# 23MAT124
# ENGINEERING MATHEMATICS-1

## Solution of system of equations & Gauss elimination method

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## System Of Linear Equations :

→ A linear system of 'm' equations in 'n' unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \quad\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow (1)$$

It is called linear because each variable is of power "one"

→ The numbers $a_{11}, a_{12}, \ldots, a_{mn}$ are called coefficients of the system

→ The numbers $b_1, b_2, \ldots, b_n$ and $a_{11}, a_{12}, \ldots, a_{mn}$ will be given.

→ If all the $b_i$'s are zeroes then the system is called homogeneous system.

→ If atleast one $b_i \neq 0$ then it is called inhomogeneous / non homogeneous system.

Let $A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$ The matrix A is called coefficient Matrix.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then the system ① can be written in the matrix notation as $AX = b$.

Let

$$\tilde{A} = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & b_2 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

The matrix $\tilde{A}$ is called the <u>Augmented Matrix</u> of the system ①

It contains complete information about ①

# Mathematical Representations

**General Form** A system of $m$ linear equations in $n$ variables, or unknowns, has the general form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned} \tag{1}$$

The **coefficients** of the variables in the linear system (1) can be abbreviated as $a_{ij}$, where $i$ denotes the row and $j$ denotes the column in which the coefficient appears. For example, $a_{23}$ is the coefficient of the unknown in the second row and third column (that is, $x_3$). Thus, $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. The numbers $b_1, b_2, \ldots, b_m$ are called the **constants** of the system. If all the constants are zero, the system (1) is said to be **homogeneous**; otherwise it is **nonhomogeneous**. For example,

this system is homogeneous

$$\begin{aligned}
5x_1 - 9x_2 + x_3 &= 0 \\
x_1 + 3x_2 \qquad &= 0 \\
4x_1 + 6x_2 - x_3 &= 0
\end{aligned}$$

this system is nonhomogeneous

$$\begin{aligned}
2x_1 + 5x_2 + 6x_3 &= 1 \\
4x_1 + 3x_2 - \phantom{6}x_3 &= 9.
\end{aligned}$$

A **solution** of (1) is a set of numbers $x_1, \cdots, x_n$ that satisfies all the $m$ equations. A **solution vector** of (1) is a vector $\mathbf{x}$ whose components form a solution of (1). If the system (1) is homogeneous, it always has at least the **trivial solution** $x_1 = 0, \cdots, x_n = 0$.

**Matrix Form of the Linear System (1).** From the definition of matrix multiplication we see that the $m$ equations of (1) may be written as a single vector equation

$$\mathbf{Ax} = \mathbf{b} \tag{2}$$

where the **coefficient matrix** $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors. We assume that the coefficients $a_{jk}$ are not all zero, so that $\mathbf{A}$ is not a zero matrix. Note that $\mathbf{x}$ has $n$ components, whereas $\mathbf{b}$ has $m$ components. The matrix

The augmented matrix denoted by

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & : b_1 \\ & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & : b_m \end{bmatrix}$$

The system of equations are consistent if $\rho(A) = \rho[A: B]$

Inconsistent if $\rho(A) \neq \rho[A: B]$ (does not possess a solution)

Unique solution if $\rho(A) = \rho[A: B] = r = n$, n being the number of unknowns.

Infinite solution if $\rho(A) = \rho[A: B] = r < n$

# Solution of the system

- A solution is a set of numbers that satisfies all the m equations. A solution vector is a vector x whose components form a solution of the system.
- If the system is homogeneous, it always has at least the trivial solution $x_1 = 0, \ldots\ldots, x_n = 0$.

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1). The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix $\mathbf{A}$ but came from vector $\mathbf{b}$. Thus, we *augmented* the matrix $\mathbf{A}$.

*Note that **the augmented matrix** $\tilde{\mathbf{A}}$ **determines the system** (1) **completely*** because it contains all the given numbers appearing in (1).

**EXAMPLE 1** **Geometric Interpretation. Existence and Uniqueness of Solutions**

If $m = n = 2$, we have two equations in two unknowns $x_1, x_2$

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

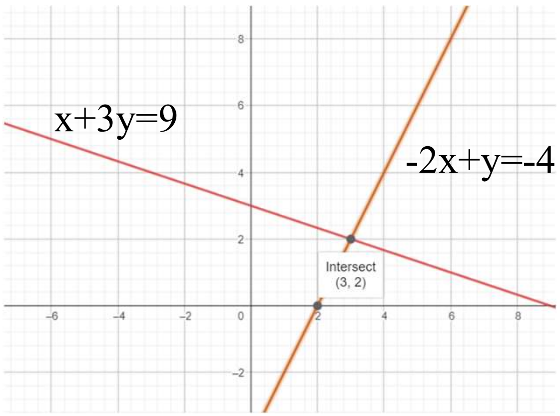
If we interpret $x_1, x_2$ as coordinates in the $x_1x_2$-plane, then each of the two equations represents a straight line, and $(x_1, x_2)$ is a solution if and only if the point $P$ with coordinates $x_1, x_2$ lies on both lines. Hence there are three possible cases (see Fig. 158 on next page):

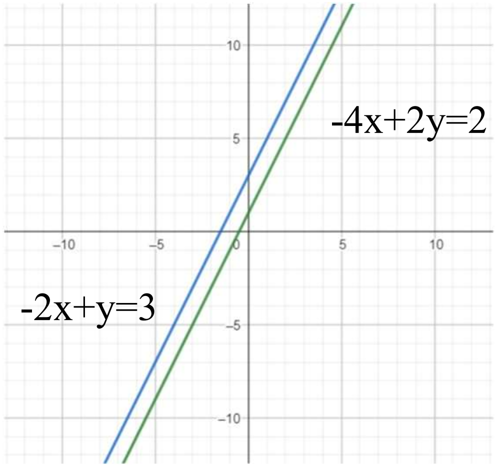
(*a*) Precisely one solution if the lines intersect

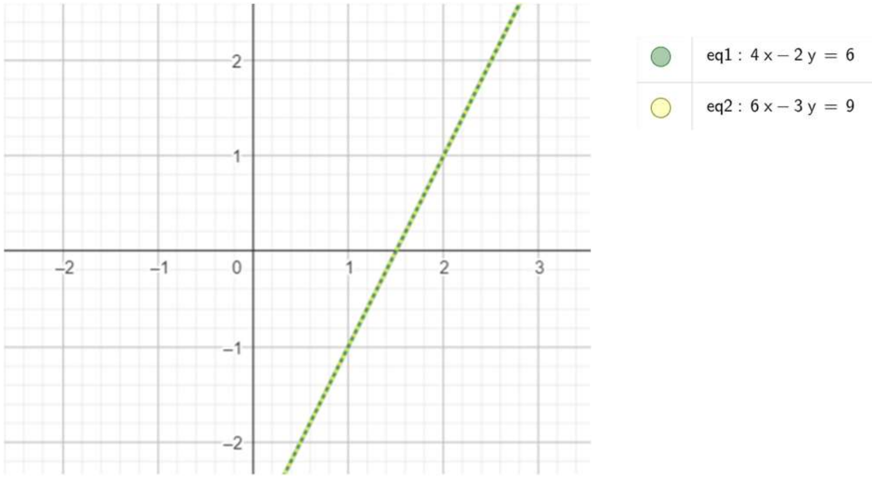
(*b*) Infinitely many solutions if the lines coincide

(*c*) No solution if the lines are parallel

# Solutions for system of linear equations




*Unique solution*

$x + 3y = 9$

$-2x + y = -4$

Lines intersect at (3, 2)

Unique solution:

$x = 3, y = 2.$

*No solution*

$-2x + y = 3$

$-4x + 2y = 2$

Lines are parallel.

No point of intersection.

No solutions.

*Many solution*

$4x - 2y = 6$

$6x - 3y = 9$

Both equations have the same graph.

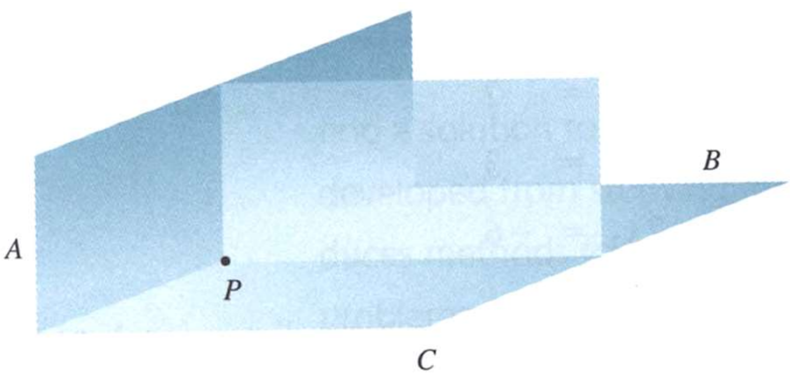
Any point on the graph is a solution.

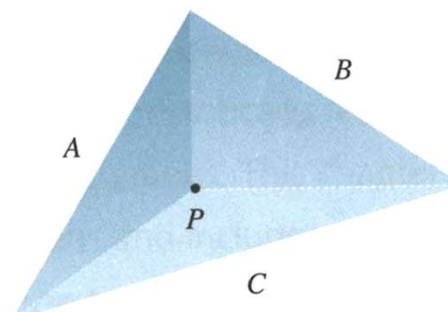
Many solutions.

# System of three linear equations in three variables

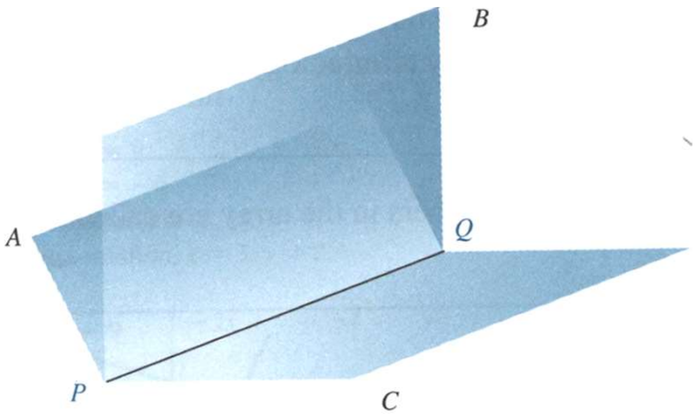
- A linear equation in three variables corresponds to a plane in three-dimensional space.

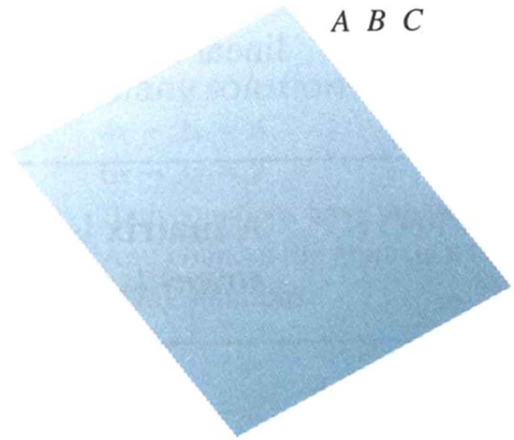
- Unique solution



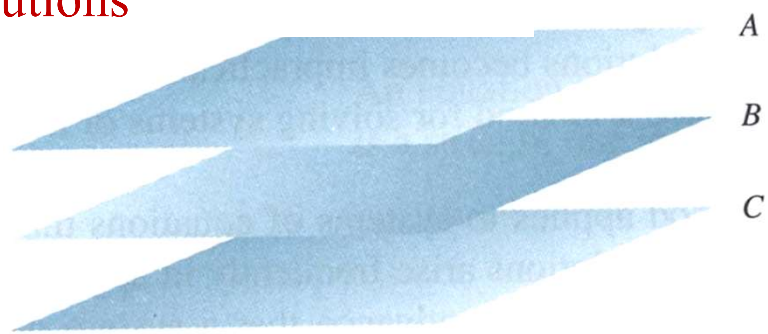



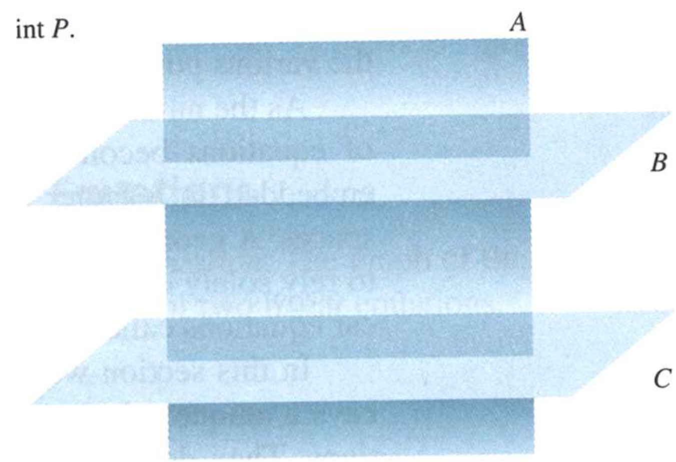
- Many solutions




- No solutions



1. Test the consistency and solve the system of linear equation

$$x + y + z = 6, \quad x - y + 2z = 5, \quad 3x + y + z = 8$$

Ans: $[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$

$R_2 \to -R_1 + R_2$

$R_3 \to -3R_1 + R_3$ $\quad [A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$

$R_3 \to -R_2 + R_3$

$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$

$$\rho(A) = 3, \qquad \rho[A : B] = 3$$

The given system of linear equations is consistent

The above matrix is converted into system of linear equations

$$x + y + z = 6,$$

$$-2y + z = -1,$$

$$-3z = -9$$

Solving we get $x = 1, \quad y = 2, \; z = 3$

2. Test the consistency and solve the system of linear equation

$$x + 2y + 3z = 14, \quad 4x + 5y + 7z = 35, \quad 3x + 3y + 4z = 21$$

Ans: $[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 4 & 5 & 7 & : & 35 \\ 3 & 3 & 4 & : & 21 \end{bmatrix}$

$R_2 \to -4R_1 + R_2$

$R_3 \to -3R_1 + R_3$ $\qquad [A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -1 \\ 0 & -3 & -5 & : & -21 \end{bmatrix}$

$R_3 \to -R_2 + R_3$ $\qquad [A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$

$$\rho(A) = 2, \qquad \rho[A{:}B] = 2$$

The given system of linear equations is consistent and will have infinite solution

The above matrix is converted into system of linear equations

$$x + 2y + 3z = 14,$$

$$-3y - 5z = -21$$

Put $z = k$ (arbitrary constant)

$-3y - 5k = -21,$ $\qquad y = 7 - \frac{5k}{3}$

$x + 27 - \frac{5k}{3} + 3k = 14,$ $\qquad x = \frac{k}{3}$

3. Solve the following linear equation by rank method

$$4x - 2y + 5z = 6$$
$$3x + 3y + 8z = 4$$
$$x - 5y - 3z = 5$$

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 3 & 3 & 8 \\ 1 & -5 & -3 \end{pmatrix} \quad B = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

$$[A, B] \sim \left(\begin{array}{ccc:c} 4 & -2 & 5 & 6 \\ 3 & 3 & 8 & 4 \\ 1 & -5 & -3 & 5 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & -5 & -3 & 5 \\ 0 & 18 & 17 & -11 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$\rho(A) = 2$ and $\rho([A|B]) = 3$.

$\therefore$ The system is inconsistent and it has no solution.

4. Solve the following linear equation by rank method

$$x + 9y - z = 27$$

$$x - 8y + 16z = 10$$

$$2x + y + 15z = 37$$

$$A = \begin{pmatrix} 1 & 9 & -1 \\ 1 & -8 & 16 \\ 2 & 1 & 15 \end{pmatrix} \qquad B = \begin{pmatrix} 27 \\ 10 \\ 37 \end{pmatrix}$$

$$[A, B] \sim \left(\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 1 & -8 & 16 & 10 \\ 2 & 1 & 15 & 37 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & -17 & 17 & -17 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here $\rho(A) = \rho([A|B]) = 2 < 3$
$\therefore$ The system is consistent and it has infinitely many solutions.

Solving the system,

$$\sim \left(\begin{array}{ccc|c} 1 & 9 & -1 & 27 \\ 0 & -17 & 17 & -17 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From the 1st row, $x + 9y - z = 27$ ---(1)

From the 2nd row, $17y + 17z = -17$ ---(2)

Dividing by 17, we get

$$y + z = -1$$

Put $z = t$

$$y = -1 - t$$

By applying the value of y and z in (1), we get

$$x = 36 - 8t$$

$$x = 36 - 8t,\ y = -1 - t \text{ and } z = t \text{ where } t \in \text{Real numbers.}$$

5. Investigate the values of $\alpha$ and $\beta$ such that the system of equations

$x+y+z=6, \quad x+2y+3z=10, \quad x+2y+\alpha z=\beta$ may have

i) Unique solution ii) Infinite solution iii) No solution

Ans: $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 3 & : & 10\\ 1 & 2 & \alpha & : & \beta\end{bmatrix}$

$R_2 \to -R_1 + R_2$

$R_3 \to -R_1 + R_3 \qquad [A:B]=$

$$\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \alpha-1 & : & \beta-6\end{bmatrix}$$

$R_3 \to -R_2 + R_3 \qquad [A:B]=$

$$\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 0 & \alpha-3 & : & \beta-10\end{bmatrix}$$

a) Unique solution : $\rho(A)=\rho[A:B]=3$

if $\rho(A)=3$ then $\alpha-3\neq 0$ since the other entries in the last row of A are zero.

If $\alpha-3\neq 0$ or $\alpha\neq 3$ irrespective of the value of $\beta$, $\rho[A:B]$ will also be 3

The system will have unique solution if $\alpha\neq 3$

b) Infinite solution: $\rho(A)=\rho[A:B]=r<3$, we must have r = 2 since first row and second row are non zero

$\rho(A)=\rho[A:B]=2$ only when the last row of $[A:B]$ is completely zero. This is possible if

$\alpha-3=0$ , $\beta-10=0$

The system will have infinite solution if $\alpha=3, \quad \beta=10$

c) No solution: we must have $(A)\neq\rho[A:B]$ , $\rho(A)=3$ if $\alpha\neq 3$ and hence if $\alpha=3$ we obtain $\rho(A)=2$, if we impose $\beta-10\neq 0$ then $\rho[A:B]$ will be 3

the system has no solution if $\alpha=3, \quad \beta\neq 10$

# Exercise problems

1) Solve the system of linear equations using the matrix method

$$x + 2y + 3z = 5$$

$$7x + 11y + 13z = 17$$

$$19x + 23y + 29z = 31$$

2) Test the consistency and solve the following system of equations

$$2x - y + 3z = 8$$

$$-x + 2y - z = 4$$

$$3x + y - 4z = 0$$

3) Find the value of $k$ so that the equations $x + y + 3z = 0$, $4x + 3y + kz = 0$, $2x + y + 2z = 0$ have a non-trivial solution.

# Exercise problems

4) Determine $b$ such that the system of homogenous equations $2x + y + 2z = 0$, $x + y + 3z = 0$, $4x + 3y + bz = 0$ has

a) Trivial solution

b) Non-trivial solution. Find the nontrivial solution by using the matrix method

5) Find the value of $\mu$, the system possesses a solution. Solve completely in each case $x + y + z = 1$, $x + 2y + 4z = \mu$, $x + 4y + 10z = \mu^2$

6) Test the following system of equations are consistent $x + 2y + 3z = 1$, $2x + 3y + 8z = 2$, $x + y + z = 3$

## Gauss Elimination Method :-

1) Given :- $2x_1 + 5x_2 = 2 \longrightarrow ①$

$$13x_2 = -26 \longrightarrow ②$$

Here, from the ② equation

we have $x_2 = \frac{-26}{13} = -2.$

substituting this value of $x_2$ in the 1st equation,

we get

$$2x_1 + 5(-2) = 2$$

$$\Rightarrow 2x_1 = 2+10$$

$$\Rightarrow 2x_1 = 12$$

$$\Rightarrow \boxed{x_1 = 6.}$$

This method is called "Backward Substitution Method".

2)

$$\left.\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30 \end{aligned}\right\} \longrightarrow (S_1)$$

Sol) The augmented matrix of $S_1$ is

Pivot ①

$$\tilde{A} = \left[\begin{array}{cc|c} \boxed{2} & 5 & 2 \\ -4 & 3 & -30 \end{array}\right]$$

Apply Row operation

$R_2 \longrightarrow R_2 + 2R_1$.

$$\sim \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right] \equiv \begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \\ x_2 &= -2 \end{aligned}$$

The new system thus obtained is in triangular form

Again as in problem ①, one can solve for $x_2$ and use Backward substitution method to get $x_1$

$$x_1 = 6.$$

# Solution of a system of Non homogenous equation:

## Gauss elimination method:

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution.

**Working procedure**:

Consider the system of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The system is equivalent to the matrix equation

AX=B

Where A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ X= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ B= $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

In this method, reducing the coefficient matrix 'A' to an upper triangular matrix.

Consider a new matrix comprising all the elements of the matrix 'A' along the elements of the column matrix 'B' such matrix denoted by $[A \vdots B]$ is called augmented matrix.

$$[A \vdots B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \vdots b_1 \\ a_{21} & a_{22} & a_{23} \vdots b_2 \\ a_{31} & a_{32} & a_{33} \vdots b_3 \end{bmatrix}$$

Step-I: Use the element $a_{11}(\neq 0)$ to make the elements $a_{21}$ and $a_{31}$ zero by elementary row transformations, this transforms $[A \vdots B]$ into the form

$$[A \vdots B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \vdots b_1 \\ 0 & a'_{22} & a'_{23} \vdots b'_2 \\ 0 & a'_{32} & a'_{33} \vdots b'_3 \end{bmatrix}$$

Step-II: Use the element $a'_{22}$ ($\neq$0) to make the elements $a'_{32}$ zero by elementary row transformations, this transforms $[A \vdots B]$ into the form

$$[A \vdots B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \vdots b_1 \\ 0 & a'_{22} & a'_{23} \vdots b'_2 \\ 0 & 0 & a''_{33} \vdots b''_3 \end{bmatrix} \ldots\ldots\ldots\ldots \text{(i)}$$

From (i) the given system of linear equations is equivalent to the system equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a'_{22}x_2 + a'_{23}\,x_3 &= b'_2 \\ a''_{33}x_3 &= b''_3 \end{aligned}$$

We get '$x_3$' from the last equation and by back substitution, we get $x_2$ and $x_1$.The values $x_1, x_2, x_3$ so obtained constitutes the exact solution of the given system of equations.

1. Solve the following system of linear equations by Gauss elimination method
4x+y+z=4, x+4y-2z=4, 3x+2y-4z=6.

**Sol**: $$[A \vdots B] = \begin{bmatrix} 4 & 1 & 1 & \vdots 4 \\ 1 & 4 & -2 & \vdots 4 \\ 3 & 2 & -4 & \vdots 6 \end{bmatrix}$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 4 & -2 & \vdots 4 \\ 4 & 1 & 1 & \vdots 4 \\ 3 & 2 & -4 & \vdots 6 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 4 & -2 & \vdots\ 4 \\ 0 & -15 & 9 & \vdots -12 \\ 0 & -10 & 2 & \vdots -6 \end{bmatrix} \qquad R_2 = R_2 - 4R_1 \text{ , } R_3 = R_3 - 3R_1$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 4 & -2 & \vdots\ 4 \\ 0 & -5 & 3 & \vdots -4 \\ 0 & -5 & 1 & \vdots -3 \end{bmatrix} \qquad R_2 = R_2/3 \text{ , } R_3 = R_3/2$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 4 & -2 & \vdots\ 4 \\ 0 & -5 & 3 & \vdots -4 \\ 0 & 0 & -2 & \vdots\ 1 \end{bmatrix} \qquad R_3 = R_3 - 4R_2$$

x+4y-2z=4
-5y+3z=-4
-2z=1
We get z=-1/2, y=1/2, x=1.

2. Apply Gauss elimination method to solve the equation x+4y-z=5, x+y-6z=-12, 3x-y-z=4.

**Sol:** AX=B

$$[A \vdots B] = \begin{bmatrix} 1 & 4 & -1 \vdots 5 \\ 1 & 1 & -6 \vdots 4 \\ 3 & -1 & -1 \vdots 4 \end{bmatrix}$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 4 & -1 \vdots & 5 \\ 0 & -3 & -5 \vdots & -17 \\ 0 & -13 & 2 \vdots & -11 \end{bmatrix} \quad R_2 = R_2 - R_1 \text{ , } R_3 = R_3 - 3R_1$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 4 & -1 \vdots & 5 \\ 0 & -3 & -5 \vdots & -17 \\ 0 & 0 & 71 \vdots & 188 \end{bmatrix} \quad R_3 = 3R_3 - 13R_1$$

x+4y-z=5

-3y-5z=-17

71z=188

By solving above equations we get z=188/71, y=89/71, x=187/71.

3. Solve the equations by using the Gauss elimination method x + y + z=9, x-2y+3z=8, 2x+y-z=3.

**Sol:** AX=B

The augmented matrix of the system is:

$$[A \vdots B] = \begin{bmatrix} 1 & 1 & 1 & \vdots 9 \\ 1 & -2 & 3 & \vdots 8 \\ 2 & 1 & -1 & \vdots 3 \end{bmatrix}$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 9 \\ 0 & -3 & 2 & \vdots & -1 \\ 0 & -1 & -3 & \vdots & -15 \end{bmatrix} \qquad R_2 = R_2 - R_1 \text{ , } R_3 = R_3 - 2R_1$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 9 \\ 0 & -3 & 2 & \vdots & -1 \\ 0 & 0 & -11 & \vdots & -44 \end{bmatrix} \qquad R_3 = 3R_3 - R_2$$

x + y + z=9, -3y+2z=-1, -11z=-44

By solving above equations we get Z=4, y=3, x=2.

4. Solve $5x_1 + x_2 + x_3 + x_4 = 4$, $x_1 + 7x_2 + x_3 + x_4 = 12$ ,$x_1 + x_2 + 6x_3 + x_4 = -5$ ,
$x_1 + x_2 + x_3 + 4x_4 = -6$ by using Gauss elimination method.

**Sol:** AX=B

The augmented matrix of the system is:

$$[A \vdots B] = \begin{bmatrix} 5 & 1 & 1 & 1 & : & 4 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 1 & 1 & 1 & 4 & : & -6 \end{bmatrix}$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 5 & 1 & 1 & 1 & : & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 6 & 0 & -3 & : & 18 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix} \quad R_2 = R_2 - R_1 \ ,$$

$R_3 = R_3 - R_1, R_4 = R_4 - 5R_1$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix}$$

$R_2 = R_2/3$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix} \quad R_4 = R_4 + 2R_2$$

$$[A \vdots B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & 0 & -117 & : & 234 \end{bmatrix} \quad R_4 = 5R_4 + 4R_3$$

$x_1 + x_2 + x_3 + 4x_4 = -6$ , $2x_2 - 4x_4 = 6$ , $5x_3 - 3x_4 = 1$ , $-117x_4 = 234$ ,
$x_4 = -2$ ,$x_3 = -1$,$x_2 = 2, x_1 = 1$

**EXAMPLE 2** **Gauss Elimination. Electrical Network**

Solve the linear system

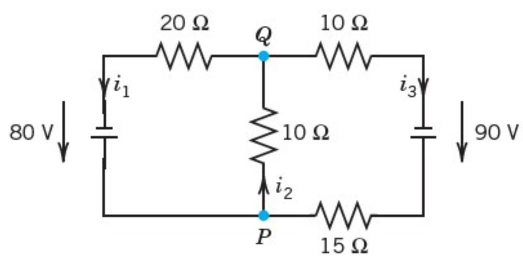
$$
\begin{aligned}
x_1 - \quad x_2 + \quad x_3 &= 0 \\
-x_1 + \quad x_2 - \quad x_3 &= 0 \\
10x_2 + 25x_3 &= 90 \\
20x_1 + 10x_2 \qquad &= 80.
\end{aligned}
$$

***Derivation from the circuit in Fig. 159 (Optional).*** This is the system for the unknown currents $x_1 = i_1$, $x_2 = i_2$, $x_3 = i_3$ in the electrical network in Fig. 159. To obtain it, we label the currents as shown, choosing directions arbitrarily; if a current will come out negative, this will simply mean that the current flows against the direction of our arrow. The current entering each battery will be the same as the current leaving it. The equations for the currents result from Kirchhoff's laws:

***Kirchhoff's Current Law (KCL).*** *At any point of a circuit, the sum of the inflowing currents equals the sum of the outflowing currents.*

***Kirchhoff's Voltage Law (KVL).*** *In any closed loop, the sum of all voltage drops equals the impressed electromotive force.*

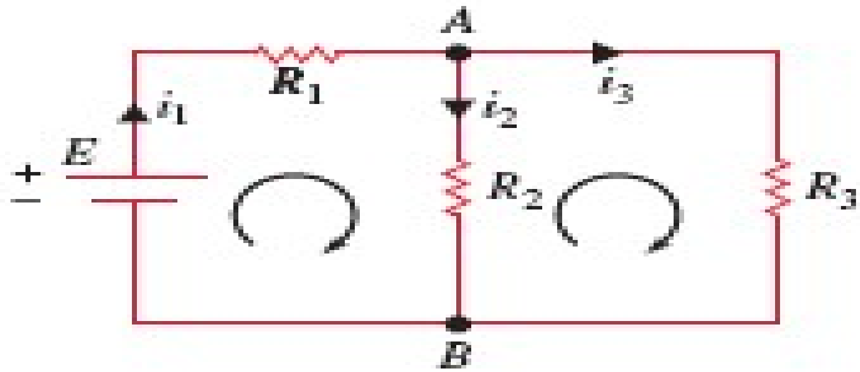
Node $P$ gives the first equation, node $Q$ the second, the right loop the third, and the left loop the fourth, as indicated in the figure.




| | |
|---|---|
| Node $P$: | $i_1 - i_2 + i_3 = 0$ |
| Node $Q$: | $-i_1 + i_2 - i_3 = 0$ |
| Right loop: | $10i_2 + 25i_3 = 90$ |
| Left loop: | $20i_1 + 10i_2 = 80$ |

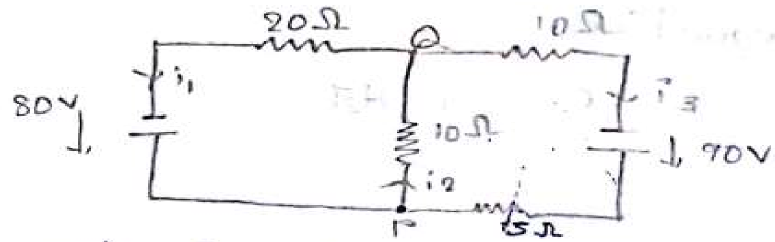
**Fig. 159.** Network in Example 2 and equations relating the currents

- ELECTRICAL NETWORK




- **CURRENT EQUATIONS FOR THE ABOVE CIRCUIT**

$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ E - i_1R_1 - i_2R_2 &= 0 \\ i_2R_2 - i_3R_3 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} i_1 \quad - i_2 \quad - i_3 \quad &= 0 \\ i_1R_1 + i_2R_2 \qquad &= E \\ i_2R_2 - i_3R_3 &= 0. \end{aligned}$$




At node P : $i_1 + i_3 = i_2 \implies i_1 - i_2 + i_3 = 0$

At node Q : $i_2 = i_1 + i_3 \implies -i_1 + i_2 - i_3 = 0.$

For the right loop : $10 i_2 + 25 i_3 = 90.$

For the left loop : $20 i_1 + 10 i_2 = 80.$

Pivot-1

$$\tilde{A} = \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$R_2 \to R_2 + R_1$ ; $R_4 \to R_4 - 20R_1$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$R_2 \longleftrightarrow R_4$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \longrightarrow 3R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95 & 190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\equiv \quad i_1 - i_2 + i_3 = 0 \longrightarrow (1)$$

$$30 i_2 - 20 i_3 = 80 \longrightarrow (2)$$

$$95 i_3 = 190 \longrightarrow (3)$$

$$i_3 = \frac{190}{95} = 2$$

$$30 i_2 = 80 + 20 i_3 = 80 + 40 = 120$$

$$i_2 = \frac{120}{30} = 4$$

$$i_1 = i_2 - i_3$$

$$= 4 - 2 = 2$$

$$\boxed{\therefore \ i_1 = 2}$$

Text book: Tenth edition

# PROBLEM SET 7.3

## 1–14 GAUSS ELIMINATION

Solve the linear system given explicitly or by its augmented matrix. Show details.

**1.** $$\begin{aligned} 4x - 6y &= -11 \\ -3x + 8y &= 10 \end{aligned}$$

**2.** $$\begin{bmatrix} 3.0 & -0.5 & 0.6 \\ 1.5 & 4.5 & 6.0 \end{bmatrix}$$

**3.** $$\begin{aligned} x + y - z &= 9 \\ 8y + 6z &= -6 \\ -2x + 4y - 6z &= 40 \end{aligned}$$

**4.** $$\begin{bmatrix} 4 & 1 & 0 & 4 \\ 5 & -3 & 1 & 2 \\ -9 & 2 & -1 & 5 \end{bmatrix}$$

**5.** $$\begin{bmatrix} 13 & 12 & -6 \\ -4 & 7 & -73 \\ 11 & -13 & 157 \end{bmatrix}$$

**6.** $$\begin{bmatrix} 4 & -8 & 3 & 16 \\ -1 & 2 & -5 & -21 \\ 3 & -6 & 1 & 7 \end{bmatrix}$$

**7.** $$\begin{bmatrix} 2 & 4 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 4 & 0 & 6 & 0 \end{bmatrix}$$

**8.** $$\begin{aligned} 4y + 3z &= 8 \\ 2x - z &= 2 \\ 3x + 2y &= 5 \end{aligned}$$

**9.** $$\begin{aligned} -2y - 2z &= -8 \\ 3x + 4y - 5z &= 13 \end{aligned}$$

**10.** $$\begin{bmatrix} 5 & -7 & 3 & 17 \\ -15 & 21 & -9 & 50 \end{bmatrix}$$

**11.** $$\begin{bmatrix} 0 & 5 & 5 & -10 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

**12.** $$\begin{bmatrix} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 5 & 15 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$$

**13.** $$\begin{aligned} 10x + 4y - 2z &= -4 \\ -3w - 17x + y + 2z &= 2 \\ w + x + y &= 6 \\ 8w - 34x + 16y - 10z &= 4 \end{aligned}$$

**14.** $$\begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 1 & -1 & 3 & -3 & 3 \\ 3 & 4 & -7 & 2 & -7 \end{bmatrix}$$

**15. Equivalence relation.** By definition, an *equivalence relation* on a set is a relation satisfying three conditions: (named as indicated)

**(i)** Each element $A$ of the set is equivalent to itself (*Reflexivity*).

**(ii)** If $A$ is equivalent to $B$, then $B$ is equivalent to $A$ (*Symmetry*).

**(iii)** If $A$ is equivalent to $B$ and $B$ is equivalent to $C$, then $A$ is equivalent to $C$ (*Transitivity*).

Show that row equivalence of matrices satisfies these three conditions. *Hint.* Show that for each of the three elementary row operations these conditions hold.

Text book: 9th edition

## EM SET 7.3

### 1–16 GAUSS ELIMINATION AND BACK SUBSTITUTION

Solve the following systems or indicate the nonexistence of solutions. (Show the details of your work.)

1. $5x - 2y = 20.9$
   $-x + 4y = -19.3$

2. $3.0x + 6.2y = 0.2$
   $2.1x + 8.5y = 4.3$

3. $0.5x + 3.5y = 5.7$
   $-x + 5.0y = 7.8$

4. $4y - 2z = 2$
   $6x - 2y + z = 29$
   $4x + 8y - 4z = 24$

5. $0.8x + 1.2y - 0.6z = -7.8$
   $2.6x + 1.7z = 15.3$
   $4.0x - 7.3y - 1.5z = 1.1$

6. $14x - 2y - 4z = 0$
   $18x - 2y - 6z = 0$
   $4x + 8y - 14z = 0$

7. $y + z = -2$
   $4y + 6z = -12$
   $x + y + z = 2$

8. $2x + y - 3z = 8$
   $5x + 2z = 3$
   $8x - y + 7z = 0$

9. $4y + 4z = 24$
   $3x - 11y - 2z = -6$
   $6x - 17y + z = 18$

10. $0.6x + 0.3y - 0.4z = -1.9$

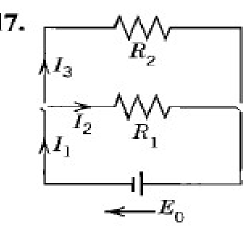
15. $3x + 7y - 4z = -46$
    $5w + 4x + 8y + z = 7$
    $8w + 4y - 2z = 0$
    $-w + 6x + 2z = 13$

16. $-2w - 17x + 4y + 3z = 0$
    $7w + 3y - 2z = 0$
    $2x + 8y - 6z = -20$
    $5w - 13x - y + 5z = 16$

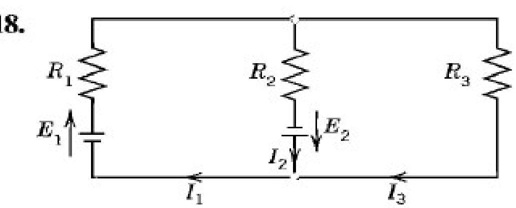
### 17–19 MODELS OF ELECTRICAL NETWORKS

Using Kirchhoff's laws (see Example 2), find the currents. (Show the details of your work.)

17.




18.




Thank
you