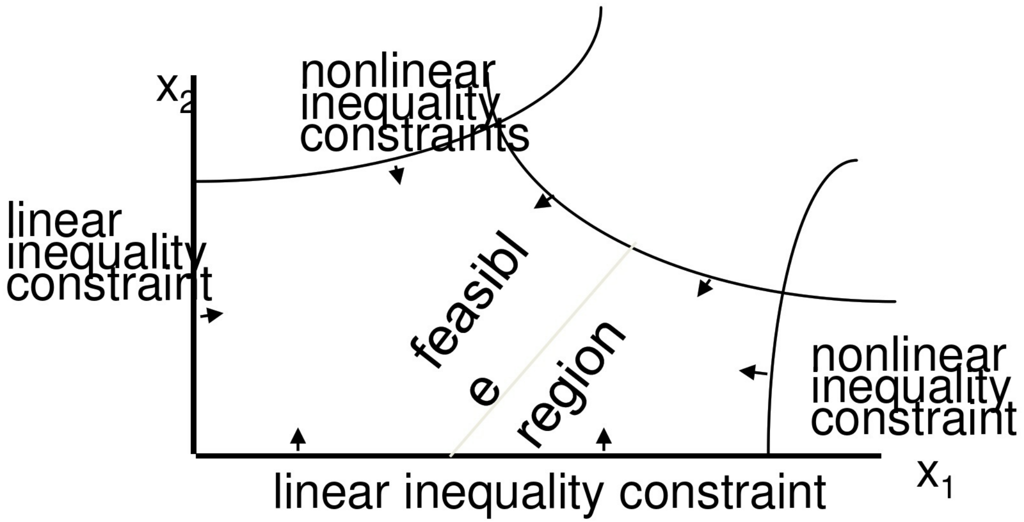
- Essential Components in an Optimization problem

  Minimize f(**x**) subject to **x** ϵ S

  x: set of variables

  f : objective function

  S: feasible region (set of all points that satisfy all the constraints)



- Solution of an optimization problem: Minimize f(**x**) subject to **x** ϵ S

  x* ϵ S such that f (**x***) ≤ f (**x**) for every **x** ϵ S

  x* : solution, f (**x***) : optimal objective function value

  x* may not be unique and may not even exist.

Types of Optimization Problems

- Single Variable and Multi variable Optimization Problems
- Constrained and Unconstrained Optimization Problems
- Linear and Nonlinear Optimization Problems
- Continuous (real variables) and Discrete optimization (binary or integer variables)
- **Single-objective** and Multi-objective optimization
- Stochastic and deterministic optimization

    Stochastic

        Some or all of the problem data are random

        In some cases, the constraints hold with some probabilities

    Deterministic

        No randomness in problem data and constraints

Some Optimization Problems

Some Optimization Problems *(Classification Example):*

- **Maximum area rectangle problem**

  Variables : x = length of rectangle, y = breadth of the rectangle

  Maximize xy

  subject to $2x + 2y \leq 100$, $x > 0$, $y > 0$.

- **Minimum perimeter rectangle problem**

  Variables : x = length of rectangle, y = breadth of the rectangle

  Minimize $2x + 2y$

  subject to $xy = 1000$, $x > 0$, $y > 0$.

Some Optimization Problems

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➢ Multivariable, Constrained, Non-linear, Continuous, Deterministic

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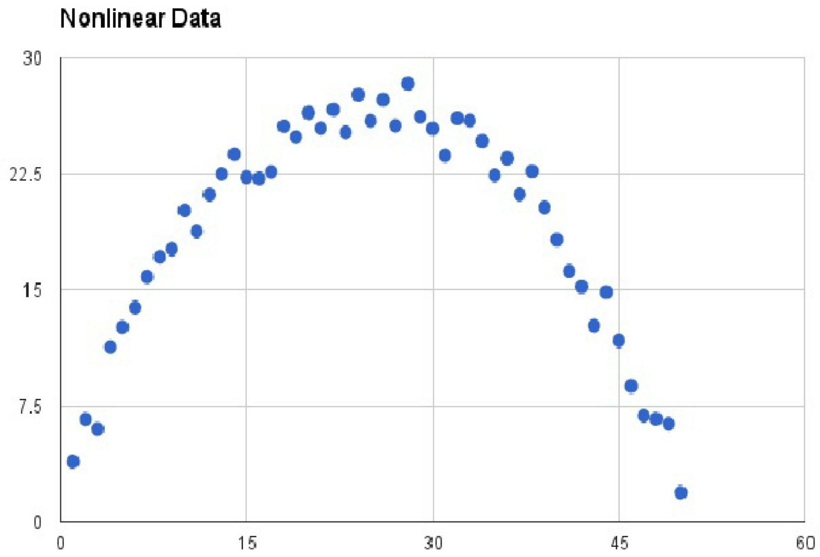
Some Optimization Problems Contd…

- **Data Fitting Problem**: Find a model that "best" fits the observed data.

  Given : (1) $\{x_i , y_i\}$ for i = 1 to n (data points)

  (2) Fit Model $f(x) = ax^2 + bx + c$



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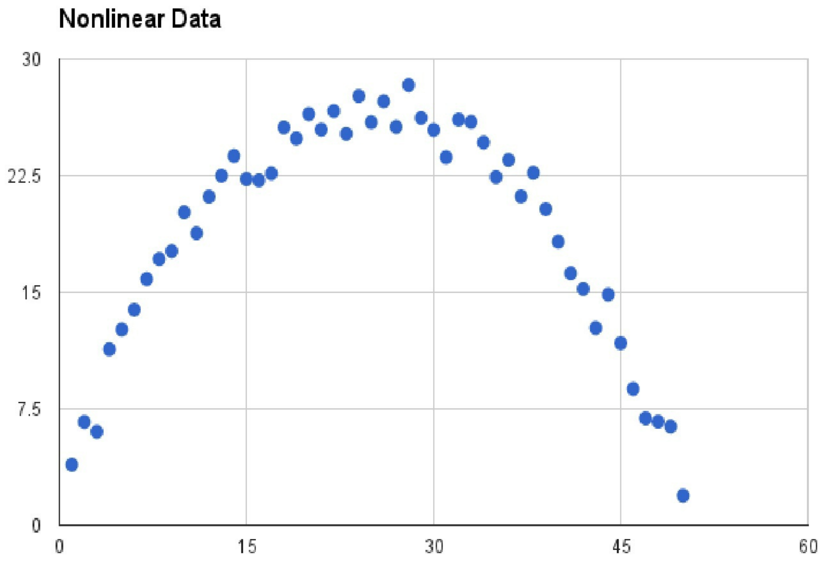

Variables : a, b, c

Objective : Minimize $\sum_{i=1}^{n}(y_i - (a\,x_i{}^2 + bx_i + c))^2$

Multivariable, Unconstrained, Non-linear, Continuous, Deterministic

Data Fitting Model Contd…

Formulate an optimization problem to fit a quadratic curve to the given data points:

X_i	Y_i
-3	8.2
-2	3.9
-1	1.1
0	0.003
1	0.99
2	4.2
3	9.8

Let the fit be $f(x) = ax^2 + bx + c$

Variables: *a, b, c*

$f(X_i) = aX_i^2 + bX_i + c$	$[Y_i - f(X_i)]^2$
9a-3b+c	$[8.2-9a+3b-c]^2$
4a-2b+c	$[3.9-4a+2b-c]^2$
a-b+c	$[1.1-a+b-c]^2$
c	$[0.003-c]^2$
a+b+c	$[0.99-a-b-c]^2$
4a+2b+c	$[4.2-4a-2b-c]^2$
9a+3b+c	$[9.8-9a-3b-c]^2$

Objective: Minimize $[8.2-9a+3b-c]^2 + [3.9-4a+2b-c]^2 + [1.1-a+b-c]^2 + [0.003-c]^2 + [0.99-a-b-c]^2 + [4.2-4a-2b-c]^2 + [9.8-9a-3b-c]^2$

Some Optimization Problems

- Two Products - Bulbs and Batteries
- Each requires two raw materials - P and Q

- One batch of bulbs needs
  - 2 units of P, 4 units of Q
- One batch of batteries needs
  - 3 units of P, 2 units of Q

- Raw material availability is limited
  - **24 units of P, 32 units of Q**
- Each product yields different profit/unit
  - bulb: $4 per unit
  - battery: $5 per unit

❖ **Problem:** How much of each product should be produced so that total Profit is maximized?

Some Optimization Problems

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(i) Aim ?
(ii) Decision Variables ?
(iii) restriction ?

Multivariable, Constrained, Linear, Discrete, Deterministic

Max: $4x + 5y$
s. to c, $2x + 3y \le 24$
$4x + 2y \le 32$
$x \geq 0, y \geq 0$

Some Optimization Problems Contd…

- **Diet Problem:** Propose a diet containing at least 2,000 (Kcal), at least 55 grams of protein and 800 (mg) of calcium with reference to the given table and additionally at minimum cost.

Food	Portion Size	Energy (Kcal)	Proteins (grams)	Calcium (mg)	Price ($/portion)	Limit (portions/day)
Oats	28 g	110	4	2	30	4
Chicken	100 g	205	32	12	240	3
Eggs	2 big ones	160	13	54	130	2
Milk	237 cc	160	8	285	90	8
Kuchen	170 g	420	4	22	200	2
Beans	260g	260	14	80	60	2

Some Optimization Problems Contd…

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Beans	260g	260	14	80	60	2

- **Variables:** Let X_i be the portion of food i to eat during a day

  (i=1 for oats, i=2 for chicken, i=3 for egg, i=4 for milk, i=5 for kuchen, i=6 for beans)

Diet Problem continued…

Food	Portion Size	Energy (Kcal)	Proteins (grams)	Calcium (mg)	Price ($/portion)	Limit (portions/day)
Oats	28 g	110	4	2	30	4
Chicken	100 g	205	32	12	240	3
Eggs	2 big ones	160	13	54	130	2
Milk	237 cc	160	8	285	90	8
Kuchen	170 g	420	4	22	200	2
Beans	260g	260	14	80	60	2

Minimize $30X_1 + 240\, X_2 + 130\, X_3 + 90\, X_4 + 200\, X_5 + 60\, X_6$

Subject to (s.t.) $110\, X_1 + 205\, X_2 + 160\, X_3 + 160\, X_4 + 420\, X_5 + 260\, X_6 \geq 2000$

$4\, X_1 + 32\, X_2 + 13\, X_3 + 8\, X_4 + 4\, X_5 + 14\, X_6 \geq 55$

$2\, X_1 + 12\, X_2 + 54\, X_3 + 285\, X_4 + 22\, X_5 + 80\, X_6 \geq 800$

$0 \leq$ $X_1 \leq 4$;

$X_2 \leq 3$;

$X_3 \leq 2$;

$X_4 \leq 8$;

$X_5 \leq 2$;

$X_6 \leq 2$;

$X_i \geq 0$ for i = 1, 2, 3, 4, 5, 6

Multivariate, Constrained,
Linear, Continuous,
Deterministic

H.W.

Some Optimization Problems Contd…

A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Note: Dimension is in cm

Some Optimization Problems Contd…

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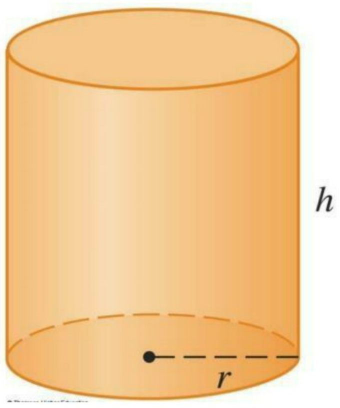
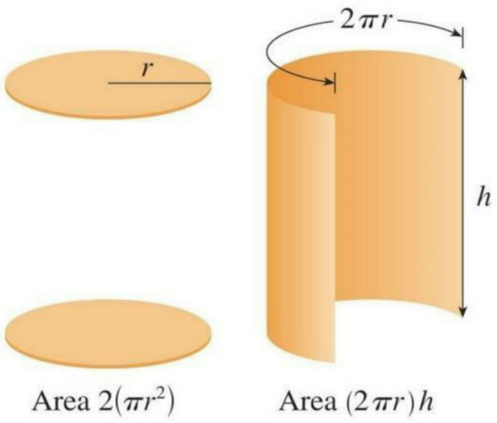




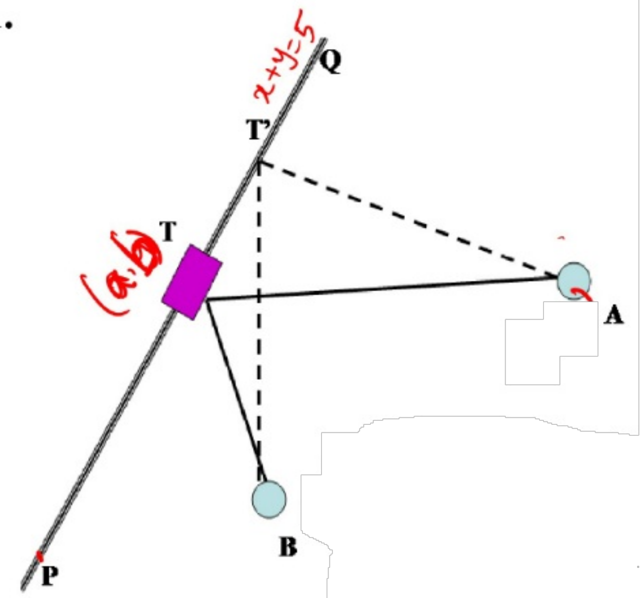
Minimize: $A = 2\pi r^2 + 2\pi rh$

Constraint: $\pi r^2 h = 1500$

Dimension is in cm

Some Optimization Problems Contd…

- **Bus Terminus Location Problem**: Find the location of the bus terminus T on the road segment PQ such that the lengths of the roads linking T with the two cities A and B is minimum.



Find the coordinates of the point T on the line PQ, $x + y = 5$, such that the lengths of the line segments AT and BT is minimum, Given, coordinate of A is (1,2) and B is (3,1).

Some Optimization Problems Contd…

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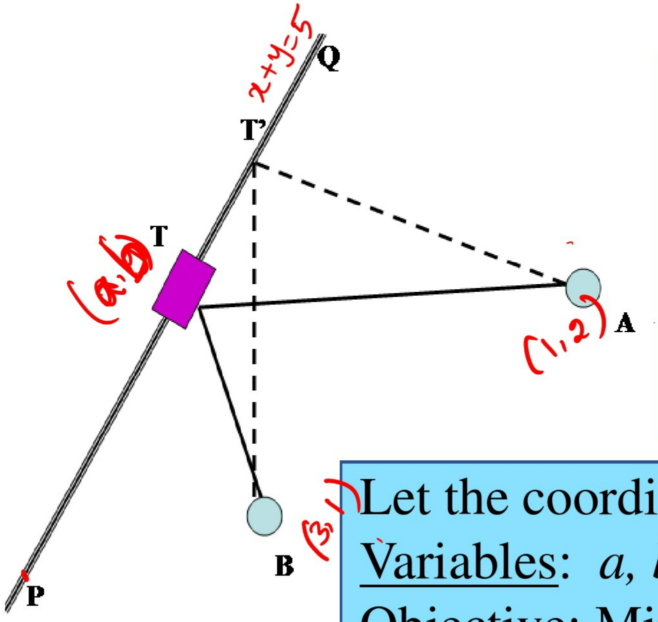

Find the coordinates of the point T on the line PQ, x + y = 5, such that the lengths of the line segments AT and BT is minimum, Given, coordinate of A is (1,2) and B is (3,1).

Let the coordinates of the point T be (a, b)
Variables: a, b
Objective: Minimize the distance AT + BT

$$\text{i.e. Minimize } \sqrt{(a-1)^2+(b-2)^2} + \sqrt{(a-3)^2+(b-1)^2}$$

$$\text{subject to } a + b = 5$$

- Multivariable, Constrained, Non-linear, Continuous, Deterministic

Some Optimization Problems Contd…

- **Portfolio Optimization**
  - **Variables:** amounts to be invested in different assets
  - **Objective:** Minimize the overall risk or return variance
  - **Constraints:** budget, max/min investment per asset, minimum return

- **Development of device in electronic circuit**
  - **Variables:** device width and length
  - **Objective:** Minimize power consumption
  - **Constraints:** manufacturing limits, timing requirements, maximum area

H.W.

Problem for practice

1. Write a mathematical model to find the dimensions of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is equal to 50 sqm. Classify the model based on all five classifications.
2. A firm manufactures two products A and B on which the profit earned per unit are Rs.3 and Rs.4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2 , while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours and 30minutes, while machine M_2 is available for 10 hours. Formulate the problem as a mathematical model, if the objective is to maximize the profit.

Problems for practice

3. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of Aspirin, 5 grains of bicarbonate and 1 grain of codein. Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codein. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codein for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem.
4. The final product of a firm has a requirement that it must weigh exactly 150 kg. The two raw materials used are product A with cost of Rs.2/unit and B with a cost of Rs.8/unit. At least 14 units of B and not more than 20 units of A must be used. Each unit of A weighs 5kg and that of B weighs 10kg. Formulate the problem so as to know how much of each type of raw materials should be used for each unit if the cost is to be minimized.

Problems for practice

5. A balanced diet for a five year old boy should contain at least 1400 Kcal of energy, 75 grams of proteins and 800 mg of calcium per day. With reference to the given table, formulate a mathematical model to decide the diet of a five year old boy, with cost minimization as the objective.

Food	Portion size	Energy (Kcal)	Proteins (gram)	Calcium (mg)	Price (Rs./portion)	Limit portion/ day
Rice	150 gram	380	14	19	25	3
Wheat	150 gram	295	21	33	42	2
Milk	250 ml	160	11	285	20	4
Dal	250 gram	180	23	52	165	2