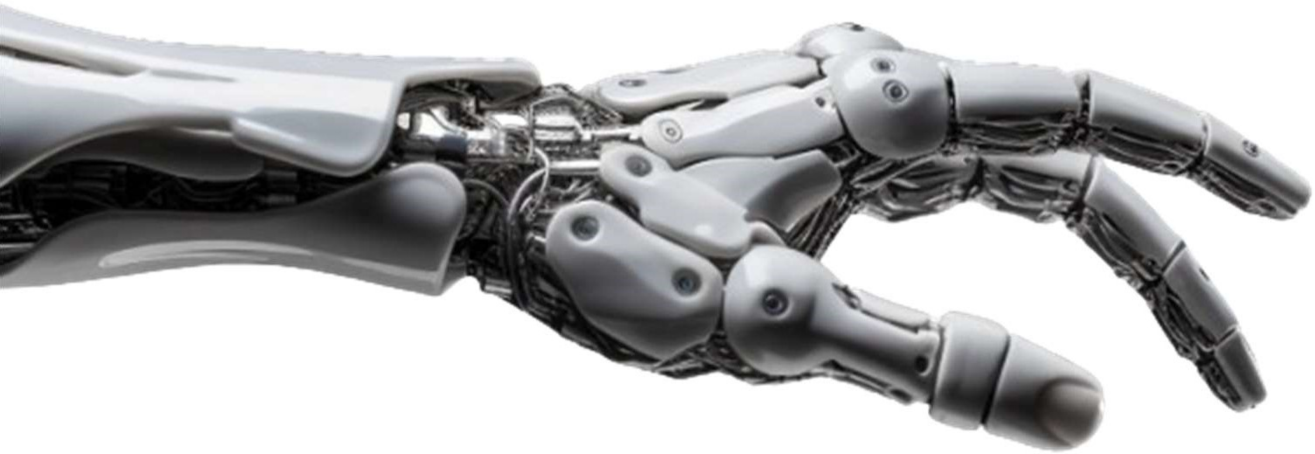

Rank of a matrix

Rank of a matrix- Definitions

Linearly independent rows or columns in a matrix

Maximum number of nonzero rows in the echelon matrix of the given matrix.

If $|A|_{n\times n} = 0$, rank(A)$\neq n$ ($< n$).
(At least one row/column vector will be dependent).

Order of the largest submatrix with non-zero determinant

Examples

Maximum number of non-zero rows in the echelon matrix of the given matrix.

$$A=\begin{bmatrix}1 & 2 & 3\\ 3 & 6 & 9\\ 0 & 0 & 0\end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$A \sim E = \begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$ $\to$ Row reduced echelon form

Here $\vec{r_1}$ is only independent

$\Rightarrow$ Only one linearly independent row in echelon form.

$A \sim E \Rightarrow R(A) = R(E) = 1$

$\vec{C_2} = 2\vec{C_1}$

$\vec{C_3} = 3\vec{C_1}$

$\Rightarrow C_1$ is linearly independent

Rank

$=$ # L.I column vectors

L.I row vectors

$\therefore$ $\boxed{\text{Rank}(A) = 1}$

Examples

Maximum number of nonzero rows in the echelon matrix of the given matrix.

$$B=\begin{bmatrix}1&2&3\\2&2&2\\1&4&3\end{bmatrix}$$

$\boxed{\text{Rank}(B)=2}$

$R_2 \to R_2-2R_1$

$R_3 \to R_3-R_1$

$B \sim \begin{bmatrix}1&2&3\\0&0&0\\0&2&0\end{bmatrix}$ > 2 Non zero rows

$\Rightarrow \text{Rank}(B)=2$

$$C=\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}$$

$\boxed{\text{Rank}(C)=1}$

$R_2 \to R_2-2R_1$

$R_3 \to R_3-3R_1$

$C \sim \begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix} = E$

$R(E)=1$

$C \sim E \Rightarrow R(C)=R(E)=1$

$\therefore \underline{R(C)=1}$

$$D=\begin{bmatrix}1&2&3\\6&1&-2\\5&-1&-5\end{bmatrix}$$

$R_2 \to R_2-6R_1$

$R_3 \to R_3-5R_1$

$\therefore D \sim \begin{bmatrix}1&2&3\\0&-11&-20\\0&-11&-20\end{bmatrix}$

$R_3 \to R_3-R_2$

$\sim \begin{bmatrix}1&2&3\\0&-11&-20\\0&0&0\end{bmatrix} \to$ Row reduced Echelon form

Two nonzero rows are there

$\Rightarrow \boxed{\text{Rank}(D)=2}$

Examples

No: of linearly independent row/column vectors in a matrix.

1. $A=\begin{bmatrix}2 & 3 & 5\\ 0 & 7 & -1\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}$ Rank(A) = 3

2. $B=\begin{bmatrix}5 & 0 & 0 & 0 & 0\\ 0 & 7 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 7 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}$ Rank(B) = 5

3. $C=\begin{bmatrix}6 & 3\\ 2 & 4\\ 1 & 2\\ 0 & 0\end{bmatrix}$ $\times 2$ Rank(C) = 2

4. $D=\begin{bmatrix}1 & -1 & 2\\ 0 & 0 & 3\\ 0 & 0 & 9\\ 0 & 0 & 12\end{bmatrix}$ $\times 3$ $\times 4$ Rank(D) = 2

5. $E=\begin{bmatrix}1 & -1 & 2\\ 0 & 0 & 3\\ 0 & 0 & 5\\ 0 & 0 & 7\end{bmatrix}$

$\vec{r_3} = \frac{5}{3}\vec{r_2}$, $\vec{r_4} = \frac{7}{3}\vec{r_2}$ — $\vec{r_3}$ and $\vec{r_4}$ are depending on $\vec{r_2}$ $\boxed{R(E)=2}$

$c_2 = -c_1$

c_1 and c_3 are L. Indep.
or
c_2 and c_3 are L. Indep $\boxed{R(E)=2}$

6. $E=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ Rank[E] = 0

Zero matrix is the only matrix with rank=0

7. $E=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ $r = 1$

Examples

If $|A|_{n\times n} = 0$, rank(A)$\neq n$ $(< n)$.

(At least one row/column vector will be dependent).

1 $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}$

$|A| = 0 \Rightarrow \text{rank}(A) \neq 3$

$\text{rank}(A) = 1$

2 $B=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$

$|B| = 4 \neq 0 \Rightarrow \text{rank}(B) = 3$

3×3

$n=3$ $\quad |B| \neq 0$ $\quad \text{rank}(B) = n = 3$ //

3 $C=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ 1 & 3 & 2\end{bmatrix}$

$|C| = 0$

$\sim \begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 0\end{bmatrix}$

$\text{Rank}(C) = 2$

Examples

Order of the largest submatrix with non-zero determinant

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}_{3\times 3}$$

Here $|A| = 0 \Rightarrow rank(A) \neq n$

$\Rightarrow rank(A) \neq 3$

Now consider the 2x2 Submatrix of A.

$\begin{vmatrix}1 & 1\\ 2 & 2\end{vmatrix} = 0$, $\begin{vmatrix}1 & 1\\ 3 & 3\end{vmatrix} = 0$, $\begin{vmatrix}2 & 2\\ 3 & 3\end{vmatrix} = 0$

Determinant of all 2x2 submatrices of A = 0

$\Rightarrow rank(A) \neq 2$

Thus $\boxed{rank(A) = 1}$

$$B=\begin{bmatrix}2 & -1 & 3 & 4\\ 1 & 0 & 2 & 1\\ -1 & 1 & -1 & -3\end{bmatrix}$$

Consider the 3x3 submatrices of B.

$\begin{vmatrix}2 & -1 & 3\\ 1 & 0 & 2\\ -1 & 1 & -1\end{vmatrix} = 0$, $\begin{vmatrix}-1 & 3 & 4\\ 0 & 2 & 1\\ 1 & -1 & -3\end{vmatrix} = 0$,

$\begin{vmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ -1 & 1 & -3\end{vmatrix} = 0 \Rightarrow Rank(B) \neq 3$

Then go for the 2x2 submatrix.

$\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix} \neq 0 \Rightarrow \boxed{Rank(B) = 2}$

Properties of rank of a matrix

- Only the rank of a null matrix is zero.
- Rank of an identity matrix of order $n \times n$ is n.
- Rank of a matrix $A_{m\times n} \leq \min(m,n)$.
- Rank$(A_{n\times n}) = n$ if $|A| \neq 0 < n$ if $|A| = 0$
- Rank$(\ A^T) = Rank(A)$
- Two matrices that are row equivalent have the same rank.

Homework problems

1. Find the rank of the following matrices just by looking at it:

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

$$B=\begin{bmatrix}0 & 5 & 3\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$C=\begin{bmatrix}0 & 0 & -1\\ 1 & 1 & 1\\ -1 & -1 & -1\\ -2 & -2 & -2\end{bmatrix}$$

$$D=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & 0 & 0\\ 3 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 3 & 0\end{bmatrix}$$

$$E=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}$$

2. What is the rank of A^T if A is the 5×5 matrix with three independent rows?

Homework problems

3. Find the rank of the matrix by reducing to row reduced echelon form

$$a)\ \begin{bmatrix} 2 & 1 & -1 & 6 & 7 \\ 1 & 3 & 2 & 0 & 2 \\ 3 & 4 & 1 & 6 & 9 \\ 1 & -2 & -3 & 6 & 5 \end{bmatrix}$$

$$b)\ \begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 6 \\ 2 & 3 & 1 \\ 5 & 0 & 10 \\ 4 & 1 & 7 \end{bmatrix}$$

Homework problems

4. What is the rank of A if A is a non-zero singular 2×2 matrix?

4. If rank of A is 99, what is the rank of (i) 6A (ii) $9A^T$?

The scalar multiplication of a matrix does not change its rank. The rank of a matrix A is determined by the number of linearly independent rows or columns, which remains the same after multiplying by a scalar. Therefore, the rank of 6A is the same as the rank of A.

Rank(6A) = Rank(A) =99.

The rank of the transpose of a matrix A^T is the same as the rank of the original matrix A. Again, scalar multiplication (here, by 9) does not change the rank.

Rank $(9A^T) = Rank(A^T)$=Rank(A)=99

4. Find all values of k for which the rank of the matrix, $A = \begin{bmatrix} 1 & -2 \\ -2 & k \end{bmatrix}$ is one.

Homework problems

7. Write a 3x3 matrix with rank 1.

8. Write a 5x4 matrix with rank 2.

9. Create a 5x5 matrix with all non-zero entries and having rank as three.

10. What should be the value of a and b such that $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 8 & a & b \end{bmatrix}$ and r(C)=1.


Thank
you