Visualizations for the Quantum Bouncer

Yen Lee Loh and Chee Kwan Gan, 2021-5-30

1 General-purpose setup

2 One-Sided Bouncer (1SB)

2.1 Eigenvalues and eigenfunctions

2.2 Classical paths (interactive)

2.3 Feynman path integral in semiclassical approximation: infographic

2.4 Wavepacket evolution (animation)

2.5 Propagator visualized as heatmap

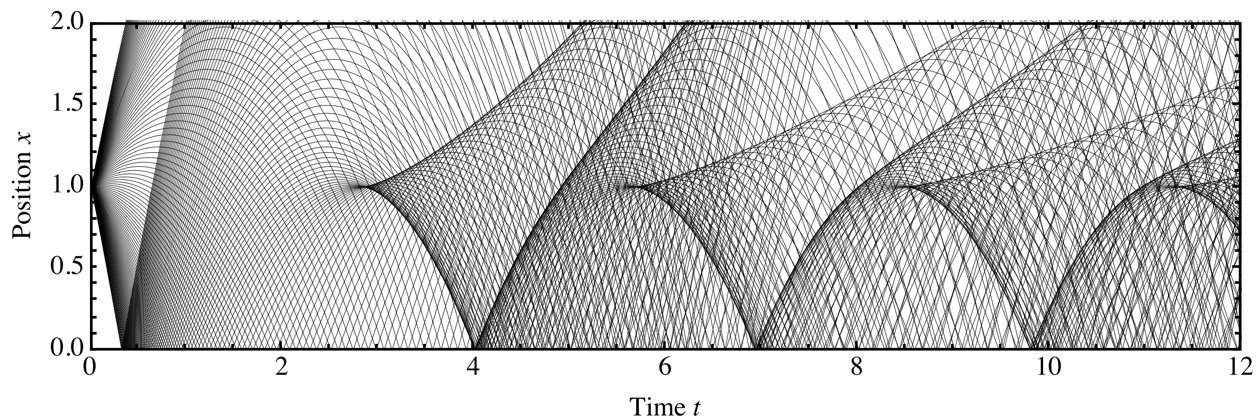
2.6 Phase along path: detailed study

2.7 Phase diagram as function of (x_f, T) for fixed x_i: obtaining caustics as loci of foci

Plotting many paths by brute force reveals that the paths overlap strongly along caustic curves:

In[273]:=
```
xi = 1;  (* initial position *)
g = 1;   (* gravitational field strength *)
viList = Range[-3, 3, .05]; (* list of initial velocities for plotting *)
Tmax = 12; (* maximum time for plotting *)
xmax = 2; (* maximum position for plotting *)
plotPathVi[xi_, vi_, T_, g_,
    style_ : Directive[AbsoluteThickness@0, Opacity[0.25, Black]] ] :=
  Module[{vm, xt, tb, t0, αmax, colorα},
   vm = Sqrt[vi^2 + 2 g xi]; t0 = vi/g; tb = 2 vm/g;   (* vm and tb are lists *)
   xt = Function[{t}, vm^2/(2 g) - g/2 (t - t0 - tb Floor[(t - t0)/tb + 1/2])^2];
   Plot[Evaluate[xt[t]], {t, 0, T}, PlotStyle → style, Exclusions → None]];
gr = Show[
  Table[  plotPathVi[xi, vi, Tmax, g,
    Directive[AbsoluteThickness@0.1, Black]]  , {vi, viList}],
  Frame → True, FrameLabel → {"Time t", "Position x"}, RotateLabel → True,
  PlotRange → {{0, Tmax}, {0, xmax}}, PlotRangePadding → 0,
  ImageSize → {648, 216}, AspectRatio → Full,
  ImagePadding → {{Automatic, Automatic}, {Automatic, 8}}]
```

Out[275]=



From our analysis of the symmetric bouncer, we find that for a path with initial position x_i, initial velocity v_i, and k bounces, there is a focus at time

$$t_k^F = \frac{2k}{g}\,\frac{v_i^2+2kv_iv_m+v_m^2}{2kv_i+v_m} = \frac{2k}{g}\,\frac{2v_i^2+2gx_i+2kv_i\sqrt{v_i^2+2gx_i}}{2kv_i+\sqrt{v_i^2+2gx_i}}.$$

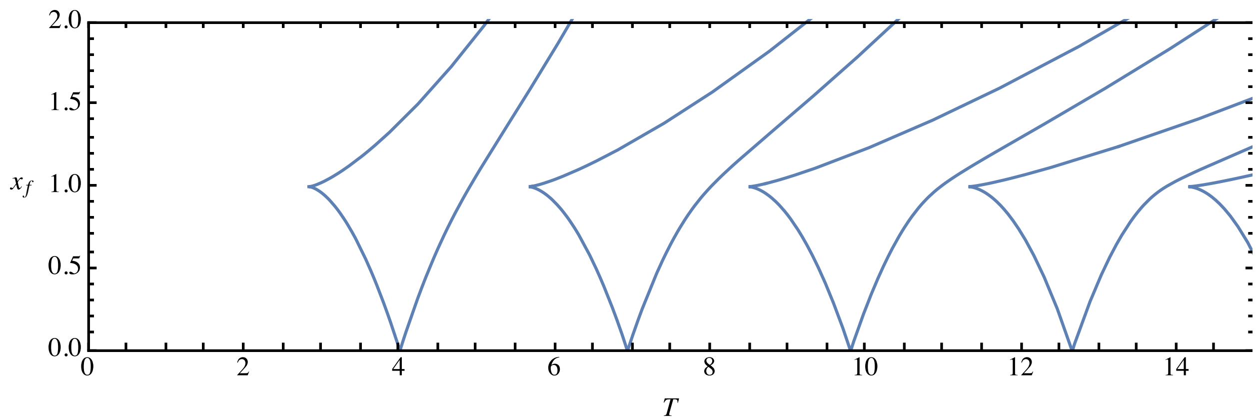
The position at this time is

$$x_k^F = X\left(t_k^F\right) \qquad \text{where } X(t) = x_i + \frac{v_i^2}{2g} - \frac{g}{2}\left(t - \frac{v_i}{g} - \frac{2\sqrt{v_i^2+2gx_i}}{g}\,k\right)^2.$$

If we loop over each value of $k = 1, 2, 3, \ldots$, loop over many values of v_i, and make a Parametric-Plot of the locus of $\left(t_k^F, x_k^F\right)$, we obtain the caustics. The caustics are the loci of the foci.

In[587]:=
```
Clear[n, vi, xi, g];
xi = 1; g = 1;
ParametricPlot[
 Table[{
   (2 n/g) * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi),
   xi + vi^2/(2 g) - g/2 ((2 n/g) * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi) - vi/g - (2 Sqrt[vi^2 + 2 g xi])/g n)^2
  }, {n, 1, 6}]
 , {vi, -5, 5}, PlotRange → {{0, 15}, {0, 2}},
 ImageSize → {648, 216}, AspectRatio → Full,
 FrameLabel → {"T", "x_f"}]
```

Out[589]=



The pattern of caustics above is universal. Changing x_f merely results in rescaling.

2.8 Phase diagram as function of (x_f, T) for fixed x_i: discriminant method

2.9 Phase diagram as function of (x_i, x_f) for fixed T – a bit old

2.10 Van Vleck determinant D_α – details of derivation

3 Symmetric Bouncer

3.1 Eigenvalues and eigenfunctions

3.2 Feynman path integral in semiclassical approximation: infographic

3.3 Phase diagram as function of (x_f, T) for fixed x_i: discriminant method

3.4 Path divergence function $f_\alpha(t)$ – slightly messy

3.5 Focal times t_k^F – slightly messy

3.6 Morse index m as a function of (T, v_i) – slightly messy

3.7 Propagator visualized as heatmap

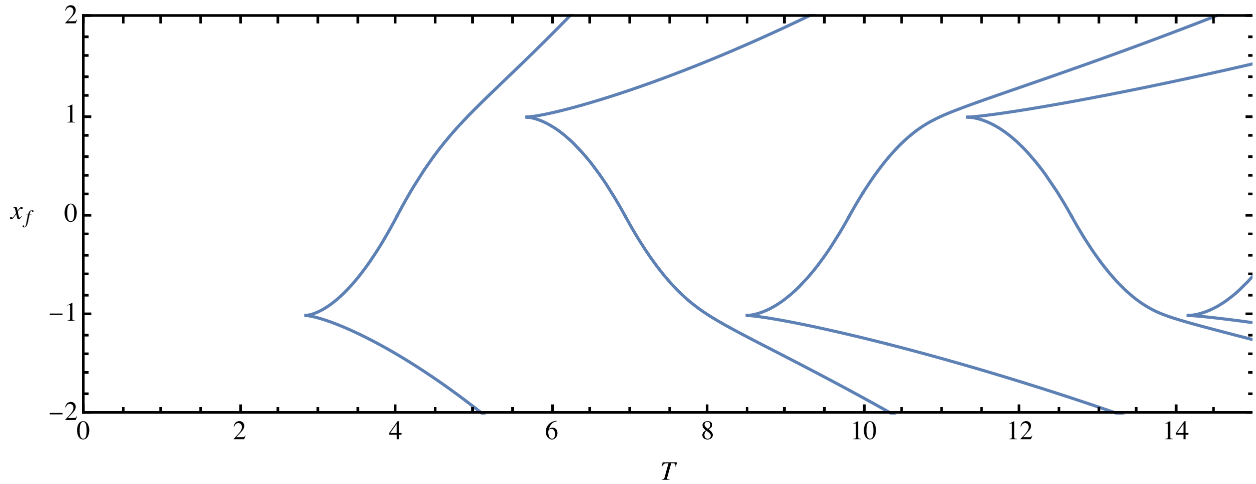
3.8 Wavepacket evolution (animation)

3.9 Phase diagram as function of (x_f, T) for fixed x_i: obtaining caustics as loci of foci

The expression for the kth focal time is surprisingly messy. (See section on focal times.) Careful consideration allows us to plot the caustics using ParametricPlot:

In[500]:=
```
Clear[n, vi, xi, g];
xi = 1; g = 1;
gr2 = Show[{
   (*-------- Plot one branch --------*)
   Table[
    ParametricPlot[{
      2 n/g * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi),
      (xi + vi^2/(2 g) -
         g/2 (2 n/g * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi) - vi/g - (2 Sqrt[vi^2 + 2 g xi])/g n)^2) (-1)^n
     }, {vi, -1/(Sqrt[2] Sqrt[n^2 + n]), 2}]
    , {n, 1, 6}],
   (*-------- Plot the other branch --------*)
   Table[
    ParametricPlot[{
      2 n/g * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi),
      (xi + vi^2/(2 g) -
         g/2 (2 n/g * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi) - vi/g - (2 Sqrt[vi^2 + 2 g xi])/g n)^2) (-1)^n
     }, {vi, -2, -1/(Sqrt[2] Sqrt[n^2 - n])}]
    , {n, 2, 6}]
  },
  PlotRange → {{0, 15}, {-2, 2}}, PlotRangePadding → 0,
  ImageSize → {648, 126 * 2}, AspectRatio → Full, Axes → False,
  FrameLabel → {"T", "x_f"}]
```

Out[502]=



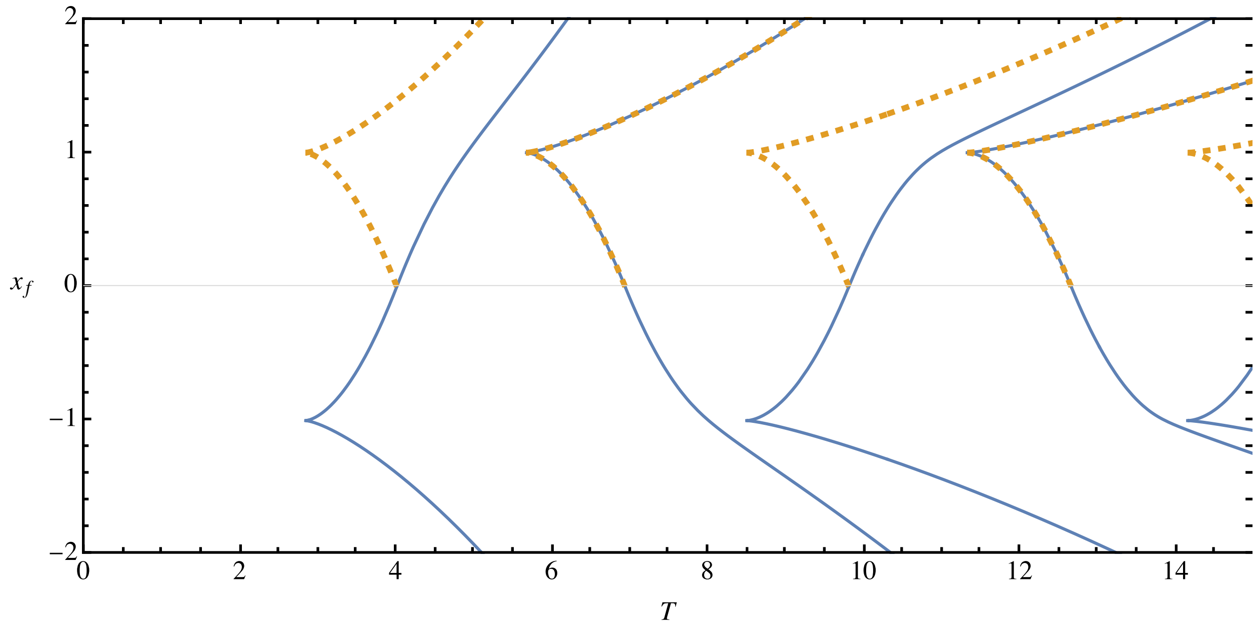
Comparison of phase diagrams of one-sided bouncer and symmetric bouncer

In[579]:=

```
Clear[n, vi, xi, g];
xi = 1; g = 1;
gr1 = Show[
   Table[
    ParametricPlot[{
      (2 n)/g * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi),
      xi + vi^2/(2 g) - g/2 (2 n/g * (2 vi^2 + 2 g xi + 2 n vi Sqrt[vi^2 + 2 g xi])/(Sqrt[vi^2 + 2 g xi] + 2 n vi) - vi/g - (2 Sqrt[vi^2 + 2 g xi])/g n)^2
     }, {vi, -1/(Sqrt[2] Sqrt[n^2 + n]), 2}, PlotStyle →
      {{Dashed, CapForm@"Butt", AbsoluteThickness@3, ColorData[97][2]}}]
    , {n, 1, 6}],
   ImageSize → {648, 162}, AspectRatio → Full,
   PlotRange → {{0, 15}, {0, 2}}, FrameLabel → {"T", "xf"}];
```

In[586]:=
```
Show[gr2, gr1, PlotRange → {{0, 15}, {-2, 2}}, ImageSize → {648, 162 * 2},
 Axes → True, AxesOrigin → {0, 0}, AxesStyle → LightGray]
```

Out[586]=



In the figure above, the dashed orange curves are the caustics (phase boundaries) of the one-sided bouncer. The solid blue curves are the caustics of the two-sided bouncer. We see that these boundaries are related to each other by reflection.

4 Roots of cubics and quartics

The solutions of the cubic equation $z^3 + bz^2 + cz + d = 0$ can be found as follows:

$$p = \left(\frac{9bc-2b^3-27d+\sqrt{27\,(4c^3+4b^3d+27d^2-b^2c^2-18bcd)}}{54}\right)^{1/3}$$

$$q = \frac{b^2-3c}{9p}$$

$$z_n = pe^{2\pi in/3} + qe^{-2\pi in/3} - \frac{b}{3} \quad \text{for } n = 0, 1, 2.$$

The solutions of the quartic equation $z^4 + bz^3 + cz^2 + dz + e = 0$ can be found as follows (taking both combinations of ± signs):

$$f = c^2 - 3bd + 12e$$

$$g = 2c^3 - 9bcd + 27d^2 + 27b^2e - 72ce$$

$$p = \left(\frac{g+\sqrt{g^2-4f^3}}{2}\right)^{1/3}$$

$$q = p + \frac{f}{p}$$

$$r = \pm\sqrt{12q + 9b^2 - 24c}$$

$$z = \frac{-3b-r\pm\sqrt{18b^2-48c-12q+54\,(b^3-4bc+8d)/r}}{12}.$$

Cubic: Verification by comparison with Solve[]

In[335]:=
```
(*-------- RUN THE FOLLOWING MANY TIMES TO MAKE SURE IT IS RELIABLE --------*)
{b, c, d} = N@RandomReal[{-9, 9}, 3];
Block[{z}, Last /@ Last /@ Solve[z^3 + b z^2 + c z + d == 0, z] // N // Sort // Chop // Print];
```

$$p = \left(\frac{9\,b\,c - 2\,b^3 - 27\,d + \sqrt{27\left(4\,c^3 + 4\,b^3\,d + 27\,d^2 - b^2\,c^2 - 18\,b\,c\,d\right)}}{54}\right)^{1/3};$$

$$q = \frac{b^2 - 3\,c}{9\,p};$$

$$\text{zList} = \text{Table}\left[p\,E^{2\,\pi\,I\,n/3} + q\,E^{-2\,\pi\,I\,n/3} - \frac{b}{3},\ \{n, \{0, 1, 2\}\}\right];$$

```
zList // N // Sort // Chop // Print
```

{0.477193 − 1.19989 i, 0.477193 + 1.19989 i, 3.93064}

{0.477193 − 1.19989 i, 0.477193 + 1.19989 i, 3.93064}

Quartic: Verification by comparison with Solve[]

In[274]:=
```
(*-------- RUN THE FOLLOWING MANY TIMES TO MAKE SURE IT IS RELIABLE --------*)
{b, c, d, e} = RandomReal[{-9, 9}, 4];
Block[{z},
  Last /@ Last /@ Solve[z^4 + b z^3 + c z^2 + d z + e == 0, z] // N // Sort // Chop // Print];
```

$$f = c^2 - 3\,b\,d + 12\,e;$$

$$g = 2\,c^3 - 9\,b\,c\,d + 27\,d^2 + 27\,b^2\,e - 72\,c\,e;$$

$$p = \left(\frac{g + \sqrt{g^2 - 4\,f^3}}{2}\right)^{1/3};$$

$$q = p + \frac{f}{p};$$

Table[

$$r = \alpha\,\sqrt{12\,q + 9\,b^2 - 24\,c};$$

$$x = \frac{-3\,b - r + \beta\,\sqrt{18\,b^2 - 48\,c - 12\,q + 54\left(b^3 - 4\,b\,c + 8\,d\right)/r}}{12}\ //\ N$$

, {α, {-1, 1}}, {β, {-1, 1}}

] // Flatten // N // Sort // Chop // Print

{−6.44643, −0.500645, 0.609211 − 0.213005 i, 0.609211 + 0.213005 i}

{−6.44643, −0.500645, 0.609211 − 0.213005 i, 0.609211 + 0.213005 i}