Rank $[A] = 1$

(The number of non zero rows in $[A]$)

Since Rank $[A \mid B] \neq$ rank $[A]$,

The system has no solution.

51. (d)

The augmented matrix for above system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 4 & & \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & \\ & & & & 1 \\ 1 & 3 & 4 & 2 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & \\ 0 & 0 & 4 & 5 \end{array}\right]$$

Now as long as ... rank(A) = rank$(A \mid B) = 3$

∴ All ... correct answer is (d)

52. (b)

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 5 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)(-5-\lambda) - 2 \times 5 = 0$

$\Rightarrow \lambda^2 + \lambda - 30 = 0$

$\lambda = 5, -6$

53. (b)

$$\begin{vmatrix} (1-\lambda) & 2 \\ 0 & (2-\lambda) \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) = 0$

$\therefore \lambda = 1, 2$

Now eigenvector eigenvalue problem is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

putting the value of $\lambda = 1$ and $X = X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow y = 0$ ...(i)

putting the value of $\lambda = 2$ and $X = X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow -x + 2y = 0$

and $0 = 0$

$\Rightarrow \frac{x}{y} = \frac{2}{1}$ ...(ii)

The ... is ...

$$= \ldots = \frac{1}{2}$$

54. (a)

Eigen values of $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-\lambda) = 0$

$\lambda = 0, 1$

Eigen values of $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$\lambda^2 = 0$

$\lambda = 0, 0$

Eigen values of $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 + 1 = 0$

$(1-\lambda)^2 = -1$

$1 - \lambda = i$ or $-i$

$\lambda = 1 - i, 1 + i$

Eigen values of matrix are $\begin{vmatrix} -\lambda & 0 \\ & -\lambda \end{vmatrix}$ ...

[illegible]

Corresponding to [illegible] eigen vector is [illegible]

Correct option is (c)

55. (c)

Sum of the eigen values of matrix is = trace of matrix = sum of diagonal values present in the matrix

[illegible] $= 3 - \lambda_2 - \lambda_3$

$\Rightarrow$ [illegible] $= 3 - \lambda_2 - \lambda_3$

$\Rightarrow$ $\lambda_2 - \lambda_3 =$ [illegible]

56. (a)

Since, $\prod \lambda_i$ [illegible]

and [illegible] of the eigen values is zero [illegible]

$\prod \lambda_i = |A| = 0$

Now [illegible] $= 0$

[illegible] $= 0$

Which is option (a)

57. (d)

[illegible] equation is

[illegible] $= 0$

[illegible] theorem [illegible]

[illegible] $= 0$

[illegible]

Multiplying by [illegible] both sides

[illegible]

58. (a)

[illegible] skew-symmetric [illegible]

[illegible] by definition

59. (a)

[illegible]

[illegible]

[illegible]

$\Rightarrow$ Compare both sides A_{12}

[illegible] $= 0$

[illegible]

60. (c)

[illegible] Trace (A) [illegible]

[illegible] ...(i)

$\prod \lambda_i = |A|$ [illegible]

[illegible] ...(ii)

Solving (i) and (ii) we get [illegible]

61. (d)

Sum of diagonal values = Tr(A) [illegible]

[illegible]

Only one option [illegible]

62. (b)

[illegible]

63. (d)

[illegible]

The augmented matrix is [illegible]

Performing gauss elimination on this we get

[illegible]

[illegible]

So, system is consistent

Since system's rank [illegible] is less than the number of the unknowns, only infinite (multiple) non-trivial solution exists

64. (a)

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = 1, 4$$

The eigen value problem is $[A - \lambda I][X] = 0$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 1$,

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \text{(i)}$$

$$x_1 + 2x_2 = 0 \quad \text{(ii)}$$

Solution is $x_2 = k$, $x_1 = -2k$

$$X_1 = \begin{bmatrix} -2k \\ k \end{bmatrix}$$

i.e. $x_1 : x_2 = -2 : 1$

Since vector (a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in same ratio of x_1 and x_2

∴ (a) is a valid eigen vector.

65. (c)

Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Since matrix is triangular, eigen values are the diagonal elements themselves. Eigen values are therefore $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Now the eigen value problem is $[A - \lambda I][X] = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 1$, we get the eigen vector corresponding to this eigen value.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations

$$x_2 = 0$$

$$x_2 + 2x_3 = 0$$

$$2x_3 = 0$$

This says that $x_3 = 0$, $x_2 = 0$, $x_1 = k$

So one eigen vector is $X_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ where $x_1 : x_2 : x_3$

$= k : 0 : 0$

Since none of the eigen vectors shown in the choices matches with this, we need to proceed further and find the other eigen values corresponding to the other Eigen vectors.

Now, corresponding to $\lambda_2 = 2$, we get by substituting $\lambda = 2$ in the eigen value problem the following set of equations

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations

$$-x_1 + x_2 = 0$$

$$2x_3 = 0$$

$$x_3 = 0$$

Solution is $x_3 = 0$, $x_1 = k$, $x_2 = k$

$$\therefore X_2 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} \text{ i.e. } x_1 : x_2 : x_3 = 1 : 1 : 0$$

Since none of the eigen vectors given in the choices matches this, we need to find the 3rd eigen vector also.

By putting $\lambda = 3$ in the eigen value problem, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0$$

$$-x_2 + 2x_3 = 0$$

Putting $x_1 = k$, we get $x_2 = 2k$ and $x_3 = x_2/2 = k$

$$\therefore X_3 = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix}$$

$x_1 : x_2 : x_3 = 1 : 2 : 1$

Only the eigen vector given in choice (c) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is of this form. So the correct answer is choice (c).

66. (c)
Eigen values of a skew symmetric matrix are either zero or pure imaginary.

67. (c)
Sum of eigen values = Trace(A) = 2 + y
Product of eigen values = |A| = 4y − 3x
∴ 4 + 8 = 2 + y ...(i)
4 × 8 = 4y − 3x ...(ii)
∴ 2 + y = 12 ...(iii)
4y − 3x = 32 ...(iv)
∴ Solving (iii) and (iv) we get x = [illegible] and y = 10

68. (c)
The augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & [illegible] \\ [illegible] & 1 & [illegible] & [illegible] \\ 1 & [illegible] & 0 & [illegible] \end{array}\right]$$

Performing gauss elimination on [A|B] we get

[illegible]

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & [illegible] \\ 0 & 1 & [illegible] & [illegible] \\ 0 & \frac{1}{2} & \frac{1}{2} & [illegible] \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{cccc} [illegible] & [illegible] & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank(A) = Rank(A | B) = 2 < 3
So infinite number of solutions will exist.

69. (c)
The augmented matrix for the system of equations is

$$[A \mid B] = \left[\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$$

$$[A \mid B] = \left[\begin{array}{cccc} 1 & 1 & 1 & 6 \\ [illegible] & [illegible] & [illegible] & [illegible] \\ 0 & 0 & \lambda - 8 & \mu - 20 \end{array}\right] \quad [R_3 \to R_3 - R_2]$$

If λ = 8 and μ ≠ 20 then
Rank (A | B) = 3 and Rank (A) = 2
∴ Rank (A | B) ≠ Rank (A)
∴ Given system of equations has no solution for λ = 8 and μ ≠ 20.

70. (c)
Eigen values of symmetric matrices are always real.

71. (a)
Since the given matrix is upper triangular, its eigen values are the diagonal elements themselves which are [illegible].

72. (c)
We need eigen values of $A = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 5 \\ 5 & 9-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(9-\lambda) - 25 = 0$$

$$\Rightarrow \lambda^2 - 12\lambda + 2 = 0$$

So eigen values are,

$$\lambda = 6 \pm \sqrt{34}$$

73. (c)
The given system is

$$x + 2y + z = 4$$
$$2x + y + 2z = 5$$
$$x - y + z = 1$$

Use Gauss elimination method as follows.
Augmented matrix is

$$[A \mid B] = \left[\begin{array}{cccc} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \left[\begin{array}{cccc} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 0 & -3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank (A) = 2
Rank (A | B) = 2
So Rank (A) = Rank (A | B) = 2
System is consistent.
Now system rank r = 2
Number of variables n = 3
r < n
So we have infinite number of solutions.

74. (c)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigen values are the roots of the characteristic polynomial given below

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 1 = 0$$

$$-(1-\lambda)(1+\lambda)-1 = 0$$
$$\lambda^2-2 = 0$$
$$\lambda = \pm\sqrt{2}$$

Eigen values of A are $\sqrt{2}$ and $-\sqrt{2}$ respectively.

The eigen values of $A^{19} = (\sqrt{2})^{19}$ and $(-\sqrt{2})^{19}$

$$= 2^{9.5} \text{ and } -2^{9.5}$$
$$= 2^{9}\cdot 2^{0.5} \text{ and } -2^{9}\cdot 2^{0.5}$$
$$= 512\sqrt{2} \text{ and } -512\sqrt{2}$$

75. (b)

$$A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation is

$$\begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$$
$$(5-\lambda)(3-\lambda)-3 = 0$$
$$\lambda^2-8\lambda+12 = 0$$
$$\lambda = 2, 6$$

Now, to find eigen vectors:

$$[A-\lambda I]\bar{X} = 0$$

Which is $\begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 2$ in above equation and we get

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives us the equation,

$$3x_1+3x_2 = 0$$
$$\text{and} \qquad x_1+x_2 = 0$$

Which is only one equation,

$$x_1+x_2 = 0$$

Whose solution is

$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix}$$

So one eigen vector is $\bar{X}_1 = \begin{bmatrix} k \\ -k \end{bmatrix}$

Which after normalization is $= \begin{bmatrix} k \\ -k \end{bmatrix}$

$$= \frac{1}{\sqrt{k^2+(-k)^2}}\begin{bmatrix} k \\ -k \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

The other eigen vector is obtained by putting the other eigen value

$\lambda = 6$ in eigen value equation

$$\begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives the equations

$$-x_1+3x_2 = 0$$
$$\text{and} \qquad x_1-3x_2 = 0$$

Which is only one equation

$$x_1-3x_2 = 0$$

Whose solution is

$$\bar{X}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3k \\ k \end{bmatrix}$$

Which after normalization is

$$\frac{\bar{X}_2}{|\bar{X}_2|} = \frac{1}{\sqrt{(3k)^2+(k)^2}}\begin{bmatrix} 3k \\ k \end{bmatrix}$$
$$= \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

Choice (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ is the only correct choice

since it is one of the two normalized vectors which is $\bar{X}_1$.

76. (c)

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} -5-\lambda & -3 \\ 2 & 0-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-\lambda)+6 = 0$$

$\Rightarrow \qquad \lambda^2+5\lambda+6 = 0$

So, $\qquad A^2+5A+6I = 0 \qquad$ (by Cayley Hamilton theorem)

$$\therefore \qquad A^2 = -5A-6I$$

Multiplying by A on both sides, we have,

$$A^3 = -5A^2-6A$$

$\Rightarrow \qquad A^3 = -5(-5A-6I)-6A$

$$= 19A+30I$$

77. Sol.

The minimum number of multiplications required to multiply [illegible]

$A_{p \times q}$ with $B_{q \times r}$ is pqr. To compute PQR, if we multiply PQ first and then R, the number of multiplications required will be [illegible] to get PQ and then [illegible] multiplications to multiply PQ with R. So total multiplications required in this method is

[illegible] $= 16$ [illegible]

To compute PQR if we multiply QR first and then P, the number of multiplications required would be [illegible] to get QR and then [illegible] multiplications to multiply P with QR. So the multiplications required in this method is

[illegible]

Therefore, the minimum number of multiplications required to compute the matrix PQR is [illegible]

78. (b)

Characteristic [illegible] of given matrix $|A|$

[illegible]

79. (a)

The given matrix can be transformed into the matrix given in options (b), (c) and (d) by elementary operations of the type [illegible] only [illegible].

Option (b):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow{C_2+C_1} \begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$$

Option (c):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow{R_1-R_2,\ R_2-R_3} \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Option (d):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Option (a): We can show that given matrix can not be converted into option (a) without doing a column exchange [illegible] change the sign of the determinant as shown below.

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & x+1 & x(x+1) \\ 1 & y+1 & y(y+1) \\ 1 & z+1 & z(z+1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$$

80. (b)

$$A = \begin{bmatrix} 2 & \cdot & 1 \\ 1 & \cdot & 0 \\ -\cdot & \cdot & -\cdot \end{bmatrix}_{3 \times 3}$$

Clearly, rank $= 2$

Rank $= 2$

$\therefore$ dimension of null space of $A = 3 - 2 = 1$.

81. (c)

[illegible] therefore [illegible] linearly dependent.

82. (d)

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

$2x_1 + 2x_2 = 2$

$x_1 + x_2 = 1$

$\Rightarrow$ [illegible]

i.e. [illegible] number of solutions.

$\Rightarrow$ Multiple solutions are there.

83. (a, d)

Eigen values [illegible]

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\therefore \lambda = \pm i$$

To find eigen vector,

$\lambda = i$

$$\begin{bmatrix} - & 1 \\ \cdot & - \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 = 0$ and $x_1 = x_2 = $ [illegible]

clearly $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = $ [illegible] and [illegible] satisfy

$$x = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 = 0$ and $x_1 = x_2 = $ [illegible]

clearly

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = $ [illegible] and [illegible] satisfy

Thus, the two eigen value of the given matrix are [illegible]

84. (c)

(i) The Eigen values of symmetric matrix ($A^T = A$) are purely real.

(ii) The Eigen value of skew-symmetric matrix ($A^T = -A$) are either purely imaginary or zeros.

85. (d)

$$AX = \lambda X$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$a - b = -1$ ...(i)

$c - d = 1$ ...(ii)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\Rightarrow a - 2b = -2$ ...(iii)

$c - 2d = 4$ ...(iv)

From equation (i) and (iii), $a = 0$ and $b = 1$

From equation (ii) and (iv), $c = -2$ and $d = -3$

$$\therefore \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

86. (a)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

$(3-\lambda)[(12-\lambda)(5-\lambda)-49] - 5[25-5\lambda-14] + 2[35-2(12-\lambda)] = 0$

[illegible]

$3\lambda^2 - 51\lambda + 33 - \lambda^3 + 17\lambda^2 - 11\lambda - 55 + 25\lambda + 22 + 4\lambda = 0$

$-\lambda^3 + 20\lambda^2 - 33\lambda = 0$

$\lambda^3 - 20\lambda^2 + 33\lambda = 0$

$\lambda(\lambda^2 - 20\lambda + 33) = 0$

$\lambda = 0, 1.82, 18.2$

So the [illegible] eigen value is [illegible]

87. (a)

Statement 1: [illegible] as shown below

[P] has a size 1×3

[Q] has a size 3×3

[R] has a size 3×3

[S] has a size 3×3

[P] has a size 3×1

So $[P]^T [Q]^{-1} [R][S][P]$ has a size 1×1. Therefore it is a scalar.

So, Statement 1 is true.

Consider Statement 2: [illegible] skew-symmetric [illegible] does not exist since $F_{3\times 3}$, $F_{\ldots}$ and $G_{\ldots}$ are not compatible for multiplication, [illegible] does not exist.

So, Statement 2 is false.

88. Sol.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$X^T A X = [1\ 2\ -1]\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$= [1\ 2\ -1]\begin{bmatrix} \cdot \\ \cdot \\ 2 \end{bmatrix} = 6 + 16 - $ [illegible]

89. (a)

Area of the triangle

$$= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\,[\text{illegible}]$$

$$= [\text{illegible}]$$

92. (d)

$$(P+Q)^2 = P^2 + PQ + QP + Q^2$$

$$= P \cdot P + P \cdot Q + Q \cdot P + Q \cdot Q$$

$$= P^2 + PQ + QP + Q^2$$

93. (d)

Matrix multiplication is not commutative.

94. Sol.

$$A = [\text{illegible}]$$

$R_1 \to R_1 - R_2 - R_3$

$$A = [\text{illegible}]$$

$R_2 \to R_2 - 2R_1$

$$A = [\text{illegible}]$$

$R_3 \to R_3 - 3R_1$

$$A = [\text{illegible}]$$

Interchanging row 1 and column 2 and taking determinant

$$A = [\text{illegible}]$$

$$= [\text{illegible}] \begin{vmatrix} [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \\ 0 & [\text{illegible}] & 4 \\ [\text{illegible}] & [\text{illegible}] & 10 \end{vmatrix}$$

$$= [\text{illegible}]$$

$$= [\text{illegible}]$$

95. (a)

Let P be the given matrix

$$A = \begin{bmatrix} [\text{illegible}] & [\text{illegible}] & 0 \\ [\text{illegible}] & [\text{illegible}] & 4 \\ [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \end{bmatrix} = (2)^3 \begin{bmatrix} 1 & [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \end{bmatrix}$$

Taking 2 common from each row,

$\therefore$ Det(A) $= (2)^3 \times [\text{illegible}]$

96. Sol.

Determinant of $A = 5$

Determinant of $B = 40$

Determinant of $AB = |A||B|$

$= 5 \times 40 = 200$

97. Sol.

Let,
$$A = \begin{vmatrix} a & x \\ x & b \end{vmatrix}$$

$\Rightarrow$ $|A| = ab - x^2$

Given trace $(A) = a + b = 14$

So $|A| = a(14 - a) - x^2$

Since x^2 is always positive, maximum value of $a(14 - a) - x^2$ will occur only when $x = 0$.

Hence, $|A| = a(14 - a) = 14a - a^2$

Now maximizing this with respect to a

$$\frac{d|A|}{da} = 14 - 2a = 0$$

$\Rightarrow$ $a = 7$

Also
$$\left.\frac{d^2|A|}{da^2}\right|_{a=7} = -2 < 0$$

At $a = 7$ we get maximum. The maximum value $= 14 \times 7 - 7^2 = 49$

98. Sol.

$$A = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} [\text{illegible}]$$

$$A = \begin{bmatrix} [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \\ 4 & [\text{illegible}] & [\text{illegible}] \\ 7 & [\text{illegible}] & 35 \end{bmatrix} \Rightarrow |A| = 0$$

99. Sol.

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 2 & 14 & 8 & 16 \\ 14 & -14 & 0 & [illegible] \end{bmatrix}$$

[illegible]

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 2 & 14 & 0 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[illegible] minor of matrix $\begin{bmatrix} 6 & 0 \\ 2 & 14 \end{bmatrix}$ is not zero.

$\therefore$ Rank = 2

100. (a)

$$A = \begin{bmatrix} [illegible] & [illegible] \\ [illegible] & [illegible] \end{bmatrix}$$

$$B = \begin{bmatrix} a^2 - [illegible] & [illegible] \\ [illegible] & [illegible] \end{bmatrix} = AA^T$$

There are three cases for the rank of A

Case I:

rank $(A) = 0$

$\Rightarrow$ A is null. So $B = AA^T$ also has to be null and hence rank (B) is also equal to 0. Therefore in this case rank $(A) =$ rank (B)

Case II:

rank $(A) = 1$

$\Rightarrow$ A cannot be null. So B also cannot be null, since $B = AA^T$

$\Rightarrow$ $|B| = |AA^T| = |A| \cdot |A^T| = |A|^2$

For rank $(A) = 1$, [illegible] since rank of A is 1, $|A| = 0$ [illegible] $|A|^2 = 0$

So rank (B) is also < 2. Now since rank $(B) \neq 0$ and < 2, therefore rank (B) must be equal to 1.

Therefore in this case also rank $(A) =$ rank (B)

Case III:

rank $(A) = 2$

So A has to be non singular i.e. $|A| \neq 0$

Therefore, $|B| = |A|^2$ [illegible] $\neq 0$. So rank $(B) = 2$

Therefore in this case also rank $(A) =$ rank (B).

Thus in all three cases rank $(A) =$ rank (B). So rank of A is [illegible] than the rank of matrix B [illegible]

101. (b)

The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 3 & [illegible] & [illegible] & [illegible] \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & [illegible] \\ 0 & -2 & -7 & [illegible] \end{array}\right]$$

Now this matrix is in echelon form. We can see that the Rank $(A) = 2$

Rank $(A|B)$ [illegible] $= 2$ [illegible] does not depend on value of [illegible].

Rank $(A) =$ Rank $(A|B) = 2 <$ Number of variables $= 3$

Therefore the system is consistent and has infinitely many solutions.

102. (a)

$$\begin{bmatrix} [illegible] & [illegible] & [illegible] & [illegible] \\ [illegible] & 2 & 1 & 4 \\ 1 & 2 & [illegible] & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & [illegible] & 4 \\ 3 & 0 & [illegible] & 2 \\ 2 & [illegible] & 7 & 9 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & [illegible] & 14 \\ 0 & [illegible] & 14 & 46 \\ 0 & -3 & -7 & -[illegible] \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_2 \begin{bmatrix} [illegible] & [illegible] & [illegible] & 12 \\ [illegible] & 6 & -[illegible] & -46 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[A|B]$ = Rank of A < order of matrix

$\Rightarrow$ Infinite number of solutions are possible.

103. Sol.

Given

$$[illegible] - 2y = 1$$

$$[illegible] + [illegible]z = 1$$

$$x + [illegible] = 0$$

$$[illegible] - 2[illegible] + [illegible]z = 0$$

$$[illegible] - [illegible] + z = 2$$

$$-x - 2y - 7z = 0$$

$$2y - 8z = 3$$

$$\Rightarrow \quad [illegible] = [illegible]$$

$$\Rightarrow \quad 2y - 4z = 2$$

$$2[illegible] + 2[illegible] = [illegible]$$

$$[illegible] - 7z = [illegible]$$

$$[illegible] + 7z = 1$$

$$z = 2$$

$$[illegible] = [illegible]$$

$$3x - 2y = [illegible]$$

$$\Rightarrow \quad y = -[illegible]$$

$$\Rightarrow \quad [illegible] + [illegible]z = 1 \quad [illegible]$$

$$y + z = 1$$
$$z = 1$$

∴ The number of solutions to this system is one: $x = 2$, $y = -1/2$ and $z = 1$ is the only solution.

104. (a)

Sum of eigenvalues = trace of matrix

= 215 + 50 + 550 = 815

105. (c)

3 × 3 real symmetric matrix with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with respective eigenvectors

$$\text{if } x_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \text{ and } x_2 \text{ then}$$

$x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32} = 0$ because they are orthogonal.

$$\therefore \quad x_1^T x_2 = 0 \qquad [\text{illegible}]$$

$$[x_{11} \; x_{21} \; x_{31}] \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = 0$$

106. (d)

The characteristic equation $|A - \lambda I| = 0$

$$\text{i.e.} \quad \begin{vmatrix} 5-\lambda & 2 \\ -9 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5 - \lambda)(6 - \lambda) + 18 = 0$

$\Rightarrow \lambda^2 - 11\lambda + 30 - 20 + 8 = 0$ [illegible]

or $\lambda^2 - \lambda - 12 = 0$

$$\text{or} \quad \lambda = \frac{1 \pm \sqrt{1+48}}{2} = \frac{1 \pm 7}{2} = 4,$$

-3

Corresponding to $\lambda = 4$, we have

$$[A - \lambda I] = \begin{bmatrix} 5-\lambda & 2 \\ -9 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\text{or} \quad \begin{bmatrix} -6 & 2 \\ -9 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives only one independent equation $-3x + 2y = 0$

$\therefore \frac{x}{2} = \frac{y}{3}$ gives eigen vector (2, 3)

Corresponding to $\lambda = -3$,

$$\begin{bmatrix} -2 & 2 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which gives $x + y = 0$ (only one independent equation)

$$\therefore \qquad \frac{x}{1} = \frac{y}{-1} \qquad \text{which gives } (1, -1)$$

So the eigen vectors are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

107. Sol.

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & [\text{illegible}] & -1 \\ 6 & 11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda[(55 - 11\lambda - 5\lambda + \lambda^2) + 66] - 1[-66 + 6\lambda + 66] = 0$ [illegible]

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$ [illegible]

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow \lambda = 1, 2, 3$

Maximum eigenvalue is 3 and minimum is 1

Ratio of maximum and minimum eigenvalue is

$$= 3 : 1 = \frac{3}{1} = 3$$

108. Sol.

Since, $A^2 = I \Rightarrow \text{eig}(A^2) = \text{eig}(I) = 1$

$\Rightarrow \quad \text{eig}(A)^2 = 1$

$\Rightarrow \quad \text{eig}(A) = \pm 1$

Therefore, the positive eigen value of A is +1.

109. Sol.

The value of the dot product of the eigenvectors corresponding to any pair of different eigen values of a 4 × 4 symmetric positive definite matrix is 0.

110. Sol.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{let } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$AX = \lambda X$

$\Rightarrow \quad x_1 + x_5 = \lambda x_1 = \lambda x_5$

$\Rightarrow \quad x_2 + x_3 + x_4 = \lambda x_2 = \lambda x_3 = \lambda x_4$

(i) $\lambda \neq 0$

$AX = \lambda_1 X$

$x_1 = x_2 = x_3 = k$

$\Rightarrow$ $x + \ldots = \ldots$

$\Rightarrow$ [illegible]

$\therefore$ $k = 2$

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

$\Rightarrow$ $\lambda = 3$

(ii) $\lambda = 0$

$\Rightarrow$ Eigen value $\lambda = 0$

$\therefore$ There are 3 distinct eigen values 0, 2, 3

Product of non zero eigen values $= 2 \times 3 = 6$

111. (a)

Since the trace of the matrix is positive, there exists at least one positive eigen value.

Trace of the matrix is positive and the determinant of the matrix is negative, this is possible only when there is odd number of negative eigen values. Hence at least one eigen value is negative.

112. (a)

Properties of determinant: If any two rows or columns are interchanged, then the magnitude of determinant remains same but sign changes.

113. Sol.

Since operations 1 and 2 are elementary operations of the type of $R_i \to kR_j$ and $C_i \to kC_j$, the determinant will be unchanged from the original determinant.

As the reduced determinant $= \begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix}$

$$\begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix} \xrightarrow{C_3 \to C_3 - 15C_1} \begin{vmatrix} 3 & 4 & 0 \\ 7 & 9 & 0 \\ 13 & 2 & 0 \end{vmatrix} = 0$$

So the reduced determinant $= 0$

114. (c)

Long Method:

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}[\text{adj } A]$$

$$= \frac{1}{1 + \tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$\text{Here } A^T A^{-1} = \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 - \tan^2 x & -2\tan x \\ 2\tan x & 1 - \tan^2 x \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \dfrac{1 - \tan^2 x}{\sec^2 x} & \dfrac{-2\tan x}{\sec^2 x} \\ \dfrac{2\tan x}{\sec^2 x} & \dfrac{1 - \tan^2 x}{\sec^2 x} \end{bmatrix}$$

$$|A^T A^{-1}| = \left(\frac{1 - \tan^2 x}{\sec^2 x}\right)^2 + \left(\frac{2\tan x}{\sec^2 x}\right)^2$$

$$= \frac{1 + \tan^4 x - 2\tan^2 x + 4\tan^2 x}{\sec^4 x}$$

$$= 1$$

(c)

Short Method:

Since $|AB| = |A||B|$

$$|A^T A^{-1}| = |A^T||A^{-1}|$$

$$= |A| \times \frac{1}{|A|} = 1$$

$$\left[\text{Note: } |A^T| = |A| \text{ and } |A^{-1}| = \frac{1}{|A|}\right]$$

115. (a)

$$A = \begin{bmatrix} 4 + 3i & -i \\ i & 4 - 3i \end{bmatrix}$$

$$B = \frac{\begin{bmatrix} 4 - 3i & i \\ -i & 4 + 3i \end{bmatrix}}{|A|}$$

$$= \frac{1}{24}\begin{bmatrix} 4 - 3i & i \\ -i & 4 + 3i \end{bmatrix}$$

116. (c)

Rank of $A = 1$

Because every row will be scalar multiple of first row. So, we will get only one non-zero row in row echelon form of A.

Alternative:

Rank of $A = 1$

Because all the minors of order 2 (2 × 2) will be zero.

117. Sol.

Given system of equations has no solution if the lines are parallel i.e. their slopes are equal.

$$\frac{2}{5} = \frac{4}{p}$$

$\Rightarrow \quad p = 10$

118. (a)

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $|A| \neq 0$ then $AX = B$ has unique solution as $X = A^{-1}B$. It contains unique solution.

If $|A| \neq 0$ and $b_1, b_2, b_3 \neq 0$ then X is non-zero.

If $|A| \neq 0$ then all the row/column vectors of A are linearly independent.

119. Sol.

$x + 2y + 3z = 1$

$x + 3y + 4z = 3$, and

$2x + 4y + 6z = k$

$$[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 1 & 3 & 4 & : & 3 \\ 2 & 4 & 6 & : & k \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & 1 & 1 & : & 2 \\ 0 & 0 & 0 & : & k-2 \end{bmatrix}$$

For infinitely many solution

$\rho(A : B) = \rho(A)$

$<$ number of variables

$\rho(A : B) = 2$

$k - 2 = 0$

$k = 2$

120. (c)

$px + qy + rz = 0$

$qx + ry + pz = 0$

$rx + py + qz = 0$

Let $A = \begin{bmatrix} p & q & r \\ q & r & p \\ r & p & q \end{bmatrix}$, the system is $AX = 0$

This is a homogeneous system. Such a system has non-trivial solution if $|A| = 0$

so $\quad \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$p(qr - p^2) - q(q^2 - pr) + r(pq - r^2) = 0$

$p^3 + q^3 + r^3 - 3pqr = 0$

$p = q = r$ satisfies the above equation.

Also if $p + q + r = 0$ then also the determinant becomes zero as shown below

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} \xrightarrow{C_1 \to C_1 + C_2 + C_3} \begin{vmatrix} p+q+r & q & r \\ p+q+r & r & p \\ p+q+r & p & q \end{vmatrix}$$

$$= (p + q + r)\begin{vmatrix} 1 & q & r \\ 1 & r & p \\ 1 & p & q \end{vmatrix} = 0$$

Therefore the correct option is (c) which is $p = q = r$ or $p + q + r = 0$.

121. (c)

Given $\quad AX = 0$

$\rho(A_{5 \times 7}) = 5$

p = Number of independent solutions = Nullity

We know that

rank + nullity = n

$5 + p = 7$

$p = 2$

122. (c)

The eigen values $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 2 \\ 4 & 4-\lambda & 6 \\ 2 & -8 & 5-\lambda \end{vmatrix} = 0$$

[illegible]

[illegible]

$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

Only 1 and 2 satisfy this equation

$\lambda = 1, 1, 2$

Hence distinct eigen values = 1 and 2

Largest eigen value = 2

123. Sol.

$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = 0$

$(4-\lambda)(3-\lambda) - 2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow \lambda = 5, 2$

Minimum value = 2

124. Sol.

$AX = \lambda X$

$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 12 \\ p+7 \\ 36 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ 3\lambda \end{bmatrix}$

$\dfrac{p+7}{12} = 2 \Rightarrow p = 17$

125. Sol.

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 10 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$(\lambda - 6)(\lambda + 1) = 0$

$\Rightarrow \lambda = 6, -1$

$\therefore$ Maximum eigen value is 6

126. (a)

Sol. $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

Given eigen value $\lambda = 1$

Let X be the vector, then $(A - \lambda I)X = 0$

$\begin{bmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{bmatrix} X = 0$

Put $\lambda = 1$

$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} x_2 + 2x_3 \\ 0 \\ x_1 + 2x_2 \end{bmatrix} = 0$

putting $x_1 = k$ we get $x_2 = -k/2$ and $x_3 = k/4$

So the eigen vector $= k\begin{bmatrix} 1 \\ -1/2 \\ 1/4 \end{bmatrix}$

Then ratio $x_1/x_3 = \dfrac{1}{1/4} = 4$

and $x_2/x_3 = \dfrac{-1/2}{1/4} = -2$

Only option (a) (4, −2, 1) has the same ratio and therefore is a correct eigen vector.

127. (d)

$A = \begin{bmatrix} 2 & -1 \\ 1 & p \end{bmatrix}$

Let λ_1 and λ_2 be the eigen values of matrix A

$\therefore \dfrac{\lambda_1}{\lambda_2} = \dfrac{3}{1}$ (given)

Sum of eigen value

$= \lambda_1 + \lambda_2 = 2 + p$ ...(i)

Product of eigen value

$= \lambda_1 \lambda_2 = 2p + 1$ ...(ii)

$\dfrac{\lambda_1}{\lambda_2} = \dfrac{3}{1}$

$\Rightarrow \lambda_1 = 3\lambda_2$

Put value in (i)

$\Rightarrow 3\lambda_2 + \lambda_2 = 2 + p$

$4\lambda_2 = 2 + p$

$\lambda_2 = \dfrac{2+p}{4}$

Put in equation (ii)

$\Rightarrow 3\lambda_2^2 = 2p + 1$

$\Rightarrow \quad 9\left(\frac{\alpha-2}{2}\right)^2 = 4\alpha$ [illegible]

$\Rightarrow \quad \alpha = 4, \frac{16}{9}$ [illegible]

[illegible]

$\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & p-\lambda \end{vmatrix} = 0$

$(2-\lambda)(p-\lambda) - 1 = 0$

$\lambda^2 - \lambda(p+2) + (2p-1) = 0$

By putting values of p from options.

By putting value (c) $\frac{14}{5}$ in above equation

gives value $5, \frac{2}{3}$ [illegible]

Hence ratio of two eigen values $= \frac{5}{5/3} = 3$ [illegible]

So option (c) is correct.

128. (b)

For singular matrix

$|A| = 0$

According to the properties of eigen value

Product of eigen values $= |A| = 0$

$\Rightarrow$ At least one of the eigen values is zero.

129. (b)

Characteristic equation $|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 0 & -2 \\ 1 & 1-\lambda & 0 \\ 4 & 3 & -2-\lambda \end{vmatrix} = 0$

$(3-\lambda)[(1-\lambda)(-2-\lambda) - 0] - 2(3 - 4(1-\lambda)) = 0$ [illegible]

$2a = -(\lambda+1)(\lambda+2)(\lambda-3)$ [illegible]

$= -(\lambda+1)(\lambda^2 - \lambda - 6)$ [illegible]

$2a = -(\lambda^3 + \ldots)$ [illegible]

$\frac{d(2a)}{d\lambda} = -(3\lambda^2 + 12\lambda + 11)$ [illegible]

$= 0$ (for extremum) [illegible]

$3\lambda^2 + 12\lambda + 11 = 0$ [illegible]

$\lambda = \frac{-12 \pm \sqrt{144 - 132}}{6} = -2 \pm \frac{1}{\sqrt{3}}$

$\lambda = -2 + \frac{1}{\sqrt{3}}$

$2a = -\left(-2+\frac{1}{\sqrt{3}}+1\right)\left(-2+\frac{1}{\sqrt{3}}+2\right)\left(-2+\frac{1}{\sqrt{3}}-3\right)$ [illegible]

$= -\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}+1\right)\left(\frac{1}{\sqrt{3}}-1\right)$ [illegible]

$2a = \frac{2}{3} \times \frac{1}{\sqrt{3}}$

$a = \frac{1}{3\sqrt{3}}$

130. (b)

For a matrix containing complex number eigen values are real and given

$A = A^{\theta} = (\bar{A})^T$

$A = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & y & 10 \end{bmatrix}$ [illegible]

$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 10 & \bar{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & 10 \end{bmatrix}$ [illegible]

By comparing these,

$x = 5 - j$

131. (d)

Trace = Sum of eigen values

$1 + a = 6$

$\Rightarrow \quad a = 5$

Determinant = Product of eigen values

$(5 - 4b) = -7$ [illegible]

[illegible]

$-4b = -12$ [illegible]

$\Rightarrow \quad b = 3$

$\therefore \quad a = 5, b = 3$

132. (c)

A is skew-symmetric

$\therefore \quad A^T = -A$

133. (c)

Given that, $M^4 = I$ or $M^{48} = I$ or [illegible]

[illegible] $= M^{44}$ [illegible]

[illegible]

134. Sol.

Trace of $A = 14$

$a + 5 + 2 + b = 14$

$a + b = 7$

$\det(A) = 100$

[illegible]

$10\,ab = 100$

$ab = 10$

From equation (i) and (ii)

either $a = 5,\ b = 2$

or $a = 2,\ b = 5$

$\therefore |a - b| = 3$

135. (a)

Recall, Rank (A|B) = Rank (A)

136. (c)

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} [illegible] \\ [illegible] \end{bmatrix}$$

$2x + y = 8$

[illegible]

$\Rightarrow 10x^2 + 10y^2 - 12xy = 1$

Ellipse with major axis along [illegible]

137. (c)

[illegible]

$a + 2c = 4$

$b = 0$

$2a + c = -3$

Therefore, $a = -\frac{2}{5}$ [illegible]

$b = 0$

$c = \frac{[illegible]}{5}$

$\therefore \ -\frac{2}{5}u_1 + 0u_2 + \frac{[illegible]}{5}u_3$

138. (b)

We can represent the above set of equations in matrix form as

[illegible]

(c)

$[A \mid B]$ = [illegible]

By performing operation $R_3 \to R_3 - (3R_1 + R_2)$

$[A \mid B]$ = [illegible]

(d)

For consistency of system, $c - 3a - b = 0$

139. (a)

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 30 \end{bmatrix}$$

$13y = 26$

$y = 2$

$2x + y = 2$

$2x = 2 - 10$

$2x = 16$

$x = 8$

140. (c)

I. [illegible] system may still be inconsistent as [illegible]

II. [illegible] rank may still be equal to [illegible] and hence [illegible] incorrect.

III. [illegible] and hence may have solution (so correct).

So only III is correct.

141. (d)

Consider a 2 × 2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \quad \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad \ldots(i)$

Putting $\lambda = 1$, we get

$1 - (a + d) + (ad - bc) = 0$

[illegible] $(1 - a)(1 - d) = 0$

[illegible]

[illegible] satisfied the eq. (i) [illegible] satisfy the eq. (i) [illegible] values of a, d.

142. Sol.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ Eigen values are 0, 0, 3

143. (a)

As Eigen values of A [illegible] are positive

[illegible]

[illegible] leading minors [illegible] greater than zero

$$\begin{vmatrix} 2 & 1 \\ 1 & k \end{vmatrix} > 0$$

$$2k - 1 > 0$$

$$2k > 1$$

$$k > \frac{1}{2}$$

144. (d)

[illegible]

The sum of the eigen values

Sum = $\alpha + \beta + \alpha + \beta$ [illegible]

$|A|$ = product of eigens

[illegible] $= (\alpha - \beta)(\alpha + \beta) = \alpha^2 - \beta^2$ [illegible]

which is positive only when [illegible]

145. Sol.

Two eigen vectors are 2 − [illegible] 3 [illegible]

The third eigen value [illegible]

$\Rightarrow$ [illegible] $= (2$ [illegible] $)(2$ [illegible] $) \times 3$

$(4 - 1) \times 3$

$= 3 \times 3 = 15$

146. (a)

$\dot{x} = Ax$

Eigen values are [illegible]

Solution is

$$x(t) = \begin{bmatrix} e^{[illegible]} & 0 \\ 0 & e^{[illegible]} \end{bmatrix}$$

By given initial conditions

$x(t) = e^{At}x(0)$

$$x(t) = \begin{bmatrix} e^{[illegible]} & 0 \\ 0 & e^{[illegible]} \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = a e^{[illegible]}$$

147. (a)

Eigen values of $A^2 - 3A + 4I$ are [illegible]

[illegible]

Note: $AX = \lambda X$

$\Rightarrow$ X is eigen vector of A corresponding to eigen value λ

X_1 and X_2 are eigen vectors of A corresponding to [illegible]

Then X_1 and X_2 are eigen vectors of [illegible] corresponding to [illegible]

148. Sol.

$$\text{Consider } |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$(2 - \lambda)^2 (3 - \lambda) = 0$

$\lambda = 2, 2, 3$

$\lambda = 3$ [illegible] one Eigen vector

$\lambda = 2$ Consider $(A - 2I)x = 0$

[illegible] The equations are $x_2 = 0$

[illegible] $x_3 = 0$

[illegible] independent

$$\therefore \text{ Eigen vector is } \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Only one linearly independent Eigen vector in the case of $\lambda = 2$

Total number of independent Eigen vectors = 2

149. Sol.

[illegible] eigen values are [illegible]

[illegible] $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 2 & 4 \\ 9 & 7-\lambda & 14 \\ -4 & -2 & -3-\lambda \end{vmatrix} = 0$$

[illegible] $= 0$

[illegible] $= 0$

[illegible] $= 0$

$\lambda = 3$ [illegible]

[illegible] $= 1$

150. Sol.

Eigen values of given matrix A are [illegible], 3

Eigen values of A^3 are	[illegible]	27
Eigen values of [illegible] are	[illegible]	27
Eigen values of $A^3 - [illegible]$ are	4	0

[illegible]

151. Sol.

Eigenvalues = 1, 2, ... ; $\lambda_1 = 1 \times 2 \times 3 = 6$ [illegible]

Now $\left|(A^{-1})^{T} \cdot A^{-1}\right| = \frac{1}{|A|^2} = \frac{1}{8} = 0.125$

152. (c)

By Cayley-Hamilton theorem

$A^2 = A$

$\lambda = 1, 1, -1$

153. Sol.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

equation $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

$-\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - \lambda - 1 = 0$

By Cayley-Hamilton Theorem

$A^2 - A - I = 0$

$A^2 = A + I$

$A^4 = A^2 \cdot A^2 = (A + I)^2 = A^2 + I + 2A = \ldots$

$= 3A + 2I$

$A^8 = (3A + 2I)^2$ [illegible]

$= \ldots = 21A + 13I$ [illegible]

$A^{12} = A^8 \cdot A^4 = 144A + 89I$

$= \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = 233$

154. (a)

Consider a random matrix which satisfy property of skew symmetric with all non-zero real values [illegible]

i.e. $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}_{2 \times 2}$ whose eigen values are $5i$ and $-5i$. Since $|A| = 25$ and sum of trace $|A| = 0$

i.e. $5 - 5 = 0 = 0$

Hence, rank of matrix is 2 and there exist a eigen value which would be zero for every skew symmetric matrix [illegible] $n \times n$ [illegible]

$\lambda_1 \cdot \lambda_2 \leq \sum_{i}\sum_{j} a_{ij}$ [illegible]

$\lambda_1 \cdot \lambda_2 \leq 5$ [illegible]

Both λ_1 and λ_2 are real when A is real and symmetric [illegible], which means atleast one eigen value would be [illegible]. But there is no example for eigen value greater than [illegible] [illegible] following.

So option (b) is not correct.

155. Sol.

The product of eigen values of any square matrix is the determinant value of the matrix.

$\lambda_1 = 10$, $\lambda_2 =$ unknown, $|A| = 50$

$\lambda_1 \cdot \lambda_2 = 50$

$10 \cdot \lambda_2 = 50$

$\therefore \lambda_2 = 5$

156. Sol.

$A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$

Eigen values of A are λ_1, λ_2

$\lambda_1 + \lambda_2 = 130$

$\lambda_1 \lambda_2 = -900$

Given $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$, $x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$

$x_1^T x_2 = [70 \quad \lambda_1 - 50] \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$

$= 70\lambda_2 - 5600 + 70\lambda_1 - 3500$

$= 70(\lambda_1 + \lambda_2) - 9100$

$= 70 \times 130 - 9100$

$= 9100 - 9100 = 0$

157. (b)

Trace = Sum of eigen values = determinant value

$2(5 - 3c) + \ldots = 8c + \ldots$ [illegible]

$= \ldots(-2) + 8 = \ldots$ [illegible] $c = 2$

158. (c)

$(PP^T) = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \ldots$ [illegible]

(PP^T) [illegible] diagonal element [illegible]

$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Since P is [illegible] its inverse is equal to its transpose.

(c) The eigen values

$$|P - \lambda I| = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1-\lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

[illegible]

$$\lambda = 1, \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$$

159. (b)

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 5 \\ 0 & 5-\lambda & 6 \\ 0 & -6 & 5-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(\lambda^2 - 10\lambda + 61) = 0$

$\lambda = 1$

$$\lambda = \frac{10 \pm \sqrt{100 - 244}}{2} = \frac{10 \pm 12i}{2} = 5 \pm 6i$$

$\lambda = 1, 5 \pm 6i$

160. (c)

Coordinate transformation matrix

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Orthogonal $\theta = 90°$

Coordinate transformation matrix for mirror image

$$= \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \sin 90° & \cos 90° \\ \cos 90° & -\sin 90° \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

161. (a)

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix} = 0$$

[illegible] $= 0$

$\lambda(\lambda^2 + 4\lambda + 3) = 0$

$\lambda = 0, (\lambda + 1)(\lambda + 3) = 0$

$\lambda = -1, -3$

$\lambda = 0, -1, -3$

162. (c)

The given matrix is symmetric and all its eigen values are distinct. Hence all its eigen vectors are orthogonal. If $\lambda = 0$ eigen vector is $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Then corresponding to [illegible] eigen vector will be orthogonal to x_1 i.e. $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$x_1^T x_2 = [1 \; 0 \; 1]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$

163. Sol.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & [illegible] \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_1$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ [illegible]

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$= \begin{bmatrix} [illegible] \end{bmatrix}$$

$R_3 \to R_3 \times ?,\ R_5 \to R_5 - R_4$

$$\begin{bmatrix} [illegible] \end{bmatrix}$$

From here,

$\therefore \quad \rho(A) = ?$

104. (c)

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & ? & ? & ? & ? \\ ? & ?-\lambda & ? & ? & ? \\ 1 & 5 & -\lambda & ? & ? \\ ? & ? & ? & ?-\lambda & 2 \\ 2 & ? & ? & ? & 1-\lambda \end{vmatrix} = 0$$

Sum of all elements in any one row must be zero

i.e. $15 - \lambda = 0$

$\lambda = 15$

105. (c)

$$A = \begin{bmatrix} ? & ? & ? \\ 1 & 0 & 2 \\ ? & ? & ? \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ ? & ? & 10 \\ ? & ? & 8 \end{bmatrix}$$

$R_2 \to R_2 - ?R_1$ and $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 10 & ? \\ 0 & 2 & ? \end{bmatrix}$$

$R_3 \to R_3 - \frac{?}{10}R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 10 & ? \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in Echelon form

rank of matrix M is

$\rho(M) = 2$

106. Sol.

$$P - Q = \begin{bmatrix} 0 & -1 & -2 \\ 4 & ? & 10 \\ ? & 6 & ? \end{bmatrix}$$

$|P - Q| = ? - ? = 0$

So, rank ≠ 3

Consider 2×2 minor $\begin{bmatrix} ? & -? \\ ? & ? \end{bmatrix} \neq 0$

So, rank of $P - Q$ is 2.

107. Sol.

$f(x) = x^3 - 4x^2 + ax + 30 = 0$

Now 2 is one of roots of this equation

So, $2^3 - 4 \times 2^2 + a \times 2 + 30 = 0$

$\Rightarrow 8 - 16 + 2a + 30 = 0$

$\therefore a = -11$

So the equation is $x^3 - 4x^2 - 11x + 30 = 0$

Now, by polynomial division we get

$$\frac{x^3 - 4x^2 - 11x + 30}{x - 2} = x^2 - 2x - 15$$

Roots of $x^2 - 2x - 15 = 0$ are

$$x = \frac{2 \pm \sqrt{4 + 60}}{2} = \frac{2 \pm 8}{2} = 5 \text{ and } -3$$

So the eigenvalues are 2, 5 and −3, the maximum absolute eigen value is 5.

108. (a)

The equation $Ax = b$ ... according to what is given

$$[a_1\ a_2 \ldots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1 + a_2 + a_3 + \ldots = 0$$

where x_i are column vectors of A. So, if we have C_i (not all zero) such that,

$$\sum C_i x_i = 0$$

It means the n column vectors are not linearly independent and hence

rank $(A) < n$.

So we have infinitely many solutions one of which will be [illegible]

169. (a)

$$|A| = [illegible]$$

$$[A - \lambda I] = \begin{bmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$(3-\lambda)(1-\lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

170. (b)

$$AB^T = [illegible]$$

171. (a)

Given that P is inverse of Q

$$P = Q^{-1} \qquad P^{-1} = Q$$

$$PQ = (Q^{-1}Q) \qquad QP = QQ^{-1}$$

$$PQ = I \qquad QP = I$$

$$\therefore \quad PQ = QP = I$$

172. (a)

$$A = [illegible]$$

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} [illegible] & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$[illegible] + 4 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda = 3, 3$$

Algebraic multiplicity of eigen value = 2, hence only one independent eigen vector exists.

173. (d)

$$x + y + z = 4 \qquad \text{...(1)}$$

$$x - y + z = 0 \qquad \text{...(2)}$$

$$2x + y - z = ? \qquad \text{...(3)}$$

Adding (1) and (2) & (2) and (3) gives

$2x + 2z = 4$ and $3x + [illegible] = [illegible]$ which gives $x = 1$, $z = 1$ and $y = 2$

All options (a), (b), (c) can be eliminated since they do not satisfy this condition. Only (d) satisfies all equation.

■■■■

2 Calculus

2.1 Limit

2.1.1 Definition

[illegible]

2.1.2 Right and Left Hand Limits

[illegible]

2.1.3 Various Formulae

[illegible]

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots, |x| < 1$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right), |x| < 1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

Remember: $\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

2.1.4 Some Useful Results

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x\to 0} \cos x = 1$ 3. $\lim_{x\to 0} \frac{\tan x}{x} = 1$ 4. $\lim_{x\to 0}(1+x)^{1/x} = e$

5. $\lim_{x\to 0}(1+ax)^{1/x} = e^a$ 6. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ 7. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

2.1.5 Indeterminate Forms

A fraction whose numerator and denominator both tend to zero as $x \to a$ is an example of an indeterminate form $0/0$ with a value [illegible]; these have no definite values. Other indeterminate forms are ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞. These forms do not have any definite number and hence are not acceptable as limits. To find limits in these cases, we use the L'Hospital's rule.

2.1.5.1 Indeterminate Forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

Use L'Hospital's Rule.

L'Hospital Rule: [illegible]

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} \phi(x) = 0$$

or $$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} \phi(x) = \infty,$$

then $$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

[illegible]

Working Rule: [illegible]

Caution: [illegible]

2.1.5.2 Indeterminate Form-II ($0 \times \infty$)

[illegible]

[illegible]

2.1.5.3 Indeterminate Form-III (0^0 or 1^∞ or ∞^0)

[illegible]

[illegible]

2.2 Continuity

2.2.1 Definition

[illegible]

1. [illegible]
2. [illegible]

Note: [illegible]

[illegible]

2.2.2 Continuity from Left and Continuity from Right

[illegible]

[illegible]

2.2.3 Continuity in an Open Interval

[illegible]

2.2.4 Continuity in a Closed Interval

[illegible]

1. [illegible]
2. [illegible]
3. [illegible]

2.3 Differentiability

[illegible]

[illegible]

[illegible]

2.3.1 Progressive and Regressive Derivatives

[illegible]

[illegible]

[illegible]

[illegible]

2.3.2 Differentiability in an Open Interval

[illegible]

2.3.3 Differentiability in a Closed Interval

[illegible]

1. [illegible] and
2. [illegible] and
3. [illegible]

2.3.4 Relationship between Differentiability and Continuity

Theorem: [illegible]

Note: [illegible]

[illegible]

[illegible]

[illegible]

2.4 Mean Value Theorems

2.4.1 Rolle's Theorem

[illegible]

1. [illegible] and
2. [illegible] and
3. [illegible]

[illegible]

Note: [illegible]

1. [illegible]
2. [illegible]
3. [illegible]

2.4.2 Geometrical Interpretation

[illegible]

[illegible]

[illegible]

There may be more than one point between A and B, the tangent at which is parallel to x-axis [as shown in Figure (b)] i.e. there may be more than one number c in (a, b) such that $f'(c) = 0$. Rolle's theorem ensures at least one such number c in (a, b) such that $f'(c) = 0$.

Remarks:

1. Rolle's theorem fails even if one of the three conditions is not satisfied by the function.
2. The converse of Rolle's theorem is not true, i.e., $f'(x)$ may vanish at a point in (a, b) without satisfying all the three conditions of Rolle's theorem.

Example 1.

Verify Rolle's theorem for the following functions:

(a) $f(x) = x^2 + x - 6$ in $[-3, 2]$

(b) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$

(c) $f(x) = (x^2 - 1)(x - 2)$ in $[-1, 2]$

Solution:

(a) Given $f(x) = x^2 + x - 6$ ...(i)

(i) Since $f(x)$ is a polynomial function, it is continuous in $[-3, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(-3, 2)$.

(iii) $f(-3) = (-3)^2 - 3 - 6 = 0$, $f(2) = 2^2 + 2 - 6 = 0 \Rightarrow f(-3) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied. Therefore, there exists at least one number $c \in (-3, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get $f'(x) = 2x + 1$

$$\text{Now } f'(c) = 0 \Rightarrow 2c + 1 = 0 \Rightarrow c = -\frac{1}{2}$$

$$\text{So there exists } -\frac{1}{2} \in (-3, 2) \text{ such that } f'\left(-\frac{1}{2}\right) = 0$$

Hence, Rolle's theorem is verified.

(b) Given $f(x) = (x - 1)(x - 2)^2$ ...(i)

(i) Since $f(x)$ is a polynomial function, it is continuous in $[1, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(1, 2)$.

(iii) $f(1) = (1 - 1)(1 - 2)^2 = 0$, $f(2) = (2 - 1)(2 - 2)^2 = 0 \Rightarrow f(1) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore there exists at least one number $c \in (1, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$\begin{aligned} f'(x) &= (x - 1) \cdot 2(x - 2) + (x - 2)^2 \cdot 1 \\ &= (x - 2)(2x - 2 + x - 2) \\ &= (x - 2)(3x - 4) \end{aligned}$$

$$\begin{aligned} \text{Now} \quad f'(c) &= 0 \\ \Rightarrow \quad (c - 2)(3c - 4) &= 0 \\ \Rightarrow \quad c &= 2, \frac{4}{3} \end{aligned}$$

But $c = 2 \notin (1, 2)$, therefore $c = \frac{4}{3}$

So there exists $\frac{4}{3} \in (1, 2)$ such that $f'\left(\frac{4}{3}\right) = 0$.

Hence, Rolle's theorem is verified.

(ii) Given $f(x) = (x^2-1)(x-2)$ ...(i)

(i) Since $f(x)$ is a polynomial function, it is continuous in $[-1, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(-1, 2)$.

(iii) $f(-1) = (1-1)(-1-2) = 0$, $f(2) = (4-1)(2-2) = 0 \Rightarrow f(-1) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore there exists at least one real number c in $(-1, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = (x^2-1)\cdot 1 + (x-2)\cdot 2x = 3x^2 - 4x - 1$$

Now $$f'(c) = 0 \Rightarrow 3c^2 - 4c - 1 = 0$$

$$\Rightarrow \quad c = \frac{4 \pm \sqrt{16 + 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{2 \pm \sqrt{7}}{3}$$

Also $-1 < \frac{2-\sqrt{7}}{3} < \frac{2+\sqrt{7}}{3} < 2 \Rightarrow \frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ both lie in $(-1, 2)$

So there exist two real numbers $\frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ in $(-1, 2)$ such that

$$f'\left(\frac{2-\sqrt{7}}{3}\right) = 0 \text{ and } f'\left(\frac{2+\sqrt{7}}{3}\right) = 0$$

Hence, Rolle's theorem is verified.

Example 3.

Verify Rolle's theorem for the following function and find point(s) in (a, b) where the derivative vanishes:

$$f(x) = \sin x + \cos x - 1 \text{ in } \left[0, \frac{\pi}{2}\right]$$

Solution:

Given $$f(x) = \sin x + \cos x - 1 \quad \text{...(i)}$$

(a) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

(b) $f(x)$ is derivable in $\left(0, \frac{\pi}{2}\right)$ and

(c) $f(0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1 = 0$

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} - 1 = 1 + 0 - 1 = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore there exists at least one real number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

Now $$f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow \tan c = 1$$

$$\Rightarrow \quad c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots \text{ but } c \in \left(0, \frac{\pi}{2}\right) \Rightarrow c = \frac{\pi}{4}$$

So there exists $c = \frac{5}{2}$ such that $f'\left(\frac{5}{2}\right) = 0$

Hence Rolle's theorem is verified and $c = \frac{5}{2}$

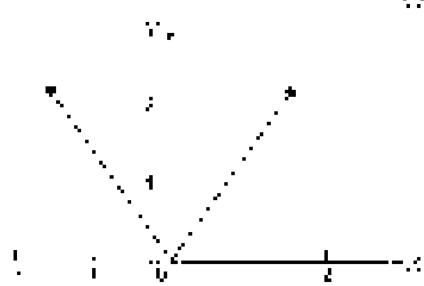
Example 3.

Discuss the applicability of Rolle's theorem for the function $f(x) = |x|$ in $[-2, 2]$

Solution:

Given $f(x) = |x|$, $x \in [-2, 2]$ ...(1)

The graph of $f(x) = |x|$ in $[-2, 2]$ is shown in figure.



(i) $f(x)$ is continuous in $[-2, 2]$

(ii) Differentiating (1) w.r.t. x we get

$$f'(x) = \frac{|x|}{x}, \; x \neq 0$$

⇒ The derivative of $f(x)$ does not exist at $x = 0$

⇒ $f(x)$ is not derivable in $(-2, 2)$

Thus, the condition (ii) of Rolle's theorem is not satisfied, therefore Rolle's theorem is not applicable to the function $f(x) = |x|$ in $[-2, 2]$.

Moreover $f(-2) = |-2| = 2$ and $f(2) = |2| = 2 \Rightarrow f(-2) = f(2)$, so the condition (iii) of Rolle's theorem is satisfied.

Further it is clear from the graph that there is no point on the curve $y = |x|$ in $(-2, 2)$ at which the tangent is parallel to x-axis.

2.4.3 Lagrange's Mean Value Theorem

If a function $f(x)$ is

1. Continuous in a closed interval $[a, b]$ and
2. Differentiable in open interval (a, b) i.e. $a < x < b$

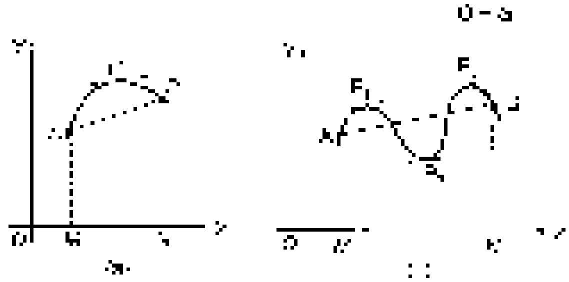
⇒ Then there exist at least one value c in the open interval $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2.4.4 Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ respectively.

Since $f(x)$ is continuous in $[a, b]$, the graph of the curve $y = f(x)$ is continuous from A to B. Again since $f(x)$ is derivable in (a, b), the curve $y = f(x)$ has a tangent at each point between A and B. Also as $x \neq a$, the slope of the chord AB exists and the slope of the chord AB $= \frac{f(b) - f(a)}{b - a}$



Thus Lagrange's Mean Value Theorem asserts that there is at least one point [illegible] at which the tangent is parallel to the chord AB. There may exist more than one such point [illegible] Lagrange's mean value theorem [illegible]

[illegible] $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Remarks:

1. Lagrange's mean value theorem [illegible] conditions.

2. The converse of Lagrange's mean value theorem may not be true [illegible] may be equal to $\dfrac{f(b) - f(a)}{b - a}$ at a point [illegible] Lagrange's mean value theorem.

Example 1.

Verify Lagrange's mean value theorem for the following functions in the given interval and find [illegible]

(a) $f(x) = x^2 + 2x + 3$ in $[4, 6]$ (b) $f(x) = px^2 + qx + r$, $p \neq 0$ in $[a, b]$

Solution:

(a) Given $f(x) = x^2 + 2x + 3$

(i) $f(x)$ being a polynomial function is continuous in $[4, 6]$ ...(i)

(ii) $f(x)$ being a polynomial function is differentiable in $(4, 6)$.

Thus all the conditions of Lagrange's mean value theorem are satisfied. Therefore there exists at least one real number $c \in (4, 6)$ such that

$$f'(c) = \frac{f(6) - f(4)}{6 - 4}$$

$$f(6) = 6^2 + 2 \cdot 6 + 3 = 51, \quad f(4) = 4^2 + 2 \cdot 4 + 3 = 27$$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2x + 2 \Rightarrow f'(c) = 2c + 2$$

$$\therefore \quad f'(c) = \frac{f(6) - f(4)}{6 - 4} \Rightarrow 2c + 2 = \frac{51 - 27}{2} = 12$$

$$\Rightarrow \quad 2c = 10 \Rightarrow c = 5$$

Thus there exists $c = 5$ in $(4, 6)$ such that $f'(5) = \dfrac{f(6) - f(4)}{6 - 4}$

Hence Lagrange's mean value theorem is verified and $c = 5$

(b) Given $f(x) = px^2 + qx + r$, $p \neq 0$

(i) $f(x)$ being a polynomial function is continuous in $[a, b]$

(ii) $f(x)$ being a polynomial function is differentiable in (a, b)

Thus all the conditions of Lagrange's mean value theorem are satisfied, therefore there exists a real number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

$$f(x) = px^2 + qx + r, \quad f'(x) = 2px + q$$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2px + q \Rightarrow f'(c) = 2pc + q$$

$$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible] and [illegible]

Thus there exists $c = \frac{a+b}{2} \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Hence Lagrange's mean value theorem is verified and $c = \frac{a+b}{2}$

Example 2.

[illegible] $y = x^3$ where the tangent is parallel to the chord joining [illegible]

Solution:

$$f(x) = x^3 \text{ in the interval } [1, 3]$$

(i) [illegible]

(ii) [illegible]

[illegible]

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = [illegible]$$

Differentiating [illegible]

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$$

[illegible] $f'(c) = \frac{f(3)-f(1)}{3-1} = 3c^2 = \frac{27-1}{2} = 3c^2 = 13$

$\Rightarrow$ $c^2 = \frac{13}{3}$ [illegible]

$\Rightarrow$ $c = \pm\frac{\sqrt{39}}{3}$

But [illegible] $c = \frac{\sqrt{39}}{3}$

When $x = \frac{\sqrt{39}}{3}$, then [illegible]

Hence the [illegible] $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ [illegible]

[illegible]

Example 3.

Does the Lagrange's mean value theorem apply to [illegible] What [illegible]

Solution:

Given $f(x) = x^{1/3}$ in $[-1, 1]$ ...(1)

(i) $f(x)$ is continuous in $[-1, 1]$.

(ii) Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}, x \neq 0 \quad ...(2)$$

$\Rightarrow$ The derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow$ $f(x)$ is not derivable in $(-1, 1)$.

Thus, the condition (ii) of Lagrange's mean value theorem is not satisfied by the function $f(x) = x^{1/3}$ in $[-1, 1]$ and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in $[-1, 1]$, i.e., Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in $[-1, 1]$.

Conclusion. However, from (2), $f'(c) = \frac{1}{3c^{2/3}}, c \neq 0$

Also $f(-1) = (-1)^{1/3} = -1$ and $f(1) = 1^{1/3} = 1$ (we have taken only real values)

$$\therefore \quad f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$\Rightarrow \quad \frac{1}{3c^{2/3}} = \frac{1 - (-1)}{1 - (-1)} = \frac{2}{2} = 1$$

$$\Rightarrow \quad c^{2/3} = \frac{1}{3} \Rightarrow c^2 = \frac{1}{27} \Rightarrow c = \pm\frac{1}{3\sqrt{3}}$$

As $-1 < -\frac{1}{3\sqrt{3}} < \frac{1}{3\sqrt{3}} < 1 \Rightarrow c = \pm\frac{1}{3\sqrt{3}}$ both lie in $(-1, 1)$.

Thus, we find that there exist two real numbers $c = \pm\frac{1}{3\sqrt{3}}$ in $(-1, 1)$ such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$.

It shows that the converse of Lagrange's mean value theorem may not be true.

2.4.5 Some applications of Lagrange's Mean Value theorem

1. If a function $f(x)$ is

   (a) continuous in $[a, b]$

   (b) derivable in (a, b) and

   (c) $f'(x) > 0$ for all x in (a, b), then $f(x)$ is strictly increasing function in $[a, b]$.

   Proof. Let x_1, x_2 be any two numbers in $[a, b]$ such that $a \le x_1 < x_2 \le b$, then $f(x)$ satisfies both the conditions of Lagrange's mean value theorem in $[x_1, x_2]$, therefore, there exists at least one real number c in (x_1, x_2) such that

   $$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

   $\Rightarrow \quad (x_2 - x_1) f'(c) = f(x_2) - f(x_1)$

   But $f'(x) > 0$ for all x in $(a, b) \Rightarrow f'(c) > 0$ (as c lies in (x_1, x_2)). Also $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

   $\Rightarrow \quad (x_2 - x_1) f'(c) > 0$

   $\Rightarrow \quad f(x_2) - f(x_1) > 0$

   $\Rightarrow f(x_1) < f(x_2)$ for all x_1, x_2 such that $a \le x_1 < x_2 \le b$.

   Hence $f(x)$ is strictly increasing in $[a, b]$.

8. If a function f(x) is
   (a) continuous in [a, b]
   (b) derivable in (a, b)
   (c) $f'(x) < 0$ for all $a < x < b$, then f(x) is strictly decreasing function in [a, b]
   (proof same as above)

2.4.6 Some Important Deductions from Mean Value Theorems

1. If a function f(x) be such that $f'(x) = 0$ throughout the interval, then f(x) must be constant throughout the interval.
2. If f(x) and $\phi(x)$ be two functions such that $f'(x) = \phi'(x)$ throughout the interval (a, b), then f(x) and $\phi(x)$ differ only by a constant.
3. If f(x) is
   (a) continuous in closed interval [a, b]
   (b) differentiable in open interval (a, b)
   (c) $f'(x) \geq 0$ for $a < x < b$, then f(x) is monotonically increasing function in the closed interval [a, b] and if $f'(x)$ is positive for $a < x < b$, then f(x) is strictly increasing function in the closed interval [a, b].

2.4.7 Some Standard Results on Continuity and Differentiability of Commonly used Functions

It is useful to remember the following facts regarding common functions while solving questions of continuity or differentiability or checking mean value theorems.

1. Constant function is continuous and differentiable everywhere [$f'(x) = 0, \forall x$].
2. Any polynomial function is continuous and differentiable everywhere.
3. The exponential function (e^x, a^x), sin x as well as cos x are also continuous and differentiable everywhere.
4. Log function, trigonometric and inverse trigonometric functions are differentiable within their domains.
5. tan x is discontinuous at $x = \pm\pi/2, \pm 3\pi/2, \ldots$
6. |x| is continuous but not differentiable at $x = 0$.
7. If $f(x) = |x - a|$ then f(x) is continuous but not differentiable at $x = a$.
8. Sum, difference, product, quotient (with nonzero denominator) of continuous and differentiable functions are continuous and differentiable.

2.5 Computing the Derivative

Rules of Differentiation:

$$(f + g)' = f' + g' \qquad \text{(Sum rule)}$$

$$(f - g)' = f' - g' \qquad \text{(Difference rule)}$$

$$(fg)' = fg' + gf' \qquad \text{(Product rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \qquad \text{(Quotient rule)}$$

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx} \qquad \text{(Chain rule)}$$

Using the above five rules, we can differentiate most of the cases where (f) is an explicit function of x.

The following is the table of derivatives of commonly occurring functions.

$g(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\left(\frac{1}{x}\right)\log_a e$
e^x	e^x
a^x	$a^x \log_e a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\cot x$	$-\text{cosec}^2 x$
$\sinh x$	$\cosh x$

$g(x)$	$f'(x)$
$\cosh x$	$\sinh x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\text{cosec}^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\lvert x \rvert$	$\begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$

Most explicit functions can be differentiated by using above rules along with the four rules of differentiation. For more complicated cases, we have a few additional more advanced methods of differentiation as given below:

1. Differentiation by substitution
2. Implicit differentiation
3. Logarithmic differentiation
4. Parametric differentiation

2.5.1 Differentiation by Substitution

There are no general rules for finding suitable substitution and the proper choice depends on the form of the function to be differentiated. However some useful suggestions are given below:

If the function contains an expression of the form:

1. $\sqrt{a^2-x^2}$, put $x = a\sin\theta$ or $x = a\cos\theta$
2. $\sqrt{a^2+x^2}$, put $x = a\tan\theta$ or $x = a\cot\theta$
3. $\sqrt{x^2-a^2}$, put $x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
4. $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, put $x = a\cos\theta$
5. $a\cos x \pm b\sin x$, put $a = r\cos\theta$ and $b = r\sin\theta$, $(r > 0)$

Example:

Differentiate the following functions by suitable substitution method:

(a) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(b) $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

(c) $\cos^{-1}\left[\frac{x-x^{-1}}{x+x^{-1}}\right]$

(d) $\tan^{-1}\left[\sqrt{1+x^2}+x\right]$

Solution:

(a) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, put $x = \tan\theta$ so $\theta = \tan^{-1} x$.

Then $$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$$

$= 2\tan^{-1}x$, differentiating w.r.t. x we get

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

(b) Let $$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right) \text{ Put } x = \tan\theta \text{ i.e. } \theta = \tan^{-1}x$$

then $$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \tan^{-1}\left[\frac{\frac{1}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$$

$= \frac{\pi}{2}-\frac{\theta}{2} = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}x$, differentiating w.r.t. x we get

$$\frac{dy}{dx} = 0 - \frac{1}{2}\cdot\frac{1}{1+x^2} = -\frac{1}{2(1+x^2)}$$

(c) Let $$y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left[\frac{x-\frac{1}{x}}{x+\frac{1}{x}}\right] = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

Put $x = \tan\theta$ i.e. $\theta = \tan^{-1}x$

then $$y = \cos^{-1}\left(\frac{\tan^2\theta-1}{\tan^2\theta+1}\right) = \cos^{-1}\left(-\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}(\cos(\pi-2\theta))$

$= \pi - 2\theta = \pi - 2\tan^{-1}x$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 - 2\cdot\frac{1}{1+x^2} = -\frac{2}{1+x^2}$$

(d) Let $y = \tan^{-1}(\sqrt{1+x^2}-x)$

put $x = \cot\theta$ i.e. $\theta = \cot^{-1}x$

then $y = \tan^{-1}(\sqrt{1+\cot^2\theta}-\cot\theta)$

$= \tan^{-1}(\operatorname{cosec}\theta - \cot\theta)$

$$= \tan^{-1}\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right] = \frac{\pi}{2}-\frac{\theta}{2}$$

$$= \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x, \text{ differentiating w.r.t. } x \text{ we get}$$

$$\frac{dy}{dx} = 0-\frac{1}{2}\left(\frac{1}{1+x^2}\right) = -\frac{1}{2(1+x^2)}$$

2.5.2 Implicit Differentiation

If y is a function of x defined by an equation such as

$$y = 3x^4 - 5x^3 + 7x^2 + \sqrt{2}x - 5 \quad \text{...(i)}$$

y is said to be defined explicitly in terms of x and we write $y = f(x)$ where

$$f(x) = 3x^4 - 5x^3 + 7x^2 + \sqrt{2}x - 5$$

However, if x and y are connected by an equation of the form

$$x^2y^3 - 5xy^2 + 7y - 8x^2 + 3 = 0 \quad \text{...(ii)}$$

i.e. $f(x, y) = 0$ then y cannot be expressed explicitly in terms of x. But still the value of y depends upon x. In such a case there may exist one or more functions f corresponding with x which satisfy equation (ii) or there may not exist any of the functions satisfying equation (ii).

For example, consider the equations

$$x^2 + y^2 - 25 = 0 \quad \text{...(iii)}$$

and

$$x^2 + y^2 + 25 = 0 \quad \text{...(iv)}$$

In equation (iii), y may be expressed explicitly in terms of x but y is not a function of x. Here we have two functions of x (at least in the sense of y) which are considered to be the dependent variables f_1 and f_2 defined as

$f_1(x) = \sqrt{25-x^2}$ and $f_2(x) = -\sqrt{25-x^2}$ which satisfy equation (iii).

In equation (iv), there are no real values of x that can satisfy it.

In cases (ii), (iii) and (iv), we say that y is an implicit function of x and in all such cases we find the derivative of y with respect to x (i.e. the derivative of y with respect to x) by the process called implicit differentiation. Or in other words, we differentiate implicitly an equation that defines one variable as an implicit function of the other variable and assume that the function is differentiable.

Example 1.

Find $\frac{dy}{dx}$ when $x^2 + xy + y^2 = 100$

Solution:

Given $x^2 + xy + y^2 = 100$

Keeping in mind that y is a function of x, differentiating both sides w.r.t. x, we get

$$2x + \left(x\frac{dy}{dx} + y\cdot 1\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-2y)\frac{dy}{dx} = -2x + y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x-y}{-x+2y}$$

Example 2.

If $x^{2/3} + y^{2/3} = a^{2/3}$, find $\frac{dy}{dx}$

Solution:

Given, $\quad x^{2/3} + y^{2/3} = a^{2/3}$ ...(i)

Differentiating both sides of (i) with respect to x, regarding y as a function of x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{x^{1/3}} + \frac{1}{y^{1/3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}$$

Example 3.

If $\sin^2 y + \cos xy = k$, find $\frac{dy}{dx}$

Solution:

Given, $\quad \sin^2 y + \cos xy = k$

Differentiating both sides of (i) w.r.t. x regarding y as function of x, we get

$$2\sin y \cos y \frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$$\Rightarrow \quad 2\sin y\cos y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Example 4.

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$, prove that $(1-2y)\frac{dy}{dx} = \sin x$

Solution:

Given, $\quad y = \sqrt{\cos x + y}$

$$\Rightarrow \quad y^2 = \cos x + y$$

$$\Rightarrow \quad y^2 - y = \cos x$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x$$

$$\Rightarrow \quad (1-2y)\frac{dy}{dx} = \sin x$$

2.5.3 Logarithmic Differentiation

In order to simplify the differentiation of some functions, we first take logarithm and then differentiate. Such a process is called logarithmic differentiation. This is useful in the following two types of problems:

1. When the given function is a product or quotient of functions, then the logarithm converts the product into a sum and this facilitates the differentiation.
2. When the variable occurs in the exponent i.e. the given function is of the form $f(x)^{g(x)}$.

   The derivative of such a function is obtained by the following method:

   Let $y = u^v$, taking logarithm on both sides, we get

   $\log y = v \log u$. Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(v \log u)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\frac{d}{dx}(v \log u) = u^v \frac{d}{dx}(v \log u)$$

Example 1.

Differentiate the following functions w.r.t. x:

(a) x^x

(b) $\cos(x^x)$

Solution:

(a) Let $y = x^x$.

Taking logarithm of both sides, we get

$$\log y = x \log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

(b) Let $y = \cos(x^x)$; differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(x^x)\frac{d}{dx}(x^x)$$

Now $\frac{d}{dx}(x^x)$ has been obtained previously in part (a).

So $\quad \frac{dy}{dx} = -\sin(x^x) \cdot x^x(1 + \log x)$

Example 2.

If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Solution:

Given $x^y = e^{x-y}$; taking logarithm on both sides, we get

$$y \log x = (x - y)\log e = (x - y) \quad [\because \log e = 1]$$

$$\Rightarrow \quad y + y \log x = x$$

$$\Rightarrow \quad (1 + \log x)y = x$$

$\Rightarrow \quad y = \dfrac{x}{1+\log x}$ (differentiating w.r.t. x, we get)

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\cdot\frac{1}{x}}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

2.5.4 Derivatives of Functions in Parametric forms

If x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e. if $x = f(t)$ and $y = g(t)$, then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

Or
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \left[\text{provided } \frac{dx}{dt} \neq 0\right]$$

Example 1.

If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{2}$.

Solution:

Given $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

Differentiating both w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

and
$$\frac{dy}{d\theta} = a(0 - (-\sin\theta)) = a\sin\theta$$

We know that
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$\therefore$
$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$\therefore$
$$\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \tan\frac{\pi}{4} = 1$$

Example 2.

Differentiate $\dfrac{x^2}{1+x^2}$ w.r.t. x^2.

Solution:

Let $y = \dfrac{x^2}{1+x^2}$ and $z = x^2$ so that $\dfrac{dy}{dz}$ is wanted.

Differentiating both w.r.t. x we get

$$\frac{dy}{dt} = \frac{(1 + t^2)\,2a - [illegible]}{(1 + t^2)^2} = \frac{[illegible]}{(1 + t^2)^2}$$

and $\frac{dx}{dt} = [illegible]$

We know that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

∴ $\frac{dy}{dx} = [illegible]$

2.6 Applications of Derivatives

There are four areas where derivatives are used:

1. Increasing and Decreasing Functions
2. Maxima and Minima
   (a) Relative maxima and minima
   (b) Absolute maxima and minima
3. Taylor's and Maclaurin's Series Expansion of Functions
4. [illegible]

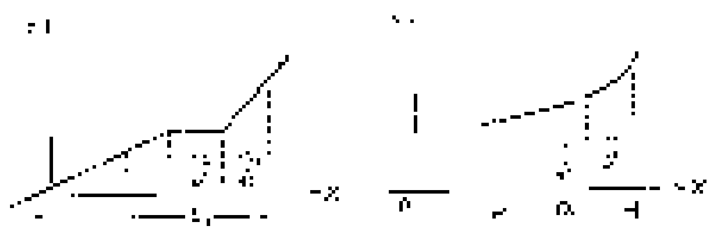
2.6.1 Increasing and Decreasing Functions

Let $f(x)$ be a real valued function defined on an interval I [illegible] increasing function in an interval I (a subset of R) if

[illegible] $x_1, x_2 \in I$
$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

and f is called strictly increasing function (or monotonically increasing function) in I if

[illegible] $x_1, x_2 \in I$
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



Increasing $f(x)$
(a)

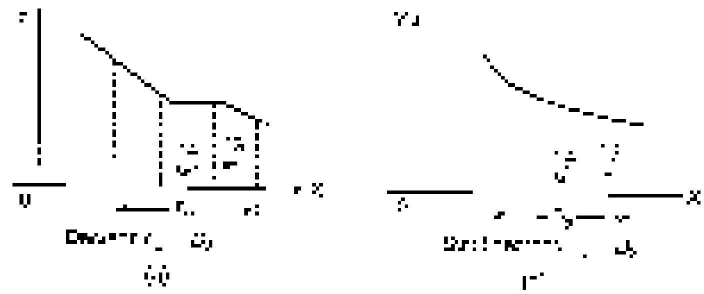
Strictly increasing $f(x)$
(b)

Analogously f is called a decreasing function in an interval I (a subset of R) if

[illegible] $x_1, x_2 \in I$
$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

and is called strictly decreasing (or monotonically decreasing) function in I if

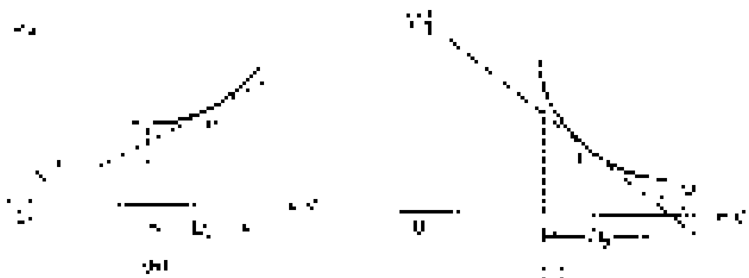
[illegible] $x_1, x_2 \in I$
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



2.6.1.1 Conditions for an Increasing or a Decreasing Function

Now we shall see how to use derivative of a function to determine where it is increasing and where it is decreasing. [illegible]

We know that the derivative [illegible] at a point P of a curve represents the slope of the tangent to the curve at P.

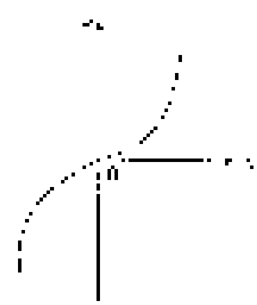


[illegible]

[illegible]

But [illegible] for example, consider the function $f(x) = x^3$, [illegible]

A [illegible] shown in figure [illegible] increasing function. However, here [illegible] $= 0$ [illegible] the slope of the tangent at $x = 0$ [illegible] zero.



[illegible] we have,

1. [illegible]

2. [illegible]

Conversely, [illegible] increasing [illegible] positive [illegible] decreasing [illegible] negative. [illegible]

Theorem 1: [illegible]

1. [illegible]

2. [illegible]

Theorem 2: If a function f is continuous in $[a, b]$ and [illegible]

1. [illegible] then f is decreasing in $[a, b]$.
2. [illegible]

Remark: [illegible] Lagrange's Mean Value Theorem.

Corollary: [illegible]

1. [illegible] in $[a, b]$.
2. [illegible] in $[a, b]$.

Example 1.

Prove that the function [illegible]

Solution:

[illegible]

Example 2.

Prove that the function [illegible] is strictly increasing on R.

Solution:

[illegible]

Example 3.

[illegible]

Solution:

[illegible]

$\Rightarrow$ the given function is strictly decreasing.

Example 4.

Prove that the function [illegible] is strictly increasing [illegible]

Solution:

[illegible]

$\Rightarrow$ [illegible] is strictly increasing function [illegible]

Example 5.

Find the intervals in which the following functions are strictly increasing or strictly decreasing:

(a) $f(x) = 10 - 6x - 2x^2$

(b) $f(x) = x^3 - 12x^2 + 36x + 17$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

Solution:

(a) Given, $f(x) = 10 - 6x - 2x^2, D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = 0 - 6 - 4x = -4\left(x + \frac{3}{2}\right)$$

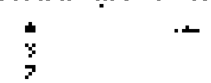
Putting $f'(x) = 0$, we get [illegible]

$$\Rightarrow \quad x + \frac{3}{2} = 0$$

$$\Rightarrow \quad x = -\frac{3}{2}$$

So there is only one critical point $x = -\frac{3}{2}$

Putting this critical point on the number line we get the following picture

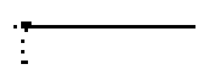


So the critical point divides the real number line into two regions which are $x \in \left(-\infty, -\frac{3}{2}\right)$ and $x \in \left(-\frac{3}{2}, \infty\right)$

Now we find $f'(0) = -6$ which is negative so in the region $x \in \left(-\frac{3}{2}, \infty\right)$ (which contains $x = 0$) the function is strictly decreasing.

Therefore the other region $x \in \left(-\infty, -\frac{3}{2}\right)$ is the region in which the function is strictly increasing.

This is shown in the following figure with the sign of $f'(x)$ in each region of the number line.



(b) Given, $f(x) = x^3 - 12x^2 + 36x + 17, D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12)$$

$$= 3(x - 2)(x - 6)$$

Putting $f'(x) = 0$, $3(x - 2)(x - 6) = 0$

$$\Rightarrow \quad (x - 2)(x - 6) = 0$$

$\therefore \quad x = 2$ and $x = 6$ are the two critical points.

Putting these critical points on number line, we get the following picture.

So the critical points divide the real number line into three regions which are $x \in (-\infty, 2)$, $x \in (2, 5)$ and $x \in (5, \infty)$.

Now we find $f'(0) = [illegible]$ which is positive and lies in the region $x \in (-\infty, 2)$ (which contains $x = 0$), so the function is strictly increasing.

Therefore in the next region i.e. $x \in (2, 5)$, the function is strictly decreasing and in the next region $x \in (5, \infty)$, the function is again strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

So the final region in which the function is strictly increasing is $x \in (-\infty, 2) \cup (5, \infty)$ and the region in which the function is strictly decreasing is $x \in (2, 5)$.

(c) Given $f(x) = -2x^3 - 9x^2 - 12x + [illegible]$

Differentiating w.r.t. x, we get

$$f'(x) = -6x^2 - 18x - 12$$
$$= -6(x^2 + 3x + 2)$$
$$= -6(x + 2)(x + 1)$$

Putting $f'(x) = 0$ i.e. $-6(x + 2)(x + 1) = 0$

$\Rightarrow (x + 2)(x + 1) = 0$

$\Rightarrow x = -2$ and $x = -1$ are the critical points.

Putting these critical points on the number line we get the following picture.

So the critical points divide the real number line into three regions which are $x \in (-\infty, -2)$, $x \in (-2, -1)$ and $x \in (-1, \infty)$.

Now we find $f'(0) = -6(0 + 2)(0 + 1) = -12$ which is negative and lies in the region $x \in (-1, \infty)$ (which contains $x = 0$), the function is strictly decreasing.

Therefore in the next adjacent region on the left i.e. $x \in (-2, -1)$ the function is strictly increasing and in the next adjacent region on the left i.e. $(-\infty, -2)$, the function is again strictly decreasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

So the final region in which the function is strictly increasing is $x \in (-2, -1)$ and the region in which the function is strictly decreasing is $x \in (-\infty, -2) \cup (-1, \infty)$.

2.6.2 Relative or Local Maxima and Minima (of function of a single independent variable)

Definitions: A function $f(x)$ is said to be a local or relative maximum at $x = a$ if there exists a positive number δ such that $f(a + h) < f(a)$ for all values of h other than zero, in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to have a local or relative minimum at $x = a$ if there exists a positive number δ such that $f(a + h) > f(a)$ for all values of h other than zero, in the interval $(-\delta, \delta)$.

[illegible]

[illegible] Turning Points.

2.6.2.1 Properties of Relative Maxima and Minima

1. [illegible]
2. Maximum and minimum values must occur alternately.
3. There may be several maximum or minimum values of same function.
4. [illegible]
5. [illegible]
6. [illegible]

2.6.2.2 Conditions for Maximum or Minimum Values

The necessary condition that $f(x)$ [illegible] $f'(x) = 0$.

2.6.2.3 Definition of Stationary Values

A function $f(x)$ is said to be stationary at $x = a$ if $f'(a) = 0$.

Thus the function $f(x)$ [illegible] maximum or minimum [illegible] $x = a$.

2.6.2.4 Sufficient Conditions of Maximum or Minimum Values

There is a maximum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ is negative.

Similarly there is a minimum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ is positive.

Note: If $f''(a)$ [illegible]

[illegible]

2.6.2.5 Working rule for Maxima and Minima of f(x)

1. Find $f'(x)$ and equate it to zero.
2. [illegible]
3. Find $f''(x)$ [illegible]
4. [illegible]

2.6.3 Working Rules for Finding (Absolute) Maximum and Minimum in Range [a, b]

[illegible]

1. Evaluate $f(x)$ [illegible] $x = a$.
2. Evaluate $f(x)$ [illegible]
3. Find $f'(x)$ [illegible]

Then the maximum of these values is the absolute maximum of the given function and the minimum of these values is the absolute minimum of the given function.

Example 1.

Find the absolute maximum and minimum values of

(a) $f(x) = 2x^3 - 9x^2 + 12x - 5$ in $[0, 3]$

(b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

Also find point of maxima and minima.

Solution:

(a) Given, $f(x) = 2x^3 - 9x^2 + 12x - 5$ ...(i)

f is continuous and differentiable in $[0, 3]$, since it is a polynomial.

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2(3x^2) - 9(2x) + 12 = 6x^2 - 18x + 12$$

Now $f'(x) = 0$

$\Rightarrow \quad 6(x^2 - 3x + 2) = 0$

$\Rightarrow \quad x^2 - 3x + 2 = 0$

$\Rightarrow \quad (x-1)(x-2) = 0$

$\Rightarrow \quad x = 1, 2$

Also 1, 2 and 0 and $3 \in [0, 3]$, thus 0, 3, 1 and 2 are the candidates for critical/turning points.

Putting, $f(1) = 2(1)^3 - 9(1)^2 + 12(1) - 5 = 2 - 9 + 12 - 5 = 0$

$f(2) = 2(2)^3 - 9(2)^2 + 12 \times 2 - 5 = 16 - 36 + 24 - 5 = -1$

$f(0) = -5$

and $f(3) = 2 \times 3^3 - 9 \times 3^2 + 12 \times 3 - 5 = 54 - 81 + 36 - 5 = 4$

Therefore, the absolute maximum value = 4 and the absolute minimum value = −5. The point of maxima is 3 and the point of minima is 0.

(b) Given, $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

Differentiating (i) w.r.t. x we get

$$f'(x) = 12\cdot\frac{4}{3}x^{1/3} - 6\cdot\frac{1}{3}x^{-2/3} = 16x^{1/3} - \frac{2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}}$$

Now $f'(x) = 0$

$$\frac{2(8x-1)}{x^{2/3}} = 0$$

$\Rightarrow \quad x = \frac{1}{8}$

As $\frac{1}{8} \in [-1, 1]$, $\frac{1}{8}$ is a critical point.

Also we note that f is not differentiable at $x = 0$.

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3} = 12\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)$$

$$= 12 \times \frac{1}{16} - 3 = \frac{3}{4} - 3 = -\frac{9}{4}$$

$f(0) = 12(0) - 6(0) = 0$

$f(-1) = 12(-1)^{4/3} - 6(-1)^{1/3} = 12(1) - 6(-1) = 18$

$f(1) = 12(1)^{4/3} - 6(1)^{1/3} = 12 - 6 = 6$

[illegible] is maximum at [illegible], $\frac{1}{6}$

Example 3.

[illegible]

Solution:

[illegible] ...(i)

[illegible]

2.6.4 Taylor's and Maclaurin's Series Expansion of Functions

2.6.4.1 Taylor's Series

[illegible] ...(i)

[illegible]

[illegible] ...(ii)

1. [illegible]

2. [illegible]

3. [illegible]

Hence [illegible]

[illegible]

Cor. 1. Taking $n = 1$ in (1), we obtain the Lagrange's Mean-value Theorem.

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{n}(\theta x) \qquad \dots(4)$$

which is known as Maclaurin's theorem with Lagrange's form of remainder.

Example

If $f(x) = \log(1 + x)$, $x > 0$, using Taylor's theorem, show that for $0 < \theta < 1$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

Solution:

Deduce that $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \qquad \dots(2)$$

Here $\qquad f(x) = \log(1+x)$, $\quad f(0) = 0$

$\therefore \qquad f'(x) = \frac{1}{1+x}$, $\quad f'(0) = 1$

$\qquad f''(x) = -\frac{1}{(1+x)^2}$, $\quad f''(0) = -1$

and $\qquad f'''(x) = \frac{2}{(1+x)^3}$, $\quad f'''(\theta x) = \frac{2}{(1+\theta x)^3}$

Substituting in (2), we get $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \qquad \dots(3)$

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or $(1+\theta x)^3 > 1$ $\quad \therefore \frac{1}{(1+\theta x)^3} < 1$

$$\therefore \quad x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $\qquad \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \qquad$ [by (3)]

3.6.4.1 Maclaurin's Series

If $f(x)$ can be expanded as an infinite series, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \infty \qquad (5)$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (3) on page 104 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (5).

Example:

Using Maclaurin's series, expand tan x upto the term containing x^5.

Solution:

Let $\quad f(x) = \tan x \qquad f(0) = 0$

$\therefore \quad f'(x) = \sec^2 x = 1 + \tan^2 x \qquad f'(0) = 1$

$$f''(x) = 2\tan x \sec^2 x = 2\tan x\,(1 + \tan^2 x)$$
$$= 2\tan x + 2\tan^3 x \qquad f''(0) = 0$$
$$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6\tan^2 x\,(1 + \tan^2 x)$$
$$= 2 + 8\tan^2 x + 6\tan^4 x \qquad f'''(0) = 2$$
$$f^{iv}(x) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x$$
$$= 16\tan x\,(1 + \tan^2 x) + 24\tan^3 x\,(1 + \tan^2 x)$$
$$= 16\tan x + 40\tan^3 x + 24\tan^5 x \qquad f^{iv}(0) = 0$$
$$f^{v}(x) = 16\sec^2 x + 120\tan^2 x \sec^2 x + 120\tan^4 x \sec^2 x$$
$$f^{v}(0) = 16$$

and so on.

Substituting the values of $f(0)$, $f'(0)$ etc. in the Maclaurin's series, we get

$$\tan x = 0 + x \cdot 1 + 0 \cdot \frac{x^2}{2!} + \frac{x^3}{3!} \cdot 2 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 16 + \ldots$$
$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

2.6.4.3 Expansion by Use of Known Series

When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series:

1. $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \ldots$
2. $\sinh\theta = \theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \dfrac{\theta^7}{7!} + \ldots$
3. $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \ldots$
4. $\cosh\theta = 1 + \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \ldots$
5. $\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{2}{15}\theta^5 + \ldots$
6. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$
7. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$
8. $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$
9. $\log(1 - x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\right)$
10. $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

Example:

Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.

Solution:

We have, $\quad e^{\sin x} = 1 + \sin x + \dfrac{(\sin x)^2}{2!} + \dfrac{(\sin x)^3}{3!} + \dfrac{(\sin x)^4}{4!} + \ldots$

$$= 1 + \left[x - \frac{x^3}{6} + \ldots\right] + \frac{1}{2!}\left[\ldots\right] + \ldots \text{[illegible]}$$

$$= 1 + \left[x - \frac{x^3}{6} + \ldots\right] + \frac{1}{2}\left[\ldots\right] + \ldots \frac{1}{24}x^4 \ldots \text{[illegible]}$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} \ldots$$

Otherwise, for [illegible] $f(0) = 1$

[illegible] $f'(0) = 1$

[illegible] $f''(0) = 1$

[illegible] $f'''(0) = 0$

[illegible] $f^{iv}(0) = -3$

[illegible]

[illegible] the values of $f(0), f'(0)$ [illegible] in the Maclaurin's series, we obtain

$$e^{\sin x} = 1 + x + 1\cdot\frac{x^2}{2!} + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \ldots$$

2.6.5 Slope Determination of Line

1. [illegible] $\frac{dy}{dx} = 1$

2. If two lines are perpendicular the product of their slopes is -1. [illegible]

$$m_1 \cdot m_2 = -1$$

3. The derivatives are also used to find slope of tangent to any curve. [illegible] $y = f(x)$ [illegible]

$$\frac{dy}{dx} = f'(x)\Big|_{(x_1, y_1)} \text{ gives slope of the tangent at point } (x_1, y_1)$$

2.7 Partial Derivatives

2.7.1 Definition of Partial Derivative

[illegible] Partial Differentiation.

[illegible]

[illegible]

2.7.2 Second order partial differential coefficients

[illegible]

We have $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right), \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ as second order partial derivatives of u and these are respectively denoted by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}$.

Note: If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial i.e. $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

2.7.3 Homogenous Functions

An expression in which every term is of the same degree is called homogenous function. Thus $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n$ is a homogenous function of x and y of degree n. This can be written as:

$$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right]$$

$= x^n f\left(\frac{y}{x}\right)$ where $f\left(\frac{y}{x}\right)$ is some function of $\frac{y}{x}$.

Note: To test whether the given function $f(x, y)$ is homogenous or not we put $x = tx$ and $y = ty$ in it.

If we get $f(tx, ty) = t^n f(x, y)$ then the function $f(x, y)$ is homogenous of degree n, otherwise $f(x, y)$ is not a homogenous function.

Note: If u is a homogenous function of x and y of degree n then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also homogenous functions of x and y each being of degree $(n - 1)$.

2.7.4 Euler's Theorem on homogenous functions

If u is a homogenous function of x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Note: Euler's theorem can be extended to a homogenous function of any number of variables. Thus if $f(x_1, x_2, \dots, x_n)$ be a homogenous function of $x_1, x_2, \dots, x_n$ of degree n then $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \dots + x_n\frac{\partial f}{\partial x_n} = nf$.

Example:

Show that $u = x^3 + y^3 - 3xy^2$ is a homogenous function of degree 3.

Solution:

Now,
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \text{ and}$$
$$\frac{\partial u}{\partial y} = 3y^2 - 6xy$$

Now
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(3x^2 - 3y^2) + y(3y^2 - 6xy)$$
$$= 3(x^3 + y^3 - 3xy^2)$$
$$= 3u$$

So, Euler's theorem shows that u is a homogenous function of degree 3.

2.8 Total Derivatives

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$.

Then,
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Here $\frac{du}{dt}$ is called the total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u.

In the same way if $u = f(x, y, z)$ where x, y, z are functions of a single variable t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

This result can be extended to any number of variables.

Corollary 1: If $u = f(x, y)$ where y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

Corollary 2: If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

and
$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

Corollary 3: If x and y are connected by an implicit function $f(x, y) = 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

2.9 Maxima and Minima (of Function of Two Independent Variables)

2.9.1 Definitions

Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively.

Then $f(a, b)$ is said to be maximum or a minimum value of $f(x, y)$ according as $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k, positive or negative, provided both of them are not equal to zero.

2.9.2 Necessary Conditions

The necessary conditions that $f(x, y)$ should have a maximum or minimum at $x = a$, $y = b$ is that

$$\left.\frac{\partial f}{\partial x}\right|_{x=a} = 0 \text{ and } \left.\frac{\partial f}{\partial y}\right|_{y=b} = 0$$

2.9.3 Sufficient Condition for Maxima or Minima

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}};\quad s = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{\substack{x=a\\y=b}};\quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$$

Case 1: $f(x, y)$ will have a maximum or minimum at $x = a$, $y = b$ if $rt - s^2 > 0$. Further, $f(x, y)$ is maximum or minimum according as r is negative or positive.

Case 2: [illegible]

Case 3: [illegible]

2.10 Theorems of Integral Calculus

1. The integral of a constant times a function is equal to the constant times the integral of the function. [illegible]

   That is, if k is a constant, then $\int k f(x)\,dx = k \int f(x)\,dx$.

2. The integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the functions. [illegible]

   $\int [f_1(x) \pm f_2(x) \pm \dots \pm f_n(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \dots \pm \int f_n(x)\,dx$

2.10.1 Fundamental Formulae

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}$
2. $\int \frac{1}{x}\,dx = \log x$
3. $\int \sin x\,dx = -\cos x$
4. $\int \cos x\,dx = \sin x$
5. $\int \sec^2 x\,dx = \tan x$
6. $\int \operatorname{cosec}^2 x\,dx = -\cot x$
7. $\int \sec x \tan x\,dx = \sec x$
8. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$
9. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x$
10. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$
11. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x$
12. $\int \cosh x\,dx = \sinh x$
13. $\int \sinh x\,dx = \cosh x$

2.10.2 Useful Trigonometric Identities

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

1. $\sin(-x) = -\sin x$
2. $\cos(-x) = \cos x$
3. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
4. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
5. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
6. $\cos(x - y) = \cos x \cos y + \sin x \sin y$

7. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

8. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

9. (i) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ (ii) $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

(iii) $\sin(\pi + x) = \sin(\pi)\cos(x) + \cos(\pi)\sin(x) = -\sin x$

(iv) $\sin(3\pi/2 + x) = \sin(3\pi/2)\cos(x) + \cos(3\pi/2)\sin(x) = -\cos x$

(v) $\sin(2\pi - x) = \sin(2\pi)\cos(x) - \cos(2\pi)\sin(x) = -\sin x$

10. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

12. $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$

13. $\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

14. $\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$

15. $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

16. $\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$

17. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

18. $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

19. $\sin^2 x = 1 - \cos^2 x$

20. $\cos^2 x = 1 - \sin^2 x$

21. $1 = \cos^2 x + \sin^2 x$

2.10.3 Methods of Integration

There are various methods of integration by which we can reduce the given integral to one of the known standard integrals. The following are principal methods of integration.

1. **Integration by substitution:** A change in the variable of integration often reduces an integral to one of fundamental integrals.

If $I = \int f(x)\,dx$, then by differentiation w.r.t. x, we have $\frac{dI}{dx} = f(x)$. Now put

$$x = \phi(t) \text{ so that } \frac{dx}{dt} = \phi'(t)$$

[illegible]

Rule to Remember:

[illegible]

Three Forms of Integrals:

(a) $\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$

Put [illegible]

$\Rightarrow$ [illegible]

[illegible]

Example:

$$\int \frac{4x^3}{1+x^4}\,dx = \log(1+x^4) \quad [illegible]$$

[illegible]

$\Rightarrow$ [illegible]

(ii) [illegible]

Some Important Formulae Based on the Above Form:

(i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ [illegible]

[illegible]

(ii) [illegible]

(iii) $\int \sec x$ [illegible]

(iv) [illegible]

(v) [illegible]

[illegible]

Formulae:

(i) $\int f(ax+b)\,dx = \dfrac{F(ax+b)}{a}$

(ii) $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2+a^2}\right]$

(iii) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$

(iv) $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2-a^2}\right]$

(v) $\int \sqrt{x^2+a^2}\,dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right)$

$= \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$

(vi) $\int \sqrt{a^2-x^2}\,dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$

2. **Integral of the product of two functions**

Integration by parts: Let u and v be two functions of x. Then we have from differential calculus

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{...(i)}$$

Integrating both sides of (i) with respect to x, we have

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\Rightarrow \quad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \quad \text{...(ii)}$$

$$\Rightarrow \quad \int u\,dv = uv - \int v\,du$$

This can also be written as $\int uv\,dx = u\int v\,dx - \int\left[u'\int v\,dx\right]dx$

The choice of which function will be u and which function will be v is very important in solving an integration by parts.

The ILATE method helps decide this.

ILATE stands for

I: Inverse trigonometric functions ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.)

L: Logarithmic functions ($\log x$, $\ln x$ etc.)

A: Algebraic functions (x, x^2, $x+1$, 2 etc.)

T: Trigonometric functions ($\sin x$, $\cos x$ etc.)

E: Exponential functions (e^x etc.)

Whichever of the two functions comes first in ILATE gets designated as u and other function gets designated as v.

Formulae Based Upon Above Method:

(i) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(b) $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

Integration by Partial Fractions

(a) $$I = \int \frac{1}{x^2-a^2}\,dx \quad (x > a)$$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$$

$$= \frac{1}{2a}[\log(x-a) - \log(x+a)] = \frac{1}{2a}\log\frac{x-a}{x+a}$$

Thus $\int \dfrac{1}{x^2-a^2}\,dx = \dfrac{1}{2a}\log\dfrac{x-a}{x+a},\ x > a$

(b) $$I = \int \frac{1}{a^2-x^2}\,dx \quad (x < a)$$

In this case $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\log\dfrac{a+x}{a-x},\ x < a$

The following is a summary of some of the integrals derived so far by using the three methods of integration.

(a) $\int \dfrac{1}{a^2+x^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

(b) $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\log\dfrac{a+x}{a-x}$

(c) $\int \dfrac{1}{x^2-a^2}\,dx = \dfrac{1}{2a}\log\dfrac{x-a}{x+a}$

(d) $\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2+a^2}\right]$

(e) $\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right)$

(f) $\int \dfrac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2-a^2}\right]$

(g) $\int \dfrac{1}{x\sqrt{x^2-a^2}}\,dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right)$

(h) $\int \sqrt{x^2+a^2}\,dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right)$

(i) $\int \sqrt{a^2-x^2}\,dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$

[illegible]

(i) $\int_0^{\pi/2} \ldots$ [illegible]

[illegible]

(ii) [illegible]

[illegible]

2.11 Definite Integrals

[illegible]

2.11.1 Fundamental Properties of Definite Integrals

1. [illegible]

   [illegible]

2. [illegible]

   [illegible]

3. [illegible]

   Note 1: [illegible]

   Note 2: [illegible]

   Thus, [illegible]

4. (a) [illegible]

   (b) [illegible]

Proof: Let $I = \int_{-a}^{a} f(x)\,dx$

Put $x = -t \Rightarrow dx = -dt$; when $x = -a$, $t = a$ and when $x = a$, $t = -a$

$\therefore$ $I = \int_{a}^{-a} f(-t)(-dt) = \int_{-a}^{a} f(-t)\,dt = \int_{-a}^{a} f(-x)\,dx$

$\therefore$ $\int_{-a}^{a} f(x)\,dx = 0$ or $2\int_{0}^{a} f(x)\,dx$ according as $f(x)$ is an odd or even function of x.

Odd and Even function

(a) A odd function: $f(-x) = -f(x)$

(b) A even function: $f(-x) = f(x)$

6. $$\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(2a - x) = f(x)$$

And $$\int_{0}^{2a} f(x)\,dx = 0 \quad \text{if } f(2a - x) = -f(x)$$

Corollary: $\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a - x)\,dx$

7. $$\int_{0}^{na} f(x)\,dx = n\int_{0}^{a} f(x)\,dx$$

If $f(x) = f(x + a)$ [Periodic function with period a]

8. $$\frac{d}{dt}\int_{\phi(t)}^{\psi(t)} f(x)\,dx = f\{\psi(t)\}\psi'(t) - f\{\phi(t)\}\phi'(t)$$

Example 1.

Evaluate the following definite integrals

(a) $\int_{-5}^{5} |x + 2|\,dx$ (b) $\int_{1}^{4} (|x| + |x - 3|)\,dx$

Solution:

(a) Since for $-5 \le x \le -2$, $x + 2 \le 0$

$\Rightarrow$ $|x + 2| = -(x + 2)$

And for $-2 \le x \le 5$, $x + 2 \ge 0$

$\Rightarrow$ $|x + 2| = x + 2$

$\therefore$ $$\int_{-5}^{5} |x + 2|\,dx = \int_{-5}^{-2} |x + 2|\,dx + \int_{-2}^{5} |x + 2|\,dx \quad \text{[Property 3]}$$

$$= \int_{-5}^{-2} -(x + 2)\,dx + \int_{-2}^{5} (x + 2)\,dx = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[(2 - 4) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - (2 - 4)\right] = 29$$

(b) Since for $1 \le x \le 3$, $x - 3 \le 0 \Rightarrow |x| = x$, $|x - 3| = -(x - 3)$

Also for $3 \le x \le 4$, $x - 3 \ge 0 \Rightarrow |x| = x$, $|x - 3| = x - 3$

$\therefore$ $$\int_{1}^{4} (|x| + |x - 3|)\,dx = \int_{1}^{3} (x - x + 3)\,dx + \int_{3}^{4} (x + x - 3)\,dx \quad \text{[Property 3]}$$

$$= [illegible]$$

$$= [illegible]$$

$$= [illegible]$$

$$= [illegible]$$

$$= [illegible]$$

Example 2

Evaluate the following definite integrals:

(a) [illegible] (b) [illegible] (c) [illegible]

Solution:

(a) [illegible] the given function is discontinuous at $x = 1$

$$\therefore \quad [illegible] \qquad \text{(Property 3)}$$

$$[illegible]$$

(b) First note that [illegible] is discontinuous at $x = 0$

$$\therefore \quad [illegible]$$

[illegible]

$$= [illegible]$$

(c) First note that $[3x]$ is discontinuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$

$$\therefore \quad [illegible]$$

$$= [illegible]$$

$$= \ldots - \frac{1}{3} = 1$$

Example 3.

By using properties of definite integral, evaluate the following.

(a) $\int_{-\pi/2}^{\pi/2} \sin^4 x \, dx$ (b) $\int_{-\pi/2}^{\pi/2} x^3 \sin^4 x \, dx$ (c) $\int_0^{2\pi} |\cos x| \, dx$

Solution:

(a) Let $f(x) = \sin^4 x$; $f(-x) = \sin^4(-x) = (\sin x)^4 = \sin^4 x = f(x)$

$$\int_{-\pi/2}^{\pi/2} \sin^4 x \, dx = 2\int_0^{\pi/2} \sin^4 x \, dx = 2\int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (3 - 4\cos 2x + \cos 4x) \, dx$$

$$= \frac{1}{4}\left[3x - 4\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(3\frac{\pi}{2} - 2\sin \pi + \frac{1}{4}\sin 2\pi\right) - \left(0 - 0 + 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{3\pi}{2} - 0 + 0\right] - [0] = \frac{3\pi}{8}$$

(b) Let $f(x) = x^3 \sin^4 x$; $f(-x) = (-x)^3 \sin^4(-x) = -x^3 \sin^4 x = -f(x)$

$\Rightarrow f(x)$ is an odd function, therefore, by property 6,

$$\int_{-\pi/2}^{\pi/2} x^3 \sin^4 x \, dx = 0$$

(c) Let $f(x) = |\cos x|$, $f(2\pi - x) = |\cos(2\pi - x)| = |\cos x| = f(x)$. Therefore by property 8,

$$\int_0^{2\pi} |\cos x| \, dx = 2\int_0^{\pi} |\cos x| \, dx \quad \ldots(i)$$

Again, $f(\pi - x) = |\cos(\pi - x)| = |-\cos x| = |\cos x| = f(x)$. Therefore by property 8,

$$\int_0^{\pi} |\cos x| \, dx = 2\int_0^{\pi/2} |\cos x| \, dx \quad \ldots(ii)$$

$\therefore$ From (i) and (ii), we get

$$\int_0^{2\pi} |\cos x| \, dx = 2 \cdot 2\int_0^{\pi/2} |\cos x| \, dx = 4\int_0^{\pi/2} \cos x \, dx$$

[illegible]

$$= 4[x]_0^{\pi/2} - 4\left[\sin\frac{\pi}{2} - \sin 0\right] \ [\text{illegible}] \ (1-0) = 4$$

Example 4.

Evaluate the following: $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

Solution:

Let $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \quad \ldots(i)$$

Then by using property 4, we get

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx \quad \ldots(ii)$$

On adding (i) and (ii) we get

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

Example 5.

Evaluate the following definite integrals:

(a) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

(b) $\int_0^{\pi/2} \sin 2x \log(\tan x)\, dx$

Solution:

(a) Let $$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \ldots(i)$$

Then by using property 4, we get

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx = \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right)^{-1} dx = \int_0^1 -\log\left(\frac{1-x}{x}\right) dx = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx$$

$$= -I$$

$$\Rightarrow \quad 2I = 0$$

$$\Rightarrow \quad I = 0$$

(b) Let $$I = \int_0^{\pi/2} \sin 2x \log(\tan x)\, dx \quad \ldots(i)$$

[illegible]

[illegible]

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

Example 6.

Evaluate the following definite integral [illegible]

Solution:

[illegible] ...(i)

Then by using property [illegible] we get

[illegible] ...(ii)

Adding (i) and (ii) we get

[illegible]

$\Rightarrow$ [illegible]

[illegible] by using property [illegible] we get

[illegible]

2.12 Applications of Integration

[illegible] where integration is applied

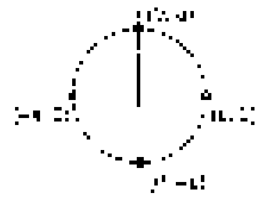
1. Areas of curves
2. Length of curves
3. Volumes of solids

2.12.1 Preliminary : Curve Tracing

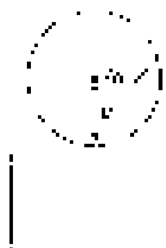
[illegible] double and triple integrals [illegible] equations

Circle : Cartesian Form:

1. $x^2 + y^2 = a^2$: Circle with centre (0, 0) and radius a.



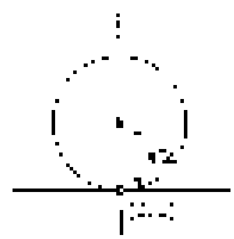
2. $(x - h)^2 + (y - k)^2 = a^2$: Circle with centre (h, k) and radius a.



Polar Form:

1. $r = a$: Circle with centre (0, 0) and radius a.

2. $r = a \sin\theta$: Circle with centre $\left(0, \frac{a}{2}\right)$ and radius $\frac{a}{2}$.

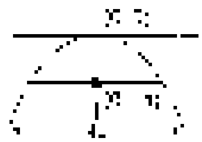


3. $r = a\cos\theta$: Circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

Parabola:

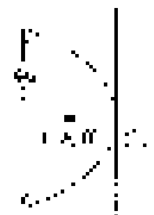
1. $y^2 = 4ax$: Parabola with vertex at (0, 0) and focus at (a, 0) and latus rectum = 4a.

2. $x^2 = -4ay$: Parabola with vertex at $(0, 0)$, focus at $(0, -a)$ and latus rectum $= 4a$.



3. $y^2 = 4ax$: Parabola with vertex at $(0, 0)$ and focus at $(a, 0)$ and latus rectum $= 4a$.

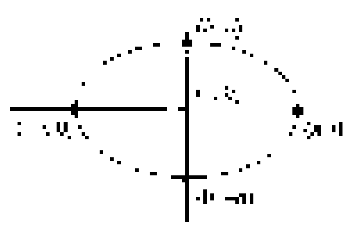
4. $y^2 = -4ax$: Parabola with vertex at $(0, 0)$, focus at $(-a, 0)$ and latus rectum $= 4a$.



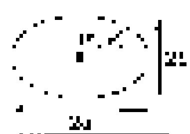
5. $(x - h)^2 = 4a(y - k)$: Parabola with centre at (h, k), focus at $(h, k + a)$ and latus rectum $= 4a$.

Ellipse:

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$: Ellipse with centre at $(0, 0)$ and major axis $= 2a$ and minor axis $= 2b$.

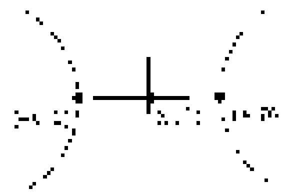


2. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$: Ellipse with centre at (h, k) and major axis $= 2a$ and minor axis $= 2b$.

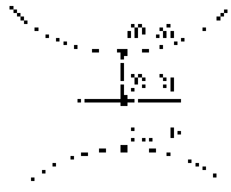


Hyperbola:

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Hyperbola with vertices at (a, 0) and (−a, 0) and centre at (0, 0)



2. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ Hyperbola with vertices at (0, b) and (0, −b) and centre at (0, 0)

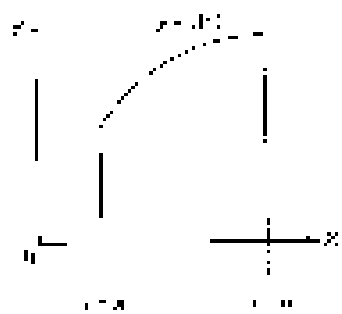


2.12.2 Areas of Cartesian Curves

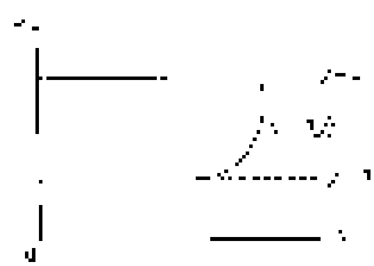
Theorem

1. Area bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$



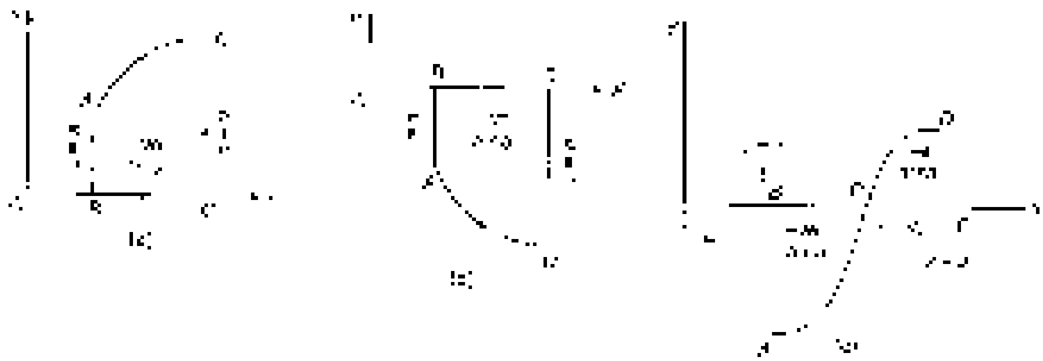
2. Interchanging x and y in the above formula, we see that the area bounded by the curve x = f(y), the y-axis and the abscissa y = a, y = b is $\int_a^b x\,dy = \int_a^b f(y)\,dy$ as shown in figure below.



Note, 1: The area bounded by a curve, the x-axis and two ordinates is called the area under the curve.

This process of finding the area of plane curves is often called quadrature.

Note. 2: Sign of an area. An area whose boundary is described in the anticlockwise direction (i.e. lies above x-axis) is considered positive (Fig. a) and an area whose boundary is described in the clockwise direction (i.e. lies below x-axis) is taken as negative (Fig. b).



In Fig. (c) where the area given by $\int_a^b y\,dx$ will not represent the sum of the areas $ABC\left(=\int_a^c y\,dx\right)$ and the area $CDE\left(=-\int_c^b y\,dx\right)$ but their difference.

Thus to find the total area in such cases the numerical values of the area of each portion must be evaluated separately with their modulus and then added afterwards.

Example:

Find the area of the region bounded by the parabola $x^2 = 8y$ and the line $x - 2y + 8 = 0$.

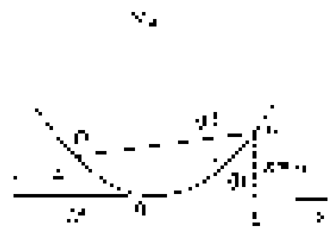
Solution:

Given curve is $\qquad x^2 = 8y \qquad \ldots(i)$

and the straight line is

$$x - 2y + 8 = 0$$

$$\Rightarrow \qquad y = \frac{x+8}{2} \qquad \ldots(ii)$$



Substituting the value of y from (ii) in (i), we get

$$x^2 = 4x + 32$$

$$\Rightarrow \qquad x^2 - 4x - 32 = 0$$

$$\Rightarrow \qquad (x-8)(x+4) = 0$$

$$\Rightarrow \qquad x = 8, -4$$

Thus (i) and (ii) intersect at P and Q where $x = -4$ and $x = 8$.

∴ Required area POQ = (area between the straight line (ii) and x-axis from $x = -4$ to $x = 8$) − (area bounded by parabola (i) and x-axis from $x = -4$ to $x = 8$)

$$= \int_{-4}^{8} y\,dx, \text{ from (ii)} - \int_{-4}^{8} y\,dx, \text{ from (i)}$$

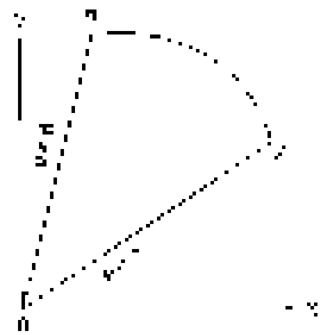
$$= \int_{-4}^{8} \frac{x+8}{2}\,dx - \int_{-4}^{8} \frac{x^2}{8}\,dx = \frac{1}{2}\left|\frac{x^2}{2} + 8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\left[(32+64) - (8-32)\right] - \frac{1}{24}(512+64) = 36$$

2.12.3 Areas of Polar Curves

Theorem: Area bounded by the curve $r = f(\theta)$ and the radii $\theta = \alpha$, $\theta = \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$



Example:

Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.

Solution:

The equations of the circles are

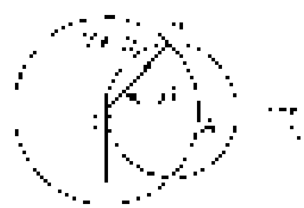
$$r = a\sqrt{2} \quad \text{and} \qquad \ldots(i)$$
$$r = 2a\cos\theta \qquad \ldots(ii)$$

(i) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$.

(ii) represents a circle symmetrical about OX with centre at (a, 0) and radius a.

The circles are shown in figure above. At their point of intersection P, eliminating r from (i) and (ii)

$$a\sqrt{2} = 2a\cos\theta \text{ i.e. } \cos\theta = \frac{1}{\sqrt{2}}$$



or $\theta = \pi/4$

$$\therefore \quad \text{Required area} = 2 \times \text{area } OAPQ \text{ (by symmetry)}$$
$$= 2(\text{area } OAP + \text{area } OPQ)$$
$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2 d\theta \text{ (for i)} + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2 d\theta \text{ (for ii)}\right]$$
$$= \int_0^{\pi/4} (a\sqrt{2})^2 d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2 d\theta$$
$$= 2a^2 [\theta]_0^{\pi/4} + 4a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$
$$= 2a^2\left(\frac{\pi}{4} - 0\right) + 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$
$$= \frac{\pi a^2}{2} + 2a^2\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right] = a^2(\pi - 1)$$

2.12.4 Derivative of arc Length d

Theorem: For the curve $y = f(x)$, we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Proof: Let $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AB. Let arc $AP = s$, arc $PQ = \delta s$.

Draw PL, QM ⊥ on the x-axis and PN ⊥ QM.

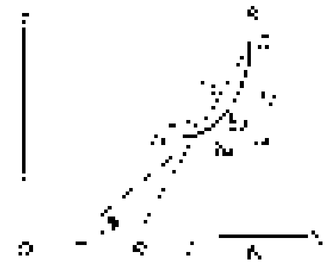
∴ From the right angled triangle PNQ

$$PQ^2 = PN^2 + NQ^2$$

i.e. $$\delta s^2 = \delta x^2 + \delta y^2$$

$$\left(\frac{\text{chord } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$$\therefore \quad \left(\frac{\delta s}{\delta x}\right)^2 = \left(\frac{\text{chord } PQ}{\delta s}\right)^{-2} \left(\frac{\text{chord } PQ}{\delta x}\right)^2 = \left(\frac{\text{chord } PQ}{\delta s}\right)^{-2} \left[1 + \left(\frac{\delta y}{\delta x}\right)^2\right]$$



Taking limit as $Q \to P$ (i.e., $\delta x \to 0$)

$$\left(\frac{ds}{dx}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right] \qquad \left[\because \lim_{Q \to P} \frac{\text{chord } PQ}{\delta s} = 1\right]$$

If s increases as x increases as in figure above, ds/dx is positive.

$$\text{Thus} \quad \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \quad \text{taking positive sign before the radical} \qquad (i)$$

Cor. 1. If the equation of the curve is $x = f(y)$, then

$$\therefore \quad \frac{ds}{dy} = \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]} \qquad (ii)$$

Cor. 2. If the equation of the curve is in parametric form $x = f(t)$, $y = \phi(t)$ then

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \cdot \frac{dx}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right)^2\right]}$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} \qquad (iii)$$

2.12.5 Lengths of Curves

Theorem: The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is

$$s = \int_a^b \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} dx$$

The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$ is

$$\int_a^b \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]} dy$$

The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = a$ and $t = b$ is

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} dt$$

the length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example:

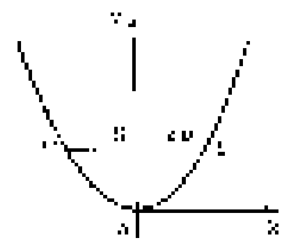
Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum.

Solution:

Let A be the vertex and L an extremity of the latus rectum so that at A, $x = 0$ and at L, $x = 2a$ as shown in figure.

Here, $y = x^2/4a$

so that $\frac{dy}{dx} = \frac{1}{4a} \cdot 2x = \frac{x}{2a}$



$$s = \text{arc } AL = \int_0^{2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{2a} \sqrt{1 + \left(\frac{x}{2a}\right)^2}\, dx$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{(2a)^2 + x^2}\, dx = \frac{1}{2a}\left[\frac{x\sqrt{(2a)^2 + x^2}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a}$$

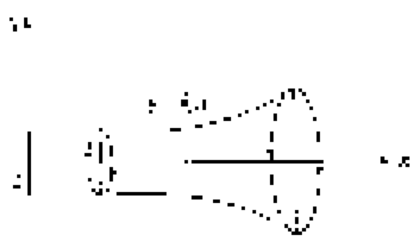
$$= \frac{1}{2a}\left[\frac{2a\sqrt{8a^2}}{2} + 2a^2 \sinh^{-1} 1\right]$$

$$= a\left[\sqrt{2} + \sinh^{-1} 1\right] = a\left[\sqrt{2} + \log(1 + \sqrt{2})\right] \qquad \left[\because \sinh^{-1} x = \log\left(x + \sqrt{1 + x^2}\right)\right]$$

2.12.6 Volumes of Revolution

1. **Revolution about x-axis:** The volume of the solid generated by the revolution about the x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b \pi y^2\, dx$

ordinate to the curve $y = f(x)$ between the ordinates AA' ($x = a$) and BB' ($x = b$).



Example:

Find the volume of a sphere of radius a.

Solution:

Let the sphere be generated by the revolution of the semicircle ABC of radius a about its diameter CA (Figure).

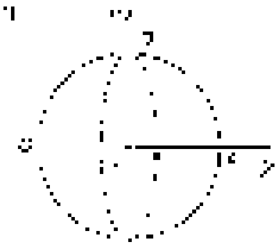
Since the sphere is generated by the revolution of the semi-circle ABC about its diameter, the equation of the circle ABC is

$$x^2 + y^2 = a^2$$

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$= 2\int_0^a \pi y^2 dx = 2\pi\int_0^a (a^2 - x^2)\,dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3}\pi a^3$$



2. **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissa $y = a$, $y = b$ is $\int_a^b \pi x^2 dy$.

Example:

Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus rectum.

Solution:

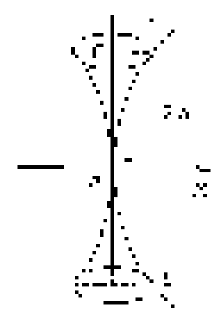
Given parabola is $x = y^2/4a$.

Let $A'A$ be the latus rectum through the focus S of the parabola (Figure)

$\therefore$ Required volume = 2 (volume generated by the revolution about the y-axis of the area $A'OL$)

$$= 2\int_0^{2a} \pi x^2 dy = 2\pi\int_0^{2a} \frac{y^4}{16a^2}\,dy$$

$$= \frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}$$



2.13 Multiple Integrals and Their Applications

1. Double integrals
2. Change of order of integration
3. Double integrals in polar coordinates
4. Areas enclosed by plane curves
5. Triple integrals

2.13.1 Double Integrals

[illegible]

$$[illegible]$$

[illegible]

$$[illegible] \qquad (i)$$

[illegible]

[illegible] as follows:

1. [illegible]

$$[illegible]$$

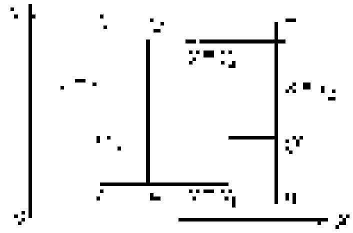
where integration [illegible] from the inner to the outer rectangle.

[illegible]

[illegible] ABCD.

2. [illegible]

[illegible]



[illegible]

Example:

[illegible]

Solution:

[illegible]

1.13.2 Change of order of Integration

[illegible]

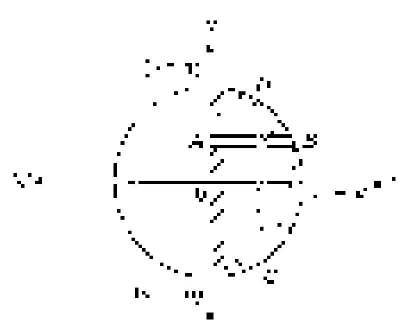
[illegible]

Example: 1

[illegible]

Solution:

[illegible]



This [illegible] the given integral

$I = \int [illegible]$

The order of integration [illegible]

[illegible]

[illegible]

$$I = \int_0 [illegible]$$

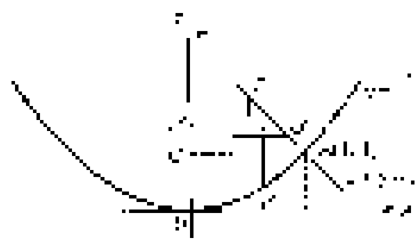
or $$= \int [illegible]$$

Example: 2

Change the order of integration in [illegible] and hence evaluate the same.

Solution:

Here the integration is first w.r.t. y along a vertical strip [illegible] which extends from [illegible] [illegible] region [illegible] in Fig. [illegible]



On changing the order of integration, we first integrate w.r.t. x along a horizontal strip [illegible] the region OAB into two parts [illegible] and the triangle ABC.

For the region [illegible] and these strips slide from $y = 0$ to $y = 1$. So the contribution [illegible] the region [illegible] is

$$[illegible] \int [illegible]$$

For the region ABC, the limits of integration [illegible] and these strips slide from [illegible] to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int [illegible]$$

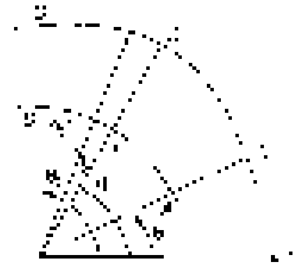
$$= \int [illegible] + \int [illegible]$$

$$= \int [illegible] \left|\frac{x^2}{2}\right| [illegible] + \int [illegible]$$

$$= \frac{1}{2}\int [illegible] - \frac{1}{2}\int [illegible] = [illegible] = \frac{3}{8}$$

2.13.3 Double Integrals in Polar Coordinates

[illegible]



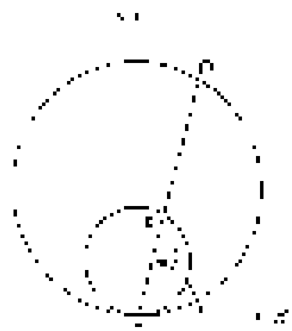
[illegible]

Example:

[illegible]

Solution:

[illegible]



[illegible]

So [illegible]

[illegible]

2.13.4 Area Enclosed by Plane Curves

[illegible]

Example:

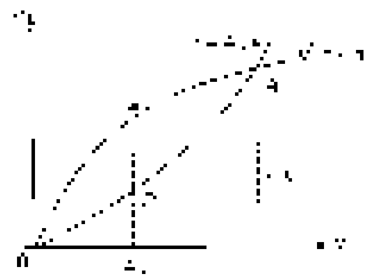
Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. $\frac{16}{3}a^2$

Solution:

The equations $y^2 = 4ax$ and $x^2 = 4ay$ intersect at O(0, 0) and A(4a, 4a). As the strip moves between these parabolas, x varies from 0 to 4a and y varies from $y = x^2/4a$ to $y = 2\sqrt{ax}$. Hence the required area

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

$$= \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx$$



$$= \left|2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\,\frac{x^3}{3}\right|_0^{4a}$$

$$\frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

2.13.5 Triple Integrals

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub division δV_r. Consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the triple integral of $f(x, y, z)$ over the region V and is denoted by

$$\iiint f(x, y, z)\,dV$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dz\,dy\,dx$$

If x_1, x_2 are constants; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows:

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus,

$$I = \int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left\{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\right\}dy\right]dx$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example:

Evaluate $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x + y + z)\,dx\,dy\,dz$.

Solution:

[illegible]

$$[illegible]$$

Example 2

Evaluate [illegible]

Solution:

We have, [illegible]

$$[illegible]$$

2.14 Vectors

2.14.1 Introduction

This chapter deals with vectors and vector functions in 3-space and extends the [illegible] [illegible] the algebra and [illegible] mechanics, [illegible] [illegible] be given as vectors.

We first explain the basic algebraic operations with vectors [illegible] physical and geometrical applications. [illegible] mechanics.

[illegible] and [illegible] important concepts related to [illegible] fields, namely the gradient, divergence, and curl. [illegible] vector integral theorems.

2.14.2 Basic Definitions

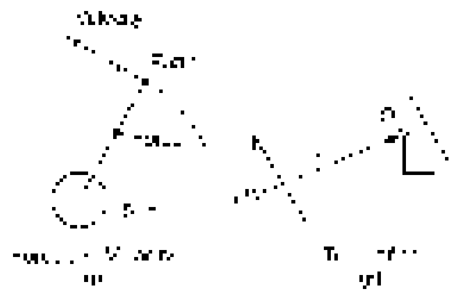
In geometry and physics and in engineering applications we use two kinds of quantities: scalars and vectors. A scalar is a quantity that is determined by its magnitude [illegible] measured on a suitable scale. [illegible], length, temperature [illegible].

A vector is a quantity that is determined by both its magnitude and its direction [illegible] arrow [illegible] directed line segment [illegible] force is a vector [illegible] velocity [illegible] the speed [illegible] direction of motion (Figure below). We denote vectors by [illegible] letters [illegible].

A vector (arrow) has a tail, called its initial point, and a tip, called its terminal point. [illegible] in Figure the triangle is translated (displaced without rotation), the initial point P of this vector is the original position of a point and the terminal point Q is [illegible] after the translation.

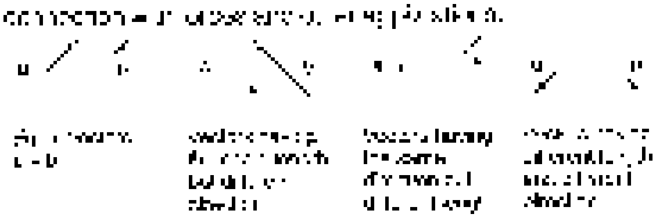
The length (or magnitude) of a vector is the length of the arrow [illegible] and is called the norm (or Euclidean norm) [illegible] is denoted by [illegible].

A vector of length 1 is called a unit vector.



2.14.3 Equality of Vectors

By definition, two vectors a and b are equal, written $a = b$, if they have the same length and the same direction (Figure below). Hence a vector can be arbitrarily translated, that is, its initial point can be chosen arbitrarily [illegible] [illegible] [illegible] possible [illegible].



Vectors

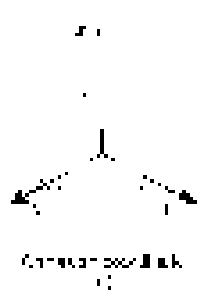
2.14.4 Components of a Vector

We [illegible] Cartesian coordinate system in space, that is, a usual rectangular coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes. [illegible] initial point $P: (x_1, y_1, z_1)$ and terminal point $Q: (x_2, y_2, z_2)$, the three numbers

1. [illegible] $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$ are called the components of the vector a with respect to that coordinate system and we write simply $a = [a_1, a_2, a_3]$.

Length in Terms of Components: By definition, the length $|a|$ of a vector a is the distance between its initial point P and terminal point Q. From the Pythagorean theorem [illegible] (Figure below) we see that,

2. $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$



Cartesian coordinate system (a)

[illegible] vector (b)

Example:

[illegible]

[illegible] has the components

$a_1 = 6 - 4 = 2,\ a_2 = -1 - 0 = -1,\ a_3 = 2 - 2 = 0$

Solution:

Hence,
$$a = [2, -1, 0],$$
$$|a| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$$

[illegible]

[illegible]

2.14.5 Position Vector

[illegible]

[illegible]

[illegible]

2.14.5.1 Vectors as Ordered Triples of Real Numbers

Theorem: [illegible]

[illegible]

[illegible]

2.14.6 Vector Addition, Scalar Multiplication

[illegible]

2.14.6.1 Definition: 1

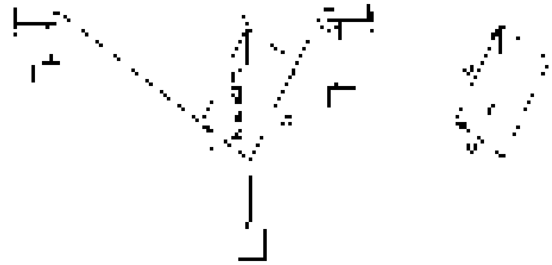
Addition of Vectors: [illegible]

$$a + b = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

[illegible]

[illegible]

[illegible]



[illegible]

[illegible]

Basic properties of Vector addition [illegible]

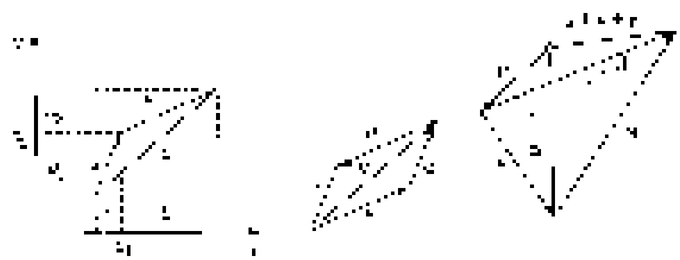
(a) $a + b = b + a$ (Commutativity)

(b) $(u + v) + w = u + (v + w)$ (Associativity)

(c) $a + 0 = 0 + a = a$

(d) $a + (-a) = 0$

[illegible]



Vector addition — Commutativity of vector addition — Associativity of vector addition

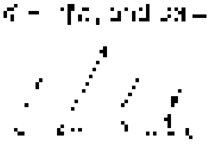
[illegible]

2.14.6.2 Definition: 2

Scalar Multiplication (Multiplication by a Number): [illegible]

$$ca = [ca_1, ca_2, ca_3]$$

[illegible]



[illegible]

[illegible]

Example:

Vector Addition and Multiplication by Scalars.

[illegible]

$$[illegible]$$

Solution:

$$[illegible]$$

2.14.7 Unit Vectors

[illegible]

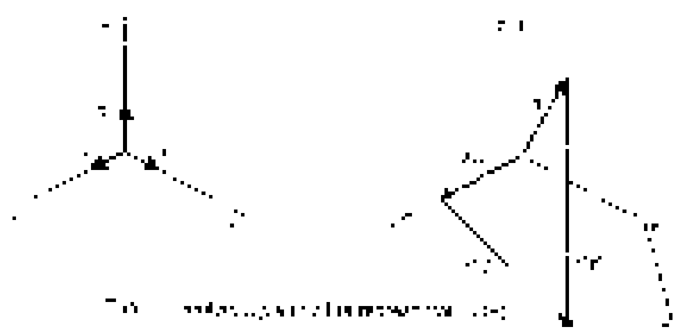
$$[illegible]$$

[illegible]

2.14.7.1 Representation of Vectors in Terms of i, j and k

$$[illegible]$$

[illegible]



[illegible]

[illegible]

Example:

[illegible]

Solution:

[illegible]

[illegible]

2.14.8 Length and Direction of Vectors

[illegible]

$$[illegible]$$

[illegible]

Example 1.

Express v as a product of its length and direction

$$v = 3i + 4j$$

Solution:

$$\text{Length of } v = |v| = \sqrt{9 + 16}$$

The unit vector in the direction of $v = \frac{v}{|v|} = \frac{3i + 4j}{5} = \frac{3}{5}i + \frac{4}{5}j$

$$v = 3i + 4j = 5\left(\frac{3}{5}i + \frac{4}{5}j\right)$$

Note that $\left|\frac{3}{5}i + \frac{4}{5}j\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

Since $\frac{3}{5}i + \frac{4}{5}j$ is a unit vector.

Example 2.

Find a unit vector in the direction of [illegible]

Solution:

The required vector is $\frac{v}{|v|} = \frac{[illegible]}{[illegible]} = \frac{4}{[illegible]}i + \frac{2}{[illegible]}j$

Example 3.

Find unit vectors tangent and normal to the curve

$$y = \frac{x^3}{2} + \frac{1}{2} \text{ at } (1, 1)$$

Solution:

Unit vector tangent to curve:

$$y' = \left[\frac{3x^2}{2}\right]_{x=1} = \frac{3}{2}$$

Any vector with slope $\frac{3}{2}$ can be written as

$$v = a(2i + 3j)$$

$$|v| = a\sqrt{2^2 + 3^2} = \sqrt{13}\,a$$

A unit vector in the direction of v

$$u = \frac{v}{|v|} = \frac{a(2i + 3j)}{\sqrt{13}\,a} = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$$

Notice also that $-u = -\frac{2}{\sqrt{13}}i - \frac{3}{\sqrt{13}}j$

is another unit vector tangent to the curve, but it is opposite to u [illegible]

Unit vector normal to curve:

$$n = -\frac{3}{\sqrt{13}}i + \frac{2}{\sqrt{13}}j$$

Any vector normal to [illegible]

$$\left(\frac{2}{[illegible]}\right)\left(-\frac{3}{[illegible]}\right) -$$

So a unit normal vector is $n = \frac{2}{\sqrt{13}}\hat{i} + \frac{3}{\sqrt{13}}\hat{j}$ and $\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}$

Note that $n' = -\frac{2}{\sqrt{13}}\hat{i} - \frac{3}{\sqrt{13}}\hat{j}$ is another unit vector normal to the line, but in opposite direction to n.

2.14.9 Inner Product (Dot Product)

We shall now define a multiplication of two vectors that gives a scalar as the product and is suggested by various applications.

Definition. Inner Product (Dot Product) of Vectors

The inner product or dot product $a \cdot b$ (read "a dot b") of two vectors a and b is the product of their lengths times the cosine of their angle, see Fig. below.

(1) $a \cdot b = |a||b| \cos\gamma$

The angle γ, $0 \le \gamma \le \pi$, between a and b is measured when the vectors have their initial points coinciding, as in Fig. shown below.

Angle between vectors Value of inner product

In components, $a = [a_1, a_2, a_3]$, $b = [b_1, b_2, b_3]$, and

(2) $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$

Some authors write $a \cdot b$ ...

Since the cosine in (1) may be positive, zero, or negative, so may be the inner product. The case that the inner product is zero is of great practical interest and suggests the following concept.

A vector a is called orthogonal to a vector b if $a \cdot b = 0$. Then b is also orthogonal to a, and we call these vectors orthogonal vectors. Clearly, this happens for nonzero vectors if and only if $\cos\gamma = 0$; thus $\gamma = \pi/2$ (90°). This proves the following important theorem.

Theorem 1 (Orthogonality)

The inner product of two nonzero vectors is zero if and only if these vectors are perpendicular.

Length and Angle in Terms of Inner Product: Equation (1) gives with $b = a$ that $a \cdot a = |a|^2$

(3) $|a| = \sqrt{a \cdot a}$

From (3) and (1) we obtain for the angle γ between two nonzero vectors

(4) $\cos\gamma = \frac{a \cdot b}{|a||b|} = \frac{a \cdot b}{\sqrt{a \cdot a}\sqrt{b \cdot b}}$

Example:

Find the inner product and the lengths of $a = [1, 2, 0]$ and $b = [3, -2, 1]$ as well as the angle between these vectors.

Solution:

$$a \cdot b = 1 \cdot 3 + 2(-2) + 0 \cdot 1 = -1$$

$$|a| = \sqrt{a \cdot a} = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

$$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$$

$$\gamma = \arccos\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \arccos(-0.1666) = 1.7382 = 99.59°$$

This gives [illegible] the angle between them [illegible] inner product has some [illegible] negative [illegible].

General Properties of Inner Products: From the definition we see that the inner product has the following properties. For any vectors $\vec{a}, \vec{b}, \vec{c}$ and scalars q_1, q_2,

(a) $(q_1\vec{a} + q_2\vec{b})\cdot\vec{c} = q_1\vec{a}\cdot\vec{c} + q_2\vec{b}\cdot\vec{c}$

(Linearity)

(b) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (Symmetry)

(c) $\vec{a}\cdot\vec{a} \geq 0$ (Positive-definiteness)

(d) $\vec{a}\cdot\vec{a} = 0$ if and only if $\vec{a} = 0$ (Positive-definiteness)

Hence dot multiplication is commutative and is distributive with respect to vector addition; in fact from (a) with $q_1 = 1$ and $q_2 = 1$ we have

5. $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$ (Distributivity)

Furthermore, from (1) and $|\cos\gamma| \leq 1$ we see that

6. $|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$ (Schwarz inequality)

7. $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (Triangle inequality)

A simple direct calculation with inner products shows that

8. $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$ (Parallelogram equality)

Equations (6)–(8) play a basic role in so-called Hilbert spaces, which are abstract inner product spaces. Hilbert spaces form the basis of quantum mechanics.

Derivation of $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ from (1): [illegible]

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Since $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors, we have from (3) $\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1$, $\hat{j}\cdot\hat{j} = |\hat{j}|^2 = 1$, $\hat{k}\cdot\hat{k} = |\hat{k}|^2 = 1$

Since they are also orthogonal (because the coordinate axes are perpendicular), we get from (2) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Now $\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

using distributive property, the first of these is a sum of nine inner products.

$$\vec{a}\cdot\vec{b} = a_1b_1\hat{i}\cdot\hat{i} + a_1b_2\hat{i}\cdot\hat{j} + \dots + a_3b_3\hat{k}\cdot\hat{k}$$

Since six of these products are zero, we obtain $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Applications of Inner Products: Typical applications of inner products are shown in the following examples.

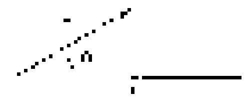
Example:

Work done by a force expressed as inner product.

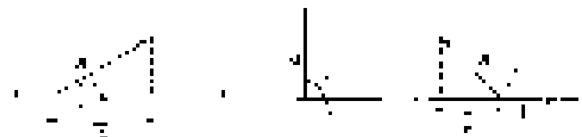
Solution:

Consider a body on which a constant force $\vec{p}$ acts. Let the body be given a displacement $\vec{d}$. Then the work done by $\vec{p}$ in the displacement is defined as

$$W = |\vec{p}||\vec{d}|\cos\alpha = \vec{p}\cdot\vec{d}$$



Vector Projection:



[illegible]

Example:

Vector projection of a on another vector b

[illegible]

Solution:

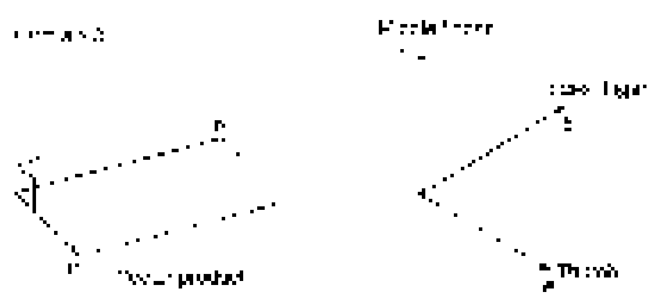
[illegible]

2.14.10 Vector Product (Cross Product)

[illegible]

Definition. Vector product (Cross product)

[illegible]



[illegible]

[illegible]

In components, [illegible] $= a \times b$

[illegible]

In terms of determinants:

[illegible]

[illegible]

2.14.10.1 **Finding a Unit Vector Perpendicular to two Given Vectors a and b**

[illegible]

Example 1

[illegible]

Solution:

[illegible]

Example 2

[illegible]

Solution:

[illegible]

Example 3.

The vectors from origin to the points A and B are $\vec{a} = [illegible]\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + [illegible]$ respectively. Find the area of

(a) the triangle OAB

(b) the parallelogram formed by $\vec{OA}$ and $\vec{OB}$ as adjacent sides.

Solution:

Given $\vec{OA} = \vec{a} = [illegible]$ and $\vec{OB} = \vec{b} = 2\hat{i} + [illegible]$

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ [illegible] & [illegible] & [illegible] \\ [illegible] & & -2 \end{vmatrix}$$

$$= [illegible]$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = \sqrt{[illegible]} = \sqrt{425} = 5\sqrt{17}$$

(a) Area of $\Delta OAB = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\sqrt{425}$ sq. units $= \frac{5}{2}\sqrt{17}$ sq. units.

(b) Area of the parallelogram formed by $\vec{OA}$ and $\vec{OB}$ as adjacent sides

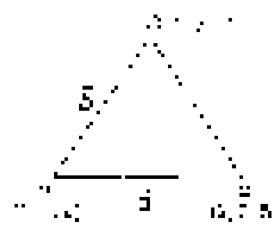
$$= |\vec{a}\times\vec{b}| = 5\sqrt{17} \text{ sq. units}$$

Example 4.

[illegible] the area of the triangle whose vertices are A(1, [illegible]), B(2, 3, 5) and C(1, 5, [illegible])

Solution:

Let the vectors $\vec{a}$ and $\vec{b}$ represent the sides AB and AC of ΔABC, then



$$\vec{a} = \vec{AB} = [illegible]$$

$$= [illegible]$$

$$= \hat{i} + 2\hat{j} [illegible]$$

and

$$\vec{b} = \vec{AC} = [illegible]$$

$$= [illegible]$$

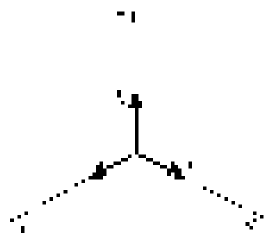
$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ [illegible] & [illegible] & [illegible] \\ 0 & [illegible] & [illegible] \end{vmatrix} = [illegible]$$

$$= [illegible]$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = \sqrt{[illegible]} = \sqrt{[illegible]}$$

$\therefore$ The area of $\Delta ABC = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\sqrt{[illegible]}$

2.14.10.2 Vector Products of the Standard Basis Vectors



Since i, j, k are orthogonal unit vectors, the definition of vector product gives some useful formulas for simplifying vector products. In right-handed coordinates these are:

$\hat{i} \times \hat{j} = \hat{k}$　　$\hat{j} \times \hat{k} = \hat{i}$　　$\hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$　　$\hat{k} \times \hat{j} = -\hat{i}$　　$\hat{i} \times \hat{k} = -\hat{j}$

2.14.10.3 General Properties of Vector Products

Vector Product has the property that for every scalar l

$$(l\vec{a}) \times \vec{b} = l(\vec{a} \times \vec{b}) = \vec{a} \times (l\vec{b})$$

It is distributive with respect to vector addition, that is

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$

$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$$

It is not commutative but anticommutative, that is

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$$

It is not associative, that is

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c} \qquad \text{in general}$$

so that the parentheses cannot be omitted.

2.14.11 Scalar Triple Product

The scalar triple product or mixed triple product of three vectors

$\vec{a} = [a_1, a_2, a_3]$,　$\vec{b} = [b_1, b_2, b_3]$,　$\vec{c} = [c_1, c_2, c_3]$

is denoted by $(\vec{a}\ \vec{b}\ \vec{c})$ and is defined by $(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$

We can write this as a third order determinant. To do this we set $\vec{b} \times \vec{c} = \vec{v} = [v_1, v_2, v_3]$. Then from the dot product in components

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \vec{v} = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

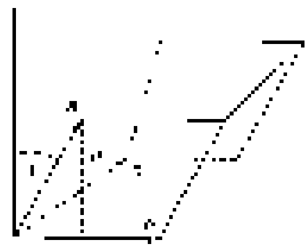
The expression on the right is the expansion of a third order determinant by its first row. Thus

$$(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric Interpretation of Scalar Triple Products

The absolute value of the scalar triple product is the volume of the parallelepiped with $\vec{a}, \vec{b}, \vec{c}$ as edge vectors (Figure). $|\vec{a} \cdot (\vec{b} \times \vec{c})| = |\vec{a}||\vec{b} \times \vec{c}||\cos\beta|$ because $|\vec{a}||\cos\beta|$ is the height h and, by (1), the base, the parallelogram with

[illegible]



[illegible]

Proof: [illegible]

[illegible]

Example

[illegible]

Solution:

[illegible]

Testing Linear Independence of 3 Vectors using Scalar Triple Product:

[illegible]

Theorem 1 (Linear Independence of Three Vectors)

Three vectors form a linearly independent set if and only if their scalar triple product is not zero.

The scalar triple product is the most important of the repeated products. Other repeated products [illegible], but are used only rarely.

2.14.12 Vector Triple Product

If a, b and c are three vectors then their vector triple product is written as $a \times (b \times c)$.

[illegible]

Example:

Let $a = i + j - k$, [illegible]

Find the vector [illegible]

Solution:

Since [illegible]

[illegible]

So [illegible]

2.14.13 Vector and Scalar Functions and Fields. Derivatives

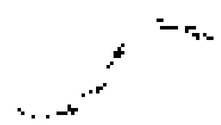
[illegible] whose values are vectors

[illegible]

depending on the points P in space, and scalar functions, whose values are scalars

[illegible]

[illegible]



[illegible]

Comment on Notation. [illegible]

[illegible]

[illegible]

Example: 1

Scalar function (Euclidean distance in space)

Solution:

The distance $f(P)$ of any point P from a fixed point P_0 in space is a scalar function whose domain of definition is the whole space. $f(P)$ defines a scalar field in space. If we introduce a Cartesian coordinate system and P_0 has the coordinates x_0, y_0, z_0, then f is given by the well-known formula

$$f(P) = f(x, y, z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

where x, y, z are the coordinates of P. If we replace the given Cartesian coordinate system by another such system, then the values of the coordinates of P and P_0 will in general change, but $f(P)$ will have the same value as before. Hence $f(P)$ is a scalar function. The direction cosines of the line through P and P_0 are not scalars because their values depend on the choice of the coordinate system.

Example: 2

Vector field (Velocity field)

Solution:

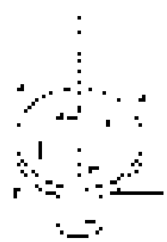
At any instant the velocity vectors $\mathbf{v}(P)$ of a rotating body B constitute a vector field, called the velocity field of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotation, then

$$\mathbf{v}(x, y, z) = \mathbf{w} \times \mathbf{r} = \mathbf{w} \times [x, y, z] = \mathbf{w} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

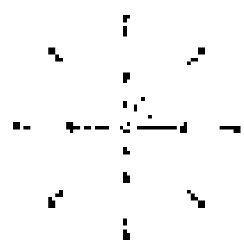
where x, y, z are the coordinates of any point P of B at the instant under consideration. If the coordinates are such that the z-axis is the axis of rotation and $\mathbf{w}$ points in the positive z-direction, then $\mathbf{w} = \omega\mathbf{k}$ and

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega[-y, x, 0] = \omega(-y\mathbf{i} + x\mathbf{j})$$

An example of a rotating body and the corresponding velocity field are shown in Figure below. Also shown is another example of vector field – the gravitational field.



Velocity field of a rotating body



Gravitational field

Vector Calculus: We shall now show that the basic concepts of calculus, such as convergence, continuity, and differentiability can be defined for vector functions in a simple and natural way. Most important is the case of the following.

Convergence: An infinite sequence of vectors $\mathbf{a}_{(n)}$, $n = 1, 2, \ldots$, is said to converge if there is a vector $\mathbf{a}$ such that

$$\lim_{n \to \infty} |\mathbf{a}_{(n)} - \mathbf{a}| = 0$$

$\mathbf{a}$ is called the limit vector of that sequence, and we write

$$\lim_{n \to \infty} \mathbf{a}_{(n)} = \mathbf{a}$$

Accordingly, a sequence of vectors is convergent if and only if the three sequences of components of the vectors converge to the corresponding components of a.

Similarly, a vector function $v(t)$ of a real variable t is said to have the limit l as t approaches t_0, if $v(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t \to t_0} |v(t) - l| = 0$$

Then we write $$\lim_{t \to t_0} v(t) = l$$

Continuity: A vector function $v(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighborhood of t_0 and

$$\lim_{t \to t_0} v(t) = v(t_0)$$

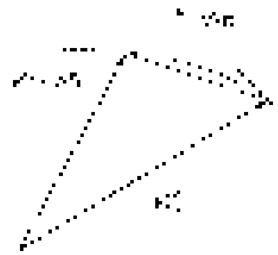
If we introduce a Cartesian coordinate system, we may write

$$v(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)i + v_2(t)j + v_3(t)k$$

Then $v(t)$ is continuous at t_0 if and only if its three components are continuous at t_0. We now state the most important of these definitions.

2.14.13.1 Derivative of a Vector Function

A vector function $v(t)$ is said to be differentiable at a point t if the following limit exists:



Derivative of a vector function

$$v'(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

This vector $v'(t)$ is called the derivative of $v(t)$. See Figure above (the curve in this figure is the locus of the tips of the arrows representing v for values of the independent variable in some interval containing t and $t + \Delta t$).

In terms of components with respect to a given Cartesian coordinate system, $v(t)$ is differentiable at a point t if and only if its three components are differentiable at t, and then the derivative $v'(t)$ is obtained by differentiating each component separately,

$$v'(t) = [v_1'(t), v_2'(t), v_3'(t)]$$

It follows that the familiar rules of differentiation yield corresponding rules for differentiating vector functions, for example

$$(cv)' = cv' \qquad (c \text{ constant})$$

$$(u + v)' = u' + v' \text{ and in particular}$$

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$(u \times v)' = u' \times v + u \times v'$$

$$[u\ v\ w]' = [u'\ v\ w] + [u\ v'\ w] + [u\ v\ w']$$

The order of the vectors must be carefully observed because cross multiplication is not commutative.

Example:

Derivative of a vector function of constant length.

Solution:

Let $v(t)$ be a vector function whose length is constant, say, $|v(t)| = c$. Then $|v|^2 = v \cdot v = c^2$, and $(v \cdot v)' = 2v \cdot v' = 0$, by differentiation. This yields the following result. The derivative of a vector function $v(t)$ of constant length is either the zero vector or is perpendicular to $v(t)$.

2.14.13.2 Partial Derivatives of a Vector Function

[illegible]

[illegible]

Similarly [illegible]

Example:

[illegible]

Solution:

[illegible]

[illegible]

2.14.14 Gradient of a Scalar Field

[illegible]

Definition of Gradient: [illegible]

1. [illegible]

[illegible]

2. [illegible]

[illegible]

2.14.15 Directional Derivative

[illegible]

$$D_b f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} \quad (s = \text{distance between } P \text{ and } Q)$$

[illegible]

[illegible]

[illegible]

[illegible]

Directional derivative

$$D_b f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'$$

[illegible]

$$D_b f = \frac{df}{ds} = b \cdot \operatorname{grad} f \qquad (|b| = 1)$$

[illegible]

$$D_a f = \frac{df}{ds} = \frac{1}{|a|}\, a \cdot \operatorname{grad} f$$ [illegible]

Example

Gradient. Directional Derivative

Find the directional derivative of $f = 2x^2 + 3y^2 + z^2$ at the point $P:(2, 1, 3)$ in the direction of the vector $a = i - 2k$

Solution:

[illegible] $P:(2, 1, 3)$, $\operatorname{grad} f = 8i + 6j + 6k$

$$D_a f = \frac{a}{|a|} \cdot \operatorname{grad} f$$

$$= \frac{1}{\sqrt{5}}(i - 2k) \cdot (8i + 6j + 6k)$$

$$= \frac{8 - 12}{\sqrt{5}} = -\frac{4}{\sqrt{5}} = [illegible]$$

[illegible]

2.14.16 Gradient Characterizes Maximum Increase

Theorem. 1 (Gradient, Maximum Increase)

[illegible]

[illegible]

7 [illegible] grad f [illegible] cos γ

[illegible]

[illegible]

Gradient as Surface Normal Vector: [illegible]

8 [illegible]

[illegible]

9 [illegible]

[illegible]

10 [illegible]

[illegible]

[illegible]

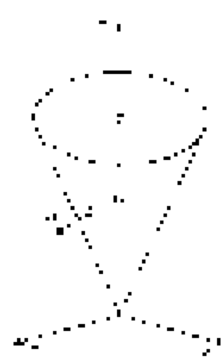
11 [illegible] = grad f [illegible] = 0

[illegible]

Theorem 2 (Gradient as Surface Normal Vector)

[illegible]

Comment. [illegible]



Example:

Gradient as Surface Normal Vector

[illegible] at the point P: (1, 0, 2).

Solution:

[illegible]

Hence by Theorem 2, grad f is a normal vector of the cone at point P.

Now a unit normal vector at point P will be,

$$n = \frac{1}{|\text{grad } f|}\,\text{grad } f = \frac{2}{\sqrt{5}}\,i - \frac{1}{\sqrt{5}}\,k$$

and the other is $-n$.

2.14.17 Vector Fields that are Gradients of a Scalar Field ("Potential")

Some vector fields have the advantage that they can be obtained from scalar fields, which can be handled more easily. Such a vector field is given by a vector function $v(P)$, which is obtained as the gradient of a scalar function, say $v(P) = \text{grad } f(P)$. The function $f(P)$ is called a potential function or a potential of $v(P)$. Such a $v(P)$ and the corresponding vector field are called conservative because in such a vector field, energy is conserved; that is, no energy is lost (or gained) in displacing a body (or a charge in the case of an electrical field) from a point P to another point in the field and back to P.

2.14.18 Divergence of a Vector Field

Vector calculus owes much of its importance in engineering and physics to the gradient, divergence and curl. Having discussed the gradient, we continue with the divergence in our following discussion.

Let $v(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of v. Then the function

1 $$\text{div } v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of v or the divergence of the vector field defined by v. Another common notation for the divergence of v is $\nabla \cdot v$,

$$\text{div } v = \nabla \cdot v$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (v_1 i + v_2 j + v_3 k) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

with the understanding that the "product" $(\partial/\partial x) \cdot v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$, etc. This is a convenient notation, but nothing more. Note that $\nabla \cdot v$ means the scalar div v, whereas ∇f means the vector grad f.

Example:

If $$v = 3xz\,i + 2xy\,j - yz^2\,k$$

then $$\text{div } v = 3z + 2x - 2yz$$

We shall see below that the divergence has an important physical meaning. Clearly, the values of a function that characterizes a physical or geometric property must be independent of the particular choice of coordinates. In other words, those values must be invariant with respect to coordinate transformations.

Theorem 1 (Invariance of The Divergence)

1. The values of div v depend only on the points in space (and, of course, on v) but not on the particular choice of the coordinates.

   Now let us turn to the more immediate practical task of gaining a feel for the significance of the divergence.

   If $f(x, y, z)$ is a twice differentiable scalar function, then

$$\text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

so $\quad \text{div}(\text{grad } f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

3. The expression on the right is the Laplacian of f. Thus

$$\text{div}(\text{grad } f) = \nabla^2 f$$

Example 1.

Gravitational force

The gravitational force p is the gradient of the scalar function $f(x, y, z) = c/r$, which satisfies Laplace's equation $\nabla^2 f = 0$. According to (3), this means that $\text{div } p = 0$ $(r > 0)$.

The following example taken from hydrodynamics shows the physical significance of the divergence of a vector field. [illegible]

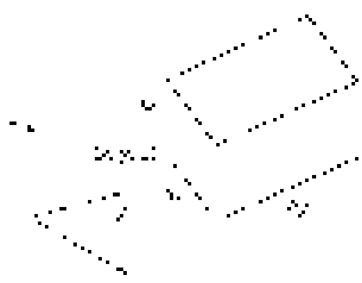
Example 2.

1. **Motion of a compressible fluid. Physical meaning of the divergence**

We consider the motion of a fluid in a region R having no sources or sinks in R, that is, no points at which fluid is produced or disappears. The concept of fluid state is meant to cover also gases and vapors. Fluids in the restricted sense, or liquids (water or oil, for instance), have very small compressibility, which can be neglected in many problems. Gases and vapors have large compressibility. Their density ρ (= mass per unit volume) depends on the coordinates x, y, z in space and may also depend on time t. We assume that our fluid is compressible.

We consider the flow through a small rectangular box W of dimensions $\Delta x, \Delta y, \Delta z$ with edges parallel to the coordinate axes (Fig. below). W has the volume $\Delta V = \Delta x \, \Delta y \, \Delta z$. Let $v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$ be the velocity vector of the motion. We set

2. $u = \rho v = [u_1, u_2, u_3] = u_1 i + u_2 j + u_3 k$ and assume that u and v are continuously differentiable vector functions of x, y, z and t, that is, they have first partial derivatives which are continuous. Let us calculate the change in the mass included in W by considering the flux across the boundary, that is, the total loss of mass leaving W per unit time. Consider the flow through the left face of W, whose area is $\Delta x \, \Delta z$. The components v_1 and v_3 of v are parallel to that face and contribute nothing to this flow. Hence the mass of fluid entering through that face during a short time interval Δt is given approximately by



$$(\rho v_2)_y \, \Delta x \, \Delta z \, \Delta t = (u_2)_y \, \Delta x \, \Delta z \, \Delta t$$

Fig. Physical interpretation of the divergence

where the subscript y indicates that this expression refers to the left face. The mass of fluid leaving the box W through the opposite face during the same time interval is approximately $(u_2)_{y+\Delta y} \, \Delta x \, \Delta z \, \Delta t$, where the subscript $y + \Delta y$ indicates that this expression refers to the right face (which is not visible in Fig. above). The difference

$$\Delta u_2 \, \Delta x \, \Delta z \, \Delta t = \frac{\Delta u_2}{\Delta y} \Delta V \, \Delta t \qquad \left[\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y\right]$$

is the approximate loss of mass. Two similar expressions are obtained by considering the other two pairs of parallel faces of W. If we add these three expressions, we find that the total loss of mass in W during the time interval Δt is approximately

$$\left(\frac{\Delta v_x}{\Delta x}+\frac{\Delta v_y}{\Delta y}+\frac{\Delta v_z}{\Delta z}\right)\Delta x\,\Delta y\,\Delta z$$

where $\Delta v_y = (v_y)_{y+\Delta y} - (v_y)_y$

and $\Delta v_z = (v_z)_{z+\Delta z} - (v_z)_z$

This loss of mass in fluid is caused by the change (decrease) in the density of the fluid and is equal to

$$-\frac{\Delta\rho}{\Delta t}\Delta x\,\Delta y\,\Delta z$$

If we equate both expressions, divide the resulting equation by $\Delta x\,\Delta y\,\Delta z$, we get

$$\frac{\Delta v_x}{\Delta x}+\frac{\Delta v_y}{\Delta y}+\frac{\Delta v_z}{\Delta z} = -\frac{\Delta\rho}{\Delta t}$$

Now we let Δx, Δy, Δz and Δt approach zero, we get

$$\operatorname{div}(\rho v) = \nabla\cdot(\rho v) = -\frac{\partial\rho}{\partial t}$$

(3) i.e., $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0$

This important relation is called as the condition for the conservation of mass or the continuity equation of a compressible fluid flow.

If the flow is steady (i.e. independent of time), then $\frac{\partial\rho}{\partial t} = 0$ and the continuity equation is

(4) $\nabla\cdot(\rho v) = 0$

If the density ρ is constant, so that the fluid is incompressible, the equation (4) becomes

(5) $\nabla\cdot v = 0$

This relation is known as the condition of incompressibility. It expresses the fact that the balance of outflow and inflow for a given volume element is zero at any time. Clearly, the assumption that the flow has no sources or sinks in R is essential to our argument.

From this discussion you should conclude and remember that, roughly speaking, the divergence measures outflow minus inflow.

If the divergence of velocity field of a fluid is zero ($\operatorname{div} v = 0$), then the fluid is called as **incompressible**. Similarly in magnetic theory, if $\operatorname{div}(v) = 0$, then the vector field v is said to be **solenoidal**.

7.14.19 Curl of a Vector Field

Gradient, divergence and curl are basic in connection with fields. We now define and discuss the curl. Let x, y, z be right-handed Cartesian coordinates, and let

$$v(x, y, z) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

be a differentiable vector function. Then the curl is

$$\operatorname{curl} v = \nabla\times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\nabla\times v = \left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\hat{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\hat{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\hat{k}$$

is called the curl of the vector function $\mathbf{v}$ or the curl of the vector field defined by $\mathbf{v}$.

Instead of curl $\mathbf{v}$, the notation rot $\mathbf{v}$ is also used. (Please see explanation for left-handed and right-handed system.)

Example 1.

With respect to right-handed Cartesian coordinates, let

$$\mathbf{v} = yz\,\mathbf{i} + 3zx\,\mathbf{j} + z\,\mathbf{k}$$

Then (1) gives

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$$= -3x\,\mathbf{i} + y\,\mathbf{j} + (3z - z)\,\mathbf{k} = -3x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

The curl plays an important role in many applications. Let us illustrate this by a basic example. (We shall say more about the physical nature of the curl in next section).

Example 2.

Rotation of a rigid body. Relation to the curl

1. The rotation of a rigid body B about a fixed axis in space can be described by a vector $\mathbf{w}$ of magnitude ω in the direction of the axis of rotation, where ω (>0) is the angular speed of the rotation, and $\mathbf{w}$ is directed so that the rotation appears clockwise if we look in the direction of $\mathbf{w}$. The velocity field of the rotation can be represented in the form

   $$\mathbf{v} = \mathbf{w} \times \mathbf{r}$$

   where $\mathbf{r}$ is the position vector of a moving point with respect to a Cartesian coordinate system having the origin on the axis of rotation. Let us choose right-handed Cartesian coordinates such that

   $$\mathbf{w} = \omega\mathbf{k}$$

   that is, the axis of rotation is the z-axis. Then

   $$\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$$

   and hence,

   $$\text{curl}\,\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k}$$

   Since $\mathbf{w} = \omega\mathbf{k}$,

2. $$\text{curl}\,\mathbf{v} = 2\mathbf{w}$$

   Hence in the case of a rotation of a rigid body, the curl of the velocity field has the direction of the axis of rotation, and its magnitude equals twice the angular speed ω of the rotation.

   Note that our result does not depend on the particular choice of the Cartesian coordinate system in space.

   For any twice continuously differentiable scalar function f,

3. $$\text{curl}(\text{grad}\,f) = 0$$

   as can easily be verified by direct calculation, as grad f is given by

   $$\text{grad}\,f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\operatorname{curl}(\operatorname{grad} f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) - j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)$$

$$= 0i + 0j + 0k = 0$$

Hence if a vector function is the gradient of a scalar function, its curl is the zero vector. Since the curl characterizes the rotation in a field, we also say more briefly that gradient fields describing a motion are irrotational. If such a field occurs in some other connection, not as a velocity field, it is usually called conservative.

If $\operatorname{curl} v = 0$, then v is said to be an irrotational field.

Example:

The gravitational field has curl $p = 0$. The field in the rotation of a rigid body example of this section is not irrotational since we saw that $\operatorname{curl} v = 2\omega \neq 0$. A similar velocity field is obtained by stirring coffee in a cup.

Other formula: another key formula for any twice continuously differentiable vector function v is

2. $\operatorname{div}(\operatorname{curl} v) = 0$

It is trivial to derive from the definitions of curl and divergence as follows. A proof of (2) follows directly from the definitions of curl and divergence as follows:

Let $v = v_1 i + v_2 j + v_3 k$

$$\operatorname{curl} v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - j\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\operatorname{div}(\operatorname{curl} v) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$$= 0$$

The curl is defined in terms of coordinates, but if it is supposed to have a physical or geometrical significance, it should not depend on the choice of those coordinates. This is true, as follows.

Theorem 4 (Invariance of The Curl)

The length and direction of curl v are independent of the particular choice of Cartesian coordinate systems in space.

2.14.19.1 Important Repeated Operations by Nabla Operator (∇)

1. $\operatorname{div}(\operatorname{grad} f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

2. $\text{curl}(\text{grad } f) = \nabla \times (\nabla f) = 0$
3. $\text{div}(\text{curl } F) = \nabla \cdot (\nabla \times F) = 0$
4. $\text{curl}(\text{curl } F) = \text{grad div } F - \nabla^2 F$
5. $\text{grad div } F = \text{curl curl } F + \nabla^2 F$

2.14.20 Vector Integral Calculus: Integral Theorems

2.14.20.1 Line Integral

The concept of a line integral is a simple and natural generalization of a definite integral

1. $\int_a^b f(x)\,dx$. In (1) we integrate the function $f(x)$ (the integrand) from $x = a$ along the x-axis to $x = b$.

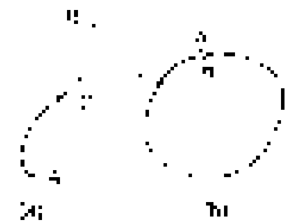
In a line integral we shall integrate a given function, called the integrand, along a curve C in space or in the plane. Hence curve integral would be a better name but line integral is standard.

We represent the curve C by a parametric representation

2. $r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k \quad (a \le t \le b)$

We call C the path of integration, $A: r(a)$ its initial point, and $B: r(b)$ its terminal point. C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow (as shown in below figure (a)). The points A and B may coincide (as shown in figure (b)). Then C is called a closed path.

We call C a smooth curve if C has a unique tangent at each of its points whose direction varies continuously as we move along C.



Directed curve

Technically: C has a representation (2) such that $r(t)$ is differentiable and the derivative $r'(t) = dr/dt$ is continuous and different from the zero vector at every point of C.

2.14.20.2 Definition and Evaluation of Line Integrals

A line integral of a vector function $F(r)$ over a curve C is defined by

3. $$\int_C F(r) \cdot dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt}\,dt$$

In terms of components, with $dr = [dx, dy, dz]$ and $' = d/dt$, formula (3) becomes

3'. $$\int_C F(r) \cdot dr = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz = \int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt$$

If the path of integration C in (3) is a closed curve, then instead of

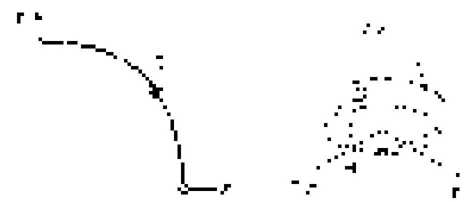
$$\int_C \quad \text{we also write} \quad \oint_C$$

We see that the integral in (3) on the right is a definite integral of a function of t taken over the interval $a \le t \le b$ on the t-axis in the positive direction: the direction of increasing t. This definite integral exists for continuous F and piecewise smooth C, because this makes $F \cdot r'$ piecewise continuous.

Example 1.

Find the value of the line integral (3) when $F(r) = [-y, -xy] = -yi - xyj$ and C is the circular arc as shown in figure from A to B shown below.

Solution:



We may represent C by

$$\mathbf{r}(t) = [\cos t, \sin t] = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \quad (0 \le t \le \pi/2)$$

Thus $\quad x(t) = \cos t, \; y(t) = \sin t$, so that

$$\mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = [-\sin t, -\cos t \sin t] = -\sin t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}$$

By differentiation $\quad \mathbf{r}'(t) = [-\sin t, \cos t] = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

So by (3)

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^{\pi/2} [-\sin t, -\cos t \sin t]\cdot[-\sin t, \cos t]\,dt$$

$$= \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\,dt$$

$$= \int_0^{\pi/2} \sin^2 t\,dt - \int_0^{\pi/2} \cos^2 t \sin t\,dt$$

$$= \int_0^{\pi/2} \left(\frac{1 - \cos 2t}{2}\right)dt - \int_1^0 u^2(-du) \quad (u = \cos t)$$

$$= \left(\frac{\pi}{4} - 0\right) - \left(\frac{1}{3}\right) \approx 0.4521$$

Example 2.

Line integral in space.

The evaluation of line integrals in space is practically the same as it is in the plane. To see this, find the value of (3) when $\mathbf{F}(\mathbf{r}) = [z, x, y] = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and C is the helix (figure above from Example 3) $\mathbf{r}(t) = [\cos t, \sin t, 3t] = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\,\mathbf{k}$ where $0 \le t \le 2\pi$.

Solution:

We have $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$.

Thus $\quad \mathbf{F}(\mathbf{r}(t)) = 3t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} (3t, \cos t, \sin t)\cdot(-\sin t, \cos t, 3)\,dt$$

$$= \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t)\,dt$$

$$= 6\pi + \pi - 0 = 7\pi$$

$$\approx 21.99$$

1. **Choice of representation:** Does this value of the line integral depend on the particular representation of C? The answer is no, as we show below.
2. **Choice of path:** Does the value change if we integrate from the same point A to the end point B along another path? The answer is yes, in general, see Example 5.

Example 3.

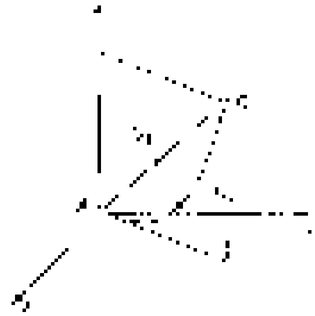
Dependence of a line integral on path (same endpoints)

Evaluate the line integral [illegible] along two different paths with the same initial point [illegible] and the same terminal point [illegible], namely (Fig.) [illegible]

(a) C_1: the straight-line segment [illegible]

and

(b) C_2: the parabolic arc [illegible] $0 \le t \le 1$.



Solution:

(a) By substituting [illegible] into [illegible]. We also need

[illegible]

Hence the integral over C_1 is

$$\int_{C_1} \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = [illegible]$$

$$= [illegible] = \frac{37}{12}$$

(b) Similarly, by substituting [illegible] over the path C_2

$$\int_{C_2} \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = [illegible]$$

The two results are different, although the endpoints are the same. [illegible]

[illegible]

Can we find conditions that guarantee independence? This is a basic question in connection with physical applications. [illegible]

2.14.20.3 General Properties of the Line Integral (3)

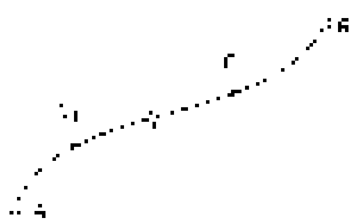
[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]



2.14.20.4 Line Integrals Independent of Path

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

[illegible]

independent. For instance, in mechanics, independence of path means that we have to do the same amount of work regardless of the path to the mountaintop, be it short and steep or long and gentle, or [illegible] the work done [illegible]. Most forces are of this type [illegible] a straight line.

We define a line integral (1) is independent of path in a domain D in space if for every pair of endpoints A, B in D the integral (1) has the same value for all paths in D that begin at A and end at B. A very practical criterion for path independence is the following.

Theorem 1 (Independence of Path)

A line integral (1) with continuous F_1, F_2, F_3 in a domain D in space is independent of path in D if and only if $F = [F_1, F_2, F_3]$ is the gradient of some function f in D,

$$(2) \qquad F = \text{grad } f$$

thus, in components,

$$(2') \qquad F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

Example 1:

Show that the line integral

$$\int_C F \cdot dr = \int_C (2x\,dx + 2y\,dy + 4z\,dz)$$

is independent of path in any domain in space and find its value if C has the initial point $A: (0, 0, 0)$ and terminal point $B: (2, 2, 2)$.

Solution:

By inspection we find that

$$F = [2x, 2y, 4z] = 2xi + 2yj + 4zk = \text{grad } f$$

where $$f = x^2 + y^2 + 2z^2$$

[illegible] more complicated [illegible] integration as in Example [illegible]. Thus we may choose the most convenient path. To find the value of the integral we can choose the convenient straight path

$$C: r(t) = [t, t, t] = ti + tj + tk, \quad 0 \le t \le 2,$$

and get $r' = [1, 1, 1]$, thus $F \cdot r' = 2t + 2t + 4t = 8t$ and from this it is

$$\int_C (2x\,dx + 2y\,dy + 4z\,dz) = \int_0^2 8t\,dt = 4t^2 \Big|_0^2 = 16$$

Proof of Theorem 1:

(a) If (2) holds for some function f in D, let C be any path in D from any point A to any point B, given by

$$r(t) = [x(t), y(t), z(t)], \quad a \le t \le b$$

Then from (2) we get

$$\int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \frac{df}{dt}\,dt = f[x(t), y(t), z(t)]\Big|_{t=a}^{t=b} = f(B) - f(A)$$

This shows that the value of the integral is simply the difference of the values of f at the two endpoints of C and is, therefore, independent of the path C.

2. [illegible]

[illegible]

3. **Potential theory** [illegible]

Example 2.

[illegible]

Solution:

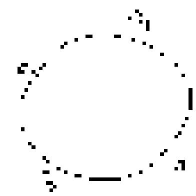
[illegible]

Theorem 2 (Independence of path)

[illegible]

Proof: [illegible]

[illegible]



[illegible]

Work, Conservative and Nonconservative (Dissipative) Physical Systems: [illegible]

[illegible]

[illegible]

[illegible]

Exactness and Independence of Path: [illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

a torus or a sphere without two points, and the interior of a cube with one space diagonal removed are not simply connected.

The criterion for path independence and exactness is then as follows:

Theorem 3 (Criterion for exactness and independence of path)

Let F_1, F_2, F_3 in the line integral

$$\int_C F(r)\cdot dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

be continuous and have continuous first partial derivatives in a domain D in space. Then:

(a) If the line integral is independent of path in D—and thus the differential form under the integral sign is exact, then in D,

F $\quad\quad$ $\operatorname{curl} F = 0$

In components, hence the formula follows from $\operatorname{curl} F = 0$, which gives

$$\text{Since} \qquad \operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl} F = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

(b) If (6) holds in D and D is simply connected, then the line integral is independent of path in D.

Proof:

(a) If the line integral is independent of path in D, then $F = \operatorname{grad} f$ and

$\operatorname{curl} F = \operatorname{curl}(\operatorname{grad} f) = 0$ $\quad$ So that (6) holds.

(b) The proof of the converse requires "Stokes's theorem" and is omitted here.

Comment For a line integral in the plane

$$\int_C F(r)\cdot dr = \int_C (F_1 dx + F_2 dy),$$

the curl has only one component (z-component), so that (6) reduces to the single relation

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Example:

Exactness and independence of path. Determination of a potential.

Using (6) show that the differential form under the integral sign of

$$I = \int_C \left[2xyz^2 dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right]$$

is exact, so that we have independence of path in any domain, and find the value of I from $A:(0, 0, 1)$ to $B:(1, \pi/4, 2)$.

Solution:

Exactness follows from (6'), which gives

$$(F_3)_y = 2x^2z + \cos yz - yz\sin yz = (F_2)_z$$
$$(F_1)_z = 4xyz = (F_3)_x$$
$$(F_2)_x = 2xz^2 = (F_1)_y$$

To find f, we integrate F_2 (which is "long," so that we save work) and then differentiate to compare with F_1 and F_3.

$$f_1 = F_1 = 2xyz^2$$
$$f_2 = F_2 = x^2z^2 + z\cos yz$$
$$f_3 = F_3 = 2x^2yz + y\cos yz$$
$$f = \int F_2\,dy = \int (x^2z^2 + z\cos yz)\,dy = x^2z^2y + \sin yz + g(x, z)$$
$$f_x = 2xz^2y + g_x = F_1 = 2xyz^2, \quad g_x = 0, \quad g = h(z)$$
$$f_z = 2x^2zy + y\cos yz + h' = F_3 = 2x^2zy + y\cos yz, \quad h' = 0$$

so that, taking $h = 0$, we have

$$f(x, y, z) = x^2yz^2 + \sin yz.$$

From (3) and (5) we get

$$I = f(B) - f(A)$$
$$= f\left(1, \frac{\pi}{4}, 2\right) - f(0, 0, 1) = \pi + \sin\frac{1}{2}\pi - 0 = \pi + 1$$

The assumptions in Theorem 3 are satisfied, therefore independence of path holds and the integral can be evaluated.

2.14.21 Green's Theorem in the Plane

Double integrals over a plane region may be transformed into line integrals over the boundary of the region and conversely. This is of practical interest because it may help to make the evaluation of an integral easier. It also helps in the theory whenever one wants to switch from one kind of integral to the other. The transformation can be done by the following theorem.

Theorem. 1 (Green's Theorem in The Plane)

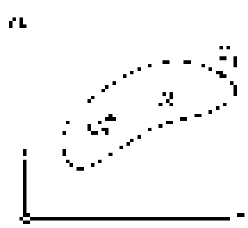
(Transformation between double integrals and line integrals)

Let R be a closed bounded region (see Sec. 6.3) in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\dfrac{\partial F_1}{\partial y}$ and $\dfrac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then

$$(1) \quad \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)$$

Here we integrate along the entire boundary C of R such that R is on the left as we advance in the direction of integration (See Fig. 1.4).

Region R whose boundary C consists of two parts: C_1 is traversed counterclockwise, while C_2 is traversed clockwise, so that R is on the left as we advance.



Comment. Formula (1) can be written in vectorial form,

$$(1') \quad \iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{k}\,dx\,dy = \oint_C \mathbf{F} \cdot d\mathbf{r} \qquad (\mathbf{F} = [F_1, F_2] = F_1\mathbf{i} + F_2\mathbf{j})$$

This follows from the fact that the third component of curl F is $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

Example.

Verification of Green's theorem in the plane.

Green's theorem in the plane will be of essential importance in our further work. Before proving it, let us get used to it by verifying it for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$.

Solution:

In (1) on the left we get

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R [(2y + 2) - (2y - 7)]\,dx\,dy = 9\iint_R dx\,dy = 9\pi$$

since the circular disk R has area π.

The circle C in the counterclockwise sense is represented by

$$r(t) = [\cos t, \sin t]$$

Then $$r'(t) = [-\sin t, \cos t]$$

On C we thus obtain

$$F_1 = y^2 - 7y = \sin^2 t - 7\sin t$$

$$F_2 = 2xy + 2x = 2\cos t \sin t + 2\cos t$$

Hence the line integral in (1) on the right becomes

$$\oint_C (F_1 x' + F_2 y')\,dt = \int_0^{2\pi} [(\sin^2 t - 7\sin t)(-\sin t) + 2(\cos t \sin t + \cos t)(\cos t)]\,dt$$

$$= 0 + 7\pi - 0 + 2\pi = 9\pi$$

This verifies Green's theorem in the plane.

2.14.22 Triple Integrals : Divergence Theorem of Gauss

In this section we first discuss triple integrals. Then we consider the first integral theorem which transforms surface integrals into volume integrals. This theorem is called as Gauss's divergence theorem because it involves the divergence of a vector function.

The triple integral is a generalization of the double integral introduced earlier. For defining this integral we consider a function $f(x, y, z)$ defined in a bounded closed region T of space. We subdivide T by planes parallel to the coordinate planes. Then we consider those boxes of the subdivision that lie entirely inside T, and number them from 1 to n. In each such box we choose an arbitrary point, say, (x_k, y_k, z_k) in box k, and form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

where ΔV_k is the volume of box k. We do this for larger and larger positive integers n arbitrarily but so that the maximum length of all the edges of those n boxes approaches zero as n approaches infinity. This gives a sequence of real numbers $J_{n_1}, J_{n_2}, \ldots$. We assume that $f(x, y, z)$ is continuous in a domain containing T, and T is bounded by finitely many smooth surfaces (see Fig. 5.4). Then it can be shown (see Ref. [GR4]) that the sequence converges to a limit that is independent of the choice of subdivisions and corresponding points (x_k, y_k, z_k). This limit is called the triple integral of $f(x, y, z)$ over the region T and is denoted by

$$\iiint_T f(x, y, z)\,dx\,dy\,dz \quad \text{or by} \quad \iiint_T f(x, y, z)\,dV$$

Triple integrals can be evaluated by three successive integrations. This is similar to the evaluation of double integrals by two successive integrations.

2.14.22.1 Divergence Theorem of Gauss

Triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely. This is of practical interest because one of the two kinds of integral is often simpler than the other.

[illegible] of a vector function $\mathbf{F} = [F_1, F_2, F_3] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is

$$(1) \qquad \operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{(Sec. 8.10)}$$

Theorem. 1 (Divergence Theorem of Gauss)

(Transformation between volume and surface integrals)

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\mathbf{F}(x, y, z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$(2) \qquad \iiint_T \operatorname{div} \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA$$

where $\mathbf{n}$ is the outer unit normal vector of S (pointing to the outside of S, as in Fig. 251).

Formula (2) in Components. Using (1) and $\mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma]$, we can write (2) as

$$(2^*) \qquad \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) \, dA$$

Since $\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_S (F_1 \, dy\,dz + F_2 \, dz\,dx + F_3 \, dx\,dy)$

Equation (2*) may be written as

$$(3) \qquad \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1 \, dy\,dz + F_2 \, dz\,dx + F_3 \, dx\,dy)$$

Example:

Evaluation of a surface integral by the divergence theorem

Before we prove the divergence theorem, let us show a typical application. By transforming to a triple integral, evaluate

$$I = \iint_S (x^3 \, dy\,dz + x^2 y \, dz\,dx + x^2 z \, dx\,dy)$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

Solution:

In (3) we have

$$F_1 = x^3, \; F_2 = x^2 y, \; F_3 = x^2 z$$

Hence, $\qquad \operatorname{div} \mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$

Introducing polar coordinates r, θ defined by $x = r\cos\theta$, $y = r\sin\theta$ (thus cylindrical coordinates r, θ, z), we have $dx\,dy\,dz = r\,dr\,d\theta\,dz$, and we obtain

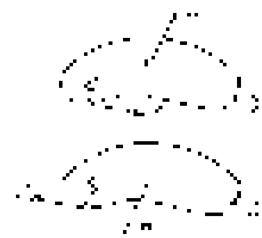
$$I = \iiint_T 5x^2 \, dx\,dy\,dz$$

$$= \int_{z=0}^{b} \int_{r=0}^{a} \int_{\theta=0}^{2\pi} 5r^2 \cos^2\theta \, r \, dr\,d\theta\,dz$$

$$= 5b \int_0^a \int_0^{2\pi} r^3 \cos^2\theta \, d\theta\,dr$$

$$= 5b \frac{a^4}{4} \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{5}{4}\pi a^4 b$$

2.14.22.2 Stokes's Theorem



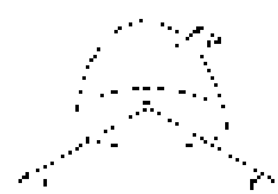
Theorem 1 (Stokes's Theorem)

Example 1.

Verification of Stokes's theorem

Solution:

$$\text{curl } F = [-1, -1, -1]$$



[illegible]

[illegible]

Example 2.

Green's theorem in the plane as a special case of Stokes's theorem

[illegible]

[illegible]

Solution:

[illegible]

[illegible]

[illegible]

■■■■

Previous GATE and ESE Questions

Q.1 $\lim_{x \to 0} \frac{[illegible]}{x}$ is equal to
(a) 0 (b) ∞
(c) 1 (d) −1
[ME, GATE-2008, 1 mark]

Q.2 If P, Q and R are three points having coordinates (3, −2, −1), (1, 3, 4), (2, 1, −2) in XYZ space, then the distance from point P to plane OQR (O being the origin of the coordinate system) is given by
(a) 3 (b) 5
(c) 7 (d) 9
[CE, GATE-2003, 1 mark]

Q.3 The value of the function $f(x) = \lim_{x \to 0} \frac{x^3 + x^2}{2x^3 - 7x^2}$ is
(a) 0 (b) $-\frac{1}{7}$
(c) $\frac{1}{7}$ (d) ∞
[CS, GATE-2004, 1 mark]

Q.4 If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then dy/dx will be equal to
(a) $\sin\left(\frac{\theta}{2}\right)$ (b) $\cos\left(\frac{\theta}{2}\right)$
(c) $\tan\left(\frac{\theta}{2}\right)$ (d) $\cot\left(\frac{\theta}{2}\right)$
[ME, GATE-2004, 1 mark]

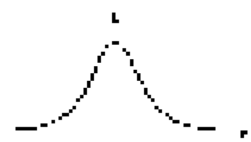
Q.5 The function $f(x) = 2x^3 - 3x^2 - 36x + 2$ has its maxima at
(a) $x = -2$ only (b) $x = 0$ only
(c) $x = 3$ only (d) both $x = -2$ and $x = 3$
[CE, GATE-2004, 2 marks]

Q.6 The volume of an object expressed in spherical coordinates is given by
$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\phi \, dr \, d\phi \, d\theta$. The value of the integral is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$
[ME, GATE-2004, 2 marks]

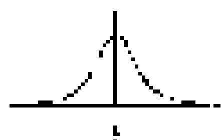
Q.7 The angle between two unit-magnitude coplanar vectors P(0.866, 0.500, 0) and Q(0.259, 0.966, 0) will be
(a) 0° (b) 30°
(c) 45° (d) 60°
[ME, GATE-2004, 1 mark]

Q.8 A rail engine accelerates from its stationary position for 8 seconds and travels a distance of 280 m. According to the Mean Value Theorem, the speedometer at a certain time during acceleration must read exactly
(a) 0 (b) 8 kmph
(c) 75 kmph (d) 126 kmph
[CE, GATE-2005, 2 marks]

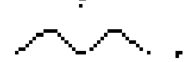
Q.9 The derivative of the symmetric function drawn in given figure will look like



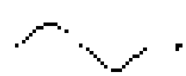
(a)



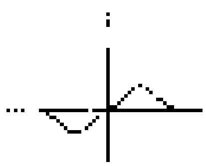
(b)



(c)



(d)



[EC, GATE-2005, 2 marks]

Q.10 The right circular cone of largest volume that can be enclosed by a sphere of 1 m radius has a height of
(a) 1/3 m (b) 2/3 m
(c) $\frac{2\sqrt{2}}{3}$ m (d) 4/3 m
[ME, GATE-2005, 2 marks]

Q. [illegible] [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2005, 2 marks]

Q.12 [illegible] values

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2005, 1 mark]

Q.13 [illegible] leads to [illegible]

(a) [illegible] (b) $16y^2$ [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2005, 1 mark]

Q.14 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) 1

[ME, GATE-2005, 2 marks]

Q.15 [illegible], magnitude [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2005, 2 marks]

Q.16 The line integral [illegible] of the vector [illegible] from the origin to the point [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) cannot be determined without specifying the path

[ME, GATE-2005, 2 marks]

Q.17 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2005, 2 marks]

Q.18 [illegible] correct

(a) [illegible] integral

(b) [illegible] integral

(c) [illegible] volume integral

(d) [illegible] surface integral

[ME, GATE-2005, 1 mark]

Q.19 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2006, 2 marks]

Q.20 As x increases [illegible], the function [illegible]

(a) [illegible]

(b) [illegible] decreases

(c) increases [illegible] value and then decreases

(d) decreases [illegible] value and then [illegible]

[EC, GATE-2006, 2 marks]

Q.21 Assuming [illegible] and [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2006, 2 marks]

Q.22 The [illegible] $\int_0^{[illegible]}$ [illegible] is given by
(a) [illegible] (c) [illegible]
(b) [illegible] (d) [illegible]
[EC, GATE-2006, 2 marks]

Q.23 What [illegible] $r = 2a \cos\theta$?
(a) [illegible] a^2 (b) [illegible] a^2
(c) [illegible] a^2 (d) [illegible] a^2
[CE, GATE-2006, 2 marks]

Q.24 The expression $V = \int_0^{[illegible]}$ [illegible] is equal to
(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]
[EE, GATE-2006, 2 marks]

Q.25 A [illegible] is integrated [illegible] point [illegible] $\sqrt{2}$ [illegible]
(a) [illegible] (b) $\frac{[illegible]}{\sqrt{[illegible]}}$
(c) [illegible] (d) [illegible]
[EE, GATE-2006, 2 marks]

Q.26 The directional derivative of
$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$
at the point P(2, 1, 3) in the direction of the vector [illegible]
(a) [illegible] (b) [illegible] 15
(c) [illegible] (d) [illegible]
[CE, GATE-2006, 2 marks]

Q.27 [illegible] function [illegible]
(a) $y = 2x$ [illegible] (b) $y = 3x - 5$
(c) [illegible] (d) [illegible] 5
[ME, GATE-2006, 2 marks]

Q.28 $\nabla \times \nabla \times \vec{P}$, where $\vec{P}$ is a vector, is equal to
(a) [illegible] (b) $\nabla^2 \vec{P} + \nabla(\nabla \times \vec{P})$
(c) [illegible] (d) [illegible]
[EC, GATE-2006, 1 mark]

Q.29 $\iint (\nabla \times \vec{P}) \cdot d\vec{s}$, where [illegible] is equal to
(a) $\oint \vec{P} \cdot d\vec{l}$ (b) $\oint$ [illegible]
(c) $\oint$ [illegible] (d) $\iiint$ [illegible]
[EC, GATE-2006, 1 mark]

Q.30 $\lim$ [illegible]
(a) [illegible] (b) [illegible]
(c) 1/2 (d) [illegible]
[ME, GATE-2007, 2 marks]

Q.31 $\lim$ [illegible]
(a) [illegible] (b) 1
(c) 2 (d) not defined
[EC, GATE-2007, 1 mark]

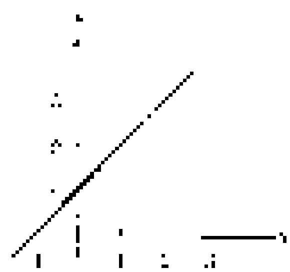
Q.32 The minimum value of [illegible] in the interval [illegible]
(a) 0 (b) [illegible]
(c) [illegible] (d) undefined
[ME, GATE-2007, 1 mark]

Q.33 Which one of the following functions is strictly bounded?
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[EC, GATE-2007, 1 mark]

Q.34 Consider the function $f(x) = x^2$ [illegible] The [illegible] value of f(x) in the closed interval [illegible] 4, 4] is
(a) [illegible] (b) [illegible]
(c) [illegible] (d) indeterminate
[EC, GATE-2007, 2 marks]

Q.35 For the function e^{-x}, the linear approximation around $x = 2$ is
(a) $(3 - x)e^{-2}$
(b) $1 - x$
(c) $[3 + 2\sqrt{2} - (1 + \sqrt{2})x]e^{-2}$
(d) e^{-2}
[EE, GATE-2007, 1 mark]

Q.36 The following plot shows a function y which varies linearly with x. The value of the integral
$$I = \int_{1}^{2} y\,dx$$



(a) 1.0 (b) 2.5
(c) 4.0 (d) 5.0
[EC, GATE-2007, 1 mark]

Q.37 The area of a triangle formed by the tips of vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is
(a) $\frac{1}{2}(\bar{a}-\bar{b})\cdot(\bar{a}-\bar{c})$ (b) $\frac{1}{2}|(\bar{a}-\bar{b})\times(\bar{a}-\bar{c})|$
(c) $\frac{1}{2}|\bar{a}\times\bar{b}\times\bar{c}|$ (d) $\frac{1}{2}(\bar{a}\times\bar{b})\cdot\bar{c}$
[ME, GATE-2007, 2 marks]

Q.38 Let x and y be two vectors in a 3 dimensional space and $\langle x, y\rangle$ denote their dot product. Then the determinant $\det\begin{vmatrix}\langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle\end{vmatrix}$
(a) is zero when x and y are linearly independent
(b) is positive when x and y are linearly independent
(c) is non-zero for all non-zero x and y
(d) is zero only when either x or y is zero
[EE, GATE-2007, 2 marks]

Q.39 A velocity vector is given as $\vec{V} = 5xy\hat{i} + 2y^2\hat{j} + 3yz^2\hat{k}$. The divergence of this velocity vector at (1, 1, 1) is
(a) 9 (b) 10
(c) 14 (d) 15
[CE, GATE-2007, 2 marks]

Q.40 Potential function ϕ is given as $\phi = x^2 - y^2$. What will be the stream function (ψ) with the condition $\psi = 0$ at $x = y = 0$?
(a) $2xy$ (b) $x^2 + y^2$
(c) $x^2 - y^2$ (d) $2x^2y^2$
[CE, GATE-2007, 2 marks]

Q.41 The value of $\lim_{x \to 8} \dfrac{x^{1/3} - 2}{(x - 8)}$
(a) $\frac{1}{16}$ (b) $\frac{1}{12}$
(c) $\frac{1}{8}$ (d) $\frac{1}{4}$
[ME, GATE-2008, 1 mark]

Q.42 $\lim_{x \to 0} \dfrac{x - \sin x}{1 - \cos x}$ equals
(a) 1 (b) -1
(c) ∞ (d) $-\infty$
[CS, GATE-2008, 1 mark]

Q.43 Consider function $f(x) = (x^2 - 4)^2$ where x is a real number. Then the function has
(a) only one minimum (b) only two minima
(c) three minima (d) three maxima
[EE, GATE-2008, 2 marks]

Q.44 A point on a curve is said to be an extremum if it is a local minimum or a local maximum. The number of distinct extrema for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is
(a) 0 (b) 1
(c) 2 (d) 3
[CS, GATE-2008, 2 marks]

Q.45 In the Taylor series expansion of e^x about $x = 2$, the coefficient of $(x - 2)^4$ is
(a) $1/4!$ (b) $2^4/4!$
(c) $e^2/4!$ (d) $e^4/4!$
[ME, GATE-2008, 1 mark]

Q.46 [illegible] have only [illegible] Laurent series expansion about the point $z = 0$
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[EC, GATE-2006, 1 mark]

Q.47 [illegible] expansion [illegible] about the point [illegible]
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[EC, GATE-2008, 2 marks]

Q.48 [illegible]
(a) 0 (b) [illegible]
(c) 1 (d) $\frac{1}{\ln 2}$
[ME, GATE-2008, 2 marks]

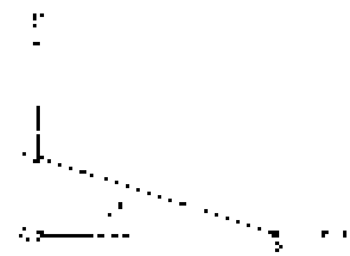
Q.49 [illegible] improper [illegible]
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[ME, GATE-2008, 2 marks]

Q.50 The length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 0$ and $x = 1$ is
(a) 0.27 (b) 0.67
(c) 1 (d) 1.22
[ME, GATE-2008, 2 marks]

Q.51 The value of the integral of the function [illegible] along the straight line segment from the point [illegible] to the point [illegible] in the x-y plane is
(a) 33 (b) [illegible]
(c) 40 (d) [illegible]
[EC, GATE-2008, 2 marks]

Q.52 The value of [illegible] is
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[CE, GATE-2008, 2 marks]

Q.53 Consider the shaded triangular region P shown in the figure. What is [illegible]



(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[ME, GATE-2008, 2 marks]

Q.54 [illegible] vectors [illegible] The angle (degrees) between the two vectors is
(a) 0 (b) 30
(c) 60 (d) 120
[CE, GATE-2008, 2 marks]

Q.55 The [illegible] of the vector field [illegible] $= [illegible] + z$ [illegible] is
(a) 0 (b) [illegible]
(c) [illegible] (d) [illegible]
[ME, GATE-2008, 1 mark]

Q.56 The directional derivative of the scalar function [illegible] at the point [illegible] in the direction of the vector $\vec{a} = $ [illegible] is
(a) −4 (b) −2
(c) [illegible] (d) 1
[ME, GATE-2008, 2 marks]

Q.57 Consider points P and Q in the x-y plane, with [illegible] and [illegible]. The line integral [illegible] along the [illegible] segment PQ is [illegible]
(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) depends on the direction (clockwise or anti-clockwise) of the [illegible]
[EC, GATE-2008, 2 marks]

Q.58 The distance between the origin and the point nearest to it on the surface $z^2 = 1 + xy$ is

(a) 1 (b) $\frac{\sqrt{3}}{2}$

(c) $\sqrt{3}$ (d) 2

[ME, GATE-2009, 2 marks]

Q.59 A cubic polynomial with real coefficients

(a) can possibly have no extrema and no zero crossings

(b) may have up to three extrema and up to 2 zero crossings

(c) cannot have more than two extrema and more than three zero crossings

(d) will always have an equal number of extrema and zero crossings

[EE, GATE-2009, 2 marks]

Q.60 The Taylor series expansion of $\frac{\sin x}{x-\pi}$ at $x = \pi$ is given by

(a) $1 + \frac{(x-\pi)^2}{3!} + \ldots$ (c) $1 - \frac{(x-\pi)^2}{3!} + \ldots$

(b) $-1 - \frac{(x-\pi)^2}{3!} + \ldots$ (d) $-1 + \frac{(x-\pi)^2}{3!} + \ldots$

[EC, GATE-2009, 2 marks]

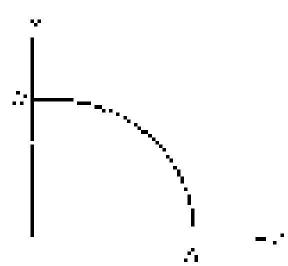
Q.61 $\int_0^{\pi/4} \frac{1-\tan x}{1+\tan x}\,dx$ evaluates to

(a) 0 (b) 1

(c) ln 2 (d) $\frac{1}{2}\ln 2$

[CS, GATE-2009, 2 marks]

Q.62 A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x + y)^2$ on path AB traversed in a counter-clockwise sense is



(a) $\frac{\pi}{2} - 1$ (b) $\frac{\pi}{2} + 1$

(c) $\frac{\pi}{2}$ (d) 1

[ME, GATE-2009, 2 marks]

Q.63 The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is

(a) $\frac{16}{3}$ (b) 8

(c) $\frac{32}{3}$ (d) 16

[ME, GATE-2009, 2 marks]

Q.64 $f(x, y)$ is a continuous function defined over $(x, y) \in [0, 1] \times [0, 1]$. Given the two constraints, $x > y^2$ and $y > x^2$, the volume under $f(x, y)$ is

(a) $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x, y)\,dx\,dy$

(b) $\int_{y=x^2}^{y=1} \int_{x=y^2}^{x=1} f(x, y)\,dx\,dy$

(c) $\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x, y)\,dx\,dy$

(d) $\int_{y=0}^{y=\sqrt{x}} \int_{x=0}^{x=\sqrt{y}} f(x, y)\,dx\,dy$

[EE, GATE-2009, 2 marks]

Q.65 For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point P(1, 2, −1) is

(a) $2\hat{i} + 6\hat{j} + 4\hat{k}$ (b) $2\hat{i} + 12\hat{j} - 4\hat{k}$

(c) $2\hat{i} + 12\hat{j} + 4\hat{k}$ (d) $\sqrt{56}$

[CE, GATE-2009, 1 mark]

Q.66 For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the directional derivative at the point P(1, 2, −1) in the direction of a vector $\hat{i} - \hat{j} + 2\hat{k}$ is

(a) −18 (b) $-3\sqrt{6}$

(c) $3\sqrt{6}$ (d) 18

[CE, GATE-2009, 2 marks]

Q.67 The divergence of the vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1, 1, 1) is equal to
(a) 7 (b) 4
(c) 3 (d) 0
[ME, GATE-2009, 1 mark]

Q.68 The $\lim_{x\to 0} \dfrac{\sin\left(\frac{2}{3}x\right)}{x}$ is
(a) 2/3 (b) 1
(c) 3/2 (d) ∞
[CE, GATE-2010, 1 mark]

Q.69 What is the value of $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$?
(a) 0 (b) e^{-2}
(c) $e^{-1/2}$ (d) 1
[CS, GATE-2010, 1 mark]

Q.70 The function $y = |2 - 3x|$
(a) is continuous $\forall x \in R$ and differentiable $\forall x \in R$
(b) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at $x = 3/2$
(c) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at $x = 2/3$
(d) is continuous $\forall x \in R$ except $x = 3$ and differentiable $\forall x \in R$
[ME, GATE-2010, 1 mark]

Q.71 Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$. The optimal value of $f(x, y)$
(a) is a minimum equal to 10/3
(b) is a maximum equal to 10/3
(c) is a minimum equal to 8/3
(d) is a maximum equal to 8/3
[CE, GATE-2010, 2 marks]

Q.72 At $t = 0$, the function $f(t) = \dfrac{\sin t}{t}$ has
(a) a minimum
(b) a discontinuity
(c) a point of inflection
(d) a maximum
[EE, GATE-2010, 2 marks]

Q.73 If $e^y = x^{1/x}$, then y has a
(a) maximum at $x = e$
(b) minimum at $x = e$
(c) maximum at $x = e^{-1}$
(d) minimum at $x = e^{-1}$
[EC, GATE-2010, 2 marks]

Q.74 The value of the integral $\int_{-\infty}^{\infty} \dfrac{dx}{1+x^2}$ is
(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) π
[ME, GATE-2010, 1 mark]

Q.75 The value of the quantity P, where $P = \int_0^1 xe^x\,dx$, is equal to
(a) 0 (b) 1
(c) e (d) 1/e
[EE, GATE-2010, 1 mark]

Q.76 A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h\dfrac{x^2}{L^2}$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is

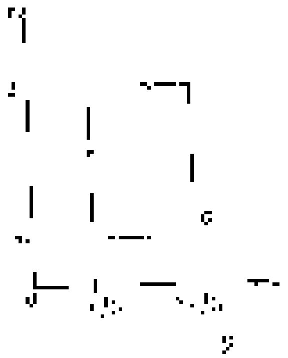
(a) $\int_0^L \sqrt{1+64\dfrac{h^2x^2}{L^4}}\,dx$ (b) $2\int_0^{L/2}\sqrt{1+64\dfrac{h^3x^2}{L^4}}\,dx$

(c) $\int_0^{L/2}\sqrt{1+64\dfrac{h^2x^2}{L^4}}\,dx$ (d) $2\int_0^{L/2}\sqrt{1+64\dfrac{h^2x^2}{L^4}}\,dx$

[CE, GATE-2010, 2 marks]

Q.77 The parabolic arc $y = \sqrt{x}$, $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is
(a) $\pi/4$ (b) $\pi/2$
(c) $3\pi/4$ (d) $3\pi/2$
[ME, GATE-2010, 1 mark]

Q.78 If $\vec A = xy\,\hat a_x + x^2\,\hat a_y$, $\oint_C \vec A \cdot d\vec l$ over the path shown in the figure is



(a) 0
(b) $\frac{2}{\sqrt{3}}$
(c) 1
(d) $2\sqrt{3}$

[EC, GATE-2011, 2 marks]

Q.79 Velocity vector of a flow field is given as $\vec V = 2xy\,\hat i - x^2z\,\hat j$. The vorticity vector at (1, 1, 1) is
(a) $4\hat i - \hat j$
(b) $4\hat i - \hat k$
(c) $\hat i - 4\hat j$
(d) $\hat i - 4\hat k$

[ME, GATE 2010, 2 marks]

Q.80 Divergence of the three-dimensional radial vector field $\vec r$ is
(a) 3
(b) $1/r$
(c) $\hat i + \hat j + \hat k$
(d) $3(\hat i + \hat j + \hat k)$

[EE, GATE 2010, 1 mark]

Q.81 What is $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ equal to?
(a) θ
(b) $\sin\theta$
(c) 0
(d) 1

[ME, GATE 2011, 1 mark]

Q.82 What should be the value of λ such that the function defined below is continuous at $x = \pi/2$?

$$f(x) = \begin{cases} \dfrac{\lambda\cos x}{\frac{\pi}{2} - x} & \text{if } x \neq \pi/2 \\ 1 & \text{if } x = \pi/2 \end{cases}$$

(a) 0
(b) $2/\pi$
(c) 1
(d) $\pi/2$

[CE, GATE-2011, 2 marks]

Q.83 The function $f(x) = 2x - x^2 + 3$ has
(a) a maxima at $x = 1$ and a minimum at $x = 5$
(b) a maxima at $x = 1$ and a minimum at $x = -5$
(c) only a maxima at $x = 1$
(d) only a minimum at $x = 1$

[EE, GATE 2011, 2 marks]

Q.84 A series expansion for the function $\sin\theta$ is
(a) $1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$
(b) $\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$
(c) $1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$
(d) $\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

[ME, GATE 2011, 1 mark]

Q.85 Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}\,dx$?
(a) 0
(b) 2
(c) $-i$
(d) i

[CS, GATE-2011, 2 marks]

Q.86 What is the value of the definite integral $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}}\,dx$?
(a) 0
(b) $a/2$
(c) a
(d) $2a$

[CE, GATE-2011, 2 marks]

Q.87 If $f(x)$ is an even function and a is a positive real number, then $\int_{-a}^{a} f(x)\,dx$ equals
(a) 0
(b) a
(c) $2a$
(d) $2\int_0^a f(x)\,dx$

[ME, GATE-2011, 1 marks]

Q.88 If $\vec a$ and $\vec b$ are two arbitrary vectors with magnitudes a and b, respectively, $|\vec a \times \vec b|^2$ will be equal to
(a) $a^2b^2 - (\vec a \cdot \vec b)^2$
(b) $ab - \vec a \cdot \vec b$
(c) $a^2b^2 + (\vec a \cdot \vec b)^2$
(d) $ab + \vec a \cdot \vec b$

[EC, GATE 2011, 2 marks]

Q.89 The two vectors $[1, 1, 1]$ and $[1, a, a^2]$, where $a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, are

(a) orthonormal (b) orthogonal
(c) parallel (d) collinear

[EE, GATE-2011, 2 marks]

Q.90 $\lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right)$ is

(a) 1/4 (b) 1/2
(c) 1 (d) 2

[ME, GATE-2012, 1 mark]

Q.91 Consider the function $f(x) = |x|$ in the interval $-1 \le x \le 1$. At the point $x = 0$, $f(x)$ is

(a) continuous and differentiable
(b) non-continuous and differentiable
(c) continuous and non-differentiable
(d) neither continuous nor differentiable

[ME, GATE-2012, 1 mark]

Q.92 At $x = 0$, the function $f(x) = x^3 + 1$ has

(a) a maximum value (b) a minimum value
(c) a singularity (d) a point of inflection

[ME, GATE-2012, 1 mark]

Q.93 The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ is

(a) 21 (b) 25
(c) 41 (d) 46

[EC, EE, IN, GATE-2012, 2 marks]

Q.94 Consider the function $f(x) = \sin(x)$ in the interval $x \in [\pi/4, 7\pi/4]$. The number and location(s) of the local minima of this function are

(a) One, at $\pi/2$
(b) One, at $3\pi/2$
(c) Two, at $\pi/2$ and $3\pi/2$
(d) Two, at $\pi/4$ and $3\pi/2$

[CS, GATE-2012, 1 mark]

Q.95 The infinite series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ corresponds to

(a) $\sec x$ (b) e^x
(c) $\cos x$ (d) $1 + \sin^2 x$

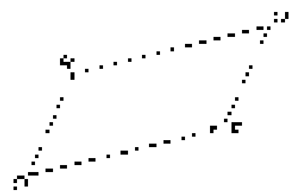
[CE, GATE-2012, 1 mark]

Q.96 The area enclosed between the straight line $y = x$ and the parabola $y = x^2$ in the x-y plane is

(a) 1/6 (b) 1/4
(c) 1/3 (d) 1/2

[ME, GATE-2012, 2 marks]

Q.97 For the parallelogram OPQR shown in the sketch, $\overline{OP} = a\hat{i} + b\hat{j}$ and $\overline{OR} = c\hat{i} + d\hat{j}$. The area of the parallelogram is



(a) $ad - bc$ (b) $ac + bd$
(c) $ad + bc$ (d) $ab - cd$

[CE, GATE-2012, 2 marks]

Q.98 For the spherical surface $x^2 + y^2 + z^2 = 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is given by

(a) $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ (b) $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

(c) $\hat{k}$ (d) $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

[ME, GATE-2012, 1 mark]

Q.99 The direction of vector A is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla \cdot A = 0$ is

(a) −2 (b) 2
(c) 1 (d) 0

[IN, GATE-2012, 2 marks]

Q.100 Which one of the following functions is continuous at $x = 3$?

(a) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x - 1, & \text{if } x > 3 \\ \frac{x+3}{3}, & \text{if } x < 3 \end{cases}$

(b) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8 - x, & \text{if } x \ne 3 \end{cases}$

(c) $f(x) = $ [illegible]

(d) [illegible]

[CS, GATE-2013, 1 Mark]

Q.101 A function $y =$ [illegible] is [illegible] in the open [illegible] interval [illegible] $(1, 2)$ [illegible]

[illegible] $\frac{dy}{dx}$ is [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2013, 2 Marks]

Q.102 A polynomial $f(x) =$ [illegible] $- 5$

(a) [illegible]

(b) no real roots

(c) no negative real root

(d) [illegible]

(e) [illegible] positive and [illegible] negative real roots

[EE, GATE-2013, 1 Mark]

Q.103 The value of $\int$ [illegible] is

(a) 0 (b) [illegible]

(c) 1 (d) [illegible]

[CE, GATE-2013, 2 Marks]

Q.104 The value of the definite integral $\int_1^e$ [illegible] dx is

(a) [illegible] $+ \frac{2}{9}$ (b) $\frac{2}{9}$ [illegible]

(c) $\frac{2}{9}$ [illegible] $- \frac{4}{9}$ (d) [illegible] $- \frac{2}{9}$

[ME, GATE-2013, 2 Marks]

Q.105 The curl of the gradient of the scalar field defined by $V =$ [illegible] is

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) 0

[EE, GATE-2013, 1 Mark]

Q.106 The divergence of the vector field [illegible]

(a) 0 (b) [illegible]

(c) 1 (d) 3

[EE, GATE-2013, 1 Mark]

Q.107 Function f is known at the following points:

[illegible]

The value of $\int_0^3 f(x)dx$ computed using [illegible] is [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[CS, GATE-2013, 1 Mark]

Q.108 For a vector E, which one of the following statements is NOT TRUE?

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[IN, GATE-2013, 1 Mark]

Q.109 Given a vector field [illegible]

the line integral $\int F \cdot dl$ evaluated along a segment [illegible] is

(a) [illegible] (b) 0

(c) [illegible] (d) [illegible]

[EE, GATE-2013, 1 Mark]

Q.110 [illegible] be evaluated [illegible] vector [illegible] coordinate system [illegible]

$$\iint \frac{1}{\pi} [illegible]$$

where S is [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2013, 2 Marks]

Q.111 Consider a vector field $\vec{A}(\vec{r})$. The closed loop line integral $\oint \vec{A} \cdot d\vec{l}$ can be expressed as

(a) $\oiint (\nabla \times \vec{A}) \cdot d\vec{s}$ over the closed surface bounded by the loop

(b) $\oiiint (\nabla \cdot \vec{A})\, dv$ over the closed volume bounded by the loop

(c) $\iiint (\nabla \cdot \vec{A})\, dv$ over the open volume bounded by the loop

(d) $\iint (\nabla \times \vec{A}) \cdot d\vec{s}$ over the open surface bounded by the loop

[EC, GATE-2015 : 1 Mark]

Q.112 $\lim_{x \to \infty} \left(\frac{x + \sin x}{x}\right)$ equals to

(a) $-\infty$ (b) 0
(c) 1 (d) ∞

[CE, GATE-2014 : 1 Mark]

Q.113 The expression $\lim_{\alpha \to 0} \frac{x^{\alpha} - 1}{\alpha}$ is equal to

(a) $\log x$ (b) 0
(c) $x \log x$ (d) ∞

[CE, GATE-2014 : 2 Marks]

Q.114 $\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x}$ is

(a) 0 (b) 1
(c) 3 (d) not defined

[ME, GATE-2014 : 1 Mark]

Q.115 $\lim_{x \to 0} \left(\frac{e^{2x} - 1}{\sin(4x)}\right)$ is equal to

(a) 0 (b) 0.5
(c) 1 (d) 2

[ME, GATE-2014 : 1 Mark]

Q.116 The value of $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$ is

(a) ln 2 (b) 1.0
(c) e (d) ∞

[EC, GATE-2014 : 1 Mark]

Q.117 The integrating factor for differential equation $\frac{dP}{dt} + k_2 P = k_1 L_0 e^{-k_1 t}$ is

(a) $e^{-k_1 t}$ (b) $e^{-k_2 t}$
(c) $e^{k_1 t}$ (d) $e^{k_2 t}$

[CE, GATE-2014 : 1 Mark]

Q.118 If a function is continuous at a point,

(a) the limit of the function may not exist at the point
(b) the function must be derivable at the point
(c) the limit of the function at the point tends to infinity
(d) the limit must exist at the point and the value of limit should be same as the value of the function at that point

[ME, GATE-2014 : 1 Mark]

Q.119 A function $f(x)$ is continuous in the interval $[0, 2]$. It is known that $f(0) = f(2) = -1$ and $f(1) = 1$. Which one of the following statements must be true?

(a) There exists a y in the interval $(0, 1)$ such that $f(y) = f(y + 1)$
(b) For every y in the interval $(0, 1)$, $f(y) = f(2 - y)$
(c) The maximum value of the function in the interval $(0, 2)$ is 1
(d) There exists a y in the interval $(0, 1)$ such that $f(y) = -f(2 - y)$

[CS, GATE-2014 : 2 Marks]

Q.120 Let the function

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

where $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $f'(\theta)$ denote the derivative of f with respect to θ. Which of the following statements is/are TRUE?

(I) There exists $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) = 0$.

(II) There exists $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) \neq 0$.

(a) I only (b) II only
(c) Both I and II (d) Neither I nor II

[CS, GATE-2014 : 1 Mark]

Q.121 The function $f(x)$ = [illegible] equation [illegible] The value of [illegible] is ____

[CS, GATE-2014 : 2 Marks]

Q.122 If y = [illegible] is the solution of $\frac{d^2y}{dx^2}$ [illegible] boundary conditions y = [illegible] and $\frac{dy}{dx}$ [illegible] at x = [illegible] ____

[EC, GATE-2014 : 2 Marks]

Q.123 [illegible] angle between [illegible]

(a) 15°
(b) [illegible]
(c) 30°
(d) 45°

[EE, GATE-2014 : 2 Marks]

Q.124 If [illegible]

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EC, GATE-2014 : 1 Mark]

Q.125 [illegible] The maximum value of the function in the interval [illegible] is

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EE, GATE-2014 : 1 Mark]

Q.126 Minimum of the real valued function [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EE, GATE-2014 : 1 Mark]

Q.127 The maximum value of the function [illegible] in the interval [illegible] is

(a) [illegible]
(b) 25
(c) [illegible]
(d) [illegible]

[EE, GATE-2014 : 2 Marks]

Q.128 [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EC, GATE-2014 : 1 Mark]

Q.129 The maximum value of the function [illegible] is ____

[EC, GATE-2014 : 1 Mark]

Q.130 The maximum value of [illegible] in the interval [illegible]

[EC, GATE-2014 : 2 Marks]

Q.131 The value of the integral [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[ME, GATE-2014 : 1 Mark]

Q.132 [illegible] is ____

[CS, GATE-2014 : 1 Mark]

Q.133 The value of the integral given below is [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[CS, GATE-2014 : 2 Marks]

Q.134 [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EC, 2014 : 2 Marks]

Q.135 The value of the integral [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[ME, GATE-2014 : 2 Marks]

Q.136 To evaluate the double integral [illegible] we make the substitution [illegible] and [illegible] The integral will reduce to [illegible]

(a) $\int_0^4\left(\int_0^2 2u\,du\right)dv$ (b) $\int_0^4\left(\int_0^1 2u\,du\right)dv$

(c) $\int_0^4\left(\int_0^1 u\,du\right)dv$ (d) $\int_0^4\left(\int_0^2 u\,du\right)dv$

[EE, GATE-2014 : 2 Marks]

Q.137 Which one of the following describes the relationship among the three vectors, $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+4\hat{k}$?

(a) The vectors are mutually perpendicular
(b) The vectors are linearly dependent
(c) The vectors are linearly independent
(d) The vectors are unit vectors

[ME, 2014 : 1 Mark]

Q.138 Curl of vector $\vec{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$
(b) $(4yz^3 + 2xy^2)\hat{i} - 2x^2z\hat{j} - 2y^2z\hat{k}$
(c) $2xz^2\hat{i} - 4xyz\hat{j} + 6y^2z^2\hat{k}$
(d) $2xz^2\hat{i} + 4xyz\hat{j} + 6y^2z^2\hat{k}$

[ME, 2014 : 1 Mark]

Q.139 The divergence of the vector field $x^2z\hat{i} + xy\hat{j} - yz^2\hat{k}$ at $(1, -1, 1)$ is

(a) 0 (b) 3
(c) 5 (d) 6

[ME, 2014 : 1 Mark]

Q.140 Let $\nabla\cdot(f\vec{v}) = x^2y + y^2z + z^2x$, where f and $\vec{v}$ are scalar and vector fields respectively. If $\vec{v} = y\hat{i} + z\hat{j} + x\hat{k}$, then $\vec{v}\cdot\nabla f$ is

(a) $x^2y + y^2z + z^2x$ (b) $2xy + 2yz + 2zx$
(c) $x + y + z$ (d) 0

[EE, 2014 : 1 Mark]

Q.141 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to

(a) e^{-2} (b) e
(c) 1 (d) e^2

[CE, GATE-2015 : 1 Mark]

Q.142 The value of $\lim_{x\to 0}\frac{1-\cos(x^2)}{2x^4}$ is

(a) 0 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) undefined

[ME, GATE-2015 : 1 Mark]

Q.143 The value of $\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right)$ is ______

[ME, GATE-2015 : 1 Mark]

Q.144 $\lim_{x\to\infty} x^{1/x}$ is

(a) ∞ (b) 0
(c) 1 (d) Not defined

[CS, GATE-2015 : 1 Mark]

Q.145 The value of $\lim_{x\to\infty}(1+x^2)^{e^{-x}}$ is

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) ∞

[CS, GATE-2015 : 1 Mark]

Q.146 Let $f(x) = x^{-(1/3)}$ and A denote the area of the region bounded by $f(x)$ and the X-axis, when x varies from -1 to 1. Which of the following statements is/are TRUE?

1. f is continuous in $[-1, 1]$
2. f is not bounded in $[-1, 1]$
3. A is nonzero and finite

(a) 2 only (b) 3 only
(c) 2 and 3 only (d) 1, 2 and 3

[CS, GATE-2015 : 2 Marks]

Q.147 A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval $[-1, 1]$. The value of x, in the open interval $(-1, 1)$ for which the mean value theorem is satisfied, is

(a) $-1/2$ (b) $-1/3$
(c) $1/3$ (d) $1/2$

[EC, GATE-2015 : 1 Mark]

Q.148 While minimizing the function $f(x)$, necessary and sufficient conditions for a point, x_0 to be a minima are:

(a) $f'(x_0) > 0$ and $f''(x_0) = 0$

(b) $f'(x_0) < 0$ and $f''(x_0) = 0$

(c) $f'(x_0) = 0$ and $f''(x_0) < 0$

(d) $f'(x_0) = 0$ and $f''(x_0) > 0$

[CE, GATE-2015 : 1 Mark]

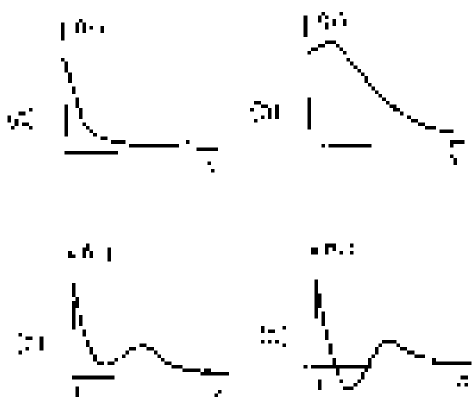
Q.149 At $x = 0$, the function [illegible] has
(a) a minimum
(b) a maximum
(c) a point of inflexion
(d) neither a maximum nor minimum

[ME, GATE-2015 : 1 Mark]

Q.150 [illegible] of a 2 × 2 symmetric matrix is [illegible] then the maximum possible value of [illegible] is ____.

[EE, GATE-2015 : 1 Mark]

Q.151 Which one of the following graphs describes the function [illegible]



[EC, GATE-2015 : 2 Marks]

Q.152 The maximum area (in square units) of a rectangle whose vertices lie on the ellipse $x^2 + 4y^2 = 1$ is ____.

[EC, GATE-2015 : 2 Marks]

Q.153 The contour on the x-y plane, where the partial derivative of [illegible] with respect to y is equal to the partial derivative of [illegible] with respect to x, is
(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EC, GATE-2015 : 1 Mark]

Q.154 [illegible] where [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[CS, GATE-2015 : 2 Marks]

Q.155 Consider [illegible] along the curve [illegible] = 4 [illegible] The [illegible] (0, 0) and [illegible] (2, 0) [illegible] ____.

[ME, GATE-2015 : 2 Marks]

Q.156 [illegible]

[illegible]

The length of the curve is ____.

[ME, GATE-2015 : 2 Marks]

Q.157 [illegible] ____.

[EE, GATE-2015 : 2 Marks]

Q.158 The double integral [illegible] is equivalent to

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[IN, GATE-2015 : 1 Mark]

Q.159 The directional derivative [illegible] ____.

[CE, GATE-2015 : 2 Marks]

Q.160 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2015 : 1 Mark]

Q.161 [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[ME, GATE-2015 : 1 Mark]

Q.162 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[IN, GATE-2015 : 1 Mark]

Q.163 The value of [illegible] [illegible] ____

[ME, GATE-2015 : 2 Marks]

Q.164 [illegible]

[ME, GATE-2015 : 2 Marks]

Q.165 [illegible] ____

[CS, GATE-2016 : 1 Mark]

Q.166 [illegible]

[CS, GATE-2015 : 1 Mark]

Q.167 [illegible] The [illegible]

[illegible] ____

[ME, GATE-2016 : 2 Marks]

Q.168 [illegible] ____

[IN, GATE-2016 : 1 Mark]

Q.169 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) 1

[ME, GATE-2016 : 1 Mark]

Q.170 [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2016 : 2 Marks]

Q.171 [illegible] ?

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[CE, GATE-2016 : 1 Mark]

Q.172 [illegible]

P: [illegible]

Q: [illegible]

R: [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[CS, GATE-2015 : 1 Mark]

Q.173 The values of x for which the function

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 3x - 4}$$

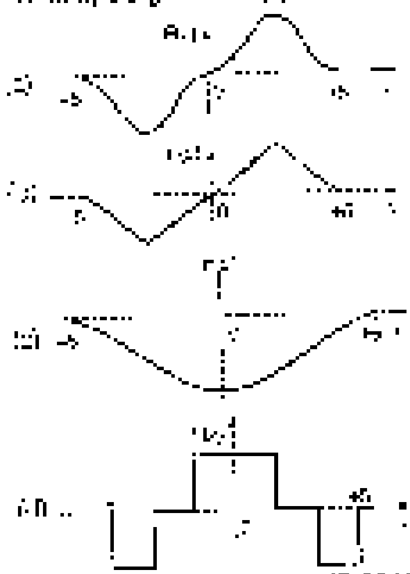
is NOT continuous are

(a) 4 and −1 (b) 4 and 1
(c) −4 and 1 (d) −4 and −1

[ME, GATE-2016 : 1 Mark]

Q.174 Consider the plot of f(x) versus x as shown below.

Suppose $F(x) = \int_{-5}^{x} f(y)\,dy$. Which one of the following is a graph of F(x)?



[EC, GATE-2016 : 1 Mark]

Q.175 Let f(x) be a polynomial and g(x) = f′(x) be its derivative. If the degree of (f(x) + f(−x)) is 10, then the degree of (g(x) − g(−x)) is ____.

[CS, GATE-2016 : 1 Mark]

Q.176 As x varies from −1 to +3, which one of the following describes the behaviour of the function $f(x) = x^3 - 3x^2 + 1$?

(a) f(x) increases monotonically.
(b) f(x) increases, then decreases and increases again.
(c) f(x) decreases, then increases and decreases again.
(d) f(x) increases and then decreases.

[EC, GATE-2016 : 1 Mark]

Q.177 Let $f : [-1, 1] \to R$, where $f(x) = 2x^3 - x^4 - 10$. The minimum value of f(x) is ____.

[IN, GATE-2016 : 2 Marks]

Q.178 The maximum value attained by the function $f(x) = x(x - 1)(x - 2)$ in the interval [1, 2] is ____.

[EE, GATE-2016 : 1 Mark]

Q.179 The optimum value of the function $f(x) = x^2 - 4x + 2$ is

(a) 2 (maximum) (b) 2 (minimum)
(c) −2 (maximum) (d) −2 (minimum)

[CE, GATE-2016 : 1 Mark]

Q.180 The quadratic approximation of $f(x) = x^3 - 3x^2 - 5$ at the point x = 0 is

(a) $3x^2 - 6x - 5$ (b) $-3x^2 - 5$
(c) $-3x^2 + 6x - 5$ (d) $3x^2 - 5$

[CE, GATE-2016 : 2 Marks]

Q.181 The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is

(a) 0° (b) 30°
(c) 45° (d) 90°

[CE, GATE-2016 : 2 Marks]

Q.182 How many distinct values of x satisfy the equation sin(x) = x/2, where x is in radians?

(a) 1 (b) 2
(c) 3 (d) 4 or more

[EC, GATE-2016 : 1 Mark]

Q.183 The value of the line integral $\oint_C \vec{F} \cdot \vec{r}\,' ds$, where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ units is ____.

Here, $\vec{F}(x, y) = y\hat{i} + 2x\hat{j}$ and $\vec{r}\,'$ is the UNIT tangent vector on the curve C at an arc length s from a reference point on the curve. $\hat{i}$ and $\hat{j}$ are the basis vectors in the x-y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction.

[ME, GATE-2016 : 2 Marks]

Q.184 A straight line of the form y = mx + c passes through the origin and the point (x, y) = (2, 6). The value of m is ____.

[IN, GATE-2016 : 1 Mark]

Q.185 The integral $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$ is equal to ____.

[EC, GATE-2016 : 1 Mark]

Q.186 The value of $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx$ is

(a) $\frac{\pi}{2}$ (b) π

(c) $\frac{3\pi}{2}$ (d) 1

[CS, GATE-2016 : 2 Marks]

Q.187 A triangle in the xy-plane is bounded by the straight lines $2x = 3y$, $y = 0$ and $x = 3$. The volume above the triangle and under the plane $x + y + z = 6$ is ____.

[EC, GATE-2016 : 2 Marks]

Q.188 The area between the parabola $x^2 = 8y$ and the straight line $y = 8$ is ____.

[EE, GATE-2016 : 2 Marks]

Q.189 The integral $\frac{1}{2\pi}\iint_D (x + y + 10)\,dx\,dy$, where D denotes the disc: $x^2 + y^2 \le 4$, evaluates to ____.

[EC, GATE-2016 : 2 Marks]

Q.190 Suppose C is the closed curve defined as the circle $x^2 + y^2 = 1$ with C oriented anti-clockwise. The value of $\oint(xy^2dx + x^2y\,dy)$ over the curve C equals ____.

[EC, GATE-2016 : 2 Marks]

Q.191 The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is

(a) $\frac{59}{6}$ (b) $\frac{9}{2}$

(c) $\frac{10}{3}$ (d) $\frac{7}{6}$

[CE, GATE-2016 : 2 Marks]

Q.192 The region specified by $\{(\rho, \phi, z) : 3 \le \rho \le 5, \frac{\pi}{8} \le \phi \le \frac{\pi}{4}, 3 \le z \le 4.5\}$ in cylindrical coordinates has volume of ____.

[EC, GATE-2016 : 2 Marks]

Q.193 Consider the time-varying vector $\vec{I} = \hat{x}\,15\cos(\omega t) + \hat{y}\,5\sin(\omega t)$ in Cartesian coordinates, where $\omega > 0$ is a constant. When the vector magnitude $|\vec{I}|$ is at its minimum value, the angle θ that $\vec{I}$ makes with the x axis (in degrees, such that $0 \le \theta \le 180$) is ____.

[EC, GATE-2016 : 1 Mark]

Q.194 The vector that is NOT perpendicular to the vectors $(\hat{i} + \hat{j} + \hat{k})$ and $(\hat{i} + 2\hat{j} + 3\hat{k})$ is ____.

(a) $(\hat{i} - 2\hat{j} + \hat{k})$ (b) $(-\hat{i} + 2\hat{j} - \hat{k})$

(c) $(0\hat{i} + 0\hat{j} + 0\hat{k})$ (d) $(4\hat{i} + 3\hat{j} + 5\hat{k})$

[IN, GATE-2016 : 1 Mark]

Q.195 Which one of the following is a property of the solutions to the Laplace equation: $\nabla^2 f = 0$?

(a) The solutions have neither maxima nor minima anywhere except at the boundaries.

(b) The solutions are not separable in the coordinates.

(c) The solutions are not continuous.

(d) The solutions are not dependent on the boundary conditions.

[EC, GATE-2016 : 1 Mark]

Q.196 The value of the line integral

$$\int_C (2xy^2dx + 2x^2y\,dy + dz)$$

along a path joining the origin (0, 0, 0) and the point (1, 1, 1) is

(a) 0 (b) 2

(c) 4 (d) 6

[EE, GATE-2016 : 1 Mark]

Q.197 The line integral of the vector field $F = 5xz\,\hat{i} + (3x^2 + 2y)\,\hat{j} + x^2z\,\hat{k}$ along a path from (0, 0, 0) to (1, 1, 1) parametrized by (t, t^2, t) is ____.

[CE, GATE-2016 : 2 Marks]

Q.198 [illegible] $f(x) = e^{-x^2}$ [illegible] The integral $g(x) = \int f(t)dt$ [illegible] is equal to

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[CS, GATE-2017 : 1 Mark]

Q.199 The divergence of the vector $-y\hat{i} + x\hat{j}$ is ____.

[ME, GATE-2017 : 1 Mark]

Q.200 The surface integral $\iint_S \vec{F}\cdot\hat{n}\,dS$ over the surface S of the sphere $x^2 + y^2 + z^2 = 9$, where $\vec{F} = (x + y)\hat{i} + (x + z)\hat{j} + (y + z)\hat{k}$ and $\hat{n}$ is the unit outward surface normal, yields ______

[ME, GATE-2017 : 2 Marks]

Q.201 The value of $\lim_{x\to 0}\frac{x^3 - \sin(x)}{x}$ is

(a) 0 (b) 3

(c) 1 (d) −1

[ME, GATE-2017 : 1 Mark]

Q.202 A parametric curve defined by $x = \cos\left(\frac{\pi u}{2}\right), y = \sin\left(\frac{\pi u}{2}\right)$ in the range $0 \le u \le 1$ is rotated about the X-axis by 360 degrees. Area of the surface generated is

(a) $\frac{\pi}{2}$ (b) π

(c) 2π (d) 4π

[ME, GATE-2017 : 2 Marks]

Q.203 For the vector $\vec{V} = 2yz\hat{i} + 3xz\hat{j} + 4xy\hat{k}$, the value of $\nabla\cdot(\nabla\times\vec{V})$ is ______

[ME, GATE-2017 : 2 Marks]

Q.204 Let I be an identity matrix of dimension 3 × 3, the [illegible] matrix $A = e^{I}$ [illegible] ______

[IN, GATE-2017 : 1 Mark]

Q.205 The angle between two vectors $x_1 = [2\ 6\ 14]^T$ and $x_2 = [-12\ 8\ 16]^T$ in radian is ______

[IN, GATE-2017 : 2 Marks]

Q.206 Let x and y be integers satisfying the following equations:

$2x^2 + y^2 = 34$

$x + 2y = 11$

The value of $(x + y)$ is ______.

[EE, GATE-2017 : 1 Mark]

Q.207 Let $y^2 - 2y + 1 = x$ and $\sqrt{x} + y = 5$. The value of $x + \sqrt{y}$ equals ______.

(Give the answer up to three decimal places)

[EE, GATE-2017 : 1 Mark]

Q.208 Let $g(x) = \begin{cases} -x, & x \le 1 \\ x + 1, & x \ge 1 \end{cases}$ and $f(x) = \begin{cases} 1 - x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

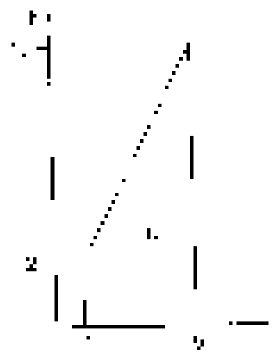
Consider the composition of f and g, i.e. $(f\circ g)(x) = f(g(x))$. The number of discontinuities in $(f\circ g)(x)$ present in the interval $(-\infty, 0)$ is:

(a) 0 (b) 1

(c) 2 (d) 4

[EE, GATE-2017 : 2 Marks]

Q.209 Let $I = c\iint_R xy^2\,dx\,dy$, where R is the region shown in the figure and $c = 6 \times 10^{-4}$. The value of I equals ______ (Give the answer up to two decimal places)



[EE, GATE-2017 : 1 Mark]

Q.210 A function $f(x)$ is defined as $f(x) = \begin{cases} e^x, & x < 1 \\ \ln x + ax^2 + bx, & x \ge 1 \end{cases}$, where $x \in \mathbb{R}$.

Which one of the following statements is TRUE?

(a) $f(x)$ is NOT differentiable at $x = 1$ for any values of a and b.

(b) $f(x)$ is differentiable at $x = 1$ for the unique values of a and b.

(c) $f(x)$ is differentiable at $x = 1$ for all values of a and b such that $a + b = e$.

(d) $f(x)$ is differentiable at $x = 1$ for all values of a and b.

[EE, GATE-2017 : 2 Marks]

Q.211 The smaller angle (in degrees) between the planes $x + y + z = 1$ and $2x - y + 2z = 0$ is ______

[EC, GATE-2017 : 1 Mark]

Q.212 The minimum value of the function $f(x) = \frac{1}{3}x(x^2-3)$ in the interval $-100 \le x \le 100$ occurs at $x =$ ___

[EC, GATE-2017 : 2 Marks]

Q.213 The values of the integrals

$$\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dy\right)dx \quad \text{and} \quad \int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dx\right)dy$$

are

(a) same and equal to 0.5
(b) same and equal to −0.5
(c) 0.5 and −0.5 respectively
(d) −0.5 and 0.5 respectively

[EC, GATE-2017 : 2 Marks]

Q.214 If the vector function

$\vec{F} = \hat{a}_x(3y - k_1 z) + \hat{a}_y(k_2 x - 2z) - \hat{a}_z(k_3 y + z)$

is irrotational, then the values of the constants k_1, k_2 and k_3 respectively, are

(a) 0.3, −2.5, 0.5
(b) 0.0, 3.0, 2.0
(c) 0.3, 0.33, 0.5
(d) 4.0, 3.0, 2.0

[EC, GATE-2017 : 2 Marks]

Q.215 Let $I = \int_C (2z\,dx + 2y\,dy + 2x\,dz)$ where x, y, z are real, and let C be the straight line segment from point $A: (0, 2, 1)$ to point $B: (4, 1, -1)$. The value of I is ___

[EC, GATE-2017 : 2 Marks]

Q.216 Let $f(x) = e^{x+x^2}$ for real x. From among the following, choose the Taylor series approximation of $f(x)$ around $x = 0$, which includes all powers of x less than or equal to 3.

(a) $1 + x + x^2 + x^3$
(b) $1 + x + \frac{3}{2}x^2 + x^3$
(c) $1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$
(d) $1 + x + 3x^2 + 7x^3$

[EC, GATE-2017 : 2 Marks]

Q.217 A three dimensional region R of finite volume is described by

$x^2 + y^2 \le z^3$; $0 \le z \le 1$,

where x, y, z are real. The volume of R (up to two decimal places) is ___

[EC, GATE-2017 : 2 Marks]

Q.218 If $f(x) = R\sin\left(\frac{\pi x}{2}\right) + S$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and

$\int_0^1 f(x)dx = \frac{2R}{\pi}$, then the constants R and S are, respectively

(a) $\frac{2}{\pi}$ and $\frac{16}{\pi}$
(b) $\frac{2}{\pi}$ and 0
(c) $\frac{4}{\pi}$ and 0
(d) $\frac{4}{\pi}$ and $\frac{16}{\pi}$

[CS, GATE-2017 : 1 Mark]

Q.219 The value of $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

(a) is 0
(b) is −1
(c) is 1
(d) does not exist

[CS, GATE-2017 : 2 Marks]

Q.220 Let $w = f(x, y)$, where x and y are functions of t. Then, according to the chain rule, $\frac{dw}{dt}$ is equal to

(a) $\frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt}$
(b) $\frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$
(c) $\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$
(d) $\frac{dw}{dx}\frac{\partial x}{\partial t} + \frac{dw}{dy}\frac{\partial y}{\partial t}$

[CE, GATE-2017 : 1 Mark]

Q.221 The divergence of the vector field $V = x^2 i + 2y^3 j + z^4 k$ at $x = 1$, $y = 2$, $z = 3$ is ___

[CE, GATE-2017 : 1 Mark]

Q.222 The tangent to the curve represented by $y = x\ln x$ is required to have 45° inclination with the x-axis. The coordinates of the tangent point would be

(a) (1, 0)
(b) (0, 1)
(c) (1, 1)
(d) $(\sqrt{2}, \sqrt{2})$

[CE, GATE-2017 : 2 Marks]

Q.223 Consider the following definite integral:

$$I = \int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$$

The value of the integral is

(a) $\frac{\pi^{[illegible]}}{24}$ (c) $\frac{\pi^2}{12}$

(b) $\frac{\pi^2}{48}$ (d) $\frac{\pi^2}{64}$

[CE, GATE-2017 : 2 Marks]

Q.224 Two cars P and Q are moving in a racing track continuously for two hours. Assume that no other vehicles are using the track during this time. The expressions relating the distance travelled d (in km) and time t (in hour) for both the vehicles are given as

P: $d = 60t$

Q: $d = 60t^2$

Within the first one hour from the start, the maximum space headway would be

(a) 15 km at 30 minutes
(b) 15 km at 15 minutes
(c) 30 km at 30 minutes
(d) 30 km at 15 minutes

[CE, GATE-2017 : 2 Marks]

Q.225 $\lim_{x \to 0}\left(\frac{\tan x}{x^2 - x}\right)$ is equal to ______

[CE, GATE-2017 : 1 Mark]

Q.226 The minimum value of the function $f(x) = \frac{1}{3}x(x^2 - 3)$ [illegible] occurs at

(a) $x =$ [illegible] (c) $x = 1$

(b) $x = -1$ (d) $x = \frac{1}{\sqrt{3}}$

[ESE Prelims-2017]

Q.227 The value of the integral $\int_0^{2\pi} \frac{8}{(2+\sin\theta)}\,d\theta$ [illegible] is

(a) $\frac{2\pi}{\sqrt{3}}$ (b) $2\sqrt{3}\pi$

(c) [illegible] (d) 2π

[ESE Prelims-2017]

■■■■

Answers Calculus

1.	[illegible]	2.	[illegible]	3.	[illegible]	4.	[illegible]	5.	[illegible]	6.	[illegible]	7.	[illegible]	8.	[illegible]	9.	[illegible]
10.	[illegible]	11.	(b)	12.	[illegible]	13.	[illegible]	14.	[illegible]	15.	[illegible]	16.	[illegible]	17.	[illegible]	18.	[illegible]
19.	[illegible]	20.	[illegible]	21.	[illegible]	22.	[illegible]	23.	[illegible]	24.	[illegible]	25.	[illegible]	26.	[illegible]	27.	[illegible]
28.	[illegible]	29.	[illegible]	30.	(b)	31.	[illegible]	32.	(b)	33.	[illegible]	34.	[illegible]	35.	[illegible]	36.	(b)
37.	[illegible]	38.	[illegible]	39.	[illegible]	40.	[illegible]	41.	[illegible]	42.	[illegible]	43.	(d)	44.	[illegible]	45.	[illegible]
46.	[illegible]	47.	(b)	48.	[illegible]	49.	[illegible]	50.	[illegible]	51.	[illegible]	52.	[illegible]	53.	[illegible]	54.	[illegible]
55.	[illegible]	56.	[illegible]	57.	[illegible]	58.	(a)	59.	[illegible]	60.	[illegible]	61.	[illegible]	62.	[illegible]	63.	[illegible]
64.	[illegible]	65.	[illegible]	66.	[illegible]	67.	[illegible]	68.	[illegible]	69.	(b)	70.	[illegible]	71.	[illegible]	72.	[illegible]
73.	[illegible]	74.	[illegible]	75.	(b)	76.	[illegible]	77.	[illegible]	78.	[illegible]	79.	[illegible]	80.	[illegible]	81.	[illegible]
82.	[illegible]	83.	[illegible]	84.	[illegible]	85.	[illegible]	86.	[illegible]	87.	[illegible]	88.	[illegible]	89.	[illegible]	90.	[illegible]
91.	[illegible]	92.	(d)	93.	[illegible]	94.	[illegible]	95.	[illegible]	96.	[illegible]	97.	[illegible]	98.	[illegible]	99.	[illegible]
100.	[illegible]	101.	(b)	102.	[illegible]	103.	[illegible]	104.	[illegible]	105.	[illegible]	106.	[illegible]	107.	[illegible]	108.	[illegible]
109.	[illegible]	110.	[illegible]	111.	[illegible]	112.	[illegible]	113.	[illegible]	114.	[illegible]	115.	[illegible]	116.	[illegible]	117.	[illegible]
118.	[illegible]	119.	[illegible]	120.	[illegible]	123.	[illegible]	124.	[illegible]	125.	[illegible]	126.	[illegible]	127.	[illegible]	129.	[illegible]
131.	[illegible]	133.	[illegible]	134.	[illegible]	135.	[illegible]	136.	[illegible]	137.	[illegible]	138.	[illegible]	139.	[illegible]	140.	[illegible]
141.	[illegible]	142.	[illegible]	143.	[illegible]	144.	[illegible]	145.	[illegible]	146.	[illegible]	147.	[illegible]	148.	[illegible]	149.	[illegible]
151.	[illegible]	152.	(d)	153.	[illegible]	154.	[illegible]	159.	[illegible]	160.	[illegible]	161.	[illegible]	162.	[illegible]	169.	[illegible]
170.	[illegible]	171.	[illegible]	172.	[illegible]	173.	[illegible]	174.	[illegible]	176.	[illegible]	178.	[illegible]	180.	[illegible]	181.	[illegible]
182.	[illegible]	183.	[illegible]	191.	[illegible]	194.	[illegible]	195.	[illegible]	196.	[illegible]	198.	[illegible]	201.	[illegible]	202.	[illegible]
208.	[illegible]	210.	[illegible]	212.	[illegible]	214.	[illegible]	216.	[illegible]	218.	[illegible]	219.	[illegible]	220.	[illegible]	222.	[illegible]
223.	[illegible]	224.	[illegible]	226.	[illegible]	227.	[illegible]										

Explanations **Calculus**

1. (a)

[illegible] $= 1 \times 1 = 0$

2. (a)

Solution by Coordinate Geometry

This problem can be done through coordinate geometry formulation [illegible]

Given P (3, −2, −1)

Q (1, 3, 4)

R (2, 1, −2)

S (0, 1, 0) [illegible]

[illegible] plane is

[illegible]

$= 0$

[illegible]

Now the distance of [illegible]

[illegible]

[illegible]

Therefore, the distance of [illegible] from plane [illegible] is given by

[illegible] $= 2$

3. (b)

[illegible]

Since this is a $\frac{0}{0}$ form, this can be found by [illegible]

[illegible]

4. (c)

Given [illegible]

[illegible]

[illegible]

5. (c)

[illegible]

$\Rightarrow$ [illegible]

Now [illegible]

and [illegible]

and [illegible]

Hence [illegible]

6. (a)

[illegible]

7. (c)

[illegible]

$\cos\theta = \dfrac{0.990 \times 0.770 - 0.5 \times 0.808}{[illegible]}$

$= 0.39$ [illegible]

$\therefore$ [illegible]

9. (d)

Since the position of rail engine $S(t)$ is continuous and differentiable function according to Lagrange's mean value theorem

At [illegible] time [illegible]

$S'(t) = v(t) = \dfrac{S(8) - S(0)}{8 - 0} = \dfrac{(280 - 0)}{(8 - 0)}$ m/sec

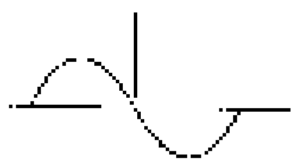
$= \dfrac{280}{8}$ m/sec

$= \dfrac{280}{8} \times \dfrac{3600}{1000}$ kmph $= 126$ kmph

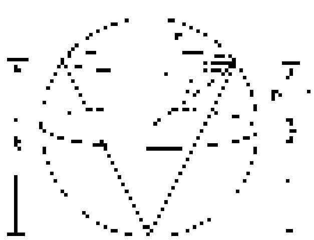
where $v(t)$ is the velocity of the rail engine.

9. (c)

Given function has equal values at the end point and [illegible] it [illegible] mean value theorem [illegible] satisfied by (c)



10. (d)



[illegible]

$r^2 + h^2 - 2h$ [illegible]

$r^2 = 2h - h^2$

Volume of the cone $V = \dfrac{1}{3}\pi r^2 h$

$V = \dfrac{\pi}{3}(2h - h^2)h = \dfrac{\pi}{3}(2h^2 - h^3)$

$\dfrac{dV}{dh} = \dfrac{\pi}{3}(4h - 3h^2)$

$\dfrac{dV}{dh} = 0$ [illegible] minima and maxima

$4h - 3h^2 = 0$

$h(4 - 3h) = 0$

$h = \dfrac{4}{3}, 0$

$V'' = \dfrac{\pi}{3}(4 - 6h)$

$h = 0$, $V'' = \dfrac{4\pi}{3} > 0$ minima

$h = \dfrac{4}{3}$; $V'' = -\dfrac{4\pi}{3} < 0$ maxima

$\therefore$ Volume is maximum when $h = \dfrac{4}{3}$

11. (4)

[illegible]

$f'(x) = $ [illegible]

$= $ [illegible] $(2x - x^2)$

Putting $f'(x) = 0$

[illegible] $(2x - x^2) = 0$

[illegible] $(2 - x) = 0$

$x = 0$ or $x = 2$ [illegible] stationary points

Now [illegible]

[illegible]

[illegible]

[illegible]

Since [illegible] minima

Now at [illegible]

[illegible]

[illegible]

12. (c)

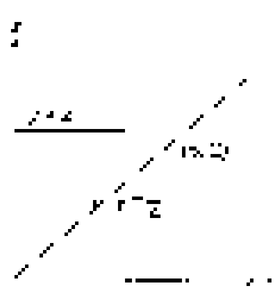
[illegible]

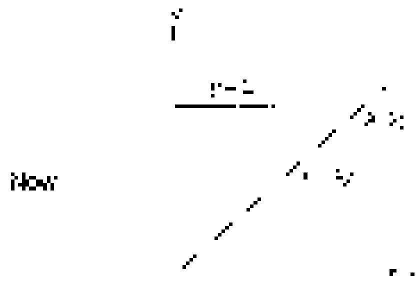
$= \left[\dfrac{1}{\infty} - \dfrac{1}{2}\right]$ [illegible]

13. (4)

Given [illegible] $\displaystyle\int\int$ [illegible] $dx\,dy\,dz$

[illegible]





Now

$$[illegible]$$

14. (a)

$$\frac{\partial u}{\partial x} = [illegible], \quad \frac{\partial u}{\partial y} = [illegible]$$

and

$$\frac{\partial v}{\partial y} = -[illegible]$$

$$\frac{\partial v}{\partial x} = \frac{1}{[illegible]}$$

$$J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = [illegible]$$

15. (c)

$$[illegible]$$

$$\text{grad } \phi = [illegible]$$

At the point [illegible]

$$|\text{grad } \phi| = [illegible] = \sqrt{5}$$

16. (a)

$$[illegible]$$

By integrating we get

[illegible]

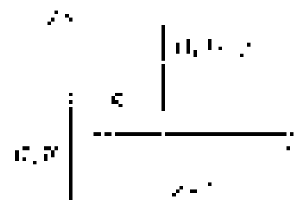
[illegible]

[illegible]

17. (c)

Green's Theorem

$$\oint [illegible] = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\,dy$$



Here $I = \oint [illegible]$

$$= [illegible]$$

$$[illegible]$$

$$\frac{\partial \psi}{\partial x} = [illegible], \quad \frac{\partial \phi}{\partial y} = [illegible]$$

Substituting in Green's theorem we get

$$I = \int [illegible]$$

$$= \int [illegible]$$

$$= [illegible] = \frac{2}{3}$$

18. (a)

[illegible]

[illegible]

19. (b)

$$\lim [illegible] \left(\frac{2x^2 - 7x - 3}{5x^2 - [illegible]} \right)$$

[illegible] form $\left(\frac{0}{0} \right)$

So, applying L-Hospital's rule

$$\lim [illegible] \left(\frac{4x - 7}{[illegible] - 12} \right) = [illegible]$$

20. (a)

$$f'(x) = \frac{[illegible]}{(1 + [illegible])^2} = \frac{e}{(1 + [illegible])^2}$$

[illegible]

[illegible]

increases

21. (a)

[illegible]

$$= \left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right] [illegible]$$

22. (c)

[illegible]

$$I = -\int_{-1}^{1} [illegible]$$

23. (d)

Area common to circles

[illegible]

24. (d)

[illegible] because [illegible] integration [illegible]

[illegible] is volume of a cone.

25. (d)

[illegible]

$$= \int_0^{2\pi} [illegible] + 5 - 3) d\theta$$

$$= \int_0^{2\pi} [illegible] d\theta$$

$= 2\sqrt{2}[\sin 2\theta$ [illegible]

$= 2\sqrt{2}(0 - 0)$ [illegible] $= 0$

26. (c)

[illegible]

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = [illegible]$$

[illegible]

directional derivative of f [illegible]

[illegible]

$$= \frac{1}{\sqrt{5}}(18 - 16 + (-36))$$

$$= \frac{[illegible]}{\sqrt{5}} = -15.2 [illegible]$$

27. (b)

Given [illegible]

Slope of tangent at point [illegible]

[illegible]

$$= \frac{1}{[illegible]} = 3$$

[illegible]

$$\Rightarrow \quad [illegible]$$

28. (a)

[illegible] vector [illegible] product

[illegible]

[illegible]

29. (a)

$\iint (x+y)\, dx\, dy$ ...

30. (b)

This is of the form of $\left(\frac{0}{0}\right)$

Applying L'Hospital rule

31. (a)

$$\lim_{x\to 0} \frac{e^{x}\sin\left(\frac{x}{2}\right)}{2x\cdot\frac{1}{2}} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x/2}{x/2} = \frac{1}{2} = 0.5$$

32. (a)

Given, $y = x^2$

$\Rightarrow \quad \frac{dy}{dx} = 2x = 0$ at $x = 0$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right]_{x=0} = 2$ which is +ve

So we have local minima at $x = 0$

at $\quad x = 0, \; y = 0$

but since $x = 0 \notin [1, 5]$

it is not a stationary point in the minima in that range.

At end point $x = 1$

$y = 1$

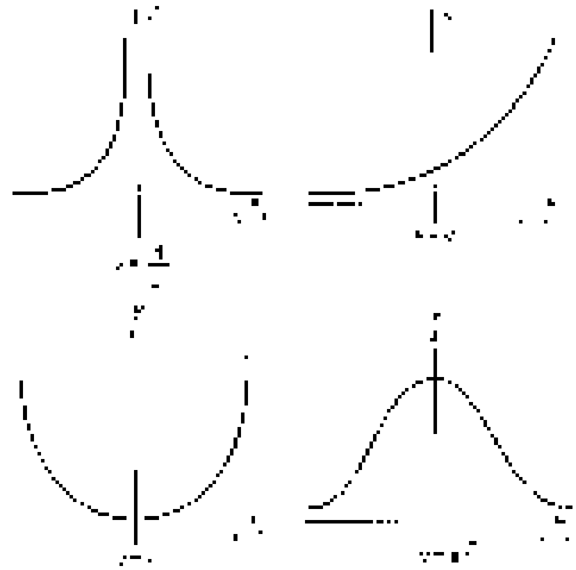
at second end point $x = 5$

$y = 25$

So absolute minimum value of function in $[1, 5]$ is 1.

33. (d)

From the graphs it is clear that only e^{-x^2} is strictly bounded.



34. (a)

$f(x) = x^2 - x - 2$ in $[-4, 4]$

$f'(x) = 2x - 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}$

$f''(x) = 2$

$f''\left(\frac{1}{2}\right) = 2 > 0$

So at $\quad x = \frac{1}{2}$,

we have a local minima so there is no local maxima in the range $[-4, 4]$

Now $\quad f(-4) = 18$

$f(4) = 10$

So maximum value in range $[-4, 4]$ is 18.

35. (a)

The Taylor series expansion of $f(x)$ about $x = 2$ is

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) \ldots$$

For linear approximation we take only the first two terms and get

$f(x) \approx f(2) + (x-2)f'(2)$

Here, $\quad f(2) = e^2$ and $f'(2) = e^2$

$\therefore \quad f(x) = e^2 + (x-2)e^2 = e^2(x-1)$

36. (b)

Equation of line with slope 1 and y-intercept 1 is

$y = x + 1$

$$= \int_1^2 x\,dx + \int_2^3 (x+1)\,dx$$

$$= \frac{(x-1)^2}{2} \ldots + \frac{1}{2}(3 - \ldots) = 2.5$$

37. (a)

From figure [illegible]

$$\sin A = \frac{|CM|}{|AC|} \Rightarrow |CM| = |AC| \sin A$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|AB| \times |CM|$$

$$= \frac{1}{2} AB \cdot AC \sin A = \frac{1}{2}|\overline{AB} \times \overline{AC}|$$

From above figure [illegible] $\overline{AB} = \bar{b} - \bar{a}$, $\overline{AC} = \bar{c} - \bar{a}$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}|(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})|$$

$$= \frac{1}{2}|(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})|$$

Option (b) is correct.

38. (b)

[illegible] $J = \begin{vmatrix} \ldots & \ldots \\ \ldots & \ldots \end{vmatrix}$

[illegible]

Now, $J = 0$

[illegible]

$$\Rightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2}$$

⇒ [illegible] functionally dependent.

∴ Dependency means $J = 0$

So, independent means $J \neq 0$

i.e. negative or positive.

However, notice that here since $J = (x_2 y_1 - x_1 y_2)^2$ it cannot be negative.

So, independence ⇒ J is positive.

39. (d)

$\vec{V} = 3xy\,\hat{i} - 2yz\,\hat{j} + 5zx\,\hat{k}$ [illegible]

$$\text{div}(\vec{V}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 3y - 2z + 5x$$

at [illegible] $\text{div}(\vec{V})$ = [illegible]

40. (a)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$2x = \frac{\partial \psi}{\partial y}$$

$$\psi = 2xy + c$$

$$\psi(0,0) = 0 \Rightarrow c = 0$$

and $\psi = 2xy$

41. (b)

[illegible]

⇒ [illegible]

[illegible]

Above form is the $\left(\frac{0}{0}\right)$ [illegible]

Applying L' Hospital rule

[illegible]

42. (c)

$$\lim \frac{x - \sin x}{x + \cos^2 x} = \lim \frac{x\left(1 - \frac{\sin x}{x}\right)}{x\left(1 + \frac{\cos^2 x}{x}\right)}$$

$$= \frac{\lim (1 - \ldots)}{\lim (1 + \ldots)}$$

$$= \frac{1 - \lim \frac{\sin x}{x}}{1 + \lim \frac{\cos^2 x}{x}} = \frac{1 - 0}{1 + 0} = 1$$

43. (b)

$$f(x) = (x^2-4)^2$$
$$f'(x) = 2(x^2-4) \times 2x = 4x(x^2-4) = 0$$

⇒ $x = 0$, $x = 2$ and $x = -2$ are stationary points.

$$f''(x) = 4[x(2x) + (x^2-4) \times 1]$$
$$= 4[2x^2 + x^2 - 4] = 4[3x^2-4]$$
$$= 12x^2 - 16$$
$$f''(0) = -16 < 0 \quad \text{(maximum at } x = 0)$$
$$f''(2) = 12(2)^2 - 16 = 32 > 0$$
$$\text{(minimum at } x = 2)$$
$$f''(-2) = 12(-2)^2 - 16 = 32 > 0$$
$$\text{(minimum at } x = -2)$$

∴ There is only one maxima and two minima for this function.

44. (d)

$$y = 3x^4 - 16x^3 - 24x^2 + 37$$
$$\frac{dy}{dx} = 12x^3 - 48x^2 - 48x = 0$$

$x(12x^2 - 48x - 48) = 0$

$$x = 0$$

or $12x^2 - 48x - 48 = 0$

$x^2 - 4x - 4 = 0$

$$x = \frac{4 \pm \sqrt{16+16}}{2}$$
$$= \frac{4 \pm \sqrt{32}}{2} = \frac{4 \pm 4\sqrt{2}}{2} = 2 \pm 2\sqrt{2}$$
$$\frac{d^2y}{dx^2} = 36x^2 - 96x - 48$$

Now at $x = 0$

$$\frac{d^2y}{dx^2} = -48 < 0$$

at $x = 2 \pm 2\sqrt{2}$, $\dfrac{d^2y}{dx^2} > 0$

∴ There are 3 extrema in this function.

45. (c)

$f(x)$ in the neighbourhood of a is

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

where, $a_n = \dfrac{f^n(a)}{n!}$

$$f^n(x) = e^x, \; f^4(2) = e^2$$

∴ Coefficient of $(x-2)^4 = a_4 = \dfrac{f^4(2)}{4!} = \dfrac{e^2}{4!}$

46. (a)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

From this, $\sin x^2 = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!}$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}$$

So, $\sin x^2$ and $\cos x^2$ have only even powers of x.

Similarly, $\sin x^3 = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} \ldots$

$$\cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} \ldots$$

So, only $\sin x^3$ has all odd powers of x.

∴ correct answer is (a).

47. (b)

$$f(x) = e^x + \sin x$$

We wish to expand about $x = \pi$

Taylor's series expansion about $x = a$ is

$$f(x) = f(a) + (x-a) f'(a)$$
$$+ \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \ldots$$

Now about $a = \pi$

$f(x) = f(\pi) + (x-\pi) f'(\pi) + \dfrac{(x-\pi)^2}{2!} f''(\pi) + \ldots$

The coefficient of $(x-\pi)^2$ is $\dfrac{f''(\pi)}{2!}$

Here, $f(x) = e^x + \sin x$

$$f'(x) = e^x + \cos x$$
$$f''(x) = e^x - \sin x$$
$$f''(\pi) = e^\pi - \sin \pi = e^\pi - 0 = e^\pi$$

The coefficient of $(x-\pi)^2$ is therefore

$\dfrac{e^\pi}{2!} = 0.5 \exp(\pi)$

48. (c)

$$r = x^y$$

Treating y as constant, we get

$$\frac{\partial r}{\partial x} = y x^{y-1}$$

Now we treat x as constant, we get

$$\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial}{\partial y}(y x^{y-1}) = x^{y-1} + y x^{y-1} \ln x$$

[illegible] value at $x = 2$

[illegible] $(1 - 2 [illegible]) = [illegible]$

49. (a)

Choice (a): $\int_0^{[illegible]} [illegible] = \log\sqrt{2}$

Choice (b): $\int_0^{[illegible]} \frac{dx}{[illegible] - 1} = \frac{\pi}{[illegible]}$

Choice (c): $\int [illegible]$

Integrating by parts taking [illegible] and [illegible] we get [illegible] and [illegible]

[illegible]

$= [illegible]$

Now $\int [illegible] = [illegible]$

Choice (d): $\int [illegible] = [illegible]$

Since [illegible] (d) is unbounded, (c) is the answer.

50. (d)

$$y = \frac{2}{5} x^{[illegible]}$$

$$\frac{dy}{dx} = [illegible]$$

Length of the curve is given by

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 [illegible]\, dx$$

$$= [illegible] = 1.2[illegible]$$

51. (a)

Equation of straight line from point (0, 0) to (1, 2) is

$$y - 0 = \frac{(2 - 0)}{(1 - 0)}(x - 0)$$

or $y = 2x$

$f(x, y) = 4x^{[illegible]} + [illegible]$

$= [illegible]$

$$\int [illegible]\, dx = \left| \frac{[illegible]}{4} + \frac{[illegible]}{5} \right|$$

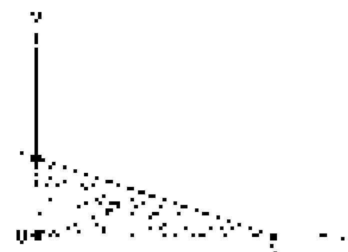
$= [illegible] + [illegible] = 33$

52. (a)

$$\int\int [illegible]\, dx\, dy = \int \left[[illegible] - \frac{y^2}{2} \right] dx$$

$$= \int_0^{[illegible]} [illegible]\, dx = [illegible]$$

53. (a)



Equation of the straight line with x-intercept $= 2$ and y-intercept $= 1$ is

$$\frac{x}{2} + \frac{y}{1} = 1$$

$\Rightarrow$ $y = 1 - \frac{x}{2}$

$\Rightarrow$ $x = 2 - 2y$

$$\int_0^{[illegible]}\int_0^{(2-2y)} [illegible]\, dx\, dy = \int_0^{[illegible]} \left[\frac{[illegible] x^2}{2} \right]_0^{2-2y} dy$$

$$= \int_0^1 \frac{y}{2}(2 - 2y)^2\, dy$$

$$= \int_0^{[illegible]} [illegible]\, dy = \frac{1}{[illegible]}$$

Alternatively, [illegible]

$$\int [illegible] = \frac{[illegible]}{[illegible]}$$

54. (c)

$\vec{F} \cdot \vec{G} = [illegible]$

$\vec{F} \cdot \vec{G} = |\vec{F}| |\vec{G}| \cos\theta$

If $\vec{F} \cdot \vec{G} = [illegible]$

$\Rightarrow |\vec{F}| |\vec{G}| \cos\theta = [illegible]$

Since, $\vec{F}$ and $\vec{G}$ are non-zero vectors

$\Rightarrow$ $\cos\theta = 0$

$\Rightarrow$ $\theta = 90°$

55. (d)

$\operatorname{div}[(x-y)\hat{i} + (y-x)\hat{j} + (x+y+z)\hat{k}]$

$= \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-x) + \frac{\partial}{\partial z}(x+y+z) = 3$

56. (b)

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 2x\hat{i} - 4y\hat{j} + \hat{k}$$

at point $P(1, 1, 2)$, grad $f = 2\hat{i} - 4\hat{j} + \hat{k}$

Now directional derivative of f at $P(1, 1, 2)$ in direction of vector $a = 3\hat{i} + 4\hat{k}$ is given as

$$\frac{a}{|a|}\cdot\text{grad } f = \left(\frac{3\hat{i} + 4\hat{k}}{\sqrt{25}}\right)\cdot(2\hat{i} - 4\hat{j} + \hat{k})$$

$$= \frac{1}{5}(6 + 4 + 0) = 2$$

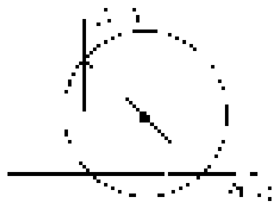
57. (b)

Taking $u = xy$ we can show that $y\,dx + x\,dy$ is exact. So, the value of the integral is independent of path

$$= 2\int (x\,dx - y\,dy)$$

$$2\int x\,dx - 2\int y\,dy$$

$$= 2\left[\frac{x^2}{2}\right] + \left[\frac{y^2}{2}\right] = 0$$



$\therefore$ Integral $= f(B) - f(A)$

$[xy]_B - [xy]_A = 0 - 0 = 0$

58. (a)

Let the point be (x, y, z) on surface $z^2 = 1 + xy$

Distance from origin $=$

$$= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{x^2 + y^2 + 1 + xy}$$

[since $z^2 = 1 + xy$ is given]

This distance is shortest when it is minimum, we need to find minima of $x^2 + y^2 + 1 + xy$

Let $f = x^2 + y^2 + 1 + xy$

$$\frac{\partial f}{\partial x} = 2x + y$$

$$\frac{\partial f}{\partial y} = 2y + x$$

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$\Rightarrow \quad 2x + y = 0$ and $2y + x = 0$

Solving simultaneously we get

$x = 0$ and $y = 0$

is the only solution and so $(0, 0)$ is the only stationary point.

Now $\quad r = \frac{\partial^2 f}{\partial x^2} = 2$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Since $\quad rt - s^2 = 2 \times 2 - 1^2 = 3 > 0$

We have local extremum, either a maximum or minimum exists at $(0, 0)$.

Now since $r = 2 > 0$, we have a minimum at $(0, 0)$.

Now at $x = 0$, $y = 0$

$$z^2 = 1 + xy = 1 + 0 = 1$$

So the points nearest to origin are

$(0, 0, \pm 1)$

The distance is $= \sqrt{0^2 + 0^2 + 1^2} = 1$

So answer is option (a).

59. (c)

An n^{th} degree polynomial bends exactly $n - 1$ times and therefore can have a maximum of $n - 1$ extrema. Also an n^{th} degree polynomial has at most n real zero crossings. So a cubic polynomial (degree 3) bends at most twice, has 2 extrema and cannot have more than 3 zero crossings.

60. (b)

Let $\quad x - \pi = t$

$x = \pi + t$

$$f(t) = \frac{-\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)}{t}$$

$$f(t) = -\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots\right)$$

$$f(t) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \cdots$$

61. (c)

$$I = \int_0^{\pi/4} \ln\left(\frac{1+\tan x}{1-\tan x}\right) dx$$

$$\because \quad \tan\left(\frac{\pi}{4} - x\right) = \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$$

$$\therefore \quad I = \int_0^{\pi/4} \ln\left[\frac{1 + \dfrac{1-\tan x}{1+\tan x}}{1 - \dfrac{1-\tan x}{1+\tan x}}\right] dx$$

$$= \int_0^{\pi/4} \frac{2\tan x}{1-\tan^2 x}\, dx = \int_0^{\pi/4} \tan 2x\, dx$$

$$= [\log \sec x]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4}\right| - \ln(\sec 0)$$

$$= \ln(\sqrt{2}) - \ln(1) = \ln(2^{1/2}) - 0$$

$$= \frac{1}{2}\ln 2$$

62. (b)

Circle $x^2 + y^2 = 1$

$x = \cos\theta$

$y = \sin\theta$

Along path θ varies from 0 to $90°$ (0 to $\pi/2$)

$$I = \int_0^{\pi/2} (\cos\theta - \sin\theta)^2 d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} (1 - \sin 2\theta)\, d\theta$$

$$= \left[\theta + \frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{1}{2}\left[\cos 2\cdot\frac{\pi}{2} - \cos 0\right]$$

$$= \frac{\pi}{2} + \frac{1}{2}[-1-1] = \frac{\pi}{2} - 1$$

63. (c)

Curve 1: $y^2 = 4x$

Curve 2: $x^2 = 4y$

Intersection of curve 1 and 2

$$y^2 = 4x = 4\sqrt{4y} = 8\sqrt{y}$$

$$y^4 = 64y \Rightarrow y(y^3 - 64) = 0$$

Solution: $y = 4$ and $y = 0$

Then $x = 4$, $x = 0$

The intersection points are $A(4, 4)$ and $O(0, 0)$. The area enclosed between curves 1 and 2 is given by

$$A = \int_0^4 y_1\, dx - \int_0^4 y_2\, dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left. 2\frac{x^{3/2}}{3/2}\right|_0^4 - \left.\frac{x^3}{3\times 4}\right|_0^4$$

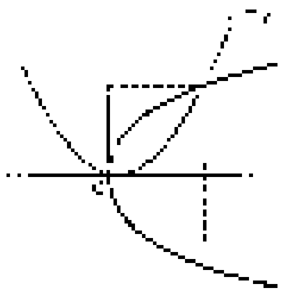
$$= \frac{4}{3}(4)^{3/2} - \frac{4^3}{3\times 4} = \frac{16}{3}$$

Alternatively the same answer could have been obtained by taking a horizontal strip ... $\rightarrow$ ...

$$\text{Required Area} = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \frac{16}{3}$$

64. (a)



$$\int \int (x, y)\,dx\,dy$$

65. (b)

$$\phi = x^2 + 3y^2 + 2z^2$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(6y) + \hat{k}(4z)$$

The gradient at P(1, 2, −1) is

$$= \hat{i}(2 \times 1) + \hat{j}(6 \times 2) + \hat{k}(4 \times -1)$$

$$= 2\hat{i} + 12\hat{j} - 4\hat{k}$$

66. (b)

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Here $f = x^2 + 3y^2 + 2z^2$

$\therefore$ $\nabla f = \hat{i}(2x) + \hat{j}(6y) + \hat{k}(4z)$

$\nabla f(1, 2, -1) = \hat{i}(2 \times 1) + \hat{j}(6 \times 2) + \hat{k}(4 \times -1)$

$= 2\hat{i} + 12\hat{j} - 4\hat{k}$

The directional derivative in direction of vector $\vec{a}$ $= \hat{i} + \hat{j} + 2\hat{k}$ is given by

$$\frac{\vec{a}}{|\vec{a}|} \cdot \text{grad}\, f = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{1^2 + 1^2 + 2^2}} \cdot (2\hat{i} + 12\hat{j} - 4\hat{k})$$

$$= \frac{1}{\sqrt{6}}[1 \times 2 + 1 \times 12 + 2 \times (-4)]$$

$$= \frac{6}{\sqrt{6}} = 2.45$$

67. (c)

Vector field,

$$\vec{F} = 3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$$

$$= F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

Divergence of vector field

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(-yz^2)$$

$$= 3z + 2x - 2zy$$

$\text{Div}\,\vec{F}\big|_{(1,1,1)} = 3(1) + 2(1) - 2(1)(1) = 3$

68. (a)

$$\lim_{x \to 0} \frac{\sin\left(\frac{2}{3}x\right)}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{2}{3}x\right)}{\frac{2}{3}x} \times \frac{2}{3}$$

$$= (1)\left(\frac{2}{3}\right) = \frac{2}{3}$$

69. (b)

$$\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)^{2n} = \lim_{n \to \infty}\left[\left(1 - \frac{1}{n}\right)^{n}\right]^{2}$$

$$= \left[\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)^{n}\right]^{2}$$

$$= e^{-2}$$

70. (c)

$y = |2 - 3x| = 2 - 3x$ for $2 - 3x \geq 0$

$= 3x - 2$ for $2 - 3x < 0$

Therefore $y = 2 - 3x$ for $x < \frac{2}{3}$

$= 3x - 2$ for $x \geq \frac{2}{3}$

Since $2 - 3x$ and $3x - 2$ are polynomials, these are continuous at all points. The only doubtful point is

$$x = \frac{2}{3}$$

Left limit $= 2 - 3 \times \frac{2}{3} = 0$

Right limit $= 3 \times \frac{2}{3} - 2 = 0$

$$f\left(\frac{2}{3}\right) = 3 \times \frac{2}{3} - 2 = 0$$

Since Left limit = Right limit $= f\left(\frac{2}{3}\right)$

Function is continuous at $\frac{2}{3}$.

$f(x)$ is therefore continuous everywhere.
Now since $2-3x$ and $3x-2$ are polynomials, they are differentiable.

Only doubtful point is $x = \frac{2}{3}$

Now at $x = \frac{2}{3}$, Left derivative $= -3$

Right derivative $= +3$

LD $\neq$ RD

$\therefore$ The function is not differentiable at $x = \frac{2}{3}$

So, $f(x)$ is continuous everywhere and differentiable for $x \in R$, except at $x = \frac{2}{3}$

71. (a)

$$f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$$

$$\frac{\partial f}{\partial x} = 8x - 8$$

$$\frac{\partial f}{\partial y} = 12y - 4$$

Putting $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$8x - 8 = 0 \text{ and } 12y - 4 = 0$$

Given $x = 1$ and $y = \frac{1}{3}$

$\left(1, \frac{1}{3}\right)$ is the only stationary point.

$$r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8$$

$$s = \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{\left(1, \frac{1}{3}\right)} = 0$$

$$t = \left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1, \frac{1}{3}\right)} = 12$$

Since $rt = 8 \times 12 = 96$

$s^2 = 0$

Since $rt > s^2$

we have either a maxima or minima at $\left(1, \frac{1}{3}\right)$

also since $r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8 > 0$, the point $\left(1, \frac{1}{3}\right)$ is a point of minima.

The minimum value is

$$f\left(1, \frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{9} - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$$

So the minimum value of $f(x, y)$ is minimum equal to $\frac{10}{3}$

72. (d)

$$f(x) = \frac{\sin x}{x}$$

$$f(x) = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x}$$

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots$$

$$f'(x) = -\frac{2x}{3!} + \frac{4x^3}{5!} \cdots$$

$$f''(x) = -\frac{2}{3!} + \frac{12x^2}{5!} \cdots$$

At $x = 0$, $f'(x) = 0$, $f''(x) < 0$

$\therefore$ $f(x)$ attains maxima.

73. (b)

$$y^x = x$$

Taking log on both sides,

$$y = \frac{1}{x} \log x$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \log x \left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}(1 - \log x)$$

Putting $\frac{dy}{dx} = 0$

$\frac{1}{x^2}(1 - \log x) = 0$

$\Rightarrow$ $\log x = 1$

$\Rightarrow$ $x = e$ is a stationary point.

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(-\frac{1}{x}\right) + (1 - \log x)\left(-\frac{2}{x^3}\right)$$

$$= -\frac{1}{x^3}[1 + 2(1 - \log x)] = -\frac{1}{x^3}(3 - 2\log x)$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=e} = -\frac{1}{e^3}(3 - 2\log e) = -\frac{1}{e^3} < 0$$

So, at $x = e$, we have a maximum.

74. (a)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{-\infty}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} - \left[\frac{-\pi}{2}\right] = \pi$$

75. (b)

$$I = \int_0^1 x e^x dx$$

Integrating by parts:

Let $u = x$,

$dv = e^x dx$

$du = dx$,

$v = \int e^x dx = e^x$

Now $\int u\,dv = uv - \int v\,du$

$\therefore \int x e^x dx = x e^x - \int e^x dx$

$= x e^x - e^x + c$

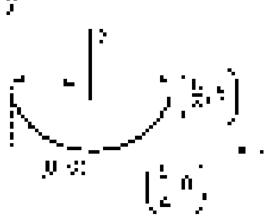
$$\int_0^1 x e^x dx = [x e^x - e^x]_0^1$$

$$= (1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0)$$

$$= 0 + 1 = 1$$

76. (d)

Length of curve $y = f(x)$ between $x = a$ and $x = b$ is given by



$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Here, $y = 4x^2$ ...(i)

$$\frac{dy}{dx} = 8x$$

when $y = 0$, $x = 0$

and $y = 1$, $x = \frac{1}{2}$

(As can be seen from equation (i) by substituting $y = 1$ and $x = 1/2$)

$\therefore \frac{1}{2}$ (Length of cable) $= \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^{1/2} \sqrt{1 + (8x)^2}\,dx$$

Length of cable $= 2\int_0^{1/2} \sqrt{1 + 64x^2}\,dx$

77. (c)

The volume of a solid generated by revolution about the x-axis, of the area bounded by curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$V = \pi \int_a^b y^2 dx$$

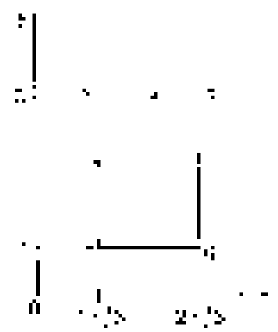
Here $a = 1$, $b = 2$ and $y = \sqrt{x} \Rightarrow y^2 = x$

$$\therefore \quad \text{Volume} = \int_1^2 \pi x\,dx$$

$$= \pi \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{\pi}{2}\left[2^2 - 1^2\right] = \frac{3}{2}\pi$$

78. (c)



$$\bar{A} = xy\,\hat{a}_x + x^2\hat{a}_y$$

$$\bar{l} = x\hat{a}_x + y\hat{a}_y$$

$$\overline{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y$$

$$\bar{A} \cdot \overline{dl} = xy\,dx + x^2\,dy$$

P → Q: $y = 1$, $dy = 0$

$$\int_P^Q \bar{A} \cdot \overline{dl} = \int_{1/\sqrt{3}}^{2/\sqrt{3}} x\,dx = \frac{x^2}{2}\bigg|_{1/\sqrt{3}}^{2/\sqrt{3}} = \frac{1}{2}$$

$Q \to R: x = \frac{2}{\sqrt{3}}, \; dx = 0$

$$\int_Q^R \vec{A}\cdot d\vec{l} = \int_1^2 \left(\frac{2}{\sqrt{3}}\right)^2 dy = \frac{4}{3}(2-1) = \frac{4}{3}$$

$R \to S: y = 2, \; dy = 0$

$$\int_R^S \vec{A}\cdot d\vec{l} = \int_{2/\sqrt{3}}^{1/\sqrt{3}} 3x\,dx$$

$$= \frac{3}{2}x^2 \Big|_{2/\sqrt{3}}^{1/\sqrt{3}} = \frac{3}{2}\left[\frac{1}{3} - \frac{4}{3}\right] = -\frac{3}{2}$$

$S \to P: x = \frac{1}{\sqrt{3}}, \; dx = 0$

$$\int_S^P \vec{A}\cdot d\vec{l} = \int_2^1 \left(\frac{1}{\sqrt{3}}\right)^2 dy = \frac{1}{3}(1-2) = -\frac{1}{3}$$

So

$$\oint \vec{A}\cdot d\vec{l} = \int_P^Q \vec{A}\cdot d\vec{l} + \int_Q^R \vec{A}\cdot d\vec{l} + \int_R^S \vec{A}\cdot d\vec{l} + \int_S^P \vec{A}\cdot d\vec{l}$$

$$= [illegible] + \frac{4}{3} - \frac{3}{2} - \frac{1}{3} = [illegible]$$

78. (c)

Velocity vector $= \vec{V} = 2xy\hat{i} - z^2\hat{j}$

The vorticity vector is twice the rotation vector

$= \text{curl}\,(\vec{V})$

$$= \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -z^2 & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-z^2)\right]\hat{i}$$

$$- \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2xy)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial y}(2xy)\right]\hat{k}$$

$$= 2z\hat{i} + [-2x - 2x]\hat{k}$$

At (1, 1, 1) i.e. substituting $x = 1$, $y = 1$ and $z = 1$, vorticity vector $= 2\hat{i} - 4\hat{k}$

80. (a)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\text{div}\,\vec{r} = \nabla\cdot\vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

81. (d)

$$\lim_{x \to 0} [illegible] = 1$$

82. (c)

If $f(x) = \lambda x$ is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}} \frac{\lambda \cos x}{\frac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \qquad \ldots(i)$$

Since the limit is in form of $\frac{0}{0}$, we can use L'Hospital's rule on LHS of equation (i) and get

$$\lim_{x \to \frac{\pi}{2}} \frac{-\lambda \sin x}{-1} = 1$$

$\Rightarrow \quad \lambda \sin\frac{\pi}{2} = 1$

$\Rightarrow \quad \lambda = 1$

83. (c)

$f(x) = 2x - x^2 + 3$

$f'(x) = 2 - 2x = 0$

$\Rightarrow \quad x = 1$ is the stationary point

$f''(x) = -2$

$\Rightarrow \quad f''(1) = -2 < 0$

So at $x = 1$ we have a relative maxima.

84. (b)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

85. (d)

$$\int_0^{\pi/4} [illegible] = [illegible]$$

$$= \frac{2}{e} - \frac{1}{e} \ldots$$

86. (b)

Let $I = \int_2^4 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}\,dx$ ...(i)

Since $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$I = \int_2^4 \frac{\sqrt{6-x}}{\sqrt{6-x}+\sqrt{x}}\,dx$ ...(ii)

(i) + (ii) ⇒ $2I = \int_2^4 \frac{\sqrt{x}+\sqrt{6-x}}{\sqrt{x}+\sqrt{6-x}}\,dx$

⇒ $2I = \int_2^4 dx$

⇒ $2I = 2$

⇒ $I = 1$

87. (d)

If $f(x)$ is even function then

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

89. (b)

Given z_1, z_2 and z_3

Note $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$\omega^3 = 1$

So the vectors are

$z_1 = 1$

and $z_2 = \omega$, $z_3 = \omega^2$

Now $z_1 + z_2 + z_3 = 1 + \omega + \omega^2$

$1 + \omega + \omega^2 = 0$

So ... [illegible]

90. (b)

$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1-\cos 0}{0} = \frac{0}{0}$

So use L'Hospital's rule

$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{0}{0}$

So use L'Hospital's rule again

$= \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$

91. (a)

$f(x) = \begin{cases} \ldots & x \geq 2 \\ \ldots & x < 2 \end{cases}$

So at $x = 2$, Left limit $= \ldots$

Right limit $= \ldots$

$f(2) = \ldots$

Since left limit = Right limit = $f(2)$

∴ $f(x)$ is continuous at $x = 2$

Now, LD = Left derivative ... $= -1$

RD = Right derivative ... $= +1$

LD ≠ RD

∴ $f(x)$ is not differentiable at $x = 2$

So $f(x)$ is continuous and not differentiable at $x = 2$

92. (d)

$f(x) = x^3 + 1$

For $f'(x) = 0$

⇒ $3x^2 = 0$

⇒ $x = 0$ is the only critical point

At critical point,

$f''(x) = 6x$

$f'''(x) = 6 \neq 0$

Now $f''(0) = 0$ and

$f'''(0) = 6$ which is non-zero

Since the first non-zero derivative value occurs at the third derivative which is an odd derivative, $x = 0$ is an inflexion point of $f(x)$ ...

95. (c)

We need absolute maximum of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$

First find local maxima by equating $f'(x) = 0$

i.e. $f'(x) = 3x^2 - 18x + 24 = 0$

i.e. $x^2 - 6x + 8 = 0$

$x = 2, 4$

Now $f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6 < 0$

∴ $x = 2$ is a point of local maximum

and $f''(4) = 24 - 18 = 6 > 0$

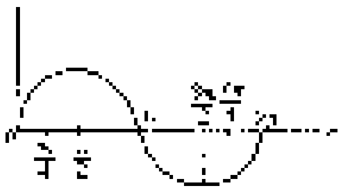
(So $x = 4$ is a point of local minimum)

Now tabulate the values of the end points of interval and of local maximum point to find absolute maximum in given range, as shown below:

x	$f(x)$
1	21
2	25
6	41

Clearly the absolute maxima is at $x = 6$ and absolute maximum value is 41.

94. (b)



From the sketch of the given function, we can easily see that in the range [0, 3π/2], there is only one local minimum at 3π/2.

95. (b)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \ldots$$

(By Maclaurin's series expansion)

96. (a)

The area enclosed is shown below as shaded.
The intersection points of $y = x$ and $y = x^2$ is obtained by solving:

$y = x$

and $y = x^2$ simultaneously

i.e. $x = x^2$

$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0$ or $x = 1$

Now, $x = 0 \Rightarrow y = 0$ which is at (0, 0)

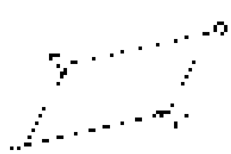
and $x = 1 \Rightarrow y = 1^2 = 1$ which is pt (1, 1)

So required area is

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

97. (a)



The area of the parallelogram OPQR in figure below is the magnitude of the vector product

$$= |\overline{OP} \times \overline{OR}|$$

$$\overline{OP} = a\hat{i} + b\hat{j}$$

$$\overline{OR} = c\hat{i} + d\hat{j}$$

$$\overline{OP} \times \overline{OR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (ad - bc)\hat{k}$$

$$|\overline{OP} \times \overline{OR}| = \sqrt{0^2 + 0^2 + (ad - bc)^2} = ad - bc$$

98. (a)

$z^2 = x^2 + y^2 - 1$

$f(x, y, z) = x^2 + y^2 - z^2 - 1 = 0$

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{grad } f|_P = \frac{2}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j} - 2(0)\hat{k}$$

$$= \sqrt{2}\hat{i} + \sqrt{2}\hat{j} + 0\hat{k}$$

$$|\text{grad } f| = \sqrt{2 + 2} = \sqrt{4} = 2$$

Then unit normal vector at point P is

$$\hat{n} = \frac{1}{|\text{grad } f|}\,\text{grad } f|_P$$

$$= \frac{1}{2}(\sqrt{2}\hat{i} - \sqrt{2}\hat{j}) = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

99. (a)

$$|\vec{A}| = r^n$$

$$\therefore \quad \vec{A} = r^n \frac{\vec{r}}{r}$$

$$\nabla \cdot \vec{A} = \nabla \cdot (r^{n-1}\vec{r}) = 0$$

We know $\nabla \cdot (\phi\vec{A}) = \phi(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla\phi$

$$\Rightarrow \left[r^{n-1}(\nabla \cdot \vec{r}) + \vec{r} \cdot \nabla r^{n-1}\right] = 0$$

$$\Rightarrow \left[3r^{n-1} + \vec{r} \cdot (n-1)r^{n-2}\frac{\vec{r}}{r}\right] = 0$$

$$\Rightarrow [3r^{n-1} + (n-1)r^{n-1}] = 0$$

$$[(n+2)r^{n-1}] = 0$$

$$n = -2$$

100. (c)

$$f(x)=\begin{cases}2 & \text{for } x=3\\ x-1 & \text{for } x>3\\ \dfrac{x+3}{3} & \text{for } x<3\end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3}\frac{x+3}{3} = 2$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} x-1 = 2$$

Also $f(3) = 2$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

So f is continuous at $x = 3$

option (c) is correct.

101. (b)

$$\frac{dy}{dx} = 10x + 10$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 20$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 30$$

y is defined open interval $x = (1, 2)$

$\therefore \quad 1 < x < 2$

$\therefore \quad 20 < \dfrac{dy}{dx} < 30$

102. (d)

Using R-H criterion

x^4	a_0	a_2	a_4
x^3	a_1	a_3	
x^2	A		
x^1	B		
x^0	a_4		

Where $A = \dfrac{a_1a_2 - a_0a_3}{a_1}$

So from the above table it is clear that there is atleast one sign change in the first column. So atleast one positive and one negative root exist.

103. (a)

Let $\quad 2\theta = t$

$2 \times d\theta = dt$

$$d\theta = \frac{dt}{2}$$

$$\theta = \frac{\pi}{4} \qquad t = \frac{\pi}{2}$$

$$\theta = 0 \qquad t = 0$$

$$I = \int_0^{\pi/2} \cos^4 t \sin^2 2t \frac{dt}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^4 t\,(2\sin t\cos t)^2\, dt$$

$$= \frac{4}{2}\int_0^{\pi/2} \cos^4 t \sin^2 t \cos^2 t\, dt$$

$$= \frac{4}{2}\int_0^{\pi/2} \cos^6 t \sin^2 t\, dt$$

$$= \frac{4}{2}\left[\frac{5\cdot 3\cdot 1\cdot 1}{8\cdot 6\cdot 4\cdot 2}\right]\frac{\pi}{2}$$

104. (c)

$$I = \int_0^1 \sqrt{x}\ln x\, dx$$

$$u = \ln x \qquad dv = \sqrt{x}\,dx$$

$$du = \frac{1}{x}dx, \quad v = \int \sqrt{x}\,dx = \frac{x^{3/2}}{\frac{3}{2}}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^1 \ln x\sqrt{x}\,dx = \left[\ln x\,\frac{x^{3/2}}{3/2}\right]_0^1 - \int_0^1 \frac{x^{3/2}}{3/2}\cdot\frac{1}{x}\,dx$$

$$= \left[\frac{2}{3}\cdot 0 - 0\right] - \frac{2}{3}\int_0^1 x^{1/2}\,dx$$

$$= -\frac{2}{3}\left[\frac{x^{3/2}}{3/2}\right]_0^1$$

$$= -\frac{2}{3}\cdot\frac{2}{3}\left[1 - 0\right]$$

$$= -\frac{4}{9}$$

105. (d)

Curl of gradient of a scalar field is always zero

$\nabla \times \nabla f = 0$

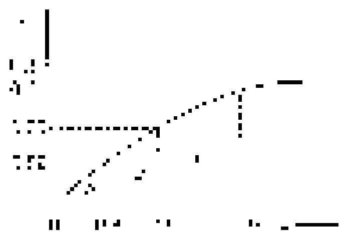
106. (c)

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1$$

$$\nabla\cdot\vec{A} = 3$$

107. (d)



Area of region I

$= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 0.03 \times 0.6$

Area of region II

$= \frac{1}{2} \times \text{height} \times (\text{base1} + \text{base2})$

$= \frac{1}{2} \times 0.3 \times (0.03 + 0.06)$

Area of region III

$= \frac{1}{2} \times \text{height} \times (\text{base 2} + \text{base 3})$

$= \frac{1}{2} \times 0.3 \times (0.06 + 0.3)$

$\int_0^1 q(y)dy = \frac{1}{2}$ [illegible]

[illegible] $= \frac{1}{2}$ [illegible]

Option (d) is correct.

108. (d)

Curl $(\nabla \times)$ is a null vector as irrotational vector has cross product as zero. Thus for vector to be irrotational $\nabla \times A = 0$

109. (b)

To find $\int \vec{F} \cdot \vec{dr}$ along a straight line [illegible] from $x = 1$ to $x = 2$

Here $y = 0$, $z = 2$, $dy = 0$ and $dz = 0$

$$\int \vec{F} \cdot \vec{dr} = \int [\text{illegible}] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int [\text{illegible}]$$

Putting $y = 0$, $z = 2$, $dy = 0$ and $dz = 0$

We get

$$\int \vec{F} \cdot \vec{dr} = 0$$

110. (c)

$\iiint_V$ [illegible]

$=$ [illegible] $= \pi$

111. (c)

According to Stokes Theorem

$$\oint_C \vec{A} \cdot d\vec{l} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{s}$$

112. (a)

$$\lim_{x \to \infty} \left[\frac{x - \sin x}{x}\right] = \frac{1 - \frac{\sin x}{x}}{1} = 1 - 0 = 1$$

Since, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

113. (a)

$\lim_{x \to 2}$ [illegible] $\frac{0}{0}$ form

Using L'Hospital's Rule (differentiating numerator & denominator w.r.t. x keeping y as constant)

$= \lim_{x \to 2}$ [illegible]

114. (a)

$$\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x}$$

Applying L'Hospital's rule

$$\lim_{x \to 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(1 - \cos x)} = \lim_{x \to 0} \frac{(1 - \cos x)}{\sin x}$$

(It is still of $\frac{0}{0}$ form)

Again applying L'Hospital's rule

$$\lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

115. (c)

$$\lim_{x \to 0} \frac{(e^{2x} - 1)}{\sin 4x} \quad \text{It is of } \left[\frac{0}{0}\right] \text{ form}$$

Applying L'Hospital's rule,

$$\lim_{x \to 0} \frac{2e^{2x}}{4 \cos 4x} = \frac{2}{4} = \frac{1}{2}$$

116. (a)

117. (d)

118. (d)

119. (a)

120. (c)

121. (b)

122. (d)

123. (c)

[illegible]

$y^2 = x^2 - [illegible]$

Also, $z = \frac{1}{2}xy$

$z^2 = \frac{x^2}{4}(y^2 - 2xy)$ [illegible]

Let, $f(x) = z^2 = \frac{x^2}{4}(y^2 - 2xy)$ [illegible]

[illegible]

$f'(x) = 0$

[illegible]

$y = \frac{a}{\sqrt{3}}$

$\theta = 60^\circ$

124. (c)

[illegible]

125. (c)

[illegible]

126. (c)

[illegible]

127. (b)

$f(x) = x^3 - 3x^2 - 24x + 100$ [illegible]

[illegible]

128. (a)

[illegible]

129. (a)

[illegible]

$x = 0$

[illegible]

130. Sol.

[illegible]

$x = 1, 2$

[illegible]

points

[illegible]

131. (b)

[illegible]

132. Sol.

[illegible]

133. (a)

[illegible]

134. (c)

[illegible]

By Stokes theorem,

[illegible]

135. (b)

[illegible]

136. (b)

[illegible]

Thus, integral becomes $\int \left[\int \ldots \right] dy$

107. (b)

For the points to be collinear the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 6 & 3 & 4 \end{vmatrix}$ must be zero.

$\therefore$ The three vertices are not collinear.

108. (a)

109. (c)

140. (a)

141. (b)

which is in the form $\frac{\infty}{\infty}$

To convert this to $\frac{0}{0}$ form, we rewrite as

Now, it is in $\frac{0}{0}$ form

Using L-Hospital's rule

$\Rightarrow \log y = \lim [illegible] = \lim [illegible] = 2$

$y = e^2$

142. (a)

$\lim_{x \to 0} \frac{1-\cos(x^2)}{2x^4}$

putting the $x \to 0$

we get $\frac{0}{0}$ form

Applying L'Hospital's rule

$\Rightarrow \lim_{x \to 0} \frac{2x \sin(x^2)}{8x^3}$

$\Rightarrow \lim \frac{\sin(x^2)}{4x^2}$

$\Rightarrow \frac{1}{4} \lim \frac{\sin(x^2)}{x^2}$

$= \frac{1}{4} \lim \frac{\sin(x^2)}{x^2} = \frac{1}{4} \times 1 = \frac{1}{4}$

143. Sol.

[illegible]

(Note: Since the function is not evaluating to $\frac{0}{0}$ no need to use L'Hospital's rule)

144. (c)

$y = \lim x^{1/x}$

$\log y = \lim \log x^{1/x}$

$\log y = \lim \frac{\log x}{x}$

Now for $x \to \infty$ L-Hospital's rule

$\log y = \lim \frac{1/x}{1}$

$\log y = 0 \Rightarrow y = 1$

145. (c)

$\lim (1+x^2)^{[illegible]}$

$\log y = \lim \log(1+x^2)^{[illegible]} = \lim \frac{\log(1+x^2)}{[illegible]}$

Now for applying L'Hospital's rule

$\Rightarrow \log y = \lim [illegible]$

$\Rightarrow \log y = \lim [illegible]$

Again we are getting [illegible] form apply L'Hospital's rule

$\log y = \lim [illegible]$

$\log y = [illegible] = 0$

$\Rightarrow y = 1$

146. (b)

$f(x) = [illegible]$

Statement 1: [illegible] continuous in $[-1, 1]$. Hence check only at $x = 0$.

We need to check continuity at $x = 0$

Left limit $= \lim [illegible]$

Right limit $= \lim [illegible]$

Left limit ≠ Right limit

∴ Statement 1 is false

Statement 2: $f(x)$ is unbounded in $[-1, 1]$. Since at $x = 0$ [illegible] the function is not bounded.

∴ Statement 2 is true

Statement 3: Area is not finite and true

$A = [illegible]$

$= [illegible] = \infty$

So Area is not finite and true

∴ Statement 3 is true

147. (b)

Since $f(x)$ in $[-1, 2]$ is [illegible] continuous [illegible]

By Lagrange mean value theorem,

$f'(c) = \frac{f(2) - f(-1)}{2-(-1)} = [illegible] = 1$

[illegible]

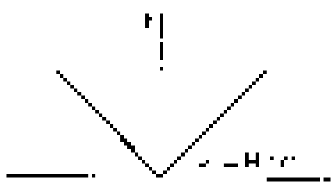
148. (d)

[illegible]

$f'(x_0) = 0$

and $f''(x_0)$ [illegible]

149. (a)



150. Sol.

[illegible] $A = \begin{bmatrix} [illegible] \end{bmatrix}$

[illegible]

$\Rightarrow$ [illegible]

[illegible]

151. (b)

[illegible]

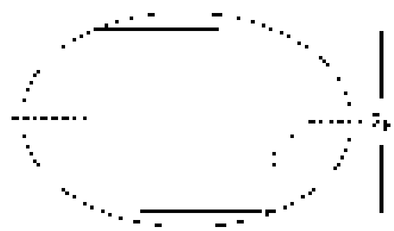
Putting [illegible] $= 0$, we get

[illegible]

[illegible]

Only curve (c) shows a single [illegible] maximum [illegible]

152. Sol.



[illegible]

We get, [illegible]

[illegible]

153. (a)

[illegible]

$2y = 4$

$\Rightarrow y = 2$

154. (a)

[illegible] ...(1)

[illegible] in equation (1)

[illegible] ...(2)

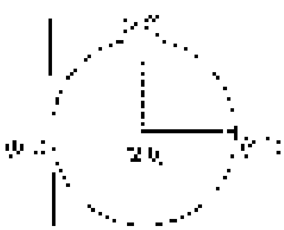
Equation (1) × [illegible] − equation (2) × [illegible]

[illegible]

[illegible]

[illegible]

155. Sol.



[illegible] is a circle of radius 2 units with centre at (2, 0)

Time to reach from (4, 0) to (2, 2) is

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

[illegible]

156. Sol.

[illegible]

157. Sol.

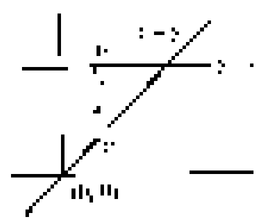
[illegible]

$= [illegible]\ 0.71828$

158. (c)

[illegible]

Limit of x:
Lower limit x = 0
Upper limit x = y
Limit of y:
Lower limit y = 0
Upper limit y = 2
By changing the order of integration limit of y:
limit of y:
Lower limit y = x
Upper limit y = 2
Limit of x:
Lower limit x = 0
Upper limit x = 2



So [illegible]

159. Sol.

$f(x, y, z) = x^2 - 3yz$ [illegible]

$\nabla f = 2x\hat{i} - 3z\hat{j} - 3y\hat{k}$ [illegible]

$\nabla f_{(\ldots)} = 4\hat{i} + \ldots\hat{j} - \ldots\hat{k}$ [illegible]

Directional derivative

$$= (4\hat{i} \ldots \hat{j} - \ldots\hat{k}) \cdot \frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$$

$$= \frac{4 \ldots 12 \ldots}{\sqrt{3}} = \frac{\ldots}{\sqrt{3}}$$

$$= \frac{\ldots\sqrt{3}}{3} = -5.715$$

160. (a)

$$\text{Curl of vector} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ldots & \ldots z^2 & \ldots \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(\ldots) - \frac{\partial}{\partial z}(\ldots)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(\ldots) - \frac{\partial}{\partial z}(\ldots)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(\ldots) - \frac{\partial}{\partial y}(\ldots)\right]$$

$$= \hat{i}[\ldots - \ldots] \ldots [\ldots - \ldots]$$

At $x = \ldots, y = 1$ and $z = 1$

Curl $= \ldots$ [illegible]

161. (c)

The Curl $\ldots$ [illegible]

[illegible]

162. (a)

$\phi(x, y) = x^2 - 3y^2$ [illegible]

[illegible] at point $P \rightarrow (\ldots, 1)$

[illegible]

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} = 2x\hat{i} \ldots y\hat{j}$$

$\nabla\phi_{(\ldots)} = \ldots\hat{i} - \ldots\hat{j}$

[illegible] vector is $\vec{a} = \ldots\hat{i} \ldots\hat{j}$

Magnitude of directional derivative along $\vec{a}$

at $(1, \ldots)$ [illegible]

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} = \ldots\hat{i} + \ldots\hat{j}$$

$\nabla\phi|_{P} = \ldots\hat{i} + \ldots\hat{j}$

$|\vec{a}| = \sqrt{\ldots} = \ldots\sqrt{\ldots}$

$$\hat{a} = \frac{\ldots\hat{i} - \ldots\hat{j}}{\ldots\sqrt{\ldots}} = \frac{\ldots}{\sqrt{\ldots}}$$

$\therefore$ Magnitude of directional derivative

$$= (\ldots\hat{i} + \ldots\hat{j}) \cdot \left(\frac{\ldots}{\sqrt{\ldots}}\right)$$

$$\frac{\ldots - \ldots}{\sqrt{2}} = \frac{\ldots}{\sqrt{\ldots}} = \ldots\sqrt{\ldots}$$

163. Sol.

$\oint_C [(3x - 8y^2)dx + (4y - 6xy)dy]$ [illegible]

Boundary of region bounded by $x = 0$, $y = \ldots$ and $x + y = 1$

Using Green's theorem

$$I = \oint_C (P\,dx + Q\,dy)$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here, $P = 3x - 8y^2$

$Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} = -6y$$

$$\frac{\partial P}{\partial y} = -16y$$

$$I = \iint (-6y + 16y)\,dx\,dy$$

$$= \iint 10y\,dx\,dy$$

$$I = 10\int_0^1 dx \int_0^{1-x} y\,dy = \ldots\int_0^1 (1 - x)^2$$

$$I = \ldots\int_0^1 (1 - x)^2 dx = 1.6666$$

164. Sol.

According to gauss divergence theorem,

$\iint_S (\ldots)\,dS = \iiint_V \text{div}(\ldots)\,dV$ [illegible]

$$= \frac{1}{\ldots}\left[\ldots \times \frac{1}{\ldots}\right] \ldots$$

$r = \ldots$ [illegible]

$$= \frac{\ldots}{\ldots} \times \ldots \frac{4}{3}\pi \ldots = 216$$

165. Sol.

[illegible]

So the required limit is [illegible] $= 1$

166. Sol.

[illegible]

So the required limit is [illegible]

167. Sol.

[illegible]

$= 729 - 1 = 728$

168. Sol.

[illegible]

$= \frac{1}{1+1} = \frac{1}{2}$

169. (a)

[illegible]

170. (c)

[illegible]

171. (c)

(i) [illegible]

(ii) [illegible]

(iii) [illegible]

which depends on m.

172. (b)

P: If $f(x)$ is continuous at $x = x_0$ then it is also differentiable at $x = x_0$

Q: If $f(x)$ is continuous at $x = x_0$, then it may or may not be differentiable at $x = x_0$

R: If $f(x)$ is differentiable at $x = x_0$ then it is also continuous at $x = x_0$

P is false

Q is true

R is true   Option (b) is correct.

173. (c)

$$f(x) = \frac{x^2 - 3x + ?}{x^2 + 3x - 4} \text{ is not continuous}$$

when

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

$x = -4, 1$

174. (c)

$F(x)$ = [illegible]

$F'(x)$ = [illegible] > 0 when $x < 0$

$\therefore$ $F(x)$ is increasing for $x < 0$

$F'(x)$ = [illegible] < 0 when $x > 0$

$\therefore$ $F(x)$ is decreasing for $x > 0$

175. Sol.

If $f(x)$ = [illegible] is degree 10

[illegible]

[illegible]

[illegible]

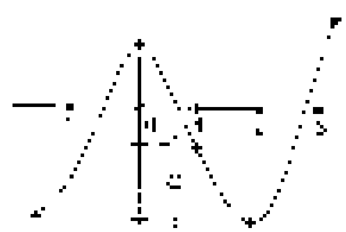
Now [illegible]

[illegible]

[illegible]

Clearly degree of $f(x) - g(x)$ is 8.

176. (b)



$f(x) = x^3 - 3x^2 - 1$

$f'(x) = 3x^2 - 6x$

$f'(x) = 0$

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0, 2$

$f''(x) = 6x - 6$

At $x = 0$ $f''(0) = -6$ negative

$x = 2$ $f''(2) = 6$ positive

177. Sol.

$f(x) = 2x^3 - x^4$ [illegible]

$f'(x) = 6x^2 - 4x^3$

For maxima or decreasing

$f'(x) = 0$

$6x^2 - 4x^3 = 0$

$2x^2(3 - 2x) = 0$

$x = 0, 0, \frac{3}{2}$

$f''(x) = 12x - 12x^2$

[illegible] $x = 0$ $f''(0) = 0$

[illegible] $x = \frac{3}{2}$ $f''\left(\frac{3}{2}\right)$ [illegible]

[illegible] $x = -1$ [illegible]

[illegible]

At $x = -1$ [illegible]

[illegible]

178. Sol.

$f(x)$ = [illegible]

$f'(x)$ = [illegible]

$f'(x) = 0$ for stationary point

stationary point is $x = \pm\frac{1}{\sqrt{3}}$

only $\frac{1}{\sqrt{3}}$ lies in [1, 2]

$f(1) = 0$

$f(2) = 0$

$f\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$

Maximum value is 0.

179. (d)

$f'(x) = 0$

$\Rightarrow$ $2x - 4 = 0$

$\Rightarrow$ $x = 2$ (stationary point)

$f''(x) = 2 > 0$

$\Rightarrow$ $f(x)$ is minimum at $x = 2$

$f(2) = 2^2 - 4(2) + 2 = -2$

$\therefore$ The minimum value is -2 [illegible]

180. (b)

The quadratic approximation of $f(x)$ at the point $x = 0$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0)$$

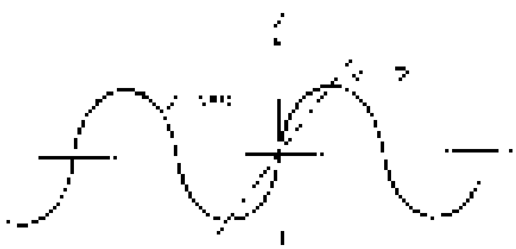
$$= [illegible]$$

$$= -3x^2 \ldots$$

181. (d)

$\Rightarrow \theta = \frac{\pi}{2} = 90°$

192. (c)



193. Sol.

$= \iint 2\,dx\,dy$

194. Sol.

passing through

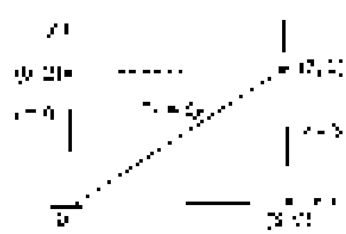
passing through

185. Sol.

186. (c)

$= \tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2}$

187. Sol.



Volume $= \iiint dz\,dx\,dy$ [illegible]

[illegible]

189. Sol.

Parabola $x^2 = 8y$

[illegible]

At the point of intersection [illegible]

$\frac{x^2}{8} = 8$

$\Rightarrow$ $x = \pm 8$ and $y = 8$

$\therefore$ Required area is [illegible]

[illegible] sq. units

189. Sol.

[illegible]

$x = r\cos\theta$

$y = r\sin\theta$

[illegible]

[illegible]

190. Sol.

By Green's theorem

[illegible]

191. (a)

At the point of intersection of the curves [illegible]

[illegible]

$\Rightarrow x^2 + x - 2 = 0$

[illegible]

[illegible] $\iint$ [illegible]

[illegible] $\frac{9}{2}$

192. Sol.

[illegible]

193. Sol.

[illegible]

194. (d)

[illegible]

195. (a)

$\int \vec{F} \cdot d\vec{r}$

where $\vec{F} = [illegible]$

[illegible]

$\vec{F} = \nabla\phi$

[illegible]

$\phi_x = 2x[illegible]$

$\phi_y = 2[illegible]$

$\phi_z = [illegible]$

$\phi = [illegible] + C$

[illegible]

196. Sol.

$= \int \vec{F} \cdot d\vec{r}$

[illegible]

$= \int_0^1 [illegible]\, dt$

[illegible]

197. (b)

[illegible]

[illegible]

198. Sol.

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}[illegible] + \frac{\partial}{\partial y}[illegible]$

$= [illegible] = 0$

200. Sol.

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}[illegible] + \frac{\partial}{\partial y}[illegible] + \frac{\partial}{\partial z}[illegible]$

By Gauss divergence theorem

$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{F}\, dV = [illegible]$

[illegible]

201. (d)

$\lim [illegible] \quad \left(\frac{0}{0}\text{ form}\right)$

[illegible]

202. (c)

[illegible]

[illegible]

203. Sol.

By vector identities

[illegible] $= 0$

204. Sol.

Since V is a non-zero vector of dimension 3×1 [illegible]

$\rho(A) \leq \min[\rho(V), \rho(V^T)]$

$\leq \min(1, 1)$

≤ 1

Since A is non-zero. Hence $\rho(A) = 1$

205. Sol.

$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ [illegible]

$\vec{b} = [illegible]\hat{i} + 5\hat{j} + [illegible]\hat{k}$

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

[illegible]

$= \frac{-21 - 40 - 224}{(15.23)(21.84)} = \frac{[illegible]}{[illegible]} = [illegible]$

$\theta = [illegible] = 2.48$ [illegible]

$= 0.733$ [illegible]

206. Sol.

$x + 2y = 11$ [illegible]

[illegible]

207. Sol.

[illegible]

$\sqrt{z} = 2$

$= 2$

[illegible] $= 4$ [illegible]

208. (a)

[illegible]

[illegible] $x < 0$

[illegible]

The [illegible] function of two continuous functions is always continuous.

[illegible] are zero

209. Sol.

[illegible]

210. (a)

[illegible]

L.H.D. = R.H.D.

[illegible]

211. Sol.

[illegible]

$x_1 = 1,\ y_1 = [illegible]$

$x_2 = 2,\ y_2 = -1,\ z_2 = 2,\ [illegible] = 0$

$\cos\theta = [illegible]$

$= \dfrac{[illegible]}{\sqrt{3}\sqrt{3}} = \dfrac{[illegible]}{\sqrt{3}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 54.73°$

212. Sol.

$f(x) = [illegible]$

$f'(x) = [illegible]$

$f'(x) = x^2 - 1 = 0$

$\Rightarrow \quad x = \pm 1$

$f''(x) = 2x$

At $x = 1$, $f''(1) = 2 > 0 \Rightarrow$ minima

At $x = -1$, $f''(-1) = -2 < 0 \Rightarrow$ maxima

Minimum value of $f(x)$ in [−100, 100] is given by

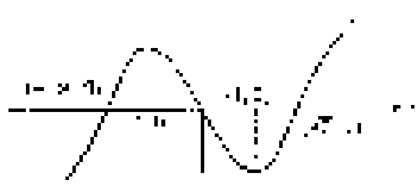
Minimum {f(−100), f(1), f(100)}

Minimum {[illegible], [illegible], [illegible]}

$= -[illegible]$

Hence the minimum value occurs at $x = -100$

Also graph of the function will be like



213. (c)

$\text{Integral}_1 = \displaystyle\iint [illegible]\, dx\, dy$

$= \displaystyle\int [illegible] \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] dx\, dy$

$= [illegible]$

$= [illegible] = 0.5$

and integral

$I = \displaystyle\int [illegible]\, dx\, dy$

$= [illegible]$

$= [illegible] = -0.5$

Option (c) is correct

214. (b)

$\vec{F} = [illegible]$

$\nabla \times \vec{F} = 0$ (given)

$\nabla \times \vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ [illegible] & [illegible] & [illegible] \end{vmatrix}$

$= [illegible]$

$\Rightarrow \quad [illegible]$

215. Sol.

A(0, 2, [illegible]) and B(4, [illegible], [illegible])

The equation of the line AB is

$\dfrac{x-0}{4-0} = [illegible] = t$ say

$[illegible]$

$dx = 4dt,\ dy = [illegible],\ dz = [illegible]$

t varies from 0 to 1

$$I = \int [illegible]$$

$$= \int_{-1} [illegible] \, dx$$

$$= \int_{-1}^{1} (-4x + [illegible])\,dx$$

$$\left(-4\frac{x^2}{2} + [illegible]\right)\Big|_{-1}^{1} = [illegible] - 1$$

216. (c)

About $x = 0$

$$[illegible]$$

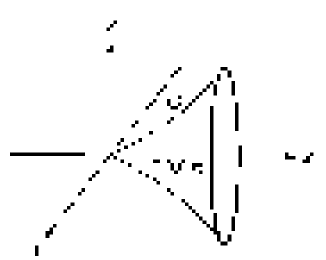
$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \ldots\right)\left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} \ldots\right)$$

$$= [illegible] + x^2 + \frac{x^3}{2} + \frac{x^4}{[illegible]} \ldots [illegible] \frac{x^6}{2} - \frac{x^7}{6} -$$

$$+ \frac{x^3}{2} - \frac{x^4}{[illegible]} - \frac{x^5}{4} + \frac{x^6}{[illegible]} \ldots \frac{[illegible]}{6} \frac{[illegible]}{6} \frac{x^7}{12} + \frac{x^8}{36} + \ldots$$

$$= [illegible] \frac{3}{2}[illegible] + \frac{7}{2}[illegible]$$

217. Sol



$[illegible] = [illegible] = 1$

$[illegible] = x^2$

Here rotation is about z axis.

Volume of region $V = \int_0 \pi [illegible]\,dz$

Here PQ is [illegible] which is given by

$$PQ = \sqrt{[illegible]}$$

$$(PQ)^2 = x^2 + y^2 = z^2$$

[illegible]

$$= \int_0 [illegible] = \int_0 [illegible] \Big|_0 = \frac{\pi}{4} = 0.7[illegible]$$

218. (c)

Given, $f(x) = [illegible]$ ...(1)

$$f\left(\frac{1}{2}\right) = \sqrt{2}$$ ...(2)

$$\int_0^1 f(x)\,dx = \frac{[illegible]}{\pi}$$ ...(3)

Now we need to find A and B

$$f(x) = A\cos\left(\frac{\pi x}{2}\right)[illegible]$$

$$f\left(\frac{1}{2}\right) = A\cos\left(\frac{\pi}{4}\right) \times \frac{\pi}{2} = \sqrt{2}$$

$$\Rightarrow \frac{A}{\sqrt{2}} \times \frac{\pi}{2} = \sqrt{2}$$

$$\Rightarrow A = \frac{1}{\pi}$$

Now $\int f(x)\,dx = \int [illegible]$

Putting $A = \frac{1}{\pi}$ we get

$$\int f(x)\,dx = \int [illegible] \left(\frac{\pi x}{2}\right) dx + \int B\,dx$$

$$= [illegible] + Bx = [illegible]\cos\left(\frac{\pi x}{2}\right) [illegible]$$

Putting limit 0 and [illegible]

$$\int_0 f(x)\,dx = \frac{[illegible]}{\pi^2}\left(\cos\frac{[illegible]}{2} - \cos(0)\right) [illegible] = \frac{2[illegible]}{\pi}$$

$$\frac{[illegible]}{\pi^2}[illegible] - 1 - [illegible] = \frac{[illegible]}{\pi}$$

But $A = \frac{4}{\pi}$ [illegible]

$B = 0$

So $A = \frac{4}{[illegible]}$ and $B = 0$ is answer.

219. (c)

$$\frac{[illegible] - 2[illegible]}{[illegible] - 3[illegible] - 2} = \frac{1 - 2[illegible]}{1 - [illegible] - 2}$$

$$= 0.5 [illegible]$$

[illegible] $= \lim \frac{[illegible]}{[illegible]}$

$$= \frac{7}{3} [illegible] - \frac{[illegible]}{3} - 1$$

220. (c)

$$w = f(x, y)$$

By chain rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

221. Sol.

$$\phi = x^2 + 2y^2 + z^4$$

$$\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^2 + 2y^2 + z^4)$$

$$= 2x\, i + 4y\, j + 4z^3\, k$$

At (1, 2, 3)

[illegible]

$$= 34$$

222. (a)

$$\tan\theta = \frac{dy}{dx} = [illegible]$$

$$\tan 45^\circ = [illegible]$$

$$1 = [illegible]$$

$\Rightarrow$ [illegible]

$\therefore$ $x = 1$

Putting $x = 1$ in the given curve

we get $y = 0$

223. (b)

Let, [illegible]

$$\frac{dx}{\sqrt{1 - x^2}} = dt$$

[illegible]

224. (b)

Speed [illegible]

[illegible]

$$\frac{ds}{dt} = [illegible] = 0$$

[illegible] 30 minutes

$$\frac{d^2s}{dt^2} = [illegible] < 0 \text{ (Maxima)}$$

[illegible]

[illegible]

225. Sol.

$$\lim \frac{[illegible]}{[illegible]} \quad \text{(Applying L' Hospital rule)}$$

[illegible]

226. (b)

$$f(x) = [illegible]$$

We can [illegible]

$$f'(x) = \frac{3x^2}{3} [illegible]$$

[illegible]

To determine [illegible]

[illegible]

[illegible]

[illegible]

Also this question can be directly solved by putting given values of x.

227. (a)

$$\int_0^{\pi/2} [illegible] \sin^2\theta\, d\theta$$

[illegible]

Let, $\tan\theta = t$ [illegible]

$\sec^2\theta\, d\theta = dt$ [illegible]

$$= \frac{12}{[illegible]} \times \int [illegible]$$

$$= [illegible] \left[\frac{1}{[illegible]} \tan^{-1} \frac{[illegible]}{\sqrt{10}}\right]$$

$$= \frac{4}{\sqrt{10}} \left[\frac{\pi}{2} - 0\right] = \frac{2\pi}{\sqrt{10}}$$

3 Differential Equations

3.1 Introduction

[illegible]

[illegible]

3.2 Differential Equations of First Order

3.2.1 Definitions

[illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

(e) [illegible] (f) [illegible]

(g) [illegible]

An ordinary differential equation [illegible]

A partial differential equation [illegible]

[illegible]

[illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(c) A differential equation such as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

is of the second order and second degree.

3.2.2 Solution of a Differential Equation

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example $\qquad y = cx^2 \qquad (1)$

is a solution of $\qquad \frac{dy}{dx} = [illegible] \qquad (2)$

The general (or complete) solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (1) is a general solution of (2) as the number of arbitrary constants (one) is the same as the order of the equation (2) (first order).

Similarly, the general solution of a second order differential equation will have two arbitrary constants.

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example, $\qquad y = 4x^2$

is the particular solution of the equation (2) as it can be derived from the solution (1) by putting $c = 4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a singular solution and is of less importance in engineering.

3.2.3 Equations of the First Order and First Degree

It is not possible to solve analytically such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations:

1. Equations where variables are separable.
2. Homogeneous equations.
3. Linear equations.
4. Exact equations.

In other cases, the particular solution may be determined numerically.

3.2.3.1 Variables Separable

If in an equation it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable. Thus the general form of such an equation is $f(y)\,dy = \phi(x)\,dx$.

Integrating both sides, we get $\int f(y)\,dy = \int \phi(x)\,dx + c$ as its solution.

Example 1.

Solve $\qquad \frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution:

[illegible]

NOTE

1. [illegible]
2. **Initial value problem:** [illegible]

Example 2.

[illegible]

Solution:

[illegible]

Note: [illegible]

Homogeneous Equations

[illegible]

Homogeneous Function: An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as $x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + \dots + a_n\left(\frac{y}{x}\right)^n\right]$.

Thus any function $f(x, y)$ which can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ is called a homogeneous function of degree n in x and y. For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3 in x and y.

To solve a homogeneous equation:

1. Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
2. Separate the variables v and x, and integrate.

Example:

Solve $(y^2 - x^2)\,dx - 2xy\,dy = 0$

Solution:

Given equation is $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ which is homogeneous in x and y. ...(i)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

∴ (i) becomes $v + x\frac{dv}{dx} = \frac{1}{2}\left(v - \frac{1}{v}\right)$

or $x\frac{dv}{dx} = \frac{1}{2}\left(v - \frac{1}{v}\right) - v = -\left[\frac{1 + v^2}{2v}\right]$

Separating the variables,

$$\frac{2v}{1 + v^2}\,dv = -\frac{dx}{x}$$

Integrating both sides,

$$\int\frac{2v\,dv}{1 + v^2} = -\int\frac{dx}{x} + c$$

or $\log(1 + v^2) = -\log x + \log c$

or $\log(1 + v^2) = \log\left(\frac{c}{x}\right)$

$$1 + v^2 = \frac{c}{x}$$

Replacing v by $\frac{y}{x}$, we get

$$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}$$

or $x^2 + y^2 = cx$

or $\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$

This general solution represents a family of circles with centres on the x-axis at $\left(\frac{c}{2}, 0\right)$ and radius $= \frac{c}{2}$, thus passing through origin at $x = 0$.

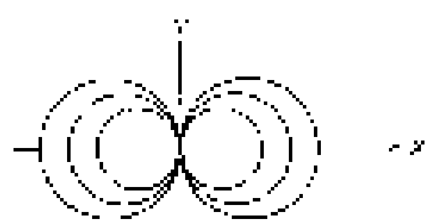


Fig. ... Solution ... circles

3.2.3.3 Linear Equations of First Order

[illegible] linear [illegible] in the first degree [illegible]

Thus the following [illegible]

1. $\frac{dy}{dx} + 4y = 2$ [illegible] 2. $x\frac{d^2y}{dx^2} + [illegible]\frac{dy}{dx} + 4y = 2$

[illegible] equation. The following [illegible]

1. $\left(\frac{dy}{dx}\right)^{2} + y = 5$ [illegible] $\frac{dy}{dx} + y^{2} = 2$ [illegible] 3. [illegible]

3.2.3.4 Leibnitz linear equation

The general form of a linear equation of the first order, [illegible] Leibnitz's linear equation, is

$$\frac{dy}{dx} + Py = Q \quad \text{where } P, Q \text{ are [illegible] functions of } x \qquad [illegible]$$

[illegible] the equation [illegible] by $e^{\int P dx}$ [illegible] we get

$$\frac{dy}{dx}e^{\int P dx} + y(Pe^{\int P dx}) = Qe^{\int P dx} \quad [illegible] \quad \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

[illegible] we get $ye^{\int P dx} = \int Qe^{\int P dx}dx + c$ [illegible]

NOTE

The factor $e^{\int P dx}$ [illegible] integrating factor (I.F.) of the linear equation (i). [illegible]

$$\text{I.F.} = e^{\int P dx}$$

and the solution is $\quad y(\text{I.F.}) = \int Q(\text{I.F.})dx + c$

3.2.3.5 Bernoulli's Equation

The equation $\frac{dy}{dx} + Py = Qy^{n}$ [illegible] (i)

where P, Q are functions of x [illegible] Bernoulli's equation.

[illegible] $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ [illegible] (ii)

Put $y^{1-n} = z$ [illegible] $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (ii) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

or $\qquad \frac{dz}{dx} + (1-n)Pz = (1-n)Q,$

which is Leibnitz's linear in z and can be solved easily.

Example

Solve $\frac{dy}{dx} = $ [illegible]

Solution.

Dividing throughout by [illegible]

$$y^{-3}\frac{dy}{dx} + \text{[illegible]} \quad \text{...(i)}$$

Put $y^{-2} = z$ so that $-2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (i) becomes $-\frac{1}{2}\frac{dz}{dx} + $ [illegible]

or $\frac{dz}{dx}$ [illegible]

which is Leibnitz's linear in z. ...(ii)

$$I.F. = e^{\int \text{[illegible]}\, dx} = \text{[illegible]}$$

$\therefore$ The solution of (ii) is $z(I.F.) = \int$ [illegible]

[illegible]

$\Rightarrow$ [illegible] $\quad (\because z = y^{-2})$

$\Rightarrow$ $y^{-2} = $ [illegible]

$\Rightarrow$ $y = $ [illegible]

3.2.3.6 Exact Differential Equations

1. **Definition.** A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if its left hand member is the exact differential of some function $u(x, y)$ i.e. $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.
2. **Theorem.** The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3. **Method of solution.** It can be shown that the equation $Mdx + Ndy = 0$ becomes

$$d\left[u + \int f(y)dy\right] = 0$$

Integrating $u + \int f(y)dy = c$.

But $\quad u = \int M dx$ and $f(y) =$ terms of N not containing x.

$\therefore$ The solution of $Mdx + Ndy = 0$ is

$\int_{(y\ \text{const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

provided of course, that the equation is exact, i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

NOTE: While finding $\int M dx$, y is treated as constant since we are integrating with respect to x.

Example:

Solve $(x^3 + 3xy^2)dx + (3x^2y - y^3)dy = 0$

Solution:

Step 1: Test for exactness

Here $M = x^3 + 3xy^2$ and $N = 3x^2y - y^3$

$$\therefore \quad \frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{y\,\text{const.}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

where $\int (x^3 + 3xy^2)dx + \int (-y^3)dy = c$

$$\Rightarrow \quad \frac{x^4}{4} + \frac{3x^2y^2}{2} - \frac{y^4}{4} = c$$

$$\Rightarrow \quad x^4 + 6x^2y^2 - y^4 = c$$

3.2.3.7 Equations Reducible To Exact Equations

Sometimes a differential equation which is not exact can be made so on multiplication by a suitable function called an integrating factor. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as given in theorems below.

In the equation $Mdx + Ndy = 0$

Theorem 1: If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x only $= f(x)$ say, then $e^{\int f(x)dx}$ is an integrating factor.

Theorem 2: If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only $= F(y)$ say, then $e^{\int F(y)dy}$ is an integrating factor.

Example 1.

Solve $(2\sin(y^2))dx + xy\cos(y^2)dy = 0$ $\quad y(2) = \sqrt{\frac{\pi}{2}}$

Solution:

Step 1: Here $M = 2\sin(y^2)$ and $N = xy\cos(y^2)$

Step 2: Test for exactness $\dfrac{\partial M}{\partial y} = 4y\cos(y^2)$ and $\dfrac{\partial N}{\partial x} = y\cos(y^2)$

$$\text{So} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

and hence, equation is not exact. So we have to find integrating factor by using either theorem 1 or theorem 2.

Step 3: Find an integrating factor by theorem 1

$$\text{Here} \quad \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{4y\cos y^2 - y\cos y^2}{xy\cos y^2} = \frac{3}{x}$$

[illegible]

$$[illegible]$$

Multiplying [illegible]

[illegible]

This equation [illegible]

Step 4: General solution

$$[illegible]$$

[illegible]

$$\frac{1}{2}[illegible]$$

Step 5: [illegible] solution of the initial value problem

Since [illegible]

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

So particular solution is [illegible]

[illegible]

Example 2

[illegible]

Solution:

Here [illegible]

$$[illegible]$$

[illegible] which is function of y only [illegible]

[illegible]

Multiplying throughout by [illegible] becomes [illegible] which is an exact equation

Its solution is [illegible]

3.2.4 Orthogonal Trajectories

3.2.4.1 Definitions

Two families of curves [illegible] angles [illegible] of each other.

[illegible]

For instance, [illegible]

[illegible]

Example 1.

Find the orthogonal trajectory of family of curves xy = Constant

Solution:

Given family of curves $\quad xy = c \quad \ldots(i)$

Differentiate w.r.t. x

$$x\frac{dy}{dx} + y = 0$$

Now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow \quad x\frac{dx}{dy} = y$$

By variable separable $\quad \int x\,dx = \int y\,dy$

$$\frac{x^2}{2} = \frac{y^2}{2} + k$$

$\Rightarrow \quad x^2 - y^2 = k$ is the orthogonal trajectory of given family of curves.

5.14.2 Orthogonal trajectory of polar curves

Example 2.

Find the orthogonal trajectory of family of curves $r^n = a^n \sin n\theta$

Solution:

Given family of curves $\quad r^n = a^n \sin n\theta \quad \ldots(i)$

Differentiate w.r.t. θ and eliminate a

$$nr^{n-1}\frac{dr}{d\theta} = na^n \cos n\theta \quad \ldots(ii)$$

Divide equation (ii) by equation (i)

$$\frac{nr^{n-1}\frac{dr}{d\theta}}{r^n} = \frac{na^n \cos n\theta}{a^n \sin n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta \quad \ldots(iii)$$

Differential equation represents given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left[-r^2\frac{d\theta}{dr}\right] = \cot n\theta$$

$$-r\frac{d\theta}{dr} = \cot n\theta$$

$$-\int\frac{1}{r}dr = \int \tan n\theta\, d\theta$$

$$\log r = -\frac{\log \sec n\theta}{n} + \log c$$

$$n\log r = \log[c^n \cos n\theta]$$

$$\Rightarrow \quad r^n = c^n \cos n\theta$$

is the required orthogonal trajectory.

3.2.4.3 Newton's Law of Cooling

Definitions

[illegible]

[illegible] $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

[illegible] $\int \frac{d\theta}{(\theta - \theta_0)} = \int -k\,dt$

$\Rightarrow$ $\log(\theta - \theta_0) = -kt + \log c$

$\theta - \theta_0 = ce^{-kt}$

[illegible]

Example 3.

[illegible]

Solution:

According to Newton's law of cooling

[illegible]

3.2.4.4 Law of Growth

[illegible]

[illegible]

Example 4.

[illegible] $\frac{1}{2}$ [illegible]

Solution:

[illegible] $\frac{dN}{dt} = kN$

[illegible] ...(1)

[illegible] ...(2)

[illegible] in equation (1)

Then, [illegible] $= 225$

3.2.4.5 Law of Decay

Definitions

[illegible]

$\therefore$ [illegible]

[illegible]

[illegible]

Example 5.

[illegible]

Solution:

[illegible] ...(1)

[illegible] ...(2)

[illegible]

[illegible] $\quad ...(iii)$

[illegible] (i)

[illegible]

[illegible]

$= 64.5$ days

3.3 Linear Differential Equations (Of n[th] Order)

3.3.1 Definitions

Linear differential equations [illegible] dependent variable [illegible] derivatives [illegible] [illegible]

$$[illegible] = 0$$

where [illegible]

Linear Differential Equations with Constant Coefficients [illegible]

$$[illegible]$$

where [illegible] [illegible]

1. Theorem: If [illegible] are only two solutions of the equation

$$[illegible] = 0 \quad ...(i)$$

Then [illegible] is also its solution.

[illegible] $\quad ...(ii)$

2. [illegible] arbitrary constants [illegible] [illegible] then [illegible] complete solution.

3. [illegible]

$$[illegible] \quad ...(i)$$

Then $\quad [illegible] = X \quad ...(ii)$

Adding (i) and (ii), we have $[illegible] = X$

Thus [illegible] complete solution of (iii).

The [illegible] complementary function (C.F.) [illegible]

Thus the complete solution (C.S.) of (i) is y = C.F. + P.I.

Thus to solve the equation (i) we first find the C.F. i.e. the complete primitive of the equation [illegible] (ii)

Operator D: Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ ... by D, D^2, D^3 ... so that

$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y, \frac{d^3y}{dx^3} = D^3y$ etc., the equation (i) above can be written in the symbolic form

$$(D^n + k_1D^{n-1} + \dots + k_n)y = X$$

i.e. $$f(D)y = X$$

where $f(D) = D^n + k_1D^{n-1} + \dots + k_n$, i.e. a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e. $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)y$$
$$= (D + 3)(D - 1)y \text{ or } (D - 1)(D + 3)y$$

3.3.2 Rules for Finding The Complementary Function

To solve the equation $\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + k_ny = 0$ ...(i)

where k's are constants.

The equation (i) in symbolic form is

$(D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n)y = 0$ ...(ii)

Its symbolic coefficient equated to zero i.e.

$D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n = 0$

is called the auxiliary equation (A.E.). Let $m_1, m_2, \dots, m_n$ be its roots. Consider following cases.

Case I. If all the roots be real and different, then (ii) is equivalent to

$(D - m_1)(D - m_2) \dots (D - m_n)y = 0$ ...(iii)

Now (iii) will be satisfied by the solution of $(D - m_n)y = 0$, i.e. by $\frac{dy}{dx} - m_ny = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_nx}$

∴ its solution is $ye^{-m_nx} = c_n$ i.e. $y = c_ne^{m_nx}$

Similarly since the factors in (iii) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)y = 0, (D - m_2)y = 0$ etc. i.e. by $y = c_1e^{m_1x}, y = c_2e^{m_2x}$ etc.

Thus the complete solution of (i) is $y = c_1e^{m_1x} + c_2e^{m_2x} + \dots + c_ne^{m_nx}$ ...(iv)

Case II. If two roots are equal (i.e. $m_1 = m_2$), then (iv) becomes

$$y = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$$
$$y = Ce^{m_1x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$$

[∵ $c_1 + c_2$ = one arbitrary constant C]

has only $n-1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows:

The part of the complete solution corresponding to the repeated root is the complete solution of $(D-m_1)^2 y = 0$.

Putting $(D-m_1)y = z$, it becomes $(D-m_1)z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$

$\therefore$ its solution is $z e^{-m_1 x} = c_2$ or $z = c_2 e^{m_1 x}$

Thus $\qquad (D-m_1)y = z = c_2 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_2 e^{m_1 x}$ ...(3)

Its I.F. being $e^{-m_1 x}$, the solution of (3) is

$$y e^{-m_1 x} = \int c_2 e^{m_1 x} e^{-m_1 x} dx + c_1 = c_2 x + c_1$$

$$\Rightarrow \qquad y = (c_1 + c_2 x) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (i.e., $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

Case III. If one pair of roots be imaginary, i.e.

$$m_1 = \alpha + i\beta$$
$$m_2 = \alpha - i\beta$$

then the complete solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

[$\because$ by Euler's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$]

$$= e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

where $\qquad C_1 = c_1 + c_2$

and $\qquad C_2 = i(c_1 - c_2)$

Case IV. If two pairs of imaginary roots be equal i.e.

$$m_1 = m_2 = \alpha + i\beta$$
$$m_3 = m_4 = \alpha - i\beta$$

then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + \dots + c_n e^{m_n x}$$

3.3.3 Inverse Operator

1. Definition: $\frac{1}{f(D)} X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e. $\qquad f(D)\left\{\frac{1}{f(D)} X\right\} = X$

Thus $y = \frac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and is, therefore, its particular integral.

Obviously, $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

2.
$$\frac{1}{D}X = \int X\,dx$$

Let
$$\frac{1}{D}X = y$$

Operating by D,
$$D\frac{1}{D}X = Dy$$

i.e.
$$X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x,
$$y = \int X\,dx$$

Thus
$$\frac{1}{D}X = \int X\,dx$$

3.
$$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx$$

Let
$$\frac{1}{D-a}X = y \quad \text{...(i)}$$

Operating by $D-a$,
$$(D-a)\frac{1}{D-a}X = (D-a)y$$

or
$$X = \frac{dy}{dx} - ay \quad \text{i.e.} \quad \frac{dy}{dx} - ay = X$$

which is a Leibnitz's linear equation.
∴ I.F. being e^{-ax}, its solution is
$$ye^{-ax} = \int Xe^{-ax}\,dx$$

no constant being added as (i) doesn't contain any constant.

Thus
$$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}\,dx$$

3.3.4 Rules For Finding The Particular Integral

Consider the equation $\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + k_ny = X$

Which in symbolic form is $(D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n)y = X$

∴
$$PI = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n}X$$

Case I. When $X = e^{ax}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
$$\dots\dots\dots$$
$$D^ne^{ax} = a^ne^{ax}$$

[illegible]

[illegible]

[illegible]

Example 1. [illegible]

[illegible]

Solution:

[illegible]

[illegible]

Example 2. [illegible]

[illegible]

Solution:

[illegible]

Case II. When $X = \sin(ax + b)$ or $\cos(ax + b)$.

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \text{ provided } f(-a^2) \neq 0 \quad \ldots(i)$$

If $f(-a^2) = 0$, the above rule fails and we proceed further.

$$\therefore \quad \frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b), \text{ provided } f'(-a^2) \neq 0 \quad \ldots(ii)$$

If $f'(-a^2) = 0$, $\dfrac{1}{f(D^2)}\sin(ax+b) = x^2\dfrac{1}{f''(-a^2)}\sin(ax+b)$, provided $f''(-a^2) \neq 0$ and so on.

Similarly, $$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \text{ provided } f(-a^2) \neq 0$$

If $f(-a^2) = 0$, $\dfrac{1}{f(D^2)}\cos(ax+b) = x\dfrac{1}{f'(-a^2)}\cos(ax+b)$, provided $f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, $\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\dfrac{1}{f''(-a^2)}\cos(ax+b)$, provided $f''(-a^2) \neq 0$ and so on.

Example 1. Solve

$$(D^2 + 4)y = \sin 2x$$

Solution:

$$(D^2 + 4)y = \sin 2x$$

Auxiliary equation is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$$C.F. = A\cos 2x + B\sin 2x$$

$$P.I. = \frac{1}{D^2+4}\sin 2x = \frac{-x\cos 2x}{2 \times 2} = -\frac{1}{4}x\cos 2x$$

Complete solution is $y = A\cos 2x + B\sin 2x - \dfrac{x}{4}\cos 2x$

Example 2. Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos 2x$$

Solution:

$$(D^2 + D + 1)y = \cos 2x$$

Auxiliary equation is $D^2 + D + 1 = 0$

$$D = \frac{-1 \pm \sqrt{3}i}{2} \quad \therefore C.F. = e^{-x/2}\left[A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right]$$

$$P.I. = \frac{1}{D^2 + D + 1}\cos 2x$$

$$= \frac{1}{(-2^2) + D + 1}\cos 2x = \frac{1}{D - 3}\cos 2x$$

$$= \frac{D+3}{D^2-9}\cos 2x = \frac{D+3}{(-2^2)-9}\cos 2x$$

$$= -\frac{1}{13}(D+3)\sin 2x = -\frac{1}{13}(\text{[illegible]})$$

Complete solution is

$$y = e^{-x/2}\left[A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right] - \frac{1}{13}(\text{[illegible]})$$

Example 5.5 Solve

$$(D^2+4)y = \cos 2x$$

Solution:

$$(D^2+4)y = \cos 2x$$

Auxiliary eq. is $\quad D^2+4=0$

$$\text{C.F.} = A\cos 2x + B\sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cos 2x = x\,\frac{1}{2D}\cos 2x = \frac{x}{2}\left(\frac{\sin 2x}{2}\right) = \frac{x}{4}\sin 2x$$

Complete solution is $\quad y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$

Case III. When $X = x^m$

Here $\quad \text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 1. Solve

the D.E. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

Solution:

Given equation in symbolic form is $(D^2+D)y = x^2+2x+4$

$$\therefore \quad \text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2-\ldots)(x^2+2x+4)$$

$$= \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3}+4x$$

Case IV. When $X = e^{ax}V$, V being a function of x.

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^2e^{ax}u + 2ae^{ax}Du + e^{ax}D^2u = e^{ax}(D+a)^2u$$

and in general $\quad D^n(e^{ax}u) = e^{ax}(D+a)^n u$

$$\therefore \quad f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both sides by $f(D)$,

$$f(D)\cdot\frac{1}{f(D)}(e^{ax}V) = \frac{1}{f(D)}e^{ax}f(D+a)X$$

$$e^{ax}V = \frac{1}{f(D)}\left[e^{ax}f(D+a)X\right]$$

Whence
$$f(D+a)X = V$$

or
$$X = \frac{1}{f(D+a)}V$$

and so
$$e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V)$$

i.e.
$$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$$

Example 1. Solve

$$(D^2-4D+4)y = x^3e^{2x}$$

Solution:

$$(D^2-4D+4)y = x^3e^{2x}$$

A.E.
$$D^2-4D+4 = 0,\ (D-2)^2 = 0 \Rightarrow D = 2, 2$$

$$CF = (C_1+C_2x)e^{2x}$$

$$PI = \frac{1}{D^2-4D+4}x^3e^{2x} = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}x^3$$

$$= e^{2x}\frac{1}{D^2}x^3 = e^{2x}\frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x}\frac{x^5}{20}$$

$$y = (C_1+C_2x)e^{2x} + e^{2x}\cdot\frac{x^5}{20}$$

Example 2. Solve

$$(D^2-5D+6)y = e^x\cos 2x$$

Solution:

$$(D^2-5D+6)y = e^x\cos 2x$$

$$D^2-5D+6 = 0$$

$$(D-2)(D-3) = 0 \text{ or } D = 2, 3$$

$$CF = C_1e^{2x}+C_2e^{3x}$$

$$PI = \frac{1}{D^2-5D+6}e^x\cos 2x$$

$$e^x\frac{1}{(D+1)^2-5(D+1)+6}\cos 2x$$

$$= e^x\frac{1}{D^2-3D+2}\cos 2x = e^x\frac{1}{-4-3D+2}\cos 2x$$

$$= e^x\frac{1}{-3D-2}\cos 2x = -e^x\frac{3D-2}{9D^2-4}\cos 2x$$

$$= -e^x\frac{3D-2}{9(-4)-4}\cos 2x = \frac{e^x}{40}(3D-2)\cos 2x$$

$$= \frac{e^x}{40}(2\sin 2x - 2\cos 2x) = \frac{e^x}{20}(\sin 2x - \cos 2x)$$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{e^x}{20}(\sin 2x - \cos 2x)$$

Case V. When X is any other function of x.

Here $\quad P.I. = \dfrac{1}{f(D)} X$

If $\quad f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$$\therefore \quad P.I. = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right] X$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \ldots + A_n \frac{1}{D - m_n} X$$

$$= A_1 e^{m_1 x} \int X e^{-m_1 x} dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx + \ldots + A_n e^{m_n x} \int X e^{-m_n x} dx$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

3.3.5 Summary: Working Procedure to Solve The Equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy}{dx} + k_n y = X$$

i.e. in symbolic form

$$(D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n) y = X$$

Step I. To Find the Complementary Function

1. Write the A.E.

$$D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n = 0$$

2. Write the C.F. as follows

Roots of A.E.	C.F.
1. $m_1, m_2, m_3 \ldots$ real and different roots	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2. $m_1, m_1, m_3 \ldots$ two real and equal roots	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
3. $m_1, m_1, m_1, m_4 \ldots$ three real and equal roots	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3 \ldots$ a pair of imaginary roots	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \ldots$ 2 pairs of equal imaginary roots	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots$

Step II: To Find the Particular Integral

From symbolic form $\quad PI = \dfrac{1}{D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n} X$

$$= \frac{1}{f(D)} X$$

1. When $\quad X = e^{ax}$

$$PI = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $\quad f'(D)$ = differentiation of $f(D)$ w.r.t. D

$f''(D)$ = differentiation of $f'(D)$ w.r.t. D etc.

2. When $X = \sin(ax + b)$ or $\cos(ax + b)$

$$PI = \frac{1}{f(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $\quad D^2 = -a^2, [f(-a^2) \neq 0]$

$$= x \frac{1}{f'(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $\quad D^2 = -a^2, [f(-a^2) = 0, f'(-a^2) \neq 0]$

$$= x^2 \frac{1}{f''(D^2)} \sin(ax + b) \qquad [\text{or } \cos(ax + b)]$$

put $\quad D^2 = -a^2, [f'(-a^2) = 0, f''(-a^2) \neq 0]$

and so on.

where $\quad f'(D^2)$ = diff. coeff. of $f(D^2)$ w.r.t. D

$f''(D^2)$ = diff. coeff. of $f'(D^2)$ w.r.t. D etc.

3. When $\quad X = x^m$, m being positive integer

$$PI = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

4. When $X = e^{ax} V$, where V is a function of x.

$$PI = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D + a)} V$$

and then evaluate $\dfrac{1}{f(D + a)} V$ as in cases 1, 2 and 3.

5. When X is any function of x:

$$\text{P.I.} = \frac{1}{f(D)}X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx$$

Step III. To find the complete solution

Then the C.S. is $y = \text{C.F.} + \text{P.I.}$

3.4 Two Other Methods of Finding P.I.

3.4.1 Method of Variation of Parameters

This method is quite general and applies to equations of the form

$$y'' + py' + qy = X \qquad \ldots(i)$$

where p, q and X are functions of x. It gives

$$\text{P.I.} = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx \qquad \ldots(ii)$$

where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ $\qquad \ldots(iii)$

and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$ is called the Wronskian of y_1, y_2.

Example 1.

Using the method of variation of parameters, solve

$$y'' + y = \sec x$$

Solution.

Given equation in symbolic form is $(D^2 + 1)y = \sec x$

(i) To find C.F.

Its A.E. is $\quad D^2 + 1 = 0$

$\therefore \quad D = \pm i$

Thus C.F. is $\quad y = c_1 \cos x + c_2 \sin x$

(ii) To find P.I.

Here $y_1 = \cos x$, $y_2 = \sin x$ and $X = \sec x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Thus

$$\text{P.I.} = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx$$

$$= -\cos x\int \frac{\sin x \sec x}{1}\,dx + \sin x\int \frac{\cos x \sec x}{1}\,dx$$

$$= -\cos x\int \tan x\,dx + \sin x\int dx$$

$$= \cos x \log \cos x + x \sin x$$

Hence the C.S. is

$$y = c_1 \cos x + c_2 \sin x + \cos x \log \cos x + x \sin x$$

3.5 Equations Reducible to Linear Equation with Constant Coefficient

Definitions

[illegible]

Let [illegible]

Now [illegible]

[illegible]

Similarly [illegible]

[illegible]

Example 1.

[illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

Solution: (a)

[illegible]

Solution of differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

$y = x^2$ is one of the solution

Alternate Solution:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

$$(x^2D^2 + xD - 4)y = 0$$

$$[\theta(\theta - 1) + \theta - 4]y = 0$$

$$(\theta^2 - \theta + \theta - 4)y = 0$$

$$(\theta^2 - 4)y = 0$$

Auxiliary equation $m^2 - 4 = 0$

$$m = \pm 2$$

CF $= c_1e^{2z} + c_2e^{-2z}$

Solution is $y = c_1e^{2z} + c_2e^{-2z} = c_1x^2 + c_2x^{-2}$

One of the independent solution is x^2

■■■■

Previous GATE and ESE Questions

Q.1 The solution of the differential equation

$\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x + c}$

(b) $y = \frac{-x^3}{3} + c$

(c) ce^x

(d) unsolvable as equation is non-linear

[ME, GATE 2005, 2 marks]

Q.2 Biomass of a certain organic compound having concentration $x(t)$ can be modeled using an ordinary differential equation $\frac{dx}{dt} + kx^2 = 0$, where k is the reaction rate constant. If $x = a$ at $t = 0$ the solution of the equation is

(a) $x = ae^{-kt}$ (b) $\frac{1}{x} = \frac{1}{a} + kt$

(c) $x = a(1 - e^{-kt})$ (d) $x = a + kt$

[CE, GATE 2004, 2 marks]

Q.3 The following differential equation has

$$3\left(\frac{d^2y}{dt^2}\right) + 4\left(\frac{dy}{dt}\right)^3 + y^2 + 2 = x$$

(a) degree = 2, order = 1

(b) degree = 1, order = 2

(c) degree = 4, order = 3

(d) degree = 2, order = 3

[EC, GATE 2005, 1 mark]

Q.4 The solution of the first order differential equation $\dot{x}(t) = -3x(t)$, $x(0) = x_0$ is

(a) $x(t) = x_0e^{-3t}$ (b) $x(t) = x_0e^{-3}$

(c) $x(t) = x_0e^{-1/3}$ (d) $x(t) = x_0e^{-1}$

[EE, GATE 2005, 1 mark]

Q.5 The transformation to linear form by substituting $v = y^{1-n}$ of the equation

$\frac{dy}{dt} + p(t)y = q(t)y^n$; $n > 0$ will be

(a) $\frac{dv}{dt} + (1 - n)pv = (1 - n)q$

(b) $\frac{dv}{dt} + (1 - n)pv = (1 + n)q$

(c) $\frac{dv}{dt} + (1 + n)pv = (1 - n)q$

(d) $\frac{dv}{dt} + (1 + n)pv = (1 + n)q$

[CE, GATE 2005, 2 marks]

Q.6 The solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$; $y(0) = 1$, $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 0$ in the range $0 < x < \frac{\pi}{4}$ is given by

(a) $e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)$

(b) $e^{x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

(c) $e^{-4x}\left(\cos x - \frac{1}{4}\sin x\right)$

(d) $e^{-4x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

[CE, GATE-2005, 2 marks]

Statement for Linked Answer Questions 7 and 8

The complete solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \text{ is } y = c_1e^{-x} + c_2e^{-3x}$$

Q.7 Then p and q are

(a) $p = 3, q = 3$
(b) $p = 3, q = 4$
(c) $p = 4, q = 3$
(d) $p = 4, q = 4$

[ME, GATE-2005, 2 marks]

Q.8 Which of the following is a solution of the differential equation $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q+1)y = 0$?

(a) e^{-3x}
(b) xe^{-x}
(c) xe^{-2x}
(d) x^2e^{-2x}

[ME, GATE-2005, 2 marks]

Q.9 A solution of the following differential equation is given by $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

(a) $y = e^{2x} + e^{-3x}$
(b) $y = e^{2x} + e^{3x}$
(c) $y = e^{-2x} + e^{3x}$
(d) $y = e^{-2x} + e^{-3x}$

[EC, GATE-2005, 1 mark]

Q.10 A spherical naphthalene ball exposed to the atmosphere loses volume at a rate proportional to its instantaneous surface area due to evaporation. If the initial diameter of the ball is 2 cm and the diameter reduces to 1 cm after 3 months, the ball completely evaporates in

(a) 6 months
(b) 9 months
(c) 12 months
(d) infinite time

[CE, GATE-2006, 2 marks]

Q.11 The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1 + x)e^{+x^2}$
(b) $(1 + x)e^{-x^2}$
(c) $(1 - x)e^{+x^2}$
(d) $(1 - x)e^{-x^2}$

[ME, GATE-2006, 1 mark]

Q.12 For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$
(b) $\frac{1}{5}e^{2x}$
(c) $3e^{2x}$
(d) $C_1e^{-x} + C_2e^{-3x}$

[ME, GATE-2006, 2 marks]

Q.13 The degree of the differential equation $\frac{d^2x}{dt^2} + 2x^3 = 0$ is

(a) 0
(b) 1
(c) 2
(d) 3

[CE, GATE-2007, 1 mark]

Q.14 The solution for the differential equation $\frac{dy}{dx} = x^2y$ with the condition that $y = 1$ at $x = 0$ is

(a) $y = e^{\frac{1}{2x}}$
(b) $\ln(y) = \frac{x^3}{3} + 4$
(c) $\ln(y) = \frac{x^2}{2}$
(d) $y = e^{\frac{x^3}{3}}$

[CE, GATE-2007, 1 mark]

Q.15 The solution of $x\frac{dy}{dx} + y = x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y = \frac{x^4}{5} + \frac{1}{x}$
(b) $y = \frac{4x^4}{5} + \frac{4}{5x}$
(c) $y = \frac{x^4}{5} + 1$
(d) $y = \frac{x^5}{5} + 1$

[ME, GATE-2007, 2 marks]

Q.16 A body originally at 60°C cools down to 40°C in 15 minutes when kept in air at a temperature of 25°C. What will be the temperature of the body at the end of 30 minutes?

(a) 35.2°C
(b) 31.5°C
(c) 28.7°C
(d) 15°C

[CE, GATE-2007, 2 marks]

Q.17 Solution of $\frac{dy}{dx} = -\frac{x}{y}$ at $x = 1$ and $y = \sqrt{3}$ is

(a) $x - y^2 = -2$
(b) $x + y^2 = 4$
(c) $x^2 - y^2 = -2$
(d) $x^2 + y^2 = 4$

[CE, GATE-2008, 2 marks]

Q.18 Which one of the following is a solution to the differential equation $\frac{dx(t)}{dt} + 3x(t) = 0$?

(a) $x(t) = 3e^{-t}$ (b) $x(t) = 2e^{-3t}$

(c) $x(t) = -\frac{3}{2}t^2$ (d) $x(t) = 3t^2$

[EC, GATE-2008, 1 mark]

Q.19 The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

(a) $y = P\cos x + Q\sin x$

(b) $y = P\cos x$

(c) $y = P\sin x$

(d) $y = P\sin^2 x$

[CE, GATE-2008, 1 mark]

Q.20 Given that $\ddot{x} + 3x = 0$, and $x(0) = 1$, $\dot{x}(0) = 0$, what is $x(1)$?

(a) −0.99 (b) −0.16

(c) 0.16 (d) 0.99

[ME, GATE-2008, 1 mark]

Q.21 It is given that $y'' + 2y' + y = 0$, $y(0) = 0$, $y(1) = 0$. What is $y(0.5)$?

(a) 0 (b) 0.37

(c) 0.62 (d) 1.13

[ME, GATE-2008, 2 marks]

Q.22 The order of the differential equation

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$$ is

(a) 1 (b) 2

(c) 3 (d) 4

[EC, GATE-2009, 1 mark]

Q.23 Solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of

(a) ellipses (b) circles

(c) parabolas (d) hyperbolas

[CE, GATE-2009, 2 marks]

Q.24 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I

A. $\frac{dy}{dx} = \frac{y}{x}$

B. $\frac{dy}{dx} = -\frac{y}{x}$

C. $\frac{dy}{dx} = \frac{x}{y}$

D. $\frac{dy}{dx} = -\frac{x}{y}$

List-II

1. Circles

2. Straight lines

3. Hyperbolas

Codes:

	A	B	C	D
(a)	2	3	3	1
(b)	1	3	2	1
(c)	2	1	3	3
(d)	3	2	1	2

[EC, GATE-2009, 2 marks]

Q.25 The solution of $x\frac{dy}{dx} + y = x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y = \frac{x^4}{5} + \frac{1}{x}$

(b) $y = \frac{4x^4}{5} + \frac{4}{5x}$

(c) $y = \frac{x^4}{5} + 1$ (d) $y = \frac{x^5}{5} + 1$

[ME, GATE-2009, 2 marks]

Q.26 The order and degree of the differential equation

$$\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3} + y^2 = 0$$ are respectively

(a) 3 and 2 (b) 2 and 3

(c) 3 and 3 (d) 3 and 1

[CE, GATE-2010, 1 mark]

Q.27 The Blasius equation, $\frac{d^3f}{d\eta^3} + \frac{f}{2}\frac{d^2f}{d\eta^2} = 0$, is a

(a) second order nonlinear ordinary differential equation

(b) third order nonlinear ordinary differential equation

(c) third order linear ordinary differential equation

(d) mixed order nonlinear ordinary differential equation

[ME, GATE-2010, 1 mark]

Q.28 The solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$ is

(a) $y = c_1e^{3x} + c_2e^{-2x}$

(b) $y = c_1e^{3x} + c_2e^{2x}$

(c) $y = c_1e^{-3x} + c_2e^{2x}$

(d) $y = c_1e^{-3x} + c_2e^{-2x}$

[CE, GATE-2010, 2 marks]

Q.29 For the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$, the solution is
(a) $x(t) = 2e^{-6t} - e^{-2t}$ (b) $x(t) = 2e^{-2t} - e^{-4t}$
(c) $x(t) = -e^{-6t} + 2e^{-4t}$ (d) $x(t) = e^{-2t} + 2e^{-4t}$
[EE, GATE-2010, 2 marks]

Q.30 A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$ where L is a constant. The boundary conditions are: $n(0) = K$ and $n(\infty) = 0$. The solution to this equation is
(a) $n(x) = K\exp(x/L)$
(b) $n(x) = K\exp(-x/\sqrt{L})$
(c) $n(x) = K^2\exp(-x/L)$
(d) $n(x) = K\exp(-x/L)$
[EC, GATE-2010, 1 mark]

Q.31 Consider the differential equation $\frac{dy}{dx} = (1 + y^2)x$. The general solution with constant c is
(a) $y = \tan\frac{x^2}{2} + \tan c$
(b) $y = \tan^2\left(\frac{x}{2} + c\right)$
(c) $y = \tan^2\left(\frac{x}{2}\right) + c$
(d) $y = \tan\left(\frac{x^2}{2} + c\right)$
[ME, GATE-2011, 2 marks]

Q.32 With K as a constant, the possible solution for the first order differential equation $\frac{dy}{dx} = e^{-3x}$ is
(a) $-\frac{1}{3}e^{-3x} + K$ (b) $-\frac{1}{3}e^{3x} + K$
(c) $-3e^{-3x} + K$ (d) $-3e^{-x} + K$
[EE, GATE-2011, 1 mark]

Q.33 The solution of the differential equation $\frac{dy}{dx} = ky$, $y(0) = c$ is
(a) $x = ce^{-ky}$ (b) $x = ke^{cy}$
(c) $y = ce^{kx}$ (d) $y = ce^{-kx}$
[EC, GATE-2011, 1 mark]

Q.34 The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x$, with the condition that $y = 1$ at $x = 1$, is
(a) $y = \frac{2}{3x^2} + \frac{x}{3}$ (b) $y = \frac{x}{2} + \frac{1}{2x}$
(c) $y = \frac{2}{3} + \frac{x}{3}$ (d) $y = \frac{2}{3x} + \frac{x^2}{3}$
[CE, GATE-2011, 2 marks]

Q.35 The solution of the ordinary differential equation $\frac{dy}{dx} + 2y = 0$ for the boundary condition $y = 5$ at $x = 1$ is
(a) $y = e^{-2x}$ (b) $y = 2e^{-2x}$
(c) $y = 10.95e^{-2x}$ (d) $y = 36.95e^{-2x}$
[CE, GATE-2012, 2 marks]

Q.36 With initial condition $x(1) = 0.5$, the solution of the differential equation $t\frac{dx}{dt} + x = t$ is
(a) $x = t - \frac{1}{2}$ (b) $x = t^2 - \frac{1}{2}$
(c) $x = \frac{t^2}{2}$ (d) $x = \frac{t}{2}$
[EC, EE, IN, GATE-2012, 1 mark]

Q.37 The partial differential equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ is a
(a) linear equation of order 2
(b) non-linear equation of order 1
(c) linear equation of order 1
(d) non-linear equation of order 2
[ME, GATE-2013, 1 Mark]

Q.38 The type of the partial differential equation $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is
(a) Parabolic (b) Elliptic
(c) Hyperbolic (d) Nonlinear
[IN, GATE-2013, 1 mark]

Q.39 The solution to the differential equation $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$ where k is a constant, subjected to the boundary conditions $u(0) = 0$ and $u(L) = U$, is

(a) $u = U\dfrac{x}{L}$ (b) $u = U\left(\dfrac{1-e^{kx}}{1-e^{kL}}\right)$

(c) $u = U\left(\dfrac{1-e^{-kx}}{1-e^{-kL}}\right)$ (d) $u = U\left(\dfrac{1+e^{kx}}{1+e^{kL}}\right)$

[ME, GATE-2013 : 2 Marks]

Q.40 The maximum value of the solution $y(t)$ of the differential equation $y(t) + \ddot{y}(t) = 0$ with initial conditions $\dot{y}(0) = 1$ and $y(0) = 1$, for $t \geq 0$ is

(a) 1 (b) 2

(c) π (d) $\sqrt{2}$

[IN, GATE-2013 : 2 Marks]

Q.41 A system is described by a linear, constant coefficient, causal, first-order differential equation has an exact solution given by $y(t)$ for $t > 0$, when the forcing function is $x(t)$ and the initial condition is $y(0)$. If one wishes to modify the system so that the solution becomes $-2y(t)$ for $t > 0$, we need to

(a) change the initial condition to $-y(0)$ and the forcing function to $2x(t)$

(b) change the initial condition to $2y(0)$ and the forcing function to $-x(t)$

(c) change the initial condition to $j\sqrt{2}\,y(0)$ and the forcing function to $j\sqrt{2}\,x(t)$

(d) change the initial condition to $-2y(0)$ and the forcing function to $-2x(t)$

[EC, GATE-2013 : 2 Marks]

Q.42 The matrix form of the linear system $\dfrac{dx}{dt} = 3x - 5y$ and $\dfrac{dy}{dt} = 4x + 8y$ is

(a) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(b) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(c) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(d) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

[ME, GATE-2014 : 1 Mark]

Q.43 The general solution of the differential equation $\dfrac{dy}{dx} = \cos(x + y)$, with c as a constant, is

(a) $y + \sin(x + y) = x + c$

(b) $\tan\left(\dfrac{x+y}{2}\right) = y + c$

(c) $\cos\left(\dfrac{x+y}{2}\right) = x + c$

(d) $\tan\left(\dfrac{x+y}{2}\right) = x + c$

[ME, GATE-2014 : 2 Marks]

Q.44 The solution of the initial value problem $\dfrac{dy}{dx} = -2xy;\ y(0) = 2$ is

(a) $1 + e^{-x^2}$ (b) $2e^{-x^2}$

(c) $1 + e^{x^2}$ (d) $2e^{x^2}$

[ME, GATE-2014 : 1 Mark]

Q.45 Which ONE of the following is a linear non-homogeneous differential equation, where x and y are the independent and dependent variables respectively?

(a) $\dfrac{dy}{dx} + xy = e^{-x}$ (b) $\dfrac{dy}{dx} + xy = 0$

(c) $\dfrac{dy}{dx} + xy = e^{-y}$ (d) $\dfrac{dy}{dx} + e^{-y} = 0$

[EC, GATE-2014 : 1 Mark]

Q.46 The solution for the differential equation $\dfrac{d^2x}{dt^2} = -9x$, with initial conditions $x(0) = 1$ and $\left.\dfrac{dx}{dt}\right|_{t=0} = 1$, is

(a) $t^2 + t + 1$ (b) $\sin 3t + \dfrac{1}{3}\cos 3t + \dfrac{2}{3}$

(c) $\dfrac{1}{3}\sin 3t + \cos 3t$ (d) $\cos 3t + t$

[EE, GATE-2014 : 1 Mark]

Q.47 Consider two solutions $x(t) = x_1(t)$ and $x(t) = x_2(t)$ of the differential equation $\dfrac{d^2x(t)}{dt^2} + x(t) = 0,\ t > 0$, such that $x_1(0) = 1$, $\left.\dfrac{dx_1(t)}{dt}\right|_{t=0} = 0$, ... The Wronskian

$$W[f] = \begin{vmatrix} y_1(t) & y_2(t) \\ \frac{dy_1(t)}{dt} & \frac{dy_2(t)}{dt} \end{vmatrix} \text{ at } t = 0 \text{ is}$$

(a) 1 (b) −1
(c) 0 (d) $\pi/2$

[ME, GATE-2014 : 2 Marks]

Q.48 The characteristic equation of the differential equation $\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots. Then the values of α are

(a) ± 1 (b) 0, 0
(c) $\pm j$ (d) $\pm 1/2$

[EC, GATE-2014 : 1 Mark]

Q.49 Which ones of the following is a homogeneous solution of linear differential equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 3t$

(a) $3e^{-t}$ (b) $ae^{-t} - 3e^{-3t}$
(c) $ae^{-t} + be^{-3t}$ (d) ae^{-t}

[EC, GATE-2014 : 1 Mark]

Q.50 Consider the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$. Which of the following is a solution to this differential equation for $x > 0$?

(a) e^x (b) x^2
(c) $1/x$ (d) $\ln x$

[EE, GATE-2014 : 1 Mark]

Q.51 Consider the following differential equation:
$\frac{dy}{dt} = -5y$; initial condition: $y = 2$ at $t = 0$
The value of y at $t = 3$ is

(a) $-5e^{-10}$ (b) $2e^{-15}$
(c) $2e^{-10}$ (d) $-15e^{2}$

[ME, GATE-2015 : 2 Marks]

Q.52 Consider the following differential equation:

$$x(y\,dx + x\,dy)\cos\frac{y}{x} = y(x\,dy - y\,dx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant)?

(a) $\frac{x}{y}\cos\frac{y}{x} = c$ (b) $\frac{x}{y}\sin\frac{y}{x} = c$

(c) $xy\cos\frac{y}{x} = c$ (d) $xy\sin\frac{y}{x} = c$

[CE, GATE-2015 : 2 Marks]

Q.53 Consider the following second-order linear differential equation

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

The boundary conditions are: at $x = 0$, $y = 5$ and $x = 2$, $y = 21$.
The value of y at $x = 1$ is ________.

[CE, GATE-2015 : 2 Marks]

Q.54 A differential equation $\frac{di}{dt} - 0.2i = 0$ is applicable over $-10 < t < 10$. If $i(4) = 10$, then $i(-5)$ is ______

[EE, GATE-2015 : 2 Marks]

Q.55 The general solution of the differential equation

$$\frac{dy}{dx} = \frac{1 + \cos 2y}{1 - \cos 2x}$$ is

(a) $\tan y - \cot x = c$ (c is a constant)
(b) $\tan x - \cot y = c$ (c is a constant)
(c) $\tan y + \cot x = c$ (c is a constant)
(d) $\tan x + \cot y = c$ (c is a constant)

[EC, GATE-2015 : 1 Mark]

Q.56 A solution of the ordinary differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is such that $y(0) = 2$ and $y(1) = -\frac{1 - 3e}{e^3}$. The value of $\frac{dy}{dt}(0)$ is ______

[EE, GATE-2015 : 2 Marks]

Q.57 The solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \text{ with } y(0) = y'(0) = 1 \text{ is}$$

(a) $(2 - t)e^{t}$ (b) $(1 + 2t)e^{-t}$
(c) $(2 + t)e^{-t}$ (d) $(1 - 2t)e^{t}$

[EC, GATE-2015 : 2 Marks]

Q.58 Consider the differential equation $\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 0$. Given $x(0) = 20$ and $x(1) = 10/e$, where $e = 2.718$, the value of $x(2)$ is ______

[EC, GATE-2015 : 2 Marks]

Q.59 Find the solution of $\frac{d^2y}{dx^2} = y$ which [illegible] [illegible] [illegible]

[illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2015 : 2 Marks]

Q.60 A function $y(x)$ [illegible] [illegible] is [illegible] of the differential equation

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$ [illegible] $= 0$. Then $y(2)$ is

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2016 : 1 Mark]

Q.61 The solution of the differential equation, [illegible] [illegible] with initial conditions [illegible] [illegible] [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2016 : 1 Mark]

Q.62 Let $y(x)$ be the solution of the differential equation [illegible] with initial conditions $y(0) = 0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 1$. Then the value of $y(1)$ is ______.

[EE, GATE-2016 : 2 Marks]

Q.63 The solution of the differential equation [illegible] given below is:

[illegible] and

$\left.\frac{dy}{dx}\right|_{x=0} = [illegible]$

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EC, GATE-2016 : 2 Marks]

Q.64 If $y = f(x)$ satisfies the boundary value problem [illegible] [illegible]

[ME, GATE-2016 : 2 Marks]

Q.65 [illegible] [illegible] [illegible]

[illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[CE, GATE-2016 : 2 Marks]

Q.66 Consider the differential equation [illegible] with initial conditions [illegible] and [illegible] The value of y at $x = 1$ is [illegible]

[ME, GATE-2017 : 2 Marks]

Q.67 The differential equation $\frac{d^2y}{dx^2} + 16y = 0$ for $y(x)$

with the two boundary conditions [illegible]

and [illegible] has

(a) no solution
(b) exactly two solutions
(c) exactly one solution
(d) infinitely many solutions

[ME, GATE-2017 : 1 Mark]

Q.68 Consider the differential equation [illegible] [illegible] [illegible]

[illegible] exists in the interval

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[EE, GATE-2017 : 2 Marks]

Q.69 The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 5y = 0$$

in terms of arbitrary constants K_1 and K_2 is

(a) $K_1 e^{(-1+\sqrt{6})x} + K_2 e^{(-1-\sqrt{6})x}$

(b) $K_1 e^{(-1+\sqrt{8})x} + K_2 e^{(-1-\sqrt{8})x}$

(c) $K_1 e^{(-2+\sqrt{6})x} + K_2 e^{(-2-\sqrt{6})x}$

(d) $K_1 e^{(-2+\sqrt{8})x} + K_2 e^{(-2-\sqrt{8})x}$

[EC, GATE-2017 : 1 Mark]

Q.70 Which one of the following is the general solution of the first order differential equation

$$\frac{dy}{dx} = (x + y - 1)^2,$$

where x, y are real?

(a) $y = 1 + x + \tan^{-1}(x + c)$, where c is a constant
(b) $y = 1 + x + \tan(x + c)$, where c is a constant
(c) $y = 1 - x + \tan^{-1}(x + c)$, where c is a constant
(d) $y = 1 - x + \tan(x + c)$, where c is a constant

[CE, GATE-2017 : 2 Marks]

Q.71 Consider the following second-order differential equation

$$y'' - 4y' + 3y = 2t - 3t^2$$

The particular solution of the differential equation is

(a) $-2 - 2t - t^2$ (b) $-2t - t^2$
(c) $2t - 3t^2$ (d) $-2 - 2t - 3t^2$

[CE, GATE-2017 : 2 Marks]

Q.72 The solution of the equation $\frac{dQ}{dt} + Q = 1$ with $Q = 0$ at $t = 0$ is

(a) $Q(t) = e^{-t} - 1$ (b) $Q(t) = 1 + e^{-t}$
(c) $Q(t) = 1 - e^{t}$ (d) $Q(t) = 1 - e^{-t}$

[CE, GATE-2017 : 2 Marks]

Q.73 A particle of mass 2 kg is travelling at a velocity of 1.5 m/s. A force $f(t) = 3t^2$ (in N) is applied to it in the direction of motion for a duration of 2 seconds, where t denotes time in seconds. The velocity (in m/s, up to one decimal place) of the particle immediately after the removal of the force is ________.

[CE, GATE-2017 : 2 Marks]

Q.74 The complete integral of [illegible] is

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) $z = ax + by + c$

[ESE Prelims-2017]

Q.75 [illegible]

(a) [illegible] times its original length be shortened

(b) [illegible] times its original length be shortened

(c) [illegible] times its original length be shortened

(d) [illegible] times its original length be shortened

[ESE Prelims-2017]

■■■■

Answers **Differential Equations**

1.	[illegible]	2.	[illegible]	3.	[illegible]	4.	[illegible]	5.	[illegible]	6.	[illegible]	7.	[illegible]	8.	[illegible]	9.	[illegible]
10.	[illegible]	11.	[illegible]	12.	[illegible]	13.	[illegible]	14.	[illegible]	15.	[illegible]	16.	[illegible]	17.	[illegible]	18.	[illegible]
19.	[illegible]	20.	[illegible]	21.	[illegible]	22.	[illegible]	23.	[illegible]	24.	[illegible]	25.	[illegible]	26.	[illegible]	27.	[illegible]
28.	[illegible]	29.	[illegible]	30.	[illegible]	31.	[illegible]	32.	[illegible]	33.	[illegible]	34.	(d)	35.	(d)	36.	(d)
37.	[illegible]	38.	[illegible]	39.	[illegible]	40.	[illegible]	41.	(d)	42.	[illegible]	43.	[illegible]	44.	[illegible]	45.	[illegible]
46.	[illegible]	47.	[illegible]	48.	[illegible]	49.	[illegible]	50.	[illegible]	51.	[illegible]	52.	[illegible]	55.	[illegible]	57.	[illegible]
59.	[illegible]	60.	[illegible]	61.	[illegible]	62.	[illegible]	65.	[illegible]	67.	[illegible]	68.	[illegible]	69.	[illegible]	70.	[illegible]
71.	[illegible]	72.	[illegible]	74.	[illegible]	75.	[illegible]										

Explanations **Differential Equations**

1\. (a)

Given differential equation [illegible]

[illegible]

On integrating, we get

[illegible]

2\. (c)

[illegible]

Integrating both sides

[illegible]

3\. (b)

[illegible]

Degree is power of highest order derivative [illegible]

So degree = 1

4\. (a)

Given, [illegible]

[illegible]

Now putting [illegible]

[illegible]

5\. (c)

Given [illegible]

putting [illegible]

[illegible]

Multiplying by [illegible]

[illegible]

6\. (a)

[illegible]

The [illegible] equation

[illegible]

$= C_1 e^{\,[illegible]} + C_2 e^{\,[illegible]}$

$= e^{[illegible]}[C_1(\cos 4x + i\sin 4x)$

$C_2(\cos 4x - i\sin 4x)]$

$= e^{[illegible]}[(C_1 + C_2)\cos 4x + i(C_1 - C_2)$ [illegible]

$\sin 4x]$

[illegible] $C_1 + C_2 = C_3$ and $(C_1 - C_2)i = C_4$

$y = e^{[illegible]}(C_3 \cos 4x + C_4 \sin 4x)$

Since $y(0) = 1$

$\Rightarrow \quad 1 = e^{0}(C_3 \cos 0 + C_4 \sin 0)$

$\Rightarrow \quad C_3 = 1$

$\frac{dy}{dx} = [illegible]$

[illegible]

[illegible]

[illegible]

$\frac{dy}{dx}(0) = [illegible]$

[illegible] $(4C_4 - C_3)e^{0} = 0$

$4C_4 = C_3$

$C_4 = \frac{C_3}{4} = \frac{1}{4}$

$\therefore \quad C_3 = 1$ and $C_4 = \frac{1}{4}$

$y = e^{[illegible]}(\cos 4x - \frac{1}{4}\sin 4x)$

7. (c)

Given equation is

$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$

$(D^2 + pD + q)y = 0$

[illegible] $m^2 + pm + q = 0$

[illegible]

[illegible]

[illegible]

[illegible]

8. (c)

Given equation is

$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q+1)y = 0$

$\Rightarrow \quad [D^2 + pD + (q+1)]y = 0$

[illegible] $p = 4$

and $q = 3$

$\therefore \quad (D^2 + 4D + 4)y = 0$

$(D^2 + 4D + 4) = 0$

$(D + 2)^2 = 0$

$\therefore \quad D = -2, -2$

$\Rightarrow \quad y = (C_1 + C_2 x)e^{-2x}$

[illegible] $y = [illegible]$

[illegible]

[illegible] putting $C_1 = [illegible]$ and $C_2 = 0$)

9. (b)

[illegible] $D^2 - [illegible]D - [illegible] = 0$

$(D - [illegible])(D - [illegible]) = 0$

$D = [illegible], 3$

$\therefore \quad y = [illegible]$

10. (a)

$\frac{dV}{dt} = [illegible]$ ...(i)

where $V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$

$\frac{dV}{dt} = \frac{1}{3}[illegible]\frac{dr}{dt} = [illegible]$

Substituting these in (i) we get

$4\pi r^2\frac{dr}{dt} = [illegible]$

$\frac{dr}{dt} = [illegible]$

$\therefore \quad [illegible]$

Integrating we get

$r = [illegible] + C$

at $t = 0, r = 1$

$\Rightarrow \quad 1 = [illegible] + C$

$\Rightarrow \quad C = 1$

$\therefore \quad r = [illegible]$ ...(ii)

Now at $t = 3$ months

$r = 0.5$ m

$\therefore \quad 0.5 = [illegible]$

$\Rightarrow \quad [illegible] = \frac{0.5}{3}$

Now substituting this value of [illegible] in equation (ii) we get,

$r = [illegible]\frac{0.5}{3}t + 1$

putting $r = 0$ [illegible]

[illegible] $\frac{0.5}{3}t = [illegible]$

$\Rightarrow \quad t = 6$ months

11. (b)

Given equation

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

This is a linear ... linear equation (i.e. a first order linear differential equation)

Integrating factor

$$IF = e^{\int 2x\,dx} = e^{x^2}$$

Solution is $y(IF) = \int Q(IF)\,dx + c$

$$ye^{x^2} = \int e^{-x^2} e^{x^2} dx + c$$

$$ye^{x^2} = x + c$$

$x = 0,\ y = 1$ (given)

$\therefore \quad 1 \cdot e^0 = 0 + c$

$\Rightarrow \quad c = 1$

So the solution is

$$ye^{x^2} = x + 1$$

$\Rightarrow \quad y = e^{-x^2}(x + 1)$

12. (c)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$$

$\Rightarrow (D^2 + 4D + 3)y = 3e^{2x}$

Particular integral

$$PI = \frac{1}{D^2 + 4D + 3}\,3e^{2x}$$

Now since $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

$$PI = 3\frac{e^{2x}}{(2)^2 + 4(2) + 3} = \frac{3e^{2x}}{15} = \frac{e^{2x}}{5}$$

13. (c)

Degree of a differential equation is the power of highest order derivative after the differential equation is made free of radicals and fractions in the power.

Hence here the degree is ... power of $\frac{d^2y}{dx^2}$

14. (d)

$$\frac{dy}{dx} = x^2y$$

This is variable separable form

$$\frac{dy}{y} = x^2dx$$

$$\int \frac{dy}{y} = \int x^2dx$$

$\Rightarrow \quad \log y = \frac{x^3}{3} + C$

$\Rightarrow \quad y = e^{\frac{x^3}{3} + C} = e^C \cdot e^{\frac{x^3}{3}}$

$$y = C_1 e^{\frac{x^3}{3}}$$

Now at $x = 0,\ y = 1$

$$1 = C_1 e^0$$

$\Rightarrow \quad C_1 = 1$

$\therefore \quad y = e^{\frac{x^3}{3}}$ is the solution

15. (a)

Given $\quad \frac{dy}{dx} = y^2$

$\Rightarrow \quad \int \frac{dy}{y^2} = \int dx$

$\Rightarrow \quad -\frac{1}{y} = x + c$

$\therefore \quad y = -\frac{1}{x + c}$

When $\quad x = 0$

$\therefore \quad y = 1$

$\therefore \quad y = \frac{1}{1 - x}$

y is infinite when

$x = 1$

$\therefore \quad x = 1$

$\therefore \quad x = 1$

16. (c)

$$\frac{d\theta}{dt} = -k(\theta - \theta_a)$$

(Newton's law of cooling)

This is in variable separable form. Separating the variables we get

$$\frac{d\theta}{\theta - \theta_a} = -k\,dt$$

$$\int \frac{d\theta}{\theta - \theta_a} = \int -k\,dt$$

$\Rightarrow \ln(\theta - \theta_a) = -kt + C_1$

$\Rightarrow \quad \theta - \theta_a = Ce^{-kt} \quad$ (where $C = e^{C_1}$)

$$\theta = \theta_a + Ce^{-kt}$$

Given $\theta_0 =$ [illegible]

Now at $t = 0$, [illegible]

$\theta = 25 + Ce^{-kt}$

$\Rightarrow$ $C =$ [illegible]

$\therefore$ $\theta = 25 + 25e^{-kt}$

At $t =$ [illegible]

$\theta =$ [illegible]

$\therefore$ [illegible]

$\therefore$ $e^{-kt} = \frac{3}{7}$ ...(i)

Now at $t =$ [illegible]

[illegible]

[illegible] $\frac{3}{7}$ [illegible]

$\theta =$ [illegible]

$=$ [illegible]

17. (b)

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

at $x = 1$

$y = \sqrt{3}$

$\therefore$ $\frac{(\sqrt{3})^2}{2} = \frac{-1}{2} + C$

$\Rightarrow$ $C = 2$

$\therefore$ Solution is $\frac{y^2}{2} = \frac{-x^2}{2} + 2$

$\Rightarrow$ $x^2 + y^2 = 4$

18. (b)

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{x} = -3dt$$

$$\int \frac{dx}{x} = \int -3dt$$

$\ln x = -3t + c$

$\Rightarrow$ $x = e^{-3t+c}$

$\Rightarrow$ $x = e^{-3t}e^{c} = c_1 e^{-3t}$ $(c_1 = e^c)$

$\Rightarrow$ $x = c_1 e^{-3t}$

19. (a)

$$\frac{d^2y}{dx^2} + y = 0$$

$D^2 + 1 = 0$

$D = \pm i = 0 \pm i$

$\therefore$ General solution is

$y = e^{0x}(C_1 \cos x + C_2 \sin x)$

$= C_1 \cos x + C_2 \sin x$

$= A \cos x + B \sin x$

where A and B are constants.

20. (b)

$y'' + 3y = 0$

Auxiliary equation is

$m^2 + 3 = 0$

$\therefore$ $m = \pm i\sqrt{3}$

$\therefore$ $y = A\cos\sqrt{3}x + B\sin\sqrt{3}x$

At $y = 1$, $x = 0$

$\Rightarrow$ $A = 1$

Now $y' = -\sqrt{3}A\sin\sqrt{3}x + \sqrt{3}B\cos(\sqrt{3}x)$

At $y' = 0$, $x = 0$

$\Rightarrow$ $B = 0$

So $y = \cos(\sqrt{3}x)$

$y(1) = \cos(\sqrt{3}) =$ [illegible]

21. (a)

$y'' + 2y' + y = 0$

$(D^2 + 2D + 1)y = 0$

$\Rightarrow$ $(D^2 + 2D + 1) = 0$

$\Rightarrow$ $(D + 1)^2 = 0$

$\Rightarrow$ $D = -1, -1$

$\therefore$ $y = (C_1 + C_2x)e^{-x}$

$y(0) = 0 \Rightarrow 0 = (C_1 + C_2 \cdot 0)e^{0}$

$\Rightarrow$ $C_1 = 0$

$y(1) = 0 \Rightarrow 0 = (0 + C_2)e^{-1}$

[illegible] $C_2 e^{-1} = 0$

$\Rightarrow$ $C_2 = 0$

$\therefore$ $y = 0$ [illegible]

$\therefore$ $y(0.5) = 0$

22. (b)

Highest derivative of differential equation is 2.

23. (a)

$$3y\frac{dy}{dx} + 2x = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-2x}{3y}$$

$$\Rightarrow \quad 3y\,dy = -2x\,dx$$

$$\Rightarrow \quad \int 3y\,dy = \int -2x\,dx$$

$$\Rightarrow \quad \frac{3}{2}y^2 = -2\frac{x^2}{2} + C$$

$$\Rightarrow \quad 3y^2 + 2x^2 = C$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{1}{2}C\right)} + \frac{y^2}{\left(\frac{1}{3}C\right)} = 1$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{1}{2}C\right)} + \frac{y^2}{\left(\frac{1}{3}C\right)} = 1$$

which is the equation of a family of ellipses.

24. (c)

A. $$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x}$$

$\log y = \log x + \log c = \log cx$

$y = cx$ ... Equation of straight lines.

B. $$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x} \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$\log y = -\log x + \log c$

$\log y + \log x = \log c$

$\log xy = \log c$

$xy = c$

$y = \frac{c}{x}$ ... Equation of hyperbola.

C. $$\frac{dy}{dx} = \frac{x}{y},\; y\,dy = x\,dx$$

$$\Rightarrow \int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c^2}{2}$$

$y^2 - x^2 = c^2$

$$\frac{y^2}{c^2} - \frac{x^2}{c^2} = 1 \quad \text{... Equation of hyperbola}$$

D. $$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c^2}{2}$$

$$\frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$$

$c^2 = y^2 + x^2$ ... Equation of circle.

25. (c)

Given differential equation is

$$x\frac{dy}{dx} - y = x^3$$

$$\Rightarrow \quad \frac{dy}{dx} - \left(\frac{y}{x}\right) = x^2 \quad \text{...(i)}$$

Standard form of linear differential equations

$$\frac{dy}{dx} + Py = Q \quad \text{...(ii)}$$

where P and Q functions of x only and solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$$

where, integrating factor I.F. $= e^{\int P\,dx}$

Here in equation (i),

$$P = -\frac{1}{x} \text{ and } Q = x^2$$

$$\text{I.F.} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

Solution is $y(1/x) = \int x^2 \cdot \frac{1}{x}\,dx + C$

$$\frac{y}{x} = \frac{x^2}{2} + C$$

given condition

$$y(1) = \frac{6}{5}$$

at $x = 1$, $y = \frac{6}{5}$

$$\Rightarrow \quad \frac{6}{5} \cdot 1 = \frac{1}{5} + C$$

$$\Rightarrow \quad C = \frac{6}{5} - \frac{1}{5} = 1$$

Therefore, $\frac{y}{x} = \frac{x^2}{5} + 1$

$$\Rightarrow \quad y = \frac{x^3}{5} + x$$

26. (a)

$$\frac{d^2y}{dx^2} + 4\sqrt{\left(\frac{dy}{dx}\right)^2 + y^2} = 0$$

Removing radicals we get

$$\left[\frac{d^2y}{dx^2}\right]^2 = 16\left[\left(\frac{dy}{dx}\right)^2 + y^2\right]$$

∴ The order of the highest differential is $\frac{d^2y}{dx^2}$

The degree of the highest differential is 2

27. (a)

$$\frac{d^2y}{dx^2} - \frac{y}{2}\frac{d^2y}{dx^2} = 0$$ is Non-linear

Non linear, since the product $y \times \frac{d^2y}{dx^2}$ is not allowed in linear differential equation

28. (c)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

$$D^2 - D - 6 = 0$$

$$(D-3)(D+2) = 0$$

$$D = 3 \text{ and } D = -2$$

∴ Solution is $y = c_1e^{3x} + c_2e^{-2x}$

29. (b)

Given, $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0$

$$y(0) = 1 \text{ and } \left.\frac{dy}{dt}\right|_{t=0} = 0$$

$$D^2 + 6D + 8 = 0$$

$$(D+4)(D+2) = 0$$

$$D = -4 \text{ and } D = -2$$

∴ Solution is $y = c_1e^{-4t} + c_2e^{-2t}$

Since, $y(0) = 1$

We have $c_1 + c_2 = 1$ ...(i)

$$\frac{dy}{dt} = -4c_1e^{-4t} - 2c_2e^{-2t}$$

Since, $\left.\frac{dy}{dt}\right|_{t=0} = 0$ we have

$$-4c_1 - 2c_2 = 0 \quad \text{...(ii)}$$

Solving (i) and (ii) we have, $c_1 = -1$ and $c_2 = 2$

So, the solution is $y = 2e^{-2t} - e^{-4t}$

30. (d)

$$\frac{d^2x(t)}{dt^2} - \frac{x(t)}{4} = 0$$

⇒ $D^2 - \frac{1}{4} = 0$

⇒ $D = -\frac{1}{2}$ and $D = +\frac{1}{2}$

∴ Solution is $x(t) = C_1e^{t/2} + C_2e^{-t/2}$

$$x(0) = C_1 + C_2 = 1$$

$$x(\infty) = C_1e^{\infty} + C_2e^{-\infty} = 0$$

⇒ $C_1e^{\infty} = 0$

⇒ $C_1 = 0$

∴ $C_2 = 1$

∴ The solution is $x = e^{-t/2}$

31. (d)

$$\frac{dy}{dx} = (1 + y^2)x$$

$$\int\frac{dy}{1+y^2} = \int x\,dx$$

$$\tan^{-1}y = \frac{x^2}{2} + C$$

$$y = \tan\left(\frac{x^2}{2} + C\right)$$

32. (a)

$$\frac{dy}{dx} = e^{-3x}$$

$$\int dy = \int e^{-3x}dx$$

$$y = \frac{e^{-3x}}{-3} + C = -\frac{1}{3}e^{-3x} + K$$

33. (c)

$$\frac{dy}{dx} = ky$$

⇒ $\frac{dy}{y} = k\,dx$

Integrating both sides

⇒ $\ln y = kx + A$

Take $x = 0$

⇒ $\ln y(0) = A$

∴ $A = \ln c$ $[\because y(0) = c]$

Hence,

⇒ $\ln y = kx + \ln c$

⇒ $\ln y - \ln c = kx$

⇒ $\ln\left(\frac{y}{c}\right) = kx$

⇒ $\frac{y}{c} = e^{kx}$

or $y = ce^{kx}$

34. (c)

$$\frac{dy}{dx} + \frac{y}{x} = x, \quad y(1) = 1$$

This is a linear differential equation $\frac{dy}{dx} + Py = Q$

with $P = \frac{1}{x}$ and $Q = x$

∴ Integrating factor

$$= e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Solution is

$$y\,(IF) = \int Q(IF)\,dx + C$$

$$\Rightarrow \quad yx = \int x \cdot x\,dx + C$$

$$\Rightarrow \quad yx = \int x^2\,dx + C$$

$$\Rightarrow \quad yx = \frac{x^3}{3} + C$$

$$\Rightarrow \quad y = \frac{x^2}{3} + \frac{C}{x}$$

Now $y(1) = 1$

$$\Rightarrow \quad \frac{1}{3} + \frac{C}{1} = 1 \Rightarrow C = \frac{2}{3}$$

So the solution is $y = \frac{x^2}{3} + \frac{2}{3x}$

35. (d)

Given $\frac{dy}{dx} + 2y = 0$ and $y(1) = 5$

$$\frac{dy}{dx} = -2y$$

$$\int \frac{dy}{y} = \int -2\,dx$$

$$\Rightarrow \quad \ln y = -2x + c$$

$$\Rightarrow \quad y = e^{-2x} \cdot e^{c} = c_1 e^{-2x}$$

$$y(1) = c_1 e^{-2} = 5 \Rightarrow c_1 = \frac{5}{e^{-2}}$$

$$\text{So,} \quad y = \frac{5}{e^{-2}} e^{-2x} = 5e^{2}e^{-2x}$$

$$= 36.95\,e^{-2x}$$

36. (c)

The given differential equation is

$t\frac{dx}{dt} + x = t$ with initial condition $x(1) = \frac{1}{2}$ which can be written as

$$\frac{dx}{dt} + \frac{x}{t} = 1$$

Which is a linear differential equation

$$\frac{dx}{dt} + Px = Q$$

Where $P = \frac{1}{t}$ and $Q = 1$

Integrating factor

$$= e^{\int P\,dt} = e^{\int \frac{1}{t}dt}$$

$$= e^{\ln t} = t$$

Solution is

$$x\,(IF) = \int Q(IF)\,dt + C$$

$$x\,t = \int 1 \cdot t\,dt + C$$

$$xt = \frac{t^2}{2} + C$$

$$x = \frac{t}{2} + \frac{C}{t}$$

But $x(1) = \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{2} + \frac{C}{1} = \frac{1}{2}$$

$$\Rightarrow \quad C = 0$$

So $\boxed{x = \frac{t}{2}}$ is the solution.

37. (d)

In the equation, dependent variable y and its derivatives are in first degree and there is no product between them. So given differential equation is non linear equation of order 2.

38. (c)

Given differential equation

$$(D^2 - 2D)y = 0$$

It is linear differential equation with constant coefficients.

∴ General solution is

$$y = CF + PI$$

CF is given by

$$f(m) = m^2 - 2m = 0$$

$$\Rightarrow \quad m = 0, 2$$

$$\therefore \quad CF = C_1 e^{0x} + C_2 e^{2x}$$

$$\therefore \quad y = C_1 + C_2 e^{2x} \quad \ldots(i)$$

Put $y = 0,\ x = 0$

We get, $C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$

Put $x = 1$ and $y = 0$

[illegible]

40. (d)

$y'' + y = 0$

$D^2 + 1 = 0$

$\therefore \quad D = \pm i$

$\therefore \quad y = C_1 e^{ix} + C_2 e^{-ix}$

$= A \cos x + B \sin x$

$y(0) = 1$

$\therefore \quad 1 = A \times 1 + B \times 0$

$A = 1$

$y' = -A \sin x + B \cos x$

$y'(0) = 1$

$\therefore \quad 1 = -A \times 0 + B \times 1$

$\therefore \quad B = 1$

So, $\quad y = \cos x + \sin x$

To maximize

$y' = -\sin x + \cos x = 0$

$\therefore \quad \tan x = 1$

$\therefore \quad x = \pi/4$

$y'' = -\cos x - \sin x$

[illegible]

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

41. (d)

$\frac{dy(t)}{dt} + 5y(t) = x(t)$

Taking Laplace transform on both sides, we have

$sY(s) - y(0) + 5Y(s) = X(s)$

$Y(s)[s + 5] = X(s) + y(0)$

$\Rightarrow \quad Y(s) = \frac{X(s)}{s+5} + \frac{y(0)}{s+5}$

Taking inverse Laplace transform we have

[illegible]

So if we want $2y(t)$ as o/p then both $x(t)$ and $y(0)$ has to be multiplied by 2 [illegible]

42. (b)

$\frac{dx}{dt} = [illegible]$

$\frac{dy}{dt} = 4x - 8y$

[illegible]

43. (d)

Let $\quad z = x + y$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\therefore \quad \frac{dz}{dx} = 1 + \cos z$

or $\quad \int \frac{dz}{1 + \cos z} = \int dx$

or $\quad \frac{1}{2}\int \sec^2\left(\frac{z}{2}\right)dz = x + c$

or $\quad \tan\left(\frac{z}{2}\right) = x + c$

$\therefore \quad \tan\left(\frac{x+y}{2}\right) = x + c$

44. (b)

[illegible] ...(i)

[illegible]

Multiplying [illegible] of equation (i)

[illegible]

[illegible]

[illegible]

45. (a)

General form of linear differential equation

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.

Only option (a) [illegible]

46. (c)

$$\frac{d^2x}{dt^2} = -9x \qquad \frac{d}{dt} = D$$

$$\frac{d^2x}{dt^2} + 9x = 0 \qquad (D^2 + 9)x = 0$$

Auxiliary equation is $m^2 + 9 = 0$

$$m = \pm 3i$$

$$x = C_1 \cos 3t + C_2 \sin 3t \qquad \dots(i)$$

$$x(0) = 1 \text{ i.e. } x = 1 \text{ when } t = 0$$

$$\boxed{1 = C_1}$$

$$\frac{dx}{dt} = -3C_1 \sin 3t + 3C_2 \cos 3t \qquad \dots(ii)$$

$$x'(0) = 1 \text{ i.e. } x' = 1 \text{ when } t = 0$$

$$1 = 3C_2 \quad C_2 = \frac{1}{3}$$

$$\therefore \qquad x = \cos 3t + \frac{1}{3}\sin 3t$$

47. (a)

Given differential equation in operator form is

$$(D^2 + 1)x = 0$$

Its A.E. is

$$D^2 + 1 = 0$$

$$\therefore \qquad D = \pm i$$

So, C.F. is

$$x_1(t) = C_1 \cos t$$

$$x_2(t) = C_2 \sin t$$

$$\therefore \qquad x_1(0) = C_1 = 1$$

$$\Rightarrow \qquad x_1(t) = \cos t \text{ because } \left.\frac{dx_1(t)}{dt}\right|_{t=0} = 0$$

$$x_2(0) = 0 = C_2 \sin(0) \qquad \text{if } C_2 \neq 0$$

$$\frac{dx_2(t)}{dt} = C_2 \cos t$$

$$\Rightarrow \left.\frac{dx_2(t)}{dt}\right|_{t=0} = C_2 = 1$$

$$\therefore \qquad x_2(t) = \sin t$$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix}$$

$$= \cos t \cdot \cos t$$

$$+ \sin t \cdot \sin t$$

$$W(t) = \cos^2 t + \sin^2 t = 1$$

48. (a)

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + y = 0$$

The characteristic equation is given as

$$m^2 + 2am + 1 = 0$$

$$m_1, m_2 = \frac{-2a \pm \sqrt{4a^2 - 4}}{2}$$

Since the roots are equal, so

$$m_1 = m_2$$

$$\frac{-2a + \sqrt{4a^2 - 4}}{2} = \frac{-2a - \sqrt{4a^2 - 4}}{2}$$

$$\sqrt{4a^2 - 4} = -\sqrt{4a^2 - 4}$$

$$2\sqrt{4a^2 - 4} = 0$$

$$4a^2 - 4 = 0$$

$$a^2 = 1$$

$$a = \pm 1$$

49. (b)

Differential equation is given as

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$$

$$y = \text{C.F.} + \text{P.I.}$$

Since there is no RHS term & zero, so there is no particular integral

$$\therefore \qquad y = \text{C.F.}$$

$$\text{Let} \qquad \frac{d}{dt} = D$$

$$\text{So, } (D^2 + 2D + 1)y = 0$$

$$\therefore \qquad (D + 1)^2 = 0$$

$$\therefore \qquad y = ae^{-t} + bte^{-t}$$

50. (c)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

$$\text{Let} \qquad x = e^z \Rightarrow z = \log x$$

$$x\frac{d}{dx} = xD = \theta = \frac{d}{dz}$$

$$x^2D^2 = \theta(\theta - 1)$$

$$[x^2D^2 + xD - 1]y = 0$$

$$[\theta(\theta - 1) + \theta - 1]y = 0$$

$$(\theta^2 - \theta + \theta - 1)y = 0$$

$$(\theta^2 - 1)y = 0$$

Auxiliary equation is $m^2 - 1 = 0$

$$m = \pm 1$$

CF = $C_1 e^{[illegible]} + C_2 x$

Solution is $y = C_1 x^{m} + C_2 x$

$= C_1 x^{-1} + C_2 x$

$= C_1 \frac{1}{x} + C_2 x$

One independent solution is $y = \frac{1}{x}$

Another independent solution is $y = x$

51. (c)

$$\frac{dy}{dt} = -5y$$

$$\int \frac{dy}{y} = -\int 5\,dt$$

$$\ln y = -5t + C$$

At $t = 0$

$y = 2$

$\ln 2 = C$

So, $\ln y = -5t + \ln 2$

$$\ln \frac{y}{2} = -5t$$

$$\frac{y}{2} = e^{-5t}$$

$$y = 2e^{-5t}$$

At $t = 3$

$$y = 2e^{-15}$$

52. (c)

$x(x\,dx + y\,dy) + [illegible] \frac{y}{x} = [illegible] \frac{y}{x}$

$$\frac{x\,dx + y\,dy}{[illegible]} = [illegible]$$

Let $[illegible]$

$[illegible]$

$$\frac{[illegible]}{[illegible]} = k \tan z$$

$$\frac{[illegible]}{[illegible]} = [illegible]$$

$$\frac{2r}{x}\frac{dr}{dz} = [illegible]$$

$$\frac{2r}{x}\frac{dr}{dz} = [illegible]$$

$$2\frac{dr}{r} = [illegible] \tan z - [illegible]\,dz$$

Integrating both sides,

$2 \log r = \log [illegible] - [illegible] + \log c$

$\Rightarrow \quad x^{[illegible]} = \frac{c \sec z}{[illegible]}$

$\Rightarrow \quad x^{2} [illegible] = [illegible] \frac{y}{x}$

$\Rightarrow \quad [illegible] \frac{y}{x} = c$

53. Sol.

$$\frac{d^2y}{dx^2} = 12x^2 - 24x + 20$$

Integrating both sides w.r.t. x

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 20x + c_1$$

Integrating both sides w.r.t. x

$y = x^4 - 4x^3 + 10x^2 + c_1 x + c_2$ ...(i)

At $x = 0,\ y = 5$

$\Rightarrow \quad c_2 = 5$

At $x = 2,\ y = 21$

$\Rightarrow \quad 21 = 16 - 32 + 40 + 2c_1 + c_2$

$2c_1 = [illegible]$

$c_1 = [illegible]$

$\Rightarrow \quad y = x^4 - 4x^3 + 10x^2 [illegible] + 5$

Put $x = 1$

$\Rightarrow \quad y = 1 - 4 + 10 [illegible] = [illegible]$

54. Sol.

$$\frac{dN}{dt} = 0.2N$$

$$\frac{dN}{N} = 0.2\,dt$$

$$\int \frac{dN}{N} = \int 0.2\,dt$$

$\log N = 0.2t + \log C$

$\log N - \log C = 0.2t$

$$\log\left(\frac{N}{C}\right) = 0.2t$$

$$\frac{N}{C} = e^{0.2t}$$

$N = Ce^{0.2t}$ ...(i)

$[illegible]$

$10 = [illegible]$

$10 = [illegible]$

$\therefore \quad C = [illegible]$

$N = ([illegible])e^{0.2t}$ ...(ii)

when, $t = 7$

$N = ([illegible])e^{[illegible]} = [illegible]$

55. (c)

$$\frac{dy}{1+\cos 2y} = \frac{dx}{1-\cos 2x}$$

$$\frac{dy}{2\cos^2 y} = \frac{dx}{2\sin^2 x}$$

$$\sec^2 y\, dy = \operatorname{cosec}^2 x\, dx$$

Integrating both sides we get

$$\tan y = -\cot x + c$$

$$\tan y + \cot x = c$$

56. Sol.

$$D^2 + 5D + 6 = 0$$

$$D = -2, -3$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Given, $y(0) = 2$

$\Rightarrow \quad c_1 + c_2 = 2 \quad \ldots(i)$

$$y(1) = \left(\frac{1-3e}{e^3}\right)$$

$$\Rightarrow \quad \frac{c_1}{e^2} + \frac{c_2}{e^3} = \frac{1-3e}{e^3}$$

$\Rightarrow \quad ec_1 + c_2 = 1 - 3e \quad \ldots(ii)$

Now solving equation (i) and (ii) we get

$$c_1 = -3$$

$$c_2 = 5$$

Substituting in $y(t)$ we get

$$y(t) = -3e^{-2t} + 5e^{-3t}$$

Now $\quad \frac{dy}{dt} = 6e^{-2t} - 15e^{-3t}$

$$\left(\frac{dy}{dt}\right)_{t=0} = 6 - 15 = -9$$

57. (c)

$$D^2 + 2D + 1 = 0$$

$$D = -1, -1$$

$$y(t) = (C_1 + C_2 t)e^{-t}$$

$$y'(t) = -C_1 e^{-t} + C_2(-te^{-t} + e^{-t})$$

$$y(0) = y'(0) = 1$$

From here, $C_1 = 1$, $C_2 = 2$

$\Rightarrow \quad y(t) = (1 + 2t)e^{-t}$

58. Sol.

$$D^2 + 3D + 2 = 0$$

$$D = -1, -2$$

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}$$

$$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$C_1 + C_2 e^{-1} = [illegible] \quad \ldots(i)$$

$$C_1 - C_2 = 20 \quad \ldots(ii)$$

From here $\quad C_1 = [illegible]$, $C_2 = [illegible]$

$$y(t) = [illegible]$$

$$= [illegible]$$

59. (c)

$$\frac{d^2 y}{dt^2} = y$$

$\Rightarrow \quad D^2 y = y \quad \left(\because D = \frac{d}{dt}\right)$

$$(D^2 - 1)y = 0$$

$$D^2 - 1 = 0$$

$$D = \pm 1$$

$$y = C_1 e^{t} + C_2 e^{-t}$$

Since path passes through $(0, 0)$

$\Rightarrow \quad 0 = C_1 + C_2$

$C_1 = -C_2 \quad \ldots(i)$

Also, path passes through $(\ln 2, 3/4)$

$$\Rightarrow \quad \frac{3}{4} = C_1 e^{\ln 2} + C_2 e^{-\ln 2}$$

$$\frac{3}{4} = 2C_1 + \frac{C_2}{2}$$

$\Rightarrow \quad C_2 + 4C_1 = 1.5 \quad \ldots(ii)$

From (i) $\quad C_1 = -C_2$ putting in (ii) we get

$\Rightarrow \quad -3C_2 = 1.5$

$$C_2 = -0.5$$

$\therefore \quad C_1 = 0.5$

$\Rightarrow \quad y = 0.5(e^t - e^{-t})$

$$y = \frac{e^t - e^{-t}}{2}$$

60. (b)

Auxiliary equation

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$y = (C_1 + C_2 x)e^{-x}$$

$$y(0) = 1$$

$$C_1 = 1$$

$$y = (1 + C_2 x)e^{-x}$$

$$y(2) = 3e^{-2}$$

$\Rightarrow \quad (1 + 2C_2)e^{-2} = 3e^{-2}$

$$C_2 = 1$$

$$y = (1 + x)e^{-x}$$

$$y(3) = 4e^{-3}$$

61. (a)

The differential equation is

...

So

Given that, $y(0) = 0$

So

So

62. Sol.

A.E. $m^2 - 4m + 4 = 0$

$m = 2, 2$

$y = (C_1 + C_2 x)e^{2x}$

$y(0) = 0 \Rightarrow C_1 = 0$

$y = C_2 x e^{2x}$

$y' = C_2 e^{2x} + 2C_2 x e^{2x}$

$y'(0) = 1$

$\Rightarrow \quad C_2 = 1$

$y = x e^{2x}$

63. (a)

$D = -3, -3$

$y = (C_1 + C_2 x)e^{-3x}$

$y(0) = 3 = C_1 \ldots$

$\therefore \quad y = \ldots$

$C_2 = 18$

$\therefore \quad y = 3e^{-3x} + 18xe^{-3x}$

64. Sol.

$y'' + 9y = 0$

$m^2 + 9 = 0$

$m = \pm 3i$

$y = C_1 \cos 3x + C_2 \sin 3x$

$C_2 = \sqrt{2}$

$\therefore \quad y = \sqrt{2} \sin 3x$

$= -1$

65. (a)

$\therefore \quad y = (C_1 + C_2 x) + C_3 \sin(\sqrt{3}x) + C_4 \cos(\sqrt{3}x)$

66. (94.08)

The differential equation is ...

The auxiliary equation is

$m^2 + 9 = 0$

$m = \pm 3i$

Solution is $y = C_1 \cos 3x + C_2 \sin 3x$

given that $y(0) = 0$

$\therefore \quad C_1 = 0$

when $x = \ldots$

67. (d)

$(D^2+16)y = 0$

Auxiliary equation $m^2+16=0$

$m = \pm 4i$

Solution is $y = c_1\cos 4x + c_2\sin 4x$

$y' = -4c_1\sin 4x + 4c_2\cos 4x$

$y'(0) = 1$

$1 = 4c_2$

$c_2 = 1/4$

$y'(\pi/8) = $ [illegible]

$1 = -4c_1\sin 2\pi + 4c_2\cos 2\pi$

$= 0 + 4c_2$

$c_2 = 1/4$

Hence the given differential equation has no solution

68. (a)

The differential equation is [illegible]

$(x^2 + \ldots)\frac{dy}{dx} + \ldots$ [illegible]

$\frac{dy}{dx} + \ldots y = \ldots$ [illegible]

[illegible]

$I.F. = e^{\int \ldots dx}$ [illegible]

$= (x^2-8)^{\frac{5}{2}}$ [illegible]

Solution is,

$y(x^2-8)^{\frac{5}{2}} = \int \ldots (x^2-8)^{\frac{5}{2}} dx$ [illegible]

$y = \frac{\int \ldots (x^2-8)^{\ldots} dx}{(x^2-8)^{5/2}} + \frac{c}{(x^2-8)^{\ldots}}$ [illegible]

The solution exists for [illegible]

[illegible]

Hence option (a) is correct

Because the remaining options are not satisfying the condition [illegible]

69. (a)

The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 5y = 0$$

$(D^2+2D-5)y = 0$

Auxiliary equation is

$m^2 + 2m - 5 = 0$

$m =$

$$\frac{-2 \pm \sqrt{4+20}}{2} = \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} = -1 \pm \sqrt{6}$$

Solution is $y = c_1 e^{(-1+\sqrt{6})x} + c_2 e^{(-1-\sqrt{6})x}$

70. (d)

$$\frac{dy}{dx} = (x+y+1)^2 \quad \ldots(i)$$

Let $x+y+1 = t \quad \ldots(ii)$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1 \quad \ldots(iii)$$

Substituting the values of (ii) and (iii) in (i)

$$\frac{dt}{dx} - 1 = t^2$$

$$\frac{dt}{t^2+1} = dx$$

Integrating both sides

$$\int \frac{1}{t^2+1}dt = \int dx$$

$\tan^{-1} t = x + c$

Since $t = x+y+1$

$\therefore \tan^{-1}(x+y+1) = x + c$

$x+y+1 = \tan(x+c)$

$y = -1 - x + \tan(x+c)$

71. (a)

$y'' - 4y' + 3y = 2e^{\ldots}$ [illegible]

$(D^2-4D+3)y = 2e^{\ldots}$ [illegible]

$P.I. = \frac{1}{D^2-4D+3}$ [illegible]

$= \frac{1}{(D-1)(D-3)}$ [illegible]

$= \left(\frac{1/2}{D-1} - \frac{1/2}{D-3}\right)$ [illegible]

[illegible]

72. (c)

$$\frac{dQ}{dt} + Q = 1$$

Comparing with linear form

$$I.F. = e^{\int 1\,dt} = e^t$$

Solution is

$$Q e^t = \int 1 \cdot e^t\,dt$$

$$= e^t + C$$

$$Q = 1 + Ce^{-t} \quad \ldots(i)$$

When $t = 0$, $Q = 0$

$\Rightarrow$ $C + 1 = 0$

$\Rightarrow$ $C = -1$

Therefore $Q(t) = 1 - e^{-t}$

73. Sol.

$m = 2$ kg, $v_0 = 10$ m/sec

$$\int_0^t F(t)\,dt = m(v - v_0)$$

[illegible]

74. (b)

[illegible]

which is in Cartesian form.

[illegible]

75. (c)

[illegible]

4 Complex Functions

4.1 Introduction

[illegible]

[illegible]

[illegible]

1. [illegible]
2. [illegible]

4.2 Complex Functions

[illegible]

[illegible]

4.2.1 Exponential Function of a Complex Variable

[illegible]

[illegible]

[illegible]

[illegible]

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)$$

$$= \cos y + i \sin y$$

Thus $\qquad e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

Also $\qquad x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

∴ Exponential form of $x + iy = re^{i\theta}$.

4.2.2 Circular Function of a Complex Variable

Since $\qquad e^{iy} = \cos y + i \sin y$

and $\qquad e^{-iy} = \cos y - i \sin y$

∴ The circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \quad \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to define the circular functions of the complex variable z by the equations:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \tan z = \frac{\sin z}{\cos z}$$

with cosec z, sec z and cot z as their respective reciprocals.

Cor. 1. Euler's Theorem. By definition

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i\,\frac{e^{iz} - e^{-iz}}{2i} = e^{iz} \qquad \text{where } z = x + iy$$

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is real or complex. This is called the Euler's theorem.

Cor. 2. De Moivre's theorem for complex numbers.

Whether θ is real or complex, we have

$$(\cos\theta + i \sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

4.2.3 Hyperbolic Functions

1. Def. If x be real or complex,

(a) $\frac{e^x - e^{-x}}{2}$ is defined as hyperbolic sine of x and is written as sinh x.

(b) $\frac{e^x + e^{-x}}{2}$ is defined as hyperbolic cosine of x and is written as cosh x.

Thus, $\qquad \sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$

Also we define $\qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Obs. $\sinh 0 = 0$, $\cosh 0 = 1$ and $\tanh 0 = 0$.

2. Relations between hyperbolic and circular functions.

Since for all real values of θ, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

[illegible]

[illegible]

Thus, [illegible] ...(i)

[illegible] ...(ii)

and [illegible] ...(iii)

Also [illegible] ...(iv)

[illegible] ...(v)

[illegible] ...(vi)

4.2.4 Inverse Hyperbolic Functions

[illegible]

[illegible]

4.2.5 Logarithmic Function of a Complex Variable

1. [illegible]

[illegible]

Obs.

(a) [illegible]

(b) [illegible]

2. Real and imaginary parts of $\log(x + iy)$

[illegible]

[illegible]

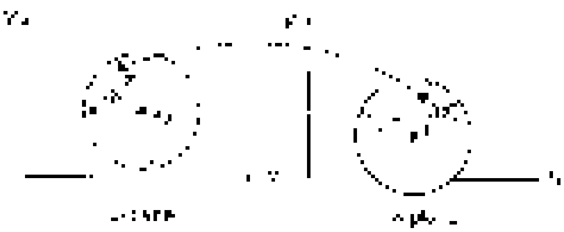
[illegible]

4.3 Limit of a Complex Function

[illegible]

[illegible]

[illegible]



Continuity of $f(z)$. A function $w = f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

[illegible]

[illegible]

4.4 Singularity

[illegible]

4.4.1 Isolated Singular Point

[illegible]

Example: The function $\frac{1}{(z-1)(z-3)}$ has two isolated singular points at $z = 1$, $z = 3$.

The function $\frac{1}{\sin \pi/z} = 0$ at, $\frac{\pi}{z} = n\pi$ or $z = \frac{1}{n}$ $[n = 1, 2, \ldots]$.

Here $z = 0$ is a non-isolated singularity.

4.4.2 Essential Singularity

If the function $f(z)$ has pole $z = a$ of infinite order, then

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

The negative power in expansion are infinite then $z = a$ is called essential singularity.

4.4.3 Removable Singularity

If $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

i.e. $$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

Here all the coefficients of negative power are zero. Then $z = a$ is called removable singularity i.e. $f(z)$ can be made analytic by redefining $f(a)$ suitably i.e., if $\lim_{z \to a} f(z)$ exists.

Example: $f(z) = \frac{\sin(z-a)}{(z-a)}$ has removable singularity at $z = a$.

4.4.4 Steps to Find Singularity

Step-1: If $\lim_{z \to a} f(z)$ exists and is finite then $z = a$ is a removable singular point.

Step-2: If $\lim_{z \to a} f(z)$ does not exist then $z = a$ is an essential singular point.

Step-3: If $\lim_{z \to a} f(z)$ is infinite then $f(z)$ has a pole at $z = a$. The order of the pole is same as the number of negative power terms in the series expansion of $f(z)$.

4.5 Derivative of f(z)

Let $w = f(z)$ be a single valued function of the variable $z = x + iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

[illegible] exists and has the same value for all the different ways in which δz approaches the zero.

Suppose that δz is real and [illegible] (Fig. above). [illegible] along any straight or curved path in the given region [illegible] conditions [illegible] the following theorem.

Theorem. The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y)$ to exist for all values of z in a region R, are —

1. $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in R;

2. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\qquad$ $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

The relations in (2) are known as Cauchy-Riemann equations or briefly C-R equations.

4.6 Analytic Functions

4.6.1 Analytic Functions

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an analytic or regular function of z in that region.

A point at which an analytic function ceases to possess a derivative is called a singular point of the function.

Thus if u and v are real single-valued functions of x and y such that $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{...(1)}$$

are both necessary and sufficient for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by

$$f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \qquad \text{...(2)}$$

or

$$f'(z) = \lim_{i\delta y \to 0}\left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right)$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y \qquad \text{...(3)}$$

The real and imaginary parts of an analytic function are called conjugate functions. The relation between two conjugate functions is given by the C-R equations as follows.

C-R equations in Polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Example 1.

Is $f(z) = z^3$ analytic?

Solution:

$$z = x + iy$$

$\Rightarrow$ $$z^2 = (x+iy)(x+iy) = x^2 - y^2 + 2ixy$$

$\Rightarrow$ $$z^3 = (x^2 - y^2 + 2ixy)(x+iy)$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Here $$u = x^3 - 3xy^2$$

$$v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2, \quad v_y = 3x^2 - 3y^2$$

$$u_y = -6xy, \quad v_x = 6xy$$

$\therefore$ $$u_x = v_y \text{ and } u_y = -v_x$$

Both C-R equations are satisfied and the partial derivatives are continuous at all points. Hence z^3 is analytic for every z.

Example 2.

If $w = \log z$, find dw/dz and determine where w is non-analytic.

Solution:

We have $$w = u + iv = \log(x + iy)$$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x}$$

So that $$u = \frac{1}{2}\log(x^2 + y^2), \quad v = \tan^{-1}\frac{y}{x}$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at $(0, 0)$. Hence w is analytic everywhere except at the origin.

$\therefore$ $$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x - iy}{(x+iy)(x-iy)}$$

$$= \frac{1}{x+iy} = \frac{1}{z} \quad (z \neq 0)$$

Obs. The real definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. Thus if a complex function is once known to be analytic, it can be differentiated just in the ordinary way.

4.6.2 Harmonic Functions

Any function which satisfies Laplace equation is known as harmonic function.

If $f(z) = u + iv$ is analytic then both u and v are harmonic functions.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating w.r.t. x Differentiating w.r.t. y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Adding these we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Therefore, both u and v are harmonic functions.

4.6.3 Orthogonal Curves

Two curves are said to be orthogonal to each other when they intersect at right angle at each of their point of intersection.

At the point of intersection, the tangents to both the curves are also perpendicular.

The analytic function $f(z) = u(x, y) + iv(x, y)$ consists of two families of curves, $u(x, y) = c_1$ and $v(x, y) = c_2$ which form an orthogonal net.

$$u(x, y) = c_1$$

$$\frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy = 0$$

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 \qquad \text{...(i)}$$

$$v(x, y) = c_2$$

$$\frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy = 0$$

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2 \qquad \text{...(ii)}$$

For orthogonality $$m_1 m_2 = \left(-\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}\right) \times \left(-\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}\right)$$

As we know that $f(z)$ is analytic function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $$m_1 m_2 = -1$$

$\Rightarrow$ The curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal.

4.7 Complex Integration

4.7.1 Line integral in the complex plane

As in calculus we distinguish between definite integrals and indefinite integrals or antiderivatives. An indefinite integral is a function whose derivative equals a given analytic function in a region. By known differentiation formulas we may find many types of indefinite integrals.

Complex definite integrals are called (complex) line integrals. They are written

$$\int_C f(z)\,dz$$

Here the integrand $f(z)$ is integrated over a given curve C in the complex plane, called the path of integration. We may represent such a curve C by a parametric representation

(1) $$z(t) = x(t) + iy(t) \qquad (a \le t \le b).$$

The sense of increasing t is called the **positive sense** on C, and we say that in this way (1) orients C.

We assume C to be a **smooth curve**, that is, C has a continuous and nonzero derivative $\dot{z} = dz/dt$ at each point. Geometrically, this means that C has a unique and continuously turning tangent.

4.7.2 Definition of the Complex Line Integral

This is similar to the method in calculus. Let C be a smooth curve in the complex plane given by (1), and let $f(z)$ be a continuous function given (at least) at each point of C. We now subdivide (we "partition") the interval $a \le t \le b$ in (1) by points

$$t_0 (= a), t_1, \ldots, t_{n-1}, t_n (= b)$$

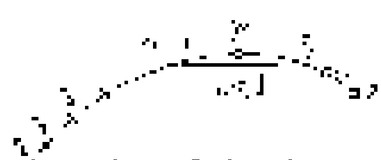
where $t_0 < t_1 < \ldots < t_n$. To this subdivision there corresponds a subdivision of C by points

$$z_0, z_1, \ldots, z_{n-1}, z_n (= Z)$$

where $z_j = z(t_j)$. On each portion of subdivision of C we choose an arbitrary point, say, a point ζ_1 between z_0 and z_1 (that is, $\zeta_1 = z(t)$ where t satisfies $t_0 \le t \le t_1$), a point ζ_2 between z_1 and z_2 etc. Then we form the sum

(2) $$S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m \qquad \text{where } \Delta z_m = z_m - z_{m-1}.$$

We do this for each $n = 2, 3, \ldots$ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest $|\Delta z_m|$ also approaches zero because it cannot exceed the length of the arc of C from z_{m-1} to z_m, and the latter goes to zero since the arc length of the smooth curve C is a continuous function of t. The limit of the sequence of complex numbers $S_2, S_3, \ldots$ thus obtained is called the line integral (or simply the integral) of $f(z)$ over the oriented curve C. This



Complex line integral

curve C is called path of integration. The line integral is denoted by

(3) $$\int_C f(z)\,dz, \quad \text{or by} \quad \oint_C f(z)\,dz$$

if C is a **closed path** (one whose terminal point Z coincides with its initial point z_0, as for a circle or an 8 shaped curve).

General Assumption. All paths of integration for complex line integrals are assumed to be piecewise smooth, that is, they consist of finitely many smooth curves joined end to end.

4.7.3 First Method: Indefinite Integration and Substitution of Limits

This method is simpler than the next one, but is less general. It is restricted to analytic functions. Its formula is the analog of the familiar formula from calculus.

$$\int_a^b f(z)\,dz = F(b) - F(a) \qquad [F'(z) = f(z)]$$

Theorem 1: (Indefinite integration of analytic functions)

Let $f(z)$ be analytic in a simply connected domain D. (A domain D is called simply connected if every simple closed curve (closed curve without self-intersections) encloses only points of D.) Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]. \tag{4}$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

This theorem will be proved in the next section.

Simple connectedness is quite essential in Theorem 1, as we shall see in Example 5. Since analytic functions are our main concern, and since differentiation formulas will often help in finding $F(z)$ for a given $f(z) = F'(z)$, the present method is of great practical interest.

If $f(z)$ is entire, we can take for D the complex plane (which is certainly simply connected).

Example 1. $$\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$

Example 2. $$\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i\sinh \pi = 23.097i$$

Example 3. $$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2\left(e^{4-3\pi i/2} - e^{4+\pi i/2}\right) = 0$$

Since e^z is periodic with period $2\pi i$.

4.7.4 Second Method: Use of a Representation of the Path

This method is not restricted to analytic functions but applies to any continuous complex function.

Theorem 2: (Integration by the use of the path)

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right) \tag{5}$$

Proof: The left side of (5) is given by (3) in terms of real line integrals, $\int_C f(z)\,dz = \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right]$, and we now show that the right side of (5) also equals the same.

We have $z = x + iy$, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write u for $u[x(t), y(t)]$ and v for $v[x(t), y(t)]$. We also have $dx = \dot{x}\,dt$ and $dy = \dot{y}\,dt$.

Consequently, in (5)

$$\int_a^b f[z(t)]\,\dot{z}(t)\,dt = \int_a^b (u + iv)(\dot{x} + i\dot{y})\,dt = \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

Steps in Applying Theorem 2

1. Represent the path C in the form $z(t)$ $(a \le t \le b)$.
2. Calculate the derivative $\dot{z}(t) = dz/dt$.
3. Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).
4. Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

Example 1: A basic result: Integral of 1/z around the unit circle

We show that by integrating $1/z$ counterclockwise around the unit circle (the circle of radius 1 and center 0) we obtain

$$(8) \qquad \oint_C \frac{dz}{z} = 2\pi i \qquad (C \text{ the unit circle, counterclockwise})$$

This is a very important result that we shall need quite often.

Solution: We may represent the unit circle C in the form

$$z(t) = \cos t + i \sin t = e^{it} \qquad (0 \le t \le 2\pi),$$

so that counterclockwise integration corresponds to an increase of t from 0 to 2π. By differentiation, $\dot{z}(t) = ie^{it}$ (chain rule) and with $f(z(t)) = 1/z(t) = e^{-it}$ we get from (10) the result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i$$

Check this result by using $z(t) = \cos t + i \sin t$.

Simple connectedness is essential in Theorem 1. Equation (9) in Theorem 1 gives 0 for any closed path because then $z_1 = z_0$, so that $F(z_1) - F(z_0) = 0$. Now $1/z$ is not analytic at $z = 0$. But any simply connected domain containing the unit circle must contain $z = 0$, so that Theorem 1 does not apply — it is not enough that $1/z$ is analytic in an annulus, say, $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected.

Example 2: Integral of integer powers

Let $f(z) = (z - z_0)^m$ where m is an integer and z_0 a constant. Integrate counterclockwise around the circle C of radius ρ with center at z_0 (Fig. below).

Solution: We may represent C in the form

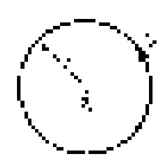
$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \qquad (0 \le t \le 2\pi).$$

Then we have

$$(z - z_0)^m = \rho^m e^{imt}, \qquad dz = i\rho e^{it}\, dt$$

and obtain

$$\oint_C (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it}\, dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t}\, dt.$$



Path in Example 2

By the Euler formula the right side equals

$$i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t\, dt + i\int_0^{2\pi} \sin(m+1)t\, dt\right].$$

If $m = -1$, we have $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. We thus obtain $2\pi i$. For integer $m \ne -1$ each of the two integrals is zero because we integrate over an interval of length 2π, equal to a period of sine and cosine. Hence the result is

(2) [illegible] $= \begin{cases} 2\pi i & [illegible] \\ 0 & [illegible] \text{ integer} \end{cases}$

Dependence on path. [illegible] If we integrate a given function $f(z)$ from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. [illegible] complex line integral depends not only on the endpoints of the path but in general also on the path itself. [illegible] example.

Example 3. Integral of a non-analytic function. Dependence on path

Integrate $f(z) = \text{Re } z$ [illegible]

(a) Along C^* in Fig. below,

(b) Along C consisting of C_1 and C_2

Solution:

(a) C^* can be represented by [illegible] calculate

$$\int_{C^*} \text{Re } z\, dz = \int_0^1 [illegible] = [illegible]$$

(b) We now have

C_1: [illegible] $\quad$ [illegible] $\quad$ [illegible]

C_2: [illegible] $\quad$ [illegible] $\quad$ [illegible]

[illegible] C_1 and C_2 [illegible]

$$\int_C \text{Re } z\, dz = \int_{C_1} \text{Re } z\, dz + \int_{C_2} \text{Re } z\, dz = [illegible]$$

Note that this result differs from the result in (a).

[illegible]

4.8 Cauchy's Theorem

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\int_C f(z)dz = 0$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that [illegible]

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \quad \dots (1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C.

Hence the Green's theorem can be applied to (1) giving

$$\oint_C f(z)dz = -\iint_R \left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]dx\,dy + i\iint_R \left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]dx\,dy \quad ...(i)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (i) vanish identically.

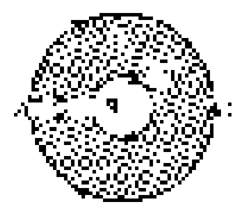
Hence $\oint_C f(z)dz = 0$.

Obs. 1 The Cauchy's theorem [illegible] independent of the path joining any two points of D.

Obs. 2 Extension of Cauchy's theorem. If $f(z)$ is analytic in the region R between two simple closed curves C and C_1 then $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in figure below, i.e., along AB – along C_1 in clockwise sense and along BA – along C in anti-clockwise sense.

i.e. $\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz + \int_C f(z)dz = 0$



But, since the integrals along AB and along BA cancel, it follows that

$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

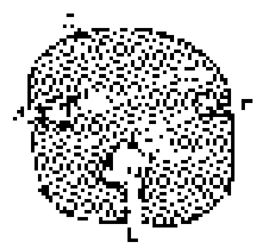
Reversing the direction of the integral around C_1 and transposing, we get

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

each integration being taken in the anti-clockwise sense.

If C_1, C_2, C_3 ... be any number of closed curves within C (Fig.) as below, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$



4.9 Cauchy's Integral Formula

If $f(z)$ is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz \qquad \left\{\begin{array}{l}\text{For any point on } C_1,\\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}d\theta\end{array}\right.$$

$$= \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}d\theta = i\int_{C_1} f(a+re^{i\theta})d\theta \quad ...(i)$$

Thus, as the circle C' shrinks to the point a, i.e. as r → 0, the integral above approaches

$$\int_{C'} f(a)\frac{dz}{z-a} = f(a)\int_0^{2\pi} i\,d\theta = 2\pi i f(a). \text{ Thus } \int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

or
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \qquad \dots(1)$$

which is the desired Cauchy's integral formula.

Cor. Differentiating both sides of (1) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz \qquad \dots(2)$$

Similarly,
$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz \qquad \dots(3)$$

And in general,
$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad \dots(4)$$

Thus it follows from the results (2), (3) and (4) that if a function f(z) is known on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C. Incidentally we have established a remarkable fact that an analytic function possesses derivatives of all orders and these are themselves all analytic.

4.10 Series of Complex Terms

1. **Taylor's series:** If f(z) is analytic inside a circle C with centre at a, then for z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots \qquad (1)$$

2. **Laurent's series:** If f(z) is analytic in the ring shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a, then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt$$

Γ being any curve in R, encircling C_1.

Obs. 1. As f(z) is analytic inside C, then $a_n = \frac{1}{2\pi i}\oint \frac{f(t)}{(t-a)^{n+1}}dt \neq \frac{f^n(a)}{n!}$

However, if f(z) is analytic inside C, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i}\oint \frac{f(t)}{(t-a)^{n+1}}dt = \frac{f^n(a)}{n!}$

and Laurent's series reduces to Taylor's series.

Obs. 2. To obtain Taylor's or Laurent's series, simply expand f(z) by binomial theorem instead of finding a_n by complex integration which is quite complicated.

[illegible]

4.11 Zeros and Singularities or Poles of an Analytic Function

4.11.1 Zeros of an Analytic Function

Definition: A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$.

If $f(z)$ is analytic in the neighbourhood of a point $z = a$ then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + \cdots \quad \text{where } a_n = \frac{f^n(a)}{n!}$$

If $a_0 = a_1 = a_2 = \cdots = a_{m-1} = 0$ but $a_m \neq 0$ then $f(z)$ is said to have a zero of order m at $z = a$.

When $m = 1$ the zero is said to be simple. In the neighbourhood of zero ($z = a$) $f(z)$ becomes

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \cdots$$
$$= (z-a)^m \phi(z)$$

where $$\phi(z) = a_m + a_{m+1}(z-a) + \cdots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

Example 1.

Poles and Essential singularities

The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

and

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}}$$

$$= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots$$

Note: The classification of singularities into poles and essential singularities is not merely a formal matter, because the behavior of an analytic function in a neighbourhood of an essential singularity is entirely different from that in the neighbourhood of a pole.

Example 2.

Find the nature of singularities of following functions:

(a) $f(z) = \dfrac{1}{z(z-2)^5} + \dfrac{3}{(z-2)^2}$ (b) $e^{1/z}$ (c) $\sin\dfrac{1}{z}$

Example 3.

Find the nature and location of singularities of the following functions:

(i) $\dfrac{z - \sin z}{z}$ (ii) $(z+1)\sin\dfrac{1}{z-2}$ (iii) $\dfrac{1}{\cos z - \sin z}$

Solution:

(a) Here $z = 0$ is a singularity.

Also
$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)\right]$$
$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots$$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

(b)
$$(z-1)\sin\frac{1}{z-2} = (t+1)\sin\frac{1}{t}, \text{ where } t = z-2$$
$$= (t+1)\left[\frac{1}{t} - \frac{1}{3!\,t^3} + \frac{1}{5!\,t^5} - \dots\right]$$
$$= \left(1 - \frac{1}{3!\,t^2} + \frac{1}{5!\,t^4} - \dots\right) + \left(\frac{1}{t} - \frac{1}{3!\,t^3} + \frac{1}{5!\,t^5} - \dots\right)$$
$$= 1 + \frac{1}{t} - \frac{1}{3!\,t^2} - \frac{1}{3!\,t^3} + \dots$$
$$= 1 + \frac{1}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{6(z-2)^3} + \dots$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ is an essential singularity.

(c) The poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, i.e., by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a simple pole of $f(z)$.

4.12 Residues

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point. Thus in the Laurent's series expansion of $f(z)$ around $z = a$ i.e. $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

Since,
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$
$$\therefore \quad a_{-1} = \operatorname{Res} f(a) = \frac{1}{2\pi i}\oint_C f(z)\,dz$$
$$\therefore \quad \oint_C f(z)\,dz = 2\pi i \operatorname{Res} f(a) \qquad \text{(i)}$$

4.12.1 Residue Theorem

If $f(z)$ is analytic in a closed curve C except at a finite number of singular points within C, then

$\oint_C f(z)\,dz = 2\pi i \times$ (sum of the residues at the singular points within C)

Let us surround each of the singular points $a_1, a_2, \dots, a_n$ by a small circle such that it encloses no other singular point. Then these circles $C_1, C_2, \dots, C_n$ together with C form a multiply connected region in which $f(z)$ is analytic.

According to Cauchy's theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \dots + \int_{C_n} f(z)dz \qquad \text{(By (i))}$$

$$= 2\pi i[\text{Res } f(a_1) + \text{Res } f(a_2) + \dots + \text{Res } f(a_n)]$$

which is the desired result.

4.12.2 Calculation of Residues

1. If $f(z)$ has a simple pole at $z = a$, then

$$\text{Res } f(a) = \underset{z \to a}{\text{Lt}} [(z-a) f(z)] \qquad \dots(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a) f(z) = c_0(z-a) + c_1(z-a)^2 + \dots + c_{-1}$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}} [(z-a) f(z)] = c_{-1} = \text{Res } f(a)$$

2. Another formula for Res $f(a)$:

Let $$f(z) = \phi(z)/\psi(z), \text{ where } \psi(z) = (z-a)F(z), F(a) \neq 0$$

Then $$\underset{z \to a}{\text{Lt}} [(z-a)\phi(z)/\psi(z)] = \underset{z \to a}{\text{Lt}} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \dots]}{\psi(a) + (z-a)\psi'(a) + \dots}$$

$$= \underset{z \to a}{\text{Lt}} \frac{\phi(a) + (z-a)\phi'(a) + \dots}{\psi'(a) + (z-a)\psi''(a) + \dots}, \text{ since } \psi(a) = 0$$

Thus, $$\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)}$$

3. If $f(z)$ has a pole of order n at $z = a$, then

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

Obs. In many cases, the residue of a pole $(z = a)$ can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|t|$ is quite small.

■■■■

Previous GATE and ESE Questions

Q.1 [illegible]

[illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[illegible], GATE-2005, 2 marks]

Q.2 [illegible]

(a) [illegible] $-4\pi i$

(b) [illegible]

(c) $\frac{4\pi}{31}-6\pi i$ [partly illegible]

(d) [illegible]

[CE, GATE-2009, 2 marks]

Q.3 [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[EC, GATE-2008, 2 marks]

Q.4 [illegible]

(a) [illegible]

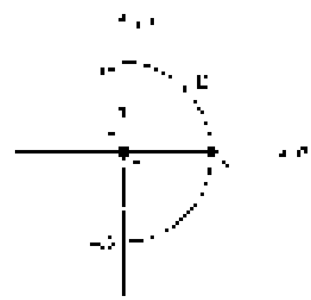
(b) [illegible]

(c) [illegible]

(d) [illegible]

[EE, GATE-2007, 2 marks]

Q.5 [illegible]



(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[EC, GATE-2007, 2 marks]

Q.6 [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[ME, GATE-2009, 2 marks]

Q.7 [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[EC, GATE-2009, 2 marks]

Q.8 [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[ME, GATE-2009, 2 marks]

Q.9 The value of the integral $\oint_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}dz$ (where C is a closed curve given by $|z| = 1$) is

(a) $-\pi i$ (b) $\frac{\pi i}{5}$

(c) $\frac{2\pi i}{5}$ (d) πi

[CE, GATE-2009, 2 marks]

Q.10 The analytic function $f(z) = \frac{z-1}{z^2+1}$ has singularities at

(a) 1 and −1 (b) 1 and i

(c) 1 and $-i$ (d) i and $-i$

[CE, GATE-2009, 1 mark]

Q.11 If $f(z) = c_0 + c_1 z^{-1}$, then $\oint_{|z|=1} \frac{1+f(z)}{z}dz$ is given by

(a) $2\pi c_1$ (b) $2\pi(1 + c_0)$

(c) $2\pi j c_1$ (d) $2\pi j(1 + c_0)$

[EC, GATE-2009, 1 mark]

Q.12 The modulus of the complex number $\left|\frac{3+4i}{1-2i}\right|$ is

(a) 5 (b) $\sqrt{5}$

(c) $1/\sqrt{5}$ (d) 1/5

[ME, GATE-2010, 1 mark]

Q.13 The residues of a complex function

$$X(z) = \frac{1-2z}{z(z-1)(z-2)}$$ at its poles are

(a) $\frac{1}{2}, -\frac{1}{2}$ and 1 (b) $\frac{1}{2}, \frac{1}{2}$ and −1

(c) $\frac{1}{2}$, −1 and $-\frac{3}{2}$ (d) $\frac{1}{2}$, −1 and $\frac{3}{2}$

[EC, GATE-2010, 2 marks]

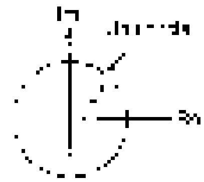
Q.14 For an analytic function, $f(x + iy) = u(x, y) + iv(x, y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v, considering K to be a constant is

(a) $3y^2 - 3x^2 + K$ (b) $6x - 6y + K$

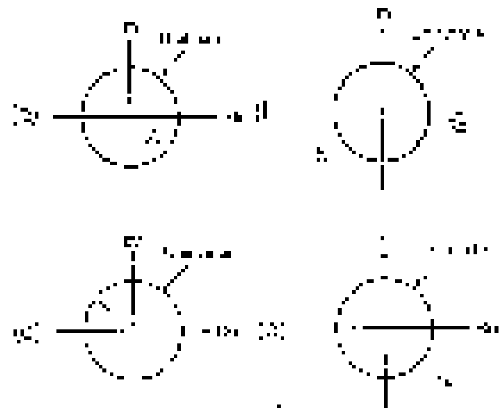
(c) $6y - 6x + K$ (d) $6xy + K$

[CE, GATE-2011, 2 marks]

Q.15 A point z has been plotted in the complex plane, as shown in figure below.



The plot of the complex number $y = \frac{1}{z}$ is



[EE, GATE-2011, 1 marks]

Q.16 The value of the integral $\oint_C \frac{-3z+4}{z^2+4z+5}dz$ where C is the circle $|z| = 1$ is given by

(a) 0 (b) 1/10

(c) 4/5 (d) 1

[EC, GATE-2011, 1 mark]

Q.17 If $x = \sqrt{-1}$, then the value of x^x is

(a) $e^{-\pi/2}$ (b) $e^{\pi/2}$

(c) x (d) 1

[EC, EE, IN, GATE-2012, 1 mark]

Q.18 Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in the z-plane such that $|z+1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)dz$ is

(a) −2 (b) −1

(c) 1 (d) 2

[EC, EE, IN, GATE-2012, 1 mark]

Q.19 Square roots of $-i$, where $i = \sqrt{-1}$, are
(a) $i, -i$
(b) $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$
(c) $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$
(d) $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$
[EE, GATE-2013, 1 Mark]

Q.20 The complex function $\tanh(s)$ is analytic over a region of the imaginary axis of the complex s-plane if the following is TRUE everywhere in the region for all integers n
(a) $\text{Re}(s) = 0$
(b) $\text{Im}(s) \neq n\pi$
(c) $\text{Im}(s) \neq \frac{n\pi}{3}$
(d) $\text{Im}(s) \neq \frac{(2n+1)\pi}{2}$
[IN, GATE-2013 : 1 Mark]

Q.21 $\oint \frac{z^2-4}{z^2+4}dz$ evaluated anticlockwise around the circle $|z-i| = 2$, where $i = \sqrt{-1}$, is
(a) -4π
(b) 0
(c) $2+\pi$
(d) $2+2i$
[EE, GATE-2013, 2 Marks]

Q.22 $z = \frac{2-3i}{-5+i}$ can be expressed as
(a) $-0.5-0.5i$
(b) $-0.5+0.5i$
(c) $0.5-0.5i$
(d) $0.5+0.5i$
[CE, GATE-2014 : 1 Mark]

Q.23 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + i\,v(x, y)$, where $i = \sqrt{-1}$. If $u = xy$, then $v(x, y)$ must be
(a) $x^2 + y^2 + \text{constant}$
(b) $x^2 - y^2 + \text{constant}$
(c) $-x^2 + y^2 + \text{constant}$
(d) $-x^2 - y^2 + \text{constant}$
[ME, GATE-2014 : 2 Marks]

Q.24 An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + i\,v(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = x^2 - y^2$, then expression for $v(x, y)$ in terms of x, y and a general constant c would be
(a) $xy + c$
(b) $\frac{x^2 + y^2}{2} + c$
(c) $2xy + c$
(d) $\frac{(x-y)^2}{2} + c$
[ME, GATE-2014 : 2 Marks]

Q.25 The argument of the complex number $\frac{1+i}{1-i}$, where $i = \sqrt{-1}$, is
(a) $-\pi$
(b) $-\frac{\pi}{2}$
(c) $\frac{\pi}{2}$
(d) π
[ME, GATE-2014 : 1 Mark]

Q.26 Let S be the set of points in the complex plane corresponding to the unit circle. (That is, $S = \{z : |z| = 1\}$). Consider the function $f(z) = zz^*$ where z^* denotes the complex conjugate of z. The $f(z)$ maps S to which one of the following in the complex plane
(a) unit circle
(b) horizontal axis line segment from origin to $(1, 0)$
(c) the point $(1, 0)$
(d) the entire horizontal axis
[EE, GATE-2014 : 1 Mark]

Q.27 All the values of the multi-valued complex function 1^i, where $i = \sqrt{-1}$, are
(a) purely imaginary
(b) real and non-negative
(c) on the unit circle
(d) equal in real and imaginary parts
[EE, GATE-2014 : 1 Mark]

Q.28 The real part of an analytic function $f(z)$ where $z = x + jy$ is given by $e^{-y}\cos(x)$. The imaginary part of $f(z)$ is
(a) $e^{y}\cos(x)$
(b) $e^{-y}\sin(x)$
(c) $-e^{y}\sin(x)$
(d) $-e^{-y}\sin(x)$
[EC, GATE-2014 : 2 Marks]

Q.29 If z is a complex variable, the value of $\int_5^{3i}\frac{dz}{z}$ is
(a) $-0.511 - 1.57i$
(b) $-0.511 + 1.57i$
(c) $0.511 - 1.57i$
(d) $0.511 + 1.57i$
[ME, GATE-2014 : 2 Marks]

Q.30 Integration of the complex function $f(z) = \frac{z^2}{z^2 - 1}$, in the counterclockwise direction, around $|z - 1| = 1$, is
(a) $-\pi i$ (b) 0
(c) πi (d) $2\pi i$
[EC, GATE-2014 : 2 Marks]

Q.31 The Taylor series expansion of $3 \sin x + 2 \cos x$ is
(a) $2 + 3x - x^2 - \frac{x^3}{2} + \ldots$
(b) $2 - 3x + x^2 - \frac{x^3}{2} + \ldots$
(c) $2 + 3x + x^2 + \frac{x^3}{2} + \ldots$
(d) $2 - 3x - x^2 + \frac{x^3}{2} + \ldots$
[EC, GATE-2014 : 2 Marks]

Q.32 The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to
(a) $2 \ln 2$ (b) $\sqrt{2}$
(c) 2 (d) e
[EC, GATE-2014 : 1 Mark]

Q.33 Given two complex numbers $z_1 = 5 + (5\sqrt{3})i$ and $z_2 = \frac{2}{\sqrt{3}} + 2i$, the argument of $\frac{z_1}{z_2}$ in degrees is
(a) 0 (b) 30
(c) 60 (d) 90
[ME, GATE-2015 : 1 Mark]

Q.34 Given $f(z) = g(z) + h(z)$, where f, g, h are complex valued functions of a complex variable z. Which one of the following statements is TRUE?
(a) If $f(z)$ is differentiable at z_0, then $g(z)$ and $h(z)$ are also differentiable at z_0
(b) If $g(z)$ and $h(z)$ are differentiable at z_0, then $f(z)$ is also differentiable at z_0
(c) If $f(z)$ is continuous at z_0, then it is differentiable at z_0
(d) If $f(z)$ is differentiable at z_0, then so are its real and imaginary parts.
[EE, GATE-2015 : 1 Mark]

Q.35 Let $z = x + iy$ be a complex variable. Consider that contour integration is performed along the unit circle in anticlockwise direction. Which one of the following statements is NOT TRUE?
(a) The residue of $\frac{z}{z^2 - 1}$ at $z = 1$ is 1/2
(b) $\oint_C z^2 dz = 0$
(c) $\frac{1}{2\pi i} \oint_C \frac{1}{z} dz = 1$
(d) $\bar{z}$ (complex conjugate of z) is an analytical function
[EE, GATE-2015 : 1 Mark]

Q.36 Let $f(z) = \frac{az + b}{cz + d}$. If $f(z_1) = f(z_2)$ for all $z_1 \neq z_2$, $a = 2$, $b = 4$ and $c = 5$, then d should be equal to ______.
[EC, GATE-2015 : 1 Mark]

Q.37 The value of $\oint \frac{1}{z^2} dz$, where the contour is the unit circle traversed clockwise, is
(a) $-2\pi i$ (b) 0
(c) $2\pi i$ (d) $4\pi i$
[IN, GATE-2015 : 1 Mark]

Q.38 If C denotes the counterclockwise unit circle, the value of the contour integral $\frac{1}{2\pi j} \oint_C \text{Re}\{z\} dz$ is ______.
[EC, GATE-2015 : 2 Marks]

Q.39 If C is a circle of radius r with centre z_0, in the complex z-plane and if n is a non-zero integer, then $\oint_C \frac{dz}{(z - z_0)^{n+1}}$ equals
(a) $2\pi nj$ (b) 0
(c) $\frac{nj}{2\pi}$ (d) $2\pi n$
[EC, GATE-2015 : 1 Mark]

Q.40 Consider the following complex function:
$$f(z) = \frac{9}{(z - 1)(z + 2)^2}$$
Which of the following is one of the residues of the above function?

(a) 1
(b) $\frac{[illegible]}{16}$
(c) 4
(d) 8

[CE, GATE-2015 : 2 Marks]

Q.41 In the neighborhood of $z = 1$, the function $f(z)$ has a power series expansion of the form $f(z) = 1 + (1 - z) + (1 - z)^2 + \ldots$ Then $f(z)$ is

(a) $\frac{1}{z}$
(b) $\frac{-1}{z-2}$
(c) $\frac{z-1}{z+1}$
(d) $\frac{1}{2z-1}$

[IN, GATE-2016 : 1 Mark]

Q.42 Consider the complex valued function $f(z) = 2z^3 + b|z|^3$ where z is a complex variable. The value of b for which the function $f(z)$ is analytic is ______.

[EC, GATE-2016 : 1 Mark]

Q.43 $f(z) = u(x, y) + iv(x, y)$ is an analytic function of complex variable $z = x + iy$ where $i = \sqrt{-1}$. If $u(x, y) = 2xy$, then $v(x, y)$ may be expressed as

(a) $-x^2 + y^2 + \text{constant}$
(b) $x^2 - y^2 + \text{constant}$
(c) $x^2 + y^2 + \text{constant}$
(d) $-(x^2 + y^2) + \text{constant}$

[ME, GATE-2016 : 1 Mark]

Q.44 A function f of the complex variable $z = x + iy$, is given as $f(x, y) = u(x, y) + iv(x, y)$, where $u(x, y) = 2kxy$ and $v(x, y) = x^2 - y^2$. The value of k, for which the function is analytic, is ______.

[ME, GATE-2016 : 1 Mark]

Q.45 Consider the function $f(z) = z + z^*$ where z is a complex variable and z^* denotes its complex conjugate. Which one of the following is TRUE?

(a) $f(z)$ is both continuous and analytic
(b) $f(z)$ is continuous but not analytic
(c) $f(z)$ is not continuous but is analytic
(d) $f(z)$ is neither continuous nor analytic

[EE, GATE-2016 : 1 Mark]

Q.46 The value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2}\, dx$$

evaluated using contour integration and the residue theorem is

(a) $\frac{-\pi \sin(1)}{e}$
(b) $\frac{-\pi \cos(1)}{e}$
(c) $\frac{\sin(1)}{e}$
(d) $\frac{\cos(1)}{e}$

[ME, GATE-2016 : 2 Marks]

Q.47 The value of the integral

$$\oint_C \frac{2z + 5}{\left(z - \frac{1}{2}\right)(z^2 - 4z + 5)}\, dz$$

over the contour $|z| = 1$, taken in the anti-clockwise direction, would be

(a) $\frac{24\pi i}{13}$
(b) $\frac{48\pi i}{13}$
(c) $\frac{24}{13}$
(d) $\frac{12}{13}$

[EE, GATE-2016 : 1 Mark]

Q.48 The value of the integral $\frac{1}{2\pi j}\oint_C \frac{z^2 + 1}{z^2 - 1}\, dz$ where z is a complex number and C is a unit circle with center at $1 + 0j$ in the complex plane is ______.

[IN, GATE-2016 : 2 Marks]

Q.49 In the following integral, the contour C encloses the points $2\pi j$ and $-2\pi j$. $-\frac{1}{2\pi}\oint_C \frac{\sin z}{(z - 2\pi j)^3}\, dz$. The value of the integral is ______.

[EC, GATE-2016 : 2 Marks]

Q.50 The values of the integral $\frac{1}{2\pi j}\oint_C \frac{e^z}{z-2}\, dz$ along a closed contour C in anti-clockwise direction for
(i) the point $z_0 = 2$ inside the contour C, and
(ii) the point $z_0 = 2$ outside the contour C,
respectively, are

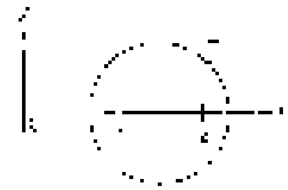
(a) (i) 2.72, (ii) 0
(b) (i) 7.39, (ii) 0
(c) (i) 0, (ii) 2.72
(d) (i) 0, (ii) 7.39

[EC, GATE-2016 : 2 Marks]

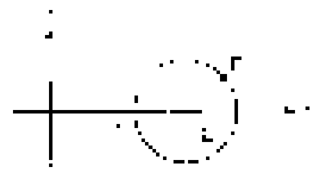
Q.51 For $f(z) = \frac{\sin(z)}{z^2}$, the residue of the pole at $z = 0$ is ______.

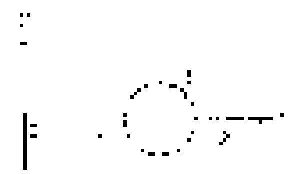
[EC, GATE-2016 : 1 Mark]

Q.52 The value of $\oint_\Gamma \frac{3z - 5}{(z-1)(z-2)}\, dz$ along a closed path Γ is equal to $(4\pi i)$, where $z = x + iy$ and $i = \sqrt{-1}$. The correct path Γ is

(a) 

(b) 

(c) 

(d) 

[ME, GATE-2016 : 2 Marks]

Q.53 If $f(z) = (x^2 - ay^2) + ibxy$ is a complex analytic function of $z = x + iy$, where $i = \sqrt{-1}$, then

(a) $a = -1, b = -1$ (b) $a = -1, b = 2$

(c) $a = 1, b = 2$ (d) $a = 2, b = 2$

[ME, GATE-2017 : 2 Marks]

Q.54 Let $z = x + iy$ where $i = \sqrt{-1}$. Then $\overline{\cos z} =$

(a) $\cos z$ (b) $\cos \bar{z}$

(c) $\sin z$ (d) $\sin \bar{z}$

[IN, GATE-2017 : 1 Mark]

Q.55 The value of the contour integral in the complex-plane

$$\oint \frac{z^3 - 2z + 3}{z - 2} dz$$

along the contour $|z| = 3$, taken counter-clockwise is

(a) $-18\pi i$ (b) 0

(c) $14\pi i$ (d) $48\pi i$

[EE, GATE-2017 : 2 Marks]

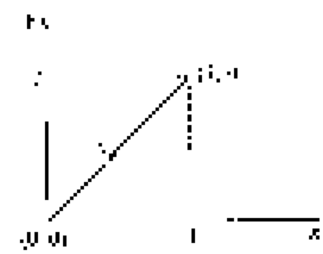
Q.56 For a complex number z, $\lim_{z \to i} \frac{z^2 + 1}{z^3 + 2z - i(z^2 + 2)}$ is

(a) $-2i$ (b) $-i$

(c) i (d) $2i$

[EE, GATE-2017 : 1 Mark]

Q.57 Consider the line integral $I = \int_c (x^2 + iy^2)\,dz$ where $z = x + iy$. The line c is shown in the figure below.



The value of I is

(a) $\frac{1}{2}i$ (b) $\frac{2}{3}i$

(c) $\frac{3}{4}i$ (d) $\frac{4}{5}i$

[EC, GATE-2017 : 2 Marks]

Q.58 The residues of a function

$$f(z) = \frac{1}{(z - 4)(z + 1)^3}$$

are

(a) $\frac{-1}{27}$ and $\frac{-1}{125}$ (b) $\frac{1}{125}$ and $\frac{-1}{125}$

(c) $\frac{-1}{27}$ and $\frac{1}{5}$ (d) $\frac{1}{125}$ and $\frac{-1}{5}$

[EC, GATE-2017 : 1 Mark]

Q.59 An integral I over a counter clockwise circle C is given by

$$I = \oint_C \frac{z^2 - 1}{z^2 + 1} e^z\,dz$$

If C is defined as $|z| = 3$, then the value of I is

(a) $-\pi i \sin(1)$ (b) $-2\pi i \sin(1)$

(c) $-3\pi i \sin(1)$ (d) $-4\pi i \sin(1)$

[EC, GATE-2017 : 2 Marks]

Q.60 If $f(z) = u + iv$ represents the complex potential for an electric field and $v = x^2 - y^2 + \frac{x}{x^2 + y^2}$, then the function u is

(a) $-2xy - \frac{y}{x^2 + y^2} + C$

(b) $2xy + \frac{y}{x^2 + y^2} + C$

(c) $-2xy + \frac{y}{x^2 + y^2} + C$

(d) $2xy + \frac{x}{x^2 + y^2} + C$

[ESE Prelims-2017]

Q.61 The residue of $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 3$ is

(a) 9 (b) $\frac{10}{13}$

(c) 1 (d) $\frac{27}{16}$

[ESE Prelims-2017]

■■■■

Answers Second Order Linear Partial Differential Equations

1.	(a)	2.	(a)	3.	[illegible]	4.	[illegible]	5.	[illegible]	6.	(c)	7.	(b)	8.	[illegible]	9.	[illegible]
10.	(d)	11.	(b)	12.	(c)	13.	(c)	14.	[illegible]	15.	[illegible]	16.	(c)	17.	(a)	18.	[illegible]
19.	(a)	20.	(d)	21.	(a)	22.	(a)	23.	(c)	24.	[illegible]	25.	[illegible]	26.	(c)	27.	(b)
28.	[illegible]	29.	[illegible]	30.	(c)	31.	[illegible]	32.	(b)	33.	(d)	34.	[illegible]	35.	[illegible]	37.	(a)
38.	(c)	40.	(a)	41.	[illegible]	46.	(a)	45.	[illegible]	48.	[illegible]	47.	(b)	50.	[illegible]	52.	[illegible]
53.	(b)	54.	(b)	55.	[illegible]	56.	(d)	57.	[illegible]	58.	[illegible]	59.	(b)	60.	[illegible]	61.	[illegible]

Explanations Second Order Linear Partial Differential Equations

1. (a)

$$\oint f(z)\,dz = \oint \frac{1}{\cos z}\,dz$$

The poles are at

$z = (2n+1)\pi/2$ [illegible]

None of these poles lie inside the unit circle $|z| = 1$

Hence, sum of residues at poles = 0

∴ singular [illegible] = [illegible]

= $2\pi i$ [sum of residues of $f(z)$ at the poles]

= $2\pi i \times 0 = 0$

2. (a)

Cauchy's integral theorem is

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

i.e. $\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

$$\text{Here } \oint_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\oint_C \frac{z^3-6}{z-\frac{i}{3}}\,dz$$

Applying Cauchy's integral theorem ($a = i/3$)

$f(z) = z^3 - 6$

$$= \frac{1}{3}2\pi i\left[\left(\frac{i}{3}\right)^3 - 6\right]$$

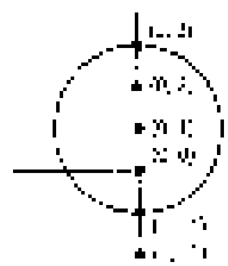
$$= \frac{1}{3}2\pi i\left[\frac{-i}{27} - 6\right] = \frac{2\pi}{81} - 4\pi i$$

$$= \frac{2\pi}{81} - 4\pi i$$

3. (d)

$$\frac{1}{z^2-4} = \frac{1}{(z+2)(z-2)}$$

Pole (2, 0) lies inside the circle $|z - i| = 2$ and pole (−2, 0) lies outside the circle $|z - i| = 2$ as can be seen from figure below.



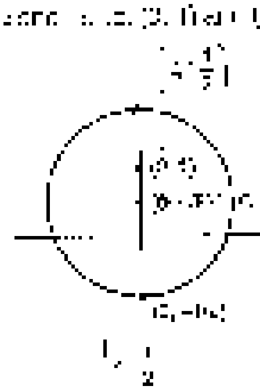
$\oint_C f(z)\,dz = 2\pi i$ [Residue at poles which lie inside C]

$$= 2\pi i \,\text{Res}\, f(2) = 2\pi i \frac{1}{(2+2)} = \frac{\pi i}{2}$$

4. (b)

[illegible]

Poles are at [illegible] and (0, −1)



[illegible]

[illegible]

So $I = 2\pi i$ Res $f(z) = $ [illegible]

5. (a)

$$I = \oint_C \frac{1}{z^2-1}dz = \oint_C \frac{1}{(z-1)(z+1)}dz$$

$= 2\pi i$ (Sum of the residues)

pole $z = -1$ is outside the circle C, but $z = 1$ is inside C.

residue at pole $z = 1$ is

$$= \lim_{z\to 1}\frac{(z-1)}{(z-1)(z+1)} = \frac{1}{2}$$

$$\therefore \oint_C \frac{1}{z^2-1}dz = 2\pi i\cdot\frac{1}{2} = \pi i$$

6. (a)

$$f(z) = \frac{\cos z}{z}$$

has simple pole at $z = 0$ and $z = 0$ is inside unit circle in complex plane.

$\therefore$ Residue of f at $z = 0$:

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0}\cos z = 1$$

$$\int_C f(z)\,dz = 2\pi i\ (\text{Residue at } z = 0)$$

$$= 2\pi i \times 1 = 2\pi i$$

7. (a)

Since $\lim_{z\to 2}[(z-2)^2 f(z)]$ is finite and non zero,

$f(z)$ has a pole of order two at $z = 2$.

The residue at a pole of order m is given as

$$\text{Res } f(a) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]_{z=a}$$

Here $z = 2$ is a pole of order 2 hence $m = 2$

$$\therefore \text{Res } f(2) = \frac{1}{1!}\left[\frac{d}{dz}\left[(z-2)^2 f(z)\right]\right]_{z=2}$$

$$= \left[\frac{d}{dz}(z-2)^2\frac{1}{(z-2)^2(z-3)}\right]_{z=2}$$

$$= \left[\frac{d}{dz}\frac{1}{(z-3)}\right]_{z=2} = \left[-1(z-3)^{-2}\right]_{z=2}$$

$$= \frac{-1}{(2-3)^2} = -1$$

8. (c)

$f(z) = u + iv$ (u is given)

$\therefore$ It must satisfy the Cauchy-Riemann equations:

$$u_x = v_y \qquad \text{(i)}$$

and $$u_y = -v_x \qquad \text{(ii)}$$

Here given $u = xy$ (given)

$\Rightarrow$ $u_x = y$ and $u_y = x$

Now substituting u_x and u_y in (i) and (ii) we get

$$v_y = y \qquad \text{(iii)}$$

and $$v_x = -x \qquad \text{(iv)}$$

Integrating (iii) w.r.t. y we can now get v as follows

$$v_y = y$$

$$\Rightarrow \quad \frac{\partial v}{\partial y} = y$$

$$\Rightarrow \quad \int dv = \int y\,dy$$

$$\Rightarrow \quad v = \frac{y^2}{2} + f(x) \qquad \ldots\text{(v)}$$

From (v) we have

$$v_x = f'(x) \qquad \ldots\text{(vi)}$$

Since (iv) = (vi) we have,

$$v_x = -x$$

Substituting this in (vi) we get,

$$f'(x) = -x$$

$$\Rightarrow \quad \frac{df}{dx} = -x$$

$$\Rightarrow \quad \int df = \int -x\,dx$$

$$\Rightarrow \quad f = -\frac{x^2}{2} + k$$

Now substituting this in (v) we get,

$$v = \frac{y^2}{2} - \frac{x^2}{2} + k \Rightarrow v = \frac{y^2-x^2}{2} + k$$

9. (c)

Let, $$I = \int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}dz$$

$$= \frac{1}{2\pi i}\oint \frac{\left[\frac{\cos(2\pi z)}{(z-2)}\right]}{\left(z-\frac{1}{2}\right)}$$

Since $z = 1/2$ is a point within $|z| = 1$ (closed curve), we can use Cauchy's integral theorem and see that

$$I = \frac{1}{2\pi i}\oint \frac{f(z)}{\left(z-\frac{1}{2}\right)}$$

where $\quad f(z) = \dfrac{\cos(2\pi z)}{(z-2)}$

[Notice $f(z)$ is analytic on and inside $|z| = 1$]

$$\therefore \qquad I = \frac{1}{2}\,\frac{\cos\left(2\pi \times \frac{1}{2}\right)}{\left(\frac{1}{2}-2\right)} = \frac{2\pi}{3}$$

10. (d)

$$f(z) = \frac{z-1}{z^2-z} = \frac{z-1}{z^2-z} = \frac{z-1}{(z-0)(z-1)}$$

$\therefore$ The singularities at $z = 0$ and 1

11. (d)

$$f(z) = c_0 + c_1 z^{-1}$$

$$\oint \frac{1+f(z)}{z}\,dz = ?$$

$\frac{1+f(z)}{z}$ has a pole at origin which is inside unit circle

So, $\oint \frac{1+f(z)}{z}\,dz = 2\pi i$ [Residue of $f(z)$ at $z = 0$]

$$= 2\pi i[1 + f(0)]$$

Since $\quad f(z) = c_0 + c_1 z^{-1} \Rightarrow f(0) = c_0$

$\therefore \quad$ Answer $= 2\pi i(1 + c_0)$

12. (b)

$$z = \frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)}$$

$$= \frac{-5+10i}{5} = -1+2i$$

$$|z| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$$

13. (c)

$$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles are $z = 0$, $z = 1$ and $z = 2$

Residue at $z = 0$

$$\text{residue} = \text{value of } \frac{1-2z}{(z-1)(z-2)} \text{ at } z = 0$$

$$= \frac{1-2\times 0}{(0-1)(0-2)} = \frac{1}{2}$$

Residue at $z = 1$

$$\text{residue} = \text{value of } \frac{1-2z}{z(z-2)} \text{ at } z = 1$$

$$= \frac{1-2\times 1}{1(1-2)} = 1$$

Residue at $z = 2$

$$\text{residue} = \text{value of } \frac{1-2z}{z(z-1)} \text{ at } z = 2$$

$$= \frac{1-2\times 2}{2(2-1)} = -\frac{3}{2}$$

$\therefore$ The residues at the poles are $\frac{1}{2}$, 1 and $-\frac{3}{2}$

14. (d)

$$f = u + iv$$

$$u = 3x^2 - 3y^2$$

for f to be analytic we have Cauchy Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{...(i)}$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \qquad \text{...(ii)}$$

From (i) we have

$$6x = \frac{\partial v}{\partial y}$$

$$\Rightarrow \qquad \int \partial v = \int 6x\,dy$$

$$v = 6xy + f(x)$$

i.e. $\qquad v = 6xy + f(x) \qquad$ ...(iii)

Now applying equation (ii) we get,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \qquad -6y = -\left[6y + \frac{df}{dx}\right]$$

$$\Rightarrow \quad 6x + \frac{\partial v}{\partial x} = 6y$$

$$\frac{\partial v}{\partial x} = 6y - 6x$$

By integrating,

$$v = 6xy - 3x^2 + k$$

Substituting in equation (iii)

$$v = 3y^2 + 6xy - 3x^2 + k$$

$$\Rightarrow \quad v = 6xy + k$$

15. (d)

Let $z = a + ib$

Since z is shown inside the unit circle in 2nd quadrant, a is $-$ve and b are both $+$ve and

$0 < \sqrt{a^2 + b^2} < 1$

Now $\frac{1}{z} = \frac{1}{a + ib}$

$$\frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

Since $a < 0$

$$\frac{a}{a^2 + b^2} < 0$$

$$\frac{-b}{a^2 + b^2} < 0$$

So $\frac{1}{z}$ is in III quadrant

$$\left|\frac{1}{z}\right| = \sqrt{\left(\frac{a}{a^2 + b^2}\right)^2 + \left(\frac{-b}{a^2 + b^2}\right)^2}$$

$$= \frac{1}{\sqrt{a^2 + b^2}}$$

Since $0 < \sqrt{a^2 + b^2} < 1$

$$\frac{1}{\sqrt{a^2 + b^2}} > 1$$

So $\frac{1}{z}$ should be outside unit circle in III quadrant.

16. (a)

$$I = \oint \frac{-3z + 4}{(z^2 + 4z + 5)}\,dz$$

$$= 2\pi i\,(\text{sum of residues})$$

Poles of $\frac{-3z + 4}{(z^2 + 4z + 5)}$ are given by

$$z^2 + 4z + 5 = 0$$

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

Since the poles lie outside the circle $|z| = 1$

So $f(z)$ is analytic inside the circle $|z| = 1$

Hence $\oint_C f(z)\,dz = 2\pi i(0) = 0$

17. (a)

$z = i$. In polar coordinates,

$$z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

Now $\quad z^i = i^i = \left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2}$

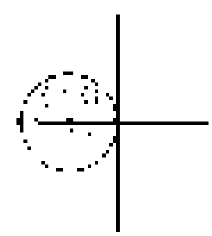
18. (c)

Given, $\quad f(z) = \frac{1}{z + 1} - \frac{2}{z + 3} = \frac{(z + 3) - 2(z + 1)}{(z + 1)(z + 3)}$

$$= \frac{-z + 1}{(z + 1)(z + 3)}$$

Poles are at -1 and -3 i.e. $(-1, 0)$ and $(-3, 0)$.

The figure above is $|z + 1| = 1$



We see that $(-1, 0)$ is inside the circle and $(-3, 0)$ is outside the circle.

From residue theorem,

$\frac{1}{2\pi i}\oint_C f(z)\,dz$ = sum of residues of poles which are inside C

So the required integral $\frac{1}{2\pi i}\oint_C f(z)\,dz$ is simply the residue of function at pole $z = -1$ (which is inside the circle)

The residue is $= \frac{-(-1) + 1}{(-1 + 3)} = \frac{2}{2} = 1$

19. (b)

$$-i = \cos\left(\frac{\pi}{2}\right) - i\sin\left(-\frac{\pi}{2}\right)$$

$$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$

$$(-i)^{1/2} = \left[\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right]^{1/2}$$

$$= \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$$

$$= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

20. (d)

Given log z ...

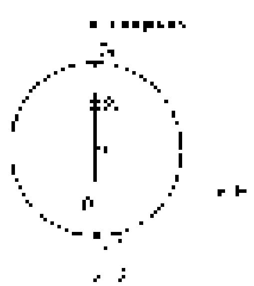
$\therefore$ $z = e^{...}$

$$z = \frac{(2\pi + 1)}{2}\pi$$

$\therefore$ $\log z = \frac{(2\pi + 1)}{2}\pi$

21. (a)

$$\frac{z^2-4}{z^2+4} = \frac{z^2-4}{(z+2i)(z-2i)}$$



Poles at $z = \pm 2i$ i.e. $(0, 2)$ and $(0, -2)$

From figure of $|z - i| = 2$ we see that only $2i$ is inside C.

While $z = -2i$ is outside C.

$$\therefore \oint \frac{z^2-4}{z^2+4}\,dz = 2\pi i \,\text{Res}\, f(2i)$$

$$= 2\pi i \left.\frac{(z-2i)(z^2-4)}{(z+2i)(z-2i)}\right|_{z=2i}$$

$$= 2\pi i \frac{(2i)^2-4}{(2i+2i)} = -4\pi$$

22. (b)

$$\frac{(2-3i)}{(-4-i)} = \frac{(2-3i)}{(-4+i)} \times \frac{(-4-i)}{(-4-i)}$$

$$= \frac{-8-2i+12i-3}{25-1} = \frac{13+16i}{25}$$

$$= 0.5 + 0.5i$$

23. (c)

Using Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y} = 2y$$

and $\quad \frac{\partial u}{\partial x} = 2x$

$$\frac{\partial v}{\partial y} = 2x$$

$\Rightarrow \quad v = 2xy + \phi(x)$

$$\frac{\partial v}{\partial x} = 2y + \phi'(x) = -2y$$

$\therefore \quad \phi(x) = -x^2 + \text{constant}$

$\therefore \quad v = y^2 - x^2 + \text{constant}$

24. (c)

Using Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$\therefore \quad \frac{\partial v}{\partial y} = 2x$

$\Rightarrow \quad v = 2xy + f(x)$

$$\frac{\partial v}{\partial x} = 2y + f'(x)$$

$\Rightarrow \quad -\frac{\partial u}{\partial y} = 2y + f'(x)$

$\Rightarrow \quad f'(x) = 0 \Rightarrow f(x) = C$

$\therefore \quad v = 2xy + C$

25. (c)

Let $z = $ [illegible]

$z = \frac{(1-i)^2 + 1}{(1-i)^2 + 1}$ [illegible]

$= \frac{2i}{2} = i$

$z = x + iy = i$

So, $x = 0$

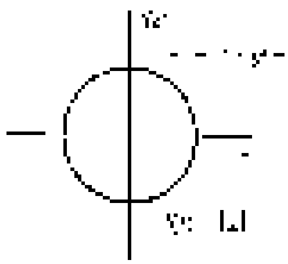
$y = 1$

$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}$ [illegible]

$= \frac{\pi}{2}$

26. (c)

$z = x + iy$



$\bar{z} = x - iy$

$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

[illegible]

$z\bar{z} = x^2 + y^2$

[illegible]

27. (b)

[illegible] $z = 1 - i$ [illegible] $z \geq 1$

$z =$ [illegible] which is purely real and non negative.

28. (b)

$Z = x + iy$

$f(z) = u + iv$

$u = e^x \cos(y)$

$\frac{\partial u}{\partial x} = e^x \sin(y)$

For analytical function

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial v}{\partial y} = $ [illegible]

Integrating w.r.t. y

$v = $ [illegible]

30. (c)

$f(z) = $ [illegible] $\frac{z^2}{}$ [illegible] $= \int f(z)$

Given circle

$|z - 1| = 1$

$\Rightarrow |(x + iy) - 1| = 1$

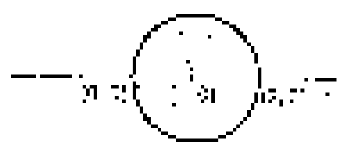
$(x-1)^2 + y^2 = 1$

[illegible] $1, y = 0, r = 1$

Poles of $f(z)$

z [illegible] $1 = 0$

[illegible]



$|z - 1|$ [illegible]

So, [illegible] outside circle

[illegible] inside circle

$\oint \frac{z^2}{(z-1)(z+1)} = 2\pi i$ [illegible] $\frac{z^2}{z}$ [illegible] $= 2\pi i\left[\frac{1}{2}\right]$ [illegible]

[illegible] $\int \frac{z^2}{(z-1)(z-1)}$ [illegible]

[illegible]

31. (a)

The Taylor's series expansion for

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$ [illegible]

and $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$ [illegible]

$\therefore 2\sin z = \ldots + \dfrac{2z^3}{3!} \ldots$

$= 2 + 3z - z^2 - \dfrac{z^3}{2} + \ldots$

32. (d)

Given

Sol. $f(z) = \sum \dfrac{1}{n!}$

$= \dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots$

$= 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \ldots$

Also we know the expansion of e^z

$e^z = 1 + z + \dfrac{1}{2}z^2 + \dfrac{1}{6}z^3 + \dfrac{1}{24}z^4 + \ldots$

Put $z = 1$ in above expression

$e^1 = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \ldots$

$= \sum \dfrac{1}{n!}$

33. (a)

$z_1 = 3 + (\sqrt{3})i \; ; \; z_2 = \ldots$

$\arg(z_1) = \theta_1 = \tan^{-1}\left(\dfrac{\sqrt{3}}{3}\right) \; ; \; \theta_1 = 30°$

$\arg(z_2) = \theta_2 = \tan^{-1}\left(\ldots\right) \; ; \; \theta_2 = 60°$

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \ldots$

35. (d)

$f(z) = \bar{z} = x - iy$

$u = x \qquad v = -y$

$\Rightarrow \quad u_x = 1 \qquad v_y = -1$

$u_y = 0 \qquad v_x = 0$

$u_x \neq v_y$, i.e. C-R not satisfied

$\Rightarrow \bar{z}$ is not analytic function

36. Sol.

$f(z_1) = \dfrac{az_1 + b}{cz_1 + d}$

$f(z_2) = \dfrac{az_2 + b}{cz_2 + d}$

$\dfrac{az_1 + b}{cz_1 + d} = \dfrac{az_2 + b}{cz_2 + d}$

$\ldots$

$\Rightarrow \quad 5z = 30$

$z = \ldots = 10$

37. (a)

Given $\oint \dfrac{1}{z^2}\,dz$ where C is the unit circle. By Cauchy's residue theorem

$\oint \dfrac{1}{z^2}\,dz = 2\pi i\,[\text{sum of residues}]$

$\because \dfrac{1}{z^2}$ is NOT analytic at $z = 0$

So, $z = 0$ is the pole of order 2

So, residue at $z = 0 = \dfrac{1}{(2-1)!}\left[\dfrac{d}{dz} z^2 \cdot \dfrac{1}{z^2}\right]_{z=0} = 0$

So $\oint \dfrac{1}{z^2}\,dz = 2\pi i(0) = 0$

38. Sol.

$\ldots = \dfrac{z+2}{z}$

Since there is no pole inside the contour

Residue at pole is zero

$\Rightarrow \dfrac{1}{2\pi i}\oint f(z)\,dz = 0$

39. (b)

By Cauchy integral formula

$\oint \dfrac{f(z)}{(z - z_0)^{n+1}}\,dz = \dfrac{2\pi i\, f^{n}(z_0)}{n!}$

$\oint \dfrac{1}{(z - z_0)^{n}}\,dz = \dfrac{2\pi i}{n!} \cdot 0 = 0$

41. (a)

$f(z)$ has poles at $z = 1, \ldots$

Residue of $f(z)$ at $z = 1$

$= \lim_{z \to 1}(z-1)f(z) = \lim_{z \to 1} \ldots 2z^2$

Residue of [illegible] at [illegible]

[illegible]

41. (a)

[illegible]

42. Sol.

[illegible]

Given that [illegible] is analytic [illegible]

[illegible] analytic

[illegible] is analytic

[illegible] $= 0$

43. (c)

$u = 2xy$

[illegible]

44. Sol.

[illegible]

45. (b)

[illegible]

C-R equation not satisfied

∴ Nowhere analytic

46. (a)

[illegible]

Poles are [illegible]

[illegible]

Residue at [illegible]

47. (c)

Singularities, [illegible]

Only [illegible]

By residue theorem

Residue [illegible]

48. Sol.

$$\frac{1}{2\pi i}\oint_C \frac{z^2}{(z-1)}\,dz = \frac{1}{2\pi i}\oint_C \frac{z^2-1}{(z-1)(z+1)}\,dz$$

Poles are at $z = 1, -1$

Given circle is $|z - 1| = 1$

pole $z = 1$ lies inside C

pole $z = -1$ lies outside C

Residue at $z = 1$ is

$$= \lim_{z\to 1}(z-1)\frac{z^2-1}{(z-1)(z+1)} = \frac{2}{2} = 1$$

Residue at $z = 1$ is $= 1$

By Cauchy's residue theorem

$$\frac{1}{2\pi i}\oint_C \frac{z^2-1}{z^2-1}\,dz = \frac{1}{2\pi i}\times 2\pi i(1) = 1$$

49. Sol.

$$I = \frac{1}{2\pi i}\oint \frac{\sin z}{(z-\pi/6)^3}\,dz$$

$$= \frac{1}{2\pi i}\cdot\frac{2\pi i\, f''(\pi/6)}{2!}$$

$f(z) = \sin z$

$f'(z) = \cos z$

$f''(z) = -\sin z$

$$I = \frac{1}{2\pi i}\times 2\pi i\frac{-\sin(\pi/6)}{2}$$

$$= -\frac{1}{2}\sin\frac{\pi}{6} = -0.25$$

50. (b)

(i) $z = 2$ lies inside C

so $\text{Res}(2) = \lim_{z\to 2}(z-2)\frac{e^z}{z-2}$

$= e^2 = 7.39$

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\cdot\frac{1}{2\pi i}(7.39) = 7.39$$

(ii) $z = 2$ lies outside C then

$\text{Res}(f) = 0$

$$\therefore \oint_C \frac{e^z}{z-2}\,dz = 2\pi i\cdot\frac{1}{2\pi i}(0) = 0$$

51. Sol.

Residue at $z = 0$ of $\frac{\sin z}{z^6}$ = Coefficient of $\frac{1}{z}$ in

$$\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^6}$$

$\therefore$ Coefficient of $\frac{1}{z}$ in $\left[\frac{1}{z^5} - \frac{1}{3!z^3} + \frac{1}{5!z}\right] - \cdots$

52. (b)

$$\oint_C \frac{3z-5}{(z-1)(z-2)}\,dz = 4\pi i$$

$$\oint_C \frac{3z-5}{(z-1)(z-2)}\,dz = 2\pi i(2)$$

Sum of residues must be equal to 2

$$\text{Res}(1) = \lim_{z\to 1}(z-1)\frac{3z-5}{(z-1)(z-2)} = \frac{-2}{-1} = 2$$

$$\text{Res}(2) = \lim_{z\to 2}(z-2)\frac{3z-5}{(z-1)(z-2)} = \frac{6-5}{2-1} = 1$$

Therefore $z = 1$ must lie inside C

$z = 2$ lies outside C

The only way to get the given integral value is with this.

53. (b)

Given that the analytic function

$f(z) = x^2 - ay^2 + i\,bxy$

i.e. $f(z) = (x^2 - ay^2) + i(bxy)$

$u = x^2 - ay^2$

$v = bxy$

$u_x = 2x \quad u_y = -2ay$

$v_x = by \quad v_y = bx$

$u_x = v_y \quad u_y = -v_x$

$2x = bx \quad -2ay = -by$

$b = 2$

$2a = b$ since $b = 2$

$2a = 2$

$a = 1$

54. (b)

$$\cos\bar{z} = \cos(x - iy)$$

$$= \cos x\cos iy + \sin x\sin iy$$

$$\overline{\cos z} = \overline{\cos x\cosh y - i\sin x\sinh y}$$

$= \cos x\cosh y + i\sin x\sinh y$

$= \cos x\cos iy + \sin x\sin iy$

$= \cos(x - iy) = \cos\bar{z}$

55. (c)

Pole $z = 2$ lies inside $|z| = 3$

$$\text{Res } f(2) = \lim_{z \to 2} (z-2)\frac{z^2 - 2z + 3}{z - 2}$$

$z = 2,$ $= 4 - 4 + 3 = 3$

By Cauchy's residue theorem

$I = 2\pi i(3) = 6\pi i$

56. (d)

$$\lim_{z \to 2} \frac{z^2 - 1}{z^3 - 2z^2 - 4z^2 + 2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\lim_{z \to 2} \frac{2z}{3z^2 - 2 - (2z)}$$

$$= \frac{2z}{3z^2 + 2 - (2z)} = \frac{2z}{3z^2 - 2z - 2} = \frac{2}{-3 - 4} \quad 2i$$

57. (b)

From the diagram, C is $y = x$

$$I = \int_C (x^2 - iy^2)\,dz$$

$$= \int (x^2 - iy^2)(dx + i\,dy)$$

$$= \int (x^2 - ix^2)(dx + i\,dx)$$

$$= \int x^2 (1 - i)(1 + i)\,dx = 2x^2\,dx$$

$$= 2\int_0^1 x^2 dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

58. (b)

Residue at $z = 4$ is

$$= \lim_{z \to 4} (z-4)\frac{1}{(z-4)(z-1)^3} = \frac{1}{(4-1)^3} = \frac{1}{27}$$

Residue at $z = 1$ is

$$= \frac{1}{2!}\lim_{z \to 1} \frac{d^2}{dz^2}\left[(z-1)^3 \frac{1}{(z-4)(z-1)^3}\right]$$

$$= \frac{1}{2}\lim_{z \to 1} \frac{2}{(z-4)^3} = \frac{1}{(1-4)^3} = -\frac{1}{27}$$

59. (d)

Poles are

$z^2 + 1 = 0$

$z = \pm i$

There poles lie inside $|z| = 3$

$z = i$ lies inside $|z| = 3$

Residue at $z = i$ is

$$= \lim_{z \to i} (z-i)\frac{e^z - 1}{(z-i)(z+i)}e^{iz} = \frac{-1-1}{2i}$$

Residue at $z = -i$ is

$$= \lim_{z \to -i} (z+i)\frac{e^z - 1}{(z-i)(z+i)}e^z$$

$$= \frac{-1-1}{-2i} = \frac{1}{i}$$

By Residue theorem

$I = 2\pi i(\text{sum of residues})$

$= 2\pi i(\ldots)$

$= \ldots$

$= \ldots$

60. (a)

$W = u + iv$

$$v = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

$$v_x = 2x + \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = -u_y$$

CR equation,

$$\psi_x = -2y - \frac{1}{(x^2 + y^2)^2}(2xy) = \phi_y$$

$$\phi = -2xy - \frac{y}{(x^2 + y^2)} + C$$

$$\phi = -2xy + \frac{y}{(x^2 + y^2)}(2x)$$

$$u_x = 2x + \frac{(x^2 - y^2) - y(2y)}{(x^2 + y^2)}$$

$$= 2x \quad \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$= \left[2x - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

61. (d)

Residue at $z = 3$ is

$$= \lim_{z \to 3} (z-3)\frac{z^2}{(z-1)(z-2)(z-3)}$$

$$= \frac{3^2}{(3-1)(3-2)} = \frac{9}{2}$$

5 Probability and Statistics

5.1 Probability Fundamentals

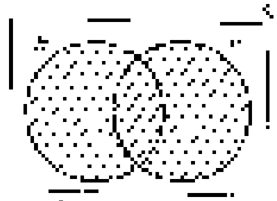
5.1.1 Definitions

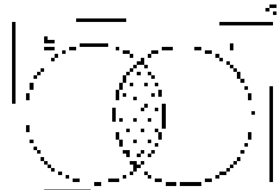
Sample Space and Event: [illegible]

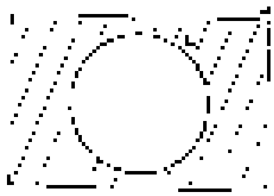
[illegible]

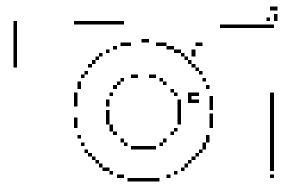
Example: [illegible]

[illegible]









5.1.2 Types of Events

5.1.2.1 Complementary Event

[illegible]

5.1.2.2 Equally Likely Events

Two events E and F are equally likely if [illegible]

$$P(E) = P(F)$$

For example [illegible]

$E = \{1, 2, 3\}$

$F = \{4, 5, 6\}$

[illegible]

5.1.2.3 Mutually Exclusive Events

[illegible]

5.1.2.4 Collectively Exhaustive Events

[illegible]

5.1.2.5 Independent Events

[illegible]

Also [illegible]

[illegible]

5.1.3 DeMorgan's Law

1. [illegible] 2. [illegible]

Example [illegible]

[illegible]

[illegible] (or both)

[illegible]

[illegible]

5.1.4 Approaches to Probability

[illegible]

1. **Classical Approach:**

[illegible]

[illegible]

Example 1.

[illegible]

Solution:

[illegible]

Example 2.

[illegible]

Grade	[illegible]	[illegible]	[illegible]	[illegible]
[illegible] Students	[illegible]	[illegible]	[illegible]	[illegible]

Solution:

[illegible]

5.1.5 Axioms of Probability

[illegible]

Axiom-1: [illegible]

Axiom-2: [illegible]

Axiom-3: [illegible]

[illegible]

Example: [illegible]

5.1.6 Rules of Probability

[illegible]

Rule 1:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This rule is also called [illegible] probability.

This formula reduces to

$$P(A \cup B) = P(A) + P(B)$$

if A and B are mutually exclusive since $P(A \cap B) = 0$ in this case.

Rule 2:

$$P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$$

where $P(A/B)$ represents the conditional probability of A given B and $P(B/A)$ represents the conditional probability of B given A.

(a) [illegible] probability.

(b) $P(A \cap B)$ is called the joint probability of A and B.

(c) If A and B are independent events, the formula reduces to

$$P(A \cap B) = P(A) \cdot P(B)$$

provided A and B are independent [illegible]

$$P(A/B) = P(A)$$

and $P(B/A)$ $$= P(B)$$

[illegible]

(d) If A and B are independent [illegible]

(e) Condition for three events to be independent

Events A, B and C are independent if

$$P(ABC) = P(A)\,P(B)\,P(C)$$

and $$P(AB) = P(A)\,P(B)$$

and $$P(AC) = P(A)\,P(C)$$ [illegible]

and $$P(BC) = P(B)\,P(C)$$

Note: [illegible]

[illegible]

Rule 3: Complementary Probability

$$P(A') = 1 - P(A)$$

[illegible]

$$P(A) = [illegible]\ P(A')$$

[illegible]

Now, P(A) [illegible]

[illegible] A and B are mutually [illegible] events [illegible].

Also, using the De Morgan's law [illegible] $A' \cap B' = (A \cup B)'$

$$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$$

$\therefore$ [illegible]

Rule 4: According to Product Rule

[illegible]

$$P(A \cap B) = P(B) \cdot P(A/B)$$

By [illegible] the conditional probability formula

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

By interchanging A and B in this formula we get

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Rule 5: Rule of Total Probability

Consider an event E which occurs via two different events A and B. [illegible] mutually exclusive and collectively exhaustive events. This situation may be represented by following [illegible]

[illegible]

Now the probability of E is given by rule of total probability as

$$P(E) = P(A \cap E) + P(B \cap E) = P(A) \cdot P(E/A) + P(B) \cdot P(E/B)$$

Rule 6: [illegible] probability

Suppose in the above [illegible] we are interested in knowing that, given that the event E has already occurred, what is the probability that it occurred via A? [illegible]

Example:

[illegible]

(a) What is the probability that [illegible]?

(b) [illegible]

Solution:

[illegible]

(a) [illegible]

(b) [illegible]

5.2 Statistics

5.2.1 Introduction

[illegible]

[illegible]

1. Measures of central tendency
2. Measures of dispersion

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

5.2.2 Arithmetic Mean

5.2.2.1 Arithmetic Mean for Raw Data

[illegible] $\bar{x} = \frac{\Sigma x}{n}$

[illegible]

Example:

[illegible]

Solution:

[illegible]

[illegible]

$$\bar{x} = \frac{\Sigma x}{n} = [illegible]$$

5.2.2.2 The Arithmetic Mean for Grouped Data (Frequency Distribution)

[illegible]

$$\bar{x} = \frac{\Sigma f x}{\Sigma f}$$

Example:

[illegible]

Weight (kg)	Midpoint of class	Number of ...	fx
[illegible]	[illegible]	[illegible]	[illegible]
Total		[illegible]	[illegible]

Solution:

[illegible]

5.2.3 Median

[illegible]

5.2.3.1 Median for Raw Data

[illegible]

Example:

[illegible]

Solution:

[illegible]

$$\text{Median} = \frac{1}{2}(\ldots) = 158.5$$

[illegible] Median for Grouped Data

1. Identify the median class which contains the middle observation $\left(\frac{N+1}{2}\right)^{th}$ observation. This can be done by observing the first class in which the cumulative frequency is equal to or more than $\frac{N+1}{2}$.
Here, $N = \Sigma f$ = total number of observations.
2. Calculate Median as follows:

$$\text{Median} = L + \left[\frac{\left(\frac{N+1}{2}\right) - F}{f_m}\right] \times h$$

Where,
L = Lower limit of median class
N = Total number of observations
F = Cumulative frequency of the class immediately preceding the median class
f_m = Frequency of median class
h = Width of median class

Example:

The given table is a grouped data using the marks obtained by students in a test [illegible]

Marks Range	No. of Students	Cumulative Frequency
0 – 20	[illegible]	[illegible]
20 – 40	[illegible]	[illegible]
40 – 60	10	[illegible]
60 – 80	15	[illegible]
80 – 100	20	[illegible]

Solution:

Here, $\frac{N}{2}$ = [illegible]

The class [illegible] is the median class as cumulative frequency is [illegible]

$$\text{Median} = [illegible]$$

∴ [illegible]

∴ [illegible]

5.2.4 Mode

Mode is defined as the value of the variable which occurs most frequently.

5.2.4.1 Mode for Raw Data

[illegible]

Example:

Find the mode of the data [illegible]

Solution:

1. Arrange in ascending order: [illegible]
2. Make a [illegible] data frequency table

Data	Frequency
50	[illegible]
60	[illegible]
[illegible]	[illegible]

[illegible] mode is 50 [illegible]

5.2.4.2 Mode for Grouped Data

Mode is [illegible] frequency is maximum. [illegible]

1. [illegible] (modal class)
2. Calculate the mode as

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

Where

L = [illegible] modal class

f_1 = Largest frequency [illegible]

f_0 = Frequency in the class preceding the modal class

f_2 = Frequency in the class next to the modal class

h = Width of the class [illegible]

Example:

Data relating to the height of 352 [illegible] students [illegible] following frequency distribution. Calculate the modal height.

Height (in feet)	Number of Students
[illegible] – 3.5	[illegible]
3.5 – [illegible]	[illegible]
4.0 – 4.5	[illegible]
4.5 – 5.0	152
5.0 – 5.5	[illegible]
5.5 – 6.0	7
Total	[illegible]

Solution:

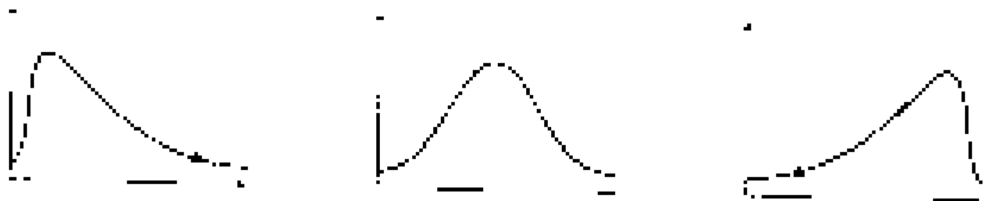
Since 152 is the largest frequency, the modal class is 4.5 – 5.0

Here $L = 4.5$, $f_1 = 152$, $f_0 = 79$, $f_2 =$ [illegible], $h = 0.5$

$$\text{Mode} = 4.5 + \frac{152 - 79}{2(152) - 79 - \text{[illegible]}} \times 0.5 = \text{[illegible] (approx.)}$$

5.2.5 Properties Relating Mean, Median and Mode

1. Empirical mode = 3 median − 2 mean
   [illegible] approximate value of mode [illegible]
2. There are three types of frequency distributions
   Positively skewed, symmetric and negatively skewed [illegible]



(a) Positively Skewed (b) Symmetric (c) Negatively Skewed

(a) In positively skewed distribution:

Mode < Median < Mean

(b) In symmetric distribution:

Mean = Median = Mode

(c) In negatively skewed distribution:

Mean < Median < Mode

5.2.6 Standard Deviation

Standard Deviation is a measure of dispersion of [illegible]

[illegible]

The [illegible] Standard Deviation [illegible]

5.2.6.1 Standard Deviation for Raw Data

[illegible] $\bar{x}$

$\bar{x} = \frac{1}{n}\sum$ [illegible]

$s^2 = \frac{1}{n}\sum$ [illegible] is the variance [illegible]

[illegible]

[illegible]

[illegible]

$$s = \sqrt{\frac{1}{n}\sum \ldots}$$ [illegible]

Example:

Consider [illegible] Find [illegible] the standard deviation of [illegible]

Solution:

Student	Marks	x_i^2
A	[illegible]	[illegible]
B	[illegible]	[illegible]
C	[illegible]	[illegible]
	[illegible]	[illegible]

Here, [illegible]

[illegible]

5.2.6.2 Standard Deviation for Grouped Data

[illegible]

Example

[illegible]

Solution:

Height (in inches)	Mid-value x	Frequency f	f x	f x²
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	16	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	70.25	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Total		150	10173.00	[illegible]

Thus [illegible]

Hence, [illegible]

Therefore, [illegible]

[illegible]

5.2.7 Variance

[illegible]

5.2.8 Coefficient of Variation

$$CV = \frac{\sigma}{\mu}$$

$$CV\% = \frac{\sigma}{\mu} \times 100$$

For example:

	μ	σ	CV%
Dataset 1	[illegible]	[illegible]	[illegible]
Dataset 2	[illegible]	[illegible]	[illegible]

5.3 Probability Distributions

5.3.1 Random Variables

Types of Random Variable: [illegible]

Discrete Random Variable: [illegible]

Example: [illegible]

Continuous Random Variable: [illegible]

Example: [illegible]

5.3.1.1 Probability Density Function (PDF)

[illegible]

5.3.1.2 Probability Mass Function (PMF)

[illegible]

5.3.2 Distributions

[illegible] discrete distribution [illegible] continuous distribution [illegible]

[illegible]

[illegible]

5.3.2.1 Properties of Discrete Distribution

$$\sum p(x) = 1$$

$$E(x) = \sum x\,p(x)$$

[illegible]

[illegible]

5.3.2.2 Properties of Continuous Distribution

[illegible]

5.3.3 Types of Distributions

Discrete Distributions

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]

[illegible]

5.3.3.1 General Discrete Distribution

[illegible]

[illegible]

Example:

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

σ_x^2 represents the variance of X [illegible]

So, $\sigma_x = \sqrt{V(x)}$ is called the standard deviation of x.

Also expected value of any function g(x) must be computed as follows:

$$E(g(x)) = \sum g(x) f(x)$$

For example,

$$E(x^2) = \sum x^2 f(x) = [illegible]$$

For the single die experiment discussed earlier,

$$\mu_x = E(x) = \sum x f(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6} = 3.5$$

and
$$\sigma_x^2 = V(x) = E(x^2) - [E(x)]^2$$
$$= \left[1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + \dots + 6^2 \times \frac{1}{6}\right] - (3.5)^2 = 2.917$$
$$\sigma_x = \sqrt{2.917} = 1.708$$

Properties of Expectation and Variance

If x and y are two random variables and a and b are constants

$$E(ax + b) = aE(x) + b \quad \text{...(1)}$$
$$V(ax + b) = a^2 V(x) \quad \text{...(2)}$$
$$E(ax + by) = aE(x) + bE(y) \quad \text{...(3)}$$
$$V(ax + by) = a^2 V(x) + b^2 V(y) + 2ab\,\mathrm{cov}(x, y) \quad \text{...(4)}$$

where cov(x, y) represents the covariance between x and y.

If x and y are independent, then cov(x, y) = 0 and the above result reduces to

$$V(ax + by) = a^2 V(x) + b^2 V(y) \quad \text{...(5)}$$

For example, from above formulas we can say

$$E(x + y) = E(x) + E(y)$$
$$E(x - y) = E(x) - E(y)$$
$$V(x + y) = V(x - y) = V(x) + V(y)$$

Formula for calculating covariance between X and Y

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

If X and Y are independent, $E(XY) = E(X)E(Y)$

and hence $\mathrm{Cov}(X, Y) = 0$

[illegible].2 Binomial Distribution

Suppose that a trial or an experiment, whose outcome can be classified as either a success or a failure is performed.

Suppose now that n independent trials, each of which results in a success with probability p and in a failure with probability 1 − p, are to be performed.

If X represents the number of successes that occur in the n trials, then X is said to be binomial random variable with parameters (n, p).

The Binomial distribution is a procedure when experiment performed several times and the probability of success is constant in each trial, and where:

1. Only 2 outcomes are possible namely success and failure.
2. Probability of success (p) and failure ($1 - p$) remains same from trial to trial.
3. The trials are statistically independent i.e. the outcome of one trial does not influence subsequent trials in the sequence.

   These assumptions are satisfied in following types of problems:

   (a) Tossing of coins.

   (b) Success and failure.

   (c) Sampling with replacement from a finite population.

   (d) Sampling without replacement from an infinite (large) population.

The probability of obtaining x successes from n trials is given by the binomial distribution formula,

$$P(X = x) = {}^nC_x\, p^x (1-p)^{n-x}$$

Where p is the probability of success in any trial and $(1 - p) = q$ is the probability of failure.

Example 1.

10 coins are tossed. What is the probability of getting exactly 5 heads?

Solution:

$$P(X = 5) = {}^{10}C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 = 0.2461$$

Example 2.

It is known that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution:

If X is the number of defective screws in a package, then X is a binomial random variable with parameters (10, 0.01). Hence, the probability that a package will have to be replaced is

$$P(X > 1) = 1 - P(X \le 1) = 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[\binom{10}{0}(0.01)^0(0.99)^{10} + \binom{10}{1}(0.01)^1(0.99)^9\right]$$

$$= 0.004$$

Hence only 0.4% of packages will have to be replaced.

For Binomial Distribution:

$$\text{Mean} = E(X) = np$$

$$\text{Variance} = V(X) = np(1-p)$$

Recurrence Relation

For binomial distribution, we have

$$P(x) = {}^nC_x\, p^x q^{n-x} \quad \text{...(i)}$$

$$P(x+1) = {}^nC_{x+1}\, p^{x+1} q^{n-x-1} \quad \text{...(ii)}$$

Dividing (ii) by (i) we get

$$\frac{P(x+1)}{P(x)} = \frac{{}^nC_{x+1}\, p^{x+1} q^{n-x-1}}{{}^nC_x\, p^x q^{n-x}}$$

[illegible]

Example 3.

[illegible]

Solution:

[illegible]

5.3.13 Hypergeometric Distribution

[illegible]

Example:

[illegible]

Solution:

[illegible]

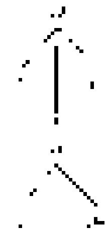
$$[illegible] = 0.5$$

[illegible]

[illegible]

[illegible]

[illegible]



[illegible]

[illegible]

[illegible]

5.3.3.4 Geometric Distribution

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Points to Remember

[illegible]

1. $E(x) = \frac{1}{p}$
2. $V(x) = \frac{q}{p^2}$
3. Cumulative distribution $F(x) = 1 - q^x$
4. [illegible]

[illegible]

[illegible]

1. [illegible]
   (a) [illegible]
   (b) [illegible]
   (c) [illegible]

Sol. 1

[illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

5.3.3.5 Poisson Distribution

[illegible] $0, 1, 2, \ldots$ [illegible] Poisson [illegible]

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

For Poisson distribution:

$$\text{Mean} = E(X) = \lambda$$

$$\text{Variance} = V(X) = \lambda$$

[illegible] Poisson [illegible]

[illegible]

[illegible]

Recurrence Relation [illegible]

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad (1)$$

$$P(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \qquad (2)$$

By dividing eq. (2) by (1)

$$\frac{P(x+1)}{P(x)} = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \cdot \frac{x!}{e^{-\lambda}\lambda^{x}} = \frac{\lambda}{x+1}$$

$$P(x+1) = \frac{\lambda}{x+1} P(x)$$

Example 1.

[illegible]

[illegible]

Solution:

[illegible] [illegible]

[illegible]

[illegible]

[illegible] [illegible]

[illegible] $\lambda = 4 \times 2 = 8$

[illegible]

$$P(X = 0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = \frac{e^{-8}8^{0}}{0!} = e^{-8}$$

[illegible] Poisson [illegible]

[illegible]

[illegible] Poisson [illegible]

Example 2.

[illegible]

[illegible]

Solution:

[illegible]

Continuous Distributions:

1. General Continuous Distribution
2. Uniform Distribution
3. Exponential Distribution
4. Gamma Distribution
5. Standard Normal Distribution

5.3.3.6 General Continuous Distribution

[illegible]

$$[illegible] = \int_{-\infty}^{\infty} [illegible] = 1$$

[illegible]

$$[illegible] = \int_{-\infty}^{\infty} [illegible]$$

$\therefore$

$$[illegible] = E(X^2) - [E(X)]^2 = \int_{-\infty}^{\infty} [illegible] - \left[\int_{-\infty}^{\infty} [illegible]\right]^2$$

$$\sigma_x^2 = [illegible]$$

[illegible]

$$[illegible] = \int_{-\infty}^{x} [illegible]$$

Note: [illegible]

$$[illegible] = \frac{dF}{dx}$$

5.3.3.7 Uniform Distribution

[illegible]

$$[illegible]$$

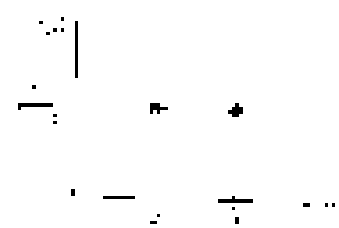
[illegible]

Graphical Representations

For Discrete Uniform Distribution

Mean $E[x] = \frac{b+a}{2}$

Variance = [illegible]



Example:

[illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

Solution:

[illegible]

5.3.3.7 Exponential Distribution

[illegible]

[illegible]

For Exponential Distribution:

$$\text{Mean} = E[X] = 1/\lambda$$

$$\text{Variance} = [illegible]$$

Example:

[illegible]

(a) [illegible]

(b) [illegible]

Solution:

[illegible]

(a) [illegible]

(b) [illegible]

5.3.3.8 Normal Distribution

[illegible]

[illegible]

[illegible]

For Normal Distribution:

$$\text{Mean} = E(X) = \mu$$
$$\text{Variance} = V(X) = \sigma^2$$

3.3.10.1 Standard Normal Distribution

[illegible]

[illegible]

[illegible]

$$Z = \frac{X - \mu}{\sigma}$$

Where Z is called standard normal variate.

For Standard Normal Distribution:

$$\text{Mean} = E(Z) = 0$$
$$\text{Variance} = V(Z) = 1$$

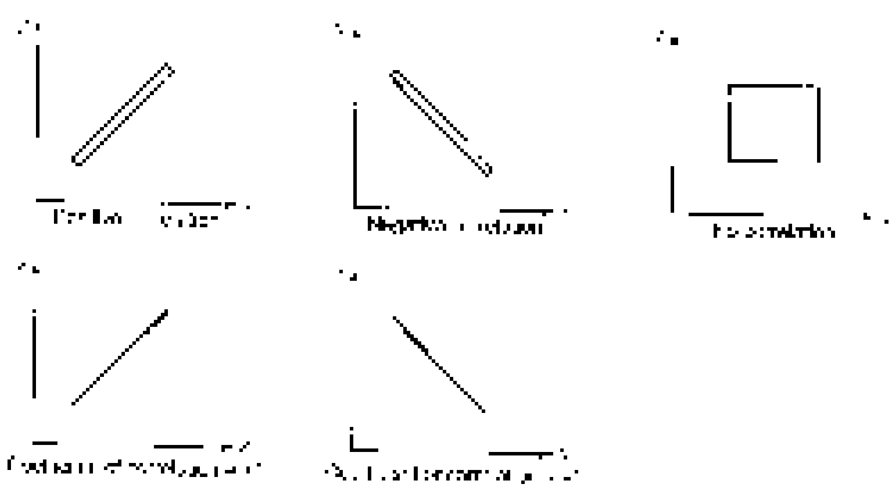
[illegible]

Correlation: [illegible]

[illegible]

[illegible]

Scatter or Dot diagram: [illegible]



Positive correlation means if value of one variable increases then value of other variable also increases. Negative correlation means if value of one variable increases then value of other variable decreases. [illegible] there is no correlation.

Coefficient of correlation

$$r = \frac{\text{Covariance}(x, y)}{\sqrt{\text{Variance}(x) \times \text{Variance}(y)}}$$

[illegible]

The correlation coefficient satisfies $-1 \le r \le 1$.

(i) [illegible] there is no correlation.

(ii) If $r = 1$, then there is perfect positive correlation.

(iii) If $r = -1$, then there is perfect negative correlation.

(iv) If $0 < r < 1$, there is positive correlation.

(v) If $-1 < r < 0$, there is negative correlation.

Regression: If the scatter diagram indicates [illegible] the curve [illegible] the scatter diagram [illegible] curve. This curve is called the curve of regression.

Regression analysis [illegible] the values of [illegible] variable.

Line of Regression: When the curve is a straight line, it is called a line of regression. A line of regression is the straight line which gives the best fit in the least square sense to the given frequency.

[illegible] regression [illegible] two variables can be regarded as [illegible] variable because we can [illegible].

[illegible] relationship between [illegible] is [illegible] regression [illegible] changes [illegible].

[illegible] line of regression is given by

$$y = a + bx \quad \ldots (1)$$

[illegible] the normal equation

$$\Sigma y = na + b\Sigma x$$

[illegible]

$$\frac{\Sigma y}{n} = a + b\frac{\Sigma x}{n}$$

or

$$\bar{y} = a + b\bar{x} \quad \ldots (2)$$

where $\bar{x}$, $\bar{y}$ are means of x and y respectively.

The line of regression passes through $(\bar{x}, \bar{y})$.

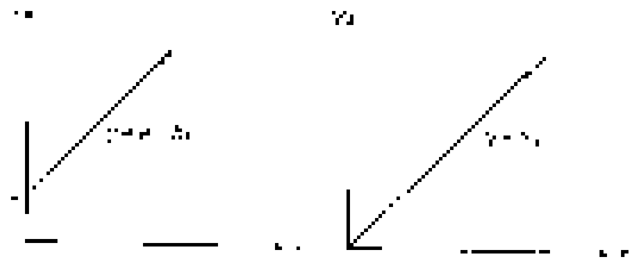
By subtracting equation (2) from equation (1), we get

$$y - \bar{y} = b(x - \bar{x})$$

$$Y = bX$$

[illegible]

[illegible]



Previous GATE and ESE Questions

Q.1 Let $P(E)$ denote the probability of the event E. Given $P(A) = 1$, $P(B) = \frac{1}{2}$, the values of $P(A|B)$ and $P(B|A)$ respectively are
(a) [illegible] (b) [illegible]
(c) $\frac{1}{2}$, 1 (d) 1, $\frac{1}{2}$
[CS, GATE-2003, 1 mark]

Q.2 A box contains 10 screws, 3 of which are defective. [illegible]
(a) 100% (b) 50%
(c) 49% (d) None of these
[CE, GATE-2000, 1 mark]

Q.3 A box contains [illegible] black and [illegible] red balls. Two balls are randomly picked one after another from the box, without replacement. The probability for both balls being red is
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[ME, GATE-2003, 2 marks]

Q.4 A program consists of two modules executed sequentially. [illegible] probability density functions of time taken to execute the two modules. The probability density function of the overall time taken to execute the program is given by
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[CS, GATE-2003, 2 marks]

Q.5 A hydraulic structure has four gates which operate independently. The probability of failure of each gate is 0.2. Given that gate [illegible] has failed, the probability that both gates [illegible] and [illegible] will fail is
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[CE, GATE-2004, 2 marks]

Q.6 An examination paper has [illegible] multiple-choice questions of one mark each, with each question having four choices. Each incorrect answer fetches [illegible] mark. Suppose [illegible] students choose all their answers randomly with uniform probability. The sum total of the expected marks obtained by all these students is
(a) 0 (b) [illegible]
(c) [illegible] (d) [illegible]
[CS, GATE-2004, 2 marks]

Q.7 If a fair coin is tossed four times. What is the probability that two heads and two tails will result?
(a) [illegible] (b) [illegible]
(c) [illegible] (d) [illegible]
[CS, GATE-2004, 1 mark]

Q.8 Two n bit binary strings, S_1 and S_2, are chosen randomly with uniform probability. The probability that the Hamming distance between these strings (the number of bit positions where the two strings differ) is equal to d is

(a) ${}^nC_d/2^n$ (b) ${}^nC_d/2^d$
(c) $d/2^n$ (d) $1/2^d$

[CS, GATE-2004, 2 marks]

Q.9 From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be Kings, if the first card is NOT replaced?

(a) $\frac{1}{26}$ (b) $\frac{1}{52}$
(c) $\frac{1}{169}$ (d) $\frac{1}{221}$

[ME, GATE-2004, 2 marks]

Q.10 A point is randomly selected with uniform probability in the X-Y plane within the rectangle with corners at (0, 0), (1, 0), (1, 2) and (0, 2). If p is the length of the position vector of the point, the expected value of p^2 is

(a) 2/3 (b) 1
(c) 4/3 (d) 5/3

[CS, GATE-2004, 2 marks]

Q.11 If P and Q are two random events, then the following is TRUE

(a) Independence of P and Q implies that probability $(P \cap Q) = 0$
(b) Probability $(P \cup Q) \geq$ Probability (P) + Probability (Q)
(c) If P and Q are mutually exclusive, then they must be independent
(d) Probability $(P \cap Q) \leq$ Probability (P)

[EE, GATE-2005, 1 mark]

Q.12 A fair dice is rolled twice. The probability that an odd number will follow an even number is

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$
(c) $\frac{1}{3}$ (d) $\frac{1}{4}$

[EC, GATE-2005, 1 mark]

Q.13 A single die is thrown twice. What is the probability that the sum is neither 8 nor 9?

(a) 1/9 (b) 5/36
(c) 1/4 (d) 3/4

[ME, GATE-2005, 2 marks]

Q.14 A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is

(a) 1/8 (b) 1/2
(c) 3/8 (d) 3/4

[EE, GATE-2005, 2 marks]

Q.15 Which one of the following statements is NOT true?

(a) The measure of skewness is dependent upon the amount of dispersion
(b) In a symmetric distribution, the values of mean, mode and median are the same
(c) In a positively skewed distribution; mean > median > mode
(d) In a negatively skewed distribution; mode > mean > median

[CE, GATE-2005, 1 mark]

Q.16 A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly 2 of the chosen items are defective is

(a) 0.0036 (b) 0.1937
(c) 0.2234 (d) 0.3874

[ME, GATE-2005, 1 mark]

Q.17 Let f(x) be the continuous probability density function of a random variable X. The probability that $a < X \leq b$, is

(a) $f(b - a)$ (b) $f(b) - f(a)$
(c) $\int_a^b f(x)dx$ (d) $\int_a^b x f(x)dx$

[CE, GATE-2005, 1 mark]

Q.18 If P and Q are two random events, then which of the following is TRUE

(a) The independence of P and Q implies that probability $(P \cap Q) = 0$
(b) Probability $(P \cup Q) \geq$ Probability (P) + Probability (Q)
(c) If P and Q are mutually exclusive then they must be independent
(d) Probability $(P \cap Q) \leq$ Probability (P)

[EC, GATE-2005, 1 mark]

Q.19 The value of the integral $I = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \exp\left(-\frac{x^2}{8}\right) dx$ is

(a) 1 (b) π
(c) 2 (d) 2π

[EC, GATE-2005, 2 marks]

Q.20 Two fair dice are rolled and the sum r of the numbers turned up is considered.
(a) Pr(r > 6) = 1/6
(b) Pr(r/3 is an integer) = 5/6
(c) Pr(r = 8 | r/4 is an integer) = 5/9
(d) Pr(r = 6 | r/5 is an integer) = 1/18

[EE, GATE-2006, 2 marks]

Q.21 Three companies X, Y and Z supply computers to a university. The percentage of computers supplied by them and the probability of those being defective are tabulated below:

Company	% of computer supplied	Probability of being supplied defective
X	60%	0.01
Y	30%	0.02
Z	10%	0.03

Given that a computer is defective, the probability that it was supplied by Y is
(a) 0.1 (b) 0.2
(c) 0.3 (d) 0.4

[EE, GATE-2006, 2 marks]

Q.22 For each element in a set of size 2n, an unbiased coin is tossed. The 2n coin tosses are independent. An element is chosen if the corresponding coin toss were head. The probability that exactly n elements are chosen is

(a) $\binom{2n}{n}/4^n$ (b) $\binom{2n}{n}/2^n$

(c) $1/\binom{2n}{n}$ (d) $\frac{1}{2}$

[CS, GATE-2006, 2 marks]

Q.23 There are 25 calculators in a box. Two of them are defective. Suppose 5 calculators are randomly picked for inspection (i.e., each has the same chance of being selected), what is the probability that only one of the defective calculators will be included in the inspection?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$

[CE, GATE-2006, 2 marks]

Q.24 A box contains 20 defective items and 80 non-defective items. If two items are selected at random without replacement, what will be the probability that both items are defective?

(a) $\frac{1}{5}$ (b) $\frac{1}{25}$

(c) $\frac{20}{99}$ (d) $\frac{19}{495}$

[ME, GATE-2006, 1 mark]

Q.25 Consider the continuous random variable with probability density function

$$f(t) = 1 + t \quad \text{for } -1 \le t \le 0$$
$$= 1 - t \quad \text{for } 0 \le t \le 1$$

The standard deviation of the random variable is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{6}}$

(c) $\frac{1}{3}$ (d) $\frac{1}{6}$

[ME, GATE-2006, 2 marks]

Q.26 A probability density function is of the form $p(x) = Ke^{-\alpha|x|}$, $x \in (-\infty, \infty)$.
The value of K is
(a) 0.5 (b) 1
(c) 0.5α (d) α

[EC, GATE-2006, 1 mark]

Q.27 A class of first year B.Tech. students is composed of four batches A, B, C and D, each consisting of 30 students. It is found that the sessional marks of students in Engineering Drawing in batch C have a mean of 6.6 and standard deviation of 2.3. The mean and standard deviation of the marks for the entire class are 5.5 and 4.2, respectively. It is decided by the course instructor to normalize the marks of the students of all batches to have the same mean and standard deviation as that of the entire class. Due to this, the marks of a student in batch C are changed from 8.5 to
(a) 6.0 (b) 7.0
(c) 8.0 (d) 9.0

[CE, GATE-2006, 2 marks]

Q.28 Suppose we uniformly and randomly select a permutation from the 20! permutations of 1, 2, 3, ..., 20. What is the probability that 2 appears at an earlier position than any other even number in the selected permutation?

(a) $\frac{1}{2}$ (b) $\frac{1}{10}$

(c) $\frac{9!}{20!}$ (d) None of these

[CS, GATE-2007, 2 marks]

Q.29 A loaded dice has following probability distribution of occurrences

Dice Value	Probability
1	1/4
2	1/8
3	1/8
4	1/8
5	1/8
6	1/4

If three identical dice as the above are thrown, the probability of occurrence of values 1, 5 and 6 on the three dice is

(a) same as that of occurrence of 3, 4, 5
(b) same as that of occurrence of 1, 2, 5
(c) 1/128
(d) 5/8

[EE, GATE-2007, 2 marks]

Q.30 An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is

(a) 0.5 (b) 0.18
(c) 0.12 (d) 0.06

[EC, GATE-2007, 2 marks]

Q.31 If the standard deviation of the spot speed of vehicles in a highway is 8.8 kmph and the mean speed of the vehicles is 33 kmph, the coefficient of variation in speed is

(a) 0.1517 (b) 0.1867
(c) 0.2666 (d) 0.3646

[CE, GATE-2007, 2 marks]

Q.32 Let X and Y be two independent random variables. Which one of the relations between expectation (E), variance (Var) and covariance (Cov) given below is FALSE?

(a) $E(XY) = E(X)\,E(Y)$
(b) $Cov(X, Y) = 0$
(c) $Var(X + Y) = Var(X) + Var(Y)$
(d) $E(X^2Y^2) = (E(X))^2\,(E(Y))^2$

[ME, GATE-2007, 2 marks]

Q.33 A person on a trip has a choice between private car and public transport. The probability of using a private car is 0.45. While using the public transport, further choices available are bus and metro, out of which the probability of commuting by a bus is 0.55. In such a situation, the probability (rounded up to two decimals) of using a car, bus and metro, respectively would be

(a) 0.45, 0.30 and 0.25
(b) 0.45, 0.25 and 0.30
(c) 0.45, 0.55 and 0.00
(d) 0.45, 0.35 and 0.20

[CE, GATE-2008, 2 marks]

Q.34 Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability that she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability that she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probability that she studies computer science on Wednesday?

(a) 0.24 (b) 0.36
(c) 0.4 (d) 0.6

[CS, GATE-2008, 2 marks]

Q.35 A coin is tossed 4 times. What is the probability of getting heads exactly 3 times?

(a) 1/4 (b) 3/8
(c) 1/2 (d) 3/4

[ME, GATE-2009, 1 mark]

Q.36 Probability density function of the random variable X is

$p_X(x) = e^{-x}$ for $x \geq 0$, and
$= 0$ otherwise. The value of [illegible]

Then the probability $P\left(\ [illegible]\ \leq \frac{1}{3}\right)$ is

(a) 0.243 (b) [illegible]
(c) [illegible] (d) 0.847

[CE, GATE-2008, 2 marks]

Q.37 Let X be a random variable following normal distribution with mean +1 and variance 4. Let Y be another normal variable with mean −1 and variance unknown. If $P(X \leq -1) = P(Y \geq 2)$, the standard deviation of Y is

(a) 3 (b) 2

(c) $\sqrt{2}$ (d) 1

[CS, GATE-2008, 2 marks]

Q.38 A fair coin is tossed 10 times. What is the probability that ONLY the first two tosses will yield heads?

(a) $\left(\frac{1}{2}\right)^2$ (b) $^{10}C_2\left(\frac{1}{2}\right)^2$

(c) $\left(\frac{1}{2}\right)^{10}$ (d) $^{10}C_2\left(\frac{1}{2}\right)^{10}$

[EC, GATE-2009, 1 mark]

Q.39 An unbalanced dice (with 6 faces, numbered from 1 to 6) is thrown. The probability that the face value is odd is 90% of the probability that the face value is even. The probability of getting any even numbered face is the same. If the probability that the face is even given that it is greater than 3 is 0.75, which one of the following options is closest to the probability that the face value exceeds 3?

(a) 0.453 (b) 0.468

(c) 0.485 (d) 0.492

[CS, GATE-2009, 2 marks]

Q.40 If two fair coins are flipped and at least one of the outcomes is known to be a head, what is the probability that both outcomes are heads?

(a) 1/3 (b) 1/4

(c) 1/2 (d) 2/3

[CS, GATE-2009, 1 mark]

Q.41 If three coins are tossed simultaneously, the probability of getting at least one head is

(a) 1/8 (b) 3/8

(c) 1/2 (d) 7/8

[ME, GATE-2009, 1 mark]

Q.42 The standard deviation of a uniformly distributed random variable between 0 and 1 is

(a) $\frac{1}{\sqrt{12}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{5}{\sqrt{12}}$ (d) $\frac{7}{\sqrt{12}}$

[ME, GATE-2009, 2 marks]

Q.43 The standard normal probability function can be approximated as

$$F(x_N) = \frac{1}{1+\exp(-1.7255\, x_N |x_N|^{0.12})}$$

where x_N = standard normal deviate. If mean and standard deviation of annual precipitation are 102 cm and 27 cm respectively, the probability that the annual precipitation will be between 90 cm and 102 cm is

(a) 66.7% (b) 50.0%

(c) 33.3% (d) 16.7%

[CE, GATE-2009, 2 marks]

Q.44 A box contains 2 washers, 3 nuts and 4 bolts. Items are drawn from the box at random one at a time without replacement. The probability of drawing 2 washers first followed by 3 nuts and subsequently the 4 bolts is

(a) 2/315 (b) 1/630

(c) 1/1260 (d) 1/2520

[ME, GATE-2010, 2 marks]

Q.45 A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

(a) 1/3 (b) 3/7

(c) 1/2 (d) 4/7

[EC, GATE-2010, 2 marks]

Q.46 Consider a company that assembles computers. The probability of a faulty assembly of any computer is p. The company therefore subjects each computer to a testing process. This testing process gives the correct result for any computer with a probability of q. What is the probability of a computer being declared faulty?

(a) $pq + (1-p)(1-q)$

(b) $(1-q)p$

(c) $(1-p)q$

(d) pq

[CS, GATE-2010, 2 marks]

Q.47 A fair coin is tossed independently four times. The probability of the event "the number of times heads shows up is more than the number of times tails shows up" is

[illegible]

[EC, GATE-2010, 2 marks]

Q.48 [illegible]

[CS, GATE-2011, 2 marks]

Q.49 [illegible]

[CS, GATE-2011, 1 mark]

Q.50 [illegible]

[EC, GATE-2011, 1 mark]

Q.51 [illegible]

[CS, GATE-2011, 2 marks]

Q.52 [illegible]

[CS, GATE-2011, 1 mark]

Q.53 [illegible]

[ME, GATE-2011, 2 marks]

Q.54 [illegible]

[EE, EE, IN, GATE-2012, 2 marks]

Q.55 [illegible]

[CS, GATE-2012, 2 marks]

Q.56 [illegible]

[CE, GATE-2012, 2 marks]

Q.57 [illegible]

[CS, GATE-2012, 1 mark]

Q.58 [illegible]

[ME, GATE-2012, 2 marks]

Q.59 [illegible]

[EE, GATE-2012, 1 mark]

Q.60 [illegible]

[CE, GATE-2012, 1 mark]

Q.61 [illegible]

[GATE-2013, 1 Mark]

Q.62 [illegible]

[CS, GATE-2012, 2 Marks]

Q.63 [illegible]

[EE, GATE-2013, 1 Mark]

Q.64 [illegible]

[IN, GATE-2013, 1 mark]

Q.65 [illegible]

[ME, GATE-2013, 1 Mark]

Q.66 [illegible]

[ME, GATE-2014, 1 Mark]

Q.67 [illegible]

[CE, GATE-2014, 1 Mark]

Q.68 [illegible]

[ME, GATE-2014, 1 Mark]

Q.69 [illegible]

[ME, GATE-2014, 1 Mark]

Q.70 [illegible]

[EE, GATE-2014, 2 Marks]

Q.71 [illegible]

[EE, GATE-2014 : 1 Mark]

Q.72 [illegible]

[CE, GATE-2014 : 1 Mark]

Q.73 [illegible]

(a) [illegible] (c) [illegible]
(b) [illegible] (d) [illegible]

[EC, GATE-2014 : 1 Mark]

Q.74 [illegible] ______.

[EC, GATE-2014 : 2 Marks]

Q.75 [illegible]

[CS, GATE-2014 : 2 Marks]

Q.76 [illegible] ______.

[CS, GATE-2014 : 2 Marks, Set-2]

Q.77 [illegible]

[CS, 2014 : 2 Marks, Set-3]

Q.78 [illegible]

x	[illegible]	[illegible]	[illegible]
$p(x)$	[illegible]	[illegible]	[illegible]

(a) [illegible] (c) [illegible]
(b) [illegible] (d) [illegible]

[ME, 2014 : 2 Marks]

Q.79 [illegible]

(a) [illegible] (c) [illegible]
(b) [illegible] (d) [illegible]

[ME, 2014 : 2 Marks]

Q.80 [illegible] ______.

[ME, GATE-2014 : 2 Marks]

Q.81 [illegible] ______.

[CS, GATE-2014 : 1 Mark]

Q.82 [illegible] ______.

[CE, GATE-2014 : 2 Marks]

Q.83 [illegible]

(a) [illegible] (c) [illegible]
(b) [illegible] (d) [illegible]

[ME, GATE-2014 : 2 Marks]

Q.84 [illegible]

$f(x) = \begin{cases} \frac{1}{5} & \text{[illegible]} \\ 0 & \text{otherwise} \end{cases}$

The probability that the [illegible]

[ME, GATE-2014 : 1 Mark]

Q.85 Let X be a random variable with probability density function

$$f(x) = \begin{cases} 0.2 & \text{for } |x| \le 1 \\ 0.1 & \text{for } 1 < |x| \le 4 \\ 0 & \text{otherwise} \end{cases}$$

The probability $P(0.5 < X < 5)$ is ______.

[EE, GATE-2014 : 2 Marks]

Q.86 [illegible] random variable [illegible] density function [illegible]

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ and σ are the [illegible] of the random variable [illegible]

[illegible] $\int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ [illegible]

(a) 1 (b) [illegible]
(c) [illegible] (d) [illegible]

[EE, GATE-2014 : 1 Mark]

Q.87 Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\bar{A}$ and $\bar{B}$ be their complements. Which one of the following statements is FALSE?
(a) $P(A \cap B) = P(A)P(B)$
(b) $P(A|B) = P(A)$
(c) $P(A \cup B) = P(A) + P(B)$
(d) $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$

[EC, GATE-2015 : 1 Mark]

Q.88 Suppose X_i for $i = 1, 2, 3$ are independent and identically distributed random variables whose probability mass functions are $Pr[X_i = 0] = Pr[X_i = 1] = 1/2$ for $i = 1, 2, 3$. Define another random variable $Y = X_1X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $Pr[Y = 0 \mid X_3 = 0]$ = ____.

[CS, 2015 : 2 Marks]

Q.89 Three vendors were asked to supply a very high precision component. The respective probabilities of their meeting the strict design specifications are 0.8, 0.7 and 0.5. Each vendor supplies one component. The probability that out of total three components supplied by the vendors, at least two will meet the design specification is ______.

[ME, GATE-2015 : 1 Mark]

Q.90 The chance of a student passing an exam is 20%. The chance of a student passing the exam and getting above 90% marks in it is 5%. Given that a student passes the examination, the probability that the student gets above 90% marks is

(a) $\frac{1}{18}$ (b) $\frac{1}{4}$

(c) $\frac{2}{9}$ (d) $\frac{5}{18}$

[ME, GATE-2015 : 2 Marks]

Q.91 If $P(X) = \frac{1}{4}$, $P(Y) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{12}$, the value of $P(Y|X)$ is

(a) $\frac{1}{4}$ (b) $\frac{4}{25}$

(c) $\frac{1}{3}$ (d) $\frac{29}{50}$

[ME, GATE-2015 : 1 Mark]

Q.92 Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

(a) $\frac{5}{11}$ (b) $\frac{1}{2}$

(c) $\frac{7}{13}$ (d) $\frac{6}{11}$

[EE, GATE-2015 : 2 Marks]

Q.93 Consider the following probability mass function (p.m.f.) of a random variable X:

$$p(x; q) = \begin{cases} q & \text{if } X = 0 \\ 1-q & \text{if } X = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $q = 0.4$, the variance of X is ______.

[CE, GATE-2015 : 1 Mark]

Q.94 Two coins R and S are tossed. The 4 joint events H_RH_S, T_RT_S, H_RT_S, T_RH_S have probabilities 0.28, 0.18, 0.30, 0.24, respectively, where H represents head and T represents tail. Which one of the following is TRUE?
(a) The coin tosses are independent
(b) R is fair, S is not
(c) S is fair, R is not
(d) The coin tosses are dependent

[EE, GATE-2015 : 2 Marks]

Q.95 The probability of obtaining at least two "SIX" in throwing a fair dice 4 times is

(a) $\frac{425}{432}$ (b) $\frac{19}{144}$

(c) $\frac{13}{144}$ (d) $\frac{125}{432}$

[ME, GATE-2015 : 2 Marks]

Q.96 The probability that a thermistor randomly picked up from a production unit is defective is 0.1. The probability that out of 10 thermistors randomly picked up, 3 are defective is
(a) 0.001 (b) 0.057
(c) 0.107 (d) 0.3

[IN, GATE-2015 : 2 Marks]

Q.97 The probability density function of a random variable, x is

$$f(x) = \frac{x}{4}(4 - x^2) \text{ for } 0 \le x \le 2 \quad = 0 \text{ otherwise}$$

The mean, μ_x of the random variable is ____

[CE, GATE-2015 : 2 Marks]

Q.98 A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value $E[X] = 2/3$, then $Pr[X < 0.5]$ is ____

[EE, GATE-2015 : 1 Mark]

Q.99 An urn contains 5 red and 7 green balls. A ball is drawn at random and its colour is noted. The ball is placed back into the urn along with another ball of the same colour. The probability of getting a red ball in the next draw is

(a) $\frac{65}{156}$ (b) $\frac{67}{156}$

(c) $\frac{79}{156}$ (d) $\frac{89}{156}$

[IN, 2016 : 2 Marks]

Q.100 The probability of getting a "head" in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a "head" is obtained. If the tosses are independent, then the probability of getting "head" for the first time in the fifth toss is ____

[EC, 2016 : 1 Mark]

Q.101 X and Y are two random independent events. It is known that $P(X) = 0.40$ and $P(X \cup Y^C) = 0.7$. Which one of the following is the value of $P(X \cup Y)$?
(a) 0.7 (b) 0.5
(c) 0.4 (d) 0.3

[CE, 2016 : 1 Mark]

Q.102 Consider the following experiment.
Step 1. Flip a fair coin twice.
Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.
Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.
The probability that the output of the experiment is Y is (up to two decimal places) ____

[CS, 2016 : 2 Marks]

Q.103 Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is ____

[CS, 2016 : 1 Mark]

Q.104 Three cards were drawn from a pack of 52 cards. The probability that they are a king, a queen, and a jack is

(a) $\frac{16}{5525}$ (b) $\frac{64}{2197}$

(c) $\frac{3}{13}$ (d) $\frac{8}{16575}$

[ME, 2016 : 2 Marks]

Q.105 Type II error in hypothesis testing is
(a) acceptance of the null hypothesis when it is false and should be rejected
(b) rejection of the null hypothesis when it is true and should be accepted
(c) rejection of the null hypothesis when it is false and should be rejected
(d) acceptance of the null hypothesis when it is true and should be accepted

[CE, 2016 : 1 Mark]

Q.106 The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53, and 49. The median speed (expressed in km/hr) is ____.
(Note: answer with one decimal accuracy)

[CE, 2016 : 1 Mark]

Q.107 If $f(x)$ and $g(x)$ are two probability density functions,

$$f(x)=\begin{cases}\dfrac{x}{a}+1 & : -a\le x<0\\ -\dfrac{x}{a}+1 & : 0\le x\le a\\ 0 & : \text{otherwise}\end{cases}$$

$$g(x)=\begin{cases}-\dfrac{x}{a} & : -a\le x<0\\ \dfrac{x}{a} & : 0\le x\le a\\ 0 & : \text{otherwise}\end{cases}$$

Which one of the following statements is true?
(a) Mean of $f(x)$ and $g(x)$ are same; Variance of $f(x)$ and $g(x)$ are same
(b) Mean of $f(x)$ and $g(x)$ are same; Variance of $f(x)$ and $g(x)$ are different
(c) Mean of $f(x)$ and $g(x)$ are different; Variance of $f(x)$ and $g(x)$ are same
(d) Mean of $f(x)$ and $g(x)$ are different; Variance of $f(x)$ and $g(x)$ are different

[CE, 2016 : 2 Marks]

Q.108 The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws and gives a guarantee of replacement if one or more screws in the packet are found to be defective. The probability that a packet would have to be replaced is ___

[ME, 2016 : 2 Marks]

Q.109 The second moment of a Poisson-distributed random variable is 2. The mean of the random variable is ____.

[EC, 2016 : 1 Mark]

Q.110 Consider a Poisson distribution for the tossing of a biased coin. The mean for this distribution is μ. The standard deviation for this distribution is given by
(a) $\sqrt{\mu}$
(b) μ^2
(c) μ
(d) $\dfrac{1}{\mu}$

[ME, 2016 : 1 Mark]

Q.111 A probability density function on the interval $[a, 1]$ is given by $1/x^2$ and outside this interval the value of the function is zero. The value of a is ___

[CS, 2016 : 1 Mark]

Q.112 Two random variables X and Y are distributed according to

$$f_{X,Y}(x,y)=\begin{cases}(x+y), & 0\le x\le 1,\ 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$

The probability $P(X+Y\le 1)$ is ___

[EC, 2016 : 2 Marks]

Q.113 Probability density function of a random variable X is given below

$$f(x)=\begin{cases}0.25 & \text{if } 1\le x\le 5\\ 0 & \text{otherwise}\end{cases}\qquad P(X\le 4)$$

(a) $\dfrac{3}{4}$
(b) $\dfrac{1}{2}$
(c) $\dfrac{1}{4}$
(d) $\dfrac{1}{8}$

[CE, 2016 : 2 Marks]

Q.114 Let the probability density function of a random variable X be given as:

$$f_X(x)=\frac{3}{2}e^{-3x}u(x)+ae^{4x}u(-x)$$

where $u(x)$ is the unit step function. Then the value of 'a' and Prob$\{X\le 0\}$, respectively, are
(a) $2, \dfrac{1}{2}$
(b) $4, \dfrac{1}{2}$
(c) $2, \dfrac{1}{4}$
(d) $4, \dfrac{1}{4}$

[EE, 2016 : 2 Marks]

Q.115 The area (in percentage) under standard normal distribution curve of random variable Z within limits from −3 to +3 is ___

[ME, 2016 : 1 Mark]

Q.116 Two coins are tossed simultaneously. The probability (upto two decimal points accuracy) of getting at least one head is ___

[ME, GATE-2017 : 1 Mark]

Q.117 A sample of 15 data is as follows: 17, 18, 17, 17, 13, 18, 5, 5, 6, 7, 8, 9, 20, 17, 3. The mode of the data is

(a) 4 (b) 13
(c) 17 (d) 20

[ME, GATE-2017 : 1 Mark]

Q.118 A six-face fair dice is rolled a large number of times. The mean value of the outcomes is ___

[ME, GATE-2017 : 1 Mark]

Q.119 Assume that in a traffic junction, the cycle of the traffic signal lights is 2 minutes of green (vehicle does not stop) and 3 minutes of red (vehicle stops). Consider that the arrival time of vehicles at the junction is uniformly distributed over 5 minute period. The expected waiting time (in minutes) for the vehicle at the junction is ___

[EE, GATE-2017 : 1 Mark]

Q.120 An urn contains 5 red balls and 5 black balls. In the first draw, one ball is picked at random and discarded without noticing its colour. The probability to get a red ball in the second draw is

(a) $\frac{1}{2}$ (b) $\frac{4}{9}$
(c) $\frac{5}{9}$ (d) $\frac{6}{9}$

[EE, GATE-2017 : 1 Mark]

Q.121 If a random variable X has a Poisson distribution with mean 5, then the expectation $E[(X+2)^2]$ equals ___

[CS, GATE-2017 : 2 Marks]

Q.122 A two-faced fair coin has its faces designated as head (H) and tail (T). This coin is tossed three times in succession to record the following outcomes: H, H, H. If the coin is tossed one more time, the probability (up to one decimal place) of obtaining H again, given the previous realizations of H, H and H, would be ___

[CE, GATE-2017 : 1 Mark]

Q.123 The number of parameters in the univariate exponential and Gaussian distributions, respectively, are

(a) 2 and 2 (b) 1 and 2
(c) 2 and 1 (d) 1 and 1

[CE, GATE-2017 : 1 Mark]

Q.124 For the function $f(x) = a + bx$, $0 \le x \le 1$, to be a valid probability density function, which one of the following statements is correct?

(a) $a = 1, b = 4$ (b) $a = 0.5, b = 1$
(c) $a = 0, b = 1$ (d) $a = 1, b = -1$

[CE, GATE-2017 : 2 Marks]

■■■■

Answers Probability and Statistics

[illegible]	[illegible]	2.	[illegible]	3	[illegible]	4.	[illegible]	5.	[illegible]	6.	[illegible]	7.	[illegible]	8	[illegible]	9	[illegible]
10	[illegible]	11.	[illegible]	12	[illegible]	13	[illegible]	14.	[illegible]	15.	[illegible]	16.	[illegible]	17	[illegible]	18	[illegible]
19	[illegible]	20.	[illegible]	[illegible]	[illegible]	22.	[illegible]	[illegible]	[illegible]	24.	[illegible]	25	[illegible]	26	[illegible]	27.	[illegible]
28.	[illegible]	29.	[illegible]	30	[illegible]	31	[illegible]	32.	[illegible]	33.	[illegible]	34	[illegible]	35	[illegible]	36.	[illegible]
37.	[illegible]	38	[illegible]	39.	[illegible]	40	[illegible]	41.	[illegible]	42.	[illegible]	43	[illegible]	44.	[illegible]	45.	[illegible]
46	[illegible]	47	[illegible]	48.	[illegible]	49	[illegible]	50.	[illegible]	51.	[illegible]	52	[illegible]	53.	[illegible]	54	[illegible]
55.	[illegible]	56	[illegible]	57.	[illegible]	58.	[illegible]	59	[illegible]	60.	[illegible]	61.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
66.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	85.	[illegible]
87	[illegible]	[illegible]	[illegible]	91.	[illegible]	[illegible]	[illegible]	94	[illegible]	95	[illegible]	[illegible]	[illegible]	99.	[illegible]	101	[illegible]
104	[illegible]	105.	[illegible]	107.	[illegible]	110.	[illegible]	[illegible]	[illegible]	114	[illegible]	117.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
124	[illegible]																

Explanations Probability and Statistics

1. (c)

Given, $P(A) =$ [illegible]

$P(B) =$ [illegible]

But [illegible] independent

So, $P(A \cap B) =$ [illegible]

$P(A|B) = \frac{P(A \cap B)}{P(B)} =$ [illegible]

$P(B|A) = \frac{P(A \cap B)}{P(A)} =$ [illegible]

2. (c)

This distribution is to be modeled by binomial distribution, since although population is finite, sampling is done with replacement and so probability does not change from trial to trial.

Here, [illegible]

[illegible]

[illegible]

So, [illegible]

[illegible]

3. (d)

Probability of drawing two red balls

= [illegible]

[illegible]

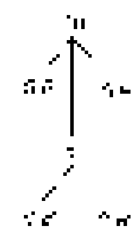
$$\frac{[illegible]}{[illegible]} \times \frac{4}{[illegible]} = \frac{[illegible]}{[illegible]}$$

Alternatively this problem can be done as [illegible]

From the tree diagram,

$P(\ldots) =$ [illegible]

$= \frac{[illegible]}{[illegible]} = \frac{[illegible]}{9}$



4. (a)

[illegible]

[illegible]

[illegible] density function

$$g(x) = \int [illegible]$$

$$= [illegible]$$

[illegible]

[illegible]

[illegible]

5. (c)
Since all the 3 gates are independent [illegible]
[illegible]
= [illegible]
= [illegible]
[illegible]
= [illegible]

6. (d)
[illegible] marks [illegible] per question [illegible]
[illegible]
[illegible] probability distribution table is given as below

X	[illegible]	[illegible]
$p(X)$	[illegible]	$3/4$

Expected marks per question
= $E(X)$ = [illegible]
= [illegible]
= [illegible] marks
Total marks expected for 150 questions
= [illegible] marks per student
Total expected [illegible] 1000 students
= [illegible] marks
So correct answer is (d).

7. (a)
The condition [illegible]
[illegible]
Given [illegible]
Applying [illegible]
[illegible]

$P(X = ?)$ = [illegible]

= [illegible] $= \frac{8}{16} = \frac{?}{?}$

8. (a)
[illegible]
[illegible]
[illegible]
[illegible] distribution [illegible]
[illegible]
= [illegible]
[illegible]
[illegible]

[illegible]
So correct answer is (a)

9. (d)
[illegible] can be solved by hypergeometric distribution as follows.

[illegible]

10. (d)
[illegible] positive [illegible]

[illegible]

Since x and y are uniformly distributed [illegible] and $0 \le y \le 2$

Probability density function [illegible] $= 1$

Probability density function [illegible]

[illegible]

= [illegible]

[illegible]

$= \frac{8}{3} = \frac{4}{?}$

$\therefore$ [illegible]

11. (d)
[illegible] independent
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]

(c) is false since independence and mutually exclusive are unrelated properties.

(d) is true

since [illegible]

⇒ [illegible]

⇒ [illegible]

12. (d)

[illegible]

13. (a)

Sample space = [illegible] = 36

Total ways in which sum is [illegible]

[illegible]

[illegible] = [illegible] ways

[illegible] = $\frac{6}{36}$ = [illegible]

[illegible]

[illegible] $\frac{3}{[illegible]}$

14. (b)

Sample space = {HHH, HHT, [illegible] TTT}

[illegible] = HTH, [illegible]

Required probability = $\frac{[illegible]}{4}$ = [illegible]

15. (d)

[illegible] negatively skewed [illegible] mean.

16. (d)

This problem [illegible] binomial distribution [illegible]

Probability of [illegible]

p = [illegible]

Probability [illegible]

[illegible]

Probability [illegible]

[illegible]

= [illegible]

= [illegible]

17. (c)

[illegible] probability density function [illegible]

[illegible] $= \int_a^b [illegible]$

18. (d)

(a) is false since [illegible]

[illegible]

which [illegible]

(b) [illegible]

[illegible]

∴ [illegible]

(c) [illegible] independence and mutually exclusive are unrelated properties.

(d) [illegible]

[illegible]

⇒ [illegible]

[illegible]

[illegible]

⇒ [illegible]

19. (a)

$I = \frac{1}{[illegible]} \int_0^{[illegible]} [illegible]$

[illegible]

[illegible]

[illegible] putting [illegible] = 0

[illegible]

Putting [illegible]

⇒ [illegible] = 2

[illegible]

⇒ [illegible] = 1

20. (c)

[illegible] probability distribution [illegible]

[illegible]

The above table is [illegible] different ways [illegible] a particular sum [illegible] 8 [illegible] can be obtained as (1, 1) (2, 3), (3, 2) and (1, 1)

∴ [illegible] = [illegible]

Now [illegible] choice (a)

$P(x > 8) = $ [illegible]

$\frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = $ [illegible]

∴ choice [illegible] is wrong

Consider choice (b)

[illegible]

$= \frac{5}{36} + \frac{3}{36} + \frac{3}{36} + \frac{1}{36} = \frac{12}{36} = \frac{1}{3}$

∴ choice (b) is [illegible]

Consider choice (c)

$p(x = 8 \mid$ [illegible] $) = \frac{1}{36}$

[illegible]

$= \frac{5}{36} + \frac{5}{36} - \frac{1}{36} - \frac{6}{36} = \frac{1}{4}$

[illegible] $= \frac{2}{36}$

[illegible] $= \frac{5/36}{1/2} = \frac{10}{36} = \frac{5}{18}$

∴ Choice (c) is correct.

21. (d)

[illegible]

Probability that the computer was [illegible]

$P(c \mid e) = \frac{P(c \cap e)}{P(e)}$

$P(c \cap e) = 0.5 \times 0.02 = $ [illegible]

$P(e) = $ [illegible]

$= $ [illegible]

$P(c \mid e) = \frac{0.005}{[illegible]} = 0.4$

22. (a)

[illegible]

[illegible] (Binomial [illegible])

$= $ [illegible]

$= $ [illegible]

23. (b)

Since population is finite, hypergeometric distribution is applicable.

[illegible]

[illegible] $= $ [illegible] $= \frac{1}{3}$

24. (d)

Problem can be solved by hypergeometric distribution.

$p(X = 2) = \frac{[illegible]}{[illegible]} = $ [illegible]

25. (b)

Mean [illegible]

[illegible]

Variance [illegible]

[illegible]

$= \int [illegible]$

$= \int [illegible]$

$= \left[\frac{[illegible]}{3} - \frac{[illegible]}{2}\right] - [illegible]$

$= \frac{1}{12} + \frac{1}{12} - [illegible]$

Standard deviation

$= \sqrt{\text{Variance}} = \frac{1}{\sqrt{[illegible]}}$

26. (c)

$[illegible] = 1$

$[illegible]$

$\Rightarrow [illegible] = 1$

$[illegible]$

$\Rightarrow X = [illegible]$

27. (d)

[illegible] mean and standard deviation of the students [illegible] respectively and the mean and standard deviation of [illegible] respectively.

$[illegible] = 83$

$[illegible] = 23$

and $[illegible] = 6.5$

$[illegible] = 4.5$

[illegible]

Thus $Z = [illegible] = \frac{[illegible] - 83}{[illegible]}$

$[illegible] = \frac{[illegible]}{2.3}$

Equating [illegible] weight

$\frac{8.5 - [illegible]}{2.5} = \frac{x - 6.5}{[illegible]}$

$\Rightarrow x = 5.080 [illegible]$

28. (c)

Number of permutations with [illegible] = [illegible]

Number of permutations with [illegible] the second position = [illegible]

[illegible]

Number of permutations with [illegible] = [illegible]

[illegible]

[illegible] satisfy the given condition [illegible] odd numbers [illegible] So the desired number of permutations [illegible] is

[illegible]

Now the probability of [illegible] is given by

$[illegible] = \frac{[illegible]}{9!} = [illegible]$

[illegible] Answer [illegible]

29. (c)

[illegible]	1	2	3	4	5	6
Probability	$\frac{1}{4}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

$[illegible] = \frac{1}{4} \times \frac{1}{6} [illegible] = \frac{[illegible]}{128}$

30. (c)

[illegible]

Given, $P(A) = 0.[illegible]$ $[illegible] = 0.7$

$P(A|B) = 0.8$

[illegible]

$P(A \cap \overline{B}) = [illegible]$

$= 0.2 \times 0.8 = [illegible]$

31. (c)

$[illegible] = \frac{9\pi}{[illegible]} = 0.7[illegible]$

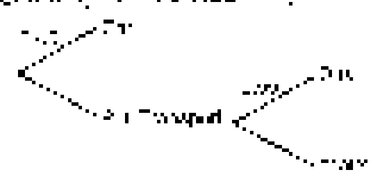
32. (d)

Since,

$E(X^2) \geq [E(X)]^2$

33. (a)

The information given in the problem can be represented by the tree diagram given below:

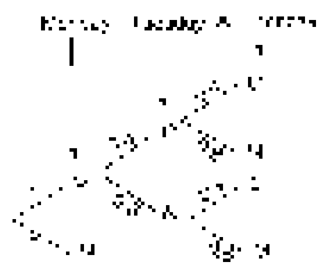


Now, writing the probabilities in the above diagram we have the tree diagram as shown below:

From above diagram,

34. (c)

Let C denote computer science students and M denote mathematics students. The tree diagram for the problem can be represented as shown below:



35. (a)

Binomial distribution with success probability

Probability of getting ...

36. (b)

Given,

Hence probability expressed in percentage

37. (a)

Given

Now, since it is known that standard normal distribution

Comparing ...

38. (c)

Now, each toss is independent.

So required probability

39. (a)

It is given that

[illegible]

Now that, Σ [illegible] $= 1$

[illegible]

[illegible]

[illegible] $= \frac{1}{3}$ [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$\Rightarrow$ [illegible] $= 0.75$

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

[illegible]

40. (a)

Sample space $=$ {[illegible]}

Favourable outcomes [illegible] $=$ {[illegible]}

Required probability $= \frac{1}{3}$

41. (d)

[illegible]

[illegible]

Here, $n = 3$

$p =$ [illegible]

[illegible]

Now, [illegible]

[illegible]

$= \frac{1}{2}$ [illegible] $\frac{7}{8}$

42. (c)

$\sigma = \sqrt{\frac{[illegible]}{12}} = \sqrt{\frac{(1-0)^2}{12}} = \frac{1}{\sqrt{12}}$

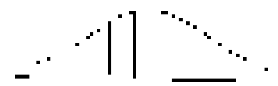
43. (b)

Here [illegible] $= 27$ [illegible]

[illegible] 27 [illegible] 27

[illegible]

This area is shown below



[illegible]

[illegible]

[illegible] $\frac{1}{[illegible]}$ [illegible]

[illegible]

[illegible]

44. (a)

[illegible]

[illegible]

[illegible]

45. (c)

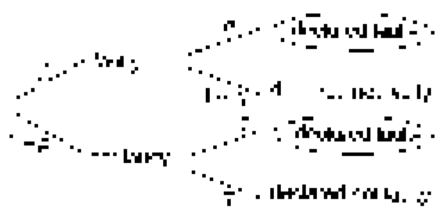
[illegible]

[illegible]

[illegible]

[illegible] $= \frac{6}{[illegible]}$

46. (a)



[illegible]

[illegible]

[illegible]

47. (a)

Coin [illegible] times

[illegible]

[illegible]

$= \frac{1}{16} = [illegible] = [illegible]$

48. [illegible]

[illegible]

49. [illegible]

[illegible]

$= \frac{4}{[illegible]} \times [illegible]$

50. [illegible]

[illegible]

$[illegible] = \frac{15}{[illegible]} = \frac{5}{[illegible]}$

51. (d)

[illegible]

52. [illegible]

[illegible]

where [illegible]

[illegible]

53. [illegible]

[illegible]

54. [illegible]

[illegible]

55. [illegible]

[illegible]

From the above diagram

[illegible]

56. (d)

[illegible]

binomial with [illegible]

[illegible]

57. (c)

The pmf of the random variable is

x	−1	+1
P(x)	0.5	0.5

The cumulative distribution function F(x) is the probability function given below:

x	−1	+1
F(x)	0.5	1.0

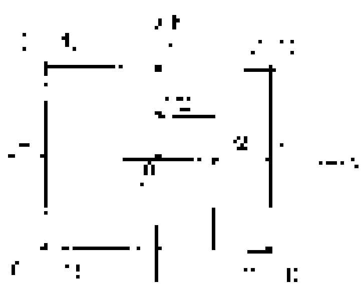
So the answer is (c).

58. (d)

The problem can be represented by the following diagram.

59. (b)

Since $0 \le x \le 1$ and $0 \le y \le 1$, this is the region for which the maximum of (x, y) is less than $\frac{1}{2}$ ... shown below as the shaded region in the rectangle.



$$P\left[\max(x,y) < \frac{1}{2}\right] = \frac{\text{Area of shaded region}}{\text{Area of the rectangle}}$$

60. (a)

The annual precipitation is normally distributed with $\mu = 1000$ mm and $\sigma = 200$ mm

$$P(X > 1300) = P\left(Z > \frac{1300-1000}{200}\right) = P(Z > 1.5)$$

where Z is the standard normal variable of the normal distribution.

Now, since $P(-1 < z < 1) = 0.68$

0.68 ... between one standard deviation from mean

... So option (a) is correct.

61. (c)

Poisson formula for $(X = x)$ given as

$$\frac{e^{-\lambda}\lambda^x}{x!}$$

... Probability of observing fewer than 3 cars

$P(X=0) + P(X=1) + P(X=2)$

... (c) is correct option.

62. Sol.

$$\int f(x)\,dx = 1$$

$$f(x) = \begin{cases} k(-x^2+3x-2), & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_1^2 k(-x^2+3x-2)\,dx = 1$$

$$\Rightarrow k\left[-\frac{x^3}{3} + 3\frac{x^2}{2} - 2x\right]_1^2 = 1$$

$$\Rightarrow k\left[-\left(\frac{8}{3}-\frac{1}{3}\right) + \frac{3}{2}(4-1) - 2(2-1)\right] = 1$$

$$\Rightarrow k\left[-\frac{7}{3} + \frac{9}{2} - 2\right] = 1$$

$$\Rightarrow k\left[\frac{-14+27-12}{6}\right] = 1$$

$$\Rightarrow k = 6$$

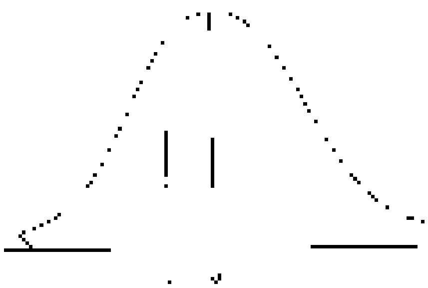
63. (a)

64. (b)

[illegible]

65. (b)

[illegible]



[illegible]

66. Sol.

Given X is normally distributed,

[illegible]

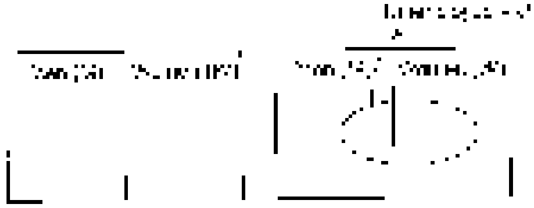
67. (b)

[illegible]

68. (a)

[illegible]

69. Sol.



[illegible]

70. (a)

[illegible] ... which is an impossible event and the required probability is zero.

71. Sol.

[illegible]

72. Sol.

[illegible]

73. (c)

[illegible]

Probability of getting exactly 3 heads in [illegible] tosses

$$^9C_3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^6 = {}^9C_3 \times \left(\frac{1}{2}\right)^9$$

and in 10th toss head must appear.

So required probability

$$= {}^9C_3 \left(\frac{1}{2}\right)^9 \times \frac{1}{2} = \frac{84}{1024} = 0.082$$

74. Sol.

Probability of [illegible] $= \frac{1}{5}$

Probability [illegible] $= \frac{4}{5} \times \frac{1}{2}$

Total probability [illegible] $= \frac{1}{5} \quad \frac{4}{10} \quad \frac{3}{5}$

Required probability $= \frac{[illegible]}{[illegible]} = \frac{1}{3} = [illegible]$

75. Sol.

[illegible] $\Rightarrow \frac{4!}{2!} = [illegible]$ ways

[illegible] $\Rightarrow \frac{4!}{2!2!} = 6$ ways

Probability of [illegible]

$$= \frac{6+[illegible]}{[illegible]} = \frac{[illegible]}{1296} = \frac{[illegible]}{1296}$$

$\Rightarrow \quad x = [illegible]$

76. Sol.

$1 \le x \le 100$

P(x is not divisible by 2, 3 or 5) $= 1 - P$(x is divisible by 2, 3 or 5)

$$1 - \left[\left[\frac{1}{2}\right] + \left[\frac{100}{3}\right] \ldots\right]$$

$$= \frac{26}{100} = 0.26$$

77. Sol.

[illegible] A and B are mutually exclusive [illegible]

[illegible]

Now [illegible] A and B are both mutually exclusive and collectively exhaustive, then $P(A) + P(B) = 1 \Rightarrow P(B) = 1 - P(A)$

Now we want to maximize $P(A)\,P(B)$

$= P(A)(1 - P(A))$

Let $P(A) = x$

Now $P(A)(1 - P(A)) = x(1 - x) = x - x^2$

Let $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x = 0 \Rightarrow x = \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = -2 < 0$$

$= -2 < 0$

[illegible] maximum at $x = 1/2$,

$$y_{max} = \frac{1}{2}\left(\frac{1}{2}\right) = 0.25$$

78. (c)

Mean

$\mu = \Sigma x\,p(x) = 1 \times 0.3 + 2 \times 0.6 + 3 \times 0.1 = 1.8$

Standard deviation

$$\sigma = [\Sigma x^2 p(x) - (\Sigma x\,p(x))^2]^{1/2}$$

$$= [1^2 \times 0.3 + 2^2 \times 0.6 + 3^2 \times 0.1 - (1.8)^2]^{1/2}$$

$= (3.6 - 3.24)^{1/2} = (0.36)^{1/2} = 0.6$

79. (a)

x	0	1	2
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$E(x) = \Sigma x\,p(x) = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1$$

$$E(x^2) = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$$

Variance $= E(x^2) - (E(x))^2 = \frac{5}{3} - 1 = \frac{2}{3}$

80. Sol.

x	R	B	G
$P(x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

$x = 3$ [illegible] red colour

$p = P(\ldots) = \frac{1}{6}$ [illegible]

$q = 1 - p = 1 - \frac{?}{6} = \frac{5}{6}$ [illegible]

Prob. of getting red colour [illegible] twice [illegible]

$= p(X = 2) = [illegible]$

$= {}^{4}C_{2} p^{2} q^{[illegible]}$ [illegible]

$= [illegible]$

$= 3 \cdot \frac{4}{36} \cdot [illegible] \frac{8}{216}$ [illegible]

$\frac{48}{216} [illegible] \frac{[illegible]}{216} = 0.2[illegible]$

81. Sol.

The tree diagram of the problem is shown below.

Required probability

$= [illegible]$

$= 0.1[illegible]0$

$\Rightarrow \quad 100\,p = 1[illegible]$

82. Sol.

Mean $\lambda = 5$ [illegible]

$P(x \leq 4) = p(x = 0) + p(x = 1) + p(x = 2) + [illegible]$

$+ p(x = 3)$ [illegible]

$= \frac{e^{-5}5^{0}}{0!} + \frac{e^{-5}5^{1}}{1!} + \frac{e^{-5}5^{2}}{2!} + \frac{e^{-5}5^{3}}{3!}$ [illegible]

$e^{-5}\left[1 + 5 + \frac{25}{2} + \frac{125}{[illegible]}\right]$ [illegible]

$= [illegible] = 0.265$

83. (a)

$P(x) = \frac{[illegible]}{x!}$ [illegible]

[illegible]

$P(x \leq [illegible]) = P(0) + P(1) = [illegible]$

$\therefore \quad [illegible] = \frac{[illegible]}{[illegible]} = 0.0[illegible]$

84. Sol.

$f(x) = \begin{cases} \frac{1}{5}, & [illegible] \\ 0, & \text{otherwise} \end{cases}$

$P([illegible] \leq x \leq 4) = \int f(x)\,dx = \int \frac{1}{5}\,dx = [illegible]$

$= \frac{1}{5}([illegible] - 4) = \frac{?}{5} = 0.4$ [illegible]

85. Sol.

Probability $(0.5 \leq x \leq 5)$ [illegible]

$= \int_{0.5}^{5} f(x)\,dx$

$= \int_{0.5}^{[illegible]} [illegible]\,dx + \int [illegible]\,dx + \int_{4}^{5} [illegible]\,dx$

$= 0.2[illegible] - 0.5] + [illegible] - 1] + 0.[illegible] - 4]$

$= [illegible] = 0.4$ [illegible]

86. (a)

In normal distribution [illegible] the normal curve [illegible]

Here, [illegible] is the mean [illegible]. The value of the integral

$[illegible] \int \frac{1}{\sqrt{2\pi}} e^{[illegible]}\,dx$ [illegible] area under the normal curve [illegible]

normal curve [illegible] mean = 0.5 [illegible]

87. (c)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [illegible]

Also

$P(A \cap B) = P(A)\,P(B)$ [illegible]

(if independent events [illegible])

So $P(A \cup B) = [illegible]$ is false

90. Sol.

X_1	X_2	X_3	[illegible]	[illegible]
[illegible]	[illegible]	0	0	0
[illegible]	[illegible]	[illegible]	0	[illegible]
1	0	[illegible]	0	[illegible]
1	0	[illegible]	[illegible]	1

[illegible] in 3 cases

X_2 = [illegible] cases

$P(Y = 0 \mid X_2 = 0) = \frac{P(Y = 0 \cap X_2 = 0)}{P(X_2 = 0)} = \frac{3}{4} = 0.75$ [illegible]

89. Sol.

Probability of atleast one [illegible]

$= 1 - [\ldots]$

$= 1 - (0.2 \times \ldots) = 0.97$

90. (a)

Given, p(passing the exam) = [illegible]

p(passing the exam and 80%) = 0.15

The probability [illegible]

= p(80% | passing the exam)

$\frac{p(\text{passing the exam and } 80\%)}{p(\text{passing the exam})} = \frac{0.05}{0.?} = \frac{1}{?}$

91. (c)

[illegible]

92. (c)

[illegible]

93. Sol.

Given, [illegible]

X	0	1
$p(X)$	0.4	0.6

Required value = $E(X^2) - [E(X)]^2$

$E(X) = \sum x_i p_i = 0 \times 0.4 + 1 \times 0.6 = 0.6$

$E(X^2) = \sum x_i^2 p_i = 0^2 \times 0.4 + 1^2 \times 0.6 = 0.6$

$\therefore V(X) = E(X^2) - [E(X)]^2 = 0.6 - 0.36 = 0.24$

94. (a)

From the given information we can make the joint probability table as follows:

[illegible]

From the table we get

[illegible]

So Coin 1 (H/T) and dice are statistically independent.

The joint probabilities are independent since the probability of heads or tails is 0.5.

Hence, [illegible]

If events are independent then all the joint probabilities will be equal to the product of the marginal probabilities.

For example

[illegible]

So events are independent.

[illegible] are dependent. So, answer (a) is true and statement (b) is true.

95. (b)

Let p be the probability that six appears on a dice.

$\therefore p = \frac{1}{6}$

$\therefore q = \frac{5}{6}$

Let X be the number of times six appears.

Probability of obtaining at least two sixes in [illegible] throws of a [illegible]

$= 1 - [P(X = 0) + P(X = 1)]$

[illegible]

$= 1 - \left[\frac{125}{144}\right] = \frac{19}{144}$

96. (a)

[illegible]

97. Sol.

$f(x) = \frac{3}{32}(4 - x^2)$, [illegible]

$= 0$, otherwise

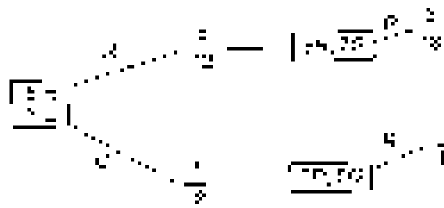
$\int_0^2 f(x)\,dx$ [illegible]

$\therefore$ Mean $(\mu_X) = \int_0^6 x \cdot \frac{x}{...}(...)dx$

$= \int ...$

$= \frac{8}{3} ... = \frac{15}{8} = ...$

98. Sol.



$P(\ldots) = \frac{5}{11} \times \frac{8}{12} + \frac{7}{12} \times \frac{5}{12} = \frac{...}{...}$

99. Sol.

$f(y) = \begin{cases} a + by & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Now given $E(Y) = \ldots$

$\Rightarrow \int_0^1 y f(y)\,dy = \frac{2}{3}$

$\Rightarrow \int_0^1 y(a + by)\,dy = \frac{2}{3}$

$\Rightarrow a\left[\frac{y^2}{2}\right]_0^1 + b\left[\frac{y^3}{3}\right]_0^1 = \frac{2}{3}$

$\Rightarrow 3a + 2b = 4 \quad \ldots(i)$

Also $\int f(y)\,dy = 1$

(Total probability is always equal to 1)

$\Rightarrow \int_0^1 (a + by)\,dy = 1$

$\Rightarrow \left[ay + \frac{by^2}{2}\right]_0^1 = 1$

$\Rightarrow a + \frac{b}{2} = 1$

$\Rightarrow 2a + b = 2 \quad \ldots(ii)$

Now solving (i) and (ii) we get

$a = 0,\ b = 2$

So $f(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Now we need $\int_0^{...} 2y\,dy = \frac{1}{2}$

100. Sol.

$P(H) = 0.3$

$P(T) = 0.7$

Since all tosses are independent,

so probability of getting head for the first time in the ... toss is

$= \ldots$

$= 0.7 \times 0.7 \times \ldots \times 0.3$

$= 0.\ldots$

101. Sol.

$P(X \cup Y) = 0.7$

$\Rightarrow P(X) + P(Y) - P(X)P(Y) = 0.7$

(Since X and Y are independent events)

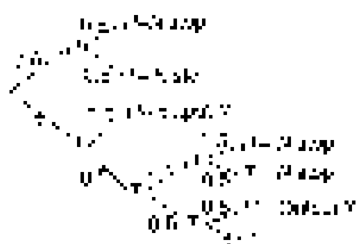
$\ldots$

$\Rightarrow P(Y) = \ldots = 0.5 \quad \ldots(ii)$

$P(X \cup Y) - P(X) = \ldots$

$= \ldots - 0.5 = 0.\ldots$

102. Sol.



The above diagram is for the ... of ...

The desired output is Y

Now probability ...

$\ldots = 0.5 \times 0.5 \times \ldots \times 0.5 \times 0.5 + \ldots$

...

$= 0.5 \times 0.5$

and $= 0.5 \times 0.5$

so, $P(\ldots) = \frac{0.5 \times 0.5}{\ldots} = \frac{0.25}{0.75}$

$= \frac{1}{3} = 0.33$ (upto 2 decimal places)

103. Sol.

[illegible]

104. (a)

[illegible]

106. Sol.

Median speed is the speed of the middle value in series of spot speeds that are arranged in ascending order. 50% of speed values will be greater than the median 50% will be less than the median.

Ascending order of spot speed studies are: [illegible]

Median speed = [illegible] km/hr

107. (b)

Mean of [illegible]

[illegible]

Variance of [illegible]

[illegible]

⇒ Variance is [illegible]

Next, mean of [illegible]

[illegible]

Variance of [illegible]

[illegible]

⇒ Variance is $\frac{a^2}{2}$

∴ Mean of [illegible] and variance of [illegible] are different.

108. Sol.

[illegible]

109. Sol.

For Poisson distribution,

Mean = [illegible]

[illegible]

Given the second moment [illegible]

[illegible]

110. (a)

[illegible]

Standard deviation = [illegible]

111. Sol.

Given [illegible]

= 0 elsewhere

So $\int f(x) = 1$

[illegible]

112. Sol.

$P(\ldots) = \int \int [\text{illegible}]\, dx\, dy$

$= \int [\text{illegible}]\, dx\, dy$

$= \int [\text{illegible}]$

$= \int [\text{illegible}]\, dx$

$= \int \left(\frac{[\text{illegible}]}{2} - \frac{[\text{illegible}]}{2}\right) dx = \left[\frac{[\text{illegible}]}{2} - \frac{[\text{illegible}]}{[\text{illegible}]}\right]$

$= \frac{[\text{illegible}]}{2} [\text{illegible}] = [\text{illegible}]$

113. (b)

$P(\ldots) = \int^{1} f(x)\, dx$

$= [\text{illegible}]$

$= [\text{illegible}] = \frac{[\text{illegible}]}{4}$

114. (b)

$f(x) = [\text{illegible}]$

$\int f(x)\, dx = 1$

$\int [\text{illegible}]\, dx + \int [\text{illegible}]\, dx = 1$

$[\text{illegible}] = 1$

$[\text{illegible}]$

$c = 2$

$P(x < [\text{illegible}]) = \int [\text{illegible}]\, dx$

$= [\text{illegible}] = \frac{1}{2}$

115. Sol.

116. Sol.

Sample space $\{HH, [\text{illegible}]\}$

Probability of getting at least one head $= \frac{3}{[\text{illegible}]}$

117. Sol.

Mode is the highest number of occurrences of [illegible] in ... 17 [illegible]

118. Sol.

[illegible]

$= [\text{illegible}]$

$\frac{1}{[\text{illegible}]}[\text{illegible}] + [\text{illegible}] = \frac{2}{3}$

119. Sol.

[illegible]

[illegible]

Then $[\text{illegible}]$

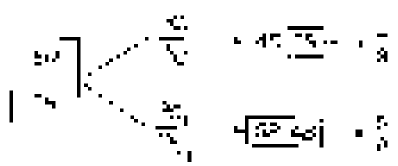
$y \rightarrow [\text{illegible}]$

$[\text{illegible}] = \frac{1}{[\text{illegible}]}$

$E(y) = \int [\text{illegible}]\, dy = \int [\text{illegible}]$

[illegible]

120. (a)



Hence, [illegible]

121. Sol.

Given, Poisson distribution $\lambda = 5$

We know that in Poisson distribution

$E(X) = V(X) = \lambda$

So here $E(X) = V(X) = 5$

Now, we need $E[(X + 2)^2]$

$= E[X^2 + 4X + 4]$

$= E(X^2) + 4E(X) + 4$

To find $E(X^2)$ we use $V(X) = E(X^2) - [E(X)]^2$

$5 = E(X^2) - 25$

So $E(X^2) = 25 + 5 = 30$

Required value $= 30 + 4 \times 5 + 4 = 54$

122. Sol.

Since the outcome of each trial of the experiment will be independent of previous outcome,

$\Rightarrow \quad P(H) = \frac{1}{2}$

123. (b)

In exponential, $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

The parameter is λ.

In Gaussian

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

The parameters are μ and σ.

Therefore, answer is (b).

124. (b)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 (a - bx)\,dx = 1$$

$$\left[ax - \frac{bx^2}{2}\right]_0^1 = 1$$

$$a - \frac{b}{2} = 1$$

Option (b) is satisfying the above equation.

6 Numerical Method

6.1 Introduction

[illegible]

1. Analytical Methods
2. Numerical Methods

6.1.1 Analytical Methods

[illegible]

Example 1.

[illegible]

Analytical solution: [illegible]

Example 2.

[illegible]

Analytical solution: [illegible]

Example 3.

[illegible]

[illegible] $y(0) = 5$

Analytical solution: [illegible]

[illegible]

$\therefore$ $y = 5e^{3x}$ [illegible]

6.1.2 Numerical Methods

[illegible]

[illegible]

[illegible]

For example, there are no [illegible] formulae available for polynomials [illegible]. Whereas Numerical methods can be used to solve polynomials [illegible] any degree.

Also numerical solutions can be used for linear as well as non-linear equations, [illegible] mostly for linear equations.

With the advent of [illegible] computers [illegible] numerical methods have [illegible] using analytical methods only.

Although Numerical Methods [illegible]

1. Solution of system of linear equations
2. Solution of algebraic and transcendental equations in single variable
3. Evaluation of definite integrals
4. Solution of ordinary differential equations

The advantage of numerical methods [illegible]

6.1.3 Errors in Numerical Methods

1. **Round-off Error:** [illegible] many significant digits.
2. **Truncation Error:** It occurs as a result of [illegible] functions.

Example:

Taylor's and Maclaurin Series [illegible] series.

Although errors [illegible] Numerical Methods [illegible]

For example, [illegible] double [illegible]

[illegible] used.

For example, [illegible] Simpson's [illegible] Numerical Methods [illegible]

[illegible] Gauss Seidel [illegible] Newton Raphson [illegible]

Quantifying Errors in Numerical Methods: [illegible]

$$\text{Error} = \text{Exact Value} - \text{Approximate Value}$$

$$\text{Absolute Error} = |\text{Exact Value} - \text{Approximate Value}|$$

$$\text{Relative Error} = \frac{|\text{Exact} - \text{Approximate}|}{\text{Exact}}$$

$$\text{Relative Error \%} = \left|\frac{\text{Exact} - \text{Approximate}}{\text{Exact}}\right| \times 100$$

6.2 Numerical Solution of System of Linear Equations

Consider the following n first degree equations containing n unknowns $x_1, x_2, \ldots, x_n$.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n = b_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nj}x_j + \ldots + a_{nn}x_n = b_n$$

In the matrix form, we have

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}_{n \times 1}$$

$$AX = B$$

By finding a solution [illegible] such that they satisfy all the given equations [illegible]. The system of equations [illegible] homogeneous [illegible] non-homogeneous system. There are a number of methods to solve the above System of Linear Equations.

These are as follows.

1. Matrix Inversion Method
2. Cramer's Rule
3. Crout's and Dolittle Method (Triangularisation Methods)
4. Gauss Elimination Method
5. Gauss Jordan's Method
6. Gauss-Seidel Iterative Method
7. Jacobi Iterative Method

In this book we shall focus on triangularisation, Gauss-Elimination and Gauss Seidel Methods only.

6.2.1 Method of Factorisation or Triangularisation Method (Dolittle's Triangularisation Method)

This method is based on the fact that a square matrix A can be expressed in the form LU, where L is unit lower triangular and U is upper triangular, if all the principal minors of A are non-singular [illegible].

We consider, for definiteness, the linear system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written in the form

$$AX = B \quad \text{...(i)}$$

Let

$$A = LU \quad \text{...(ii)}$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{...(iii)}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \text{...(iv)}$$

(i) becomes $\quad LUX = B \quad$ ...(v)

If we set $\quad UX = V \quad$ ...(vi)

then (v) may be written as $\quad LV = B \quad$ ...(vii)

which is equivalent to the system $v_1 = b_1$

$$l_{21}v_1 + v_2 = b_2$$
$$l_{31}v_1 + l_{32}v_2 + v_3 = b_3$$

and can be solved for v_1, v_2, v_3 by the forward substitution. Once V is known, the system (vi) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1$$
$$u_{22}x_2 + u_{23}x_3 = v_2$$
$$u_{33}x_3 = v_3$$

which can be solved by backward substitution.

We shall now describe a scheme for computing the matrices L and U, and illustrate the procedure with a matrix of order 3. From the relation (ii), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating the corresponding elements of both sides, we get

$$u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21}$$

$$\Rightarrow \quad l_{21} = \frac{a_{21}}{u_{11}}$$

$$l_{21}u_{12} + u_{22} = a_{22}$$

$$\Rightarrow \quad u_{22} = a_{22} - l_{21}u_{12}$$

$$l_{21}u_{13} + u_{23} = a_{23}$$

$$\Rightarrow \quad u_{23} = a_{23} - l_{21}u_{13}$$

$$l_{31}u_{11} = a_{31}$$

$$\Rightarrow \quad l_{31} = \frac{a_{31}}{u_{11}}$$

$$l_{31}u_{12} + l_{32}u_{22} = a_{32}$$

$$\Rightarrow \qquad l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

Lastly, $\quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$

$$\Rightarrow \qquad u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Thus the variables are obtained in the order:

First u_{11}, u_{12}, u_{13}

then l_{21}, l_{31}, u_{22}

finally l_{32}, u_{23}, u_{33}

Example:

Solve the equations

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

by the factorisation method.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Clearly $\quad u_{11} = 2, u_{12} = 3, u_{13} = 1$

and $\quad l_{21}u_{11} = 1$ so that $l_{21} = 1/2$

$$l_{21}u_{12} + u_{22} = 2$$

$$\Rightarrow \qquad u_{22} = 2 - l_{21}u_{12} = 1/2$$

$$l_{21}u_{13} + u_{23} = 3$$

from which we obtain $\quad u_{23} = 5/2$

$$l_{31}u_{11} = 3$$

$$\Rightarrow \qquad l_{31} = 3/2$$

$$l_{31}u_{12} + l_{32}u_{22} = 1$$

$$\Rightarrow \qquad l_{32} = -7$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$$

$$\Rightarrow \qquad u_{33} = 18$$

It follows that $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$

[illegible]

$$\begin{bmatrix} 1 & 0 & 0 \\ [illegible] & 1 & 0 \\ [illegible] & [illegible] & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & [illegible] & [illegible] \\ 0 & 0 & [illegible] \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} [illegible] \\ [illegible] \\ 8 \end{bmatrix}$$

[illegible]

[illegible]

$$z = \frac{3}{2}$$

[illegible]

[illegible]

[illegible]

$$x = [illegible], \quad y = [illegible], \quad z = [illegible]$$

Note: [illegible]

[illegible]

[illegible]

8.2.2 Gauss Seidel Method

[illegible]

[illegible]

[illegible]

[illegible] "Successive displacements" [illegible] "Gauss-Seidel method".

Note: [illegible] method"

6.3 Numerical Solutions of Nonlinear Algebraic and Trans-cendental Equations by Bisection, Regula-Falsi, Secant and Newton-Raphson Methods

[illegible]

[illegible]

6.3.1 Roots of Algebraic Equations

[illegible]

[illegible]

3. [illegible]

4. [illegible]

5. [illegible]

[illegible]

6. [illegible]

Example:

[illegible]

Solution:

[illegible]

[illegible]

1. [illegible]

2. [illegible]

3. [illegible]

4. [illegible]

[illegible]

6.3.2 Descarte's Rule of Signs

[illegible]

Example:

[illegible]

Solution:

[illegible]

6.3.3 Numerical Methods for Root Finding

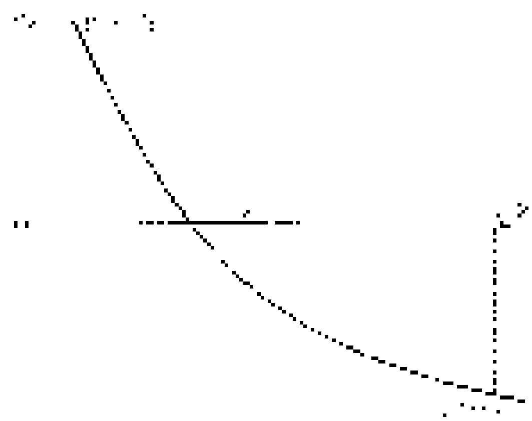
[illegible]

1. Bisection Method
2. Regula Falsi Method
3. Secant Method
4. Newton-Raphson Method

6.3.3.1 Bisection Method

[illegible]

This method is shown graphically as below.



The formula for bisection method is $x_2 = \dfrac{x_0 + x_1}{2}$ or more generally $x_{n+1} = \dfrac{x_{n-1} + x_n}{2}$

Example:

Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution:

Since $f(1)$ is negative and $f(2)$ is positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2$.

Then $f(x_0) = \dfrac{27}{8} - \dfrac{3}{2} - 1 = \dfrac{7}{8}$, which is positive. Hence the root lies between 1 and 1.5 and we obtain $x_1 = (1 + 1.5)/2 = 1.25$. We find $f(x_1) = -19/64$, which is negative. We, therefore, conclude that the root lies between 1.25 and 1.5. It follows that $x_2 = (1.25 + 1.5)/2 = 1.375$.

The procedure is repeated and the successive approximations are $x_3 = 1.3125$, $x_4 = 1.34375$, $x_5 = 1.328125$, etc.

6.3.3.3 Regula-Falsi Method

The method starts by taking two guess values x_0 and x_1 such that $f(x_0)f(x_1) < 0$. The iteration formula in Regula-Falsi method is almost similar to secant method and it is

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

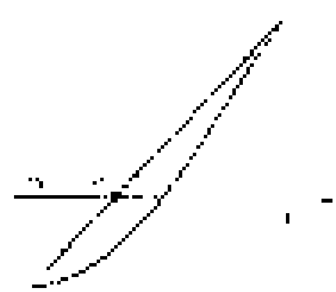
or more generally

$$x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}$$

Graphically this method is equivalent to drawing a line between $(x_0, f(x_0))$ and $(x_1, f(x_1))$ and taking the point of intersection of this line with x-axis as x_2, as shown in Figure.

Since the root lies in the interval between x_0 and x_2 or between x_2 and x_1.

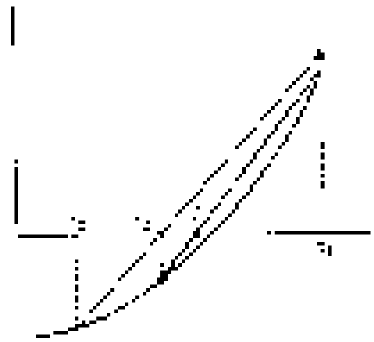
Using this method the next guess depends on the sign of $f(x_2)$.



[illegible]

This is illustrated graphically as follows:

[illegible]



6.3.3.3 Secant Method

[illegible]

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

In general we have $$x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}$$

[illegible]

$$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1}$$

[illegible]

6.3.3.4 Newton-Raphson Method

[illegible]

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

Neglecting the second and higher order derivatives, we have $f(x_0) + h f'(x_0) = 0$

which gives $$h = -\frac{f(x_0)}{f'(x_0)}$$

A better approximation to the root is given by x_1 where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$

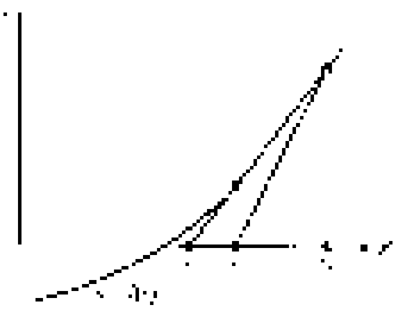
where $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots(i)$$

which is Newton-Raphson formula.

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} \quad \ldots(ii)$$

So that the Newton-Raphson process has a second order or quadratic convergence.

Geometrically, in Newton-Raphson method, a tangent to curve is drawn at point $[x_0, f(x_0)]$ and the point of intersection of this tangent and x-axis is taken as approximation to the root which is closer to root than x_0. It can be used for solving both algebraic and transcendental equations and it can also be used when the roots are complex.



The method converges very rapidly to the root with a second order convergence. The number of significant digits approximately gets doubled after each iteration of N-R method.

Following is a list of Common Newton Raphson Iterative formulas along with the Newton-Raphson iteration equation for solving the problem.

1. The inverse of N is the root of the equation $f(x) = \frac{1}{x} - N = 0$

   Iteration Equation: $x_{n+1} = x_n(2 - Nx_n)$

2. The inverse square root of N is the root of equation $f(x) = \frac{1}{x^2} - N = 0$

   Iteration Equation: $x_{n+1} = \frac{1}{2}x_n(3 - Nx_n^2)$

3. The kth root of a given number N is root of equation $f(x) = x^k - N = 0$

   Iteration Equation: $x_{n+1} = \frac{(k-1)x_n^k + N}{kx_n^{k-1}}$

Note: The orders of Bisection, Regula Falsi and Secant Method and Newton Raphson Method are given below:

S. No.	Method	Order
1	Bisection	1
2	Regula Falsi	1
3	Secant Method	1.62
4	Newton Raphson	2

6.4 Numerical Integration (Quadrature) by Trapezoidal and Simpson's Rules

[illegible]

6.4.1 Trapezoidal Rule

[illegible]

For the next interval $[x_1, x_2]$, we deduce similarly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}[y_1 + y_2] \qquad \text{...(ii)}$$

and so on. For the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n] \qquad \text{...(iii)}$$

Combining all these expressions, we obtain the rule

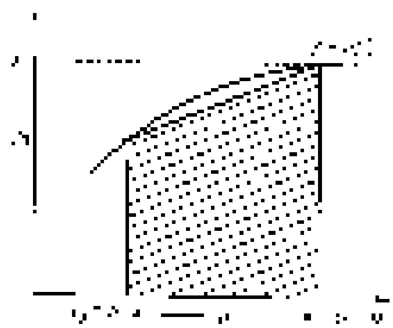
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$$

which is known as trapezoidal rule.

The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2); ...; (x_{n-1}, y_{n-1}) and (x_n, y_n).

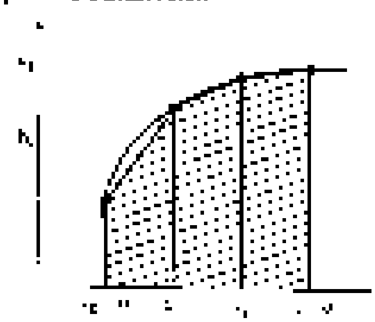
The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$ and the x-axis is then approximately equivalent to the sum of the areas of n trapeziums obtained.

Simple Trapezoidal Rule



Shaded Area = Area of Trapezium $\approx \int_a^b f(x)dx$

Compound Trapezoidal Rule (with 4 pts and 3 intervals):



Shaded Area = Sum of Areas of 3 trapeziums $\approx \int_a^b f(x)dx$

6.4.2 Simpson's Rules

6.4.2.1 Simpson's 1/3 Rule

This rule is obtained by putting $n = 2$ in general quadrature formula, i.e., by replacing the curve by $n/2$ arcs of second degree polynomials or parabolas. We have then

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly $\int_{x_2}^{x_4} y\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$

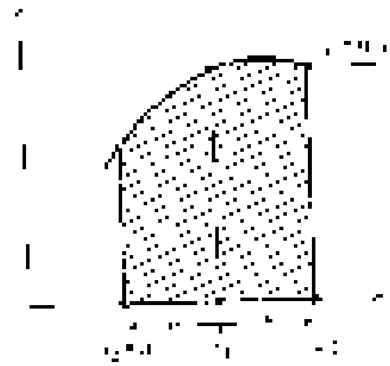
and finally $\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

Summing up, we obtain,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2}) + y_n]$$

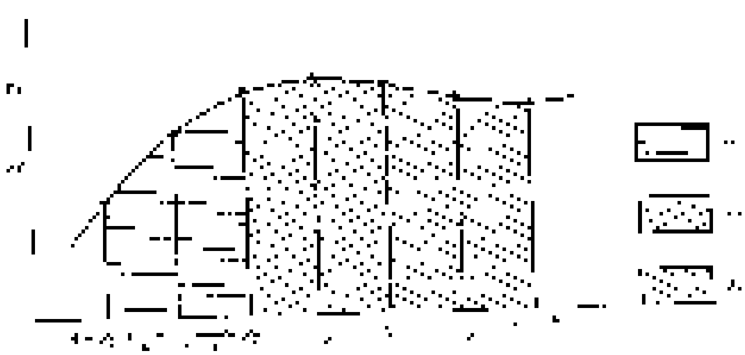
which is known as "Simpson's 1/3 rule" or simply "Simpson's rule". It should be noted that this rule requires the division of the whole range into an even number of subintervals of width h.

Simple Simpson's Rule:



$$\text{Simple } (A) = \int_a^b f(x)\,dx$$

Compound Simpson's Rule:



$$I = \int f(x)\,dx = A_1 + A_2 + A_3$$

6.4.2.2 Simpson's 3/8 Rule

Putting $n = 3$ in general formula we observe that all differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly $$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

and so on. Summing up all these, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \dots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

This is called Simpson's 3/8 rule; it requires that n be a multiple of 3 [illegible].

Example:

Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx$ correct to three decimal places using (a) Trapezoidal rule and (b) Simpson's rule, taking $h = 0.5$ and compare with the actual value.

Solution:

We solve this question by using the Trapezoidal and Simpson's rule with $h = 0.5$. The values of x and $y = \frac{1}{1+x}$ are tabulated below.

x	0	0.5	1.0
$y = \frac{1}{1+x}$	1.0000	0.6667	0.5

(a) Trapezoidal rule gives:

$$I = \frac{1}{4}\left[1.0000 + 2(0.6667) + 0.5\right] = 0.7084$$

(b) Simpson's rule gives:

$$\frac{1}{6}\left[1.0000 + 4(0.6667) + 0.5\right] = 0.6945$$

The exact value of the integral by analytical method is:

$$I = \int_0^1 \frac{1}{1+x}\,dx = \left[\log_e(1+x)\right]_0^1 = \log_e 2 = 0.6931$$

Clearly, Simpson's rule is closer to the exact value and gives better result compared to trapezoidal rule.

6.4.3 Truncation Error Formulae for Trapezoidal and Simpson's Rule

Let h be the step size used in integration.

The truncation error formula for simple trapezoidal rule with 2 points is given by

$$T_t = -\frac{h^3}{12} f''(\xi)$$

For composite trapezoidal rule with N intervals

$$T_{CTrap} = -\frac{h^3}{12} N f''(\xi)$$

The absolute truncation error bound for simple trapezoidal rule is given by

$$|T_t|_{max} = \max\left|-\frac{h^3}{12} f''(\xi)\right|$$

$$= \frac{h^3}{12}\max|f''(\xi)| \qquad \text{where } x_0 \le \xi \le x_1$$

For Composite rule with N intervals

$$|T_t|_{max} = \max\left|-\frac{h^3}{12} N f''(\xi)\right|$$

$$= \frac{h^3}{12} N \max|f''(\xi)| \qquad \text{where } x_0 \le \xi \le x_n$$

The truncation error for simple Simpson's rule with 3 points is given by

$$T_s = -\frac{h^5}{90} f^{iv}(\xi)$$

For composite Simpson's rule with N intervals, the truncation error bound is given by

$$T_{CSimp} = -\frac{h^5}{90} f^{iv}(\xi) N_s$$

where N_s is number of Simpson's intervals

Since $N_s = \frac{N}{2}$

So $T_{CSimp} = -\frac{h^5}{90}\left(\frac{N}{2}\right) f^{iv}(\xi)$

The absolute truncation error bound for simple Simpson's rule is given by

$$|T_s|_{max} = \max\left|-\frac{h^5}{90} f^{iv}(\xi)\right|$$

$$= \frac{h^5}{90}\max|f^{iv}(\xi)| \qquad \text{where } x_0 \le \xi \le x_2$$

The absolute truncation error bound for composite Simpson's rule with N intervals is given by

$$|T_{CSimp}|_{max} = \max\left|-\frac{h^5}{90}\left(\frac{N}{2}\right) f^{iv}(\xi)\right| = \frac{h^5}{90}\left(\frac{N}{2}\right)\max|f^{iv}(\xi)|$$

$$= \frac{h^5}{180} N \max|f^{iv}(\xi)| \qquad \text{where } x_0 \le \xi \le x_n$$

[illegible]

Important Note:

1. [illegible]
2. [illegible]

6.5 Numerical Solution of Ordinary Differential Equations

6.5.1 Introduction

[illegible]

[illegible]

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \quad \dots (1)$$

[illegible]

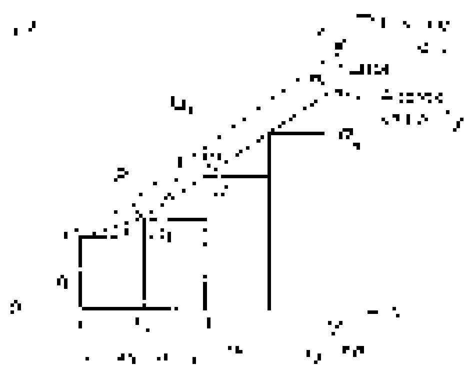
[illegible]

[illegible]

1. Euler's Method
2. Modified Euler's Method
3. Runge-Kutta Method of Fourth Order (Classical Runge-Kutta Method)

6.5.2 Euler's Method

Consider the equation, $\frac{dy}{dx} = f(x, y)$ ...(1)



[illegible]

[illegible]

[illegible]

$$= y_0 + h\left(\frac{dy}{dx}\right)_0$$

$$= y_0 + h f(x_0, y_0)$$

[illegible] the curve in the next interval by a line through $P_1(x_1, y_1)$ with slope $f(x_1, y_1)$. Then repeating this process n times, we finally reach an approximation MP_n of NQ given by

$$y_n = y_{n-1} + h f(x_0 + (n-1)h, y_{n-1})$$

In general we have

$$y_{n+1} = y_n + h f(x_n, y_n)$$

This is Euler's method of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, i.e., by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines also deviates considerably from the curve of solution. Hence there is a modification of this method which is given in the next section, called 'modified Euler's method', which is more accurate.

Example:

Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

x	y	$x + y = dy/dx$	old $y + 0.1\,(dy/dx)$ = new y
0.0	1.00	1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89) = 2.48
0.8	2.48	3.28	2.48 + 0.1 (3.28) = 2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$ when $x = 1$.

Obs. In this example, the true value of y from its exact solution at $x = 1$ is

$$y = 2e^x - x - 1$$

$$= 3.44$$

whereas by Euler's method $y = 3.18$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very slow and [illegible] can be considered as [illegible].

Example:

Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Solution:

We start with $x_0 = 0.0$, $y_0 = 1.0000$ and we have $f(x, y) = \frac{y-x}{y+x}$, $h = 0.02$. The values are calculated as shown below.

x	y	$f(x,y) = \frac{y-x}{y+x}$	$y_{n+1} = y_n + h f(x_n, y_n)$
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.9261	1.0392 + 0.02(0.9261) = 1.0577
0.06	1.0577	0.8926	1.0577 + 0.02(0.8926) = 1.0756
0.08	1.0756	0.8615	1.0756 + 0.02(0.8615) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

6.5.3 Modified Euler's Method

In Euler's method: $\qquad y_{n+1} = y_n + h f(x_n, y_n)$

In Backward Euler's method: $\qquad y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$ ...(i)

As in (i), since the unknown y_{n+1} appears on both RHS and LHS of the above equation, this method is called an implicit method. So Backward Euler's method is an implicit method, while Euler's method is explicit since y_{n+1} appears only on LHS side of the above equation.

In Backward Euler's method, we need to necessarily solve (i) for y_{n+1} before proceeding further.

Example:

Using Backward Euler's Method, find an approximate value of y corresponding to $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$, take step size $h = 0.1$.

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$
$$y_{n+1} = y_n + h(x_{n+1} + y_{n+1})$$

Solution:

Solving for y_{n+1} we get $\qquad y_{n+1} = \frac{y_n + h x_{n+1}}{1-h}$

Now the calculations are shown below:

i	x_i	y_i	Comments
0	0.0	1.00	Initial condition given
1	0.1	1.122	$y_1 = \frac{y_0 + h x_1}{1-h} = \frac{1 + 0.1 \times 0.1}{1 - 0.1} = 1.122$
2	0.2	1.2689	$y_2 = \frac{y_1 + h x_2}{1-h} = \frac{1.122 + 0.1 \times 0.2}{1 - 0.1} = 1.2689$

So the approximate value of y at $x = 0.2$ is 1.2689.

Notice that this same problem when solved with forward Euler's method, gave a slightly different answer for y which was $y = 1.22$ at $x = 0.2$.

The advantage of backward Euler's method is its stability. Backward Euler's method is more stable compared to forward Euler's method.

A method is stable if the error at any stage is bounded, independent of the number of points.

6.5.4 Runge-Kutta Method

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the order of that method. Euler's method, Modified Euler's method and Runge's method are the Runge-Kutta methods of the first, second and third order respectively.

The fourth order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta method' only.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \text{ is as follows:}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4.)

Note: One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example:

Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.2 \times f(0.2, 1.2440) = 0.2888$$

[illegible]

$$= \frac{1}{8}[1.4088 - 0.2428]$$ [illegible]

Now, $y_1 = y_0 + h$ [illegible]

$$= 1 + 0.2428 = 1.2428$$

Hence, the required approximate value of y is 1.2428

6.5.5 Stability Analysis

[illegible]

Using a general single-step method equation

$$y_{n+1} = E\,y_n \qquad \ldots(i)$$

Condition for absolute stability is

$$|E| \le 1$$

Using a test equation $y' = \lambda y$

let us find the condition for stability for Euler's method.

Euler's method equation is $y_{n+1} = y_n + hf(x_n, y_n)$

$$= y_n + h\lambda y_n$$

$$= (1 + h\lambda)\,y_n$$

Now, comparing with (i) we get

$$E = 1 + h\lambda$$

Condition for stability is $|E| \le 1$

$$|1 + h\lambda| \le 1$$

[illegible]

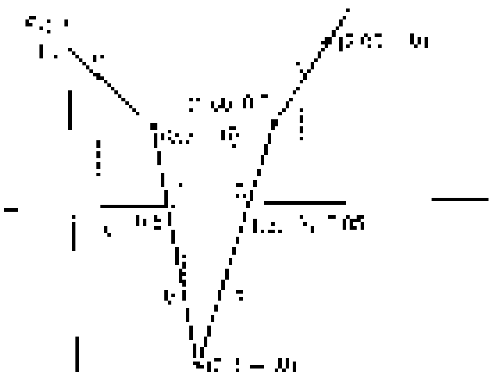
So, condition for stability is

$$-2 \le h\lambda \le 0$$

■■■■

Previous GATE and ESE Questions

Q.1 A piecewise linear function f(x) is plotted using thick solid lines in the figure below (the plot is drawn to scale).



If we use the Newton-Raphson method to find the roots of $f(x) = 0$ using x_0, x_1 and x_2 respectively as initial guesses, the roots obtained would be
(a) 1.3, 0.6 and 0.6 respectively
(b) 0.6, 0.6 and 1.3 respectively
(c) 1.3, 1.3 and 0.6 respectively
(d) 1.3, 0.6 and 1.3 respectively
[CS, GATE-2003, 2 marks]

Q.2 The accuracy of Simpson's rule quadrature for a step size h is
(a) $O(h^2)$ (b) $O(h^3)$
(c) $O(h^4)$ (d) $O(h^5)$
[ME, GATE-2003, 1 mark]

Statement for Linked Answer Questions 3 and 4
Given $a > 0$, we wish to calculate its reciprocal value $\frac{1}{a}$ by using Newton-Raphson method for $f(x) = 0$.

Q.3 The Newton Raphson algorithm for the function will be
(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$
(b) $x_{k+1} = \left(x_k + \frac{a}{2}x_k^2\right)$
(c) $x_{k+1} = 2x_k - ax_k^2$
(d) $x_{k+1} = x_k - \frac{a}{2}x_k^2$
[CE, GATE-2005, 2 marks]

Q.4 For $a = 7$ and starting with $x_0 = 0.2$, the first two iterations will be
(a) 0.11, 0.1299 (b) 0.12, 0.1392
(c) 0.12, 0.1416 (d) 0.13, 0.1428
[CE, GATE-2005, 2 marks]

Q.5 Starting from $x_0 = 1$, one step of Newton-Raphson method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value (x_1) as
(a) $x_1 = 0.5$ (b) $x_1 = 1.406$
(c) $x_1 = 1.5$ (d) $x_1 = 2$
[ME, GATE-2005, 2 marks]

Q.6 Match List-I with List-II and select the correct answer using the codes given below the lists:
List-I
A. Newton-Raphson method
B. Runge-Kutta method
C. Simpson's Rule
D. Gauss elimination
List-II
1. Solving non-linear equations
2. Solving linear simultaneous equations
3. Solving ordinary differential equations
4. Numerical integration
5. Interpolation
6. Calculation of Eigenvalues
Codes:

	A	B	C	D
(a)	6	3	4	2
(b)	1	6	4	3
(c)	1	3	4	2
(d)	5	3	4	1

[EC, GATE-2005, 2 marks]

Q.7 A 2nd degree polynomial, f(x) has values of 1, 4 and 15 at $x = 0$, 1 and 2 respectively. The integral $\int_0^2 f(x)\,dx$ is to be estimated by applying the trapezoidal rule to this data. What is the error (defined as "true value – approximate value") in the estimate?

(a) $\frac{1}{3}$ (b) $-\frac{4}{3}$

(c) 1 (d) $\frac{2}{3}$

[CE, GATE-2006, 2 marks]

Q.8 The differential equation $\frac{dy}{dx} = 0.25y^2$ is to be solved using the backward (implicit) Euler's method with the boundary condition $y = 1$ at $x = 0$ and with a step size of 1. What would be the value of y at $x = 1$?

(a) 1.33 (b) 1.67

(c) 2.00 (d) 2.33

[CE, GATE-2006, 1 mark]

Q.9 Given that one root of the equation

$$x^3 - 10x^2 + 31x - 30 = 0 \text{ is } 5,$$

the other two roots are

(a) 2 and 3 (b) 2 and 4

(c) 3 and 4 (d) −2 and −3

[CE, GATE-2007, 2 marks]

Q.10 The following equation needs to be numerically solved using the Newton-Raphson method.

$$x^3 + 4x - 9 = 0$$

The iterative equation for this purpose is (k indicates the iteration level)

(a) $x_{k+1} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$

(b) $x_{k+1} = \frac{3x_k^2 + 4}{2x_k^2 + 9}$

(c) $x_{k+1} = x_k - 3x_k^2 + 4$

(d) $x_{k+1} = \frac{4x_k^2 + 3}{9x_k^2 + 2}$

[CE, GATE-2007, 2 marks]

Q.11 The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton-Raphson method. If $x = 2$ is taken as the initial approximation of the solution, then the next approximation using this method will be

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$

(c) 1 (d) $\frac{3}{2}$

[EC, GATE-2007, 2 marks]

Q.12 Consider the series $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$, $x_0 = 0.5$ obtained from the Newton-Raphson method. The series converges to

(a) 1.5 (b) $\sqrt{2}$

(c) 1.6 (d) 1.4

[CS, GATE-2007, 2 marks]

Q.13 A calculator has accuracy up to 8 digits after decimal place. The value of $\int_0^{2\pi} \sin x \, dx$ when evaluated using this calculator by trapezoidal method with 8 equal intervals, to 5 significant digits is

(a) 0.00000 (b) 1.0000

(c) 0.00500 (d) 0.00025

[ME, GATE-2007, 2 marks]

Q.14 The differential equation $\frac{dx}{dt} = \frac{(1-x)}{\tau}$ is discretised using Euler's numerical integration method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensure stability of the solution of the corresponding discretised equation?

(a) 1 (b) $\tau/2$

(c) τ (d) 2τ

[EE, GATE-2007, 2 marks]

Q.15 Equation $e^x - 1 = 0$ is required to be solved using Newton's method with an initial guess $x_0 = -1$. Then, after one step of Newton's method, estimate x_1 of the solution will be given by

(a) 0.71828 (b) 0.36784

(c) 0.20587 (d) 0.00000

[EE, GATE-2008, 2 marks]

Q.16 The recursion relation to solve $x = e^{-x}$ using Newton-Raphson method is

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

(c) $x_{n+1} = (1 + x_n)\frac{e^{-x_n}}{1 + e^{-x_n}}$

(d) $x_{n+1} = \frac{x_n^2 - e^{-x_n}(1 + x_n) - 1}{x_n - e^{-x_n}}$

[EC, GATE-2008, 2 marks]

Q.17 The Newton-Raphson iteration $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to compute the

(a) square of R
(b) reciprocal of R
(c) square root of R
(d) logarithm of R

[CS, GATE-2008, 2 marks]

Q.18 The minimum number of equal length subintervals needed to approximate $\int_1^2 xe^x\,dx$ to an accuracy of at least $\frac{1}{3} \times 10^{-6}$ using the trapezoidal rule is

(a) 1000e
(b) 1000
(c) 100e
(d) 100

[CS, GATE-2008, 2 marks]

Q.19 Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton-Raphson's method is given by

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$
(b) $x_{k+1} = x_k - \frac{117}{x_k}$
(c) $x_{k+1} = x_k - \frac{x_k}{117}$
(d) $x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

[EE, GATE-2009, 2 marks]

Q.20 Newton-Raphson method is used to compute a root of the equation $x^2 - 13 = 0$ with 3.5 as the initial value. The approximation after one iteration is

(a) 3.575
(b) 3.677
(c) 3.607
(d) 3.667

[CE, GATE-2010, 1 mark]

Q.21 The table below gives values of a function $F(x)$ obtained for values of x at intervals of 0.25.

x	0	0.25	0.5	0.75	1.0
$F(x)$	1	0.9412	0.8	0.64	0.50

The value of the integral of the function between the limits 0 to 1 using Simpson's rule is

(a) 0.7854
(b) 2.3562
(c) 3.1416
(d) 7.5000

[CE, GATE-2010, 2 marks]

Q.22 Torque exerted on a flywheel over a cycle is listed in the table. Flywheel energy (in J per unit cycle) using Simpson's rule is

Angle (degree)	0	60°	120°	180°	240°	300°	360°
Torque (N-m)	0	1066	−323	0	323	−355	0

(a) 542
(b) 993
(c) 1444
(d) 1986

[ME, GATE-2010, 2 marks]

Q.23 Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial condition $y(0) = 0$. Using Euler's first order method with a step size of 0.1, the value of $y(0.3)$ is

(a) 0.01
(b) 0.031
(c) 0.0631
(d) 0.1

[EC, GATE-2010, 2 marks]

Q.24 The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into a product of a lower triangular matrix [L] and an upper triangular matrix [U]. The properly decomposed [L] and [U] matrices respectively are

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$
(b) $\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$
(d) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

[EE, GATE-2011, 2 marks]

Q.25 The cube root of a number N is to be computed by applying the Newton-Raphson procedure to the equation $x^3 - N = 0$. The correct iteration formula is

(a) $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$
(b) $x_{n+1} = \frac{1}{3}\left(2x_n - \frac{N}{x_n^2}\right)$
(c) $x_{n+1} = \frac{1}{3}\left(x_n - \frac{N}{x_n^2}\right)$
(d) $x_{n+1} = \frac{1}{3}\left(x_n + \frac{N}{x_n^2}\right)$

[CS, GATE-2011, 2 marks]

Q.26 Roots of the algebraic equation $x^3 + x^2 + x + 1 = 0$ are
(a) $(+1, +j, -j)$ (b) $(+1, -1, +1)$
(c) $(0, 0, 0)$ (d) $(-1, +j, -j)$
[EE, GATE-2011, 1 marks]

Q.27 Solution of the variables x_1 and x_2 for the following equations is to be obtained by employing the Newton-Raphson iterative method.
equation (i) $10x_2 \sin x_1 - 0.8 = 0$
equation (ii) $10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$
Assuming the initial values $x_1 = 0.0$ and $x_2 = 1.0$, the jacobian matrix is

(a) $\begin{bmatrix} 10 & -0.8 \\ 0 & -0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -0.8 \\ 10 & -0.6 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 0 \\ 10 & -10 \end{bmatrix}$
[EE, GATE-2011, 2 marks]

Q.28 A numerical solution of the equation $f(x) = x + \sqrt{x} - 3 = 0$ can be obtained using Newton-Raphson method. If the starting value is $x = 2$ for the iteration, the value of x that is to be used in the next step is
(a) 0.306 (b) 0.739
(c) 1.694 (d) 2.306
[EC, GATE-2011, 2 marks]

Q.29 The integral $\int_1^3 \frac{1}{x} dx$, when evaluated by using Simpson's 1/3 rule on two equal subintervals each of length 1, equals
(a) 1.000 (b) 1.098
(c) 1.111 (d) 1.120
[ME, GATE-2011, 2 marks]

Q.30 The bisection method is applied to compute a zero of the function $f(x) = x^4 - x^3 - x^2 - 4$ in the interval [1, 9]. The method converges to a solution after ___ iterations.
(a) 1 (b) 3
(c) 5 (d) 7
[CS, GATE-2012, 2 marks]

Q.31 The estimate of $\int_{0.5}^{1.5} \frac{dx}{x}$ obtained using Simpson's rule with three-point function evaluation exceeds the exact value by
(a) 0.235 (b) 0.068
(c) 0.024 (d) 0.012
[CE, GATE-2012, 1 mark]

Q.32 The error in $\frac{d}{dx} f(x)\Big|_{x=x_0}$ for a continuous function estimated with $h = 0.03$ using the central difference formula $\frac{d}{dx} f(x)\Big|_{x=x_0} \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$, is 2×10^{-3}. The values of x_0 and $f(x_0)$ are 19.78 and 500.01, respectively. The corresponding error in the central difference estimate for $h = 0.02$ is approximately
(a) 1.3×10^{-4} (b) 3.0×10^{-4}
(c) 4.5×10^{-4} (d) 9.0×10^{-4}
[CE, GATE-2013, 2 marks]

Q.33 When the Newton-Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess value as $x_0 = 1.2$ is
(a) −0.82 (b) 0.49
(c) 0.705 (d) 1.69
[EE, GATE-2013, 2 Marks]

Q.34 The magnitude of the error (correct to two decimal places) in the estimation of following integral using Simpson 1/3 rule. Take the step length as 1.
$$\int_0^4 (x^4 + 10)\, dx$$
[CE, GATE-2013, 2 Mark]

Q.35 Match the correct pairs.

Numerical Integration Scheme	Order of Fitting Polynomial
P. Simpson's 3/8 Rule	1. First
Q. Trapezoidal Rule	2. Second
R. Simpson's 1/3 Rule	3. Third

(a) P-2, Q-1, R-3 (b) P-3, Q-2, R-1
(c) P-1, Q-2, R-3 (d) P-3, Q-1, R-2
[ME, GATE-2013, 1 Mark]

Q.36 While numerically solving the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$ using Euler's predictor-corrector [illegible] method with a step size of [illegible], the value of y after the first step is [illegible]

(a) 1.00 (b) 1.03

(c) 0.97 (d) 0.96

[IN, GATE-2013 : 2 Marks]

Q.37 Match the application to appropriate numerical method.

Application

P1: Numerical integration

P2: Solution to a transcendental equation

P3: Solution to a system of linear equations

P4: Solution to a differential equation

M1: Newton Raphson Method

M2: Runge Kutta Method

M3: Simpson's 1/3 rule

M4: Gauss Elimination Method

(a) P1—M3, P2—M2, P3—M4, P4—M1

(b) P1—M3, P2—M1, P3—M4, P4—M2

(c) P1—M4, P2—M1, P3—M3, P4—M2

(d) P1—M2, P2—M1, P3—M3, P4—M4

[EC, GATE-2014 : 1 Mark]

Q.38 The real root of the equation [illegible] (up to two decimal accuracy) is [illegible]

[ME, GATE-2014 : 2 Marks]

Q.39 The function [illegible] is to be solved using Newton-Raphson method. If the initial value of x_0 is taken as 1.0, then the absolute error observed at 2nd iteration is ______ .

[EE, GATE-2014 : 2 Marks]

Q.40 In the Newton-Raphson method, an initial guess of $x_0 = 2$ is made and the sequence $x_0, x_1, x_2, \ldots$ is obtained for the function

[illegible]

Consider the statements

(I) [illegible]

(II) The method converges to a solution in a finite number of iterations.

Which of the following is TRUE?

(a) Only I (b) Only II

(c) Both I and II (d) Neither I nor II

[CS, GATE-2014 (Set-2) : 2 Marks]

Q.41 The value of $\int [illegible] \, dx$ calculated using the trapezoidal rule with [illegible] is ______ .

[ME, GATE-2014 : 2 Marks]

Q.42 The value of the integral $\int \frac{[illegible]}{x} dx$ is evaluated [illegible] by trapezoidal rule with [illegible]. The correct answer is ______ .

[ME, GATE-2014 : 1 Mark]

Q.43 Using the trapezoidal rule, and dividing the interval of integration into three equal intervals, the definite integral $\int |x| \, dx$ is ______ .

[ME, GATE-2014 : 2 Marks]

Q.44 With respect to the numerical evaluation of the definite integral, $K = \int_a^b x^2 dx$, where a and b are given, which of the following statements is/are TRUE?

(I) The value of K obtained using the trapezoidal rule is always greater than or equal to the exact value of the definite integral.

(II) The value of K obtained using the Simpson's rule is always equal to the exact value of the definite integral.

(a) I only (b) II only

(c) Both I and II (d) Neither I nor II

[CS, GATE-2014 : 2 Marks]

Q.45 Consider an ordinary differential equation $\frac{dx}{dt} = [illegible]$, $x = x_0$ at $t = 0$. The increment in x calculated using Runge-Kutta fourth order method with a step size of $\Delta t = [illegible]$ is

(a) 0.22 (b) 0.46

(c) 0.99 (d) 0.98

[ME, GATE-2014 : 2 Marks]

Q.46 In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is ______ .

[CS, GATE-2015 : 1 Mark]

Q.47 [illegible] which one of the following statements is TRUE?

[illegible]

[EE, GATE-2015 : 1 Mark]

Q.48 The quadratic equation [illegible] is to be solved numerically starting with the initial guess x_0 = [illegible]. The Newton-Raphson method is applied [illegible] method is applied [illegible] is ____

[CE, GATE-2015 : 2 Marks]

Q.49 In Newton-Raphson iterative method, the initial guess value (x_0) [illegible] finding the roots of the equation: [illegible] The corresponding [illegible] is ____

[CE, GATE-2015 : 1 Mark]

Q.50 Newton-Raphson method is used to find the roots of [illegible]. If the initial guess is x_0 = [illegible], then the value of x after 2nd iteration is ____

[ME, GATE-2015 : 2 Marks]

Q.51 [illegible] Newton-Raphson method [illegible] taking [illegible] is ____

[EC, GATE-2015 : 2 Marks]

Q.52 The secant method [illegible] equation [illegible] estimates x_0 and x_1 for the root [illegible]. The pseudocode is given below. Observe that [illegible] which is [illegible] of the secant method.

Secant
Initialize: x_0, x_1, q_0
ϵ = convergence indicator
$q_0 = f(x_0)$
[illegible] maximum number of iterations
$i = 0$
while [illegible] do
$i = i + 1$ [illegible]
[illegible]
[illegible]
$q_0 = f(x_0)$ [illegible]
end while
[illegible]
(A) $x_0 = x_0 + \dots$ [illegible]
(B) $x_0 = \dots$ [illegible]
(C) [illegible]
(D) [illegible]

[CS, GATE-2015 : 2 Marks]

Q.53 The integral [illegible] is to be evaluated [illegible]

(a) [illegible]
(b) [illegible]
(c) [illegible]
(d) [illegible]

[EE, GATE-2015 : 1 Mark]

Q.54 [illegible] using Simpson's 1/3 rule is ____

[illegible]

[CE, GATE-2015 : 2 Marks]

Q.55 [illegible] ____

[ME, GATE-2015 : 1 Mark]

Q.56 [illegible]

[illegible]

[ME, GATE-2015 : 1 Mark]

Q.57 [illegible]

[illegible]

[ME, GATE-2015 : 2 Marks]

Q.58 [illegible]

[illegible]

[CS, GATE 2015 : 2 Marks]

Q.59 Gauss-Seidel method is used to solve the [illegible]

[illegible]

[ME 2016 : 2 Marks]

Q.60 [illegible] Newton Raphson method [illegible]

[illegible]

[ME, 2016 : 1 Mark]

Q.61 [illegible] Newton Raphson [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) 1.000

[ME, 2016 : 1 Mark]

Q.62 Newton-Raphson method is [illegible] to find root of equation [illegible]

[CE, 2016 : 1 Mark]

Q.63 [illegible]

is

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, 2016 : 1 Mark]

Q.64 The error in numerically computing the integral [illegible]

[ME 2015 : 2 Marks]

Q.65 [illegible]

[illegible]

[EC 2015 : 2 Marks]

Q.66 [illegible]

[EC, 2016 : 1 Mark]

Q.67 [illegible]

(c) [illegible] (d) [illegible]

(b) [illegible] (d) [illegible]

[ME, GATE-2017 : 2 Marks]

Q.69 [illegible]

[illegible table]

(a) 1 (b) [illegible]

(c) [illegible] (d) [illegible]

[IN, GATE-2017 : 2 Marks]

Q.68 [illegible] (Give the answer up to [illegible])

[EE, GATE-2017 : 2 Marks]

Q.70 [illegible] ______

[EC, GATE-2017 : 2 Marks]

Q.71 [illegible] ______

[CE, GATE-2017 : 2 Marks]

Answers Numerical Methods

[illegible answer key: entries 1.–68.; legible only 30. (b) 31. (d) 32. (d) 33. (c)]

Explanations Numerical Methods

1. (c)

[illegible]

3. (a)

[illegible]

4. (b)

[illegible]

6. (a)

From Newton-Raphson [illegible]

[illegible]

7. (c)

[illegible]

$$f(x) = a_0 + a_1x + a_2x^2$$
$$f(0) = 1$$
$$\Rightarrow \quad a_0 + 0 + 0 = 1$$
$$\Rightarrow \quad a_0 = 1$$
$$f(1) = 4$$
$$\Rightarrow \quad a_0 + a_1 + a_2 = 4$$
$$\Rightarrow \quad 1 + a_1 + a_2 = 4$$
$$\Rightarrow \quad a_1 + a_2 = 3 \quad \ldots(i)$$
$$f(2) = 15$$
$$\Rightarrow \quad a_0 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow \quad 1 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow \quad 2a_1 + 4a_2 = 14 \quad \ldots(ii)$$

Solving (i) and (ii)

$$a_1 = -1 \text{ and } a_2 = 4$$
$$\therefore \quad f(x) = 1 - x + 4x^2$$

Now exact value

$$\int_0^2 f(x)dx = \int_0^2 (1 - x + 4x^2)dx$$
$$= \left[x - \frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$$

Error = Exact − Approximate value

$$= \frac{32}{3} - 12 = -\frac{4}{3}$$

8. (c)

$$\frac{dy}{dx} = 0.25y^2 \qquad (y = 1 \text{ at } x = 0)$$
$$h = 1$$

Iterative equation for backward (implicit) Euler method for above equation will be

$$y_{k+1} = y_k + h f(x_{k+1}, y_{k+1})$$
$$y_{k+1} = y_k + h \times 0.25\, y_{k+1}^2$$
$$\Rightarrow 0.25h\, y_{k+1}^2 - y_{k+1} + y_k = 0$$

Putting $h = 1$ in the above equation

$0.25\, y_1^2 - y_1 + y_0 = 0$

Here $y_0 = 1$ and we get

$0.25\, y_1^2 - y_1 + 1 = 0$

$$\Rightarrow \quad y_1 = \frac{1 \pm \sqrt{0}}{2 \times 0.25} = 2$$
$$\Rightarrow \quad y_1 = 2$$

9. (a)

Since $x = 5$ is a root, $f(x)$ is divisible by $x - 5$. Now dividing $f(x)$ by $x - 5$ we get,

$$x - 5 \overline{)\, x^3 - 10x^2 + 31x - 30\,} \left(x^2 - 5x + 6 \right.$$
$$\underline{x^3 - 5x^2}$$
$$-5x^2 + 31x - 30$$
$$\underline{-5x^2 + 25x}$$
$$6x - 30$$
$$\underline{6x - 30}$$
$$0$$

$$\therefore \quad x^3 - 10x^2 + 31x - 30 = (x - 5)(x^2 - 5x + 6)$$

Solving $x^2 - 5x + 6 = 0$, $x = 2, 3$

∴ The other two roots are 2 and 3.

10. (a)

$$f(x) = x^3 + 4x - 9 = 0$$
$$f'(x) = 3x^2 + 4$$

Newton-Raphson iteration is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
$$f(x_k) = x_k^3 + 4x_k - 9$$
$$f'(x_k) = 3x_k^2 + 4$$
$$x_{k+1} = x_k - \frac{x_k^3 + 4x_k - 9}{3x_k^2 + 4}$$
$$= \frac{3x_k^3 + 4x_k - x_k^3 - 4x_k + 9}{3x_k^2 + 4}$$
$$x_{k+1} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

11. (b)

Here $\quad x_0 = 2$

$$f(x) = x^3 - x^2 + 4x - 3$$
$$f'(x) = 3x^2 - 2x + 4$$
$$f(x_0) = f(2) = 9$$
$$f'(x_0) = f'(2) = 12$$
$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{9}{12} = \frac{5}{4}$$

12. (a)

Given, $x_{k+1} = \dfrac{x_k}{2} + \dfrac{9}{8x_k}$, $x_0 = 0.5$

As $k \to \infty$ then the sequence converges

$x_{k+1} = x_k = \alpha$ = root of equation

$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha}$$
$$\alpha = \frac{4\alpha^2 + 9}{8\alpha}$$

$\Rightarrow \quad \theta = \frac{9}{4}$

$\lambda = \frac{3}{2} = 1.5$

13. (c)

$h = \frac{b-a}{n} = \frac{\pi}{4}$

$y_0 = \sin(0) = 0$

1	$\frac{\pi}{4}$	0.7071
2	$\frac{\pi}{2}$	1
3	$\frac{3\pi}{4}$	0.7071
4	π	0
5	$\frac{5\pi}{4}$	0.7071
6	$\frac{3\pi}{2}$	1
7	$\frac{7\pi}{4}$	0.7071
8	2π	0

$y_1 = \sin\left(\frac{\pi}{4}\right) = 0.7071$

$y_2 = \sin\left(\frac{\pi}{2}\right) = 1$

$y_3 = \sin\left(\frac{3\pi}{4}\right) = 0.7071$

$y_4 = \sin(\pi) = 0$

$y_5 = \sin\left(\frac{5\pi}{4}\right) = -0.7071$

$y_6 = \sin\left(\frac{3\pi}{2}\right) = -1$

$y_7 = \sin\left(\frac{7\pi}{4}\right) = -0.7071$

$y_8 = \sin\left(\frac{8\pi}{4}\right) = 0$

Trapezoidal rule

$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$

$= 0.00000$

14. (c)

Here, $\frac{dy}{dx} = \frac{1-y}{x}$

Euler's Method formula is

$y_{n+1} = y_n + h f(x_n, y_n)$

15. (a)

Here $f(x) = e^x - 1$

The Newton-Raphson formula is

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(x) = e^x - 1$

$f'(x) = e^x$

$x_1 = 0.71828$

16. (c)

The given equation can be solved as

Which can be rewritten as

$f(x) = x - e^{-x} = 0$

$f'(x) = 1 + e^{-x}$

The Newton-Raphson iterative formula is

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\therefore$ The Newton-Raphson iterative formula is

$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$

17. (c)

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{[illegible]}{x_n}\right]$$

[illegible]

$$x_{n+1} = [illegible]$$

$$= \frac{1}{2}[illegible]$$

$$2\alpha = \alpha + \frac{7}{\alpha} = \frac{\alpha^2 + 7}{\alpha}$$

$$2\alpha^2 = \alpha^2 + 7$$

$$\alpha^2 = 7$$

$$\alpha = \sqrt{7}$$

So the iterative scheme converges to [illegible]

Correct option is (c).

18. (a)

Hence the function being integrated is

$$f(x) = xe^x$$

$$f'(x) = xe^x + e^x = e^x(x+1)$$

$$f''(x) = xe^x + e^x + e^x = e^x(x+2)$$

Since [illegible]

maximum value of $f''(\xi)$ in interval $1 \le \xi \le 2$ occurs at $\xi = 2$

So

$$\max|f''(\xi)| = e^2(2+2) = 4e^2$$

In trapezoidal rule [illegible]

$$\frac{h^2}{12}(b-a)\max f''(\xi)$$

Where N is number of sub intervals

$$h = \frac{b-a}{N}$$

$$\therefore [illegible] = \frac{h^2}{12}(b-a)\max|f''(\xi)|$$

$$= \frac{h^2}{12}(2-1)\max f''(\xi),\ 1 \le \xi \le 2$$

$$\frac{h^2}{12}(2-1)(4e^2) = \frac{h^2}{3}e^2$$

Now putting

$$\text{Error} \le \frac{1}{2}\times 10^{-4}$$

We get

$$\frac{h^2}{3}e^2 \le \frac{1}{2}\times 10^{-4}$$

$$\Rightarrow \qquad h^2 \le \frac{[illegible]\,10^{-4}}{e^2}$$

$$\Rightarrow \qquad h \le [illegible]$$

Now

Number of intervals $= N$

$$= \frac{b-a}{h} = \frac{2-1}{[illegible]} \approx [illegible]$$

19. (a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 117}{2x_n}$$

$$= \frac{1}{2}\left[x_n + \frac{117}{x_n}\right]$$

20. (d)

The equation is $f(x) = x^2 - 13 = 0$

Newton Raphson iteration equation is

$$x_{k+1} = x_k - \left[\frac{f(x_k)}{f'(x_k)}\right]$$

$$f(x_k) = x_k^2 - 13$$

$$f'(x_k) = 2x_k$$

$$\therefore \qquad x_{k+1} = x_k - \left[\frac{x_k^2 - 13}{2x_k}\right] = \frac{x_k^2 + 13}{2x_k}$$

For $x_0 = 3.5$ (begin with)

$$x_1 = \frac{3.5^2 + 13}{2\times 3.5} = 3.607$$

∴ The approximate value after one iteration is $x_1 = 3.607$

21. (c)

$$I = \frac{1}{3}[illegible]$$

$$= \frac{1}{3}[illegible]$$

$$[illegible]$$

$$= [illegible]$$

22. (b)

Power delivered by $= \int_0^{\pi} T(\theta)\,d\theta$, where $T(\theta)$ is torque exerted

The integral by using Simpson's rule is

$$I = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

$$h = 60 \text{ degrees} = \frac{\pi}{3} \text{ radians}$$

$$I = \frac{\pi}{3}\times\frac{1}{3}\times[illegible]$$

$$= [illegible]$$

$$= [illegible]$$

23. (c)

[illegible]

step size = h = 0.1

Improved Euler's formula is

[illegible]

Now [illegible]

Now [illegible]

Now [illegible]

Corresponding answer is option (c)

24. (d)

[illegible]

25. (a)

[illegible]

26. (c)

[illegible]

27. (c)

[illegible]

28. (c)

[illegible]

$x = 2, \quad f(x_0) = 2 \ldots$

$f'(x) = 1 + \frac{1}{2\sqrt{x}}$

$f'(2) = 1 + \frac{1}{2\sqrt{2}}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\sqrt{2} - 1}{1 + \frac{1}{2\sqrt{2}}}$

$= 1.634$

29. (c)

$I = \int \ldots = 4h \ldots$

30. (b)

If bisection method is applied ... $a = 1$ and $b = 9$

After 1 iteration $x_2 = \frac{1+9}{2} = 5$

Now $a = 1$ and $b = 5$

and ... $x_3 = \frac{1+5}{2} = 3$

... and $x_4 = 2$ ... iteration

$x_5 = \frac{1+3}{2} = 2$

31. (d)

$= 1 \ldots$

Approximate value ... plain ...

Here ... $= \ldots$

The table is ...

Approximate value $= 1 \ldots 11 - 1.0986$

$= 0.012487 \ldots 012$

32. (d)

...

$2 \times 10^{-} \ldots \frac{\ldots}{(0.003)^4} \ldots 24$

33. (c)

$f(x) = 3x - 2$

$f(x_0) = \ldots$

$f'(x_0) = \ldots$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \ldots$

$= 0.\ldots$

34. Sol.

Using Simpson's ... rule, the estimated value of the integral $\int_0^1 \ldots$

[illegible]

The exact value of integral

$\int [illegible] dx = \left[[illegible] \right]$

$= [illegible]$

∴ Magnitude of error = |exact value − [illegible] value|

$= [illegible]$

36. (a)

$\frac{dy}{dx} + 2xy^2 = 0$

$\frac{dy}{dx} = -2xy^2$

[illegible]

$y_1 = [illegible]$

$= [illegible]$

$y_2 = [illegible]$

$= [illegible]$

$= [illegible]$

37. Sol.

$f(x) = [illegible]$

$f'(x) = [illegible]$

By Newton Raphson's formula

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

[illegible]

$= [illegible]$

Again

$x_2 = [illegible]$

$= [illegible]$

$x_3 = [illegible]$

$= [illegible]$

∴ The root is $= [illegible]$

38. Sol.

Given $f(x) = [illegible]$

$f'(x) = [illegible]$

Using Newton Raphson method we have

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = \left[x_0 - \frac{f(x_0)}{f'(x_0)} \right]$ ...(i)

Now, $f(x_0) = [illegible]$

and $f'(x_0) = [illegible]$

Putting the values in eq. (i)

$x_1 = [illegible]$

$= [illegible]$

Also $x_2 = \left[x_1 - \frac{f(x_1)}{f'(x_1)} \right]$ ...(ii)

Now $f(x_1) = [illegible]$

$f'(x_1) = [illegible]$

Using the values we get

$x_2 = \left[[illegible] \right]$

$= [illegible]$

Here [illegible]

[illegible]

Absolute error [illegible]

$= \left| \frac{\text{Exact value} - \text{Approximate value}}{\text{Exact value}} \right|$

$= [illegible]$

$= [illegible]$

40. Sol.

[illegible] using Newton Raphson iteration

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

[illegible]

$\Rightarrow$ $x_0 = 2$, $x_1 = 0$, $x_2 = $ [illegible]

[illegible] converge.

41. Sol.

[illegible]

$I = \int$ [illegible]

$= \frac{h}{2}$ [illegible]

[illegible]

42. Sol.

[illegible]

$= \frac{1}{2}$ [illegible]

43. Sol.

[illegible]

44. Sol.

While evaluating [illegible] $x^4 dx$

Error = Exact value − Approximate value

For trapezoidal rule

Error = [illegible]

[illegible]

Similarly for Simpson's rule

Error = [illegible]

[illegible]

Therefore both (i) and (ii) are correct.

45. (a)

[illegible]

$k_1 = $ [illegible]

$= 0.25(0.1)\times 0.2 = 0.05$ [illegible]

[illegible]

[illegible]

46. (5)

[illegible]

$\Rightarrow$ $u_{33} = 5$

47. (c)

[illegible]

48. Sol.

[illegible]

By Newton-Raphson method

[illegible]

49. Sol.

[illegible]

By Newton Raphson Method

[illegible]

50. Sol.

[illegible]

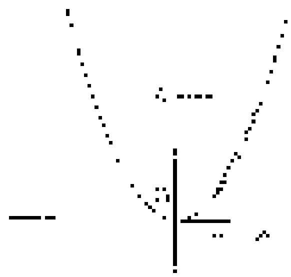
51. Sol.

[illegible]

52. (a & c)

[illegible]

53. (a)



Here

54. Sol.

By Simpson's 1/3 Rule

$= 1.357$

55. Sol.

$= 0.693$

56. Sol.

57. Sol.

$= 32$

58. Sol.

[illegible]	[illegible]	4	[illegible]	8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	10	[illegible]	[illegible]	[illegible]	[illegible]	20	[illegible]	[illegible]	[illegible]	[illegible]	2	

59. Sol.

The equations are

[illegible] ...(1)

[illegible] ...(2)

[illegible] ...(3)

By rearranging the equations [illegible]

[illegible]

$$x_1 = \frac{[illegible]}{5} \quad ...(1)$$

$$x_2 = \frac{[illegible]}{5} \quad ...(2)$$

$$x_3 = \frac{[illegible]}{7} \quad ...(3)$$

[illegible]

[illegible] 1.555

$x_3 = [illegible]$

60. Sol.

[illegible]

$= [illegible] = 5.07$

[illegible] $= 1.0828$

61. (c)

[illegible]

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$= 1 - [illegible] = 0.750$

62. Sol.

According to Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

[illegible]

$$\Rightarrow \quad x_1 = x_0 - \frac{f(0.333)}{f'(0.333)}$$

$= 0.333 - [illegible]$

$x_1 = 0.35$

63. (a)

Trapezoidal rule gives the best result in single variable function [illegible] the function is linear (degree 1)

64. Sol.

[illegible]

By Trapezoidal

[illegible]

$= [illegible] = 1.7912$

[illegible]

Error = Exact value − Approx value

$= 2 - 1.79[illegible] = 0.1[illegible]$

65. Sol.

$$\frac{dy}{dx} = -2y + 2; \quad y(0) = 1$$

[illegible] Euler's [illegible] solution

[illegible]

[illegible]

66. Sol.

[illegible]

Exact value

[illegible]

67. (a)

[illegible]

By Trapezoidal rule

[illegible]

By Simpson's rule

[illegible]

The difference between the two results will be [illegible]

68. (d)

[illegible]

69. Sol.

[illegible]

$\therefore$ $n = 13$

70. Sol.

[illegible]

By Newton Raphson method

[illegible]

71. Sol.

$$\frac{dy}{dx} = 3x^2 + 1$$

$f(x, y) = 3x^2 + 1$

$x_0 = 1$

$y_0 = 2$

$h = 2$ [illegible]

By Euler's method

$y_1 = y_0 + h f(x_0, y_0)$ [illegible]

$= y_0 + h[3x_0^2 + 1]$ [illegible]

$= 2 + 2[3(1)^2 + 1]$ [illegible]

$= 2$ [illegible]

Absolute error [illegible]

$$\frac{dy}{dx} = 3x^2 + 1$$

$$dy = (3x^2 + 1)dx$$

$$\int dy = \int_1^2 (3x^2 + 1)dx$$

$$y = [x^3 + x]_1^2$$ [illegible]

$= 8 + 2 = 10$ [illegible]

Absolute error = Exact value − Approx value

$= 10 - 2$ [illegible]

$= 8$

■■■■

7 Transform Theory

7.1 Laplace Transform

[illegible]

1st step: [illegible]

2nd step: [illegible]

3rd step: [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

7.2 Definition

[illegible]

$$[illegible] \tag{1}$$

[illegible]

Example

[illegible]

7.3 Transforms of Elementary Functions

[illegible]

1. $L(1) = \frac{1}{s}$ $(s > 0)$

2. $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3, ...$ [illegible]

3. $L(e^{at}) = \frac{1}{s-a}$ $(s > a)$

4. $L(\sin at) = \frac{a}{s^2 + a^2}$ $(s > 0)$

5. $L(\cos at) = \frac{s}{s^2 + a^2}$ $(s > 0)$

6. $L(\sinh at) = \frac{a}{s^2 - a^2}$ $(s > |a|)$

7. $L(\cosh at) = \frac{s}{s^2 - a^2}$ $(s > |a|)$

7.4 Properties of Laplace Transforms

7.4.1 Linearity Property

[illegible]

$$L[af(t) + bg(t) - ch(t)] = aL[f(t)] + bL[g(t)] - cL[h(t)]$$

7.4.2 First Shifting Property

If $L[f(t)] = \bar{f}(s)$, then

$$L[e^{at}f(t)] = \bar{f}(s-a)$$

[illegible]

1. $L(e^{at}) = \frac{1}{s-a}$ [illegible]

2. $L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$ [illegible] [illegible]

3. $L(e^{at}\sin bt) = \frac{b}{(s-a)^2 + b^2}$ [illegible]

4. $L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2 + b^2}$ [illegible]

5. $L(e^{at}\sinh bt) = \frac{b}{(s-a)^2 - b^2}$ [illegible]

6. $L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$ [illegible]

[illegible]

7.4.3 Change of Scale Property

[illegible]

Proof: [illegible]

[illegible]

7.4.4 Existence Conditions

[illegible]

[illegible]

[illegible]

[illegible]

7.4.5 Transforms of Derivatives

1. [illegible]

[illegible]

2. [illegible]

[illegible]

7.4.5.1 Differential Equations. Initial Value Problems

[illegible]

[illegible] ...(i)

[illegible]

1st Step: [illegible]

[illegible]

2nd Step: We solve the subsidiary equation algebraically for Y. Division by $s^2 + as + b$ and use of the so-called transfer function

$$Q(s) = \frac{1}{s^2 + as + b}$$

gives the solution

$$Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s) \qquad (4)$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence Q is the quotient

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

and this explains the name of Q. Note that Q depends neither on $r(t)$ nor on the initial conditions (but only on a and b).

3rd Step. We reduce (4) (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ of (1).

Example 1.

Initial problem: Explanation of the basic steps

Solve $y'' - y = t$, $y(0) = 1$, $y'(0) = 1$

Solution:

1st Step.

By Laplace transformation of the given equation we get the subsidiary equation

$$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2} \qquad \text{thus } (s^2 - 1)Y = s + 1 + 1/s^2$$

where $Y = \mathcal{L}(y)$

2nd Step.

The transfer function is $Q = 1/(s^2 - 1)$, and

$$Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

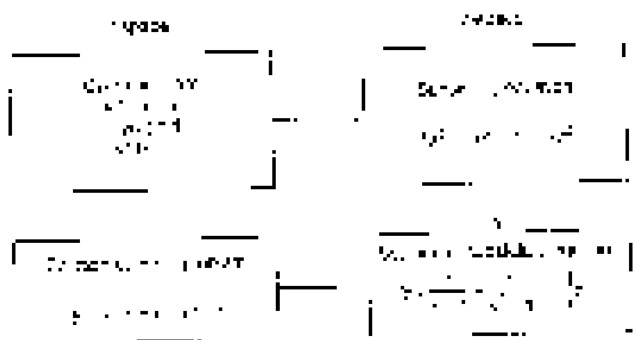
3rd Step

From this expression for Y we obtain the solution by Laplace method, as shown below

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t$$

$$y = e^t + \frac{e^t - e^{-t}}{2} - t = \frac{3e^t - e^{-t}}{2} - t$$

The diagram in Fig. below summarizes our approach.



Laplace transform method

Comparison with the usual method

[illegible]

Note: [illegible]

7.4.6 Transforms of Integrals

[illegible]

7.4.7 Multiplication By t

[illegible]

7.4.8 Division By t

[illegible]

7.5 Evaluation of Integrals by Laplace Transforms

Example:

[illegible]

Solution:

(a) [illegible]

$$\ldots = \frac{4}{25}$$

(b) Since [illegible]

$\therefore$ [illegible]

Now since [illegible]

$\therefore$ [illegible]

Now [illegible]

Thus [illegible]

[illegible]

(c) Since [illegible]

$\therefore$ [illegible] property

Thus [illegible]

Example:

[illegible]

Solution:

[illegible]

[illegible]

(**Note:** [illegible])

[illegible]

7.6 Inverse Transforms – Method of Partial Fractions

[illegible]

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]
5. [illegible]
6. [illegible]
7. [illegible]
8. [illegible]
9. [illegible]
10. [illegible]
11. [illegible]
12. [illegible]

[illegible]

Note on Partial Fractions: [illegible]

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]

[illegible]

7.7 Unit Step Function

[illegible]

Def. [illegible]

$$[illegible]$$

[illegible]

7.7.1 Transform of Unit Function

$$[illegible]$$

Thus [illegible]

[illegible]

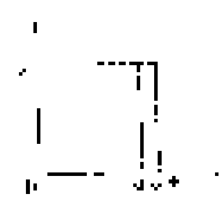
[illegible]

7.8 Second Shifting Property

$$[illegible]$$

Proof. [illegible]

7.9 Unit Impulse Function



[illegible]

$$[illegible]$$

otherwise

[illegible]

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$f(x) = w_0/\varepsilon \qquad a < x < a + \varepsilon$$
$$= 0 \qquad \text{Otherwise}$$

i.e. $$f(x) = w_0 \delta(x - a)$$

7.9.1 Transform of Unit Impulse Function

If $\delta(t-a)$ be the function of unit impulse at $t = a$ then

$$\int_0^\infty e^{-st}\delta_\varepsilon(t-a)\,dt = \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt = e^{-s(a+\theta\varepsilon)}, \text{ where } 0 < \theta < 1$$

by Mean value theorem of integrals.

As $\varepsilon \to 0$ we get $\int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$

In particular, putting $a = 0$ in the above integral

we have $$\int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$$

Now LHS is nothing but $L\{\delta(t-a)\}$

$$L\{\delta(t-a)\} = e^{-as}$$

7.10 Periodic functions

If $f(t)$ is a periodic function with period T i.e. $f(t + T) = f(t)$ then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

Example:

$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$ is a periodic function with time period $2\pi/\omega$. Determine the Laplace transform of $f(t)$.

Solution:

Laplace transform of periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-\pi s/\omega}}{s^2+\omega^2}\,\omega + \frac{\omega}{s^2+\omega^2}\right] = \frac{\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})} = \frac{\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

7.11 Fourier Transform

Fourier series is the approximation process of a very general periodic function as a discrete and infinite sum of harmonically related sinusoids. It is used in the frequency domain representation.

If the signal is periodic, the Fourier series represents the signal in the entire interval $(-\infty, \infty)$; the Fourier series can be generalized for aperiodic signals also.

Definition: Suppose f(x) a piecewise continuous periodic function of period 2L, then this is Fourier series representation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

Where the coefficients an and bn are given by the Euler-Fourier formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

7.12 Dirichlet's Conditions

The sufficient condition for the convergence of a Fourier series are called Dirichlet's conditions:

1. f(x) is periodic, single valued and finite.
2. f(x) has a finite number of finite discontinuities in any one period.
3. f(x) has a finite number of maxima and minima.

7.12.1 Fourier Cosine and Sine Series

If f is an even periodic function of period 2L, then its Fourier series contains only cosine (including, possibly, the constant term) terms, that is, all bn are zero. Then its Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Its Fourier coefficients are determined by

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3$$

If f is an odd periodic function of period 2L, then its Fourier series contains only sine terms, it will not have any cosine terms. That is its Fourier series is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

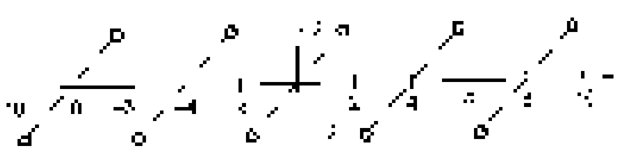
Its Fourier coefficients are determined by

$$a_n = 0, \qquad n = 0, 1, 2, 3$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

Example:

Find a Fourier series for $f(x) = x$, $-2 < x < 2$, $f(x + 4) = f(x)$.



Solution:

First note that $T = 2L = 4$, so that $L = 2$

[illegible]

$$a_0 = \frac{1}{L}\int f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int \ldots \text{[illegible]}$$

The [illegible] coefficients, for $n = 1, 2, 3, \ldots$ are

[illegible]

Hence, [illegible]

The sine coefficients for $n = 1, 2, 3, \ldots$ are

[illegible]

Therefore, $f(x) = $ [illegible] $\sin\frac{n\pi x}{2}$

Example:

Find the Fourier series for [illegible] previous example.

Solution:

$$a_0 = \frac{1}{2}\int_0^{2} x\,dx = [illegible]$$

For $n = 1, 2, 3, \ldots$

$$a_n = \frac{1}{2}\int_0^{2} x\cos\frac{n\pi x}{2}\,dx = [illegible] = 0$$

$$b_n = \frac{1}{2}\int_0^{2} x\sin\frac{n\pi x}{2}\,dx = [illegible] = \frac{-1}{n\pi}$$

Consequently,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2} + b_n\sin\frac{n\pi x}{2}\right) = [illegible] - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

Comment: Just because a Fourier series could have infinitely many [illegible] terms [illegible] that many terms [illegible] periodic function [illegible] can be expressed by finitely many terms [illegible] Fourier series [illegible] the polynomial [illegible] finitely many coefficients.

Example: The Fourier series of period 2π representing $f(x)$ [illegible]

$$[illegible]$$

Example: The Fourier series of period 2π representing [illegible] double angle formula [illegible]

Consequently, the Fourier series is [illegible]

7.12.2 The Cosine and Sine Series Extensions

[illegible] even periodic function [illegible] cosine series [illegible] odd periodic function of period $2L$ [illegible] sine series [illegible] half range expansions.

Even (cosine series) extension of $f(x)$

Given $f(x)$ defined on $[0, L]$ [illegible] even extension of period $2L$ is

$$F(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x)$$

where

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}, \qquad \text{such that}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots$$

Odd (sine series) extension of f(x)

Given $f(x)$ defined on $[0, L]$, its odd extension of period $2L$ is

$$F(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0, L \\ -f(-x) & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x)$$

Where
$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad \text{such that}$$

$$a_n = 0 \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \qquad n = 1, 2, 3, \ldots$$

Example:

A $f(x) = x$, $0 < x < 2$. Find the cosine and sine series extensions of period 4.

Solution:

Cosine series
$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}$$

Sine series
$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

■■■■

Previous GATE and ESE Questions

Q.1 If L defines the Laplace transform of a function, $L[\sin(at)]$ will be equal to

(a) $\frac{a}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$

(c) $\frac{s}{s^2+a^2}$ (d) $\frac{s}{s^2-a^2}$

[CE, GATE-2003, 2 marks]

Q.2 Laplace transform of the function $\sin \omega t$ is

(a) $\frac{s}{s^2+\omega^2}$ (b) $\frac{\omega}{s^2+\omega^2}$

(c) $\frac{s}{s^2-\omega^2}$ (d) $\frac{\omega}{s^2-\omega^2}$

[ME, GATE-2003, 2 marks]

Q.3 A delayed unit step function is defined as

$u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}$ Its Laplace transform is

(a) $a e^{-as}$ (b) $\frac{e^{-as}}{s}$

(c) $\frac{e^{as}}{s}$ (d) $\frac{e^{as}}{a}$

[ME, GATE-2004, 2 marks]

Q.4 A solution for the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$ with initial condition $x(0^-) = 0$ is

(a) $e^{-2t}u(t)$ (b) $e^{2t}u(t)$

(c) $e^{-t}u(t)$ (d) $e^{t}u(t)$

[EC, GATE-2006, 1 mark]

Q.5 If $F(s)$ is the Laplace transform of function $f(t)$, then Laplace transform of $\int_0^t f(\tau)\,d\tau$ is

(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s) - f(0)$

(c) $sF(s) - f(0)$ (d) $\int F(s)\,ds$

[ME, GATE-2007, 2 marks]

Q.6 Evaluate $\int_0^\infty \frac{\sin t}{t}\,dt$

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

[CE, GATE-2007, 2 marks]

Q.7 Laplace transform for the function $f(x) = \cosh(ax)$ is

(a) $\frac{a}{s^2-a^2}$ (b) $\frac{s}{s^2-a^2}$

(c) $\frac{a}{s^2+a^2}$ (d) $\frac{s}{s^2+a^2}$

[CE, GATE-2009, 2 marks]

Q.8 The inverse Laplace transform of $\frac{1}{(s^2+s)}$ is

(a) $1+e^{t}$ (b) $1-e^{t}$

(c) $1-e^{-t}$ (d) $1+e^{-t}$

[ME, GATE-2009, 1 mark]

Q.9 The Laplace transform of a function $f(t)$ is $\frac{1}{s^2(s+1)}$. The function $f(t)$ is

(a) $t-1+e^{-t}$ (b) $t+1+e^{-t}$

(c) $-1+e^{-t}$ (d) $2t+e^{t}$

[ME, GATE-2010, 2 marks]

Q.10 Given $f(t) = L^{-1}\left[\frac{3s+1}{s^3+4s^2+(K-3)s}\right]$. If $\lim_{t\to\infty} f(t) = 1$,

then the value of K is

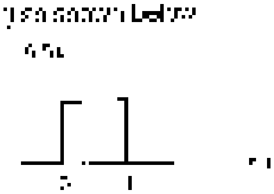
(a) 1 (b) 2

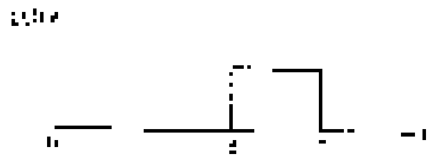
(c) 3 (d) 4

[EC, GATE-2010, 2 marks]

Common Data Questions 11 and 12

Given $f(t)$ and $g(t)$ as shown below:





Q.11 $g(t)$, can be expressed as

(a) $g(t) = f(2t - 3)$ (b) $g(t) = f\left(\frac{t}{2} - 3\right)$

(c) $g(t) = f\left(2t - \frac{3}{2}\right)$ (d) $g(t) = f\left(\frac{t}{2} - \frac{3}{2}\right)$

[EE, GATE-2015, 2 marks]

Q.12 The Laplace transform of $g(t)$ is

(a) $\frac{1}{s}\left(e^{3s} - e^{5s}\right)$ (b) $\frac{1}{s}\left(e^{-5s} - e^{-3s}\right)$

(c) $\frac{e^{-3s}}{s}\left(1 - e^{-2s}\right)$ (d) $\frac{1}{s}\left(e^{5s} - e^{3s}\right)$

[EE, GATE-2015, 2 marks]

Q.13 The inverse Laplace transform of the function $F(s) = \frac{1}{s(s+1)}$ is given by

(a) $f(t) = \sin t$ (b) $f(t) = e^{-t} \sin t$
(c) $f(t) = e^{-t}$ (d) $f(t) = 1 - e^{-t}$

[ME, GATE-2012, 2 Marks]

Q.14 Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \delta(t) \text{ with}$$

$$y(t)\big|_{t=0^-} = -2 \text{ and } \left.\frac{dy}{dt}\right|_{t=0^-} = 0$$

The numerical value of $\left.\frac{dy}{dt}\right|_{t=0^+}$ is

(a) -2 (b) -1
(c) 0 (d) 1

[EC, IN GATE-2012, 2 marks]

Q.15 The function $f(t)$ satisfies the differential equation $\frac{d^2f}{dt^2} + f = 0$ and the auxiliary conditions, $f(0) = 0$, $\frac{df}{dt}(0) = 4$. The Laplace transform of $f(t)$ is given by

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$

(c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^4+1}$

[ME, GATE-2013, 2 Marks]

Q.16 Laplace transform of $\cos(\omega t)$ is $\frac{s}{s^2+\omega^2}$. The Laplace transform of $e^{-2t}\cos(4t)$ is

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$

(c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

[ME, GATE-2014, 1 Mark]

Q.17 Let $X(s) = \frac{3s+5}{s^2+10s+21}$ be the Laplace Transform of a signal $x(t)$. Then $x(0^+)$ is

(a) 0 (b) 3
(c) 5 (d) 21

[EE, GATE-2014, 1 Mark]

Q.18 With initial values $y(0) = y'(0) = 1$, the solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at $x = 1$ is ____.

[EC, GATE-2014, 2 Marks]

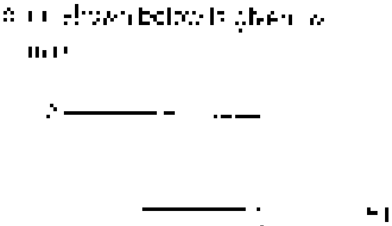
Q.19 The Laplace transform of e^{i5t} where $i = \sqrt{-1}$, is

(a) $\frac{s-5i}{s^2-25}$ (b) $\frac{s+5i}{s^2+25}$

(c) $\frac{s+5i}{s^2-25}$ (d) $\frac{s-5i}{s^2+25}$

[ME, GATE-2015, 1 Mark]

Q.20 Laplace transform of the function $f(t)$ is given by $F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$. Laplace transform of the function shown below is given by



(a) $\frac{1-e^{-2s}}{s}$ (b) $\frac{1-e^{-s}}{2s}$

(c) $\frac{2-2e^{-s}}{s}$ (d) $\frac{1-2e^{-s}}{s}$

[ME, GATE-2015, 2 Marks]

Q.21 If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform $F(s)$ is defined as

(a) $\int_0^{\infty} e^{st} f(t)\,dt$ (b) $\int_0^{\infty} e^{-st} f(t)\,dt$

(c) $\int_0^{\infty} e^{ist} f(t)\,dt$ (d) $\int_0^{\infty} e^{-ist} f(t)\,dt$

[ME, 2016 : 1 Mark]

Q.22 Consider the function $f(x) = 2x^3 - 3x^2$ in the domain $[-1, 2]$. The global minimum of $f(x)$ is ______.

[ME, 2016 : 2 Marks]

Q.23 Laplace transform of $\cos(\omega t)$ is

(a) $\frac{s}{s^2 + \omega^2}$ (b) $\frac{\omega}{s^2 + \omega^2}$

(c) $\frac{s}{s^2 - \omega^2}$ (d) $\frac{\omega}{s^2 - \omega^2}$

[ME, 2016 : 1 Mark]

Q.24 Solutions of Laplace's equation having continuous second-order partial derivatives are called

(a) biharmonic functions
(b) harmonic functions
(c) conjugate harmonic functions
(d) error functions

[ME, 2016 : 1 Mark]

Q.25 The Laplace transform of $f(t) = e^{2t}\sin(5t)\,u(t)$ is

(a) $\frac{5}{s^2 - 4s + 29}$ (b) $\frac{5}{s^2 + 5}$

(c) $\frac{s - 2}{s^2 - 4s + 29}$ (d) $\frac{5}{s + 5}$

[EE, 2016 : 1 Mark]

Q.26 Consider a causal LTI system characterized by differential equation $\frac{dy(t)}{dt} + \frac{1}{6}y(t) = 3x(t)$. The response of the system to the input $x(t) = 3e^{-\frac{t}{3}}u(t)$, where $u(t)$ denotes the unit step function, is

(a) $9e^{-\frac{t}{3}}u(t)$
(b) $9e^{-\frac{t}{6}}u(t)$
(c) $9e^{-\frac{t}{3}}u(t) - 6e^{-\frac{t}{6}}u(t)$
(d) $54e^{-\frac{t}{6}}u(t) - 54e^{-\frac{t}{3}}u(t)$

[EE, 2016 : 1 Mark]

Q.27 The Fourier series of the function,

$$f(x) = 0, \quad -\pi < x \leq 0$$
$$= \pi - x, \quad 0 < x < \pi$$

in the interval $[-\pi, \pi]$ is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right]$$
$$+ \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]$$

The convergence of the above Fourier series at $x = 0$ gives

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

[CE, 2016 : 1 Mark]

Q.28 The Laplace transform of te^t is

(a) $\frac{s}{(s+1)^2}$ (b) $\frac{1}{(s-1)^2}$

(c) $\frac{1}{(s+1)^2}$ (d) $\frac{s}{s-1}$

[ME, GATE-2017 : 1 Mark]

Q.29 For the function

$$f(x) = \begin{cases} 2, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

The value of a_0 in the Fourier series expansion will be

(a) 2
(b) 1 (c) 4
(c) 0 (d) 2

[ESE Prelims 2017]

■■■■

Answers **Transforms**

1.	(b)	2.	(d)	3.	(c)	4.	(c)	5.	(a)	6.	(b)	7.	(c)
8.	(c)	9.	(c)	10.	(c)	11.	(b)	12.	(b)	13.	(d)	14.	(b)
15.	(c)	16.	(d)	17.	(b)	19.	(c)	20.	(a)	21.	(b)	23.	(a)
24.	(b)	25.	(a)	26.	(d)	27.	(c)	28.	(b)	29.	(a)		

Explanations Transform Theory

2. (b)

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

3. (c)

$$L[\delta(t-4)] = \int_0^\infty e^{-st}\,\delta(t-4)\,dt$$

$$= \int_0^4 e^{-st}\cdot 0\,dt + \int_4^\infty e^{-st}\,\delta(t-4)\,dt$$

$$= 0 + \int_4^\infty e^{-st}\,\delta(t-4)\,dt = e^{-4s}$$

4. (a)

$y'(t) + 2y(t) = \delta(t)$

Taking Laplace transforms

$sY(s) - y(0) + 2Y(s) = 1$

$Y(s)(s+2) = 1$

$$Y(s) = \frac{1}{s+2}$$

$$y(t) = e^{-2t}u(t)$$

5. (d)

$$L\left[\int_a^t f(u)\,du\right] = \frac{1}{s}F(s)$$

In the problem $a = 0$

So

$$L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}F(s)$$

6. (b)

Since,

$$L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2} = F(s), \text{ say}$$

$$L\left\{\frac{\sin \omega t}{t}\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty \frac{\omega\,ds}{s^2+\omega^2}$$

$$= \left[\tan^{-1}\frac{s}{\omega}\right]_s^\infty$$

or by Definition

$$\int_0^\infty e^{-st}\frac{\sin \omega t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{\omega}$$

Now $\lim_{s\to 0}\tan^{-1}\left(\frac{s}{\omega}\right) = 0$ if $\omega > 0$ and π if $\omega < 0$

Thus taking limits as $s \to 0$ we get

$$\int_0^\infty \frac{\sin \omega t}{t}\,dt = \frac{\pi}{2} \text{ if } \omega > 0 \text{ and } -\frac{\pi}{2} \text{ if } \omega < 0$$

In this problem $\omega = 1$ which is > 0 hence the answer is $\frac{\pi}{2}$

7. (b)

It is a standard result that

$$L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

8. (c)

$$L^{-1}\left(\frac{1}{s^2+s}\right) = ?$$

$$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}\left(\frac{1}{s^2+s}\right) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$= 1 - e^{-t}$ [using standard formulas]

Standard formula:

$$L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

9. (a)

$$f(t) = L^{-1}\left[\frac{1}{s^2(s+1)}\right]$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$\frac{1}{s^2(s+1)} = \frac{As(s+1) + B(s+1) + Cs^2}{s^2(s+1)}$$

Matching coefficients of s^2, s and constant in numerator we get

$A + C = 0$ ...(i)

$A + B = 0$ ...(ii)

10. (c)

11. (c)

12. (a)

13. (d)

14. (a)

15. (c)

$s^2F(s) - 1 + F(s) = 0$

$(s^2 + 1)F(s) = 1$

$F(s) = \frac{1}{s^2+1}$

$F(t) = \frac{1}{s^2+1}$

16. (c)

$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}$

$a = -2, b = 3$

$\therefore L[e^{-2t}\cos 3t] = \frac{s+2}{(s+2)^2+9}$

17. (b)

Given, $F(s) = \left[\frac{2s+1}{s^2+10s+2}\right]$

Using initial value theorem

$f(0^+) = \lim_{s\to\infty} sF(s)$

$\therefore f(0^+) = \lim_{s\to\infty}\left[\frac{s(2s+1)}{s^2+10s+2}\right]$

$= \lim_{s\to\infty}\left[\frac{2+\frac{1}{s}}{1+\frac{10}{s}+\frac{2}{s^2}}\right] = \frac{2}{1} = 2$

18. Sol.

Given

$y(0) = y'(0) = \ldots$

$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$ ...(i)

Taking the Laplace transform on both sides of (i), we get:

[illegible]

$(s^2 + 4s + 4)Y(s) = \ldots$

$(s^2 + 4s + 4)Y(s) = \ldots$

$Y(s) = \frac{s+\ldots}{(s+2)^2} = \frac{s+\ldots}{(s+2)^2}$

$= \frac{1}{(s+2)} + \frac{\ldots}{(s+2)^2}$

$y(t) = e^{-2t} + \ldots te^{-2t}$

[illegible]

19. (a)

[illegible]

$L[\ldots] = \frac{\ldots}{s^2+\ldots} \ldots = \frac{\ldots}{s^2+\ldots}$

20. (c)

$F(s) = \int_0^\infty f(t)e^{-st}dt$

$= \int_0^1 \ldots e^{-st}dt + \int_1^\infty 0 \cdot e^{-st}dt$

$= \ldots\left[\frac{e^{-st}}{-s}\right]_0^1 = \ldots$

$= \frac{\ldots(1-e^{-s})}{s} = \frac{2 - 2e^{-s}}{s}$

21. (b)

$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$

22. Sol.

$f(x) = 2x^3 - 3x^2 \ldots$

$f'(x) = 6x^2 - 6x \ldots$

$f'(x) = 0$

[illegible]

$f''(x) = 12x - 6$

$f''(0) = -6$ Max.

$f''(1) = 6$ Min.

[illegible]

23. (a)

$L[\cos at] = \frac{s}{s^2+a^2}$

24. (b)

[illegible]

Second order partial differential equation

$\therefore \ldots = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

[illegible]

24. (a)

[illegible] $= \dfrac{5}{s+25}$ [illegible]

[illegible]

25. (d)

[illegible]

[illegible] $-\dfrac{1}{5}$ [illegible]

[illegible]

So, [illegible]

[illegible]

So, [illegible]

27. (c)

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] $\dfrac{1}{2}$ [illegible]

where [illegible]

[illegible]

$\dfrac{\pi}{2} =$ [illegible] $\left[\ldots\right]$

[illegible] $\dfrac{\pi^2}{8}$

28. (b)

[illegible]

[illegible]

[illegible]

29. (c)

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] $= 0$

[illegible]

[illegible]

$\Rightarrow$ [illegible] $= 0$

8 Second Order Linear Partial Differential Equations

Introduction

[illegible]

8.1 Classification of Second Order Linear PDEs

[illegible]

Example:

[illegible]

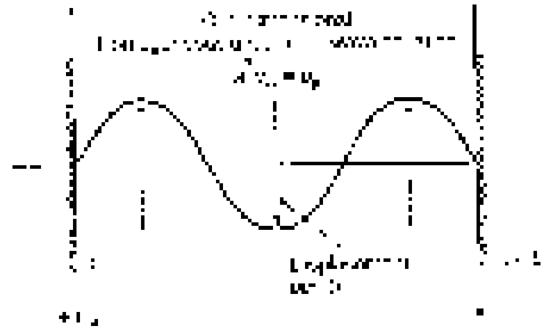
Solution:

[illegible]

8.2 Undamped One-Dimensional Wave Equation; Vibrations of an Elastic String

[illegible]

[illegible]



Hence, our problem is the following initial-boundary-value problem:

(Wave equation) $c^2 u_{xx} = u_{tt}$, $0 < x < L$, $t > 0$

(Boundary conditions) $u(0, t) = 0$ and $u(L, t) = 0$

(Initial conditions) $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$

We first apply $u = X(x)T(t)$ [illegible]. Substituting $u_{xx} = X''T$ and $u_{tt} = XT''$ into the wave equation, it becomes

$$c^2 X''T = XT''$$

Dividing both sides by $c^2 XT$:

$$\frac{X''}{X} = \frac{T''}{c^2 T}$$

As in the heat conduction equation, it is customary to consider [illegible] the separation constant and [illegible] equation:

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$$

$$\frac{X''}{X} = -\lambda \quad \Rightarrow \quad X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$$

$$\frac{T''}{c^2 T} = -\lambda \quad \Rightarrow \quad T'' = -c^2\lambda T \quad \Rightarrow \quad T'' + c^2\lambda T = 0$$

The boundary conditions also separate:

$$u(0, t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0 \quad \text{or} \quad T(t) = 0$$

$$u(L, t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0 \quad \text{or} \quad T(t) = 0$$

As before, in order to obtain nontrivial solutions, we need to choose $X(0) = 0$ and $X(L) = 0$ as the new boundary conditions. The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0$$

$$T'' + c^2\lambda T = 0$$

The next step is to solve the eigenvalue problem

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0$$

The solutions are [illegible]

Eigenvalues: $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Therefore, [illegible]

[illegible]

Special case: [illegible]

[illegible]

Therefore, [illegible]

ILLUSTRATIVE EXAMPLES

Example

[illegible]

Solution:

[illegible]

[illegible]

Hence, the particular solution is

[illegible]

Example.

Solve the vibrating string problem

$$9u_{xx} = u_{tt} \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0 \quad \text{and} \quad u(5, t) = 0$$
$$u(x, 0) = 0$$
$$u_t(x, 0) = 4$$

Solution:

As in the previous example, $a^2 = 9$ ($a = 3$), and $L = 5$. The general solution is

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5}\right)\sin\frac{n\pi x}{5}$$

Now $f(x) = 0$, consequently all $A_n = 0$. [illegible] $g(x)$ [illegible] constant function [illegible] period $T = 10$ [illegible] then:

$$B_n = \frac{2}{3n\pi}\int_0^5 g(x)\sin\frac{n\pi x}{5}\,dx = \frac{2}{3n\pi}\int_0^5 4\sin\frac{n\pi x}{5}\,dx$$

$$= \begin{cases} \dfrac{80}{3n^2\pi^2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Therefore,
$$u(x, t) = \sum_{n=1}^{\infty}\frac{80}{3(2n-1)^2\pi^2}\sin\frac{3(2n-1)\pi t}{5}\sin\frac{(2n-1)\pi x}{5}$$

8.2.1 Summary of Wave Equation: Vibrating String Problems

The vertical displacement of a vibrating string of length L, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous one-dimensional wave equation initial-boundary value problem:

$$a^2 u_{xx} = u_{tt} \qquad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(L, t) = 0,$$
$$u(x, 0) = f(x), \quad \text{and} \quad u_t(x, 0) = g(x)$$

The general solution is

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi a t}{L} + B_n \sin\frac{n\pi a t}{L}\right)\sin\frac{n\pi x}{L}$$

The particular solution can be found by the formulas

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \text{ and}$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

The solution, in the form of a wave, has a constant velocity (or constant propagation speed) in both directions of the x-axis. The vibrating string has [illegible] given by $g(x)$ [illegible] string.

Exercise.

1. Solve the vibrating string problem of the given initial conditions.

$$4u_{xx} = u_{tt} \qquad 0 < x < \pi, \quad t > 0,$$
$$u(0, t) = 0, \quad u(\pi, t) = 0$$

(b) $u(x, 0) =$ [illegible] ; $u_t(x, 0) = 0$

(c) $u(x, 0) = 0$; $u_t(x, 0) = 0$

(d) $u(x, 0) = 0$; $u_t(x, 0) =$ [illegible]

2. Solve the vibrating string problem

$$100\, u_{xx} = u_{tt}, \quad 0 < x < 2, \quad t > 0$$
$$u(0, t) = 0, \quad u(2, t) = 0,$$
$$u_t(x, 0) = \text{[illegible]}$$
$$u(x, 0) = \text{[illegible]}$$

3. Solve the vibrating string problem

[illegible]

$u(0, t) = 0$ and $u(\text{[illegible]}, t) = 0$

$u(x, 0) =$ [illegible]

$u_t(x, 0) = 0$

4. Verify that the D'Alembert form solution $u(x, t) = \frac{1}{2}[f(x - at) + f(x + at)]$ [illegible]

$a^2 u_{xx} = u_{tt}$, [illegible] $t > 0$

$u(0, t) = 0$, $u(\text{[illegible]}, t) = 0$

$u(x, 0) = f(x)$, $u_t(x, 0) = 0$

5. [illegible]

$a^2 u_{xx} = u_{tt}$, $0 < x < L$, $t > 0$

$u(0, t) = 0$, $u(L, t) = 0$

$u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$

6. What is the steady state displacement [illegible]

Answers

1. (a) [illegible]

(b) $u(x, t) =$ [illegible]

5. (a) The general solution is $u(x, t) =$ [illegible] $\sum_{n=1}^{\infty}$ [illegible]

The coefficients [illegible] by the formulas:

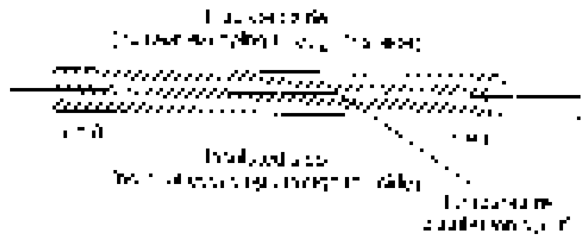
$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, [illegible] and [illegible]

6. The steady state displacement [illegible]

8.3 The One-Dimensional heat Conduction Equation

Consider [illegible] of length L [illegible]. Suppose [illegible]

[illegible]



[illegible]

8.3-1 Conduction Problem

[illegible]

ILLUSTRATIVE EXAMPLES

Example

[illegible]

[illegible]

Solution:

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

Hence,

$$[illegible]$$

Example

[illegible]

$$[illegible]$$

Solution

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

The Steady-State Solution

[illegible]

Therefore [illegible]

[illegible]

ILLUSTRATIVE EXAMPLES

Example

[illegible]

1. [illegible]
2. [illegible]

Solution:

1. [illegible]
2. [illegible]

Hence, [illegible]

[illegible]

8.4 Laplace Equation for a Rectangular Region

[illegible]

Dividing both sides by [illegible]

[illegible]

The boundary conditions also separate:

[illegible]

Thus the corresponding eigenfunctions are

[illegible]

[illegible]

Previous GATE and ESE Questions

Q.1 The solution of the partial differential equation [illegible]

$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is of the form [illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

[CH, 2016 : 1 Mark]

Q.2 The type of the partial differential equation [illegible]

[illegible] $= 0$ is

(a) elliptic (b) parabolic

(c) hyperbolic (d) [illegible]

[PI, 2016 : 1 Mark]

Q.3 Consider the following partial differential equation [illegible] with the constant [illegible]

[illegible] $= 0$

Solution of this equation is [illegible]

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ME, GATE-2017 : 1 Mark]

Q.4 Consider a function [illegible] given by [illegible]

The partial derivative of this function with respect to [illegible] at the point [illegible] $y = 1$ and $z = 3$ is ____

[EE, GATE-2017 : 1 Mark]

Q.5 Consider the following partial differential equation

[illegible] $= 0$

For the equation to be classified as parabolic, the value of [illegible] must be ____.

[CE, GATE-2017 : 1 Mark]

Q.6 The solution of the following partial differential equation [illegible] is

(a) [illegible] (b) [illegible]

(c) [illegible] (d) [illegible]

[ESE Prelims-2017]

■■■■

Answers Second Order Linear Partial Differential Equations

1. (c) 2. (c) 3. (b) 4. (b) 5. (c)

Explanations Second Order Linear Partial Differential Equations

1. (a)

The PDE $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$ ...(i)

Solution of (i) is

[illegible]

Put [illegible]

[illegible]

Putting values of [illegible] in eq. (i)

[illegible]

2. (c)

Comparing the given equation with the general form of second order partial differential equation we have [illegible]

∴ PDE is [illegible]

3. (b)

[illegible]

$\frac{\partial v}{\partial y}$ = [illegible]

[illegible]

∴ $\frac{\partial v}{\partial y} + c\frac{\partial u}{\partial x} = 0$

4. Sol.

[illegible]

[illegible]

5. Sol.

Given that the partial differential equation is parabolic

[illegible]

[illegible]

$b^2 = 0$

6. (b)

[illegible]

■■■■