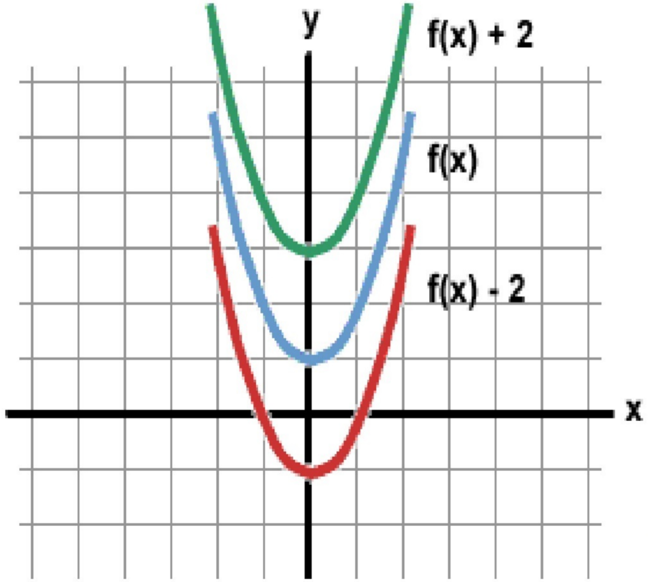
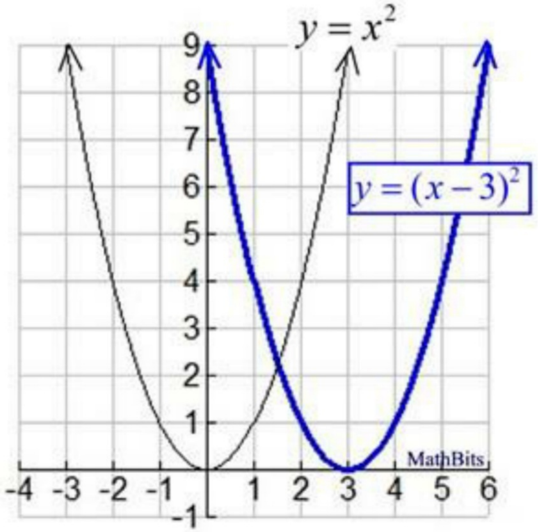
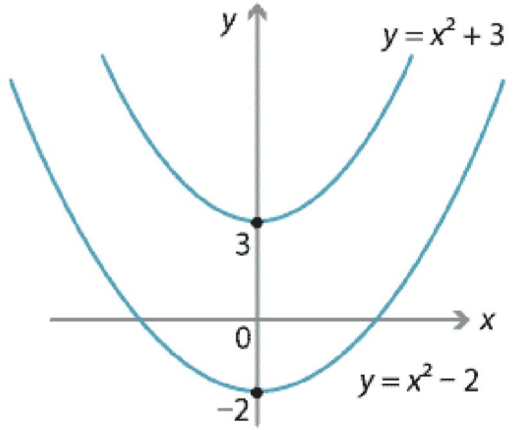
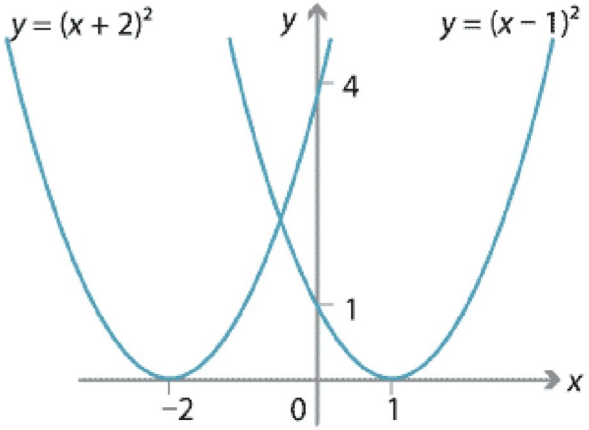
Graphical Method – S.V.O.P. – Using Shifting

- If x^* is the minimum of f(x), then x^* will also have the minimum of f(x)+a and Minimum value of [f(**x**)+a] = Minimum value of f(**x**) + a.
- If x^* is the minimum of f(x), then minimum of f(x+a) will be at x^*- a. The minimum values of f(x) and f(x+a) is the same.









Exercise

The function $f(x)$ has a global minimum at $x = 123$ and the minimum function value is 27.

Find: (a) the minimum point and minimum function value of

$$g(x) = 900 + f(x)$$

(b) the minimum point and minimum function value of

$$y(x) = f(x - 100)$$

(c) the maximum point and maximum function value of

$$h(x) = 500 - f(x)$$

~~H.W.~~ (d) the maximum point and maximum function value of

$$h(x) = -f(x + 1)$$

~~H.W.~~ (e) the maximum point and maximum function value of

$$h(x) = 9 - f(x - 100)$$

Exercise

The function $f(x)$ has a global minimum at $x = 123$ and the minimum function value is 27.

Find: (a) the minimum point and minimum function value of

$$g(x) = 900 + f(x)$$

Minimum at x=123, minimum value is 927

(b) the minimum point and minimum function value of

$$y(x) = f(x - 100)$$

Minimum at x=223, minimum value is 27

(c) the maximum point and maximum function value of

$$h(x) = 500 - f(x)$$

Maximum at x = 123, maximum value is 473

(d) the maximum point and maximum function value of

$$h(x) = -f(x + 1)$$

Maximum at x = 122, maximum value is -27

(e) the maximum point and maximum function value of

$$h(x) = 9 - f(x - 100)$$

Maximum at x = 223, maximum value is -18

Multivariable Unconstrained Optimization Problems

Minimize $f(x_1, x_2, \ldots, x_n)$

– Graphical method

– Analytical methods (or classical methods)

Graphical Method for solving constrained optimization problems

Steps involved:

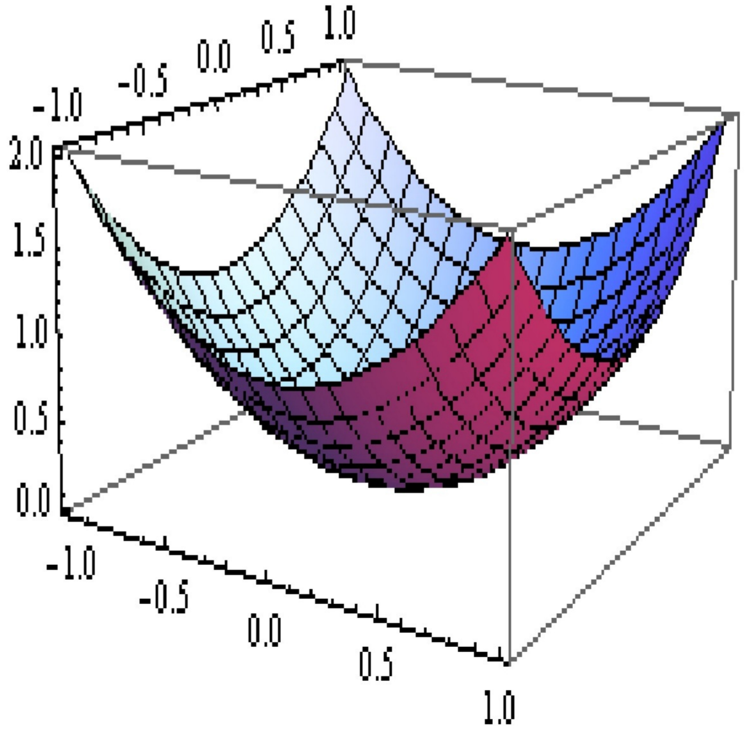
1. **Obtain the feasible region** – the region consisting of all the points that satisfy all the constraints.
2. Using **Objective contours** find the point(s) in the feasible region that gives the minimum or maximum value for the objective function

Surface Plots of some two variable functions

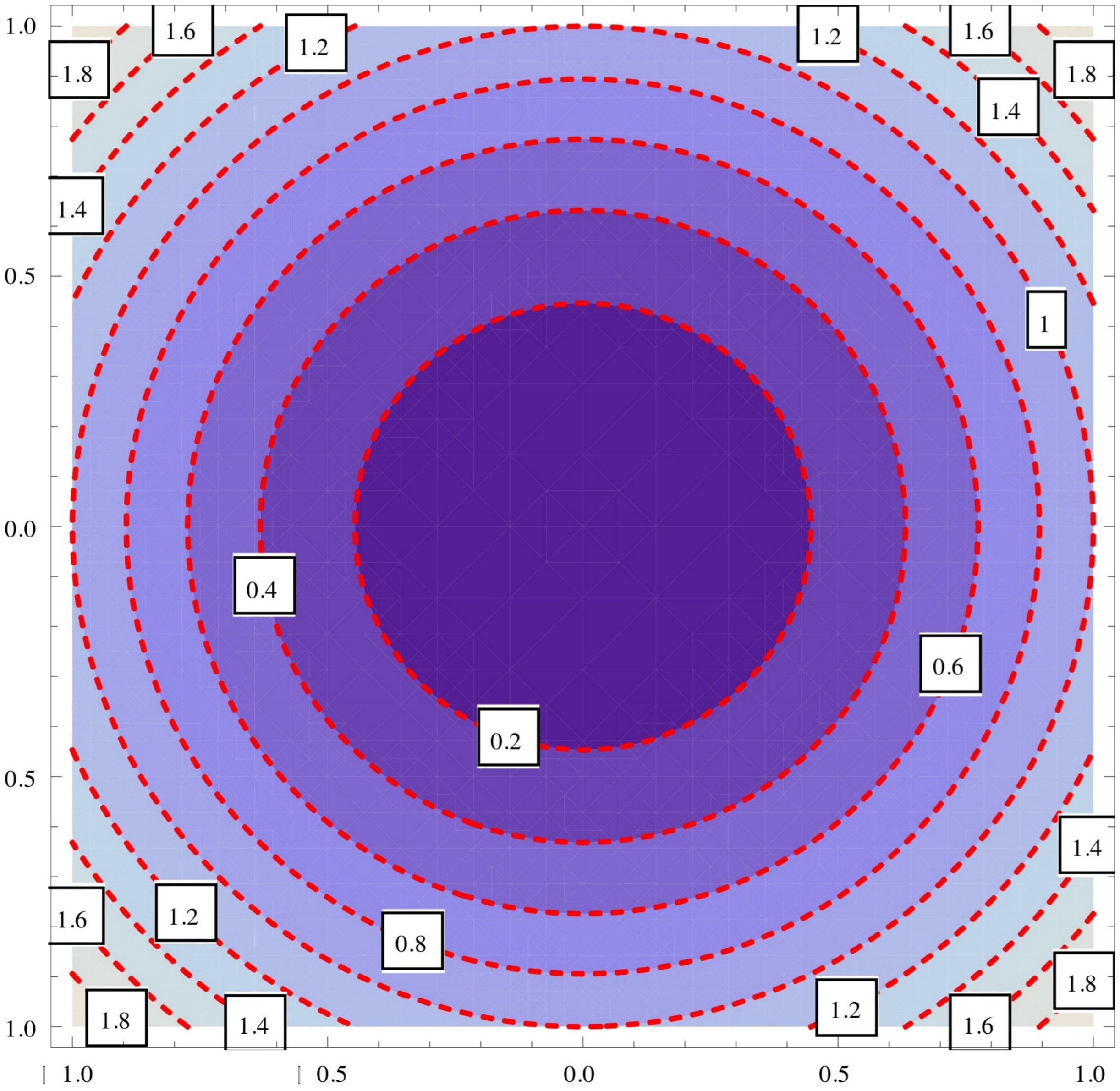
$f(x,y) = x^2 + y^2$

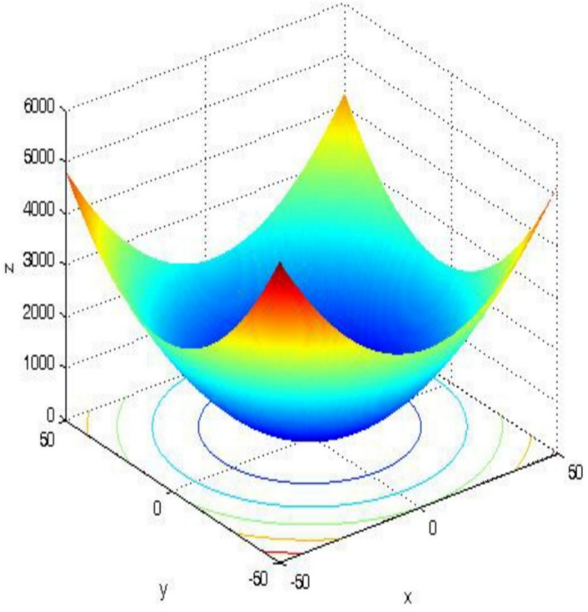
Local minimum and global minimum at $x^*=0$, $y^*=0$

Minimum value = $f(x^*, y^*)=0$



Contour Plot of $f(x,y) = x^2 + y^2$

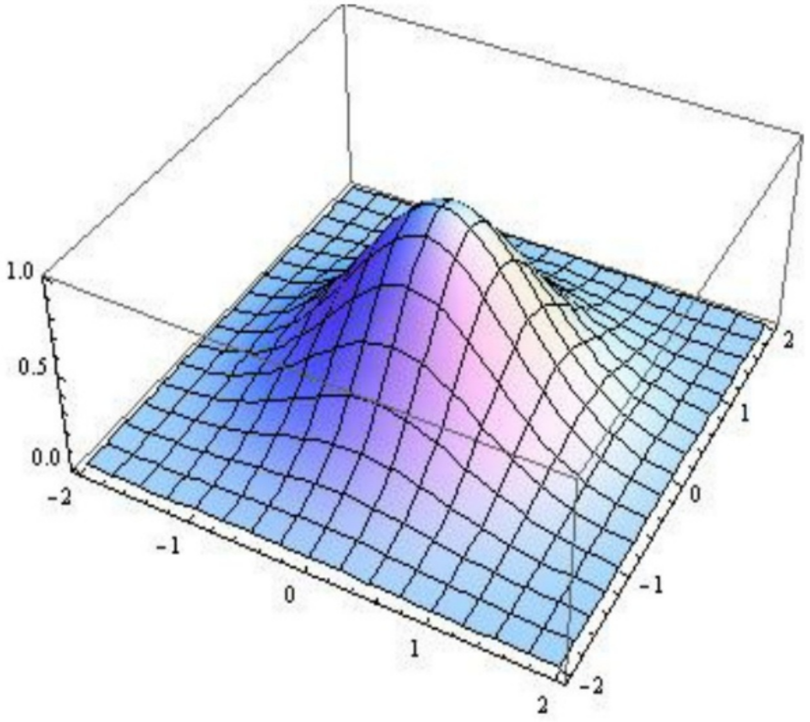


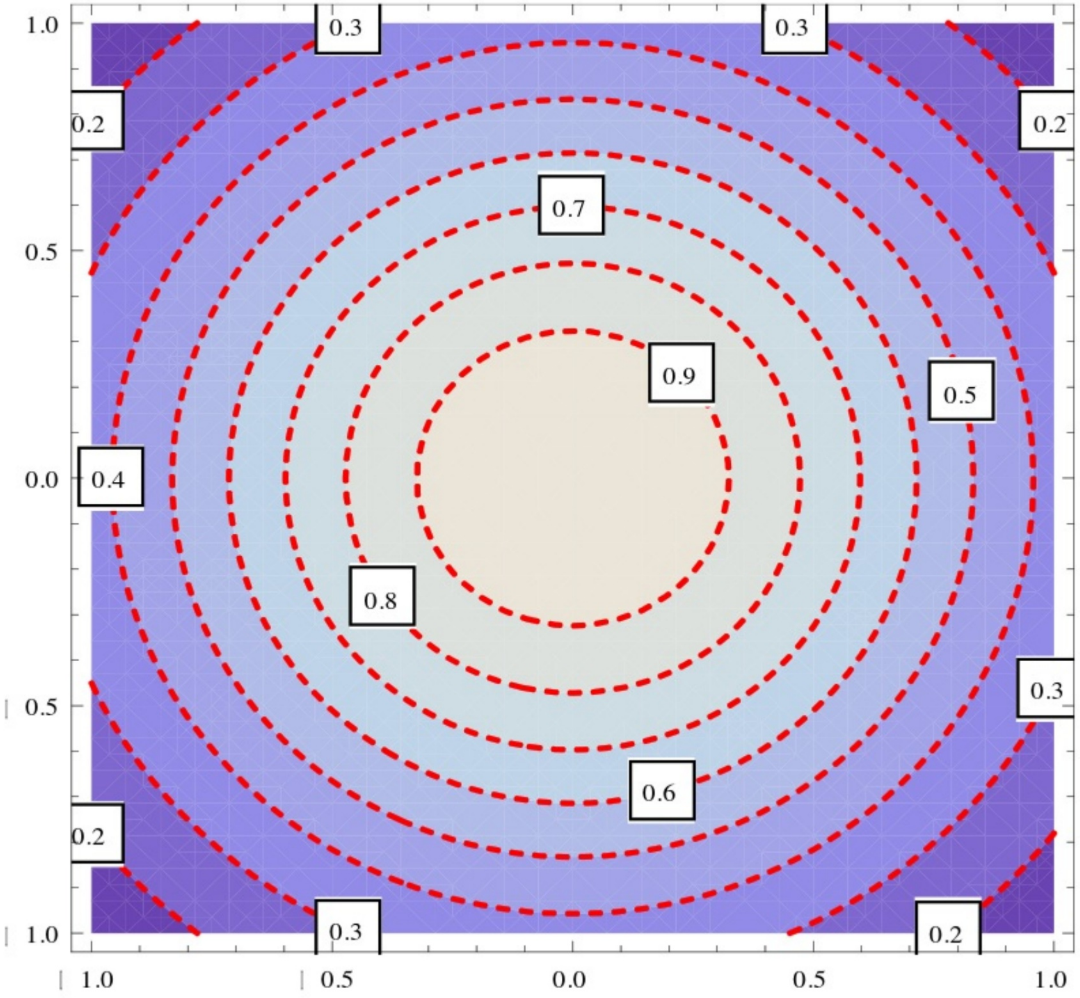


Surface Plots and Contour Plots of f(x,y) =Exp[-(x^2 + y^2)]

Local maximum and global maximum at $x^*=0$, $y^*=0$

maximum value = $f(x^*, y^*)=1$





Graphical Method – Linear Optimization Problems

1. Maximize $x + y$

subject to $x \leq 3$,

$y \leq 7$,

$x \geq 0$,

$y \geq 0$.

Graphical Method – Linear Optimization Problems

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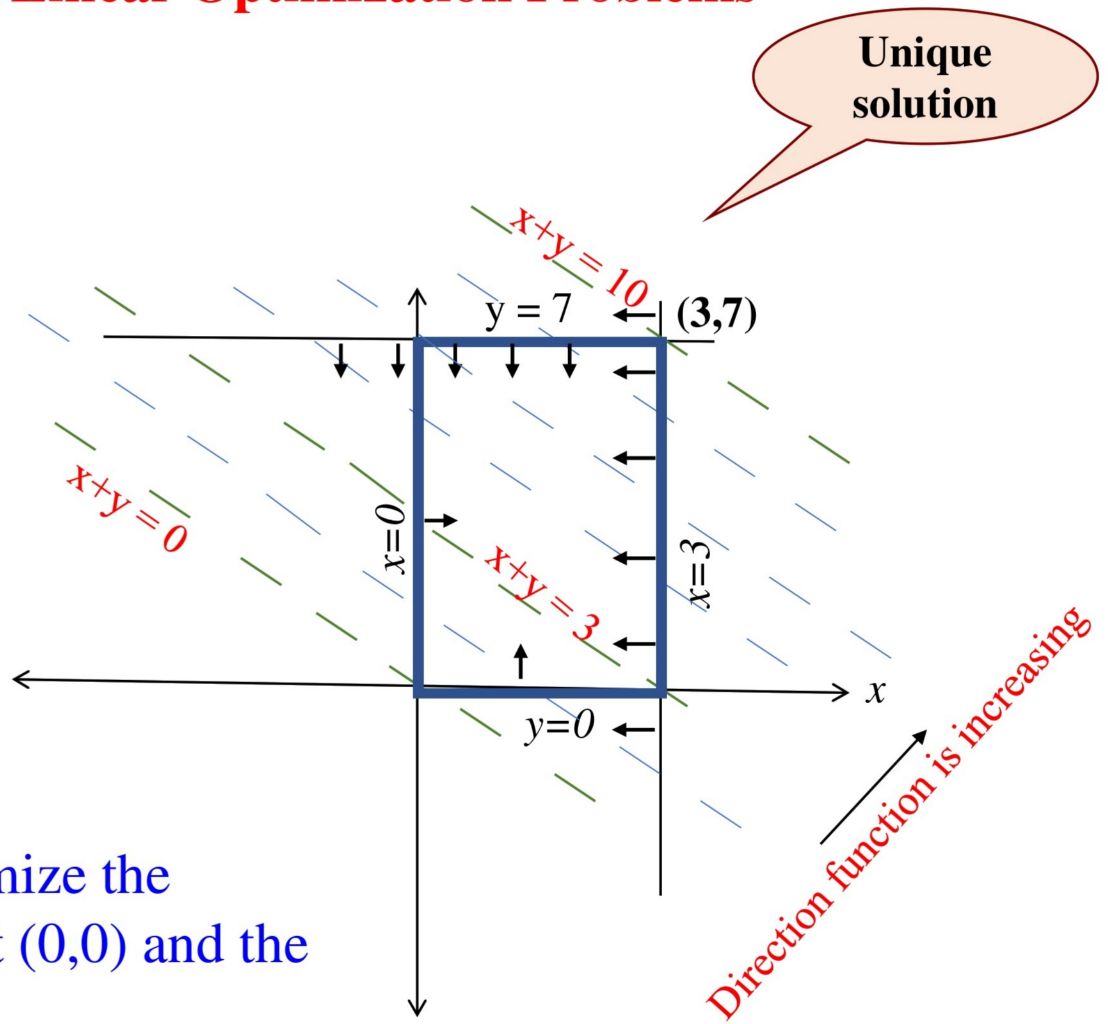

Maximum value is 10
at the point (3,7)

If the question was minimize the minimum point will be at (0,0) and the minimum value is 0

Note: The optimal point in a linear programming model always occurs at an extreme (corner) point on the boundary of the feasible solution area.

2. Maximize $2x - 3y$

subject to $x \leq 3$,

$y \leq 7$,

$x \geq 0$,

$y \geq 0$.

2. Maximize 2x - 3y

subject to $x \le 3$,

$y \le 7$,

$x \ge 0$,

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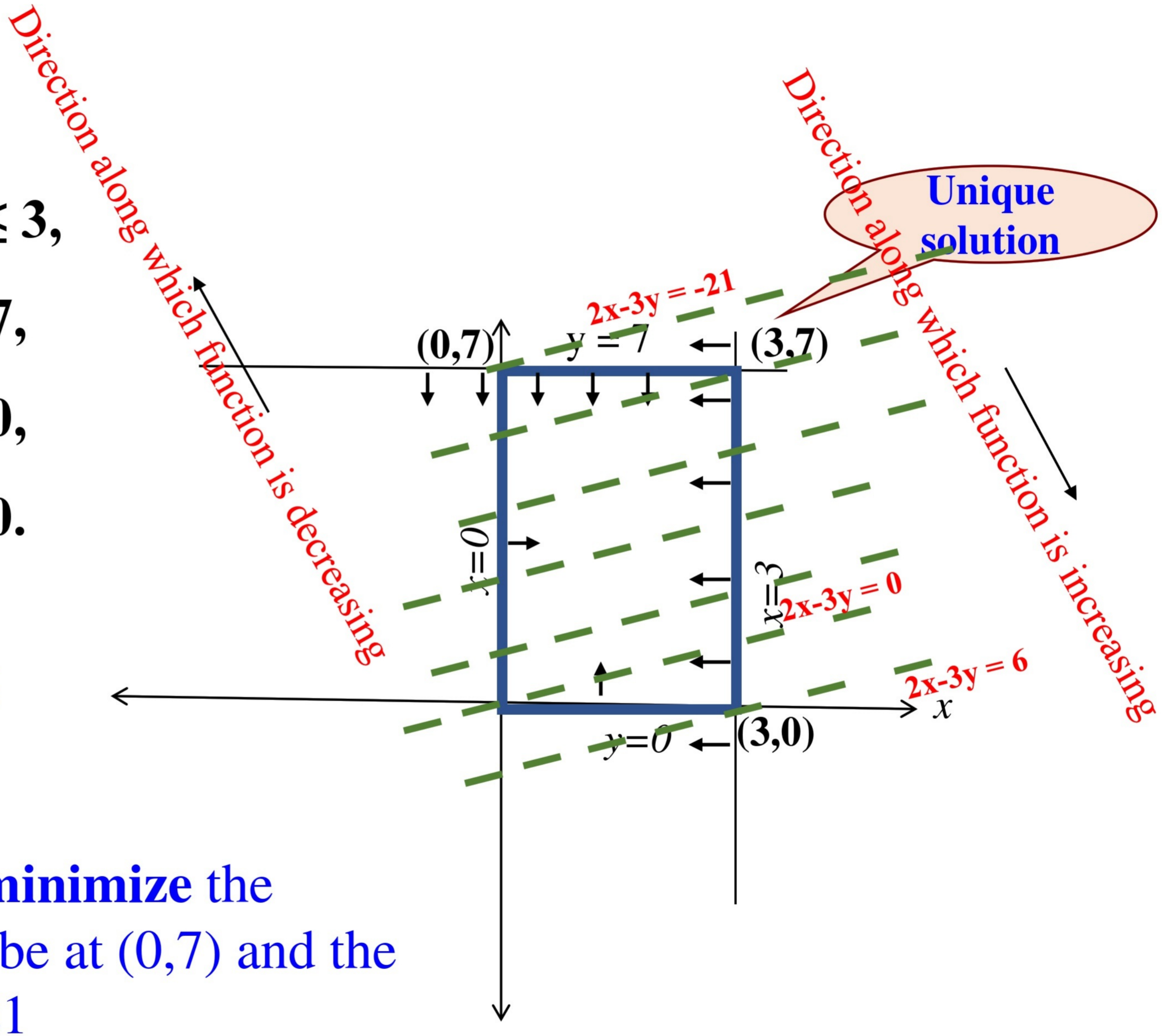


Maximum value is 6
at the point (3,0)

If the question was **minimize** the minimum point will be at (0,7) and the minimum value is -21

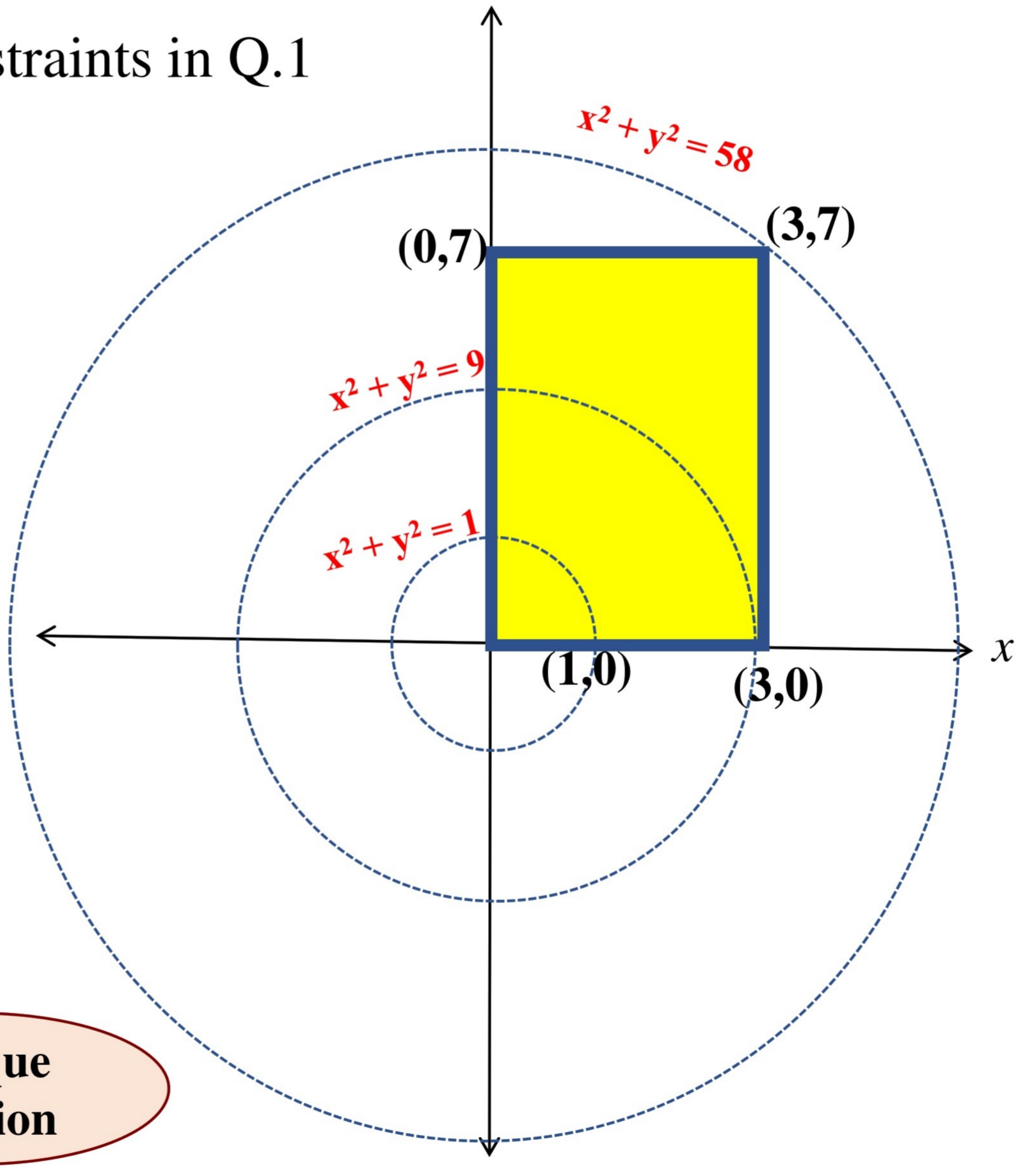
3. Minimize $x^2 + y^2$ for constraints in Q.1

Graphical Method

3. Minimize $x^2 + y^2$ for constraints in Q.1

Minimum value is 0
at the point (0,0)

If the question was **maximize** the Maximum point will be at (3,7) and the maximum value is 58



Unique solution

4. a. Maximize $x + y$

subject to the constraints,

$$y \leq 5,$$

$$x+y \leq 10$$

$$x \geq 0, \quad y \geq 0.$$

Graphical Method (Con….)

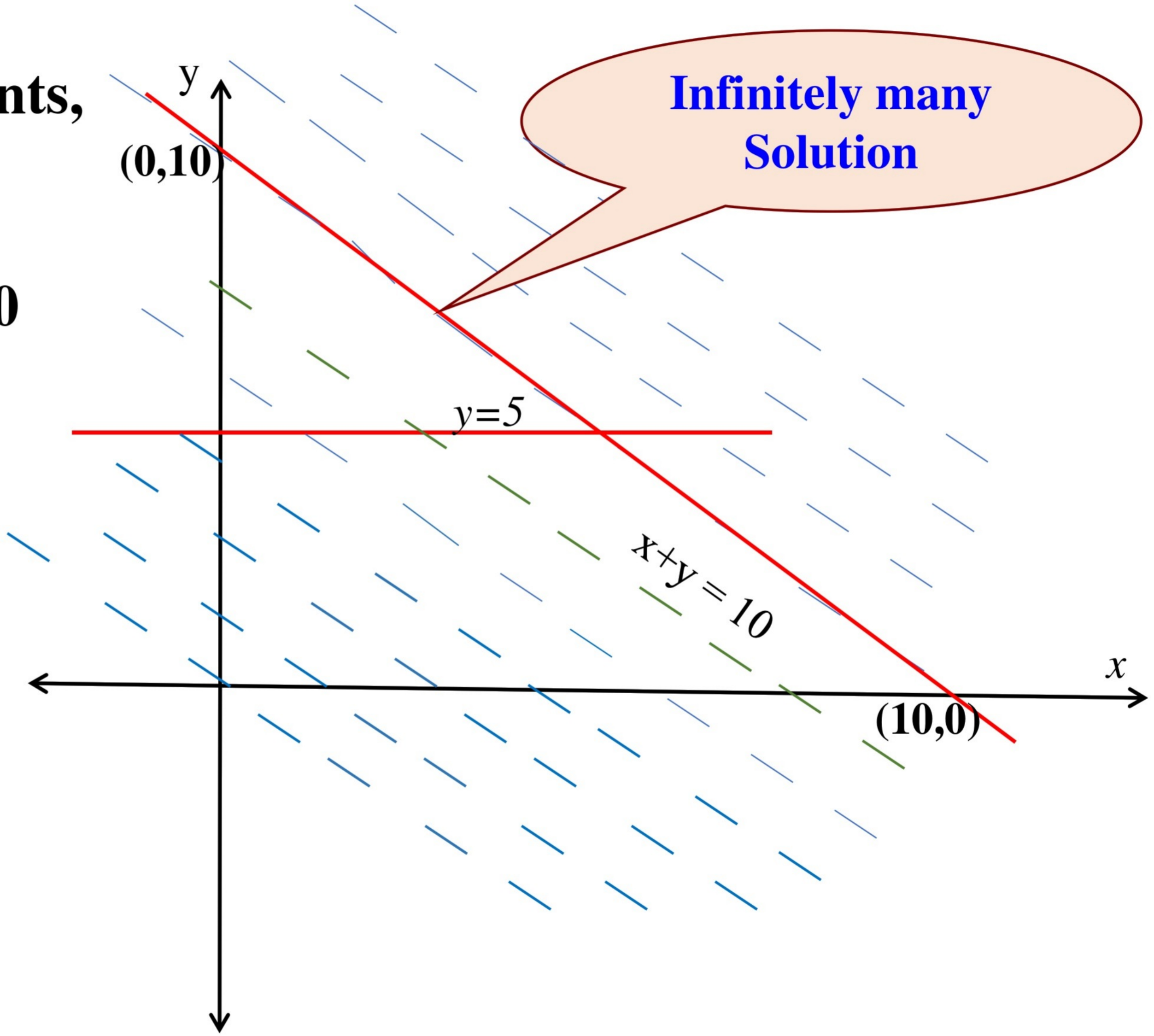
4. a. Maximize x + y

subject to the constraints,

$$y \le 5,$$

$$x + y \le 10$$

$$x \ge 0,\ y \ge 0.$$



In this case, contour line coincides with the side of the trapezium.

Optimal solution is on $\overline{BC}$

All points on $\overline{BC}$ [Infinitely many solutions]

$\therefore$ (5,5) (10,0)

$(1-\lambda)\underline{B} + \lambda(\underline{C}) \rightarrow$ all points on $\overline{BC}$ $0 \le \lambda \le 1$

Optimal solution is at $(1-\lambda)(5,5) + \lambda(10,0)$

$= [(5-5\lambda, 5-5\lambda)] + [10\lambda, 0]$

$= [5+5\lambda, 5-5\lambda], \ 0 \le \lambda \le 1$.

4. b. Maximize $3y$

subject to $x \leq 3$,

$$-x+y \leq 4,$$

$$x+y \leq 6$$

$$x \geq 0,\ \ y \geq 0.$$

Graphical Method

4. b. Maximize 3y

subject to $x \le 3$,

$-x+y \le 4$,

$x+y \le 6$

$x \ge 0, \; y \ge 0$.

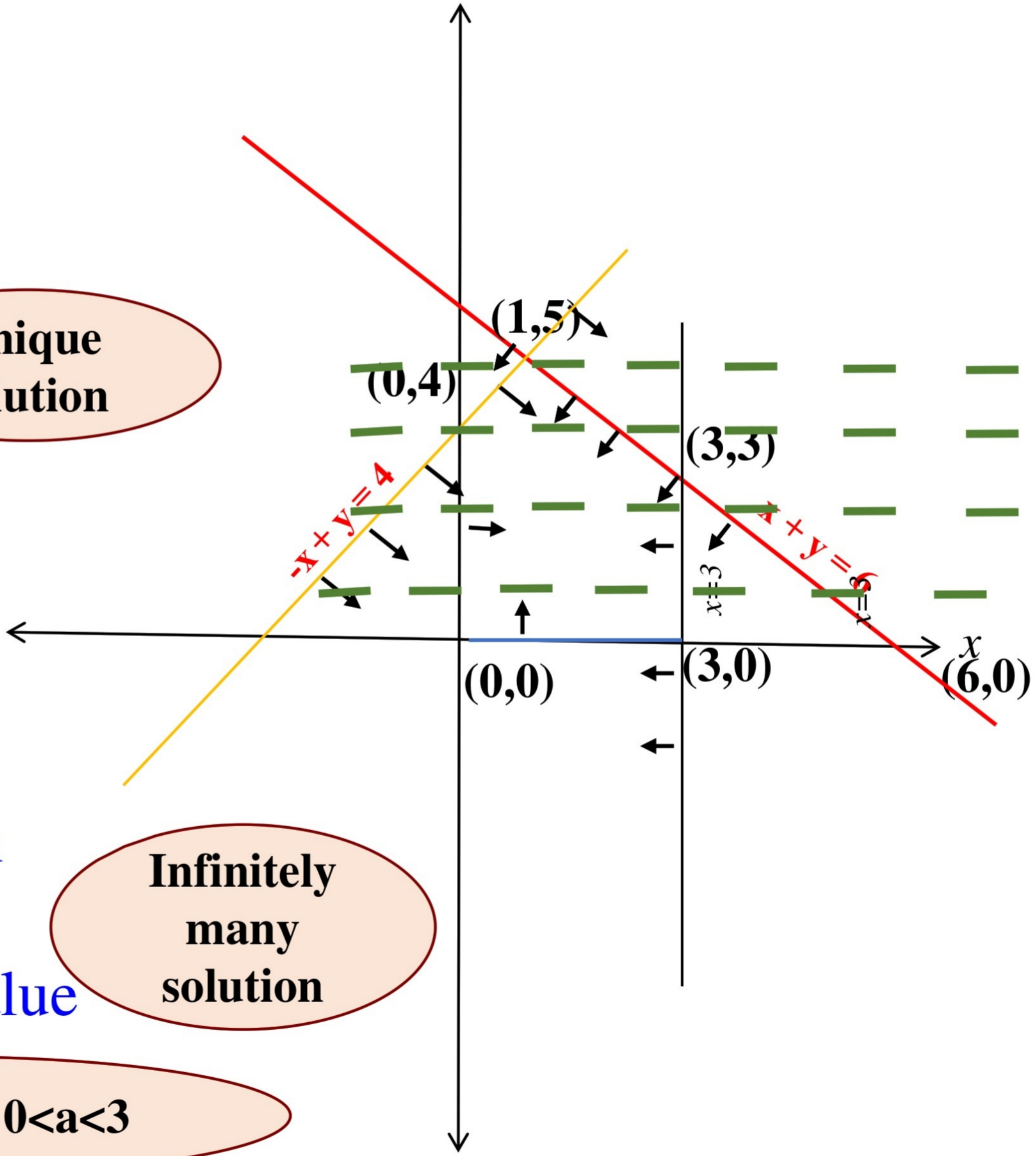
Unique solution

Maximum value is 15
at the point (1,5)

If the question was **minimize** the minimum point will be on the line segment joining (0,0) and(3,0) and the minimum value is 0.

Infinitely many solution

(x,y)= (a,0), 0<a<3



5. Maximize $x+y$

subject to $x \le 3$,

$$-x+y \le 4,$$

$$x+y \le 6$$

$$x \ge 0, \ y \ge 0.$$

Graphical Method

5. Maximize x+y

subject to $x \le 3$,

$-x+y \le 4$,

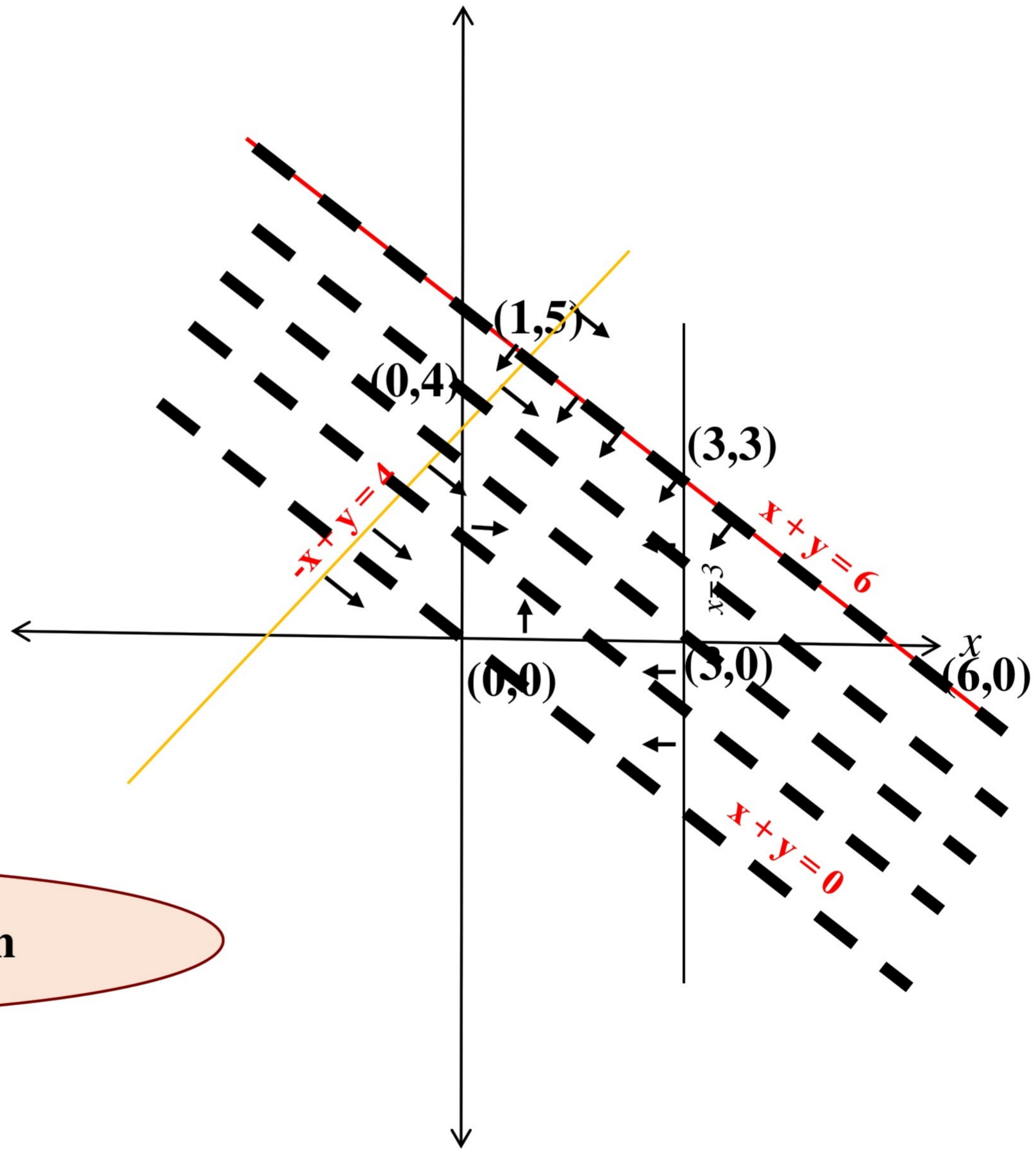
$x+y \le 6$

$x \ge 0$, $y \ge 0$.

Maximum value is 6 at every point on the line segment joining (1,5) and (3,3)

$(x,y)=(1-a)(1,5)+a(3,3)$

$=(1+2a, 5-2a), 0<a<1$



Infinitely many solution

6. Maximize $x + y$

subject to $x \leq 3$,

$x \geq 0$,

$y \geq 0$.

Graphical Method (con…)

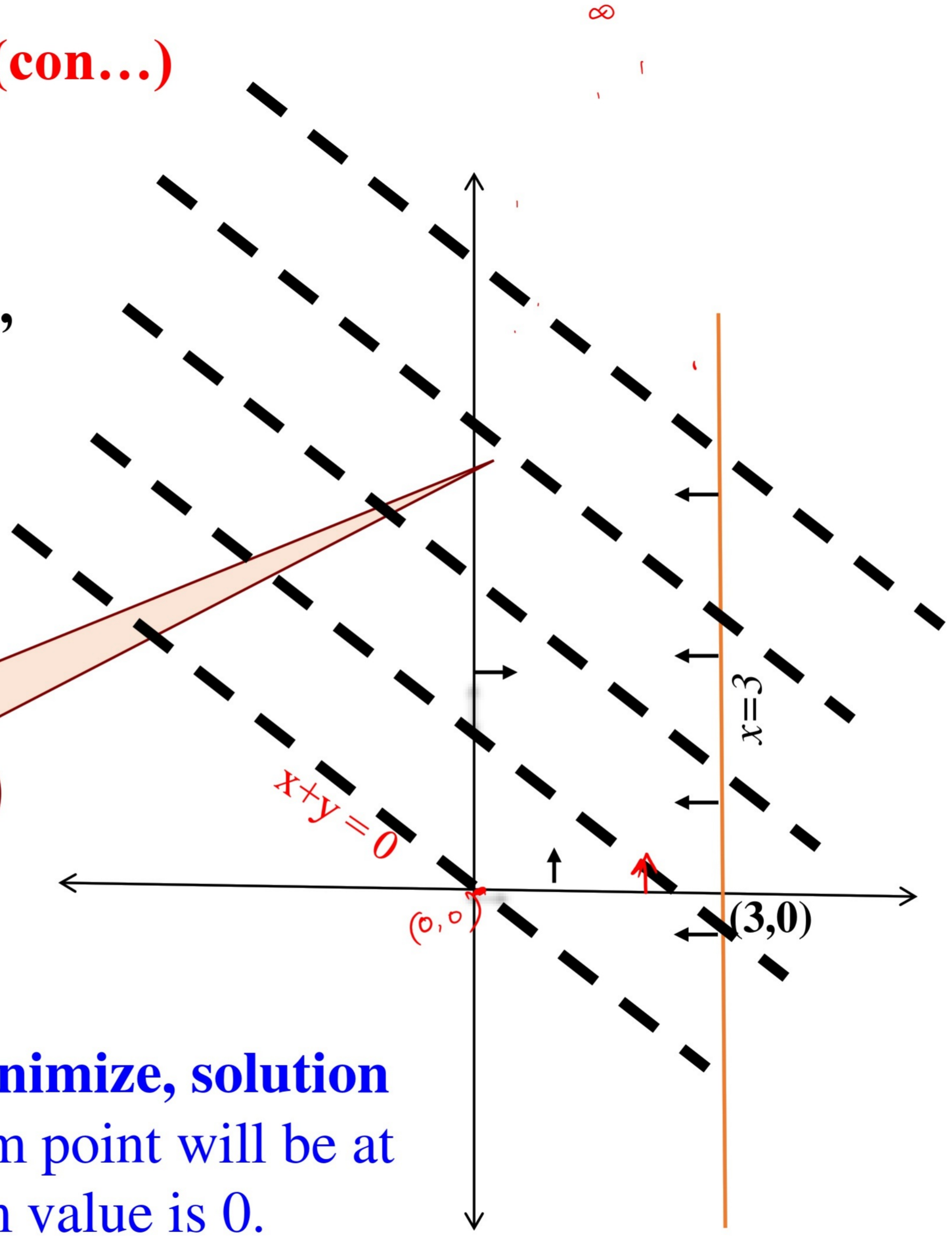
6. Maximize x + y

subject to $x \le 3$,

$x \ge 0$,

$y \ge 0$.

Unbounded Solution



If the question was **minimize, solution is unique** the minimum point will be at (0,0) and the minimum value is 0.

7. Minimize $3x + 5y$

subject to the constraints,

$$x + y \leq 2,$$

$$x + y \geq 3,$$

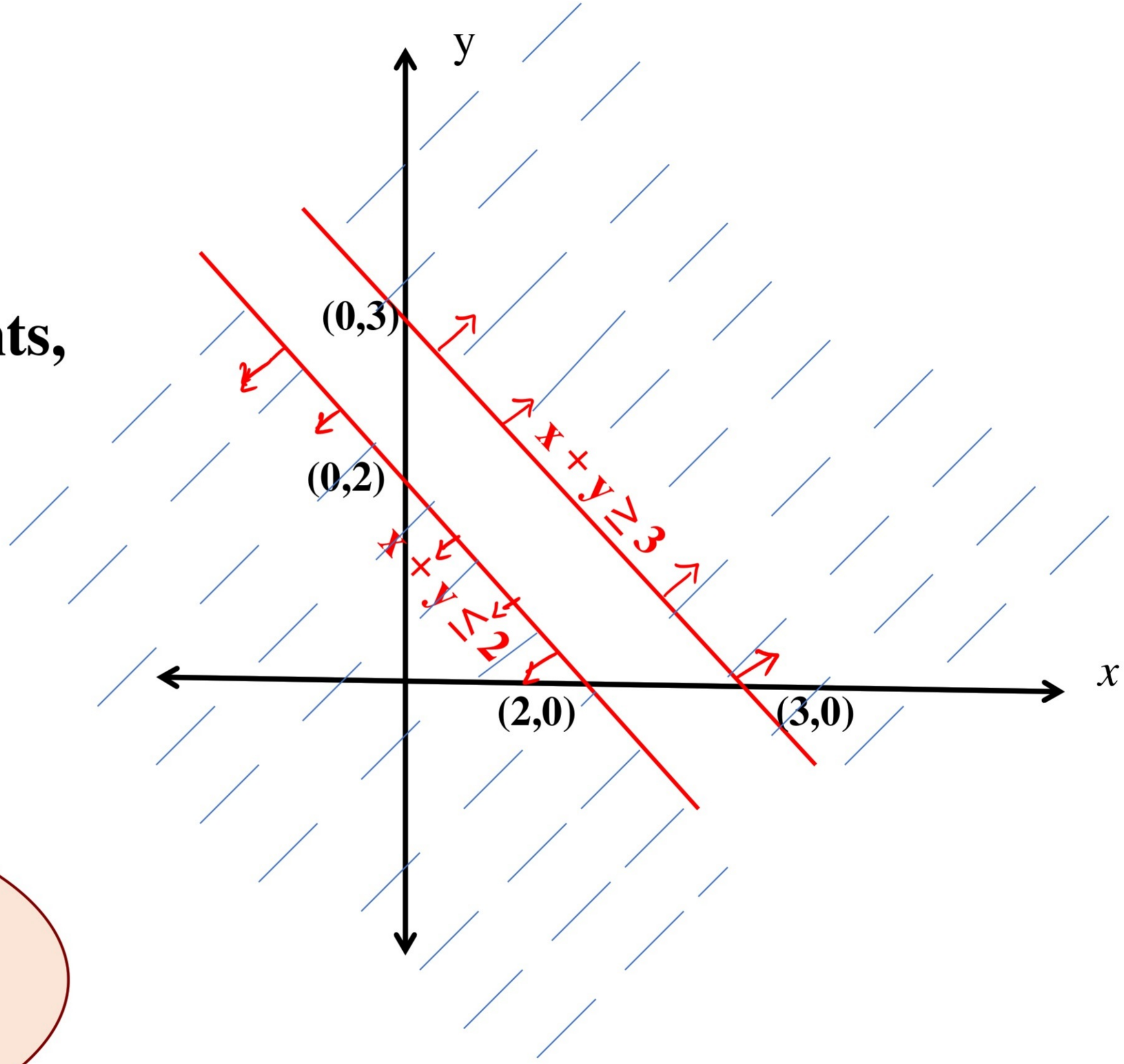
Graphical Method (Con…)

7. Minimize 3x + 5y

subject to the constraints,

$$x + y \leq 2,$$

$$x + y \geq 3,$$



No Feasible Region
No Solution /
Infeasible Solution

8. Maximize $x + y$

$$
\begin{aligned}
\text{subject to } y &\le 2,\\
x^2 - y &\le 0,\\
x &\ge 0,\\
y &\ge 0.
\end{aligned}
$$

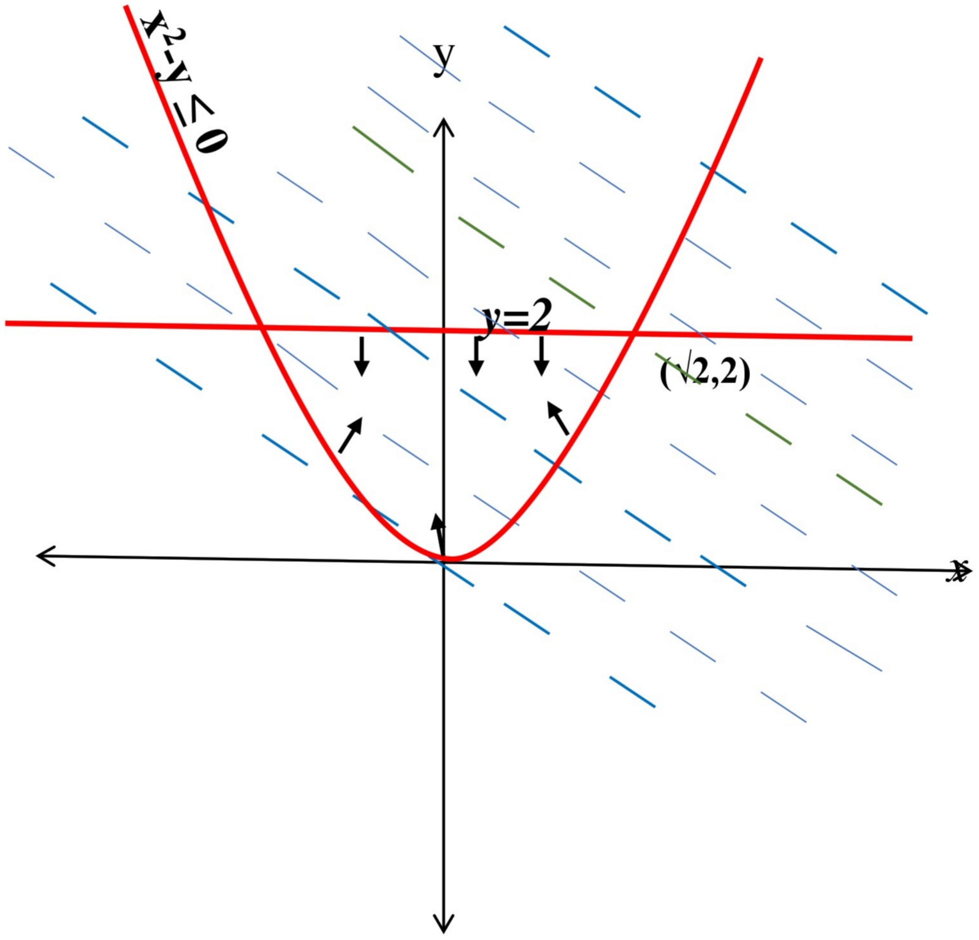
8. Maximize $x + y$

subject to $y \leq 2$,

$x^2 - y \leq 0$,

$x \geq 0$,

$y \geq 0$.



Maximum value is $2 + \sqrt{2}$
at the point $(\sqrt{2}, 2)$

9. Maximize $6x$

subject to

$$x^2 + y^2 \le 6,$$

$$y - x^2 \ge 0,$$

$$y \ge 0.$$

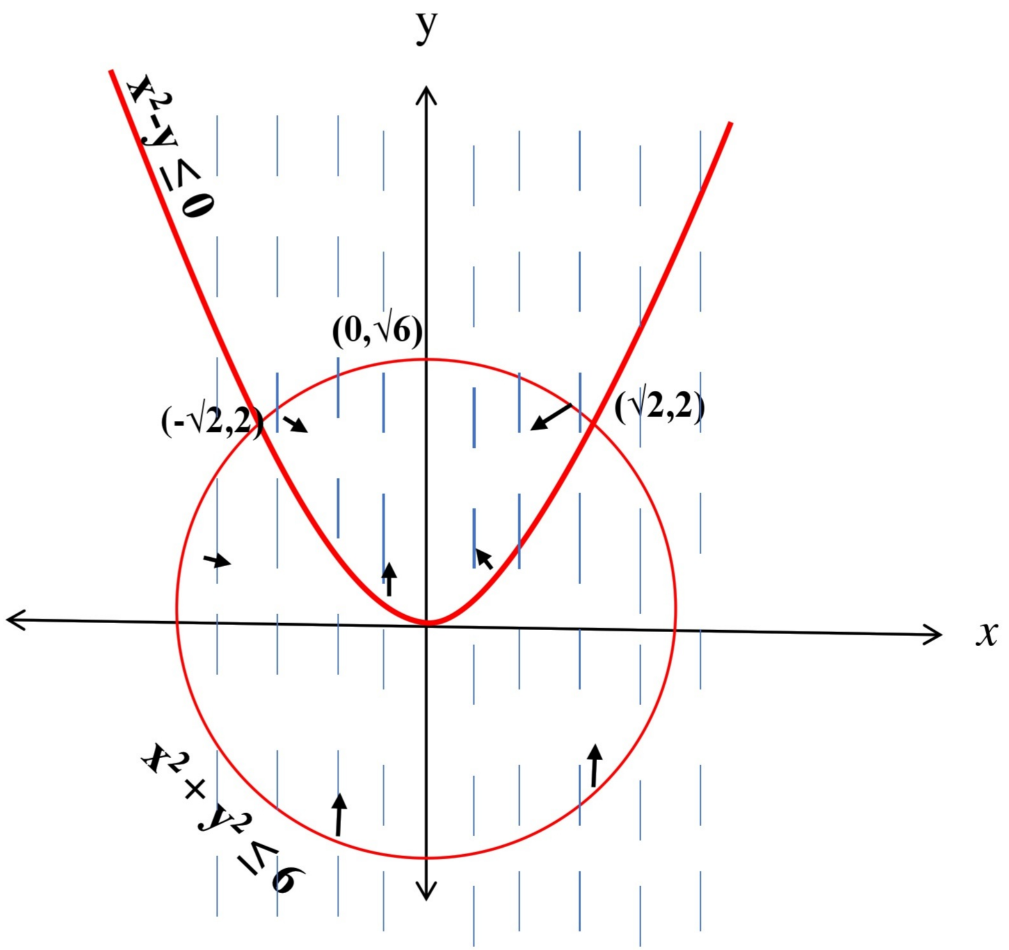
9. Maximize 6x

subject to

$$x^2 + y^2 \leq 6,$$

$$y - x^2 \geq 0,$$

$$y \geq 0.$$



Maximum value is $6\sqrt{2}$
at the point $(\sqrt{2},2)$

Graphical Method (Con….)

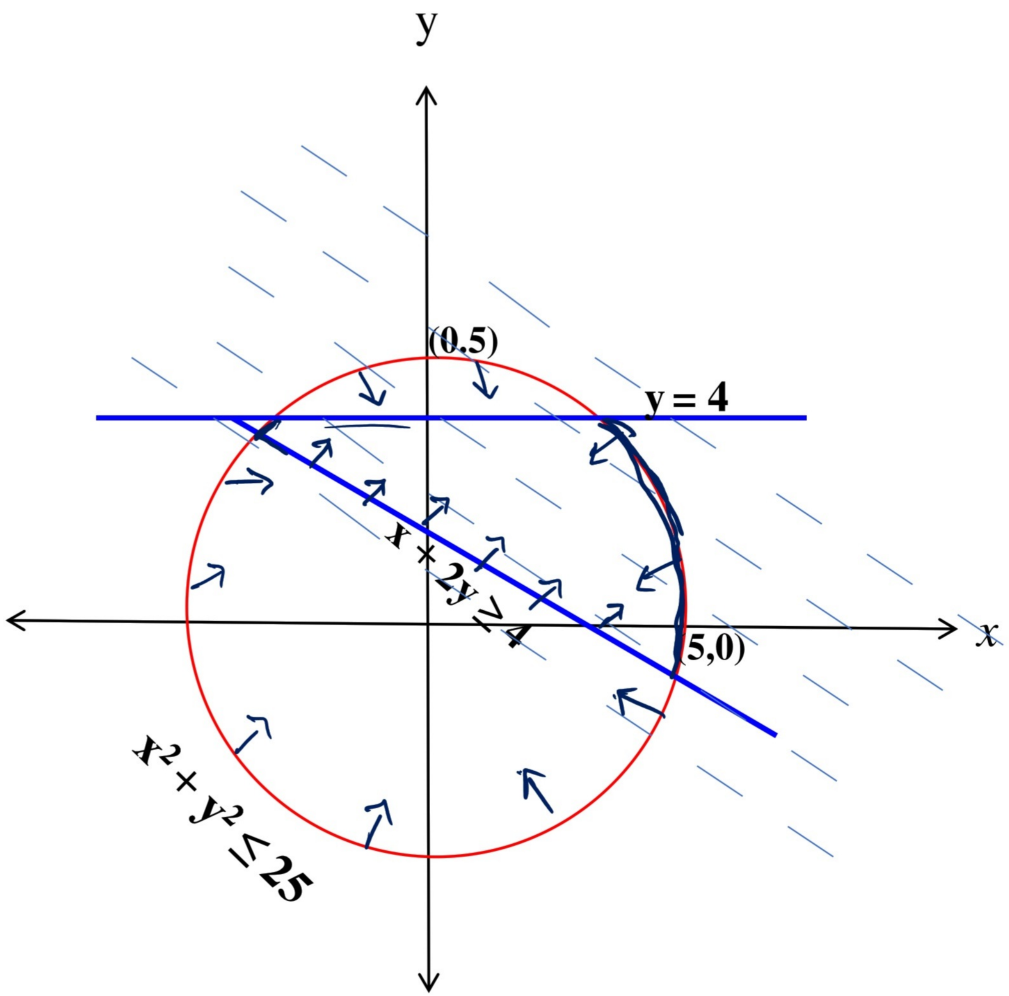
10. Minimize x + y

subject to

$$x^2 + y^2 \leq 25,$$

$$x + 2y \geq 4,$$

$$x \geq 0\ ,\ y \leq 4.$$



Minimum is at (0,2)

Graphical Method – Linear Optimization Problems

1. Maximize $3x + 2y$
   subject to $x + y \le 5$,
   $x - y \le 2$,
   $x \ge 0,\ y \ge 0$.
2. Maximize $x + y$
   subject to $y \le 5$,
   $x + y \le 10$.
   $x \ge 0, y \ge 0$.
3. Maximize $2x + y$
   subject to $x + y \ge 3$,
   $x - y \le 2$.
   $x \ge 0, y \ge 0$.
4. Minimize $x-9y$
   subject to $x + y \ge 3$,
   $x + y \le 2$.

Exercise

5. Minimize $x^2 + y^2$

subject to $x + y \geq 1$

$x, y \geq 0$

6. Consider the optimization problem,

Minimize $x_1 + x_2$

subject to ${x_1}^2 + {x_2}^2 = 1$

$x_1 \geq 0, \quad x_2 \geq 0$

(a) Solve this problem graphically.

(b) Is the solution unique?

7. Maximize $x + y$

subject to $x + y \geq 5, \quad x^2 + y^2 \leq 25, y \geq 0, \quad x \geq 0$

8. Minimize the objective in Q.8.

9. Minimize $x^2 + y^2$

subject to $x - y \geq 1$, $x \leq 3$, $y \geq 0$

Exercise

10. If the maximum of a function g(x) is at x=10 and the maximum function value is 75, the point of minimum for the function f(x) =9 –g(x), is at ____________ and the minimum function value is _____________.

11. If the minimum of a function f(x) is at x=15 and the minimum function value is -35, the point of maximum for the function g(x) = –f(x), is at ____________ and the maximum function value is ___________.

12. The function $f(x)$ has a global minimum at $x = 2$ and the minimum function value is -7.
    Find:
    (a) the minimum point and minimum function value of $g(x) = 9 + f(x)$
    (b) the minimum point and minimum function value of $y(x) = f(x - 4)$
    (c) the maximum point and maximum function value of $h(x) = 8 - f(x)$