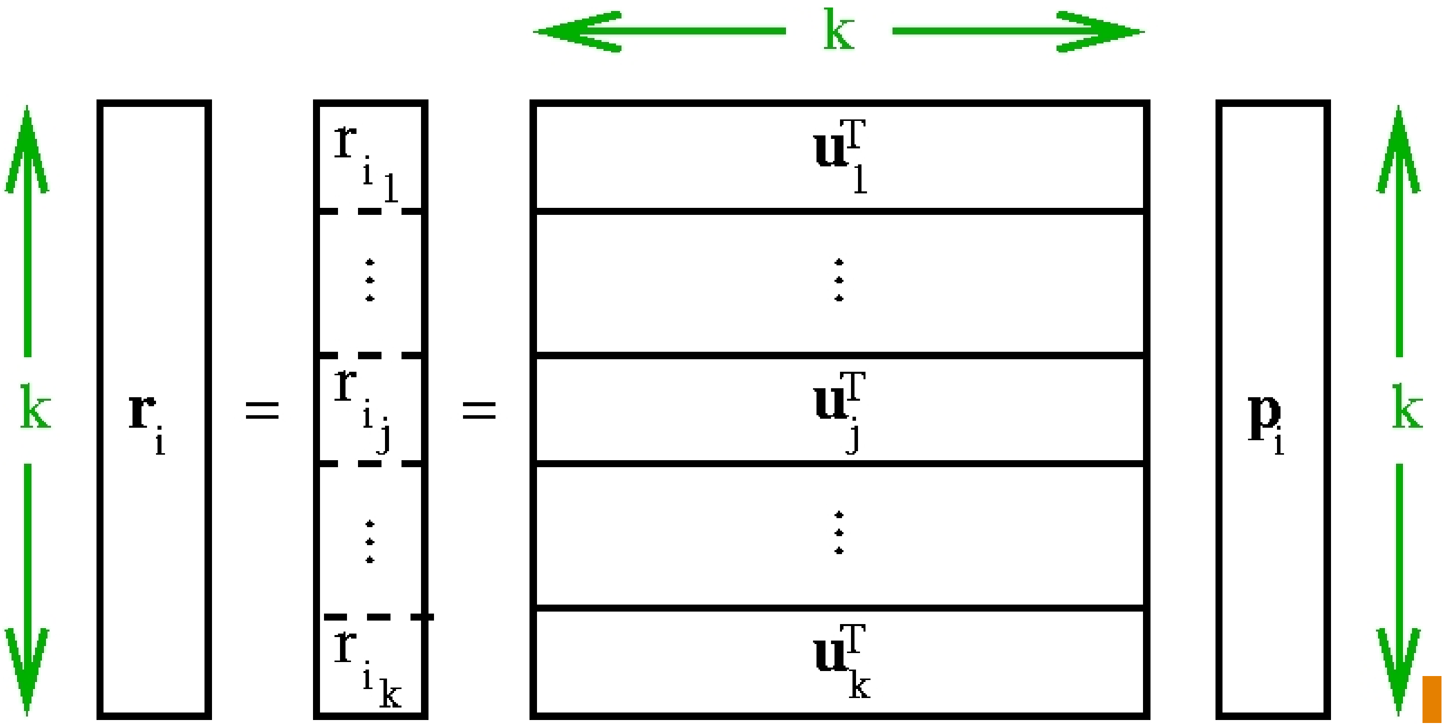
The K-L Transform

$$\mathbf{r}_i \triangleq \mathbf{U}^T \mathbf{p}_i \qquad \mathbf{R} \triangleq \mathbf{U}^T \mathbf{P}$$



$\mathbf{r}_{ij} = \mathbf{p}_i \cdot \mathbf{u}_j \rightarrow \mathbf{p}_i$'s component along $\mathbf{u}_j$.

Home Page

Title Page

Contents

Page 41 of 55

Go Back

Full Screen

Close

Quit

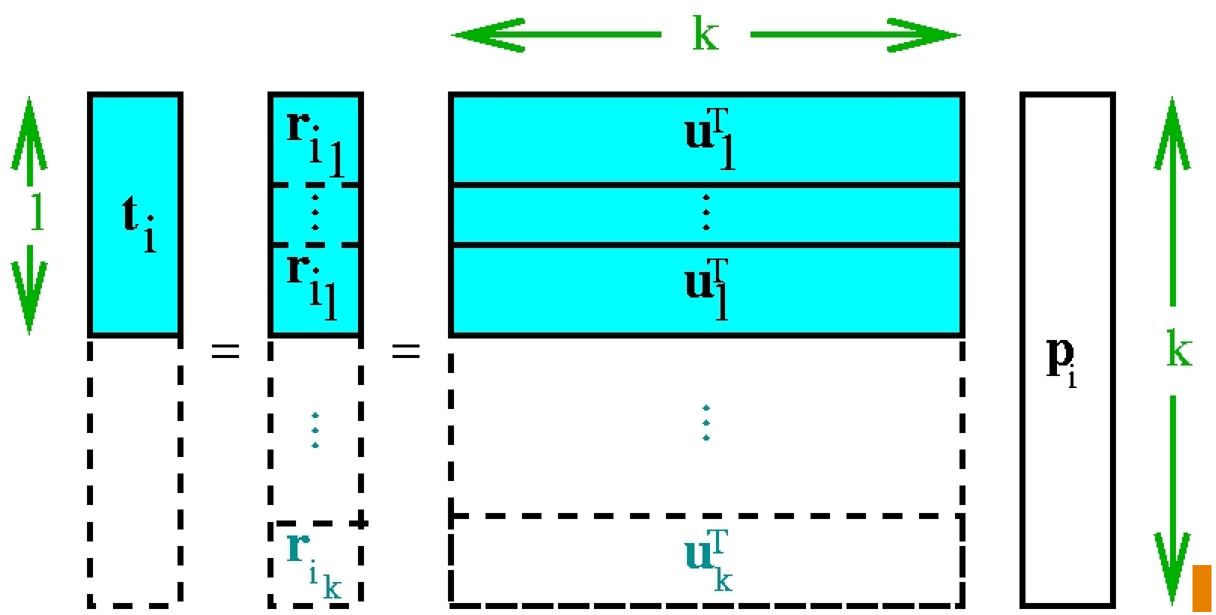
The Covariance Matrix

$$\hat{\mathbf{A}} \triangleq \frac{1}{n}\,\mathbf{R}\,\mathbf{R}^T$$
$$= \frac{1}{n}\,\mathbf{U}^T\,\mathbf{P}\,(\mathbf{U}^T\,\mathbf{P})^T = \mathbf{U}^T\,\frac{1}{n}\,\mathbf{P}\,\mathbf{P}^T\,\mathbf{U}$$
$$= \mathbf{U}^T\,\mathbf{A}\,\mathbf{U} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U} = \mathbf{\Lambda} \text{ (Diagonalisation)}$$

In a Nutshell. . .

- Orthonormal matrix: Rotation connotation
- 'Rotated' patterns $\mathbf{r}_i$ line up with the EigenVectors
- The 'rotated' patterns are uncorrelated
- The average spreads of the 'rotated' patterns are the EigenValues themselves

Dimensionality Reduction



$$\mathbf{t}_i \triangleq \mathbf{V}^T \mathbf{p}_i \qquad \mathbf{T} \triangleq \mathbf{V}^T \mathbf{P}$$

$$\tilde{\mathbf{A}} \triangleq \frac{1}{n} \mathbf{T}\,\mathbf{T}^T = \frac{1}{n} \mathbf{V}^T \mathbf{P}\,(\mathbf{V}^T \mathbf{P})^T$$

$$= \mathbf{V}^T \mathbf{A}\,\mathbf{V} = \mathbf{V}^T \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^{-1}\,\mathbf{V} \text{ (Diagonalisation)}$$

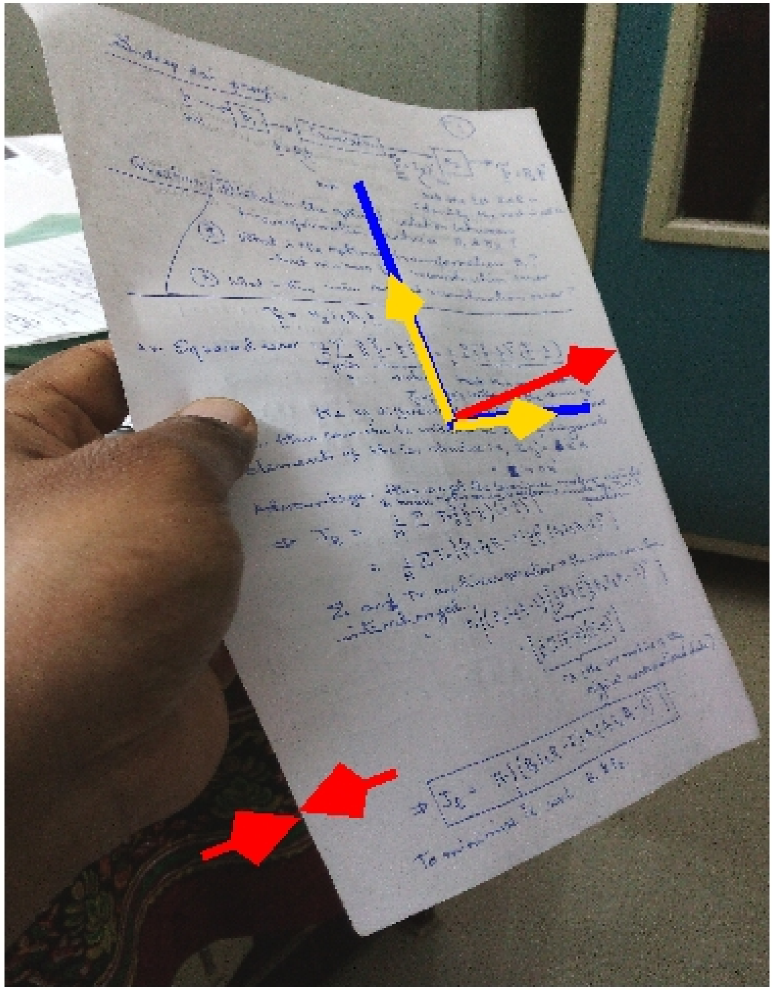
$$= (\mathbf{V}^T \mathbf{U})\,\mathbf{\Lambda}\,(\mathbf{V}^T \mathbf{U})^T = \mathbf{\Lambda}_l = diag(\lambda_1, \ldots \lambda_l)$$

How many eigenvectors (l)?

e.g., min to make up 95% energy. $\min l : \frac{\sum_{i=1}^{l} \lambda_i}{\sum_{i=1}^{k} \lambda_i} \geq 0.95$

Illustration: Dim Reduction

2-D sheet of paper in 3-D space



Home Page

Title Page

Contents

Page 50 of 55

Go Back

Full Screen

Close

Quit

The Singular Value Decomp

- Generalises the KLT $\forall$ cases, including $k > n$
- A construction: always exists!
- KL-Transform: square matrices, SVD: rectangular
- Can't diag any square matrix, SVD always exists!
- Key observation: $\mathbf{P}_{k\times n}$ not amenable for eigendecomposition, but not $\mathbf{P}\mathbf{P}^T{}_{k\times k}$ & $\mathbf{P}^T\mathbf{P}_{n\times n}$
- $\mathbf{P}\mathbf{P}^T{}_{k\times k}$ & $\mathbf{P}^T\mathbf{P}_{n\times n}$: Positive Semi-Definite: non-negative eigenvalues! How?

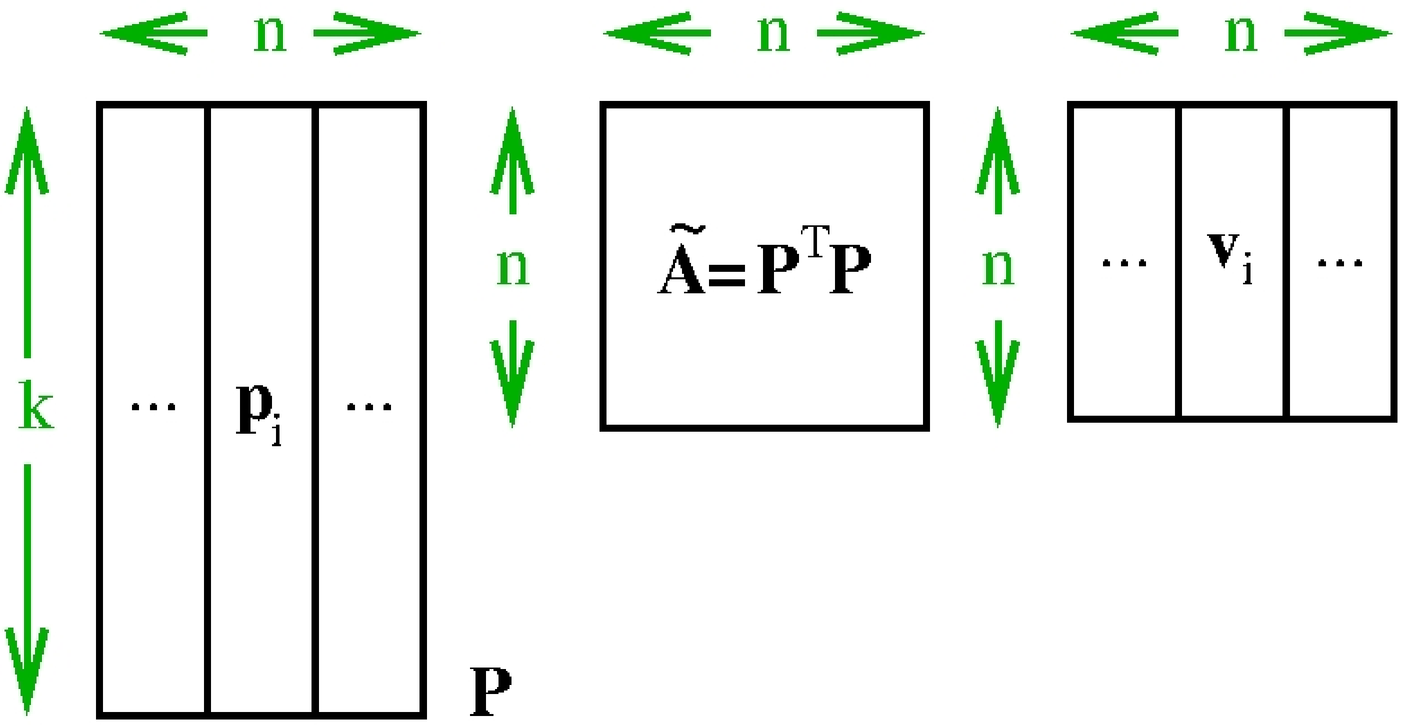
  $(\mathbf{P}\mathbf{P}^T)_{k\times k}$ PSD: $\mathbf{x}^T_{1\times k}(\mathbf{P}\mathbf{P}^T)_{k\times k}\mathbf{x}_{k\times 1} = (\mathbf{x}^T\mathbf{P})(\mathbf{P}^T\mathbf{x}) = (\mathbf{x}^T\mathbf{P})_{1\times n}(\mathbf{x}^T\mathbf{P})^T_{n\times 1} = \mathbf{y}^T\mathbf{y} = \sum y_i^2 \geq 0$

  $(\mathbf{P}^T\mathbf{P})_{n\times n}$ PSD: $\mathbf{x}^T_{1\times n}(\mathbf{P}^T\mathbf{P})_{n\times n}\mathbf{x}_{n\times 1} = (\mathbf{x}^T\mathbf{P}^T)(\mathbf{P}\mathbf{x}) = (\mathbf{x}^T\mathbf{P}^T)_{1\times k}(\mathbf{P}\mathbf{x})_{k\times 1} = \mathbf{y}^T\mathbf{y} = \sum y_i^2 \geq 0$

  Eigenvalues of $\mathbf{P}\mathbf{P}^T$: $\mathbf{P}\mathbf{P}^T\mathbf{u} = \lambda\mathbf{u}$: $\mathbf{u}^T\mathbf{P}\mathbf{P}^T\mathbf{u} = \lambda\mathbf{u}^T\mathbf{u}$

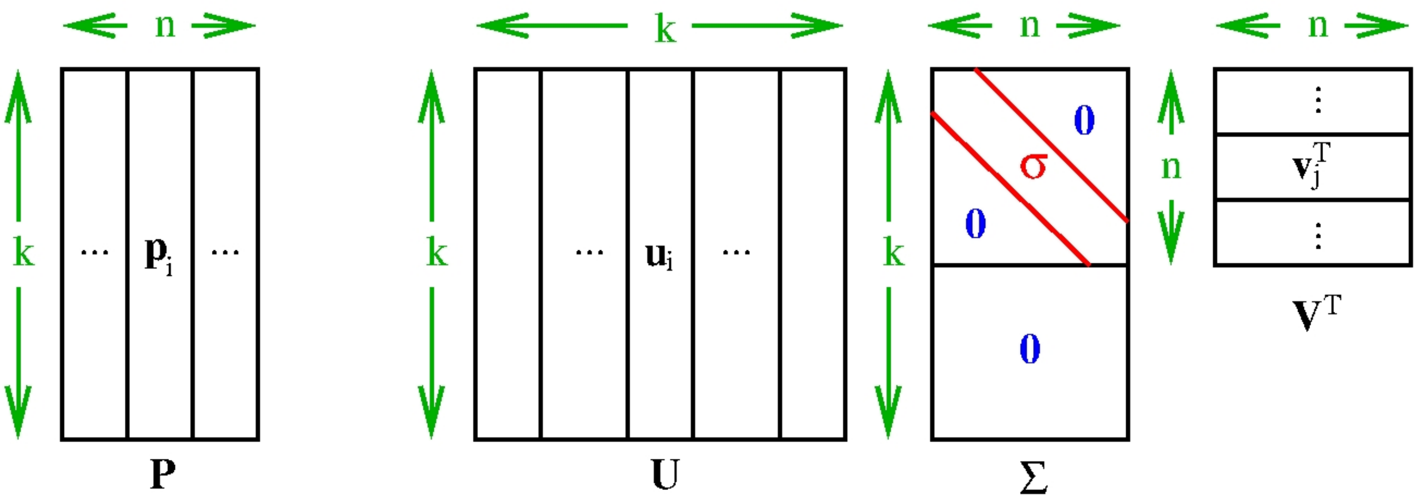
  LHS ≥ 0, $\mathbf{u}^T\mathbf{u} \geq 0$ (e'vecs) $\implies \lambda \geq 0$. $\|$ly, $\mathbf{P}^T\mathbf{P}$

- $k > n$: difficult to numerically calculate the eigenvalues and eigenvectors of $\mathbf{A}_{k\times k} = \frac{1}{n}\mathbf{P}\mathbf{P}^T$. Trick: consider the pseudo-covariance matrix $\tilde{\mathbf{A}} = \mathbf{P}^T\mathbf{P}$:
- n $k-$dim vectors $\equiv$ k $n-$dim vectors
- Let $rank(\mathbf{P}) = r$, where r is at most $\min(k,n)$.
- Let $\tilde{\mathbf{A}}$ have r non-zero eigenvalues $\lambda_1, \ldots \lambda_r$ and corresponding eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_r$ (each $n \times 1$). Stack these together to form a $n \times r$ matrix
- Append $n-r$ orthonormal vectors to get $\mathbf{V}_{n\times n}$ (all Euclidean bases: related by $\mathbf{R}, \mathbf{t}$)
- Definiton $\sigma_i \triangleq \sqrt{\lambda_i}$. $\boxed{(\mathbf{u_i})_{k\times 1} \triangleq \frac{1}{\sigma_i}\mathbf{P}_{k\times n}(\mathbf{v_i})_{n\times 1}}$, $i \in \{1,r\}$ Stack these together to form a $k \times r$ matrix
- Append $k-r$ orthonormal vectors to get $\mathbf{U}_{k\times k}$

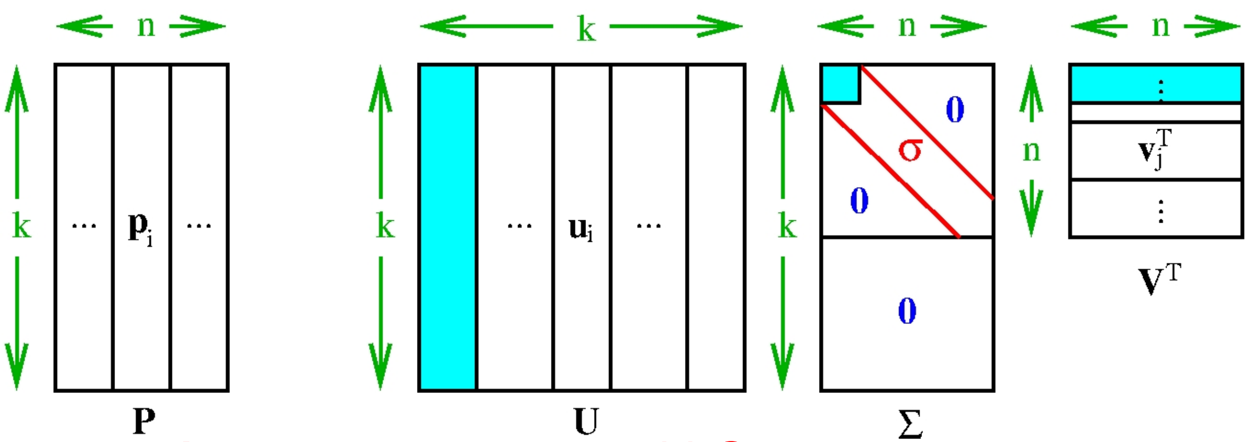


- $\mathbf{u}_i$ orthonormal; $\mathbf{U}_{k\times k}$: orthonormal basis matrix
  $\mathbf{u}_i^T\mathbf{u}_j = \frac{1}{\sigma_i}\mathbf{v}_i^T\mathbf{P}^T\frac{1}{\sigma_j}\mathbf{P}\mathbf{v}_j = \frac{1}{\sigma_i\sigma_j}\mathbf{v}_i^T(\mathbf{P}^T\mathbf{P})\mathbf{v}_j = \frac{\lambda_j}{\sigma_i\sigma_j}\mathbf{v}_i^T\mathbf{v}_j$
  $\mathbf{v}_i$ orthonormal. $i = j$, RHS = 1; $i \neq j$, RHS = 0.
- $\mathbf{u}_i^T\mathbf{P}\mathbf{v}_j = \frac{1}{\sigma_i}\mathbf{v}_i^T\mathbf{P}^T\mathbf{P}\mathbf{v}_j = \frac{1}{\sigma_i}\mathbf{v}_i^T\lambda_j\mathbf{v}_j$. $\mathbf{v}_i$ orthonormal $i = j$: value = σ_i; $i \neq j$: value = 0 $\boxed{\mathbf{U}^T\mathbf{P}\mathbf{V} = \mathbf{\Sigma};\ \mathbf{P} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T}$

Home Page
Title Page
Contents
Page 53 of 55
Go Back
Full Screen
Close
Quit



Dimensionality Reduction: Take $l < n < k$ orthonormal basis vectors $\mathbf{u}_i$: $\mathbf{P}_{k\times n} \approx \mathbf{U}_{k\times l}\mathbf{\Sigma}_{l\times l}\mathbf{V}^T_{l\times n}$



How many singular values (l)?

e.g., min to make up 95% energy. $\min l : \frac{\sum_{i=1}^{l} \sigma_i}{\sum_{i=1}^{k} \sigma_i} \geq 0.95$

Home Page

Title Page

Contents

Page 54 of 55

Go Back

Full Screen

Close

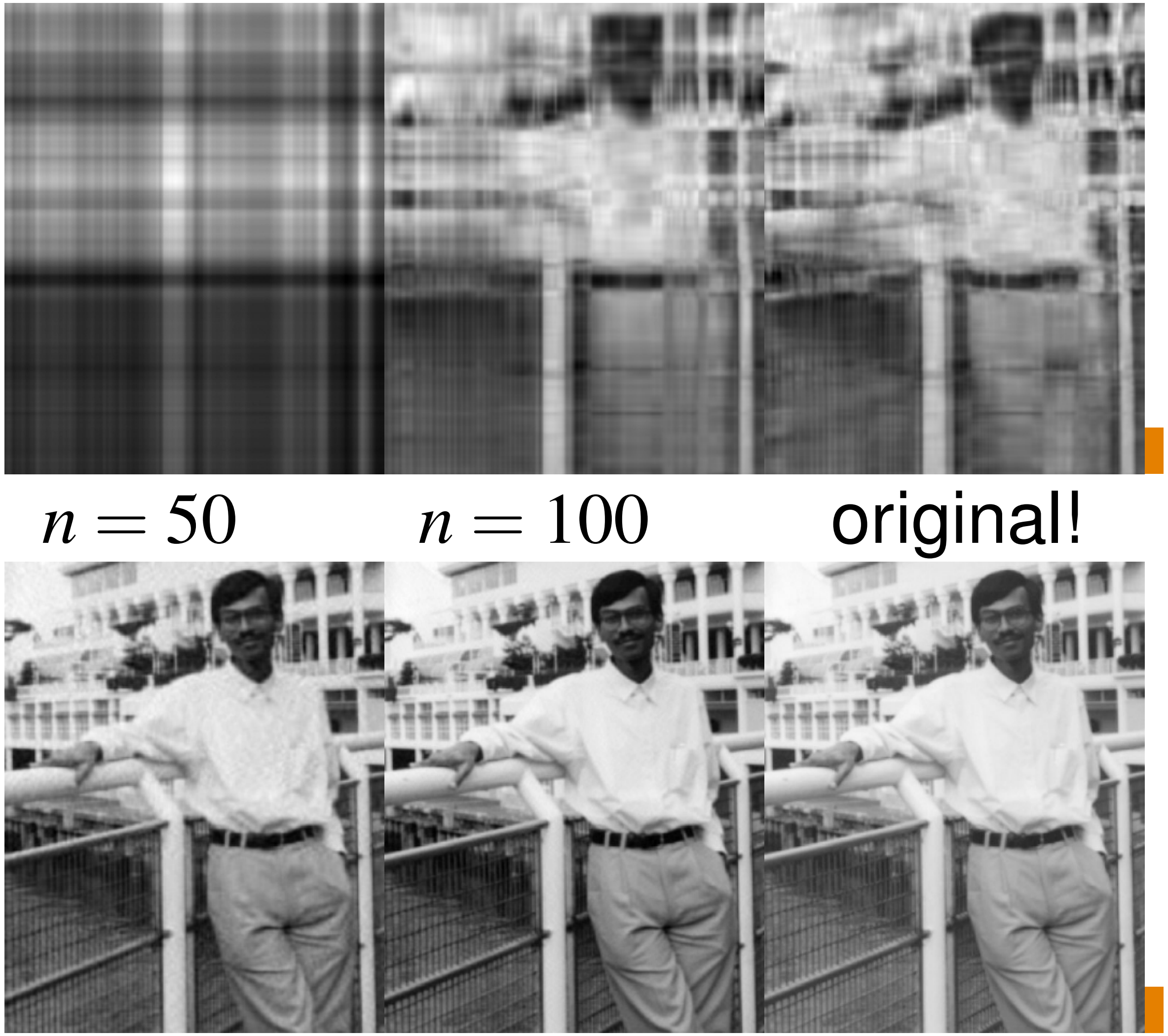
Quit

Reconstruction

Reconstruction with n EigenVectors

$n = 1$ $n = 5$ $n = 10$

$n = 50$ $n = 100$ original!



Home Page

Title Page

Contents

Page 55 of 55

Go Back

Full Screen

Close

Quit

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