

Algorithms for Topological and Metric Surfaces



Loïc Dubois

Computational Geometry

Design algorithms for geometric problems

This thesis

Focus on surfaces

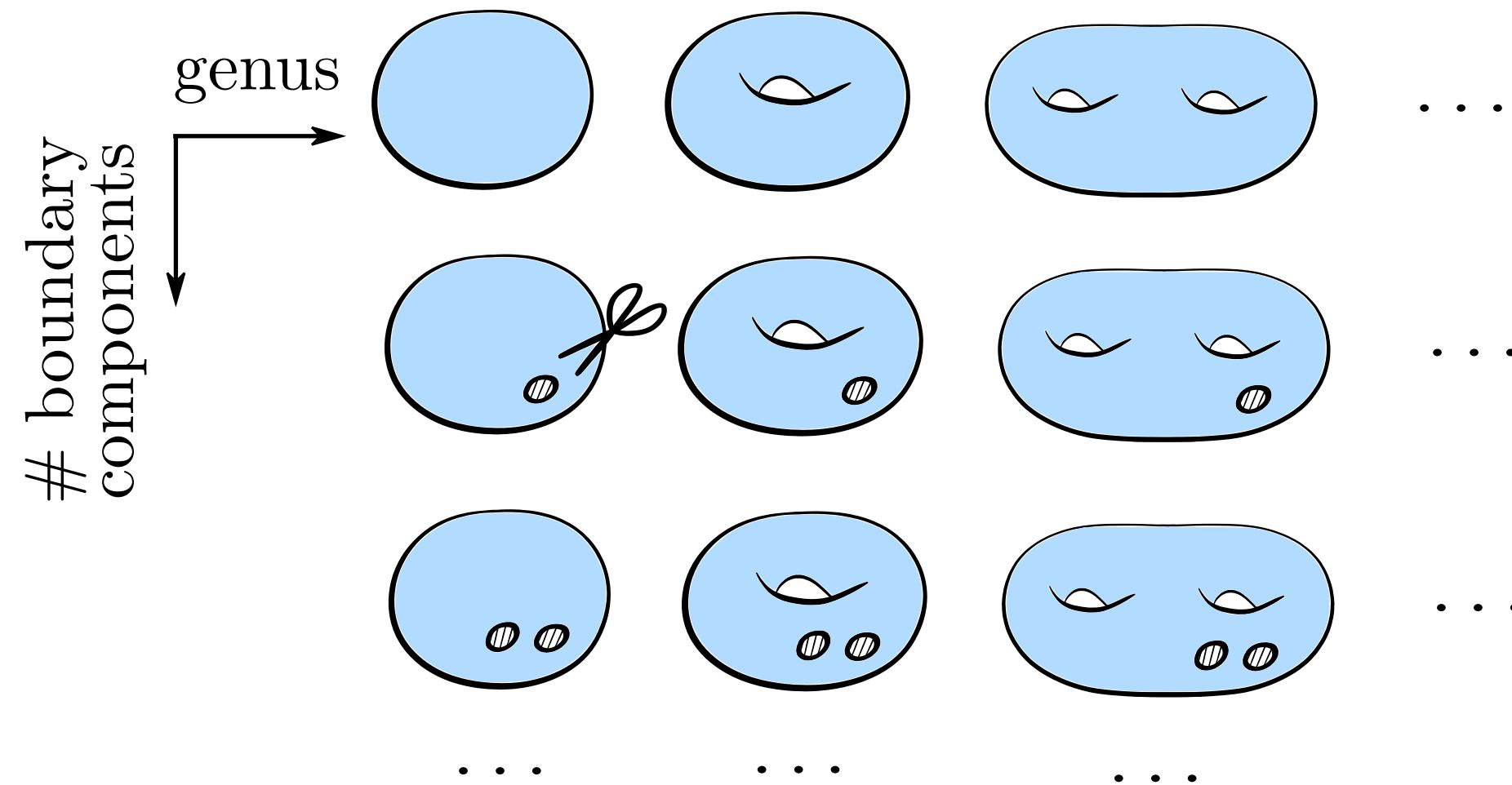


Topology

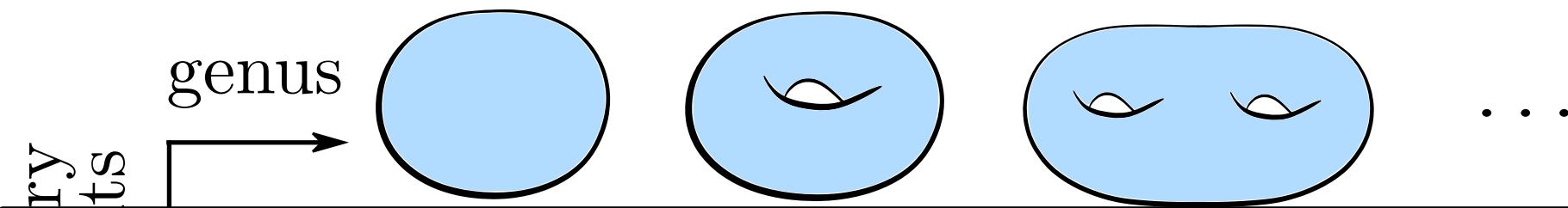


image by Crane and Segeiman

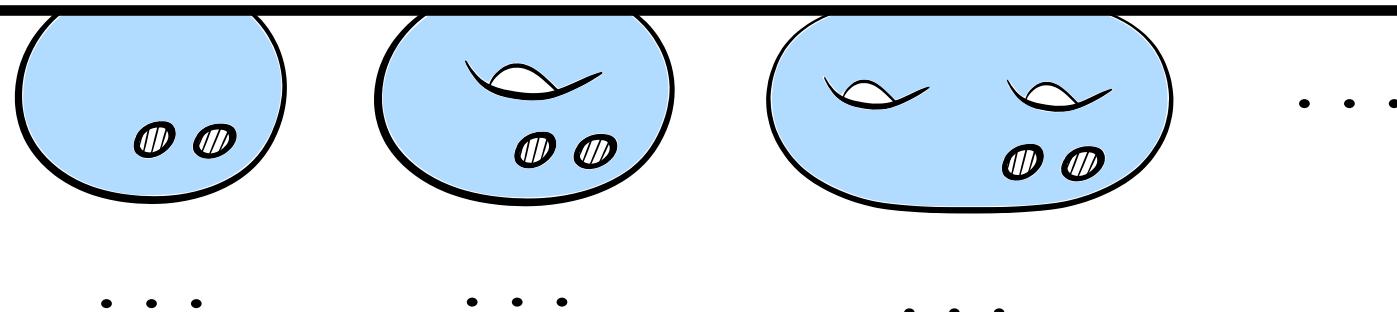
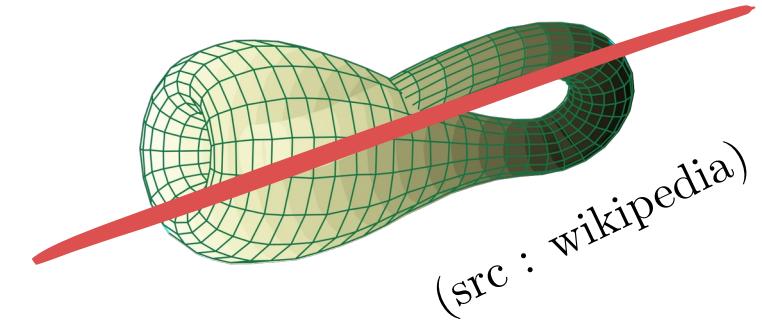
Topological Surfaces



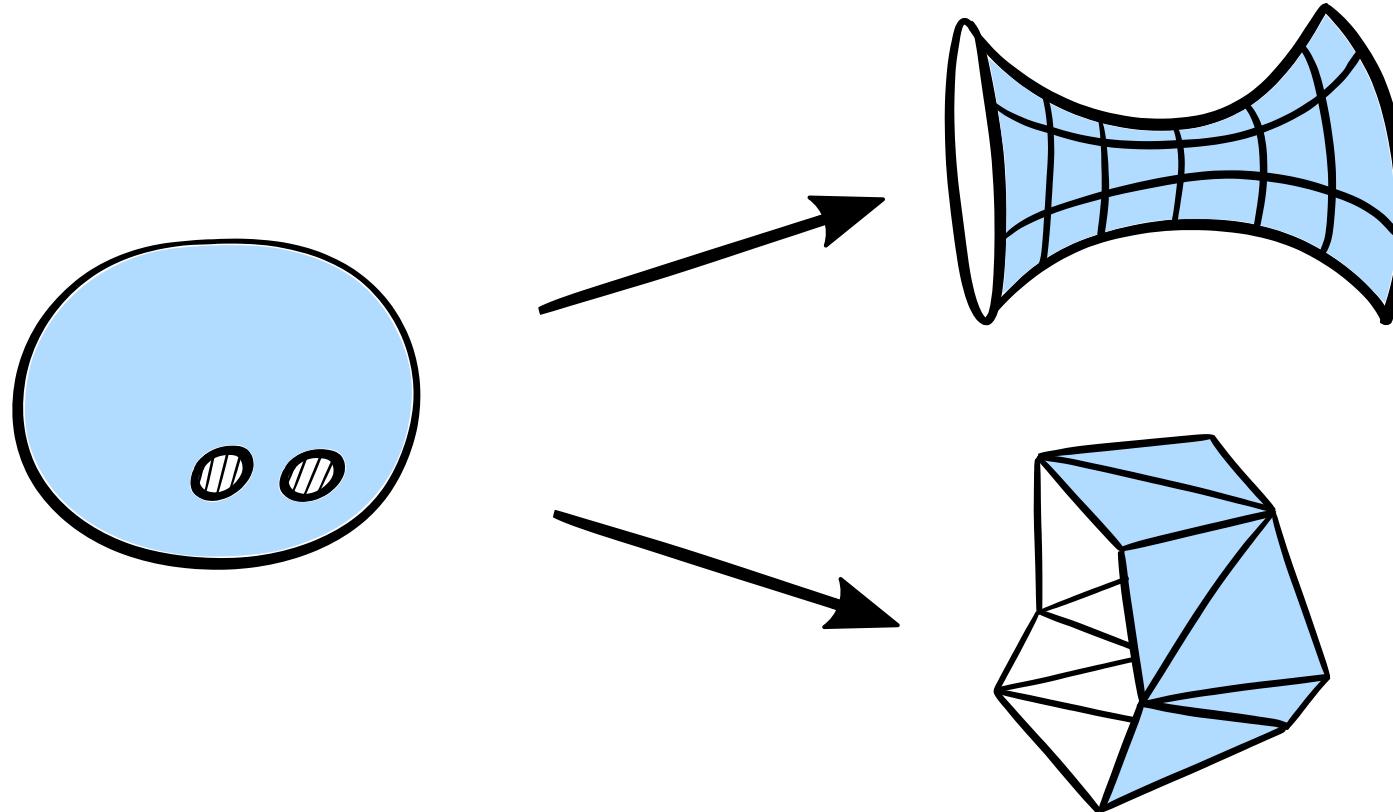
Topological Surfaces



Only **orientable** surfaces today !



Metrics on surfaces



Untangling Graphs

Computing Delaunay Triangulations

Conclusion

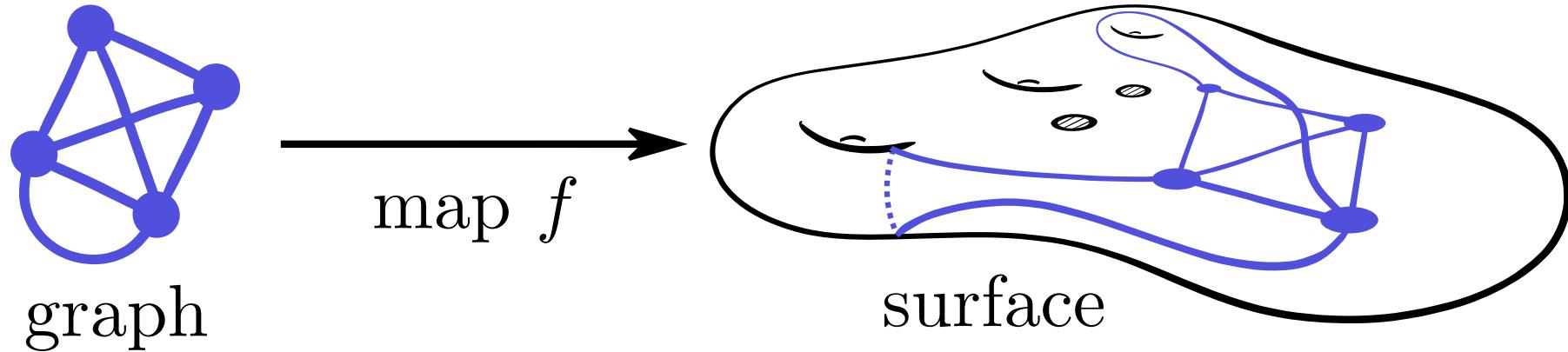
Untangling Graphs

Computing Delaunay Triangulations

Conclusion

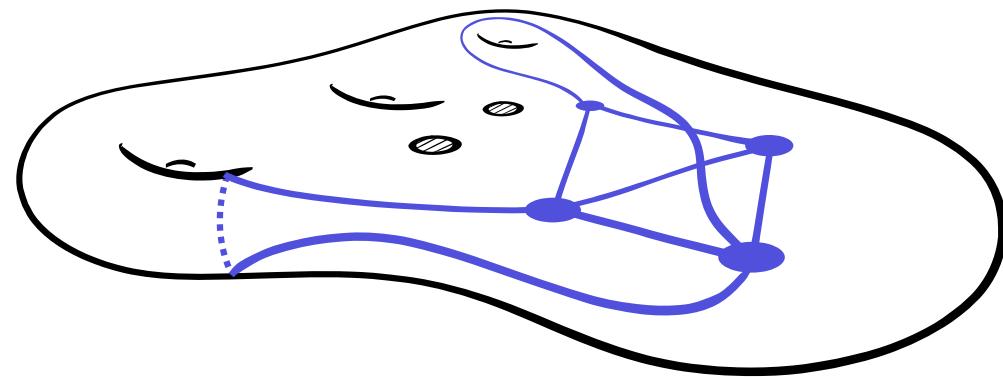
Problem: untangling graphs

Input:

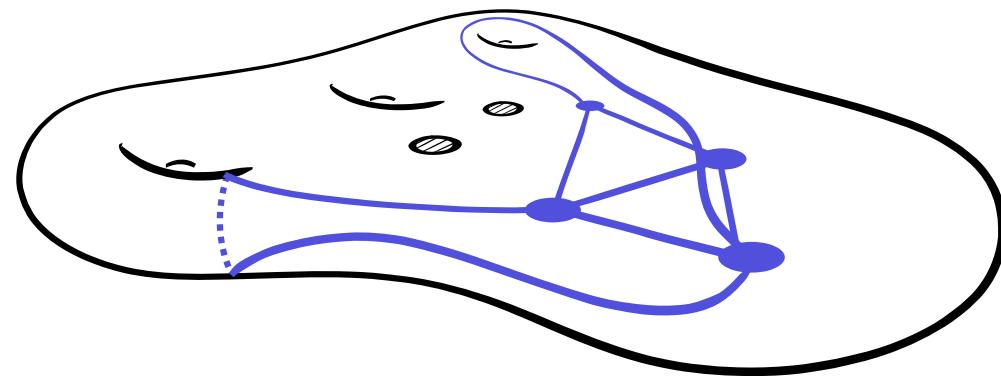


Goal: remove all crossings by deforming f

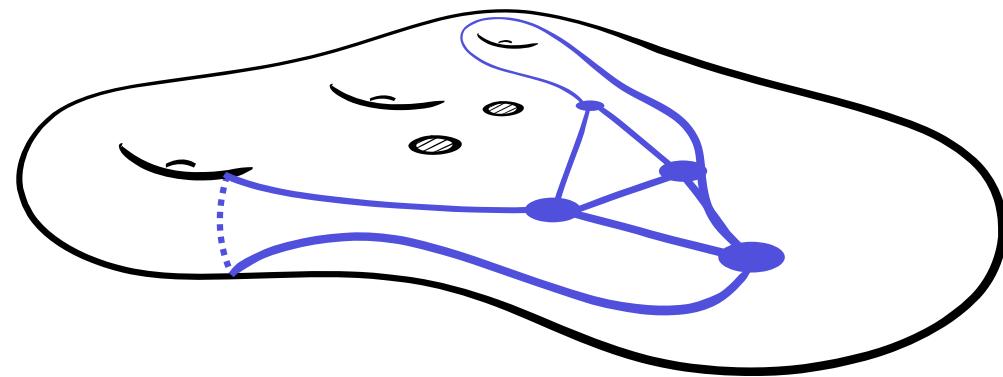
Problem: untangling graphs



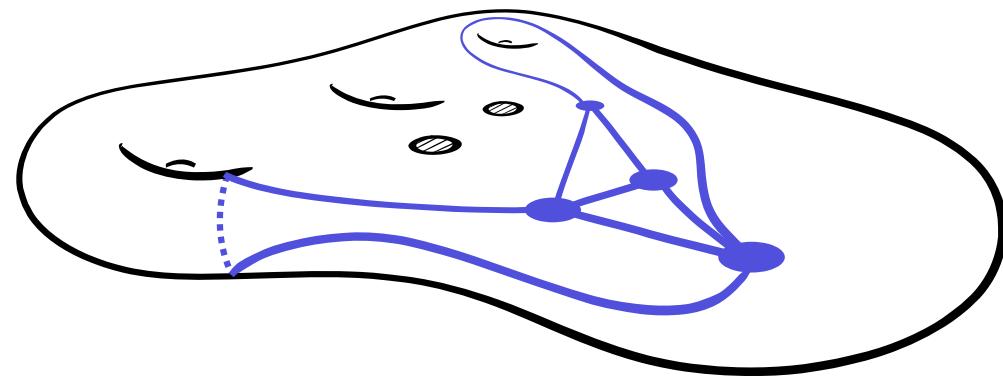
Problem: untangling graphs



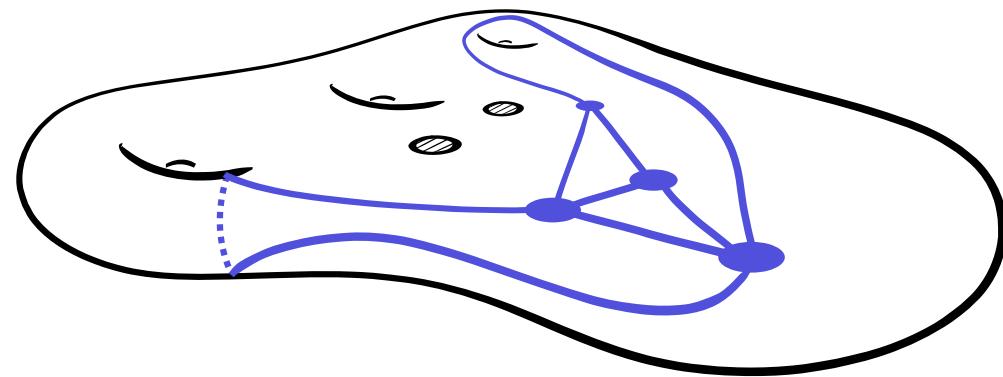
Problem: untangling graphs



Problem: untangling graphs



Problem: untangling graphs

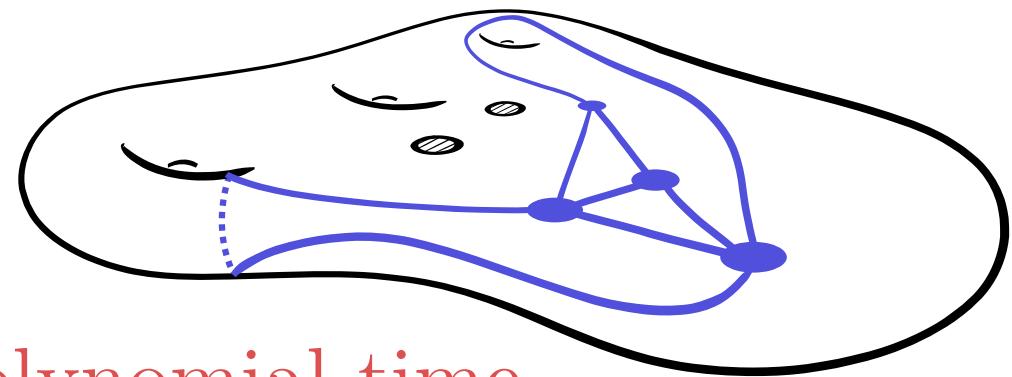


Problem: untangling graphs



Output: Yes (+ untangled drawing) or No

Problem: untangling graphs

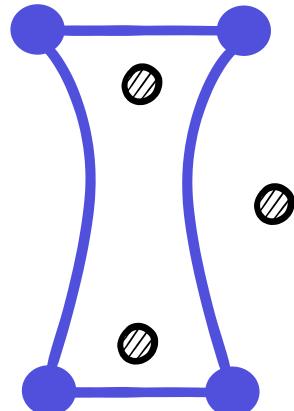


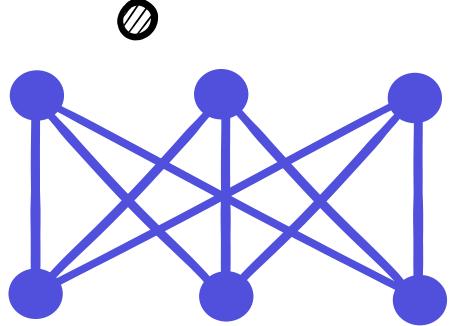
We obtain the first polynomial time
algorithms for this problem

Output: Yes (+ untangled drawing) or No



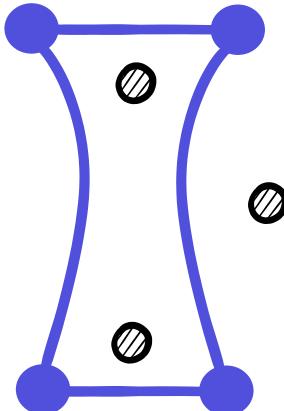
Yes:

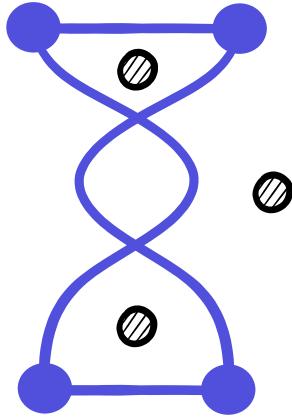




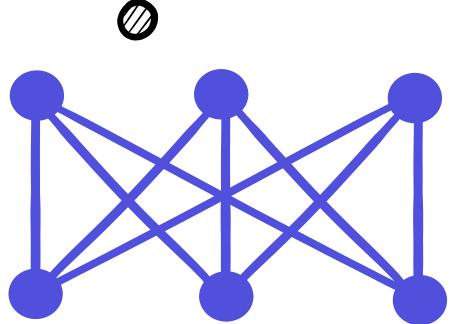
Yes:

No





Yes:



No

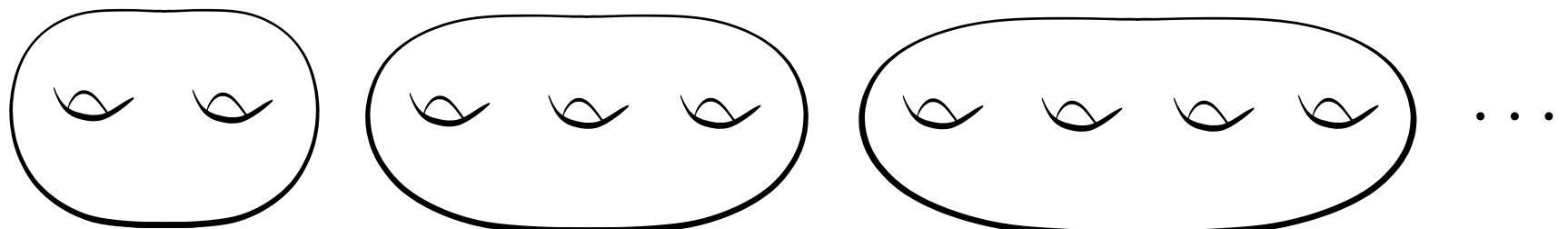


No



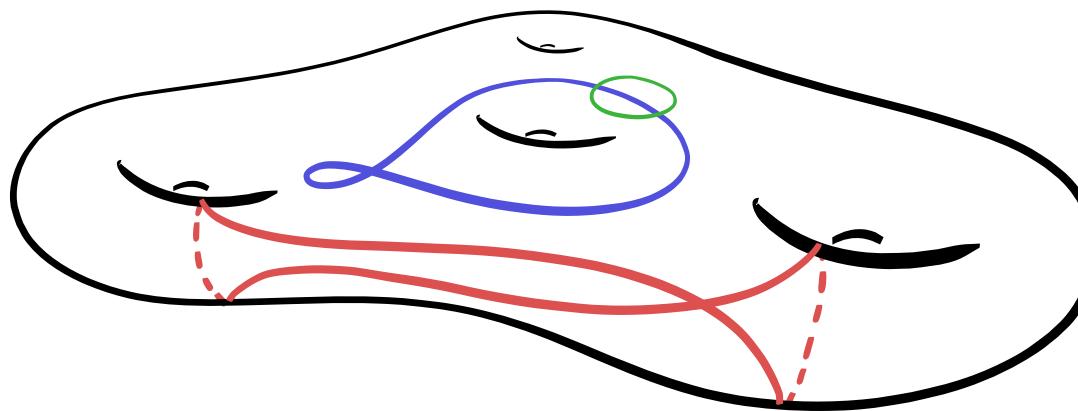
Related works

We focus on surfaces without boundary
of genus ≥ 2



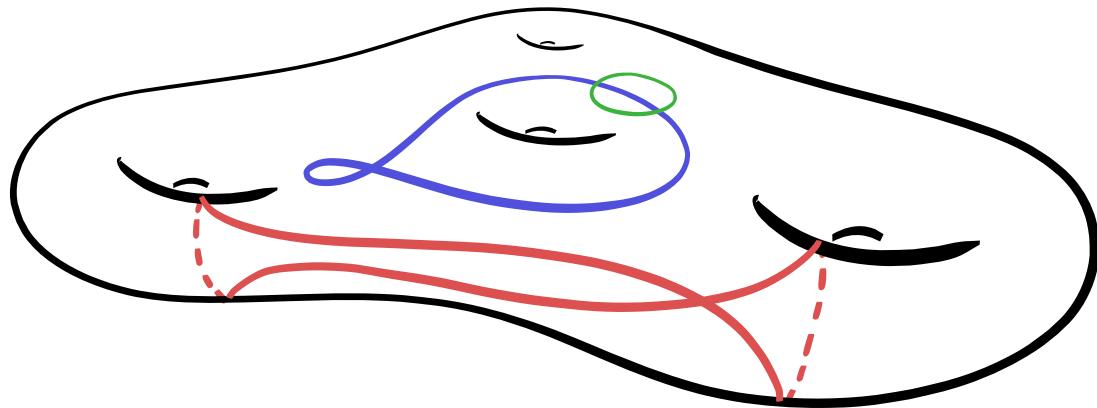
Related problem: making curves cross minimally

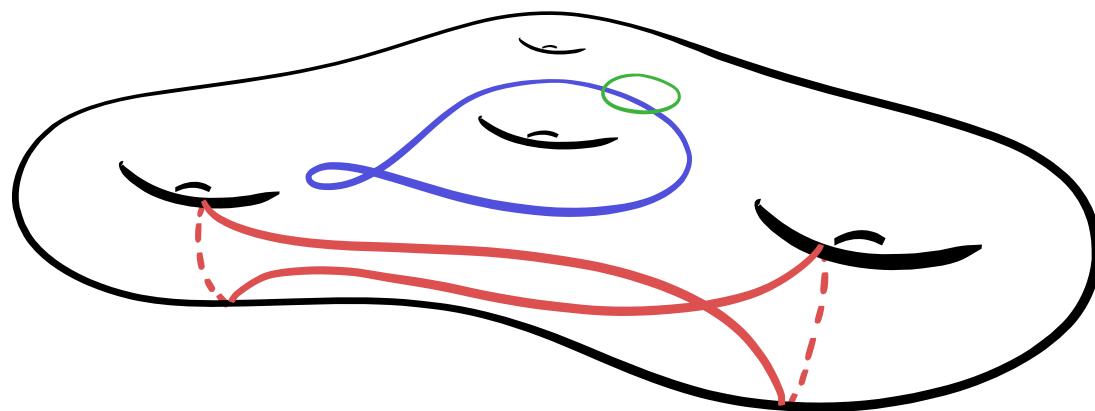
Input: closed curves on a surface

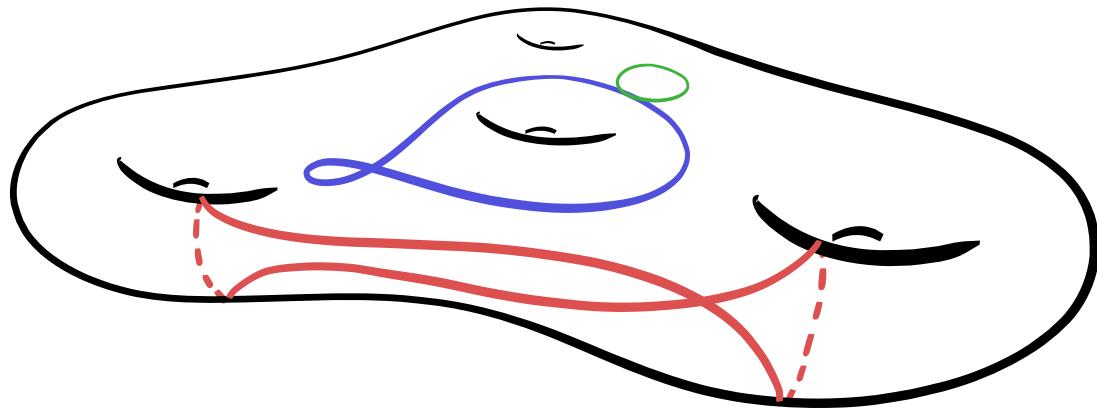


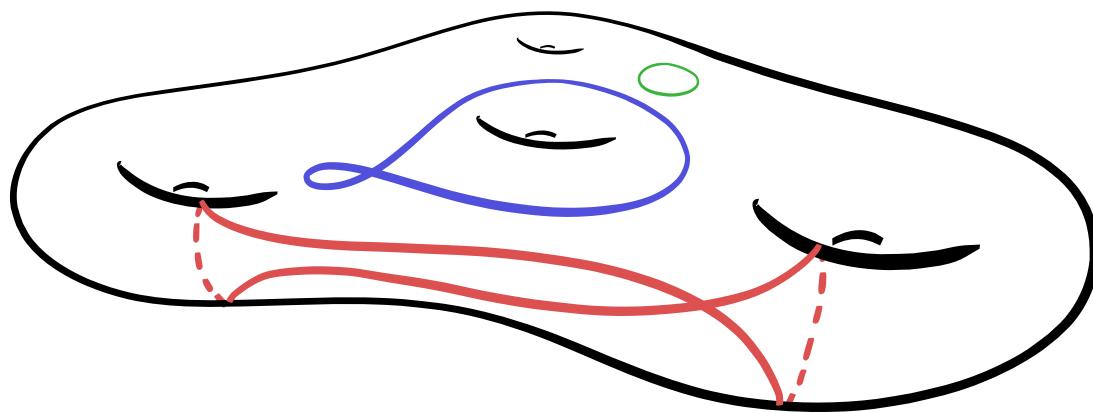
Goal:

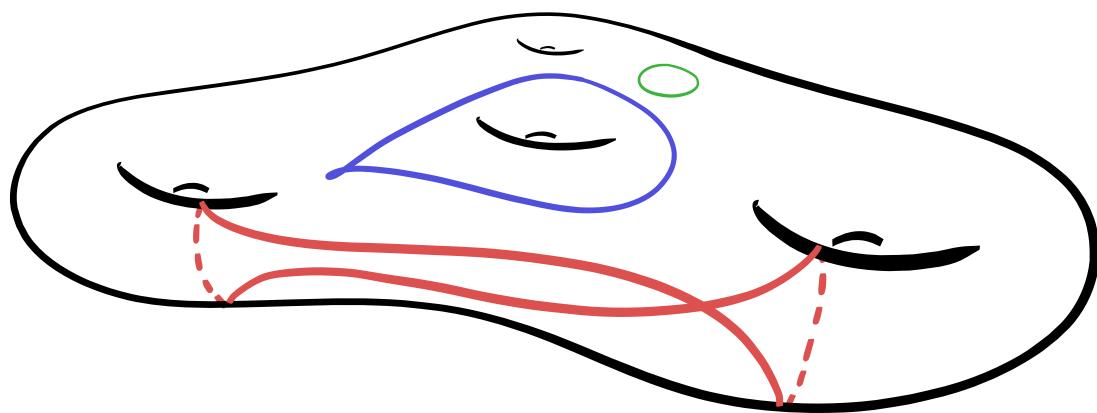
minimize the # crossings by deforming the curves

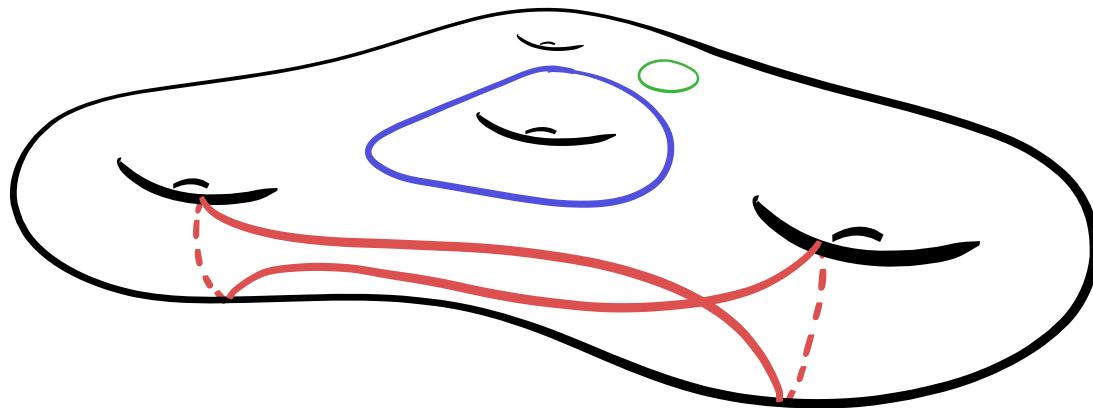


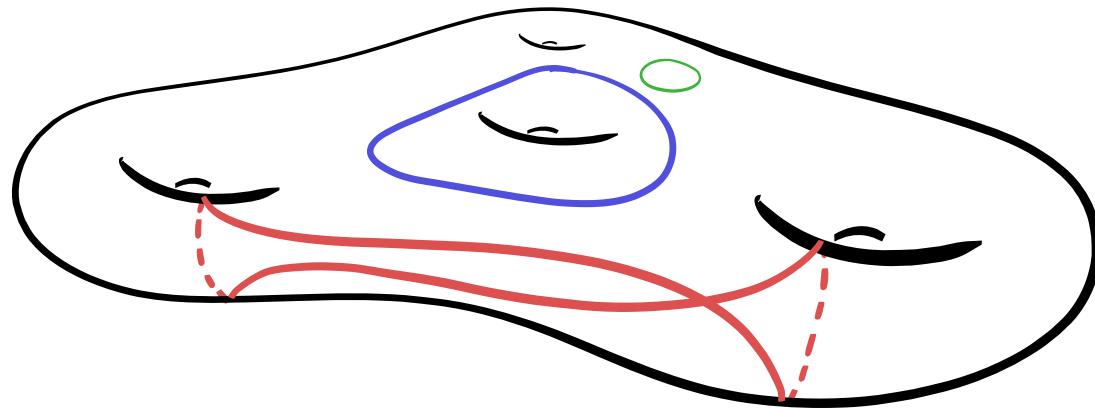












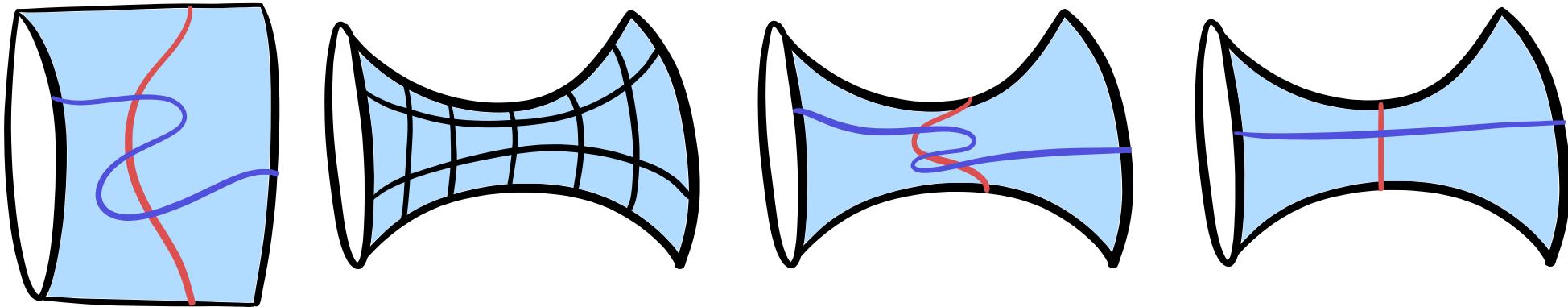
Output: $\min \#$ of crossings (+ optimal curves)

Many works related to making curves cross minimally!

- | | |
|-------------------------|-------------------------------|
| Poincaré, 1905 | de Graaf and Schrijver, 1987 |
| Dehn, 1911 | Dynnikov, 2002 |
| Dehn, 1912 | Paterson, 2002 |
| Reinhart, 1962 | Gonçalves et al., 2005 |
| Zieschang, 1965 | Schaefer et al., 2008 |
| Chillingworth, 1969 | Lazarus and Rivaud, 2012 |
| Zieschang, 1969 | Erickson and Whittlesey, 2013 |
| Chillingworth, 1971 | Arettines, 2015 |
| Turaev, 1979 | Chang et al., 2018 |
| Birman and Series, 1984 | Despré and Lazarus, 2019 |
| Cohen and Lustig, 1984 | Fulek and Tóth, 2020 |
| Hass and Scott, 1985 | Chang and de Mesmay, 2022 |
| Lustig, 1987 | Lackenby, 2024 |

Method for making curves cross minimally

Poincaré, 1905



1. give special shape to surface
2. straighten the curves

The special shape

negative curvature:

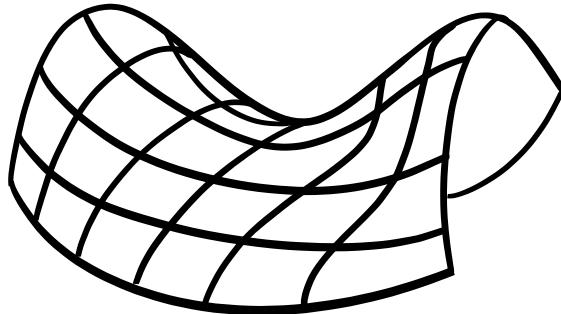


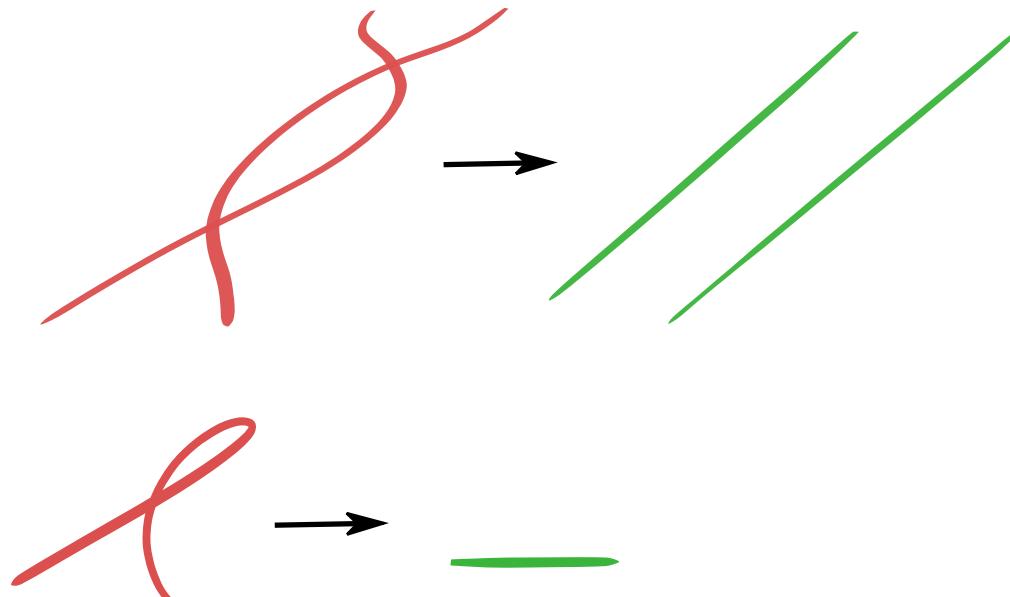
image by Susan Lombardo

almost all surfaces
can be curved negatively

The key property

On a negatively curved surface,

the geodesics cross minimally



(does not hold
on all surfaces)



The key property

On a negatively curved surface,

the geodesics cross minimally

The key property

On a negatively curved surface,

the geodesics cross minimally



even
co

The key property

On a negatively curved surface,

the geodesics cross minimally



every homo
contains a

The key property

On a negatively curved surface,

the geodesics cross minimally



every homotopy class contains a unique

The key property

On a negatively curved surface,

geodesics cross minimally



every homotopy class of paths contains a unique geodesic

The key property

On a negatively curved surface,

cross minimally



every homotopy class of paths
contains a unique **geodesic**

The key property

On a negatively curved surface,

mally



every homotopy class of paths
contains a unique **geodesic**

The key property

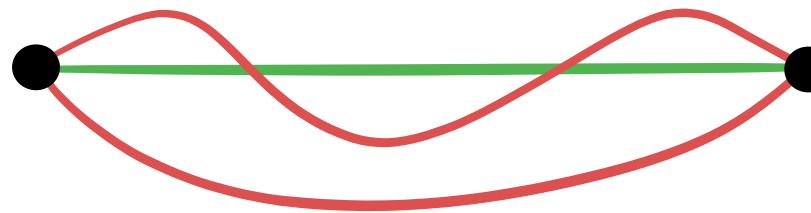
On a negatively curved surface,

every homotopy class of paths
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The key property

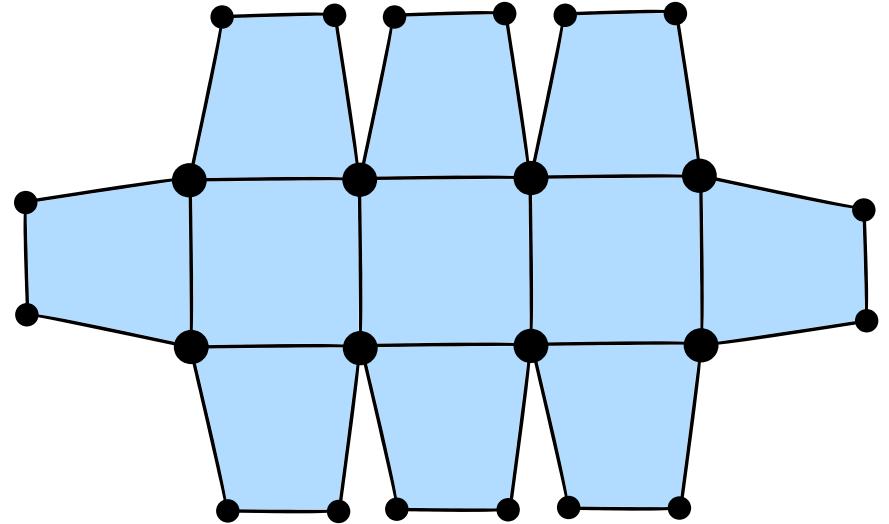
On a negatively curved surface,

every homotopy class of paths
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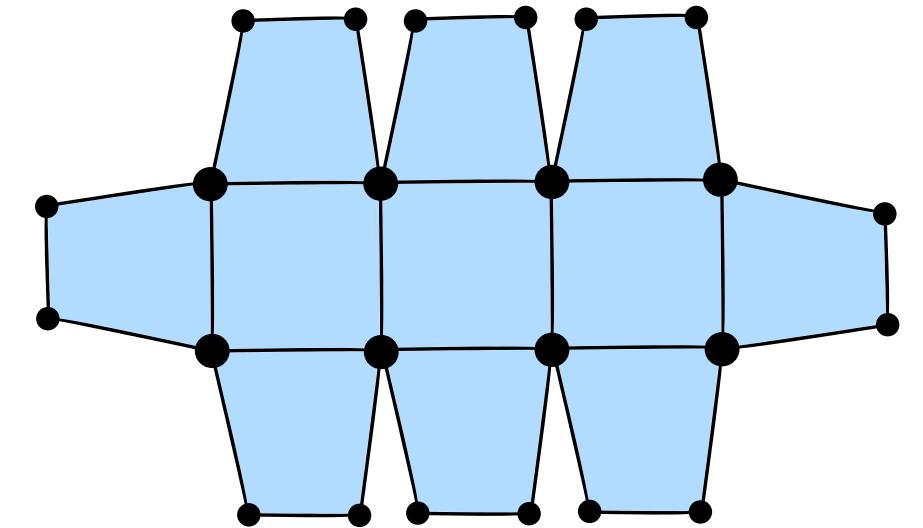


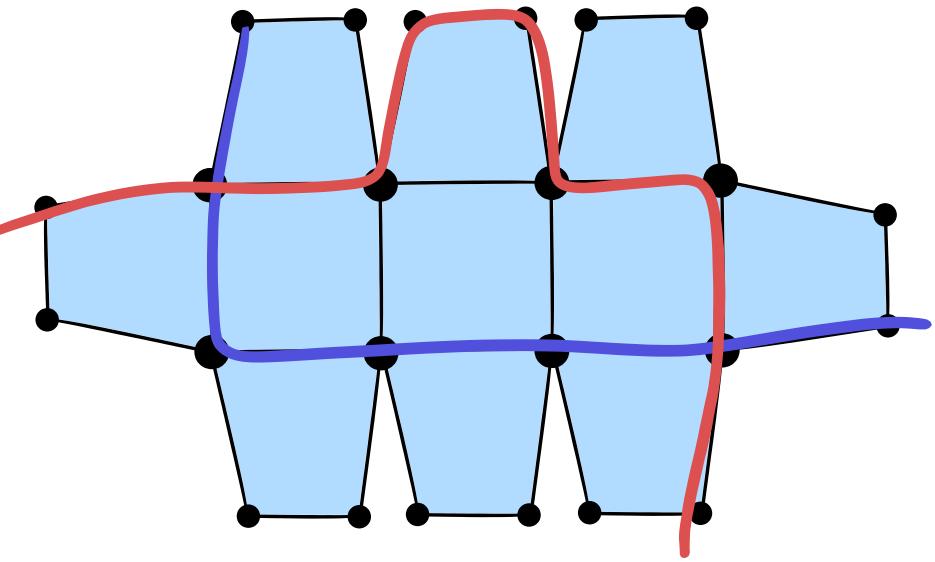
Discrete model of negatively
curved surfaces?

Lazarus and Rivaud, 2012
Erickson and Whittlesey, 2013

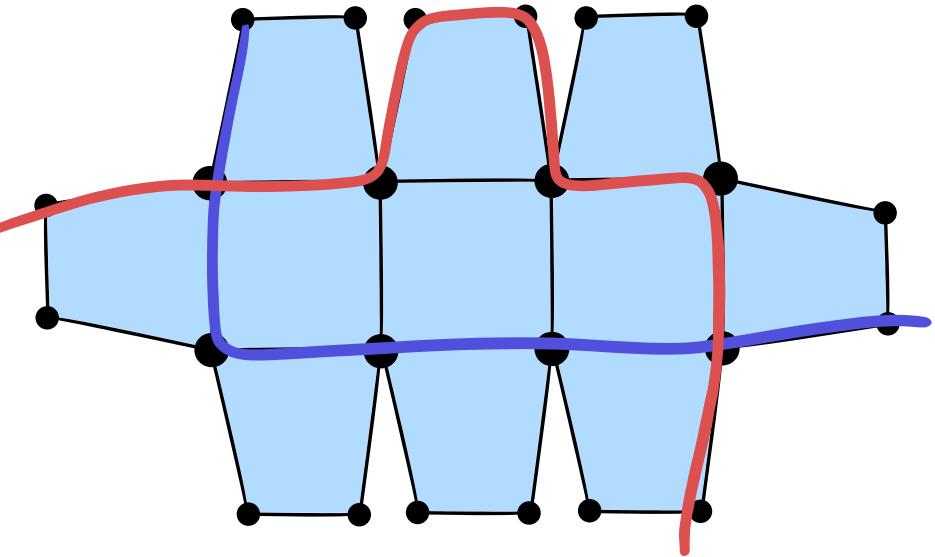


System of quads

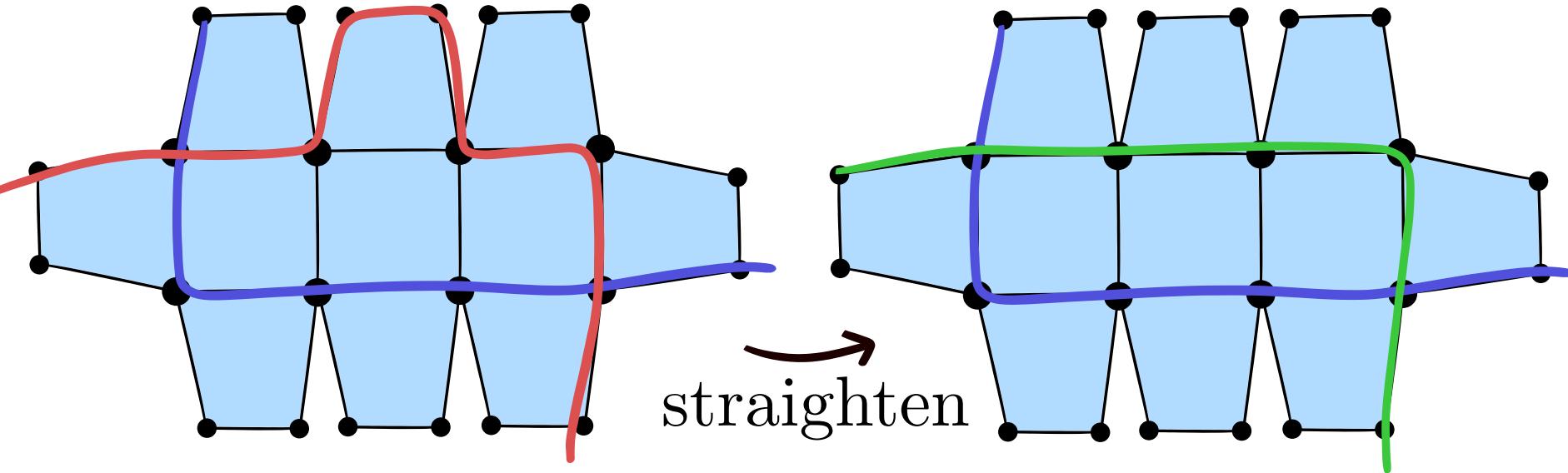




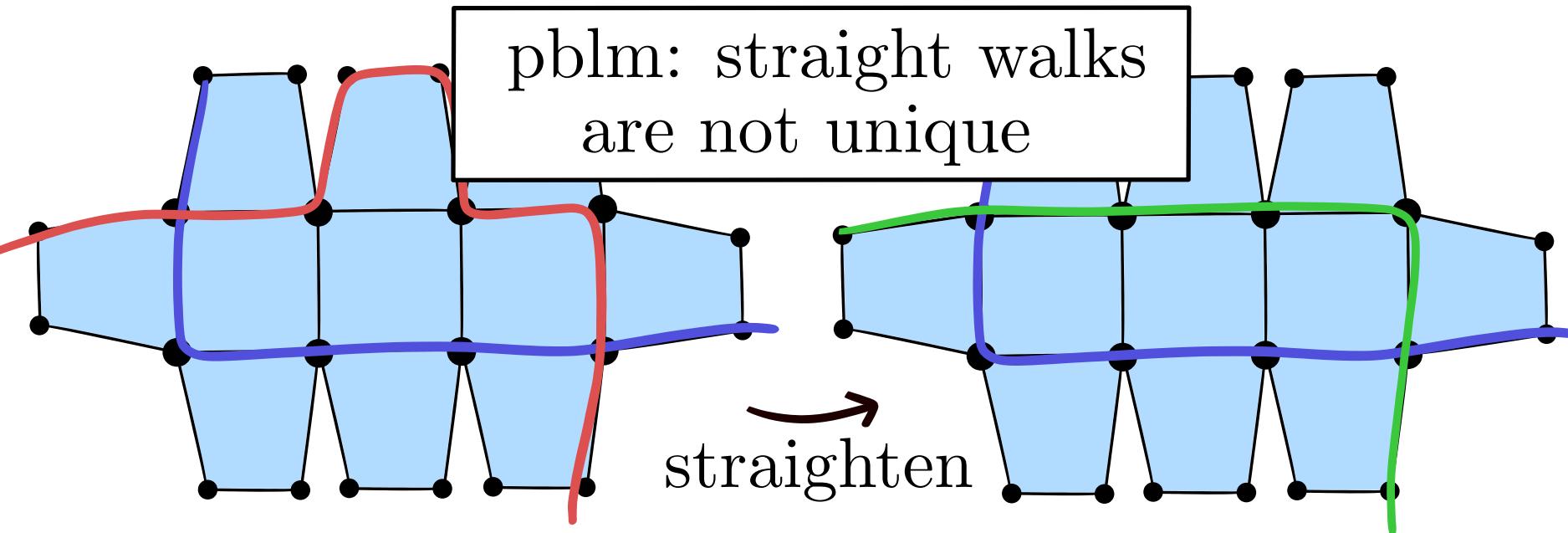
Algo for making curves cross minimally



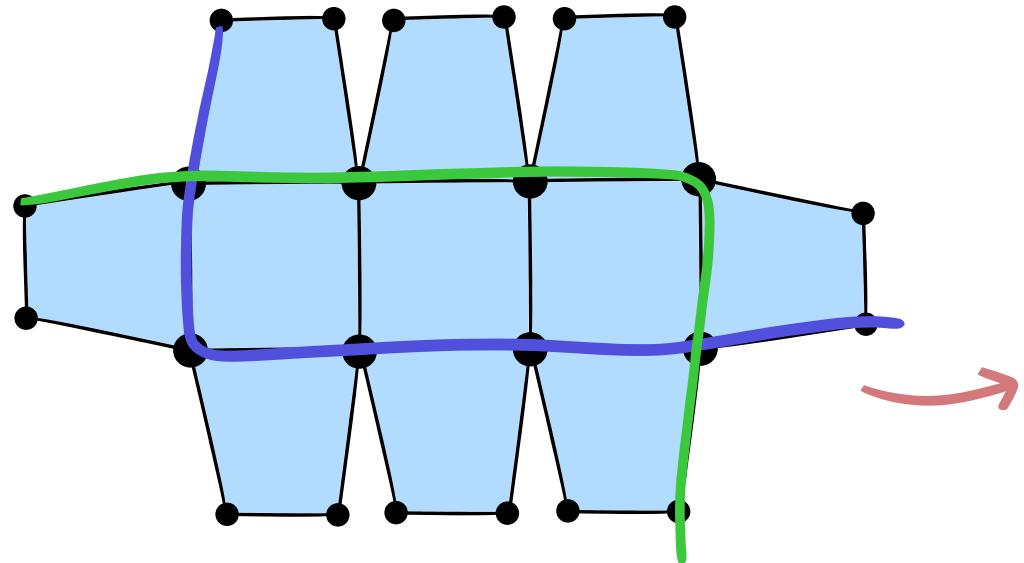
Algo for making curves cross minimally



Algo for making curves cross minimally



Algo for making curves cross minimally



Despré and Lazarus, 2019

What about
untangling graphs?

Method for untangling graphs

Tutte, 1963

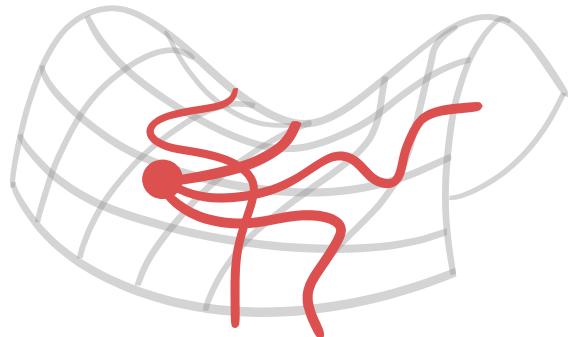
Y. Colin de Verdière, 1991



Method for untangling graphs

Tutte, 1963

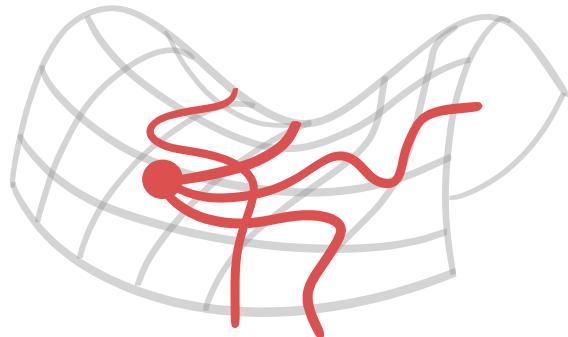
Y. Colin de Verdière, 1991



Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight

Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight

Method for untangling graphs

Tutte, 1963



Y. Colin de Verdière, 1991

make edges straight
make vertices barycentric

Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight
make vertices barycentric

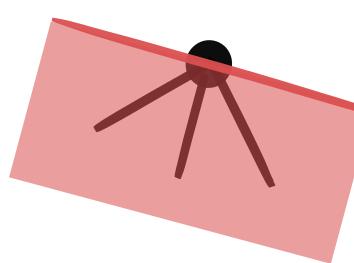
Method for untangling graphs

Tutte, 1963

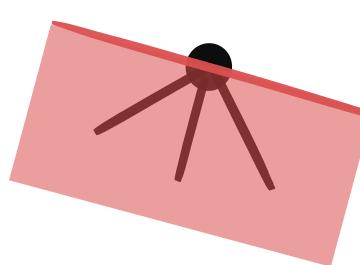


Y. Colin de Verdière, 1991

make edges straight
make vertices barycentric

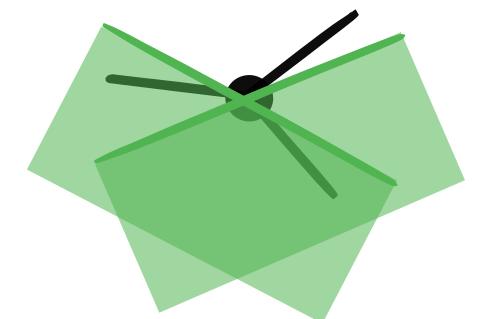
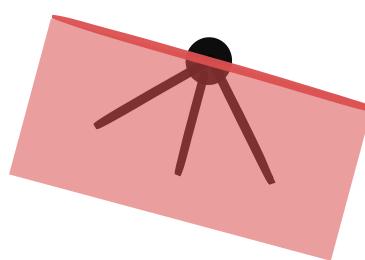


edges straight
vertices barycentric

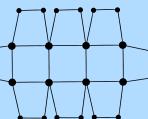


Tutte drawings

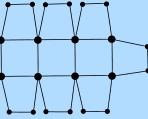
edges straight
vertices barycentric



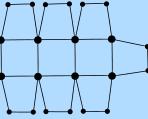
Summary

	Curves	Graphs
Method	negatively curved surface	
Algo	system of quads	

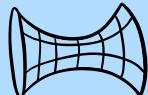
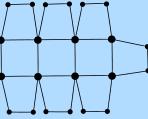
Our results

	Curves	Graphs
Method	negatively curved surface	
Algo	system of quads	

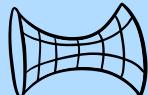
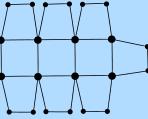
Our results

	Curves	Graphs
Method	negatively curved surface	
Algo	system of quads	
reducing triangulations		

Our results

	Curves	Graphs
Method	negatively curved surface 	
Algo	system of quads 	
	reducing triangulations	
	improved algos for making curves cross minimally	first algos for untangling graphs

Our results

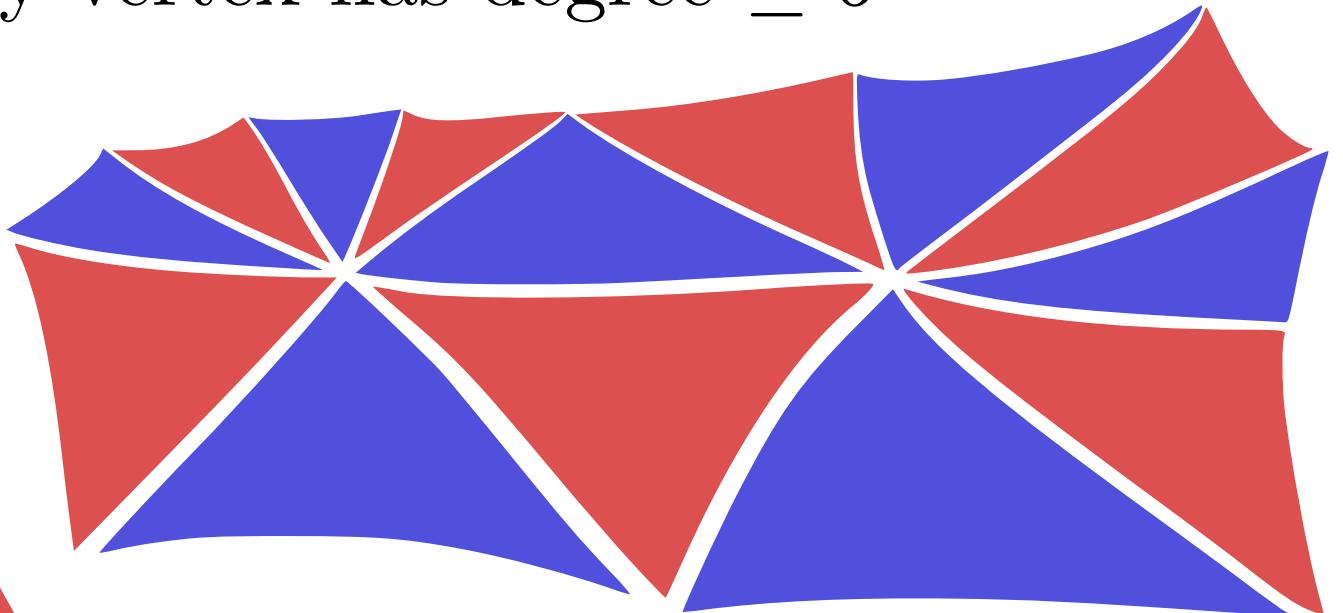
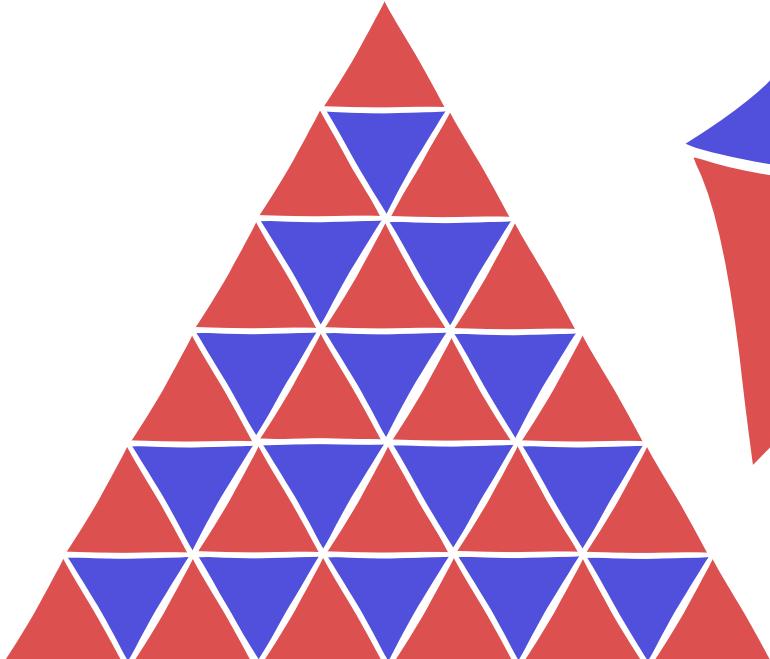
	Curves	Graphs
Method	negatively curved surface 	
Algo	system of quads 	
	reducing triangulations	
	improved algos for making curves cross minimally	first algos for untangling graphs
		Discrete analogue of Tutte drawings

A new tool:

Reducing triangulations

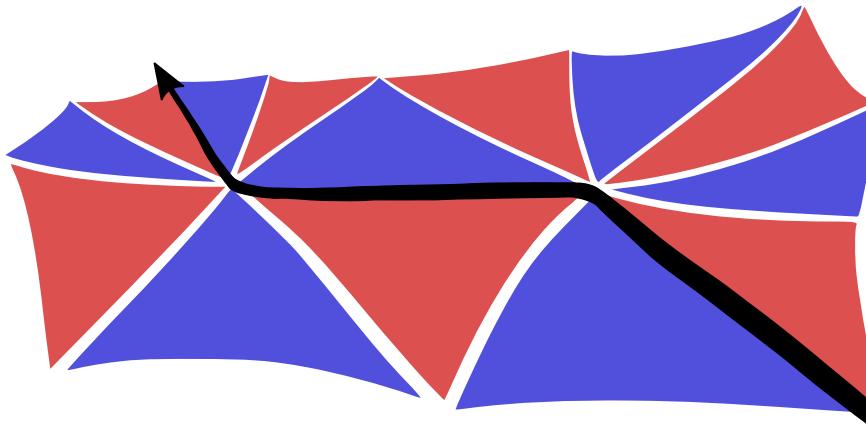
Reducing triangulations

dual is bipartite and
every vertex has degree $\geq 6^*$

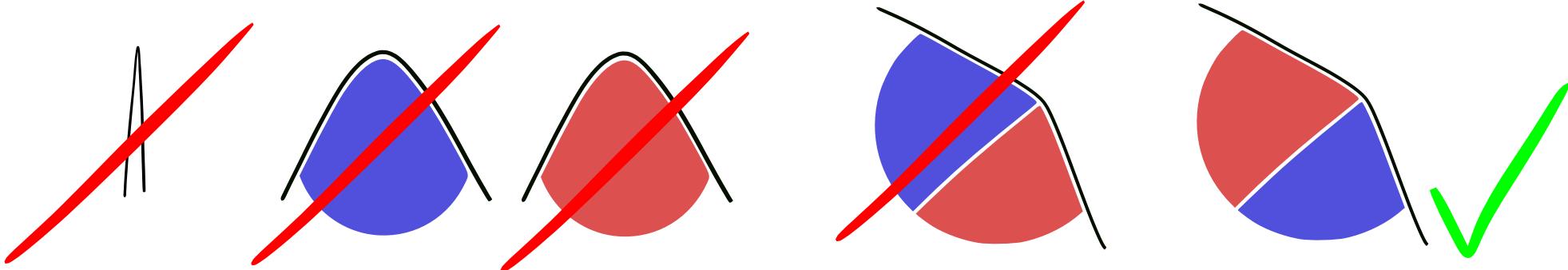


*sometimes 8

Reduced walks

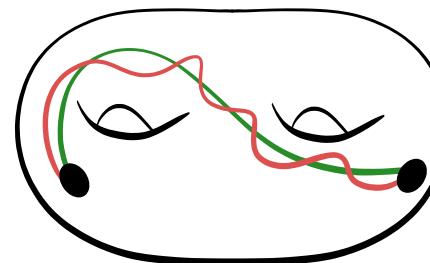
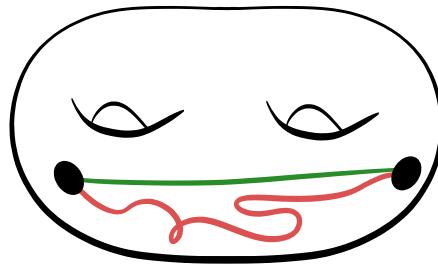


no bad turn



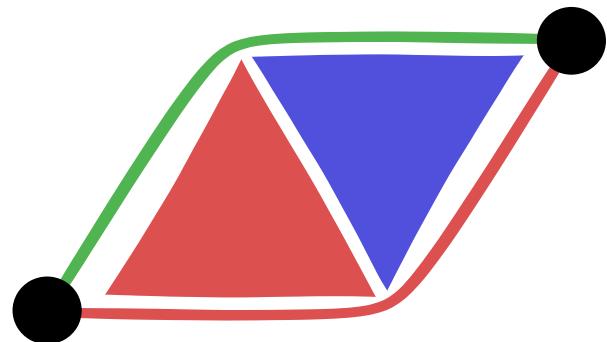
Properties of reduced walks

every walk can be deformed into a unique reduced walk, computable in linear time

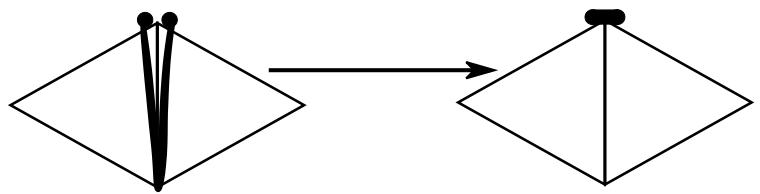


reduced walks are stable upon reversal and subwalk

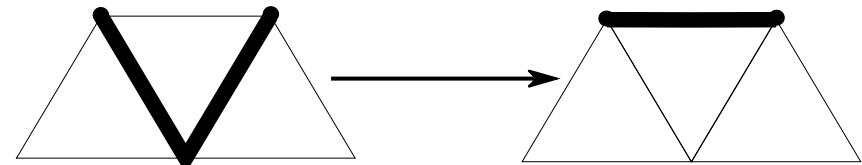
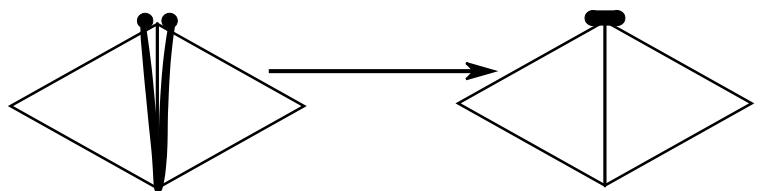
Purpose of the coloring



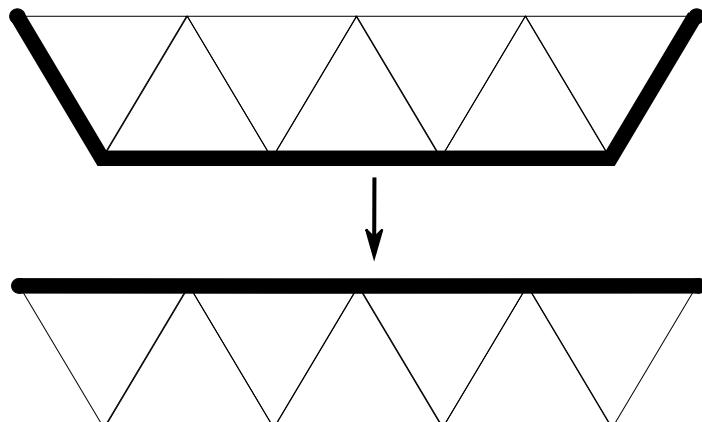
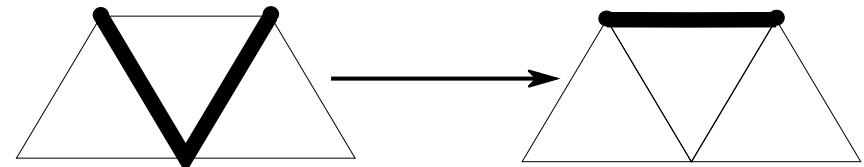
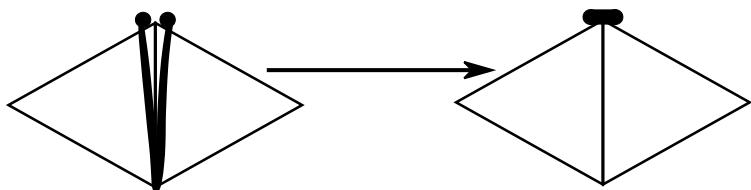
Reducing a walk



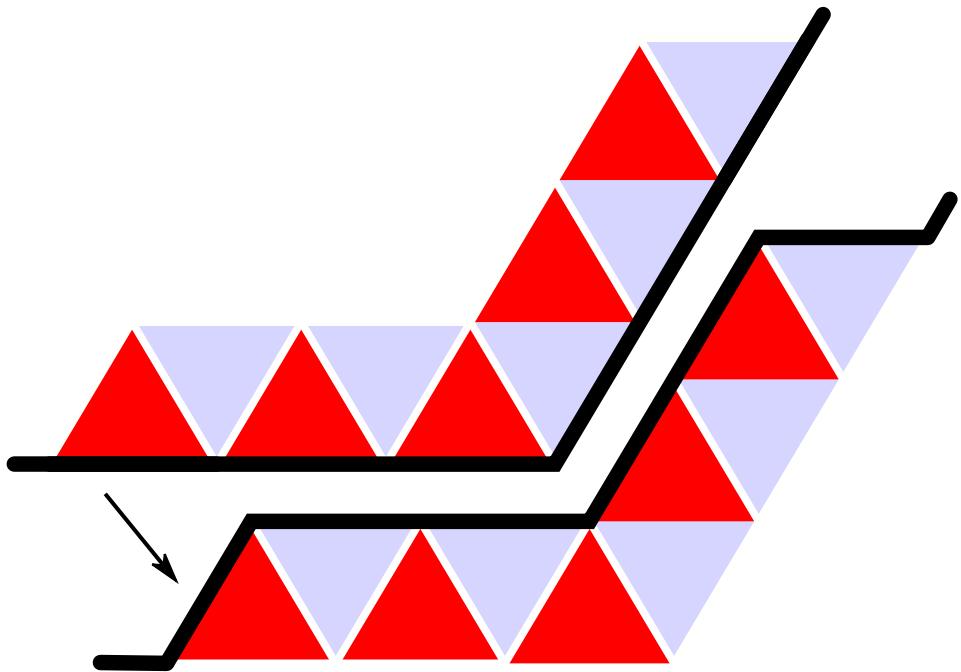
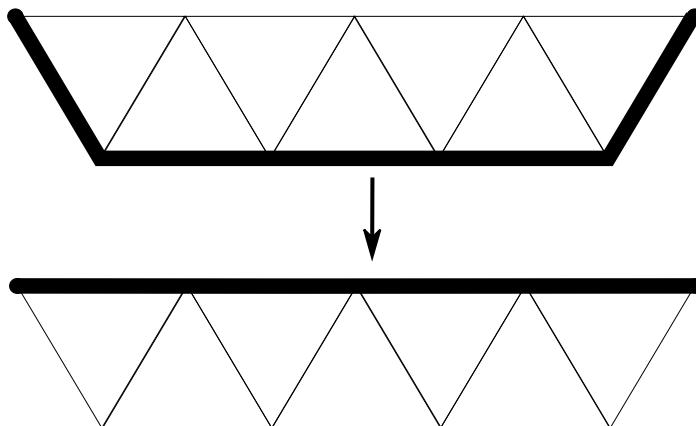
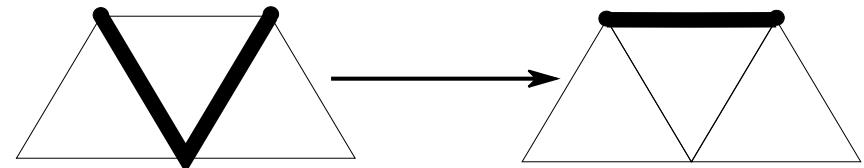
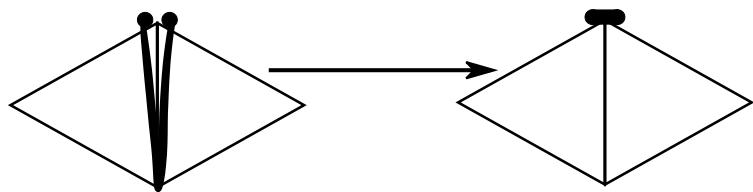
Reducing a walk



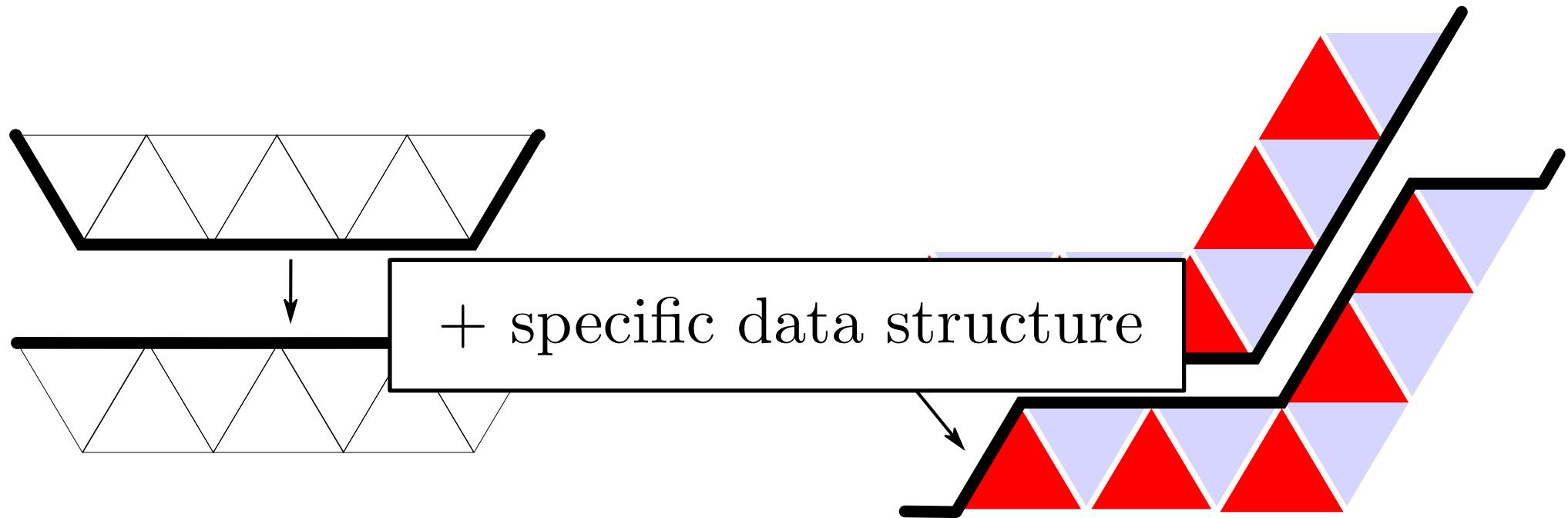
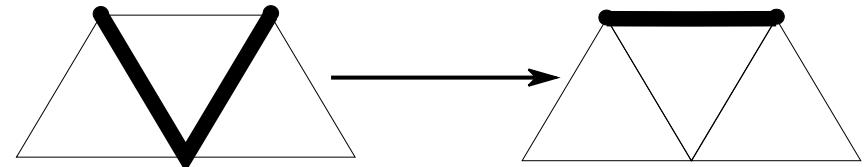
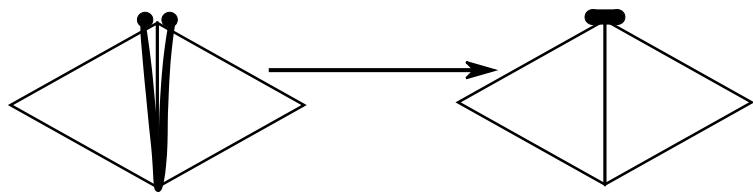
Reducing a walk



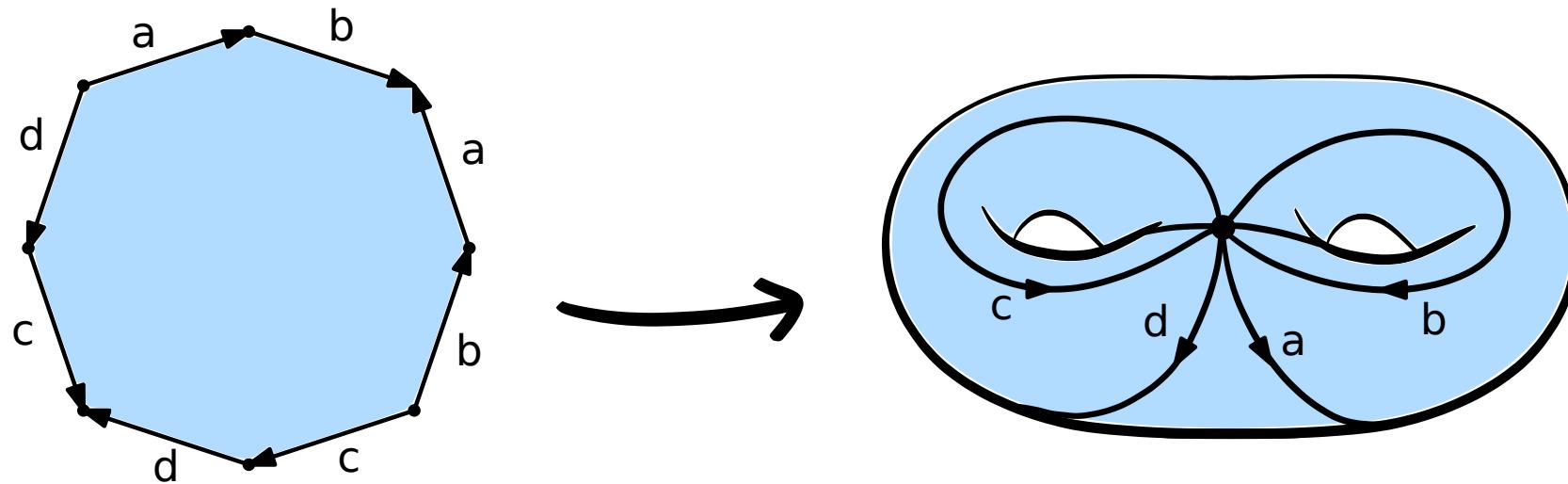
Reducing a walk



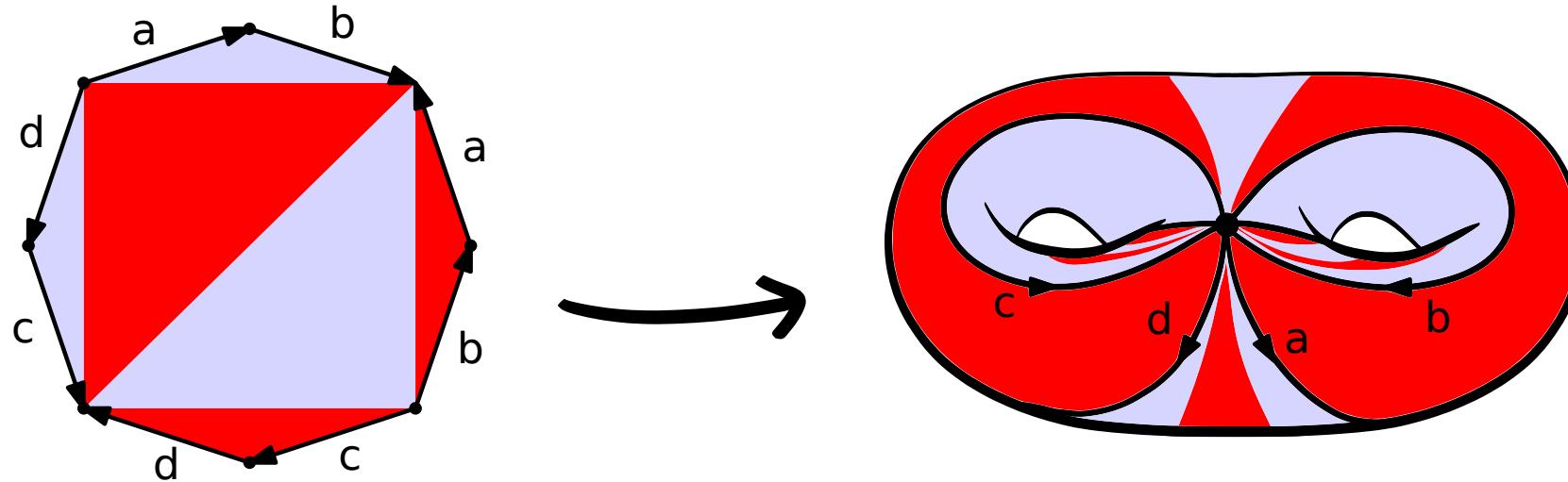
Reducing a walk



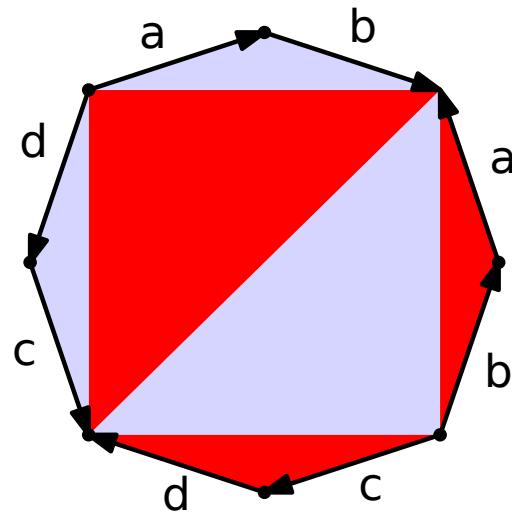
Constructing reducing triangulations



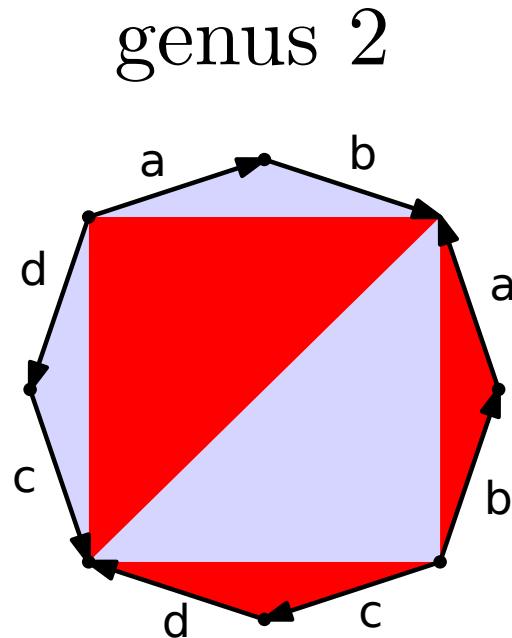
Constructing reducing triangulations



Constructing reducing triangulations

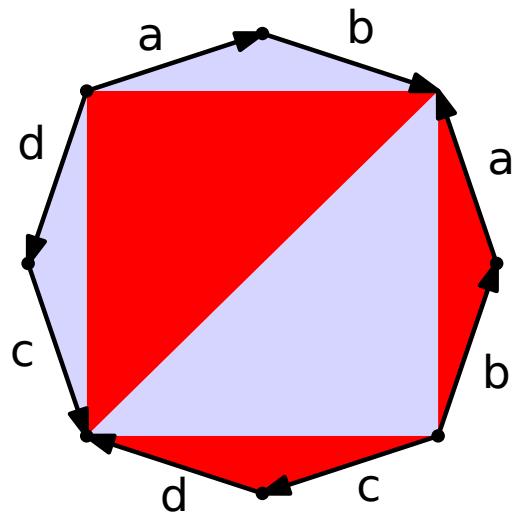


Constructing reducing triangulations

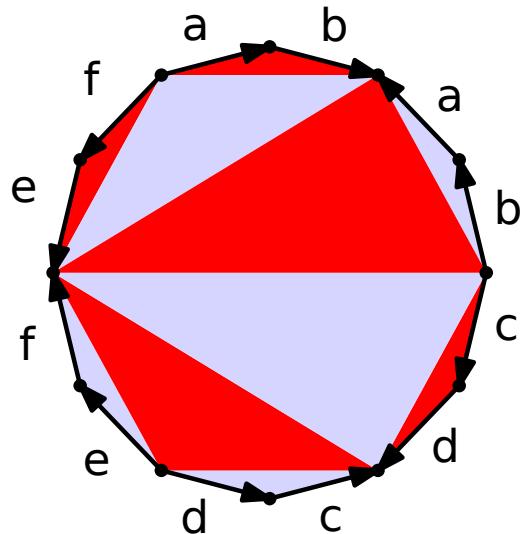


Constructing reducing triangulations

genus 2

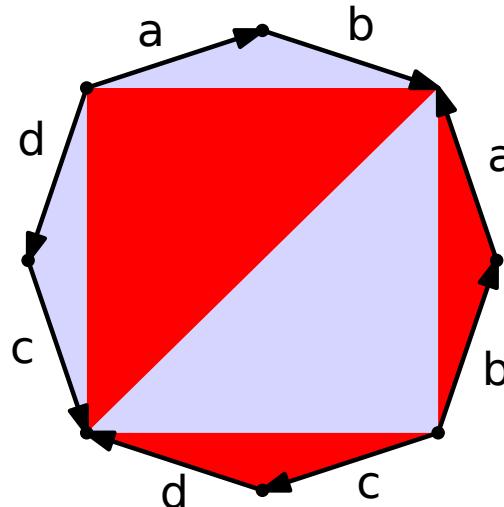


genus 3

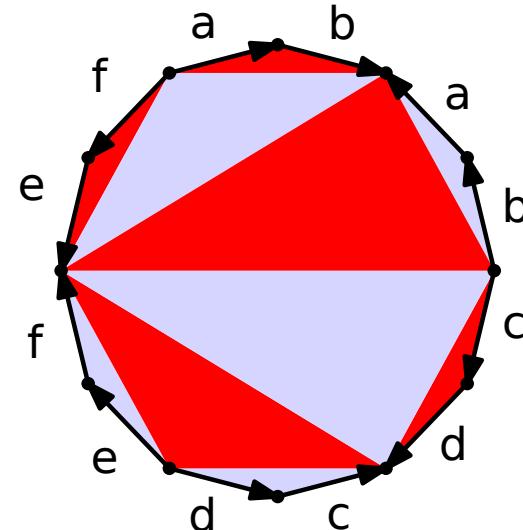


Constructing reducing triangulations

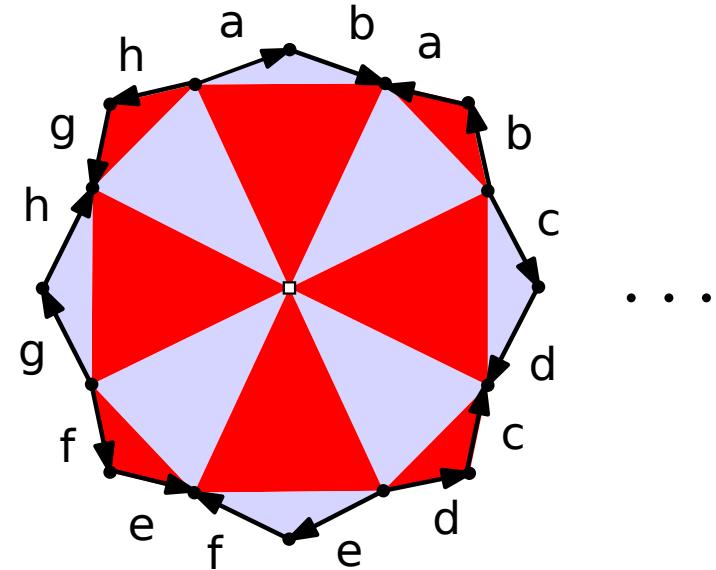
genus 2



genus 3

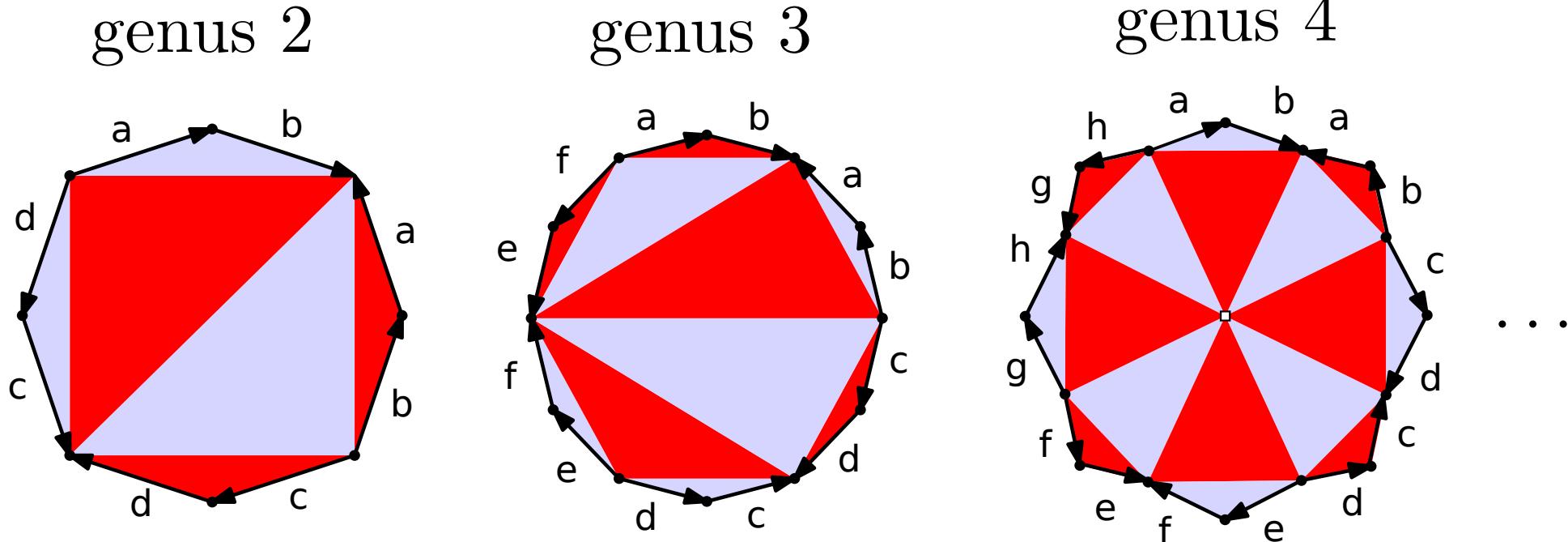


genus 4



...

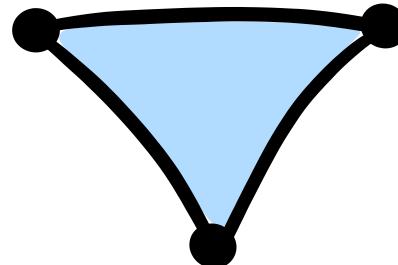
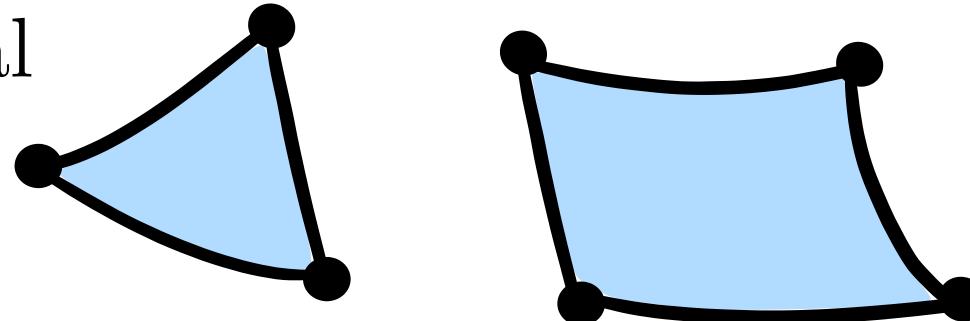
Constructing reducing triangulations



Untangling graphs using
reducing triangulations

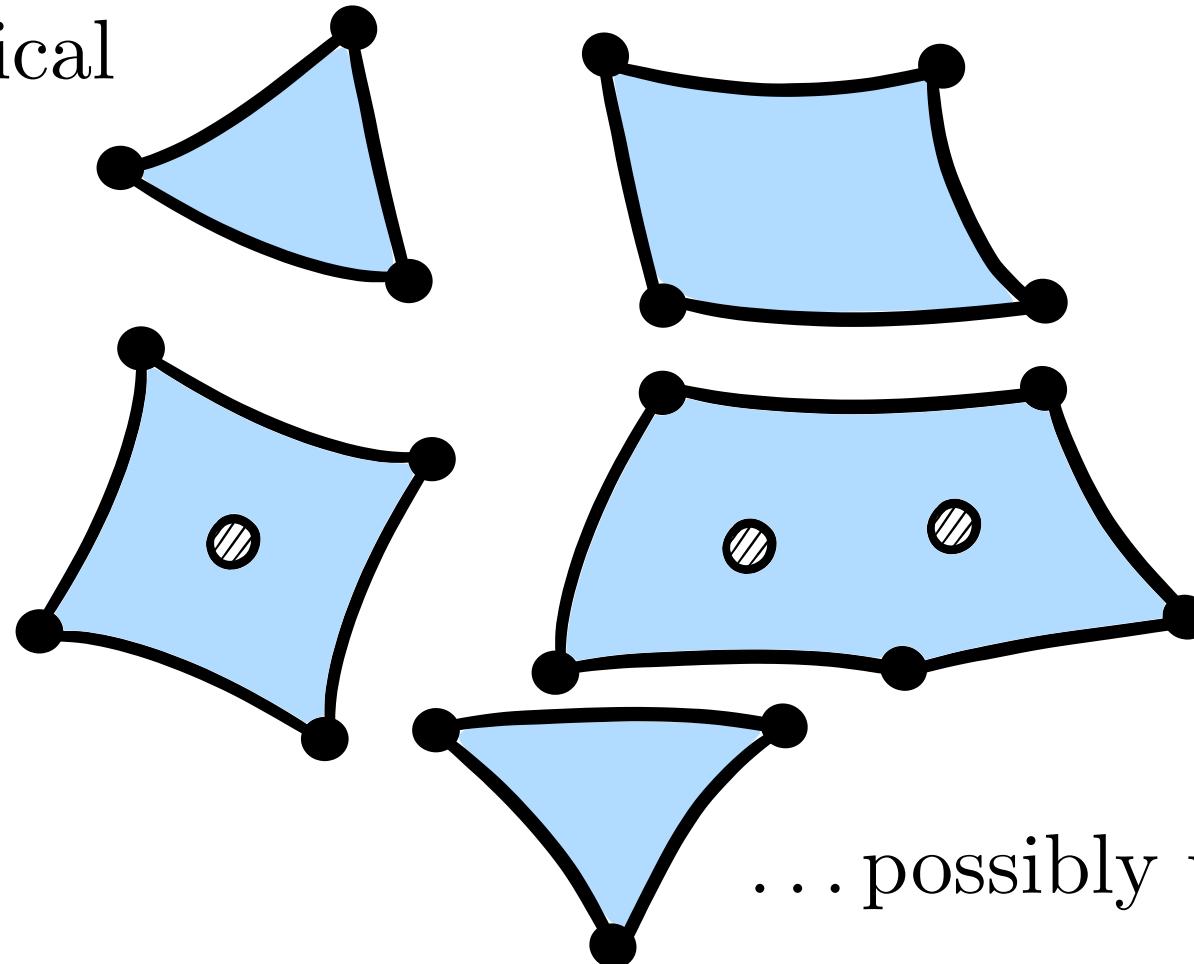
Input

take topological
polygons...



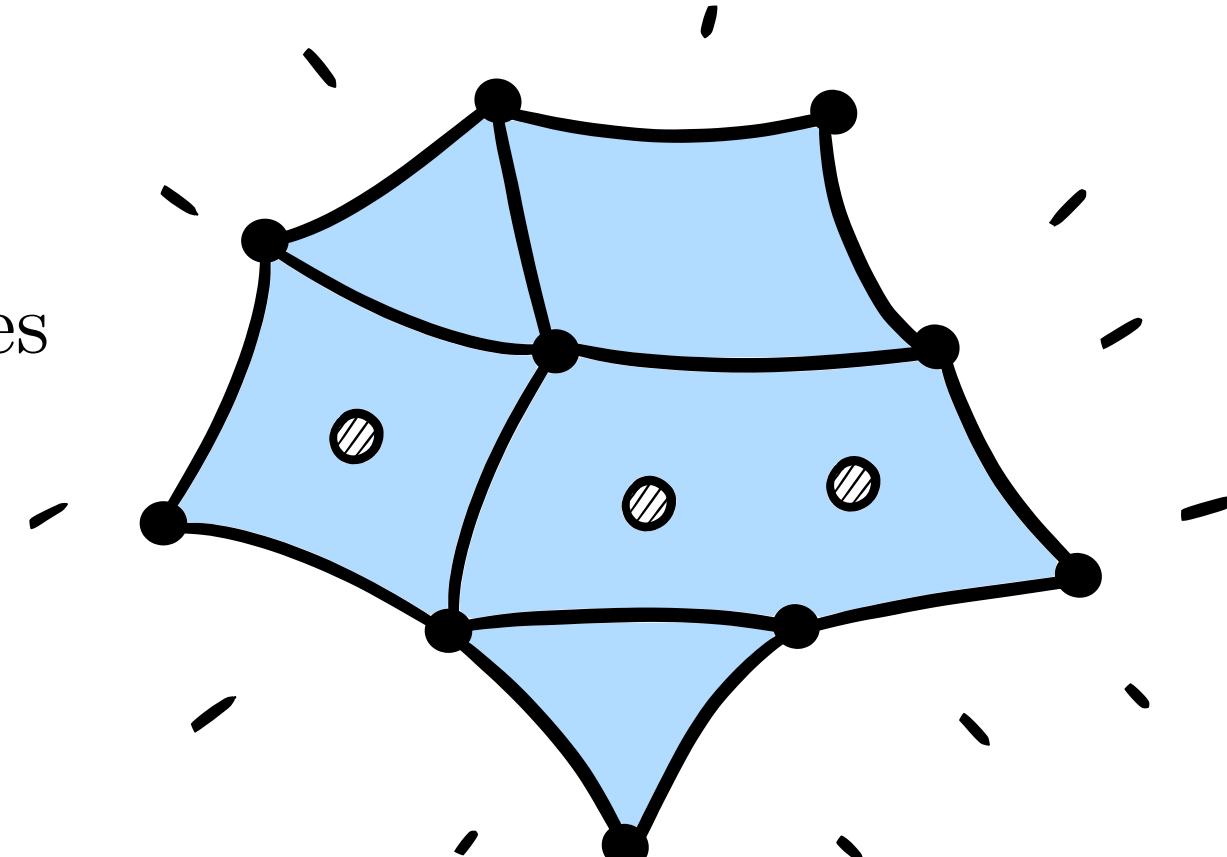
Input

take topological
polygons...



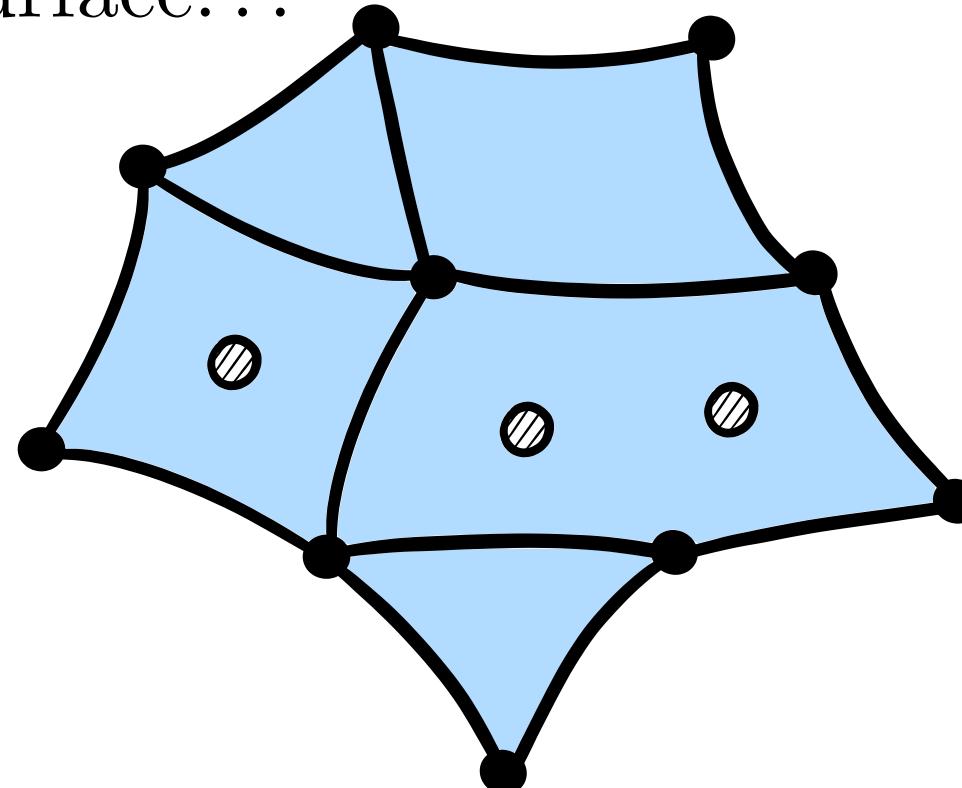
Input

glue edges



Input

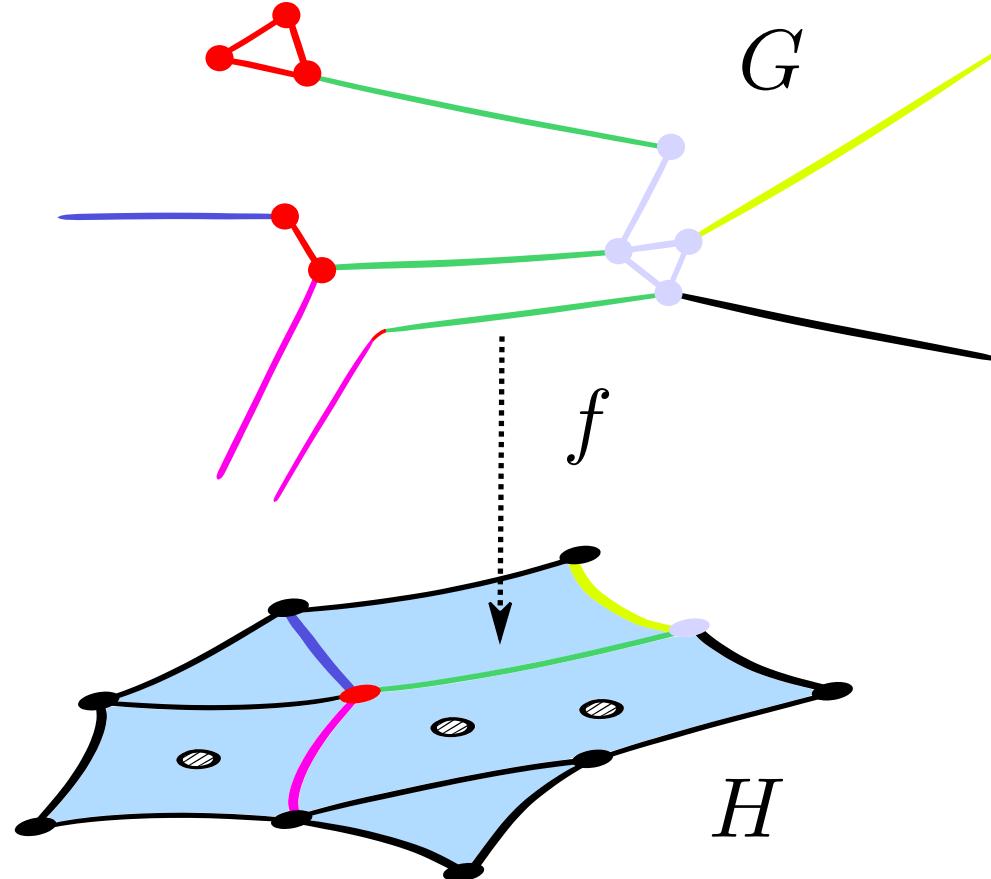
that encodes the surface . . .



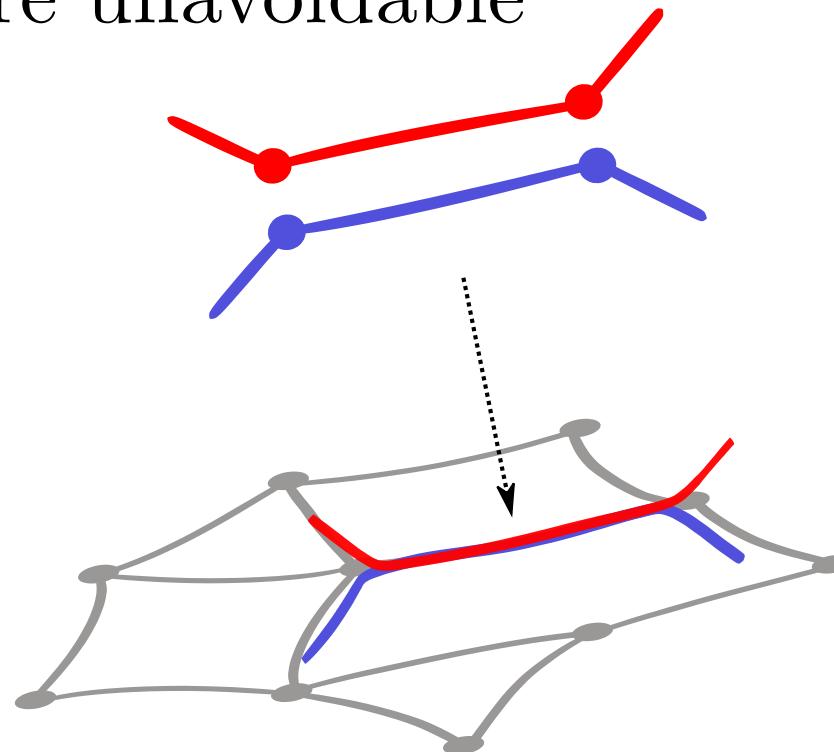
. . . and a graph H on it

Input

draw G in H

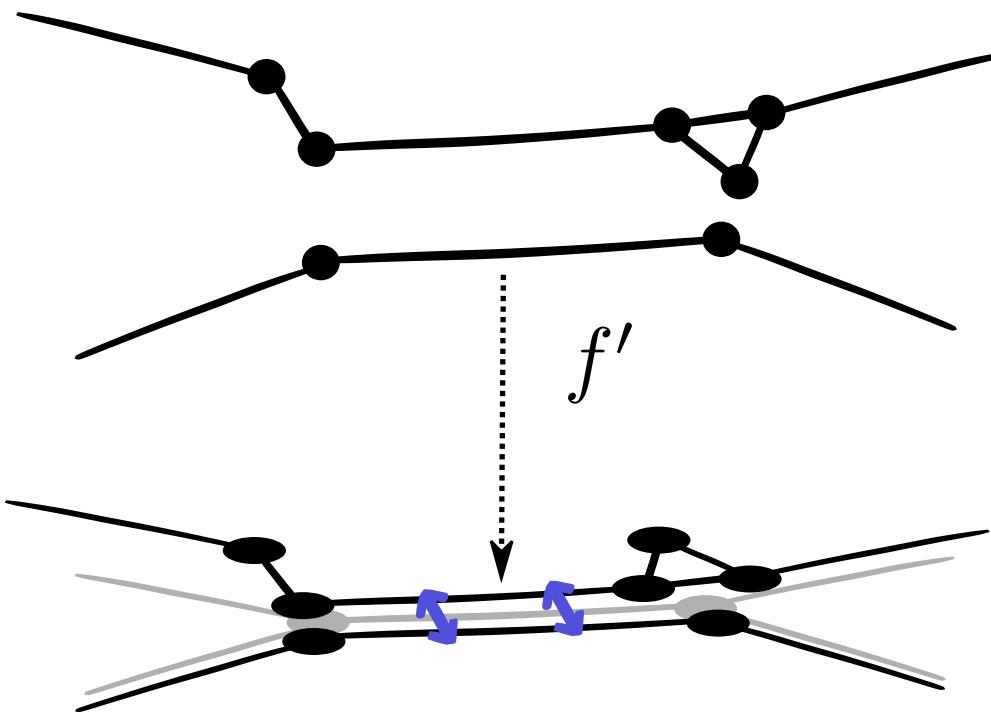


In this model overlaps
are unavoidable



Output

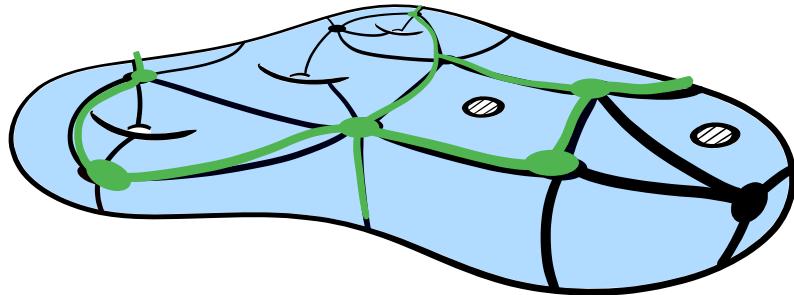
weak embedding: drawing f' that can be untangled by infinitesimal perturbation



Akitaya, Fulek, and
Tóth, 2019

algo to determine if
 f' is weak embedding,
and if so to perturb f'

Result



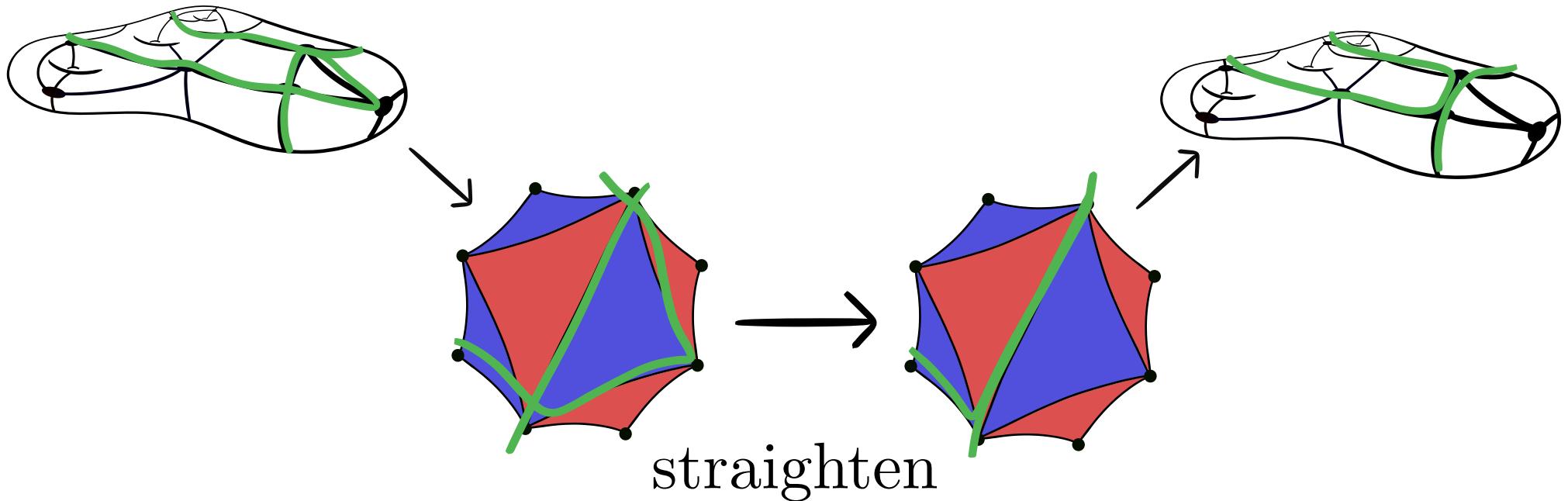
n : # times f uses an edge
or vertex of H
 m : # vertices and edges of H
 s : genus + # holes

Colin de Verdière, Despré, D., 2023

We can decide if f can be untangled,
in $O(m + s^2 n \log(s n))$ time.
If so, we can compute a weak embedding homotopic
to f in additional $O(s^2 m n^2)$ time.

Algorithm overview

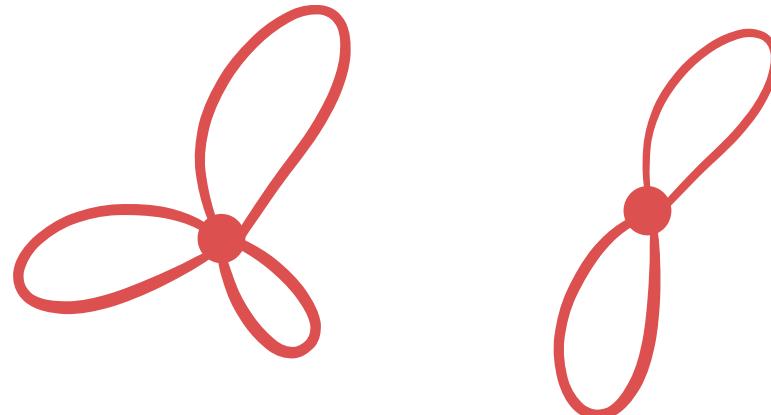
(assuming that the surface has no boundary)



Straightening a graph

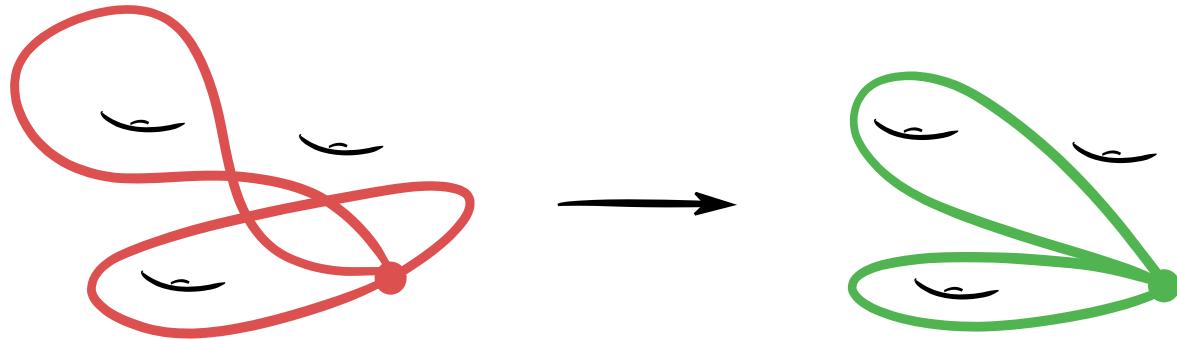
loop graph

Straightening a ~~graph~~



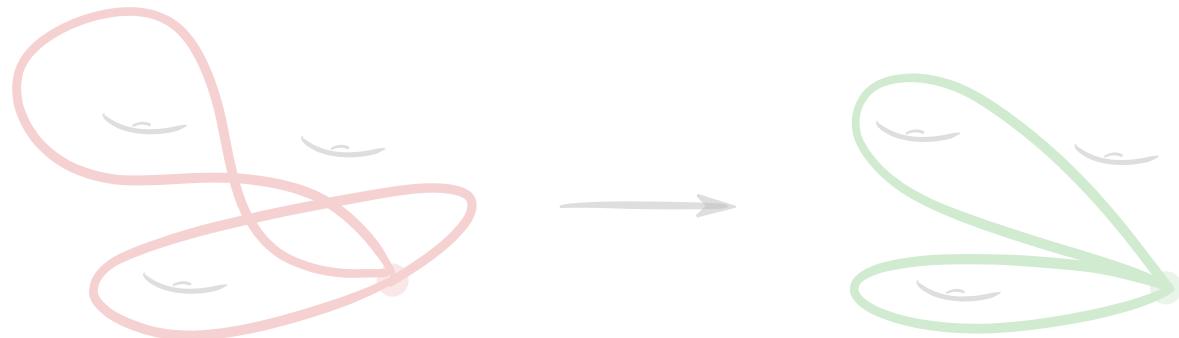
Straightening a loop graph

First attempt: reduce the loops, vertex fixed

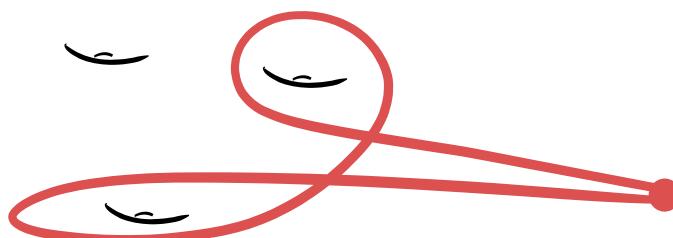


Straightening a loop graph

First attempt: reduce the loops, vertex fixed



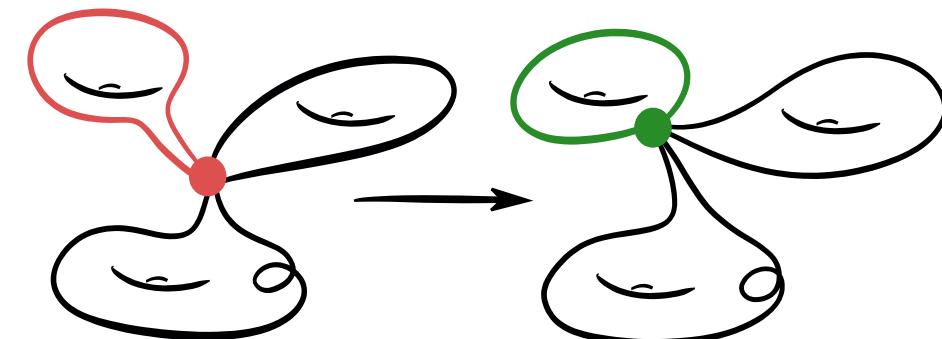
Problem:



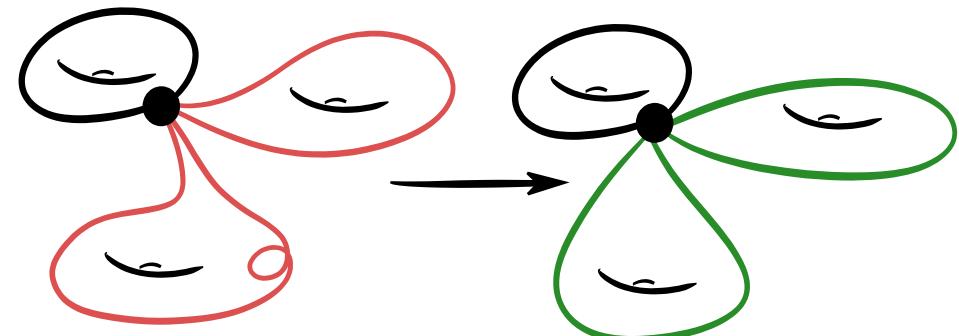
→ vertex must move

Straightening a loop graph

Solution:



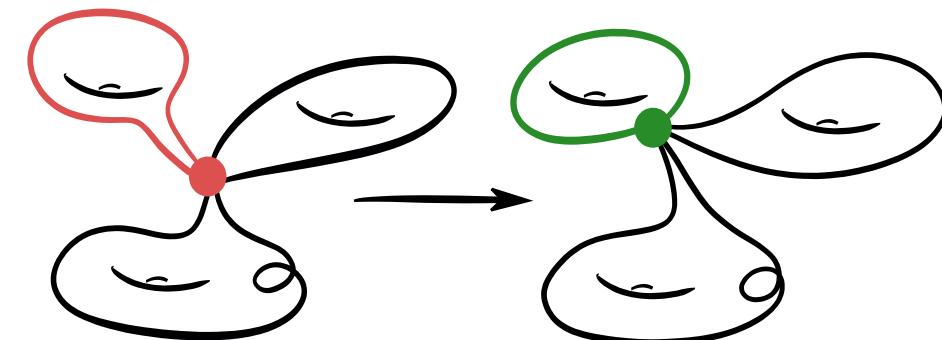
1. Reduce 1 loop cyclically



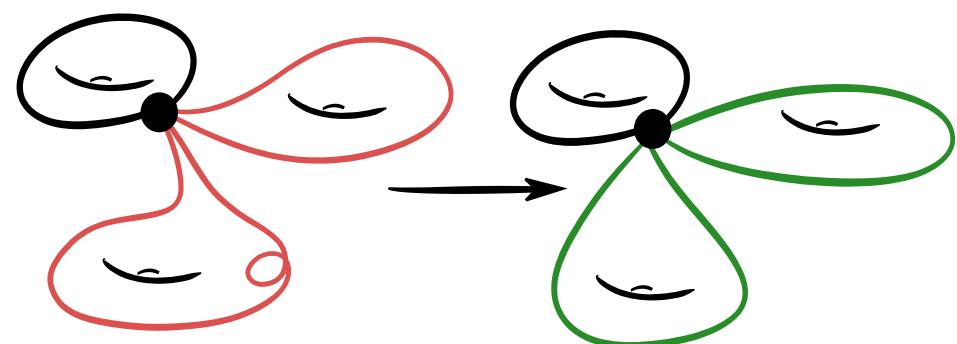
2. Reduce the other loops linearly

Straightening a loop graph

A straightened loop graph is a weak embedding
or cannot be untangled



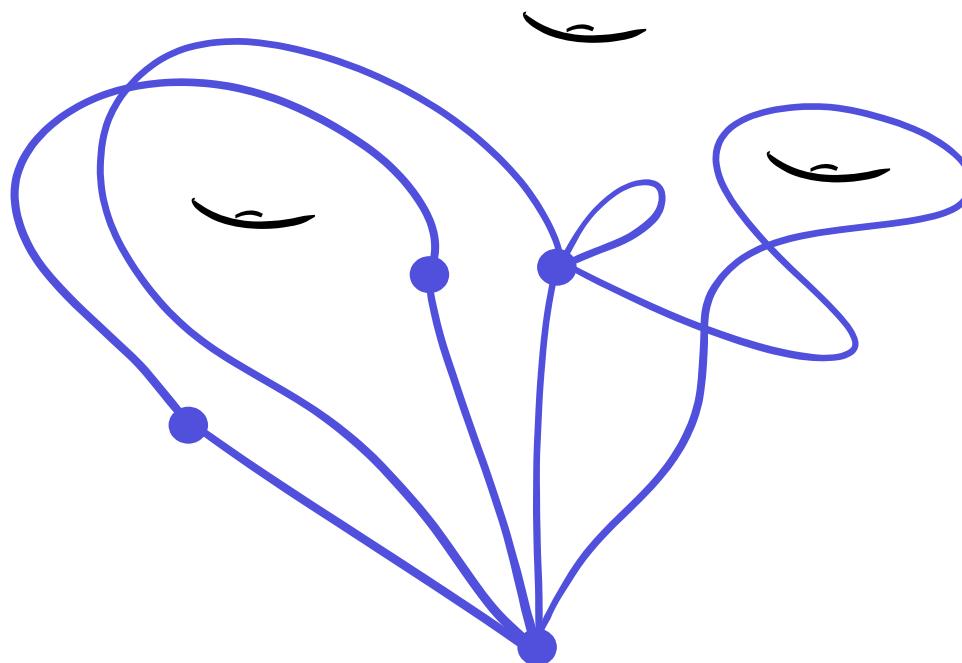
1. Reduce 1 loop cyclically



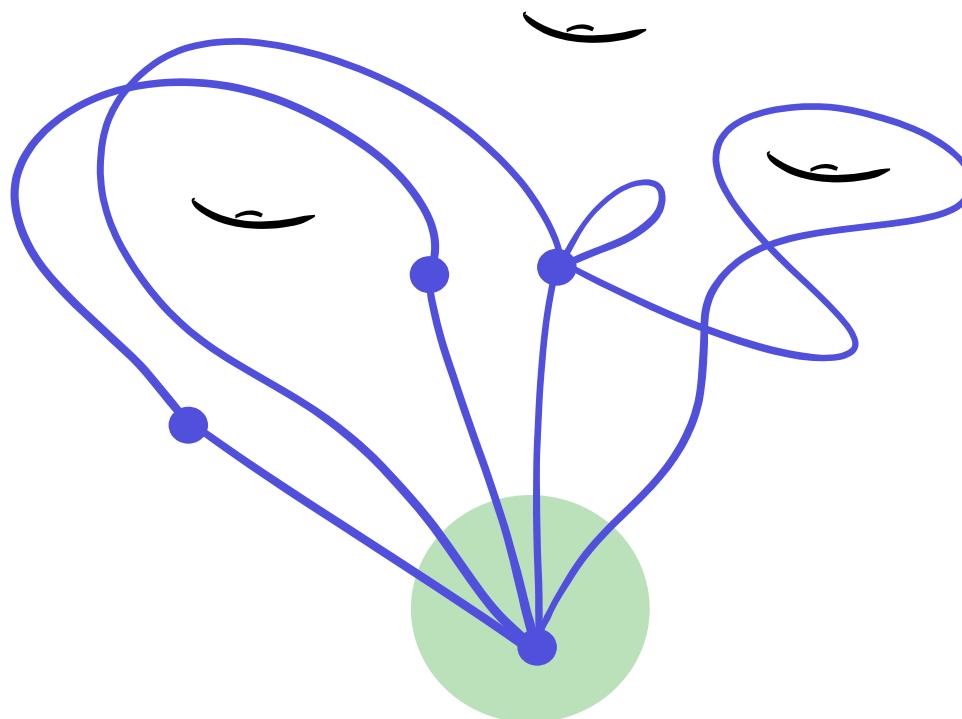
2. Reduce the other loops linearly

Straightening a graph

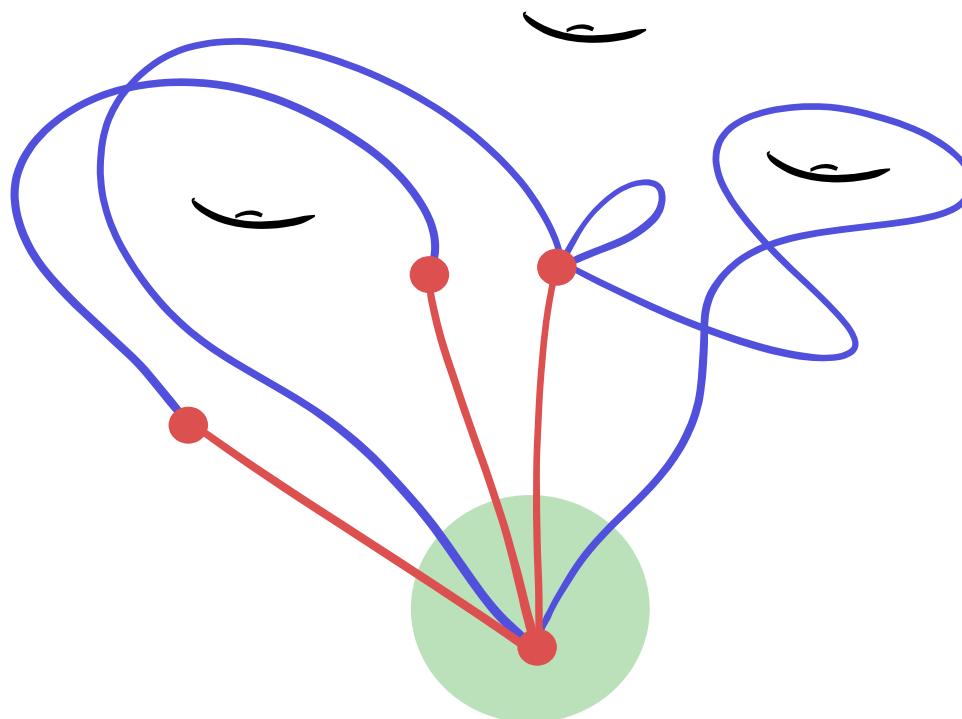
Straightening a graph



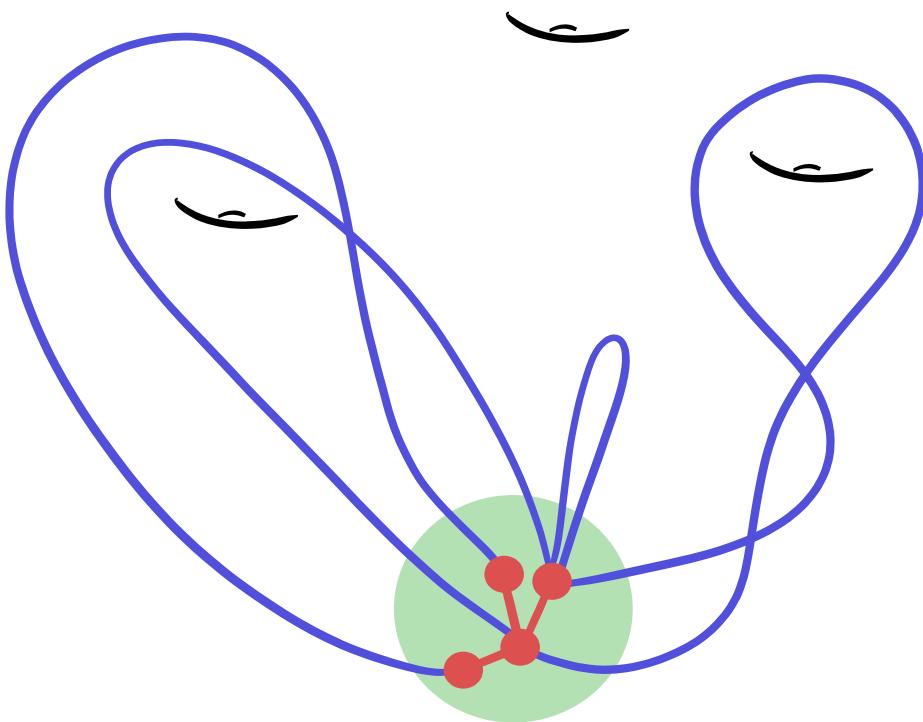
Straightening a graph



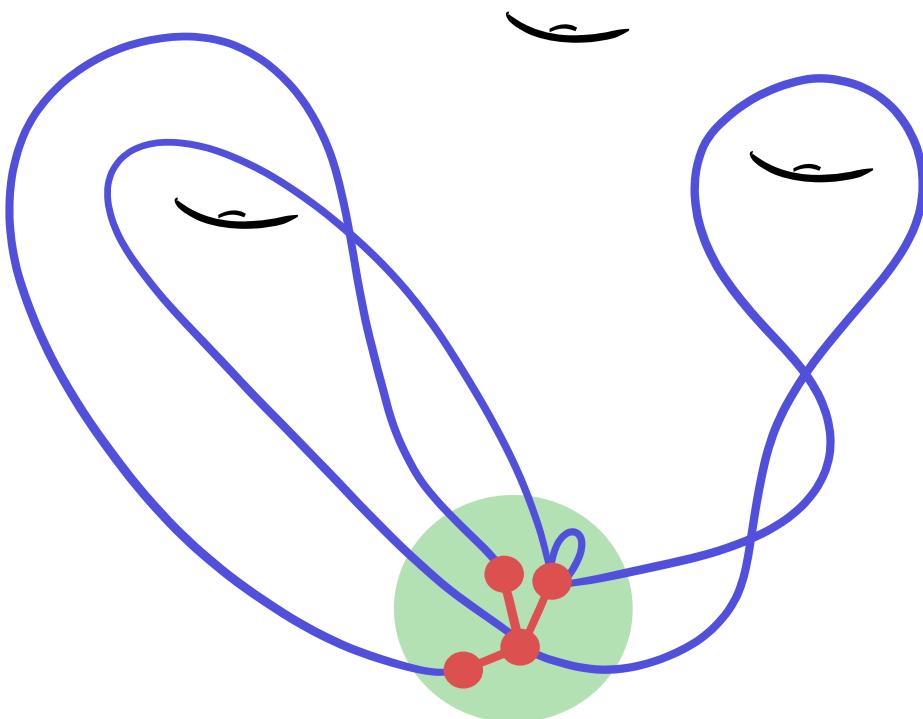
Straightening a graph



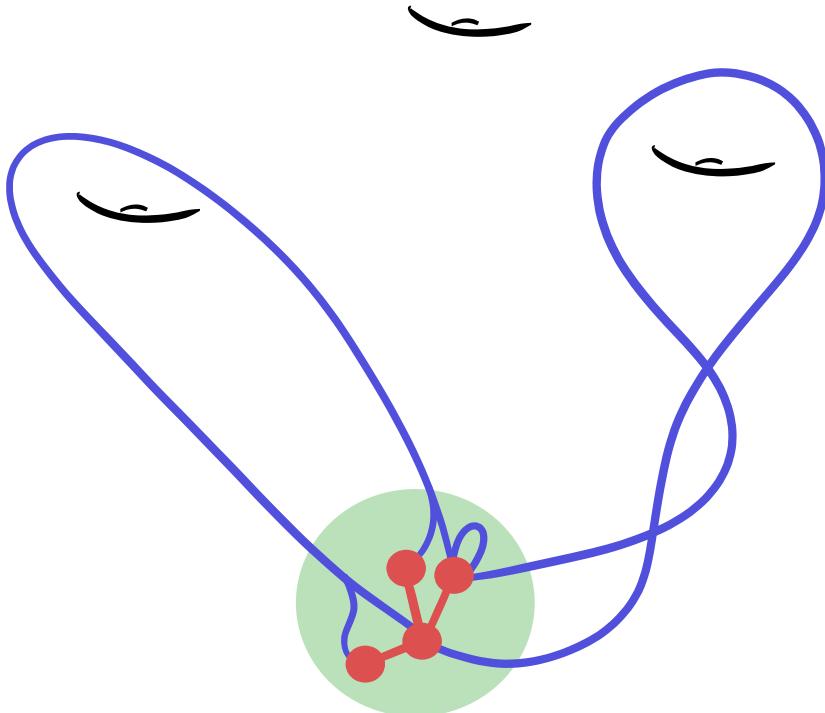
Straightening a graph



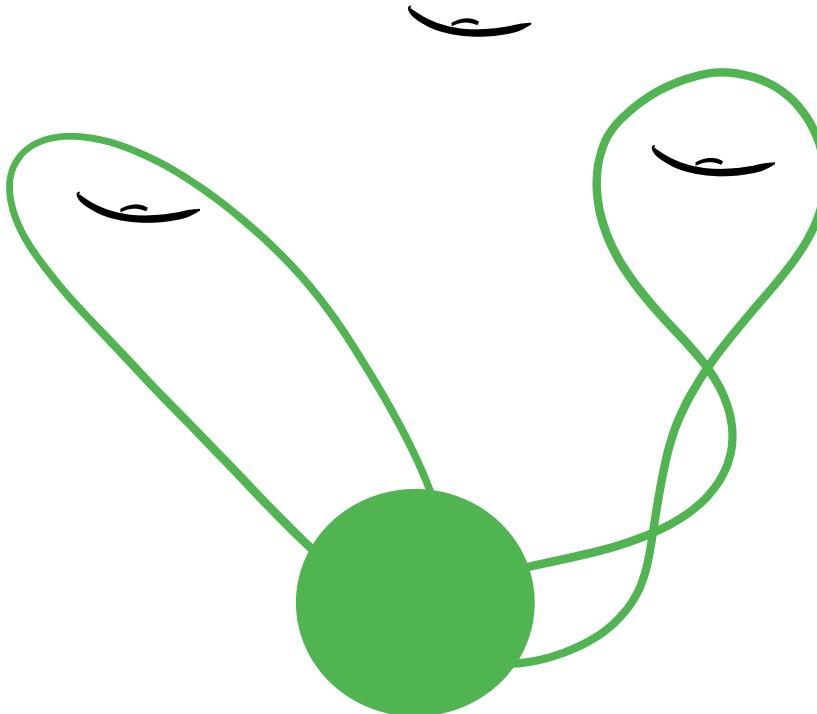
Straightening a graph



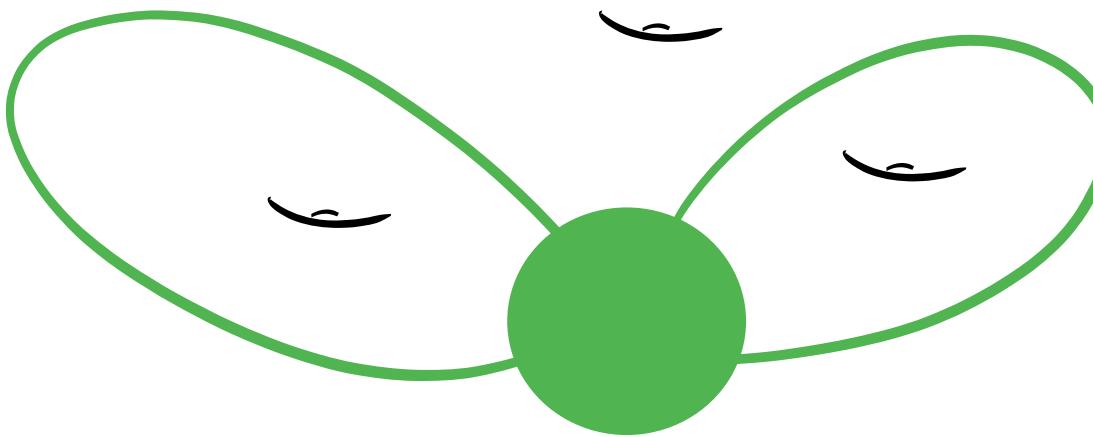
Straightening a graph



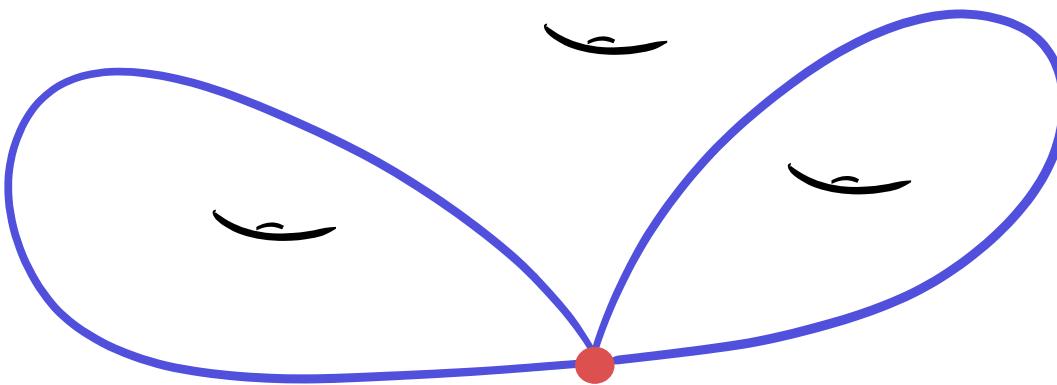
Straightening a graph



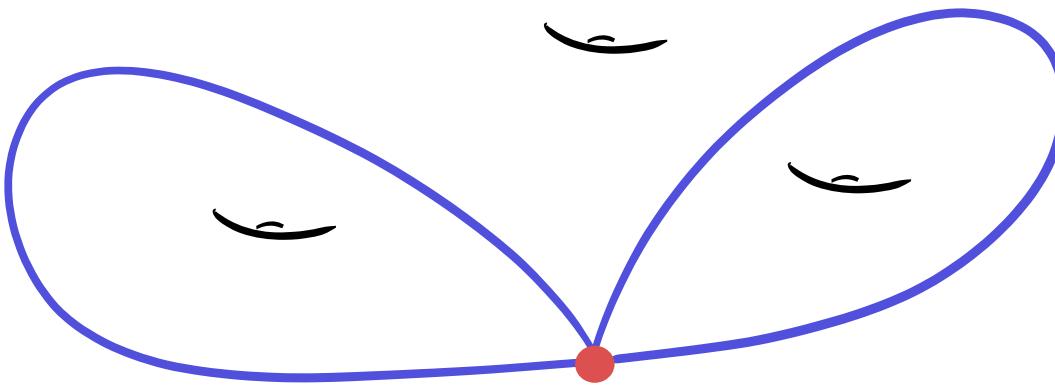
Straightening a graph



Straightening a graph

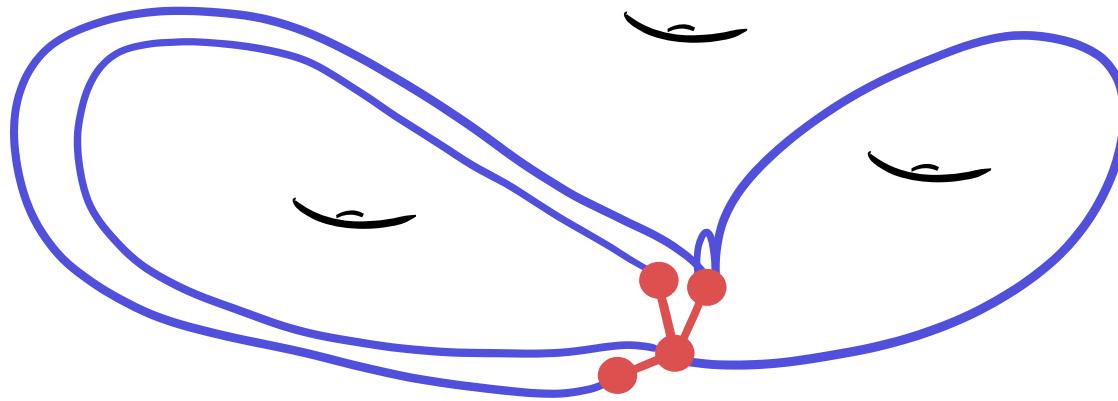


Straightening a graph



A straightened graph is a weak embedding
or cannot be untangled

Straightening a graph

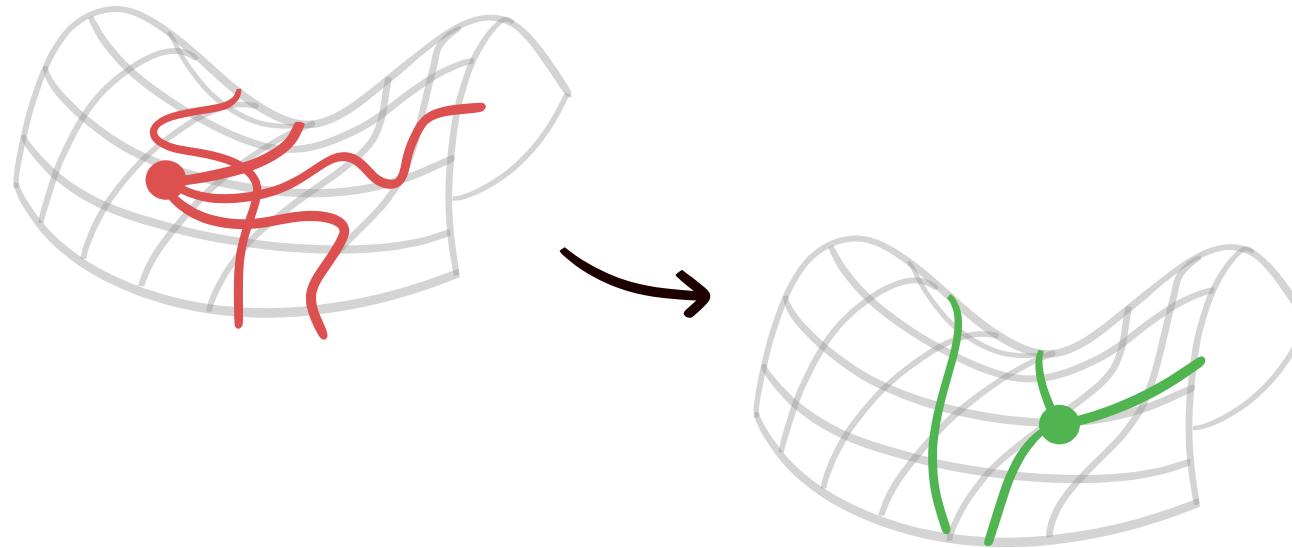


A straightened graph is a weak embedding
or cannot be untangled

(+ tricks and data structures for achieving
the announced time complexities)

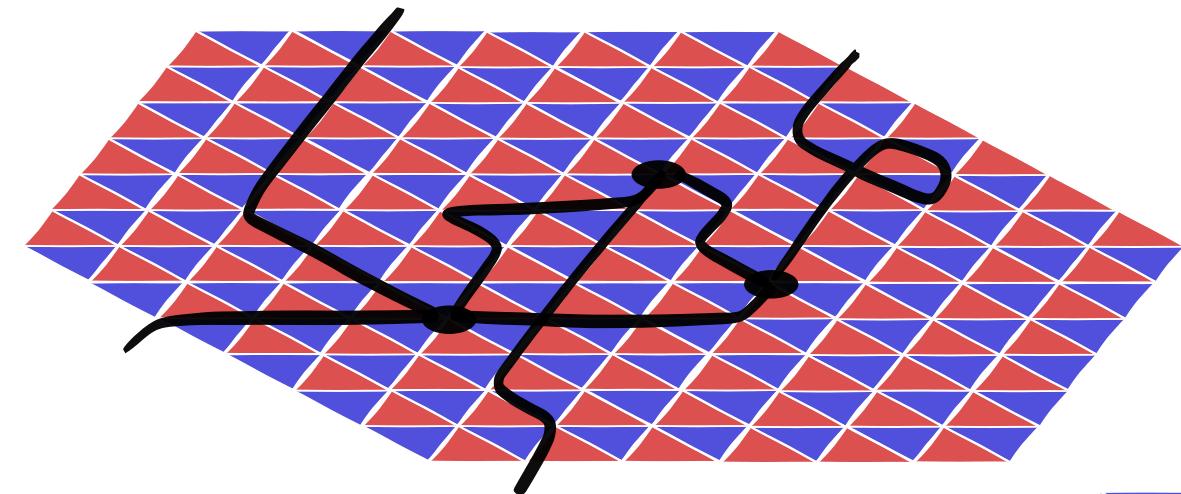
Discrete analogue of
Tutte embeddings

Recall: Tutte drawings

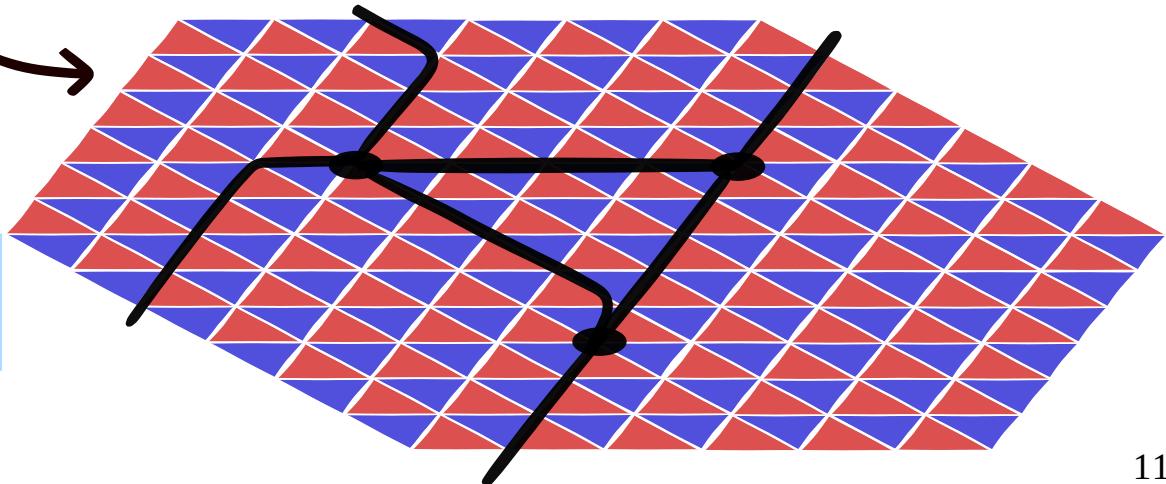


edges straight and
vertices “barycentric”

Harmonious drawings

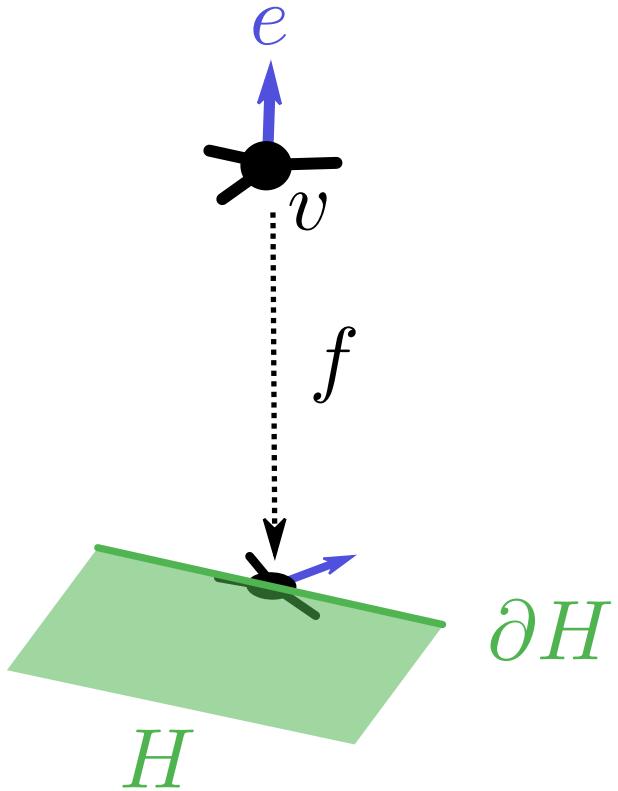


edges reduced and
vertices “barycentric”



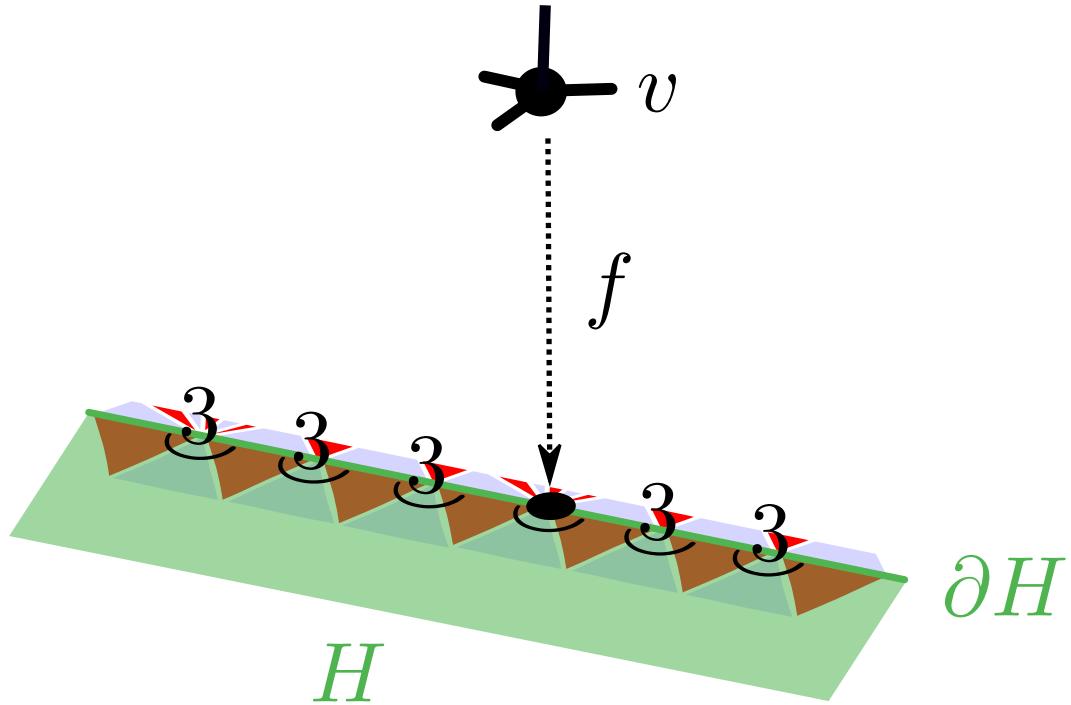
vertices “barycentric”

$$\begin{aligned} f(v) \in \partial H \\ \implies \exists e \quad f(e) \text{ escapes } H \end{aligned}$$



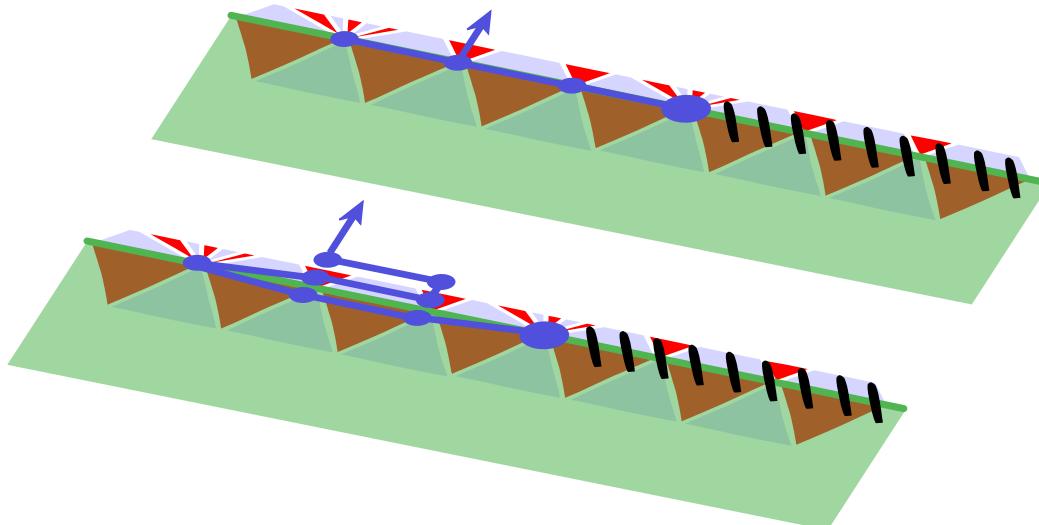
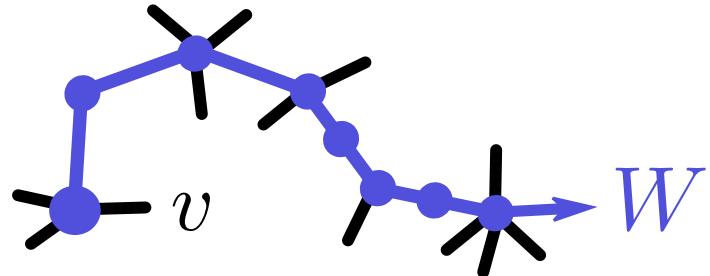
vertices “barycentric”

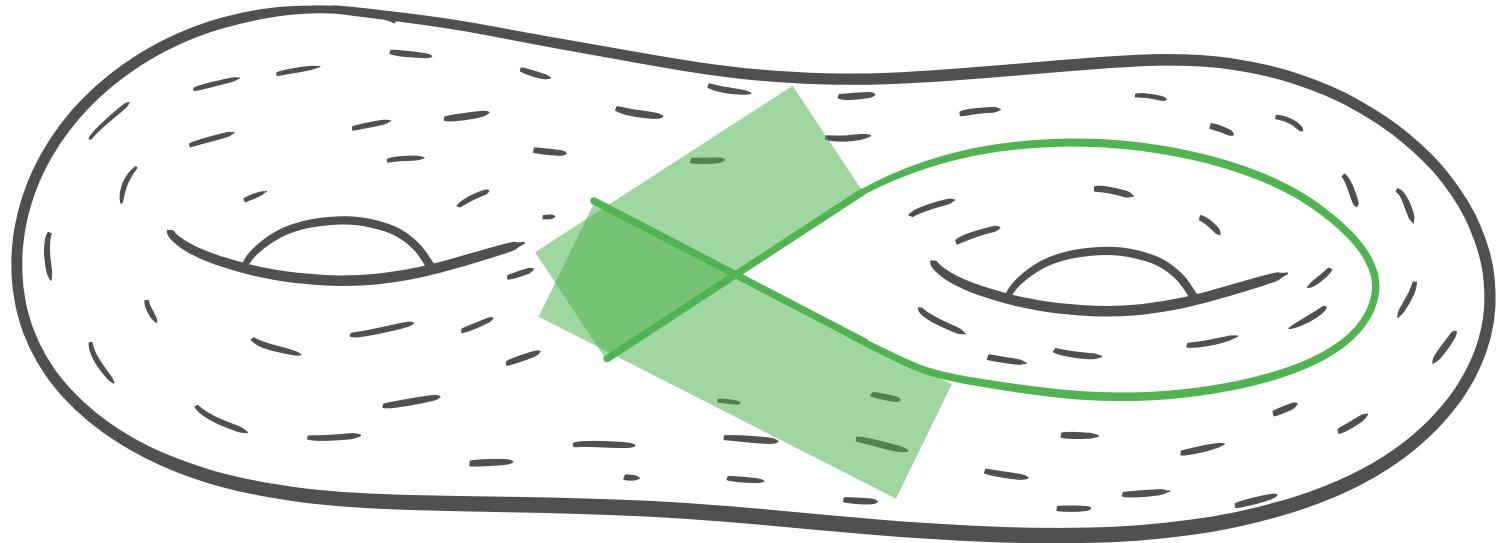
$$f(v) \in \partial H$$



vertices “barycentric”

$$f(v) \in \partial H \\ \xrightarrow{} \\ \exists W f \circ W \text{“escapes”} H$$





this definition generalizes to surfaces

Results

graph G

reducing triangulation T with m edges

$f : G \rightarrow T^1$ of size n

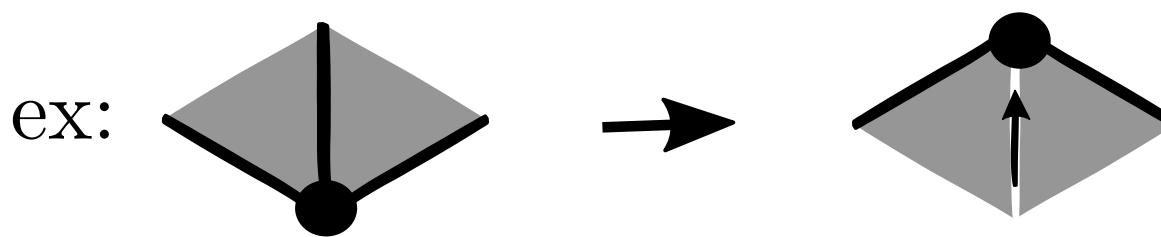
Definition of *harmonious* drawings

f harmonious and f can be untangled
 $\Rightarrow f$ weak embedding

Algo to make f harmonious in $O((m+n)^2 n^2)$ time,
without increasing any edge length

Harmonizing a drawing

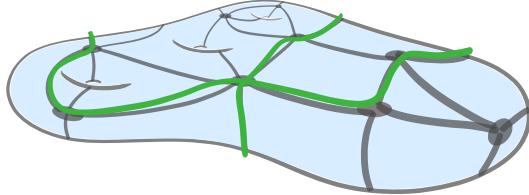
- 1 we define monotonic moves to apply to f



- 2 some moves do not seem to decrease any potential so we combine the moves carefully

Making curves cross
minimally

Related work

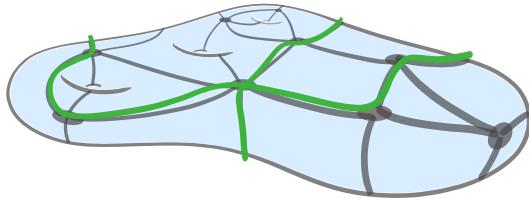


closed walks of total length n
on a graph of size m

Despré, Lazarus, 2019

- Put a single curve in minimal position in $O(m + n^4)$ time
- Compute the min. nb. of crossings in $O(m + n^2)$ time

Result



closed walks of total length n
on a graph of size m

simpler algos and proofs!

$$m^3n + mn \log(mn)$$

D., 2024

- Put a single curve in minimal position in $O(\cancel{m} + \cancel{n}^4)$ time
- Compute the min. nb. of crossings in $O(\cancel{m} + \cancel{n}^2)$ time

$$m^2 + mn \log(mn)$$

Untangling Graphs

Computing Delaunay Triangulations

Conclusion

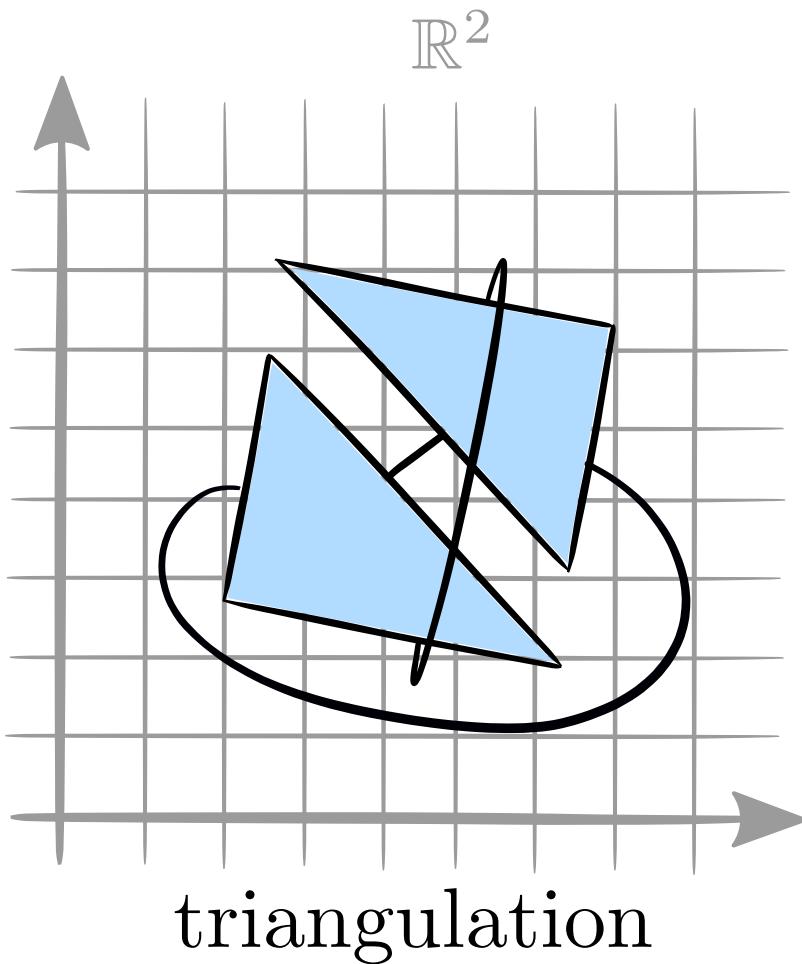
Untangling Graphs

Computing Delaunay Triangulations

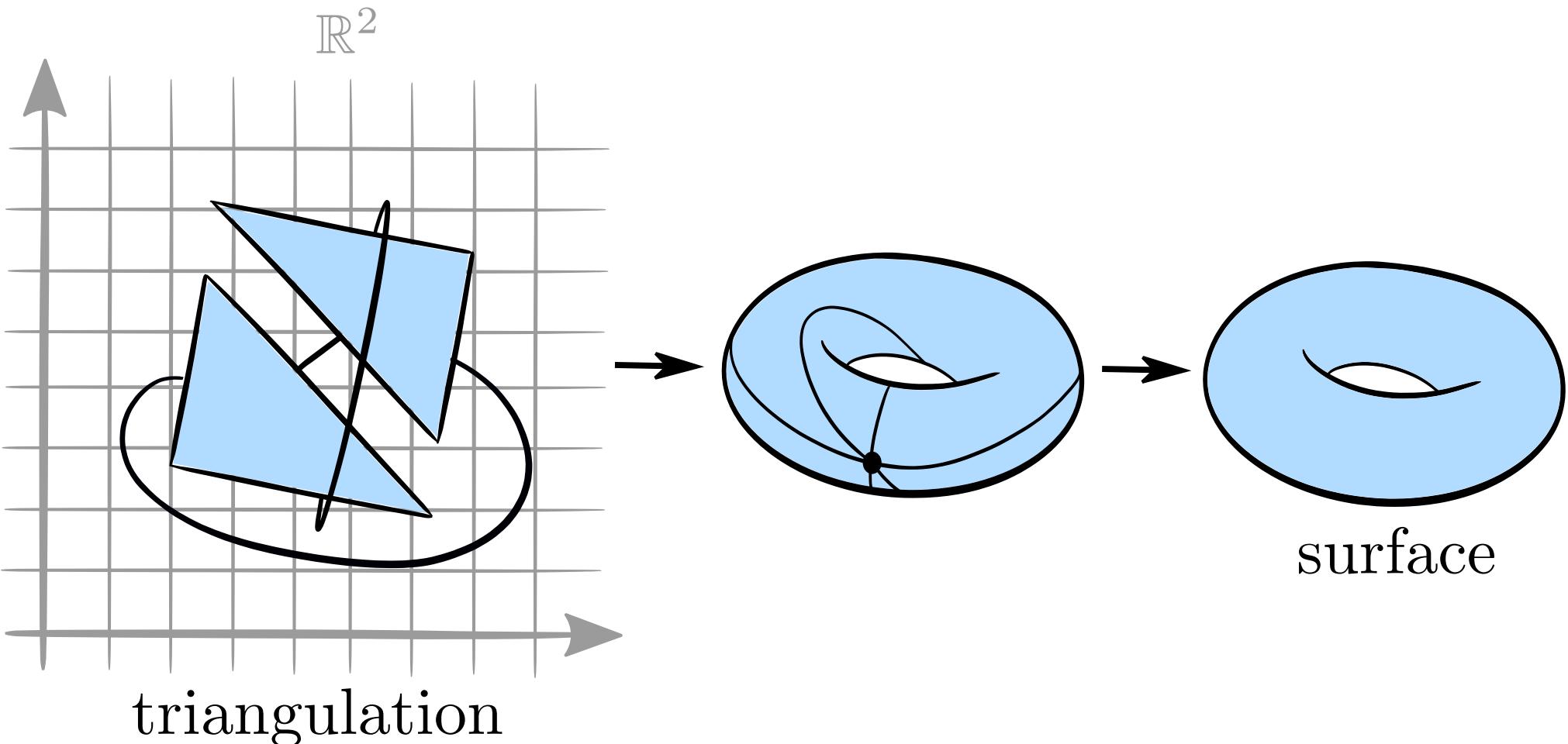
Conclusion

Triangulations of polyhedral surfaces

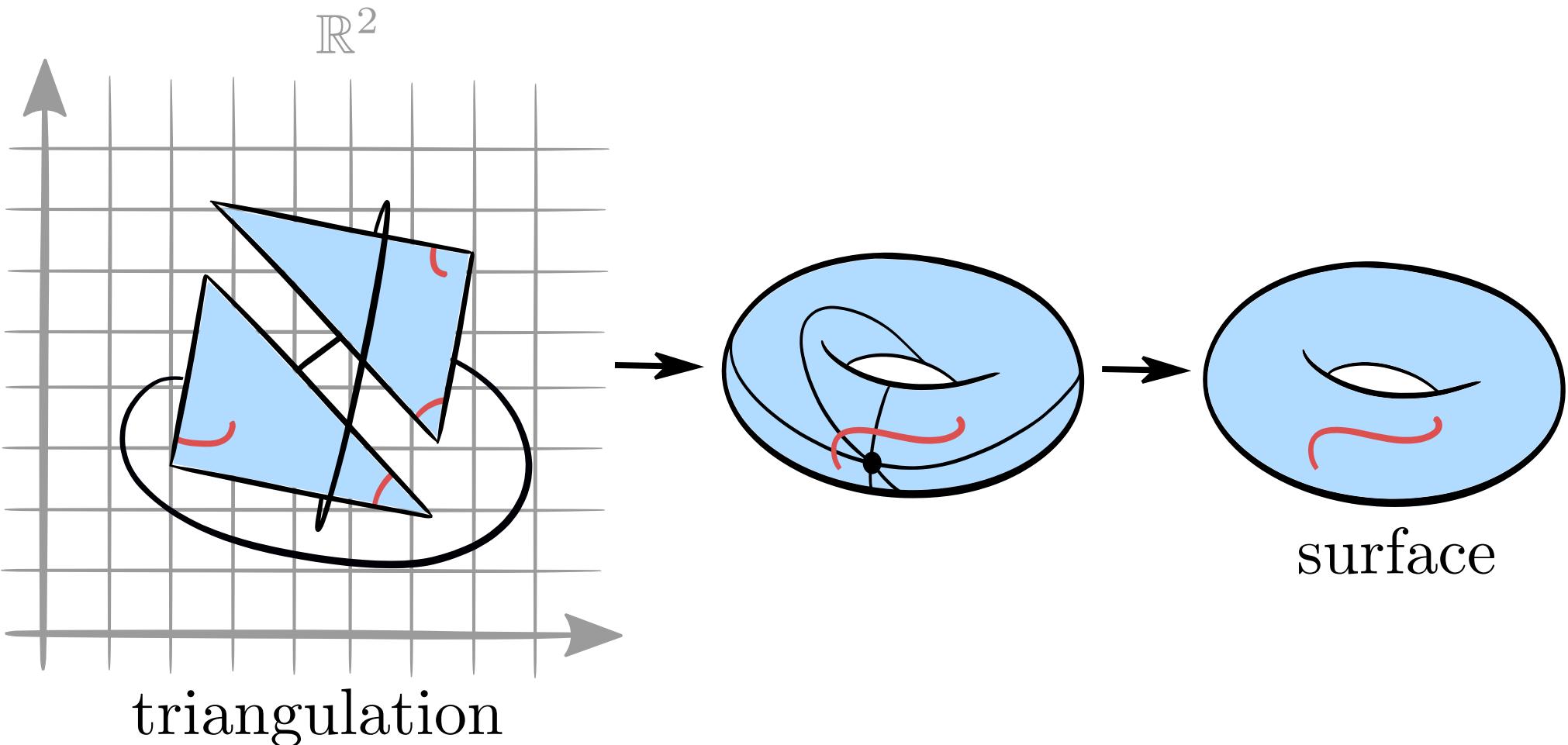
Triangulation of polyhedral surface



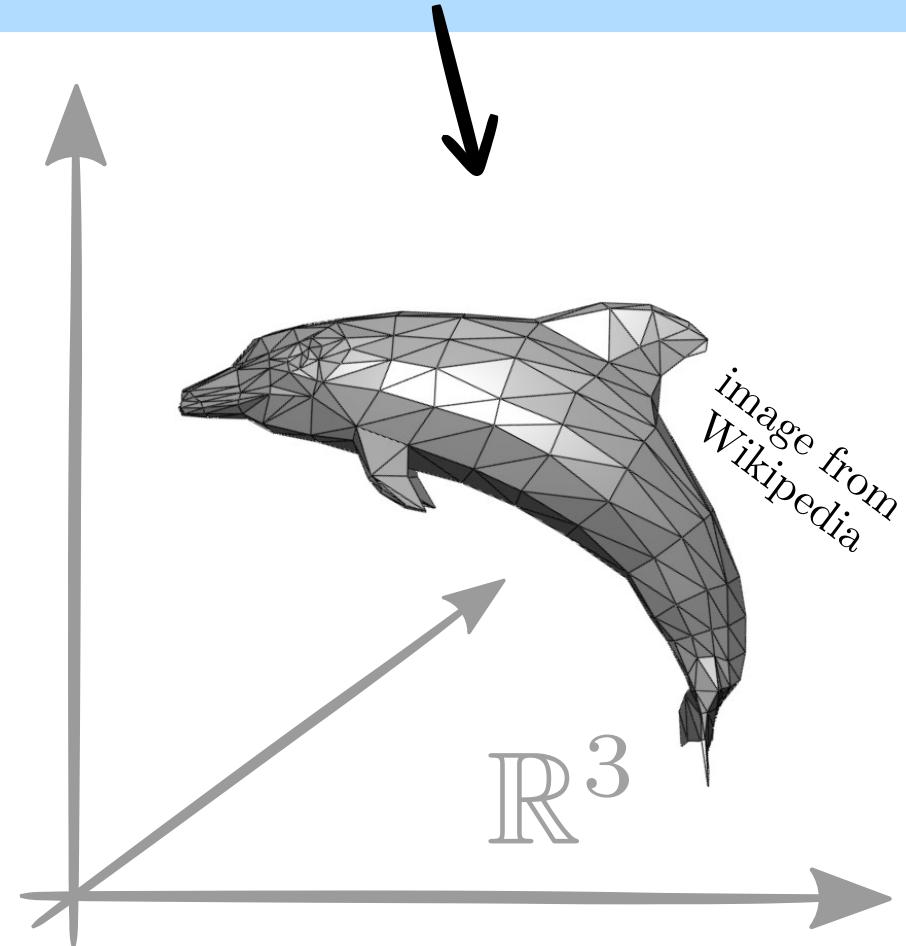
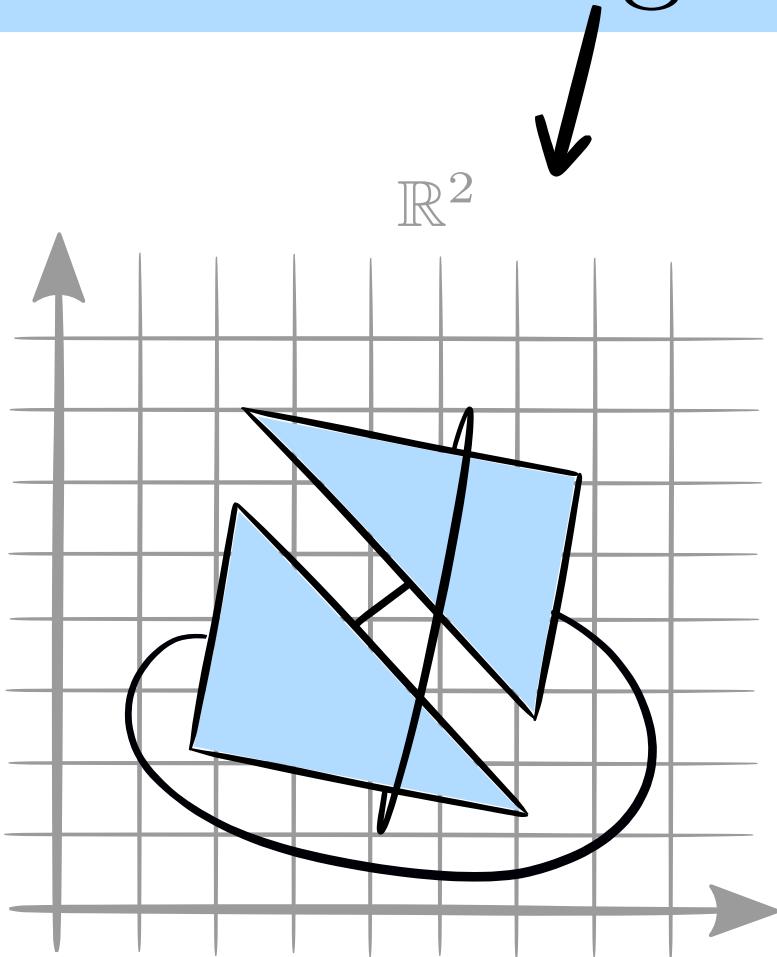
Triangulation of polyhedral surface



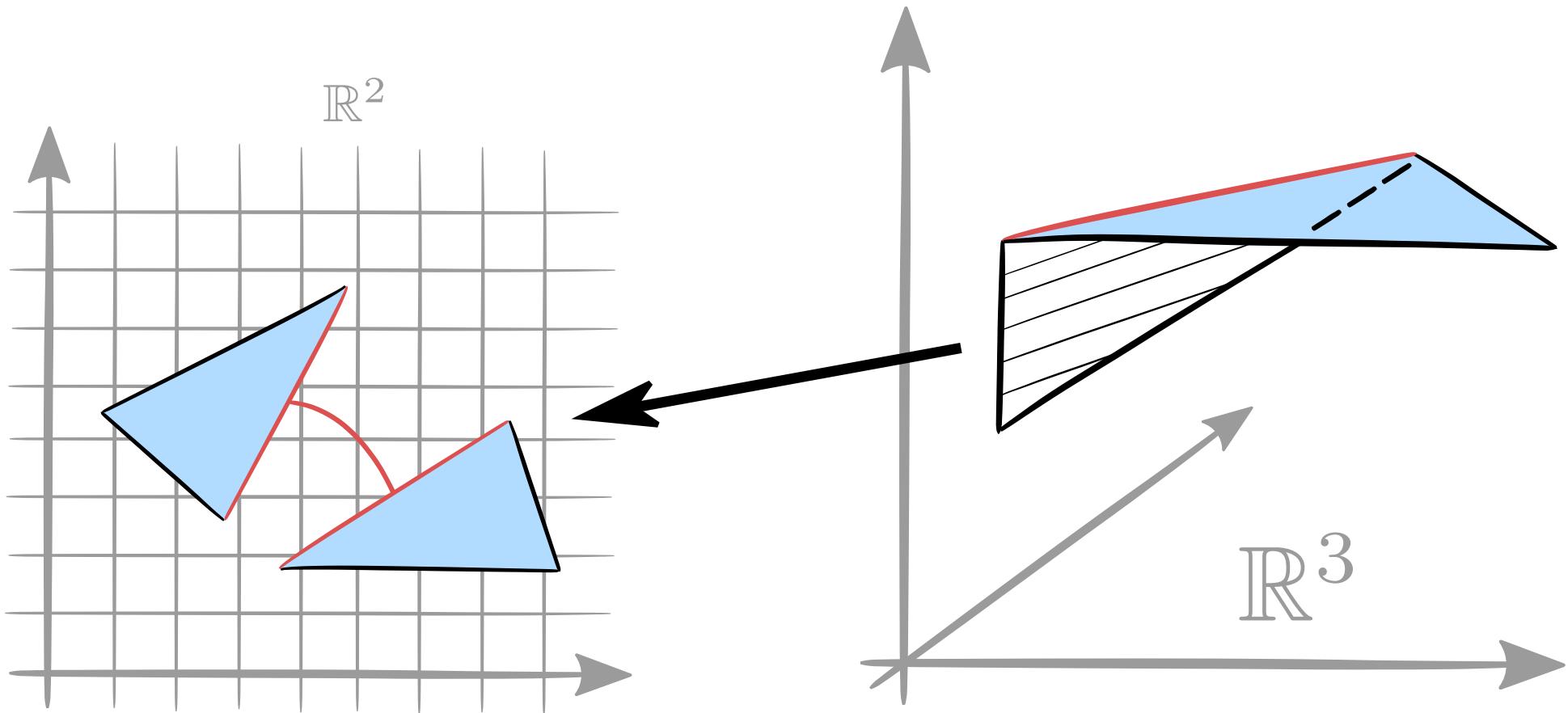
Triangulation of polyhedral surface



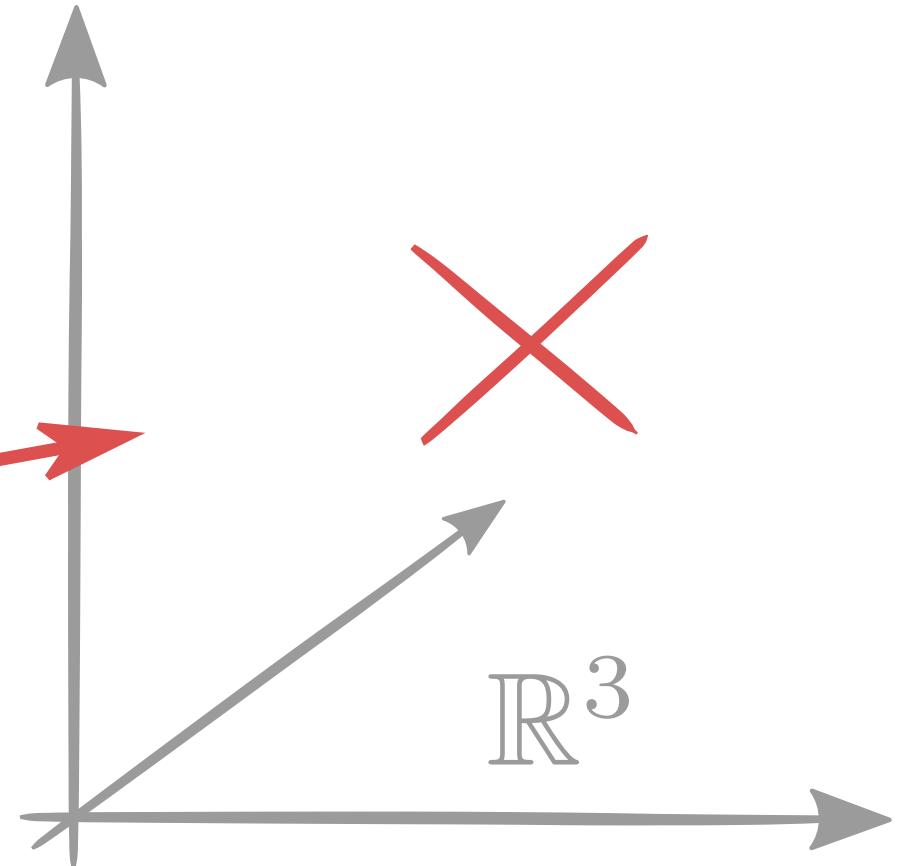
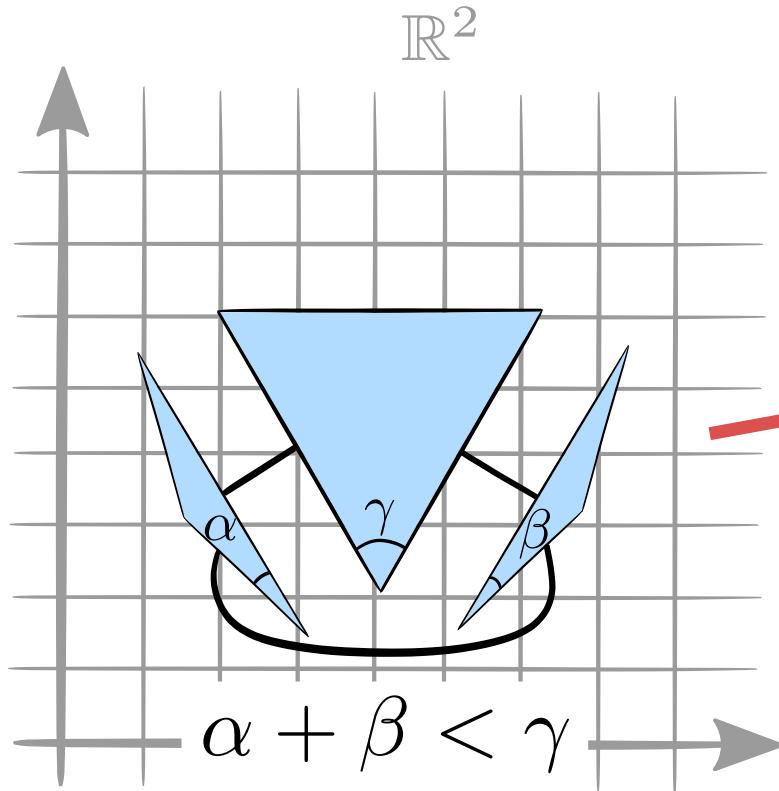
Triangulation vs. Mesh



Every mesh gives a triangulation



Every mesh gives a triangulation
but converse is false!

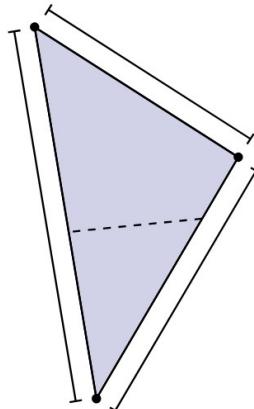
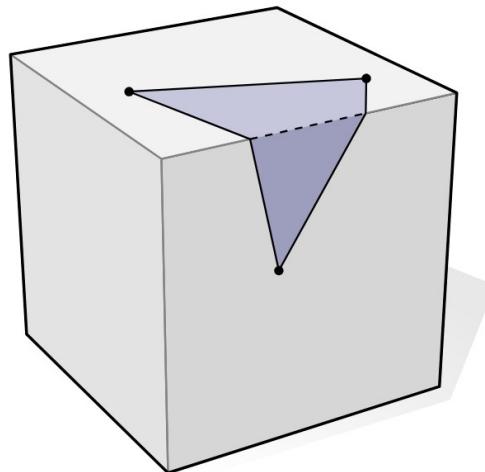


Some recent algos operate on triangulations not issued of a mesh

Liu et al., 2023

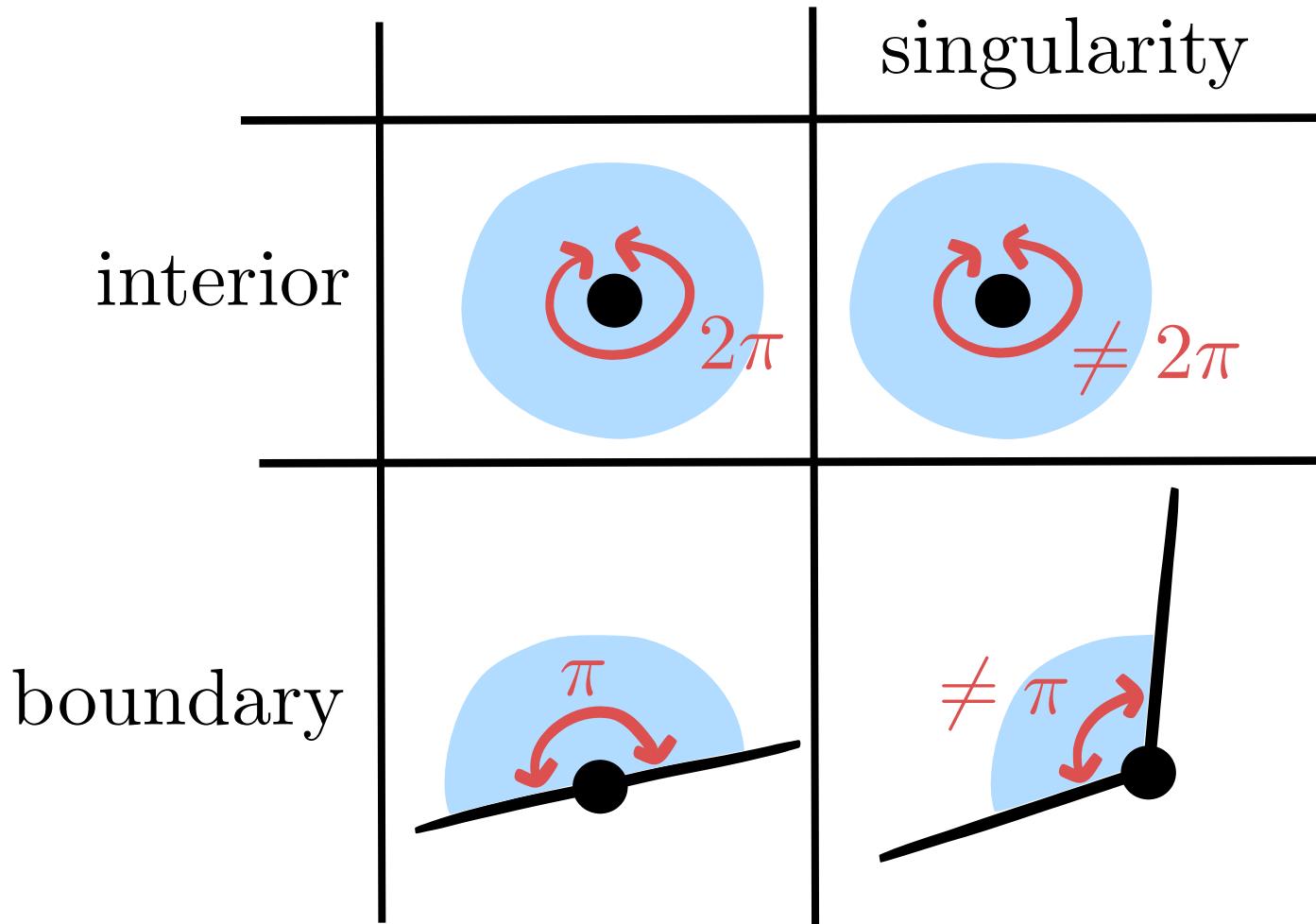
Takayama, 2022

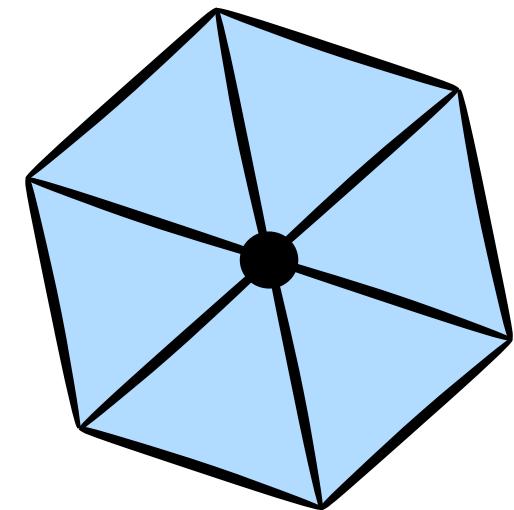
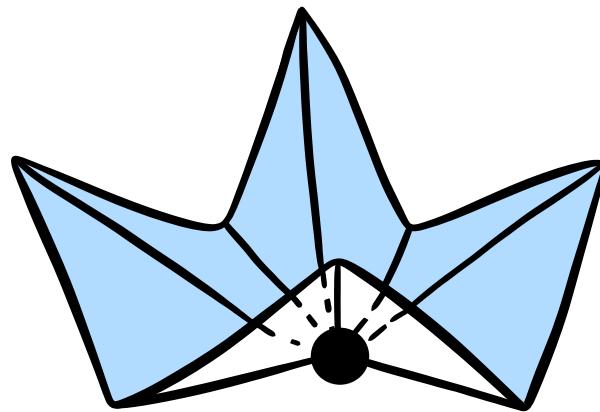
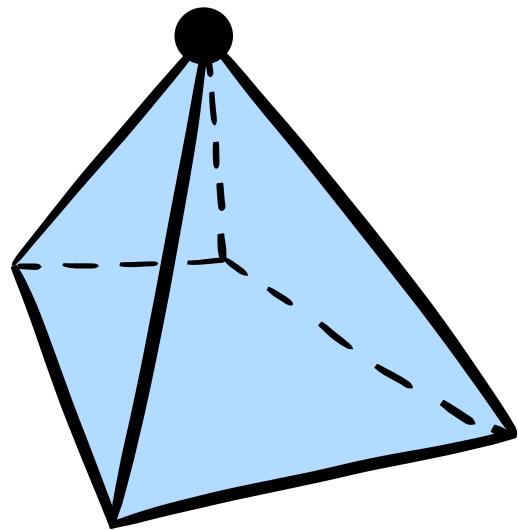
Sharp and Crane, 2020

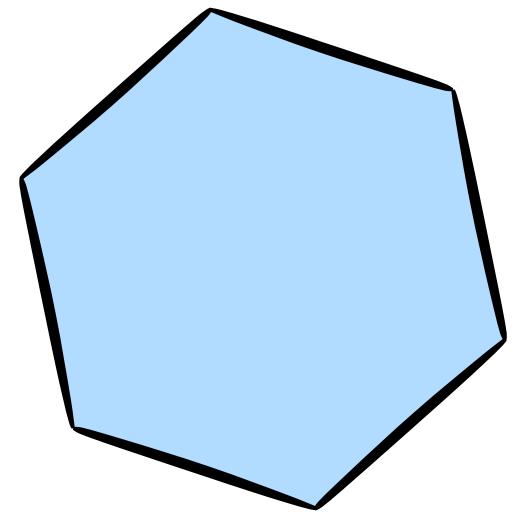
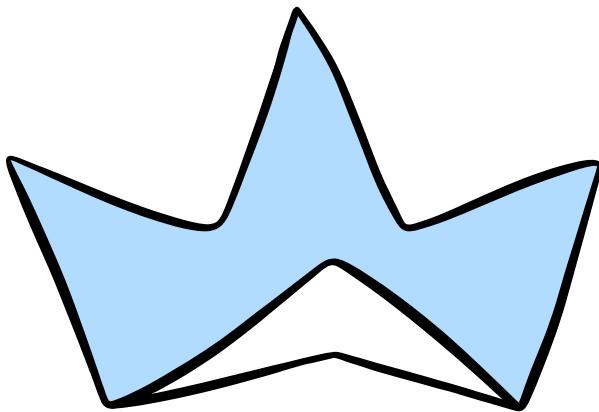
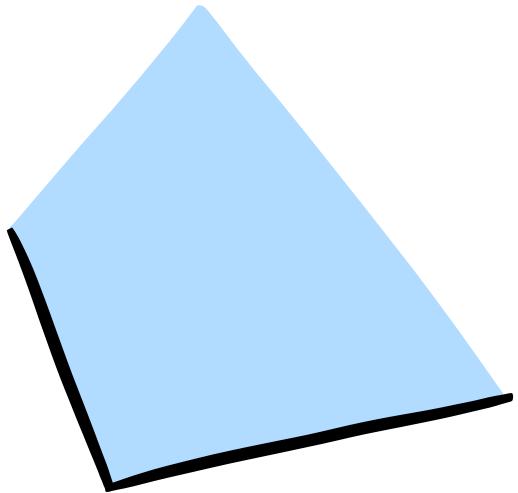


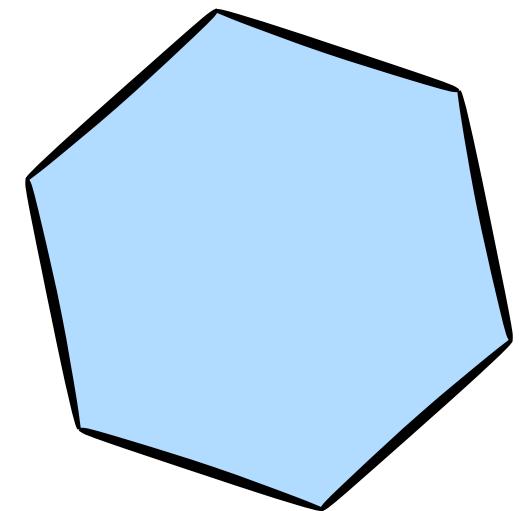
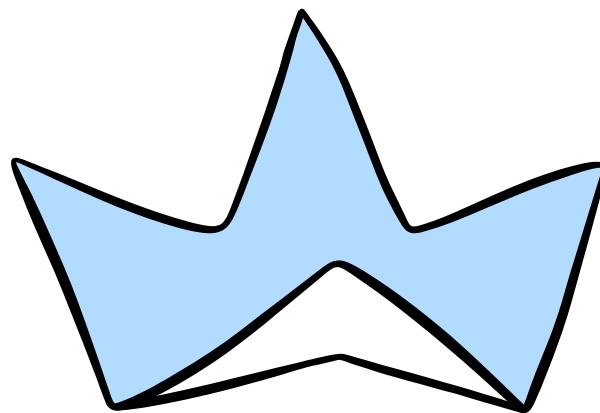
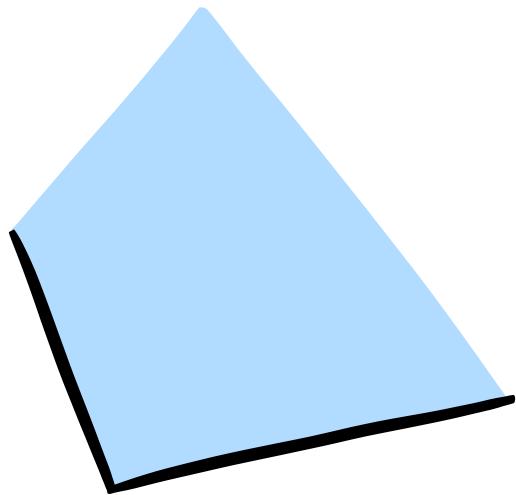
Geometry processing with
intrinsic triangulations
Sharp, Gillespie, Krane, 2021

Types of points on the surface

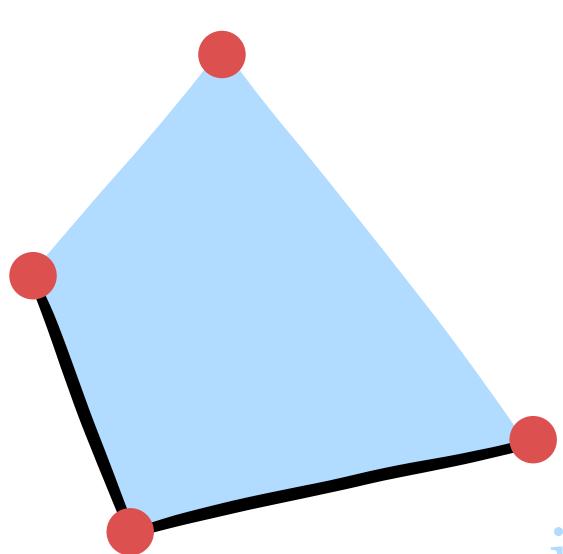




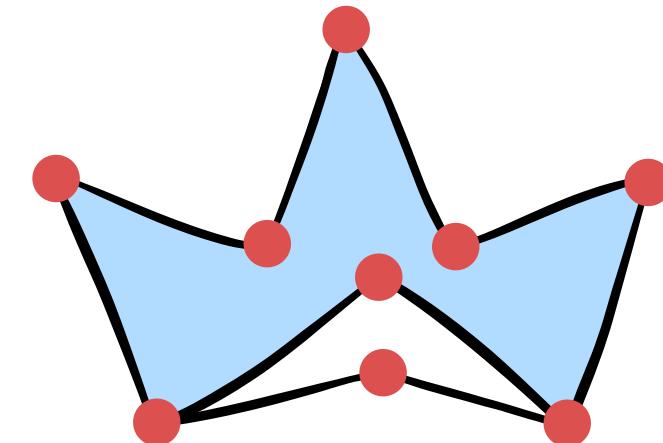




interior
boundary



interior
boundary

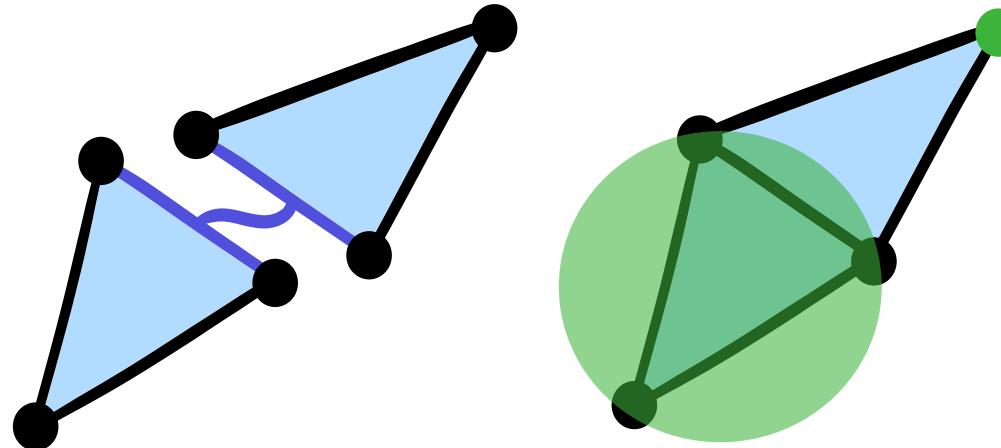


singularities

Problem

Delaunay triangulation

triangulation in which
every edge is Delaunay



The Delaunay triangulation

Generically, every surface has a **unique**
Delaunay triangulation
whose vertices are the singularities

The Delaunay triangulation

Generically, every surface has a **unique**
Delaunay triangulation
whose vertices are the singularities

Problem

Given triangulation T , compute
“the” Delaunay triangulation
of the surface of T

Motivations

Motivations

- isometry testing

Motivations

- isometry testing
- shortest paths

- shortest paths

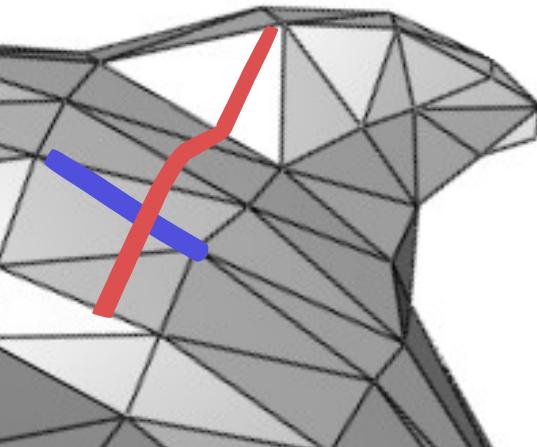
Shortest paths on meshes

On a mesh M with n triangles...

a shortest path cannot cross an edge twice

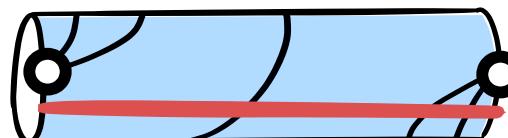
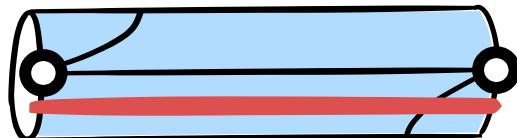
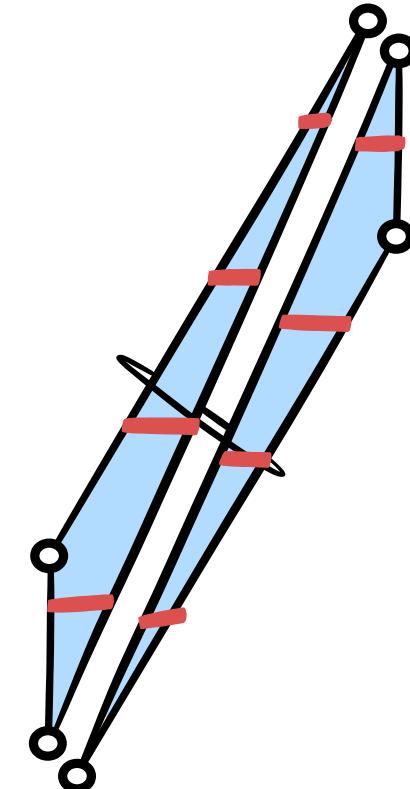
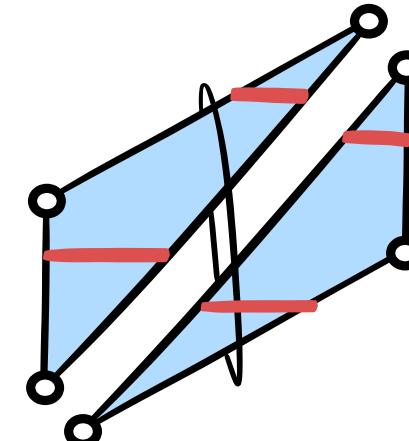
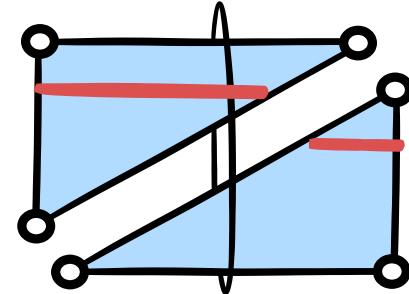
→ shortest path can be computed in $O^*(n^2)$ time

Mitchel, Mount, Papadimitriou, 1987



Shortest paths on triangulations

they can cross edges arbitrarily many times

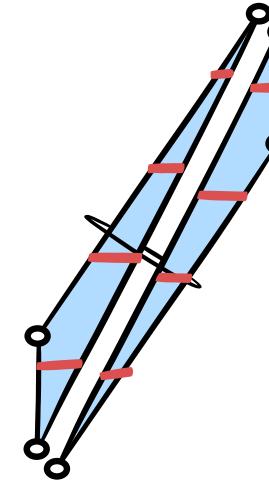


Erickson, 2006

Shortest paths on triangulations

Löffler, Ophelders, Staals, Silveira, 2023

happiness h : max number of times
a shortest path visits a triangle



→ shortest path can be computed in $O^*(n^2 h)$ time

Shortest paths on triangulations

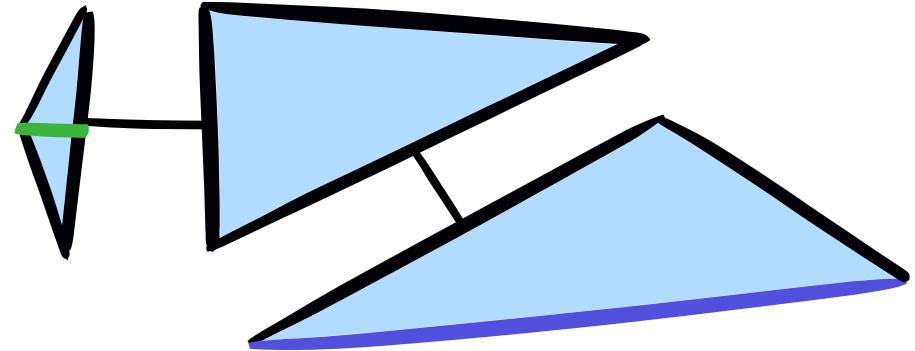
Löffler, Ophelders, Staals, Silveira, 2023

Delaunay triangulations have happiness $O(1)$

Result

Result

aspect ratio =
 $\frac{\text{maximum side length}}{\text{minimum height}}$



D. 2025

Given triangulation T of n triangles, of aspect ratio r , whose surface has no boundary, we can compute Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Result

aspect ratio =
 $\frac{\text{maximum side length}}{\text{minimum height}}$



D. 202 Previous techniques do not lead to better than $O(\text{Poly}(n, r))$

Given n triangles, of aspect ratio r , whose surface has no boundary, we can compute Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Result

aspect ratio =
 $\frac{\text{maximum side length}}{\text{minimum height}}$

D. 202

Given t,

whose surface has no boundary, we can compute
Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time



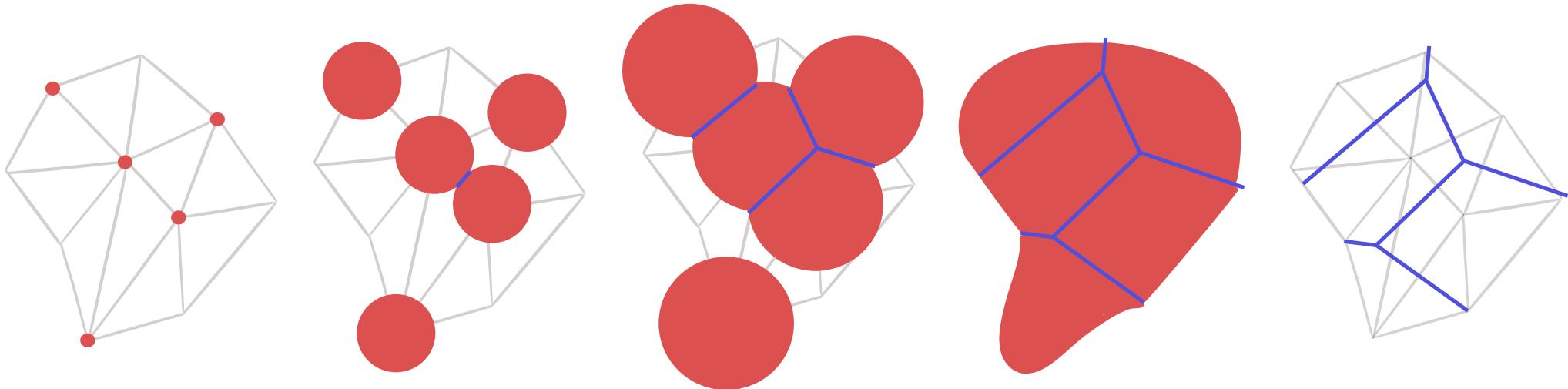
Previous techniques do not lead to
better than $O(\text{Poly}(n, r))$
triangles, or

Now backed by a
lower bound!

Algorithm overview

Classical method

compute the Voronoi diagram
by propagating waves



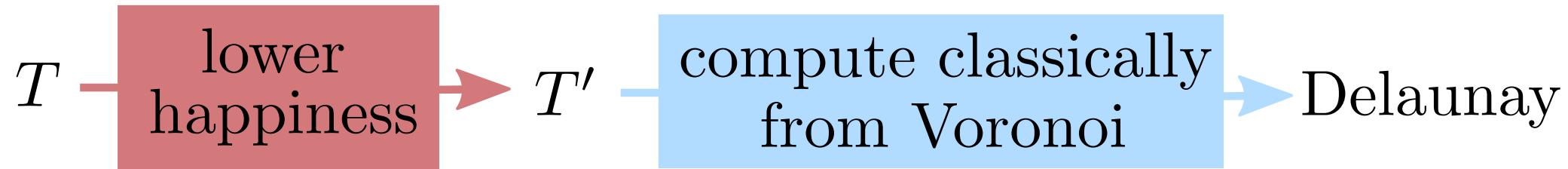
then derive Delaunay from it

Algorithm

T ← compute classically
from Voronoi → Delaunay

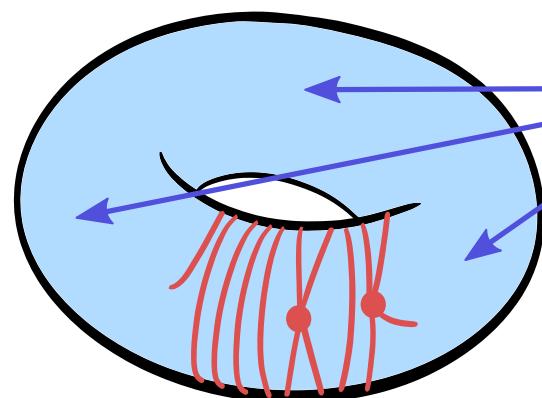
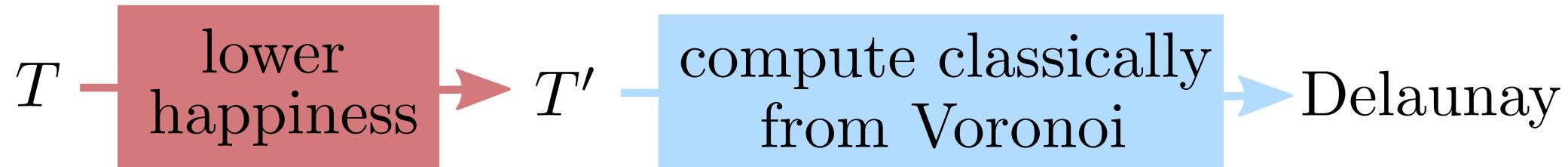
Algorithm

D. 2025



Algorithm

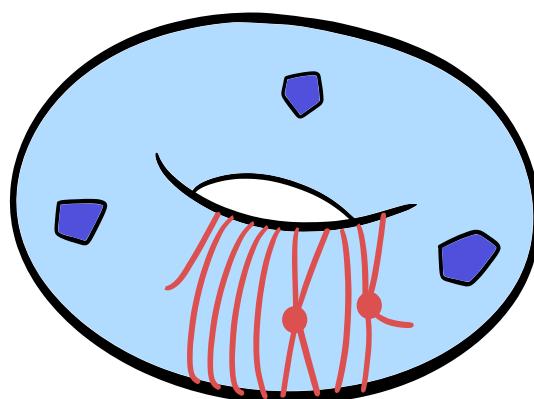
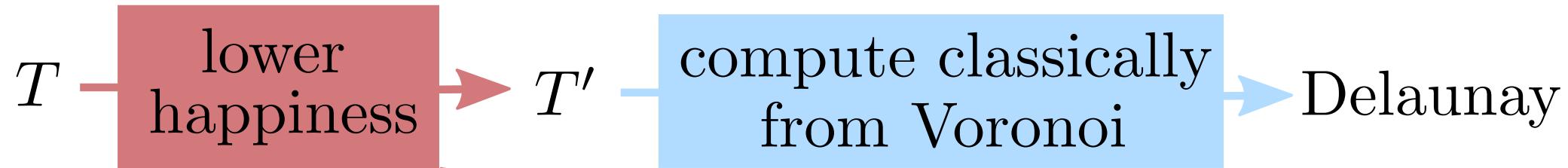
D. 2025



consider the singularities

Algorithm

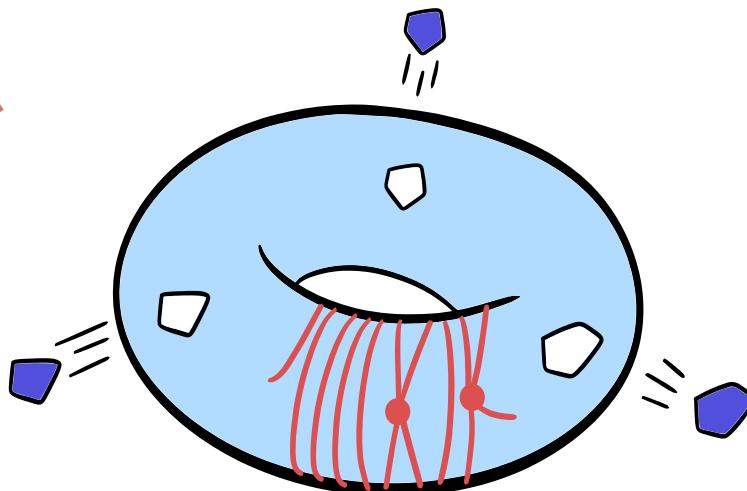
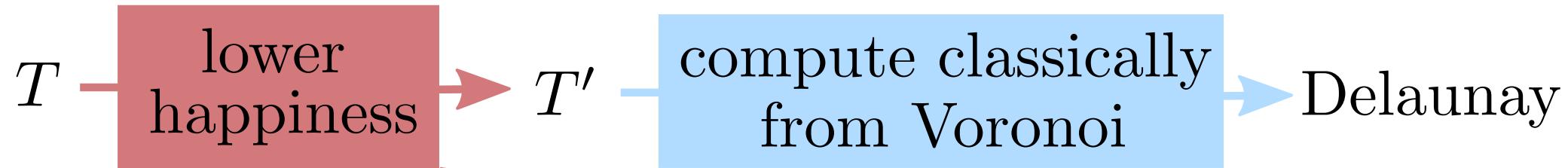
D. 2025



cut out caps around the singularities

Algorithm

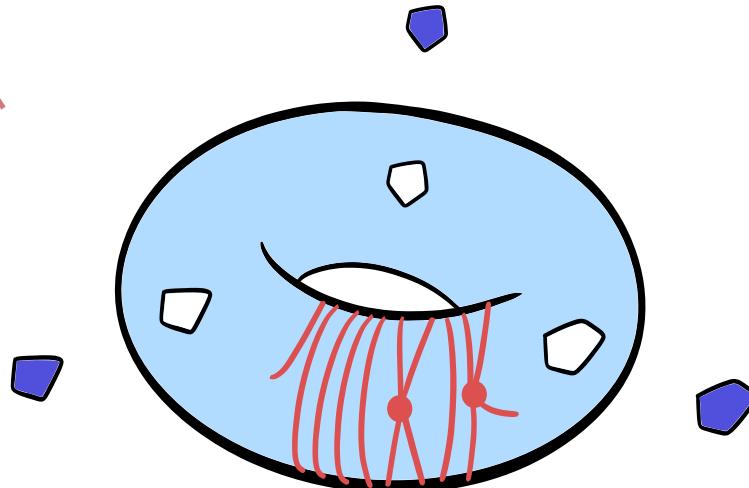
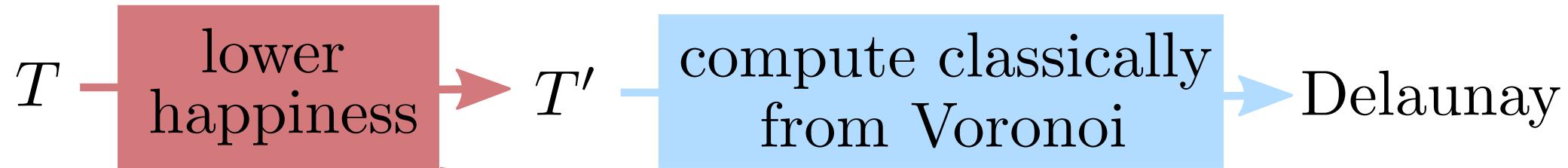
D. 2025



cut out caps around the singularities

Algorithm

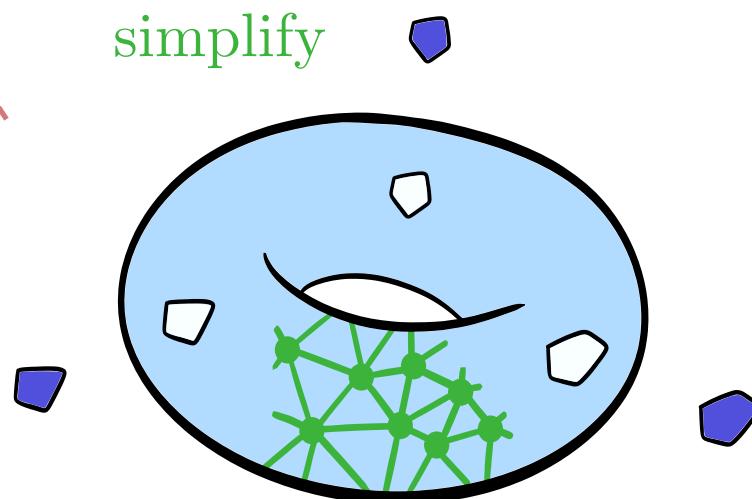
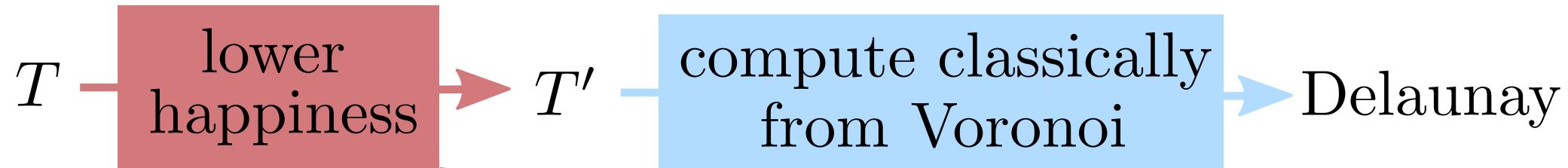
D. 2025



cut out caps around the singularities

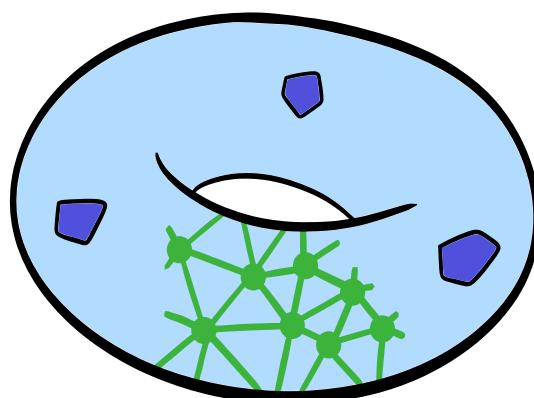
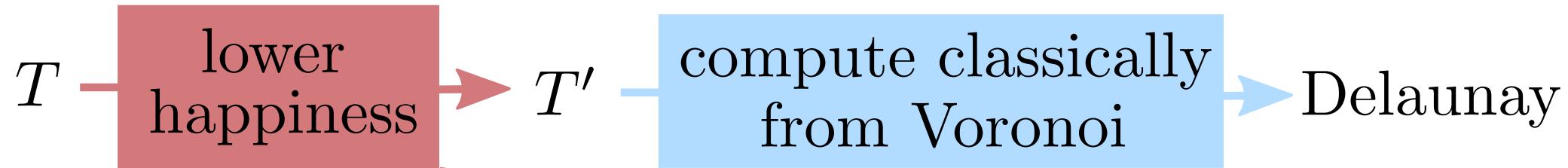
Algorithm

D. 2025



Algorithm

D. 2025



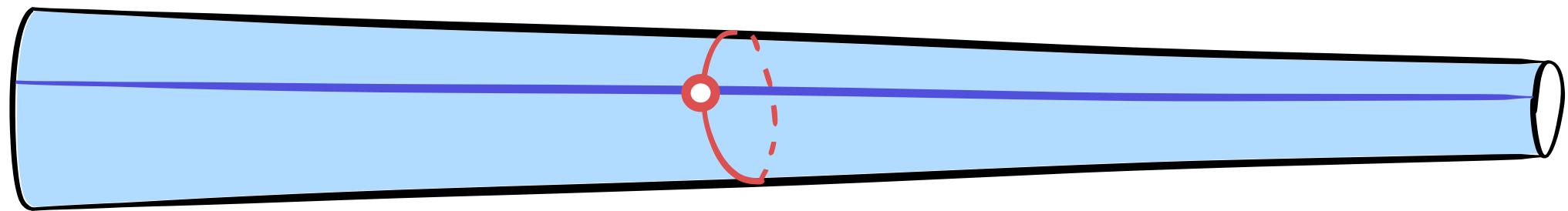
put the caps back

Simplification algorithm

Tuned combination of elementary operations, like
inserting vertices in edges
inserting edges in faces
deleting vertices
repeated many times

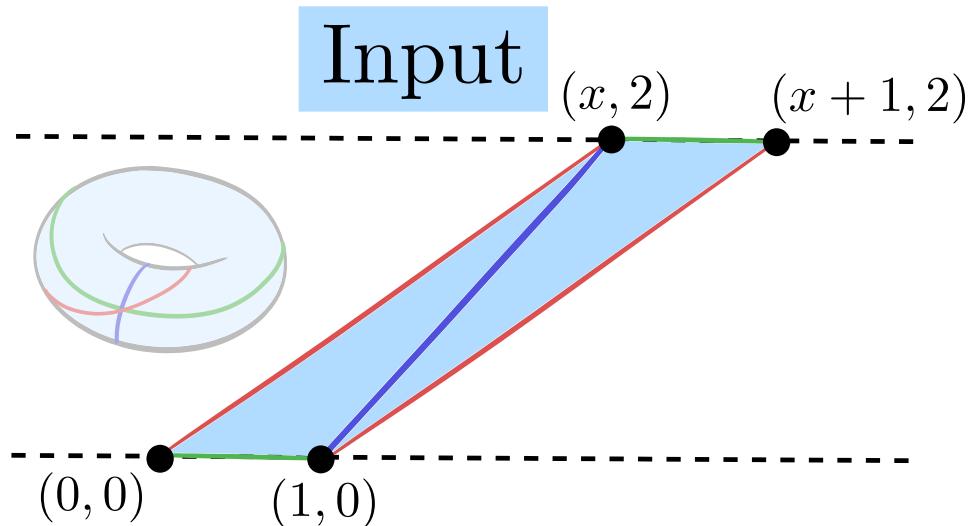
some simplify the geometry,
others decrease # vertices

Enclosure
D. 2025

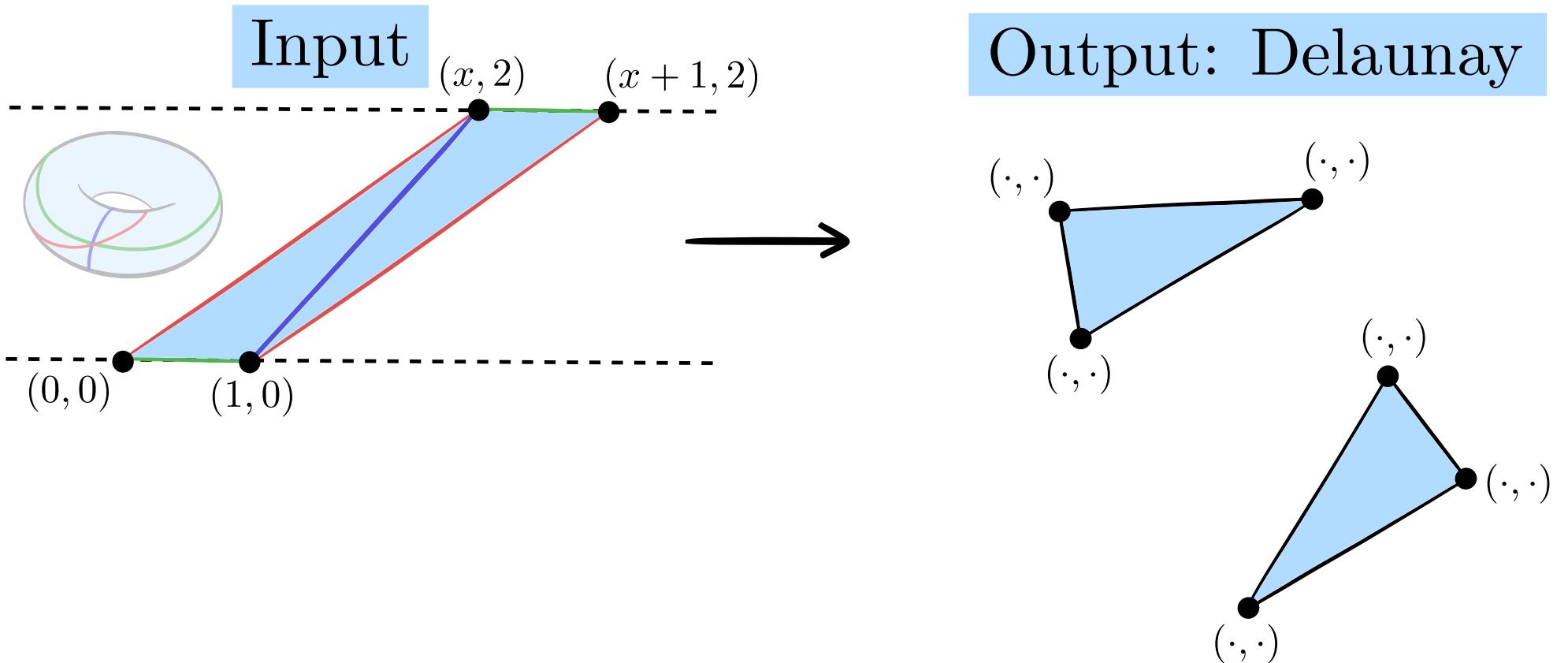


Lower bound

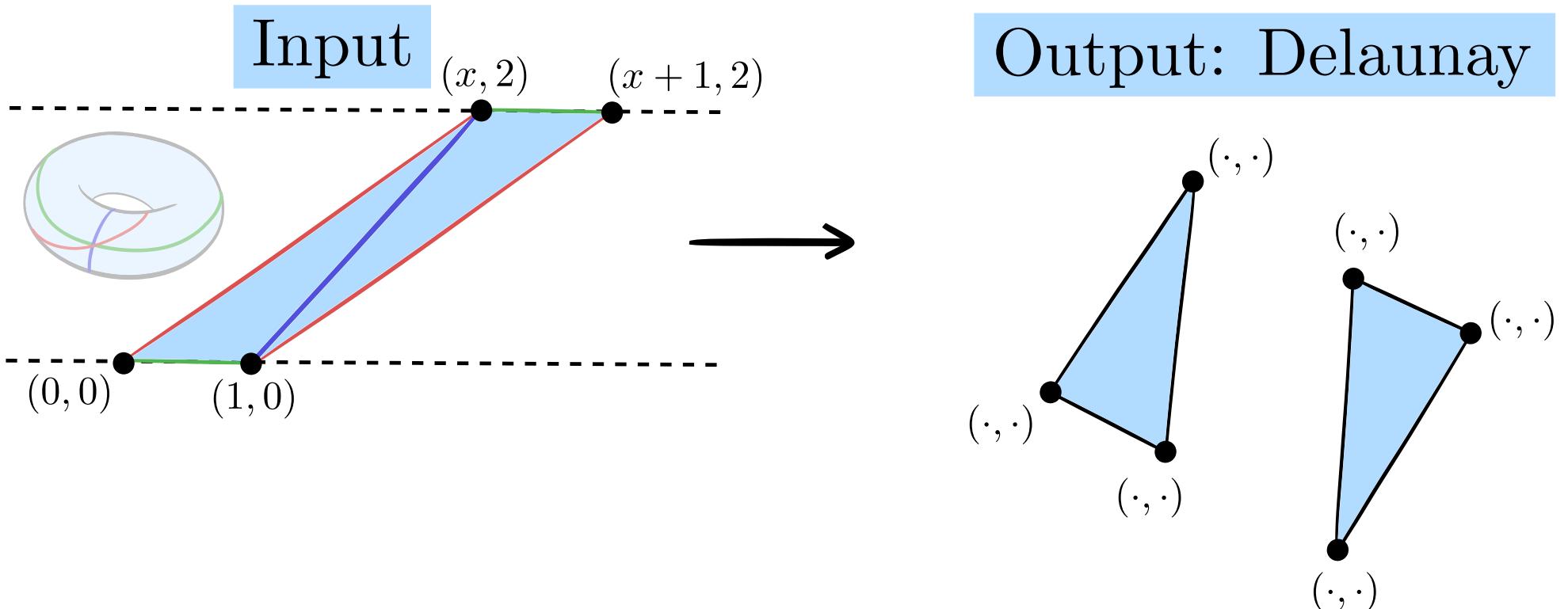
Lower bound



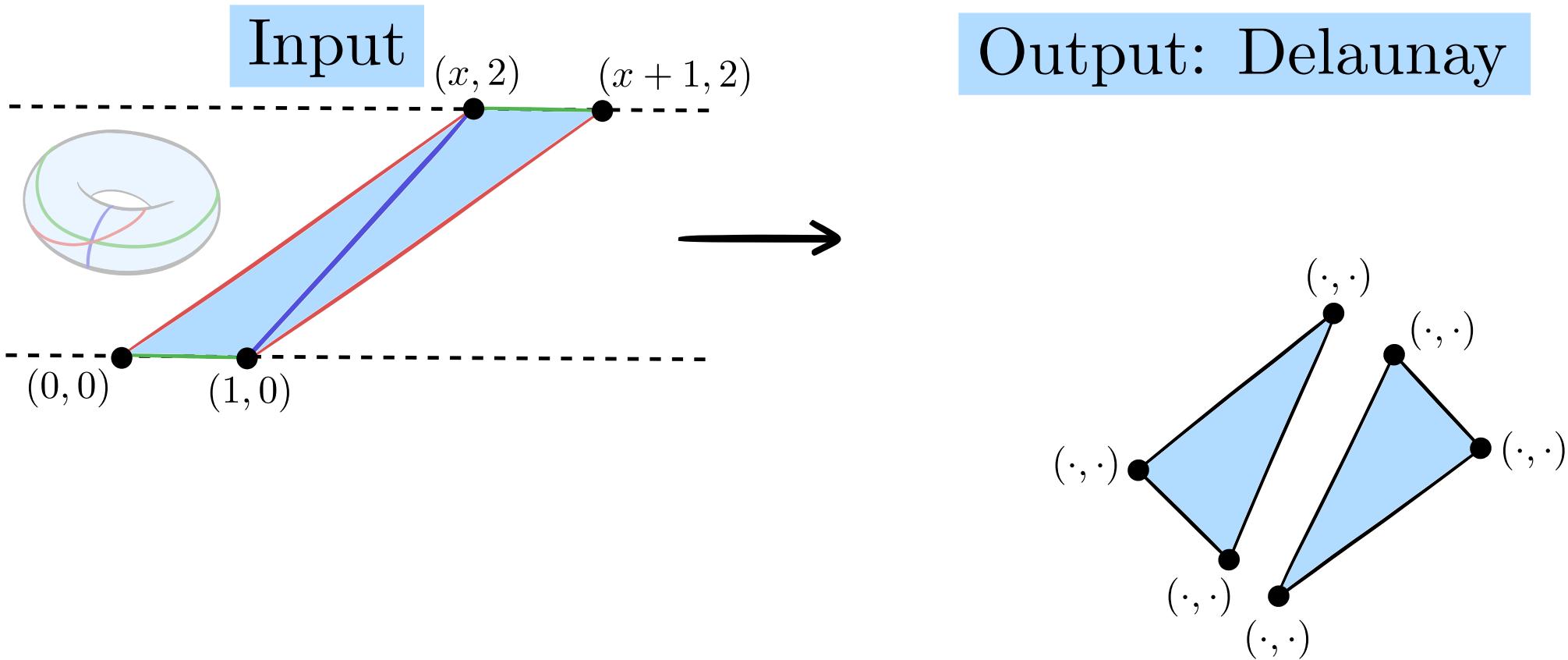
Lower bound



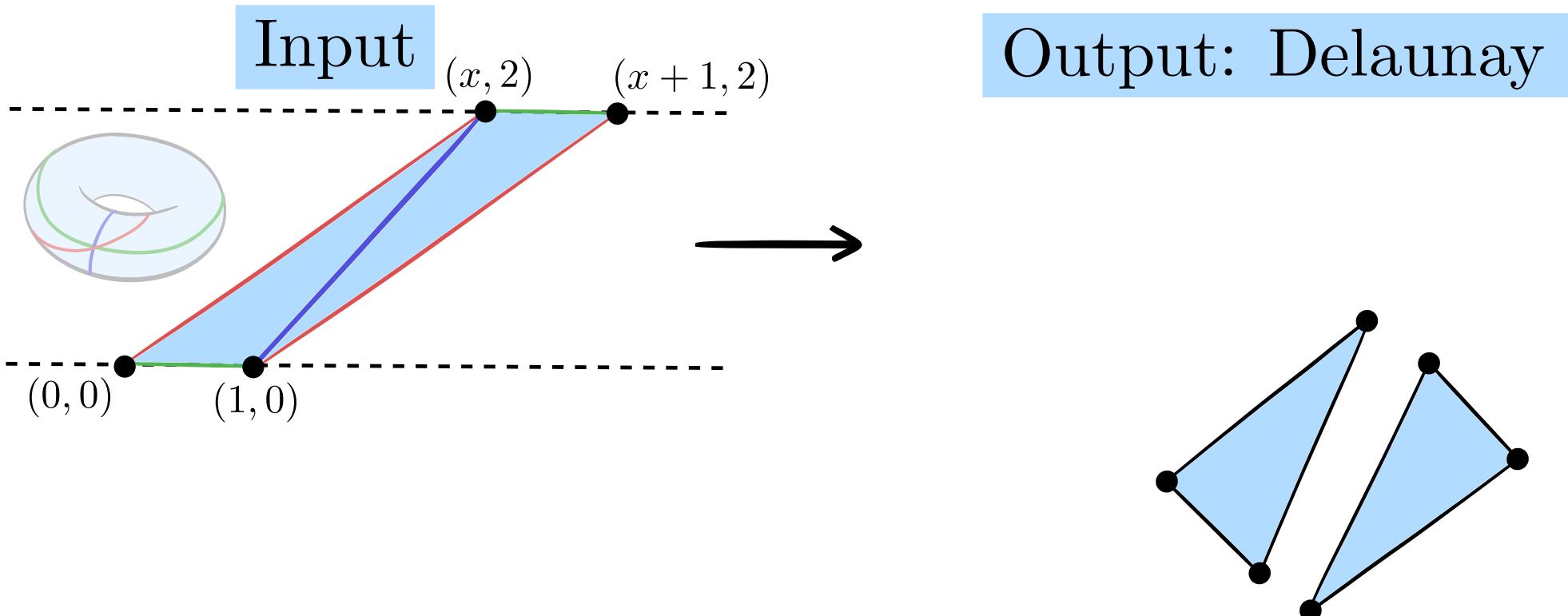
Lower bound



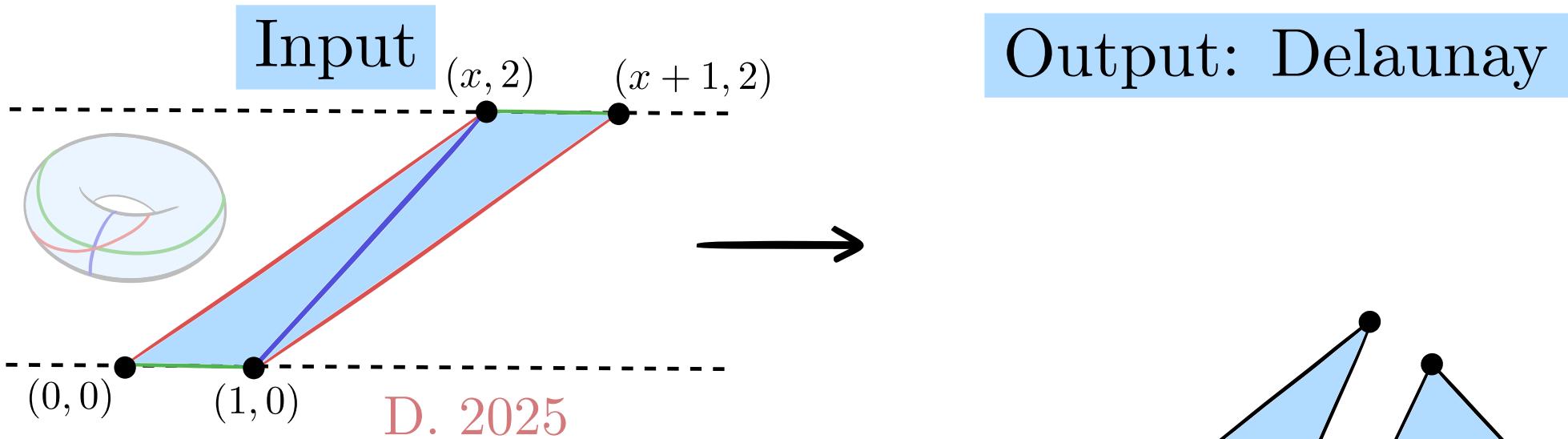
Lower bound



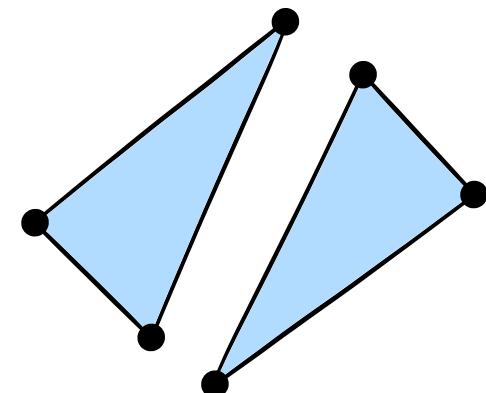
Lower bound



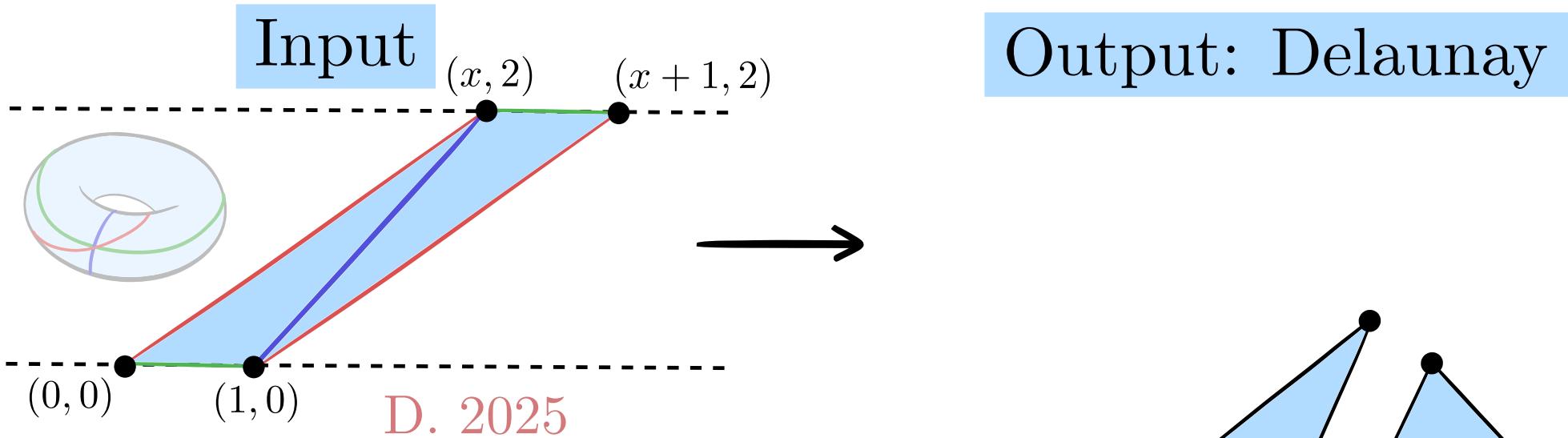
Lower bound



No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

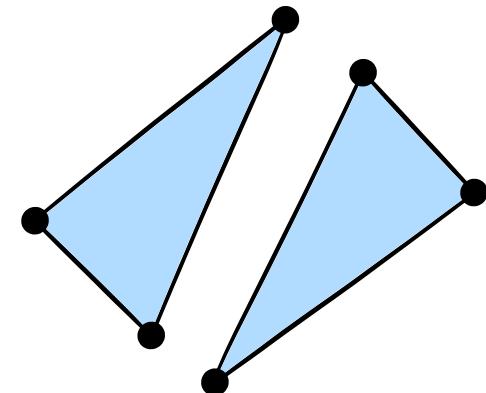


Lower bound

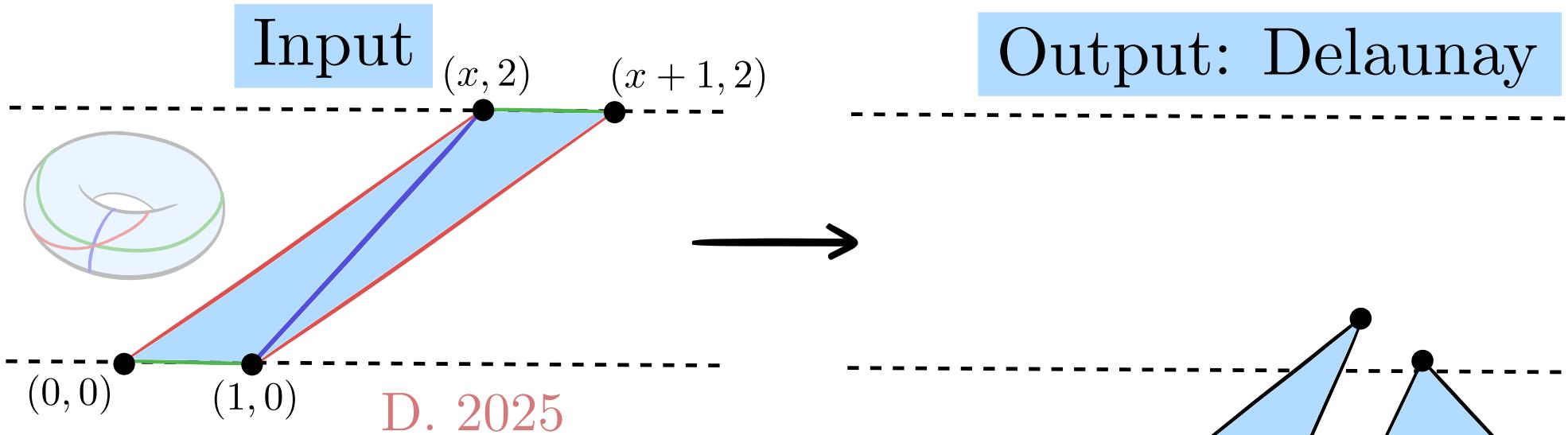


No Real RAM algo can compute
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→ Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time,
which is impossible [Blum, Shub, and Smale, 1989]



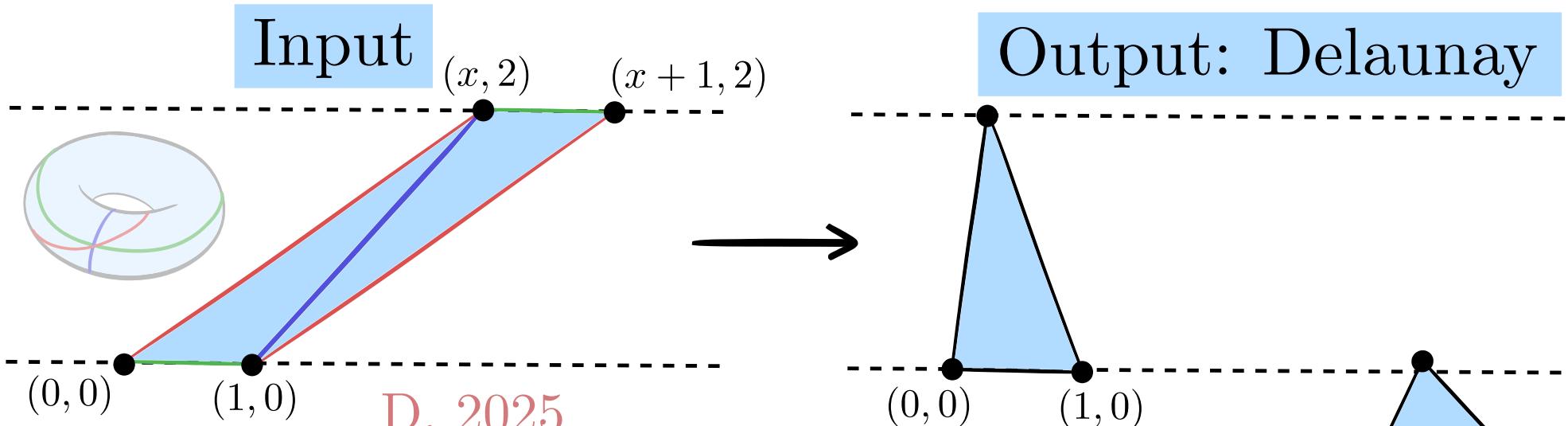
Lower bound



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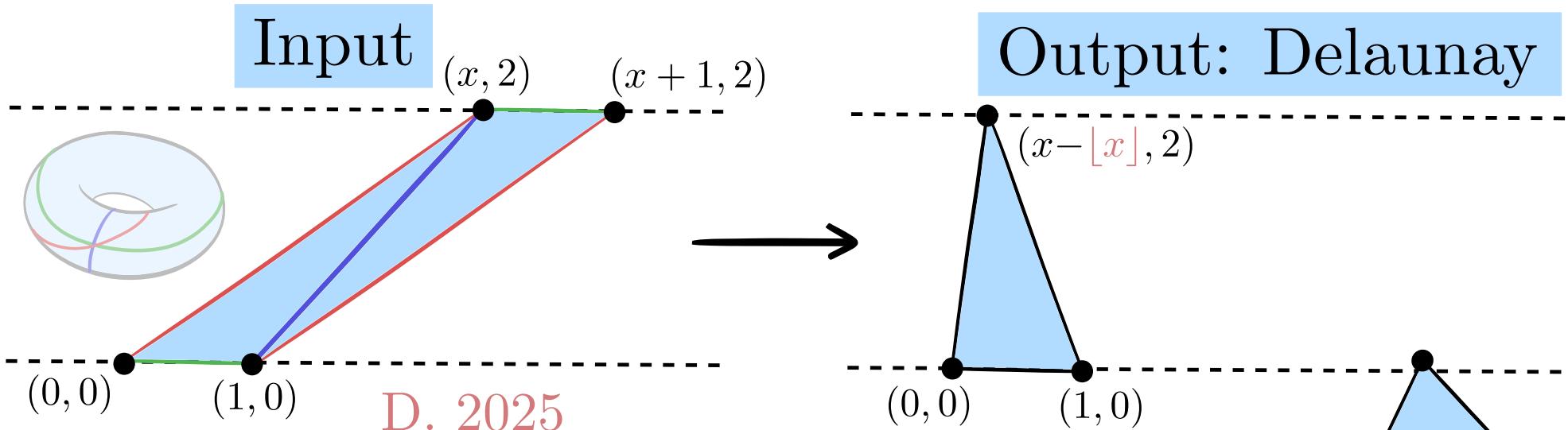
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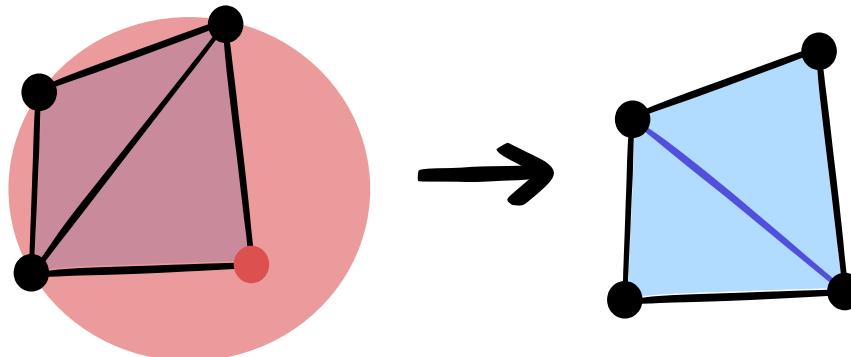
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Other results

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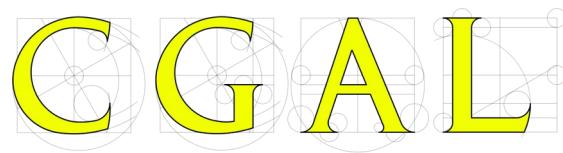
D., 2022-23

Upper bound for # Delaunay flips on flat tori,
tight up to constant factor



Other results

Despré, D., Pouget, and Teillaud, 2025



package for computing with
hyperbolic surfaces

generation (genus 2 only)

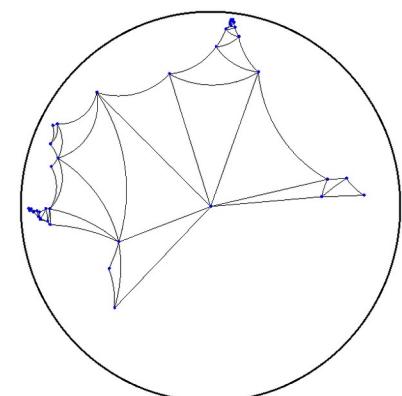
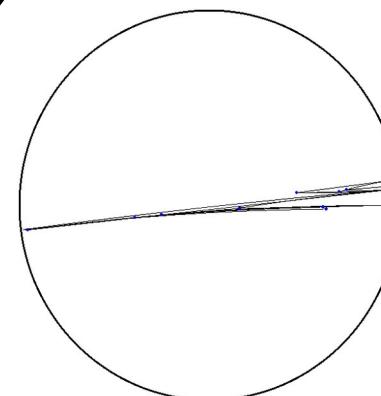


Triangulated hyperbolic
surface

Delaunay flip



↓
visualization



Other results

Despré, D., Pouget, and Teillaud

Exact computations!
package for computing
hyperbolic surfaces

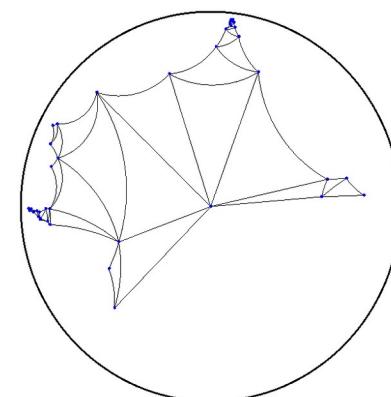
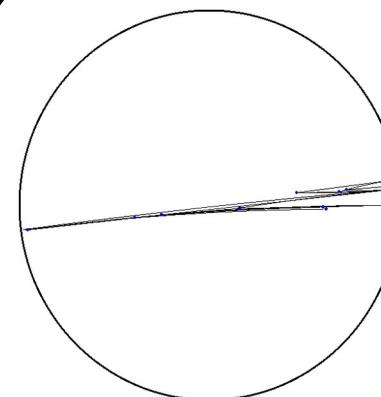
generation (genus 2 only)



Triangulated hyperbolic
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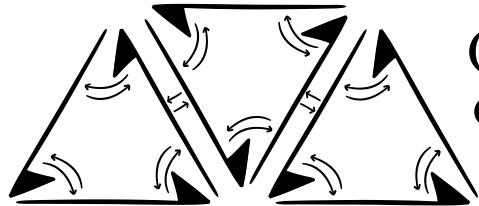
Delaunay flip

↓
visualization



Triangulated hyperbolic
surface

Combinatorics

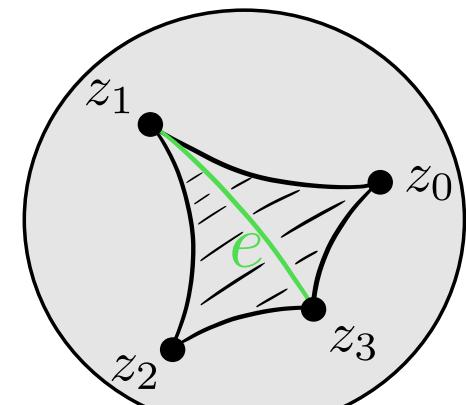


CGAL
combinatorial maps

Triangulated hyperbolic
surface

Geometry

- each edge is decorated with a complex number (cross ratio)



$$e \rightarrow \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

Untangling Graphs

Computing Delaunay Triangulations

Conclusion

Untangling Graphs

Computing Delaunay Triangulations

Conclusion

Improving the untangling algos

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- if the drawing can be untangled, return a sequence of untangling moves

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- if the drawing can be untangled, return a sequence of untangling moves
- otherwise, return a smallest sub-drawing that can't be untangled
- extension to non orientable surfaces
- improve the complexity using systems of quads?
or by better conversion of model?

Related problems

Related problems

- is computing the minimum number of crossings of a graph up to homotopy FPT or $W[1]$ -hard?

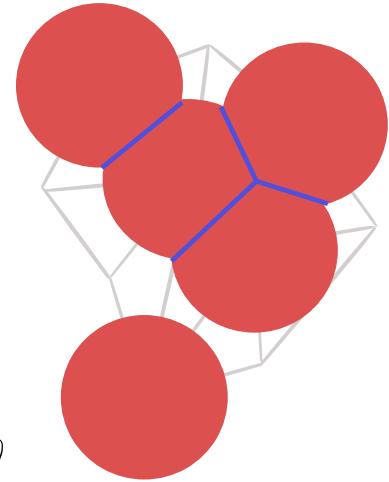
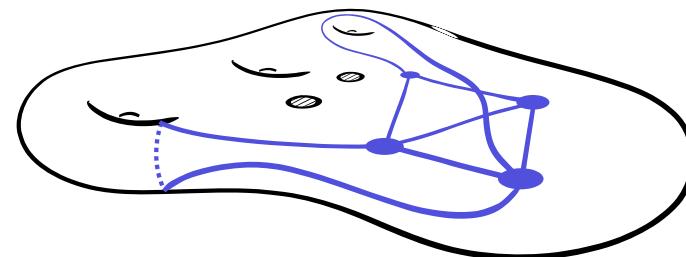
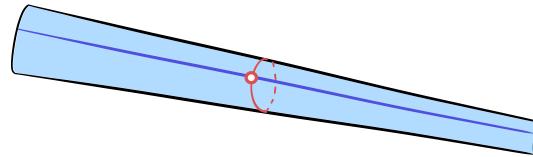
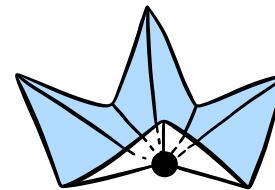
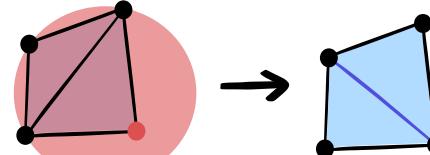
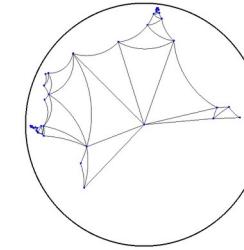
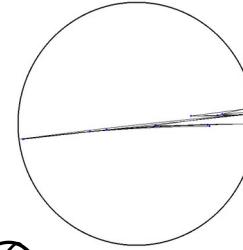
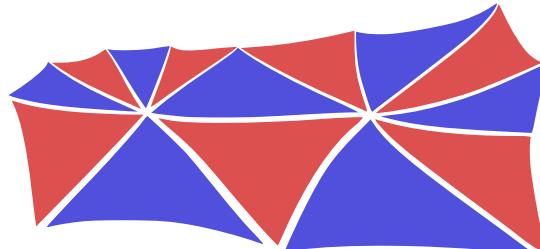
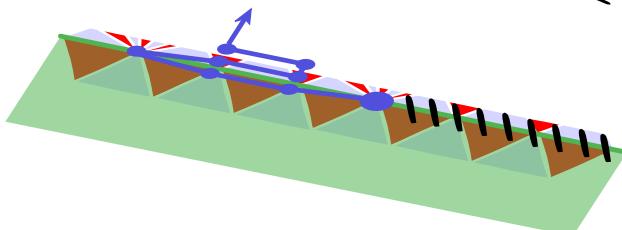
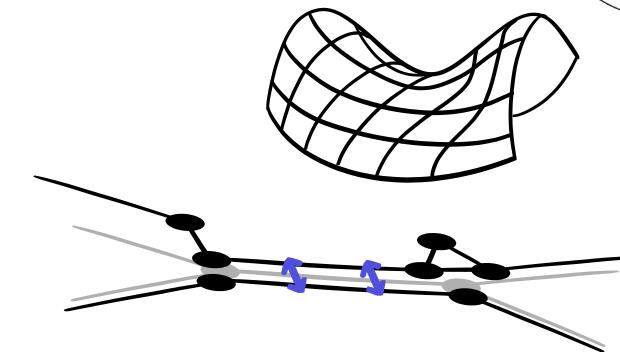
Related problems

- is computing the minimum number of crossings of a graph up to homotopy FPT or $W[1]$ -hard?
- how unique reducing triangulations are?

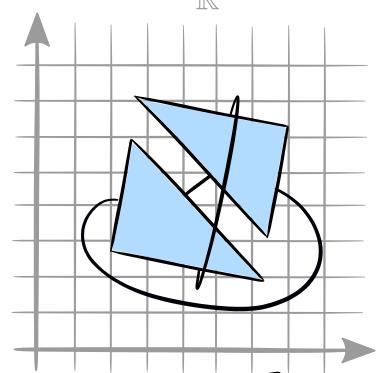
Related problems

- is computing the minimum number of crossings of a graph up to homotopy FPT or $W[1]$ -hard?
- how unique reducing triangulations are?
- what is the complexity of Delaunay flips algo?

CGAL



\mathbb{R}^2



Thank you

