

Algorithms for Topological and Metric Surfaces



Loïc Dubois

Computational Geometry

Design algorithms for geometric problems

This thesis

Focus on surfaces

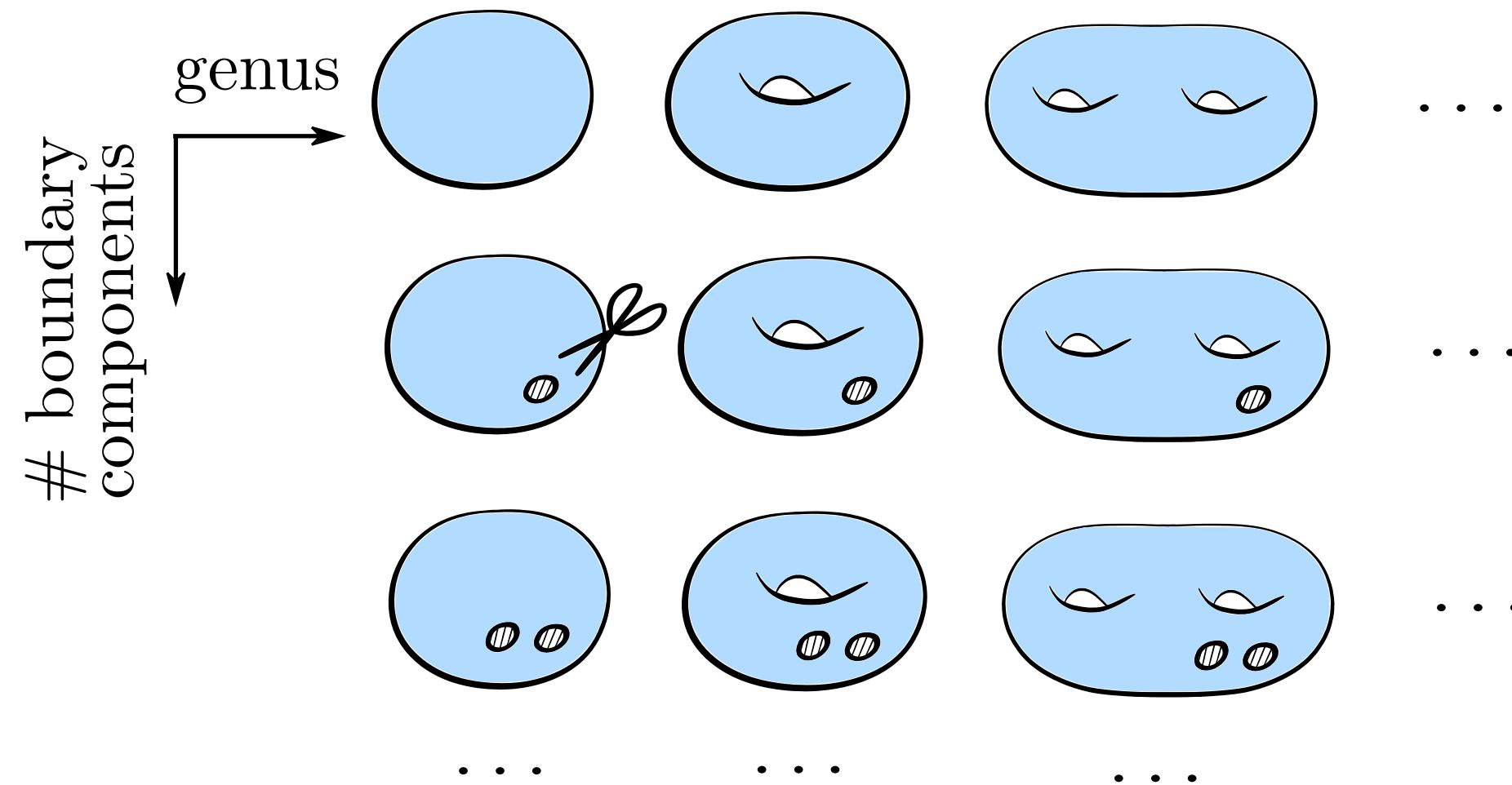


Topology

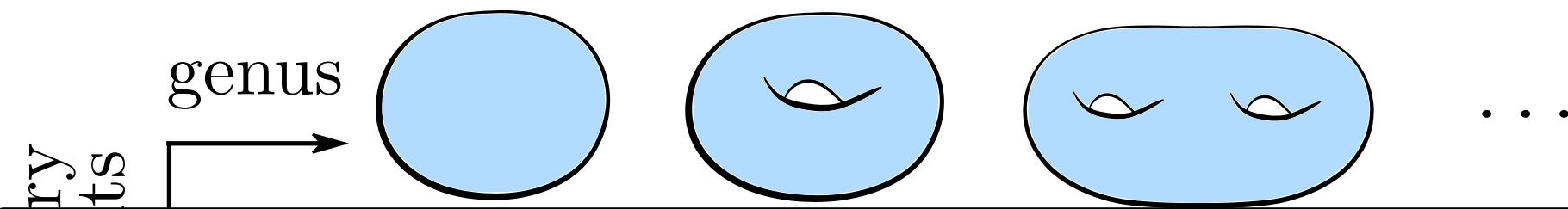


image by Crane and Segerman

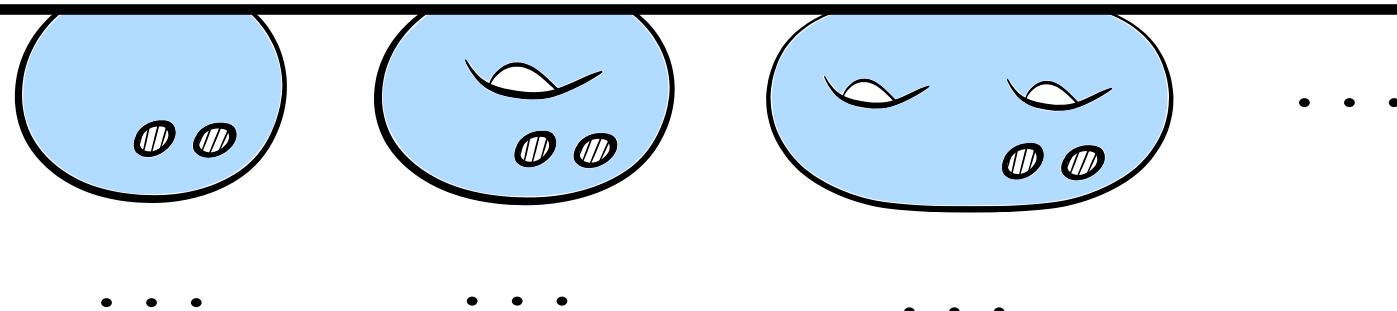
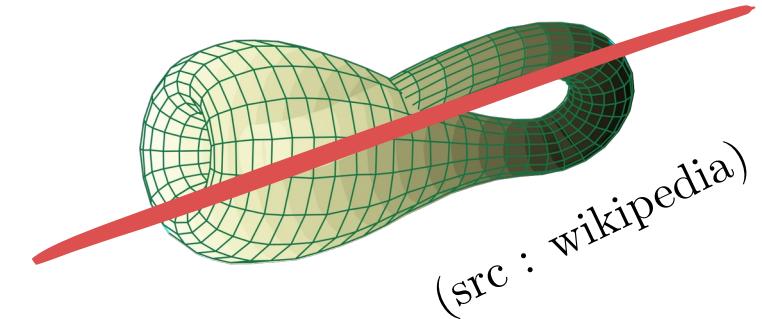
Topological Surfaces



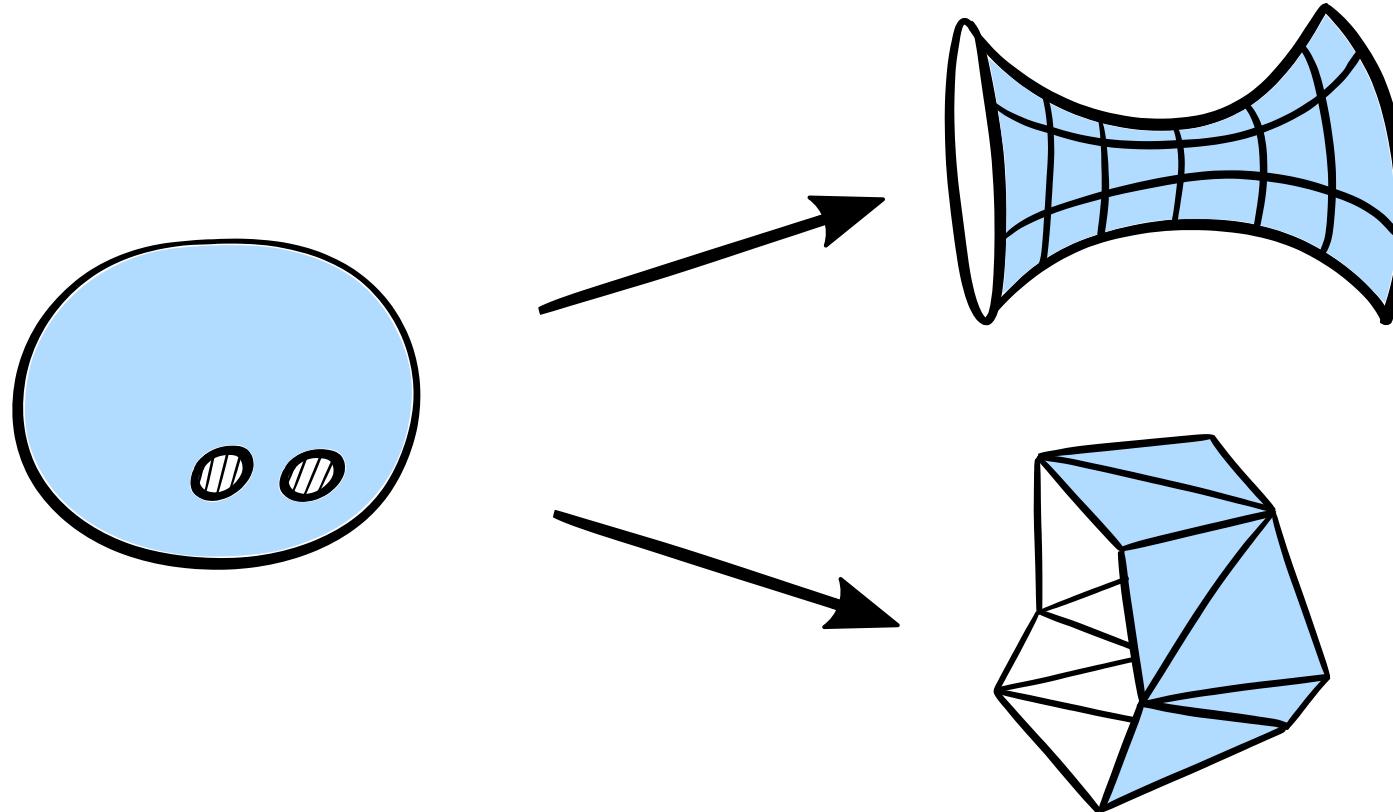
Topological Surfaces



Only **orientable** surfaces today !



Metrics on surfaces



Untangling Graphs

Computing Delaunay Triangulations

Conclusion

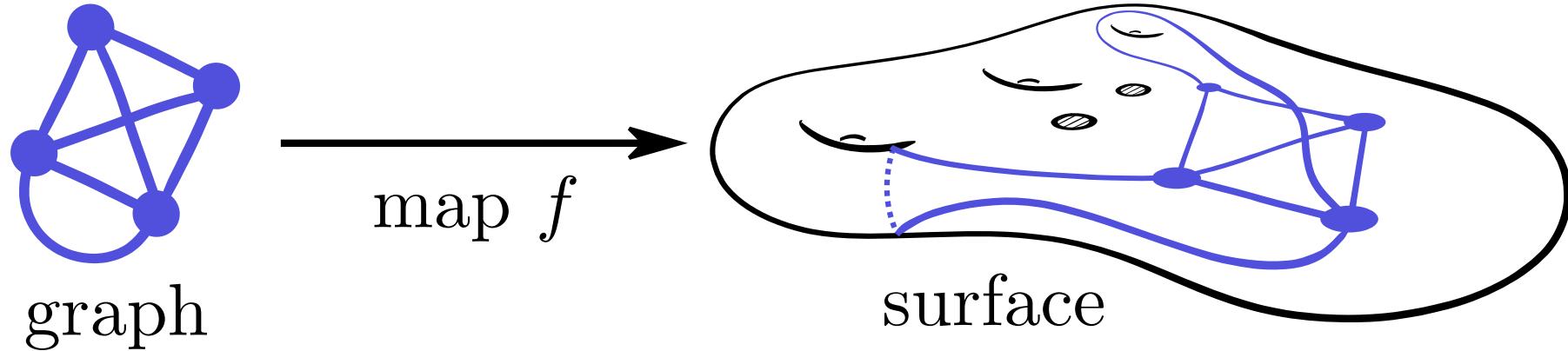
Untangling Graphs

Computing Delaunay Triangulations

Conclusion

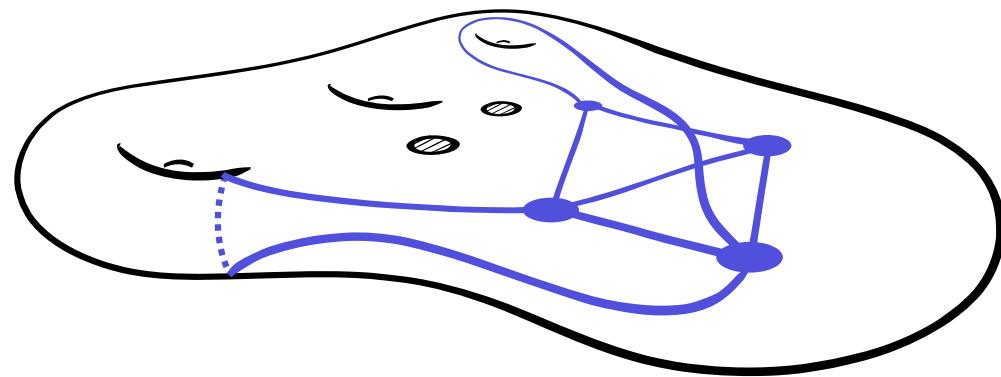
Problem: untangling graphs

Input:

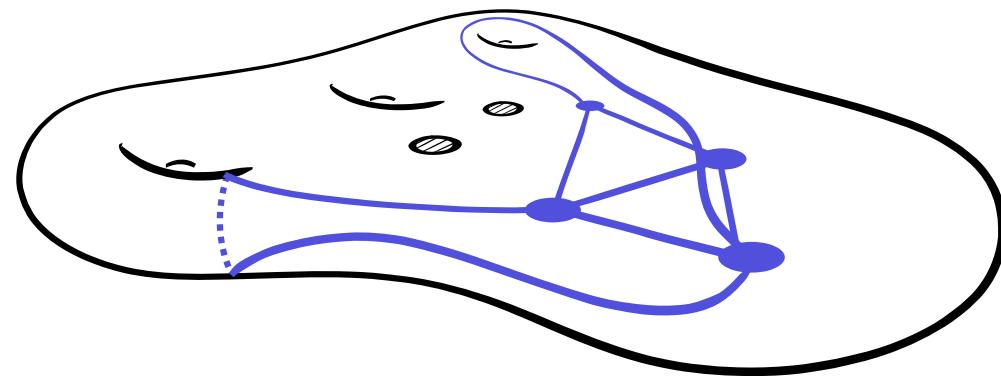


Goal: remove all crossings by deforming f

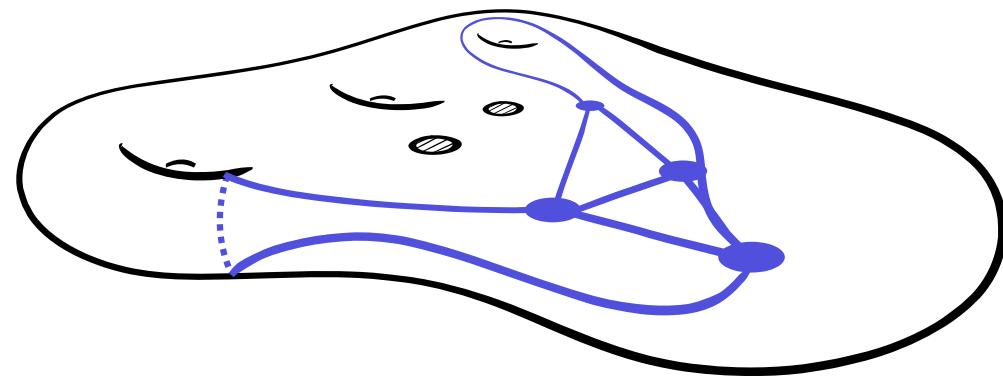
Problem: untangling graphs



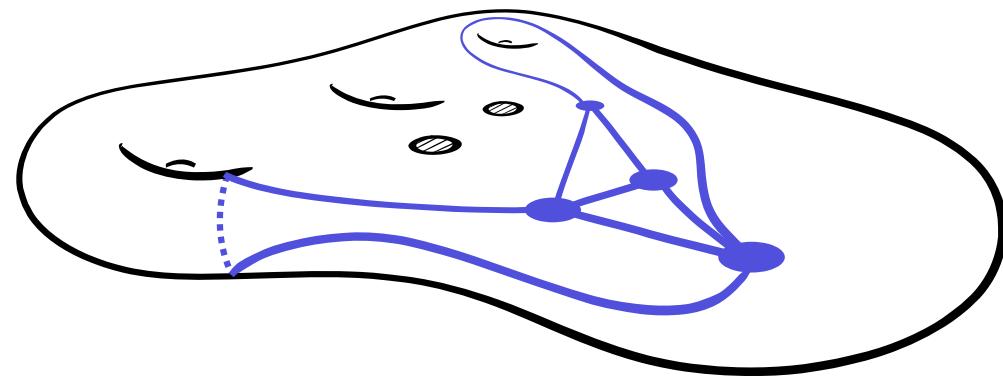
Problem: untangling graphs



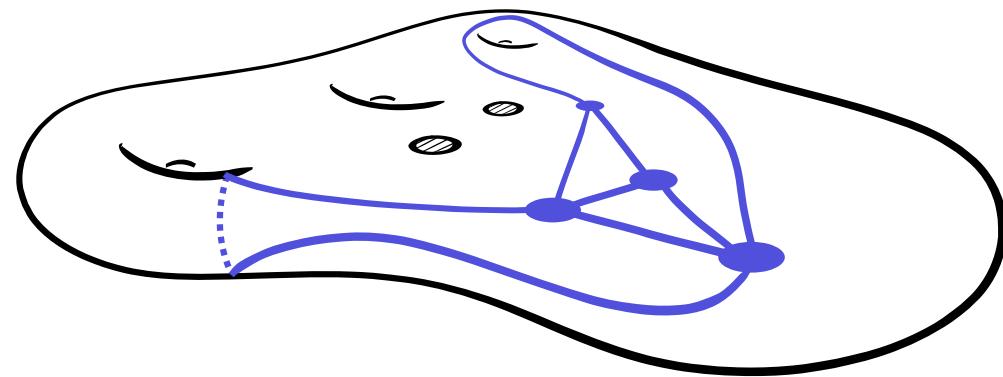
Problem: untangling graphs



Problem: untangling graphs



Problem: untangling graphs

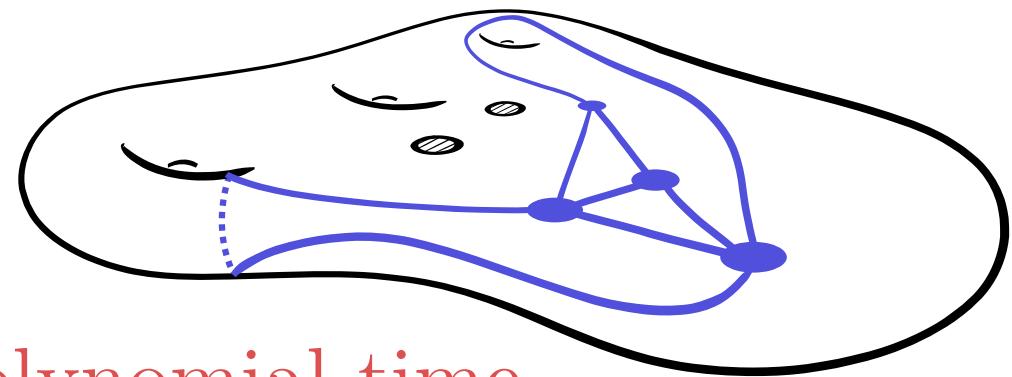


Problem: untangling graphs



Output: Yes (+ untangled drawing) or No

Problem: untangling graphs

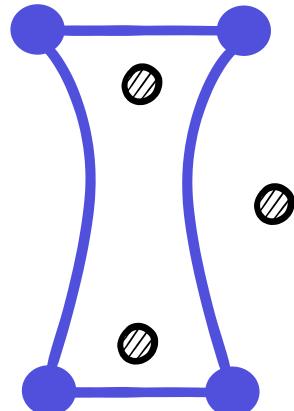


We obtain the first polynomial time
algorithms for this problem

Output: Yes (+ untangled drawing) or No

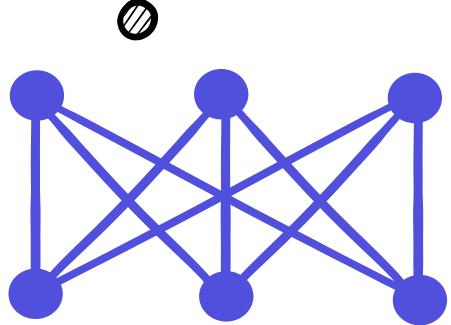


Yes:

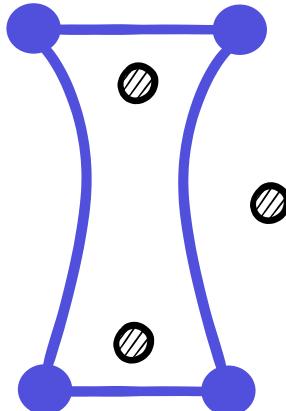


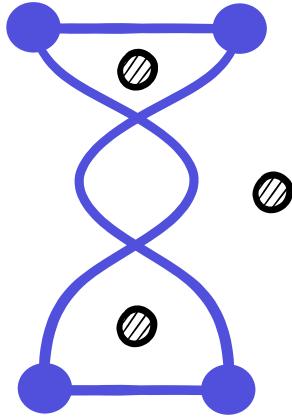


Yes:

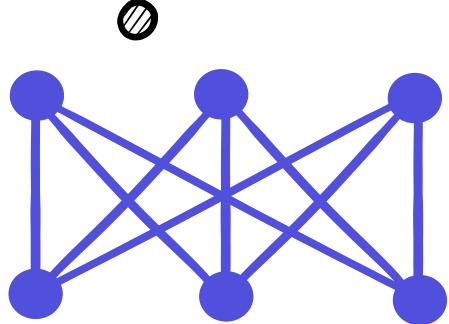


No





Yes:



No

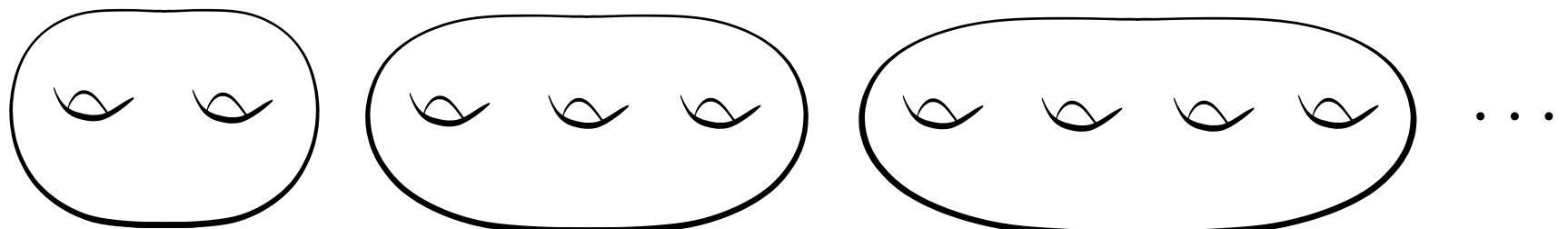


No



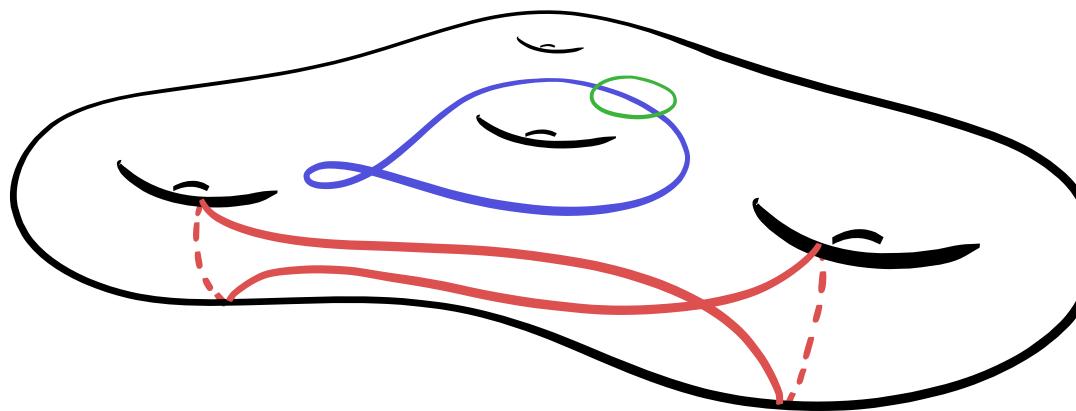
Related works

We focus on surfaces without boundary
of genus ≥ 2



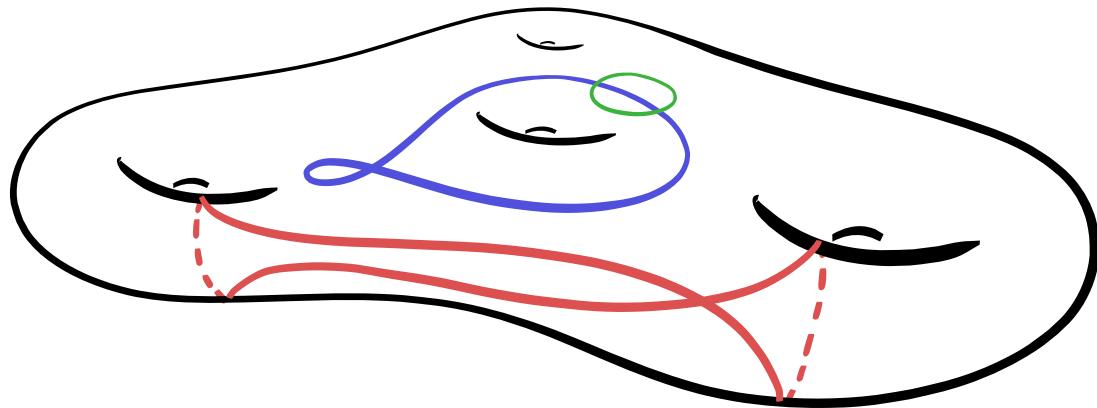
Related problem: making curves cross minimally

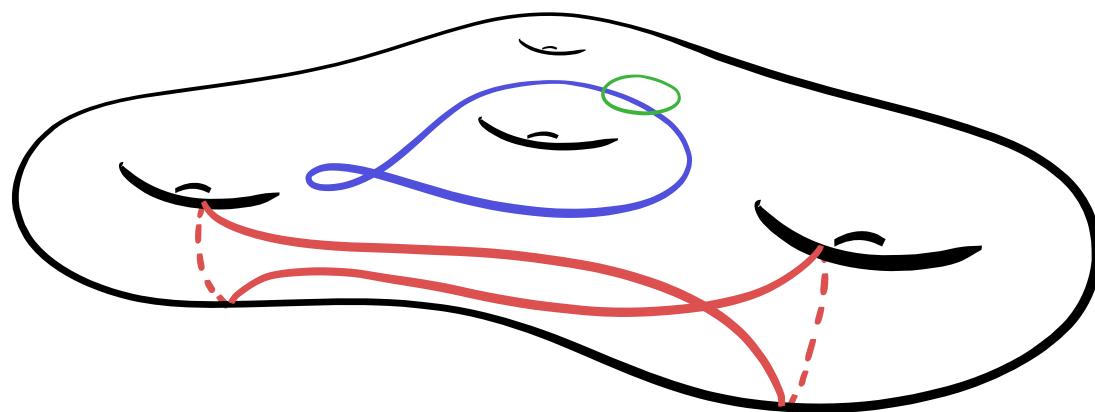
Input: closed curves on a surface

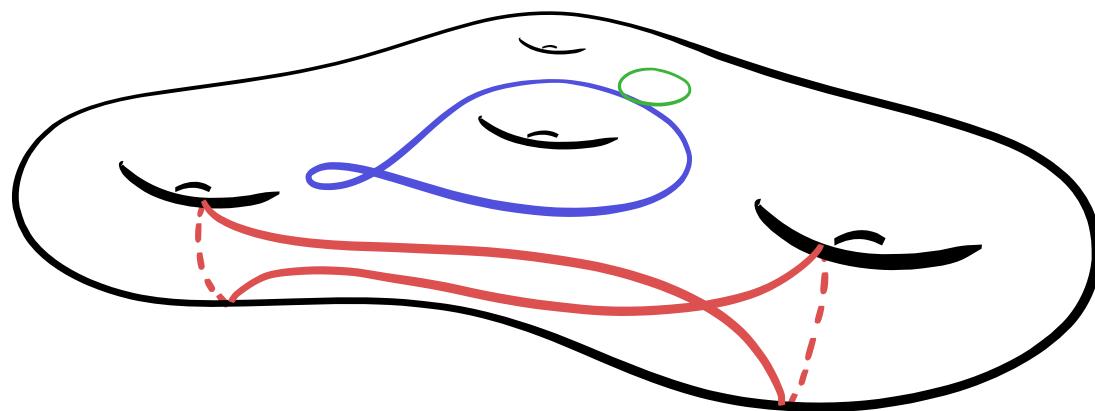


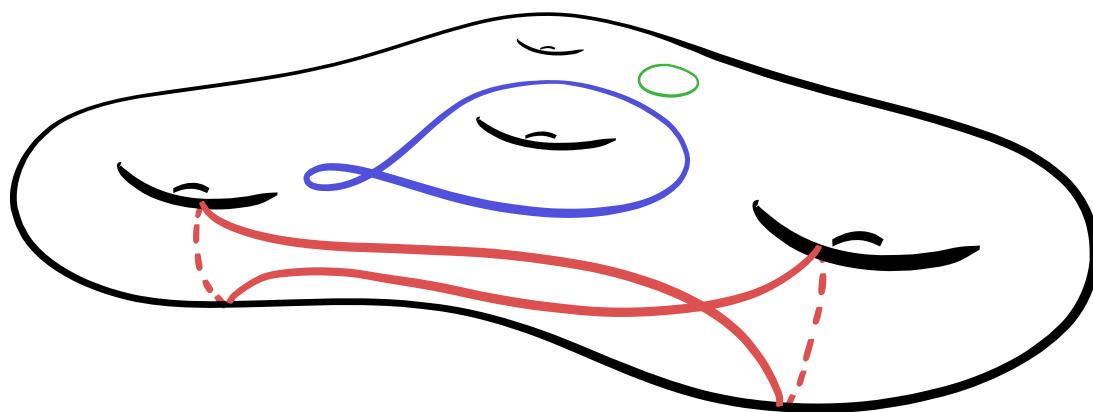
Goal:

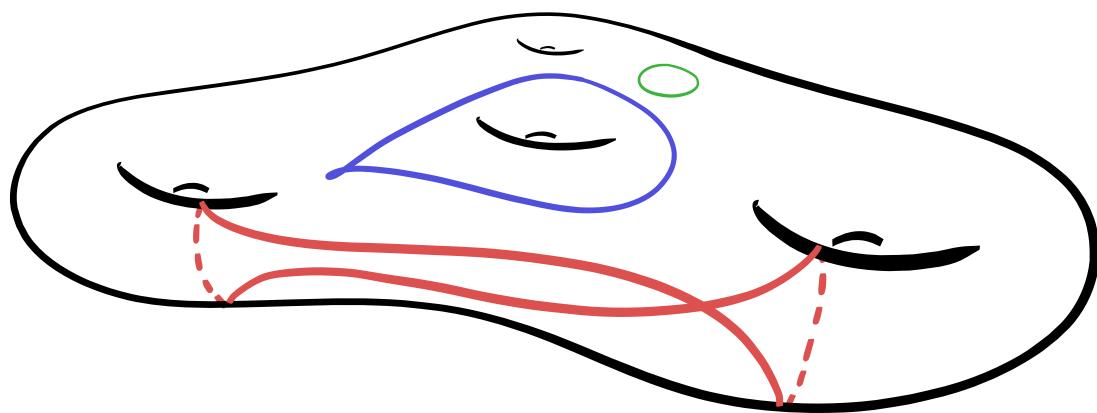
minimize the # crossings by deforming the curves

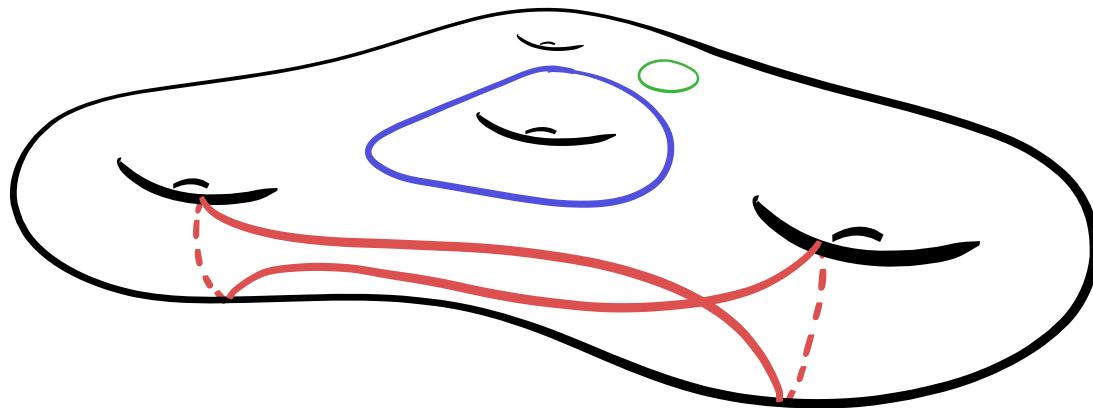


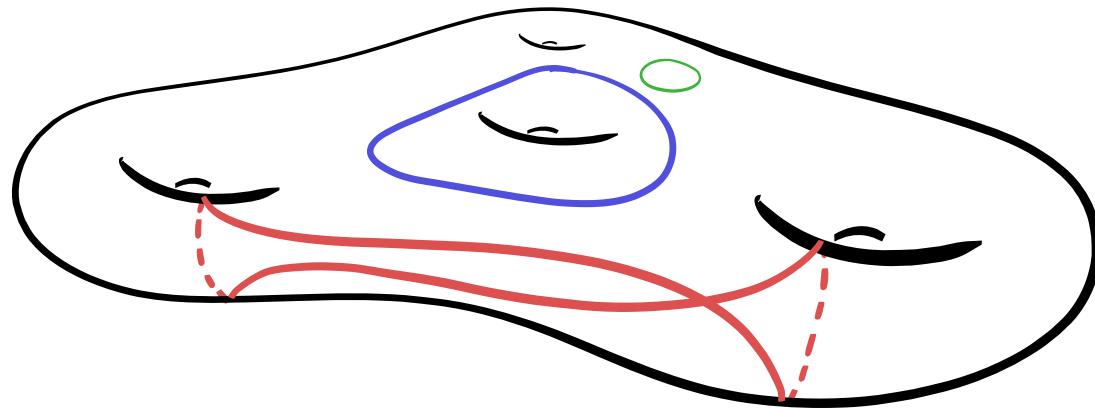












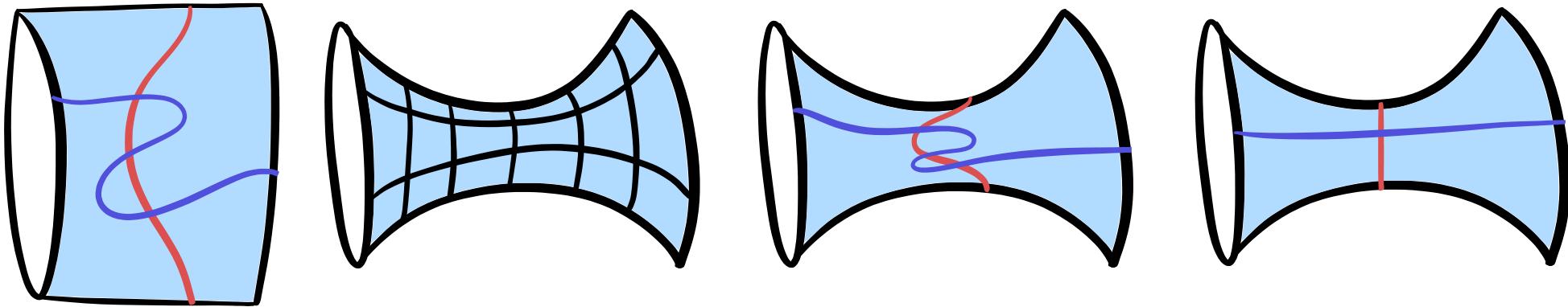
Output: $\min \#$ of crossings (+ optimal curves)

Many works related to making curves cross minimally!

- | | |
|-------------------------|-------------------------------|
| Poincaré, 1905 | de Graaf and Schrijver, 1987 |
| Dehn, 1911 | Dynnikov, 2002 |
| Dehn, 1912 | Paterson, 2002 |
| Reinhart, 1962 | Gonçalves et al., 2005 |
| Zieschang, 1965 | Schaefer et al., 2008 |
| Chillingworth, 1969 | Lazarus and Rivaud, 2012 |
| Zieschang, 1969 | Erickson and Whittlesey, 2013 |
| Chillingworth, 1971 | Arettines, 2015 |
| Turaev, 1979 | Chang et al., 2018 |
| Birman and Series, 1984 | Despré and Lazarus, 2019 |
| Cohen and Lustig, 1984 | Fulek and Tóth, 2020 |
| Hass and Scott, 1985 | Chang and de Mesmay, 2022 |
| Lustig, 1987 | Lackenby, 2024 |

Method for making curves cross minimally

Poincaré, 1905



1. give special shape to surface
2. straighten the curves

The special shape

negative curvature:

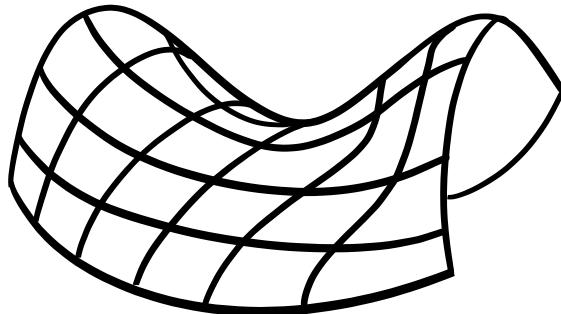


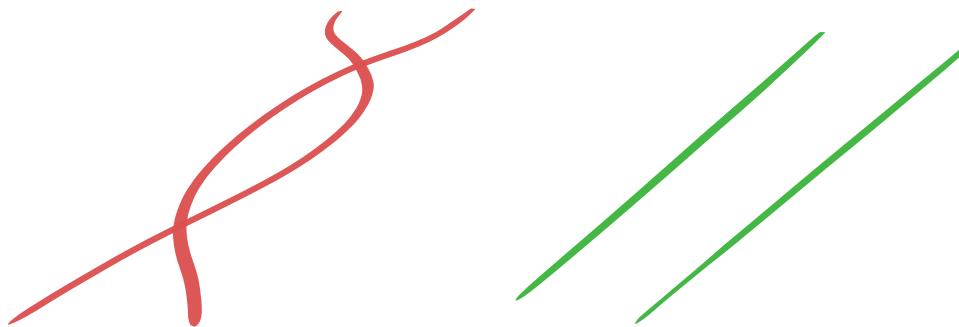
image by Susan Lombardo

almost all surfaces
can be curved negatively

The key property

On a negatively curved surface,

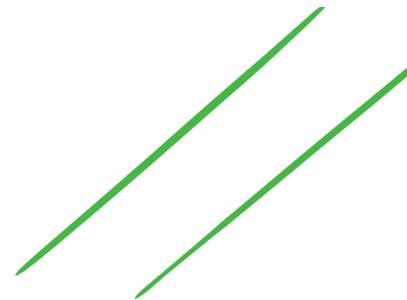
straight curves cross minimally



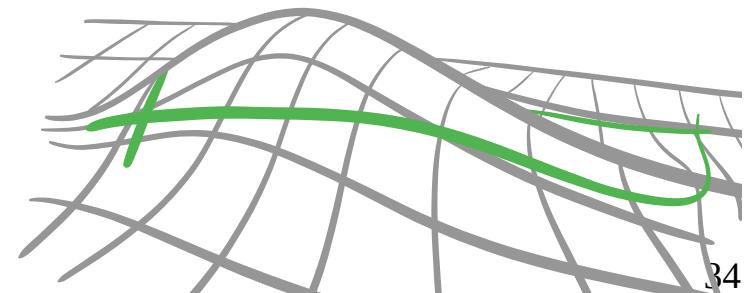
The key property

On a negatively curved surface,

straight curves cross minimally



(does not hold
on all surfaces)



The key property

On a negatively curved surface,

straight curves cross minimally



ever
into

The key property

On a negatively curved surface,

straight curves cross minimally



every path
into a unit

The key property

On a negatively curved surface,

straight curves cross minimally



every path can be
into a unique straight curve

The key property

On a negatively curved surface,

straight curves cross minimally



every path can be deformed
into a unique straight pa

The key property

On a negatively curved surface,

cross minimally



every path can be deformed
into a unique **straight** path

The key property

On a negatively curved surface,

minimally



every path can be deformed
into a unique **straight** path

The key property

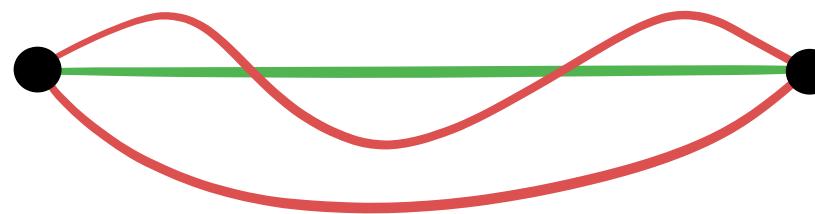
On a negatively curved surface,

every path can be deformed
into a unique **straight** path

The key property

On a negatively curved surface,

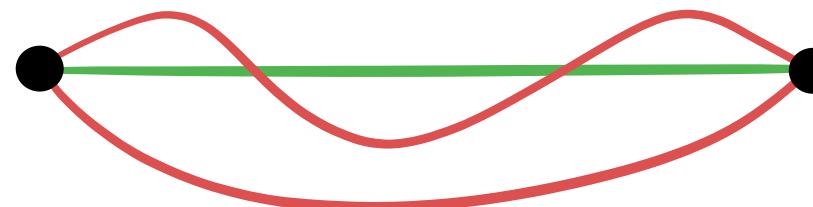
every path can be deformed
into a unique **straight** path



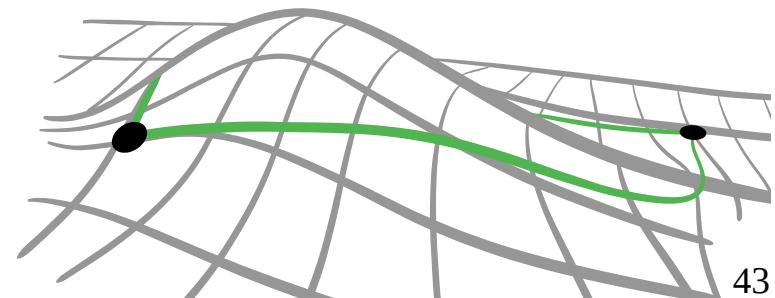
The key property

On a negatively curved surface,

every path can be deformed
into a unique **straight** path

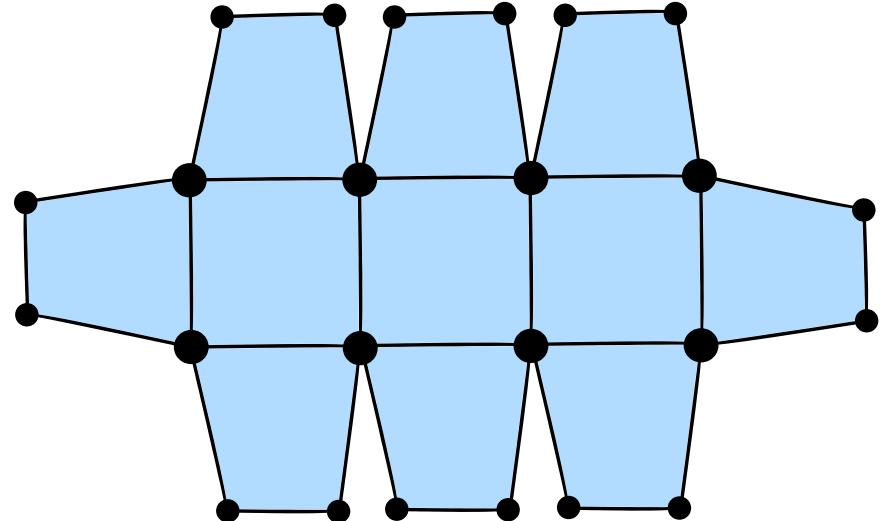


(does not hold
on all surfaces)

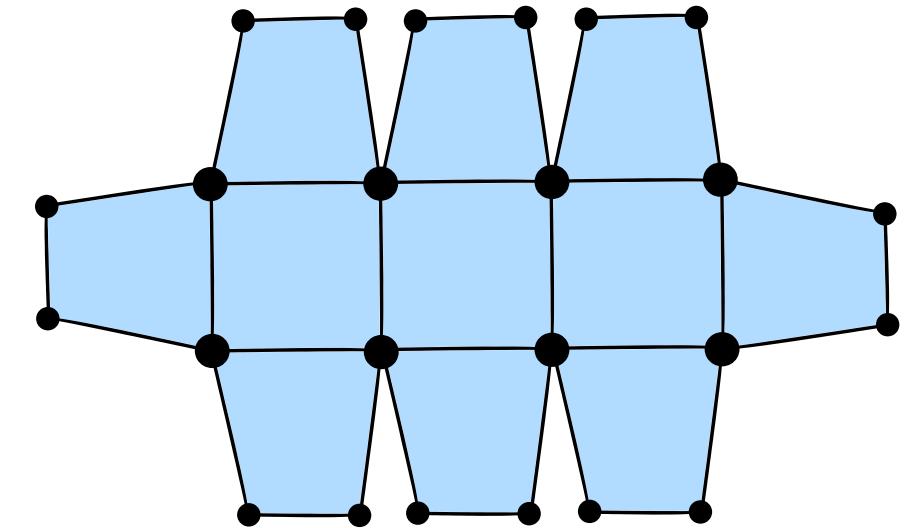


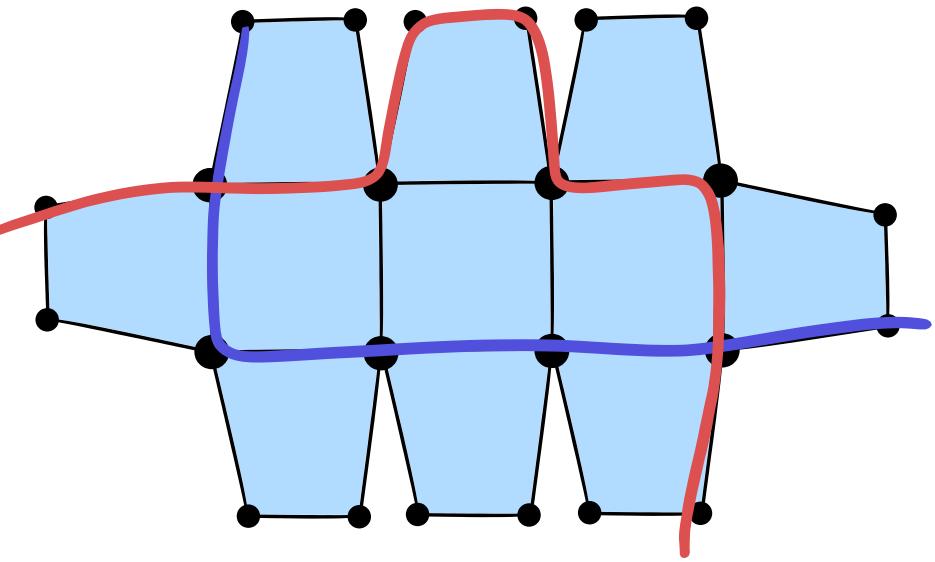
Discrete model of negatively
curved surfaces?

Lazarus and Rivaud, 2012
Erickson and Whittlesey, 2013

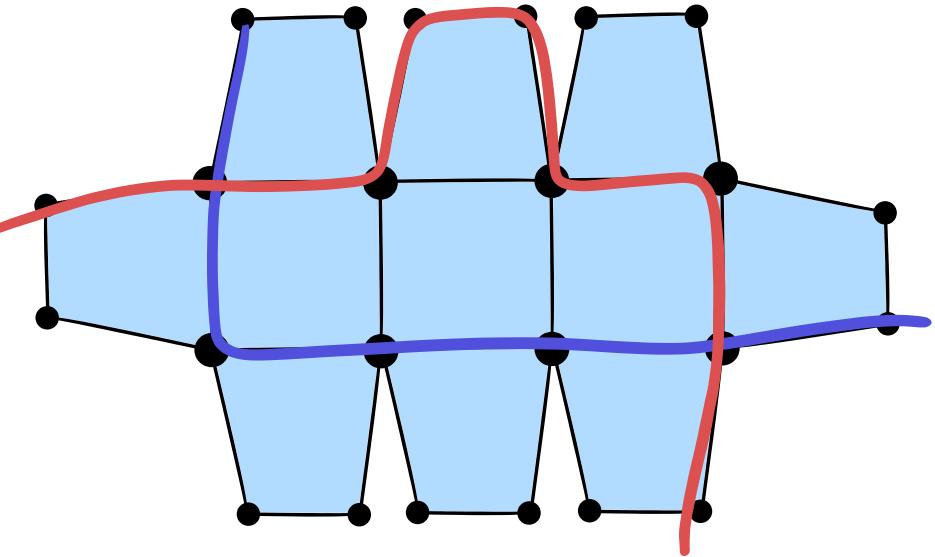


System of quads

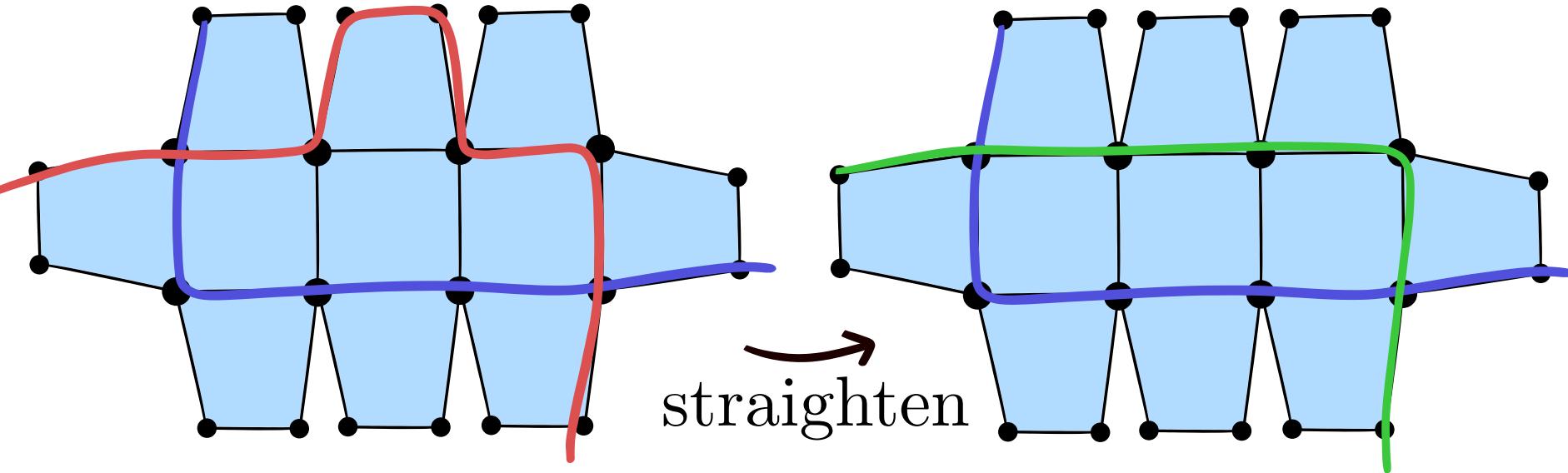




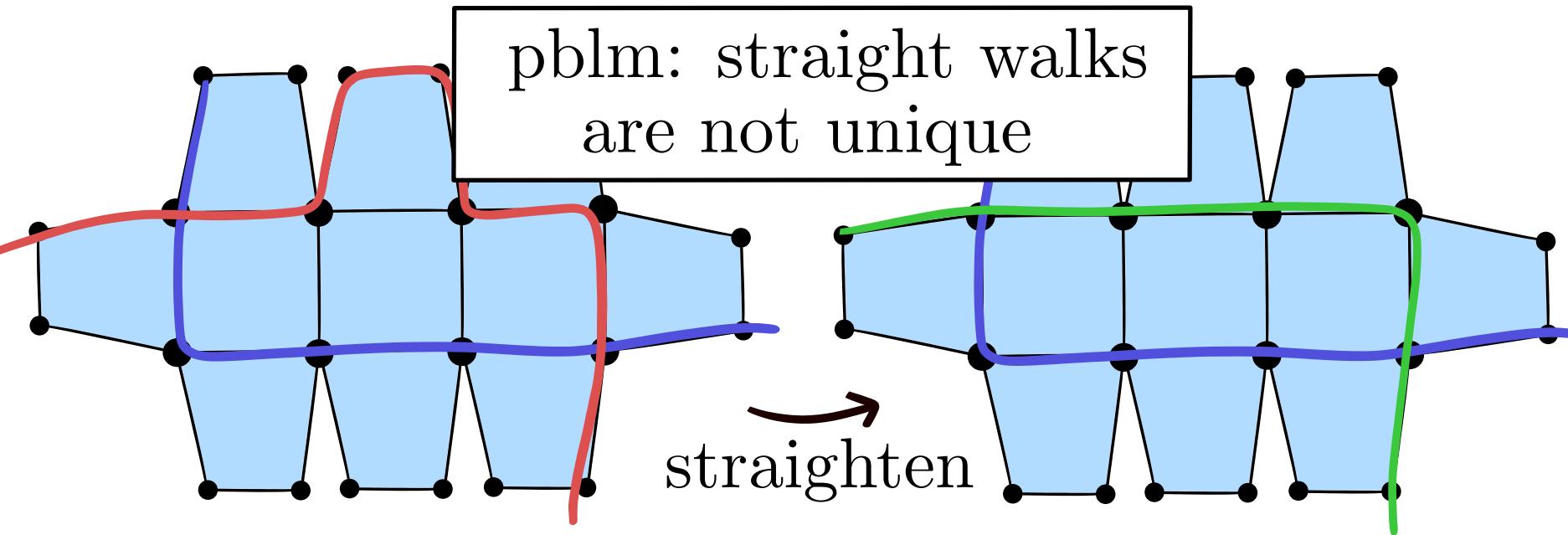
Algo for making curves cross minimally



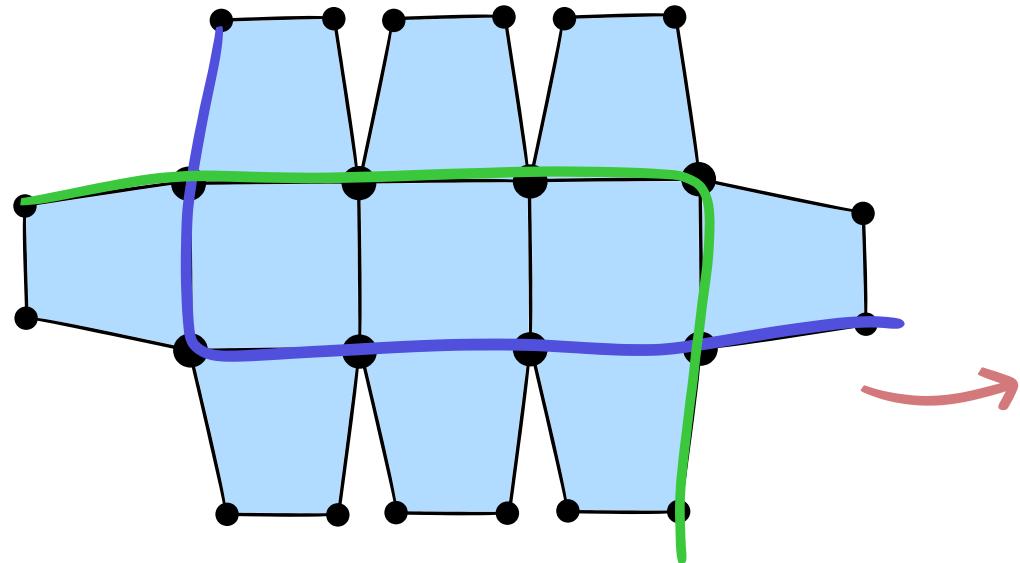
Algo for making curves cross minimally



Algo for making curves cross minimally



Algo for making curves cross minimally



Despré and Lazarus, 2019

What about
untangling graphs?

Method for untangling graphs

Tutte, 1963

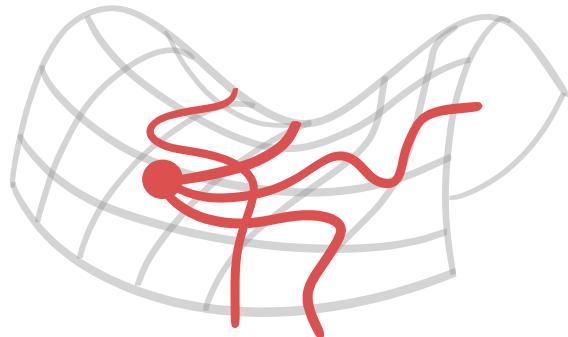
Y. Colin de Verdière, 1991



Method for untangling graphs

Tutte, 1963

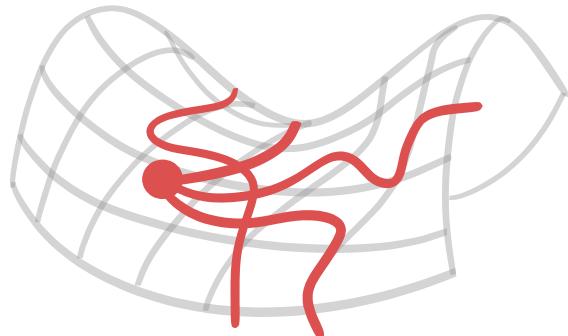
Y. Colin de Verdière, 1991



Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight

Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight

Method for untangling graphs

Tutte, 1963



Y. Colin de Verdière, 1991

make edges straight
make vertices barycentric

Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight
make vertices barycentric

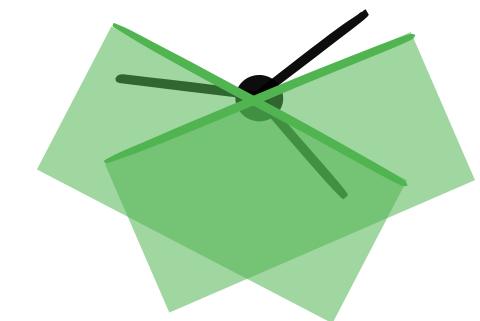
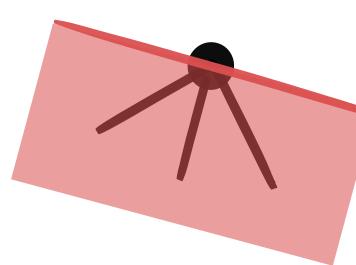
Method for untangling graphs

Tutte, 1963

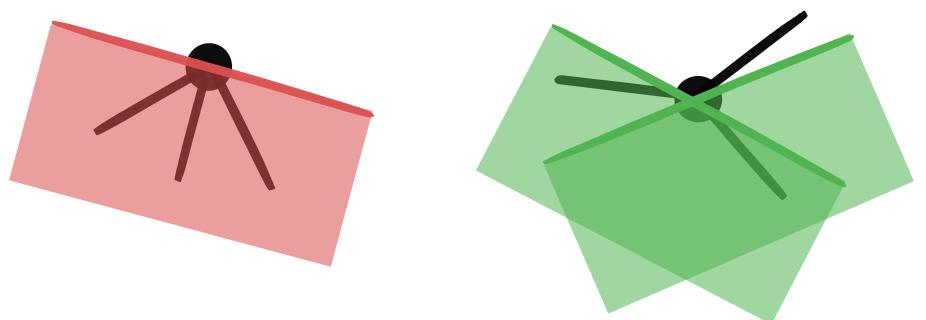


Y. Colin de Verdière, 1991

make edges straight
make vertices barycentric

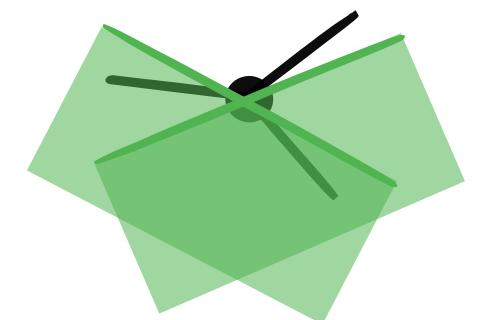
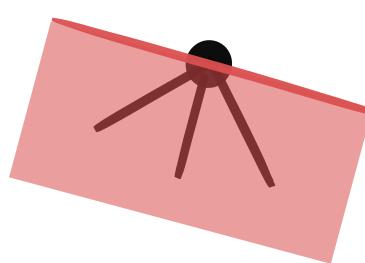


edges straight
vertices barycentric



Tutte embeddings

edges straight
vertices barycentric



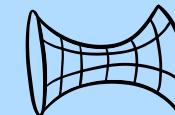
Summary

Curves

Graphs

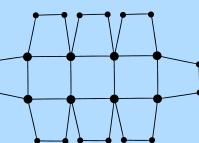
Method

negatively curved surface

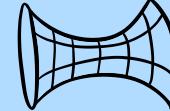


Algo

system of quads



Our results

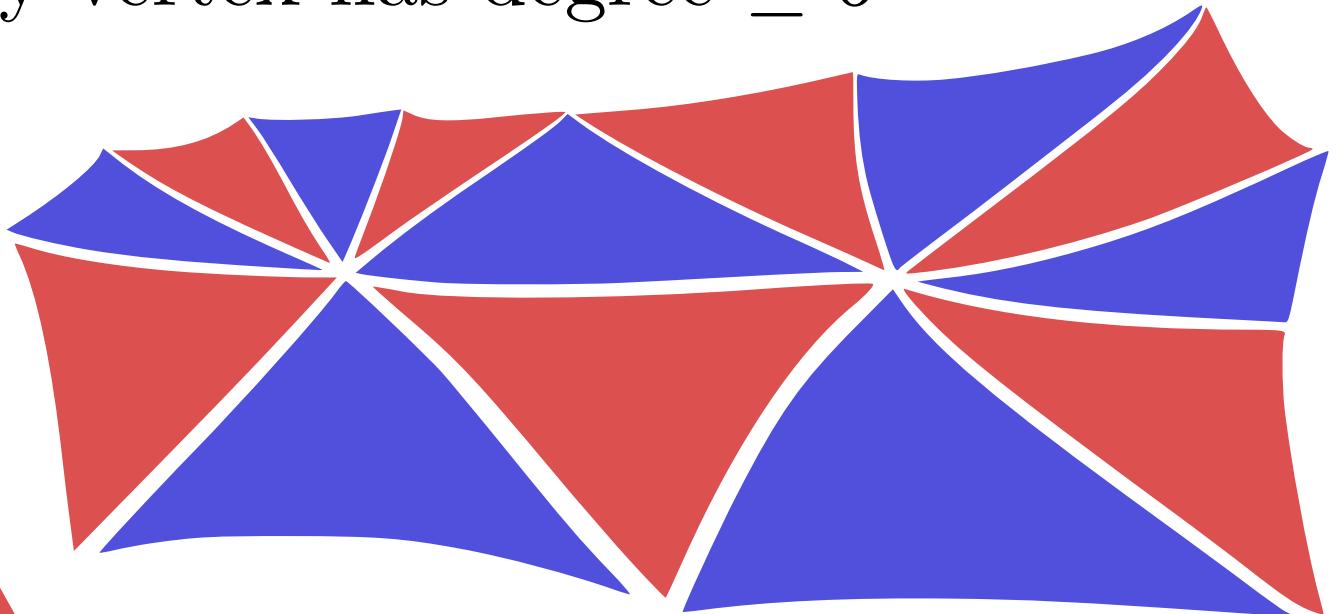
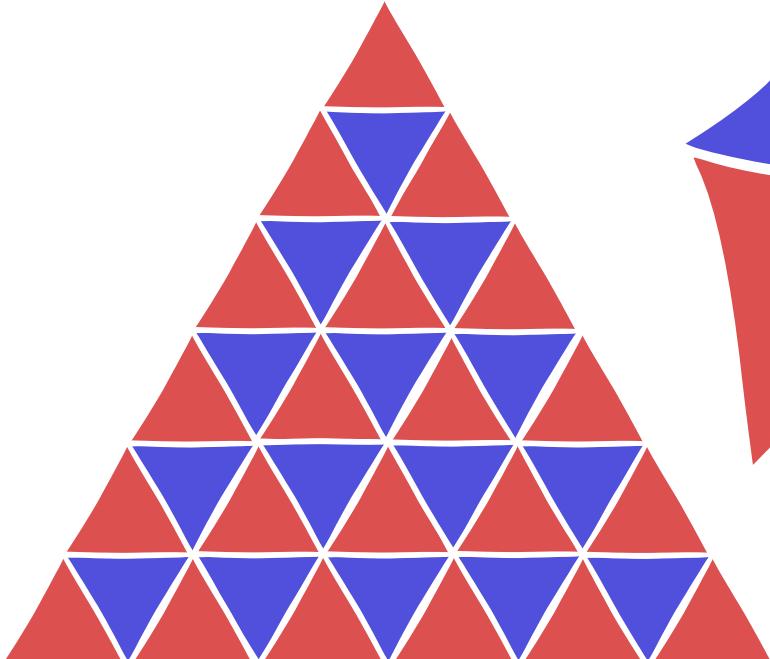
| | Curves | Graphs |
|--------|---|--|
| Method | negatively curved surface  | |
| Algo | reducing triangulations | |
| | improved algos for making curves cross minimally | first algos for untangling graphs |
| | | Discrete analogue of Tutte embeddings |

A new tool:

Reducing triangulations

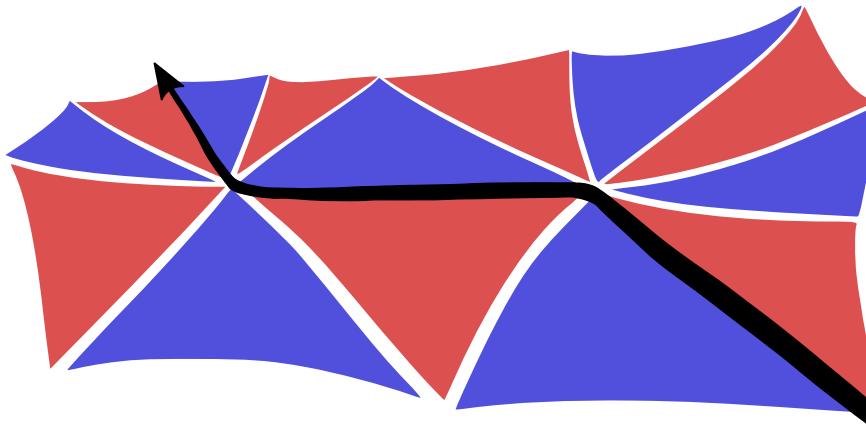
Reducing triangulations

dual is bipartite and
every vertex has degree $\geq 6^*$

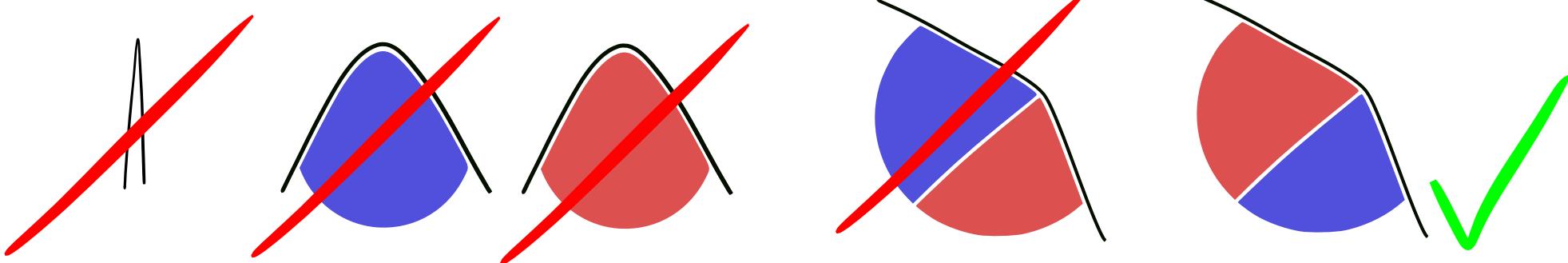


*sometimes 8

Reduced walks

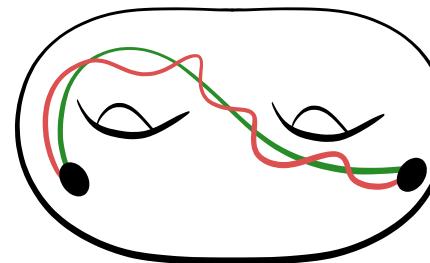
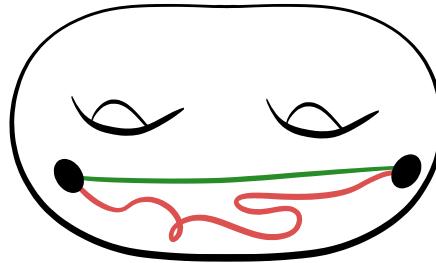


no bad turn



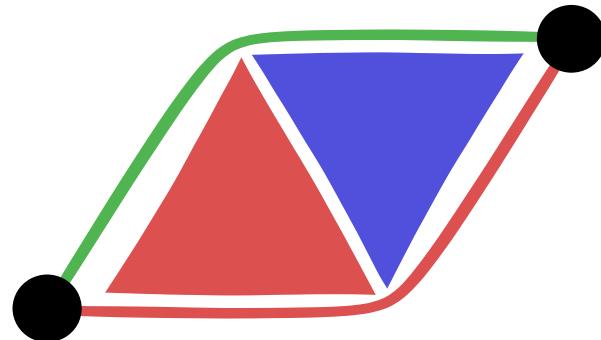
Properties of reduced walks

every walk can be deformed into a unique reduced walk, computable in linear time

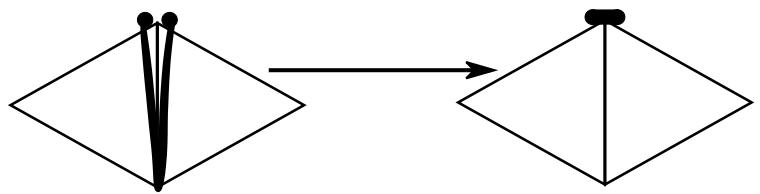


reduced walks are stable upon reversal and subwalk

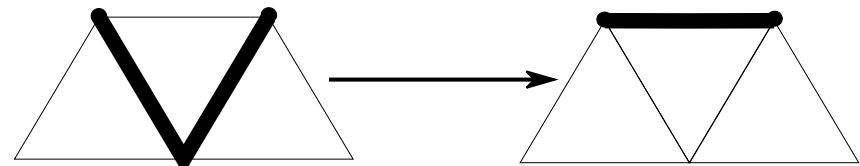
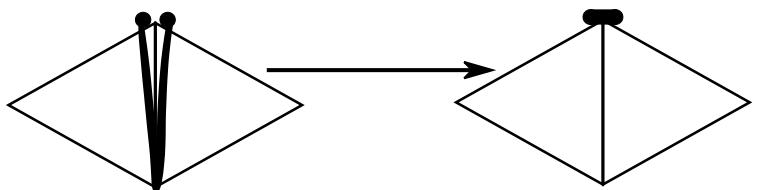
Purpose of the coloring



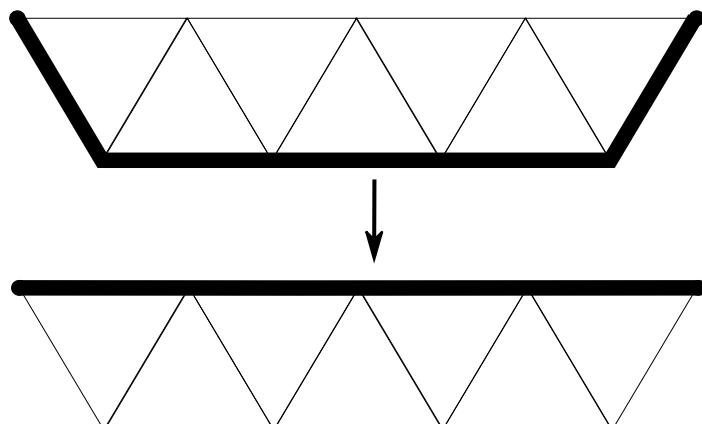
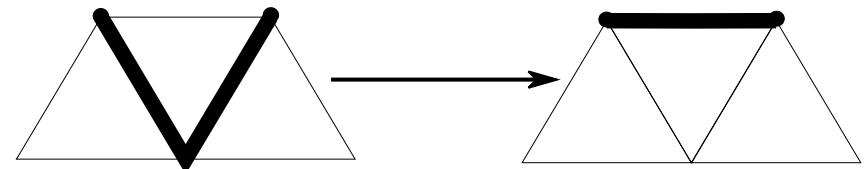
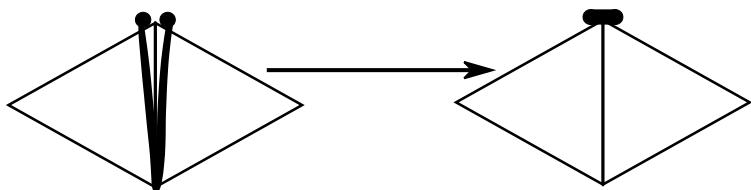
Reducing a walk



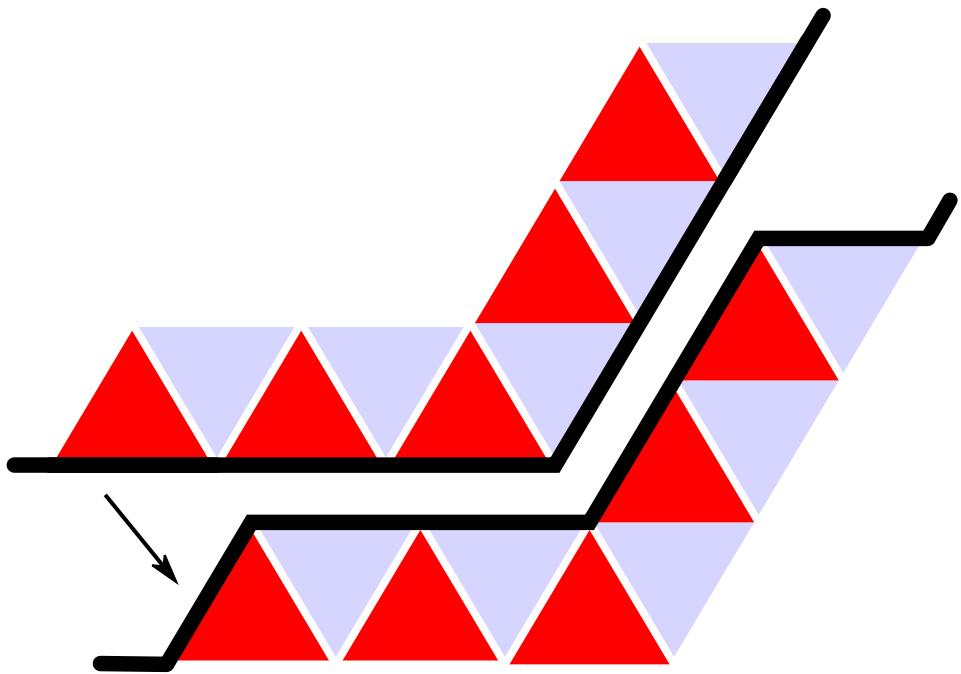
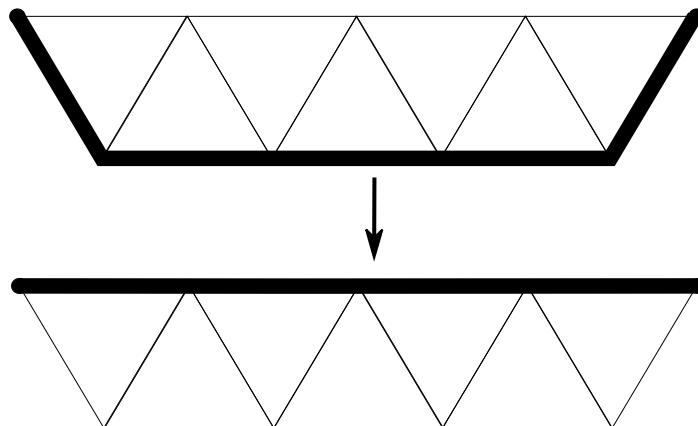
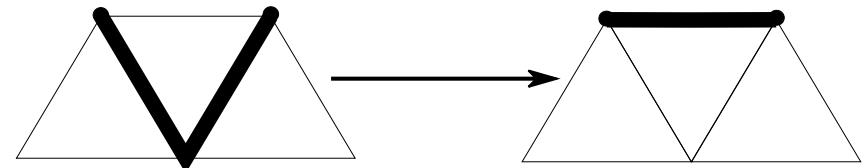
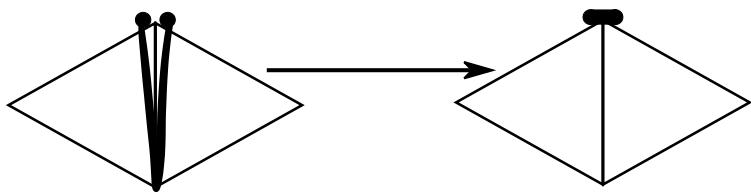
Reducing a walk



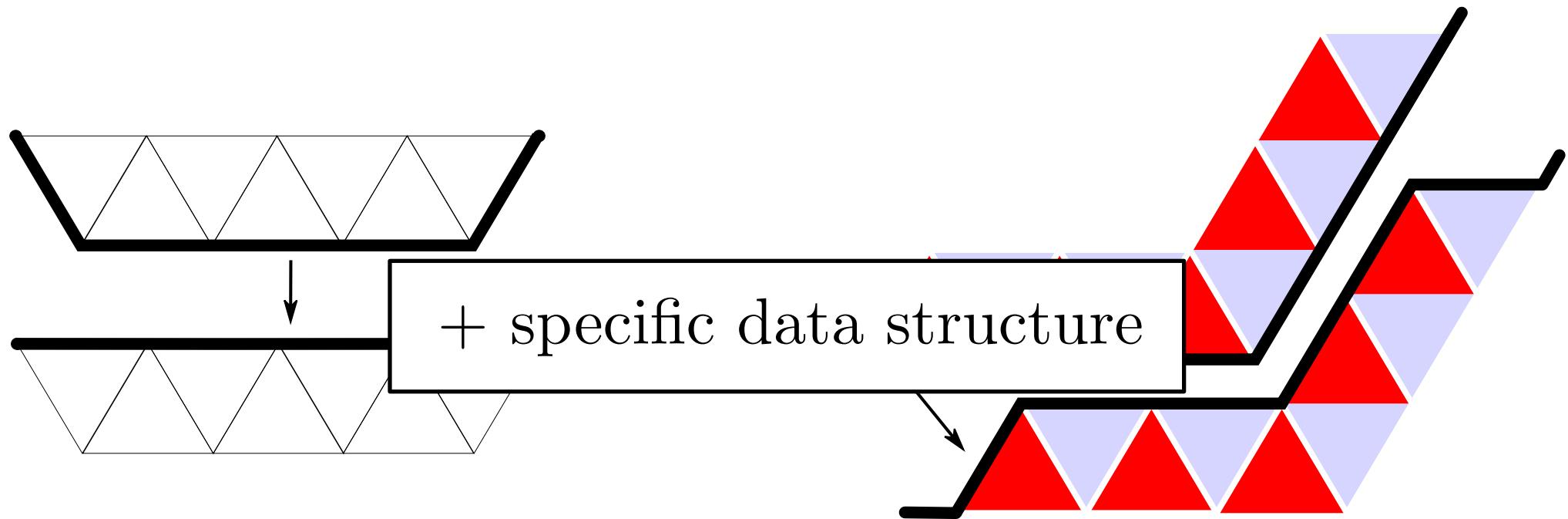
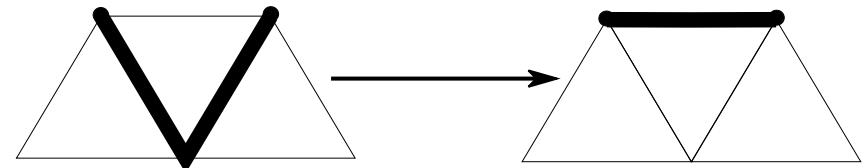
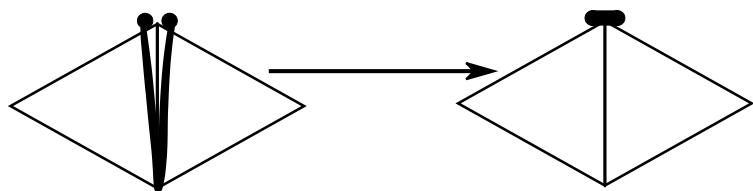
Reducing a walk



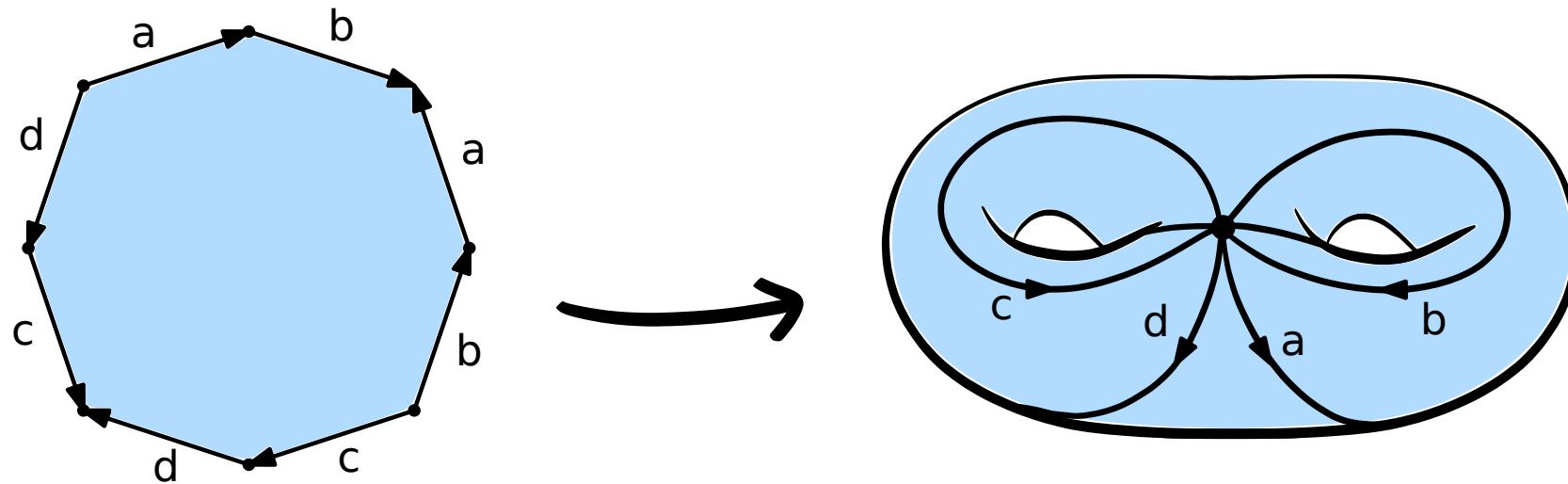
Reducing a walk



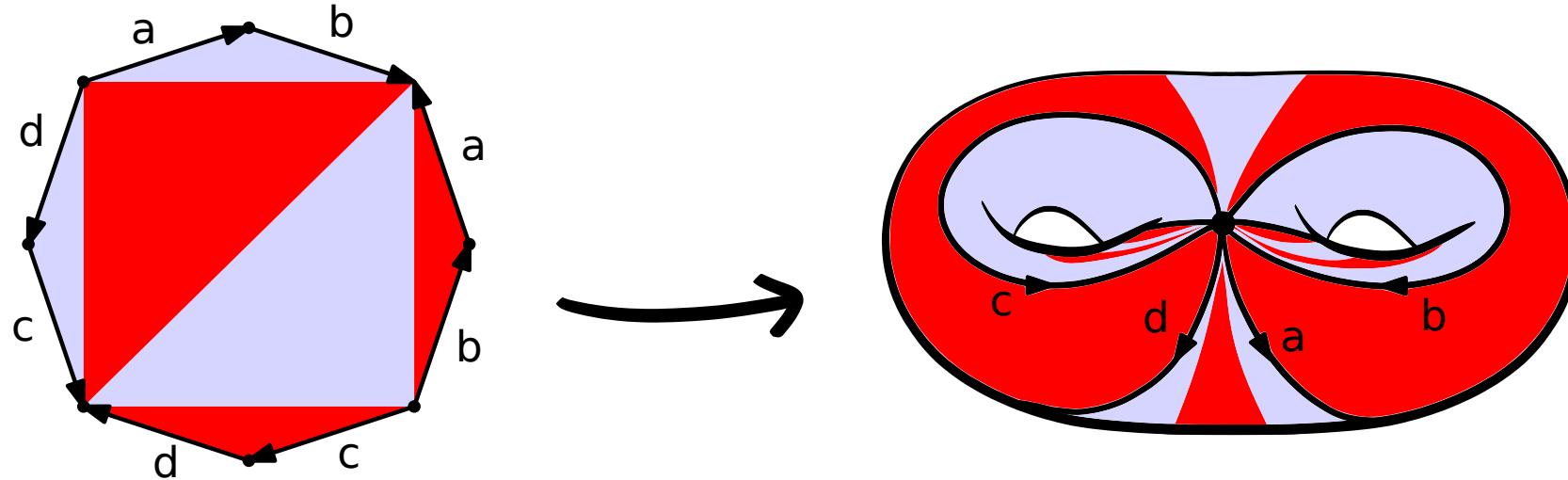
Reducing a walk



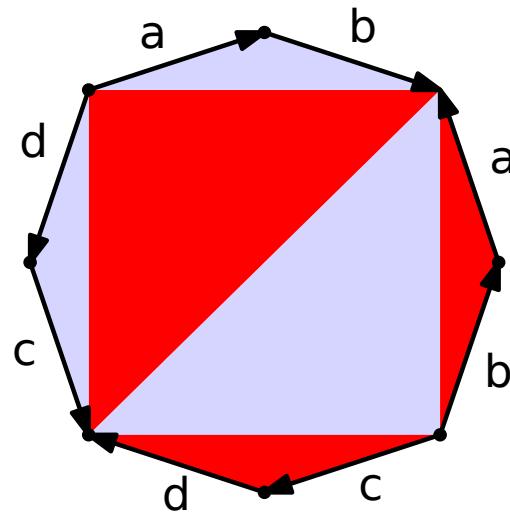
Constructing reducing triangulations



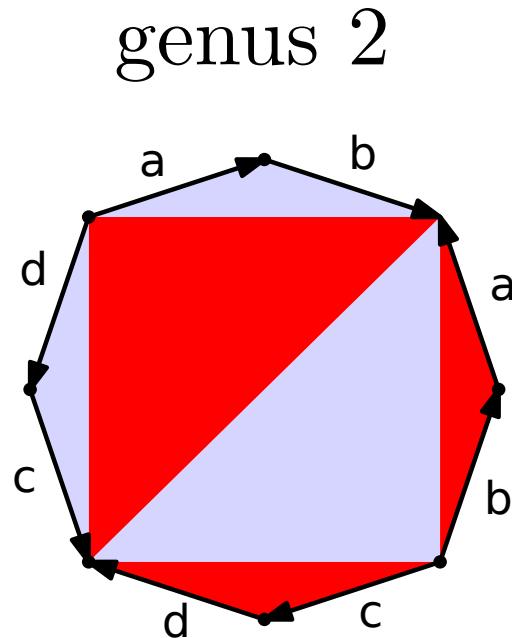
Constructing reducing triangulations



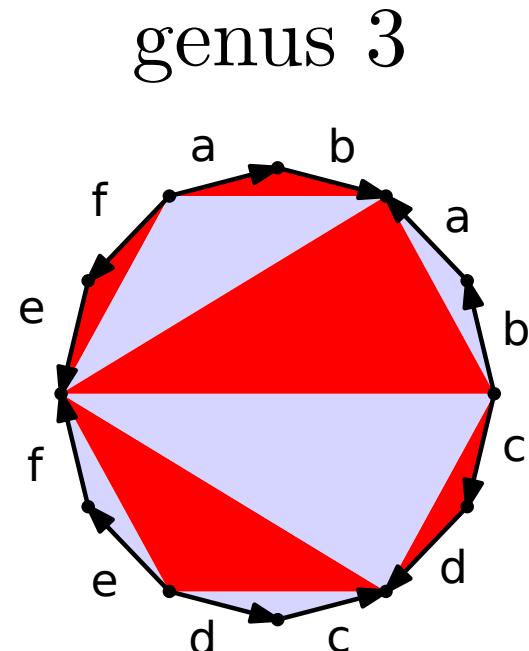
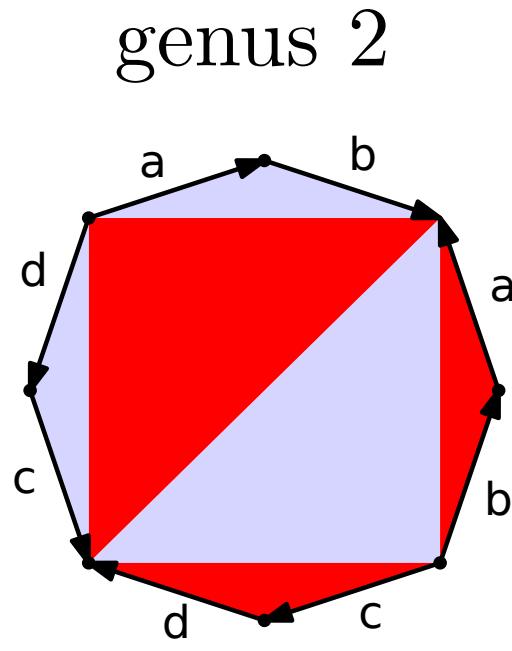
Constructing reducing triangulations



Constructing reducing triangulations

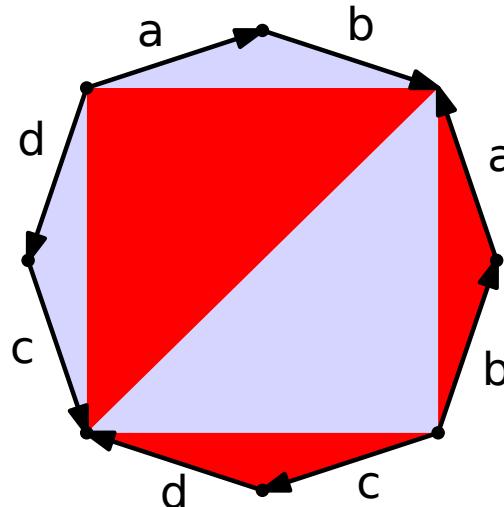


Constructing reducing triangulations

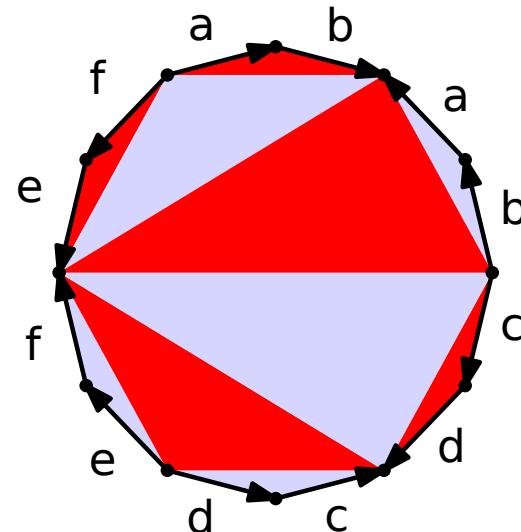


Constructing reducing triangulations

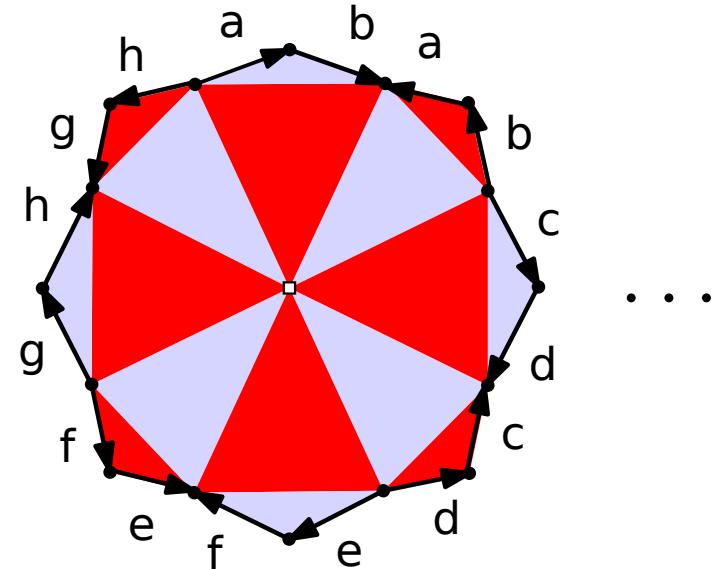
genus 2



genus 3

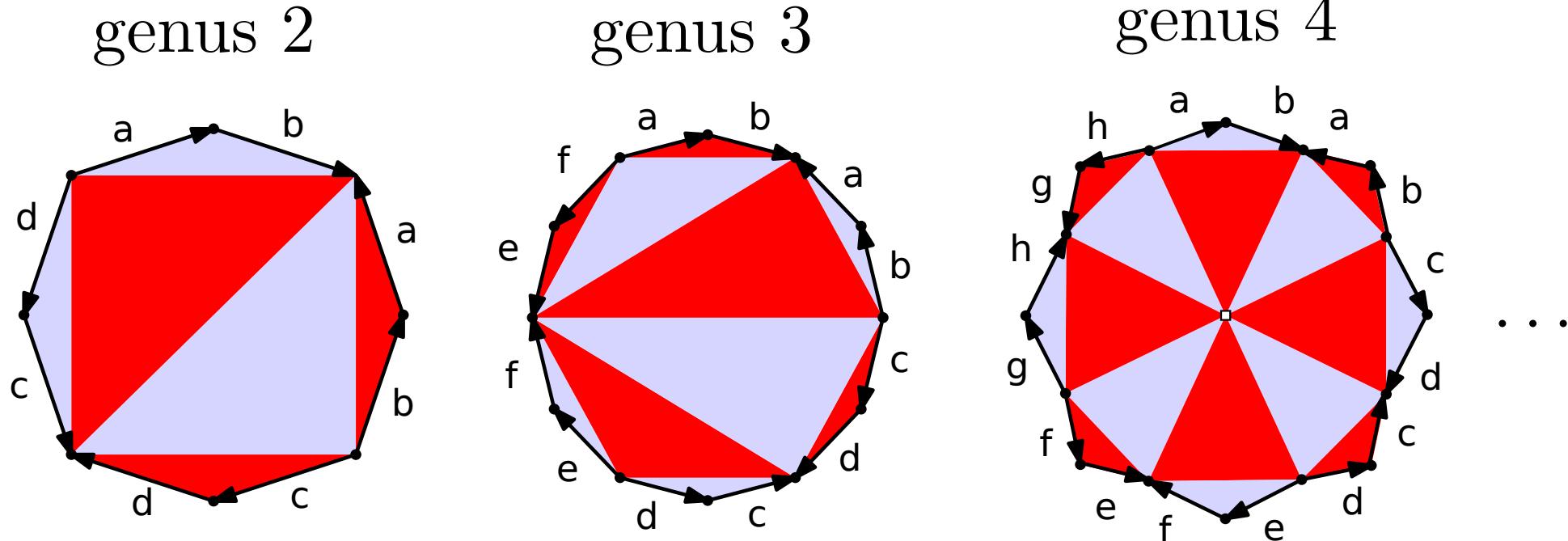


genus 4



...

Constructing reducing triangulations



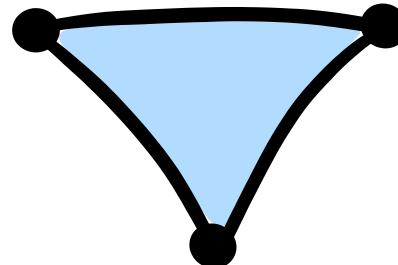
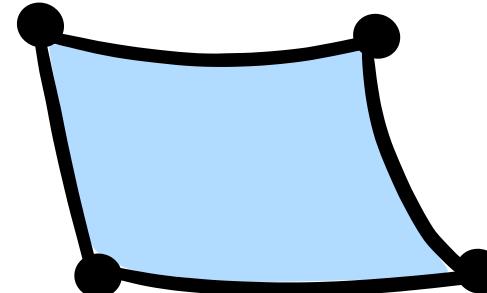
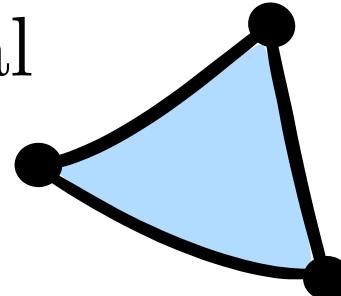
then subdivide at will:



Untangling graphs using
reducing triangulations

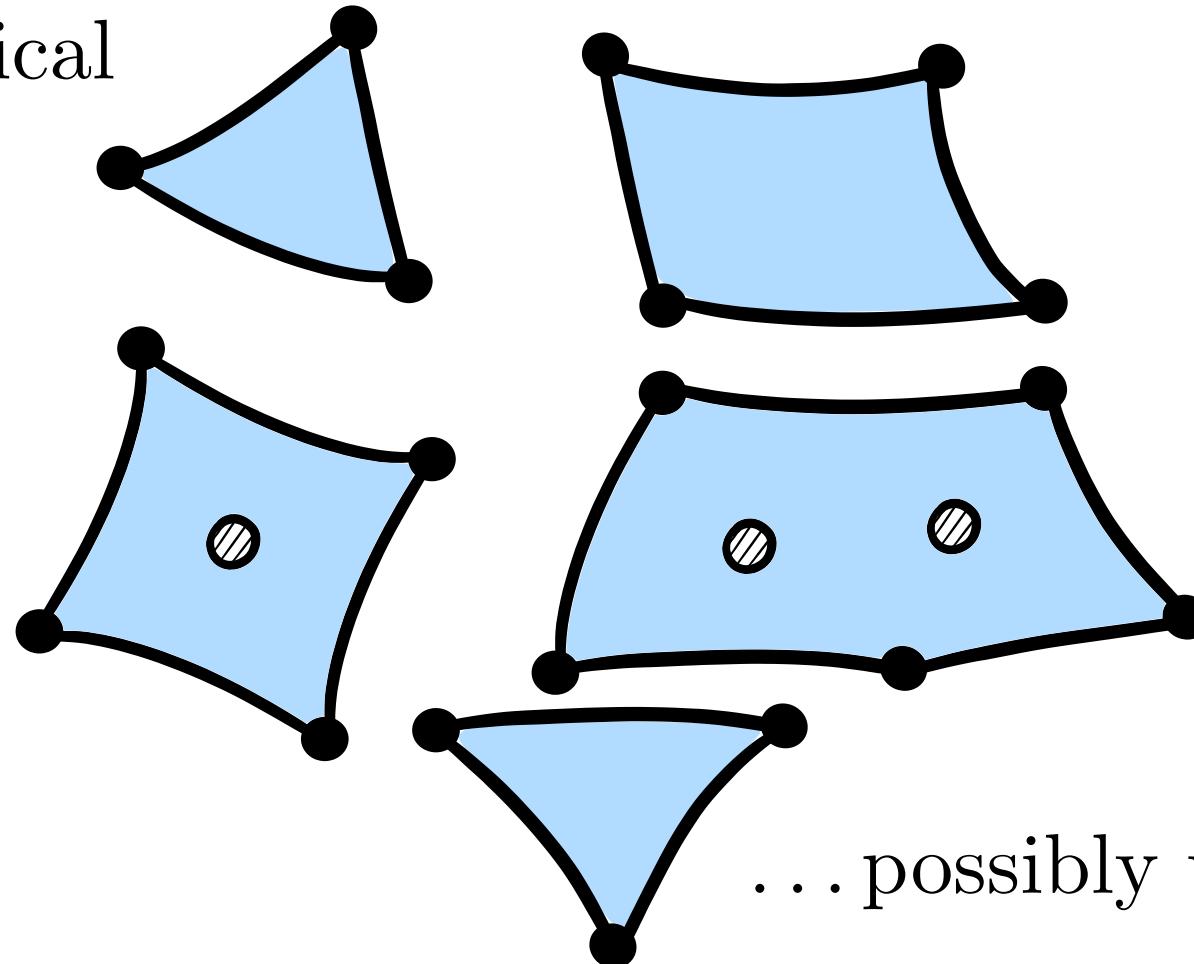
Input

take topological
polygons...



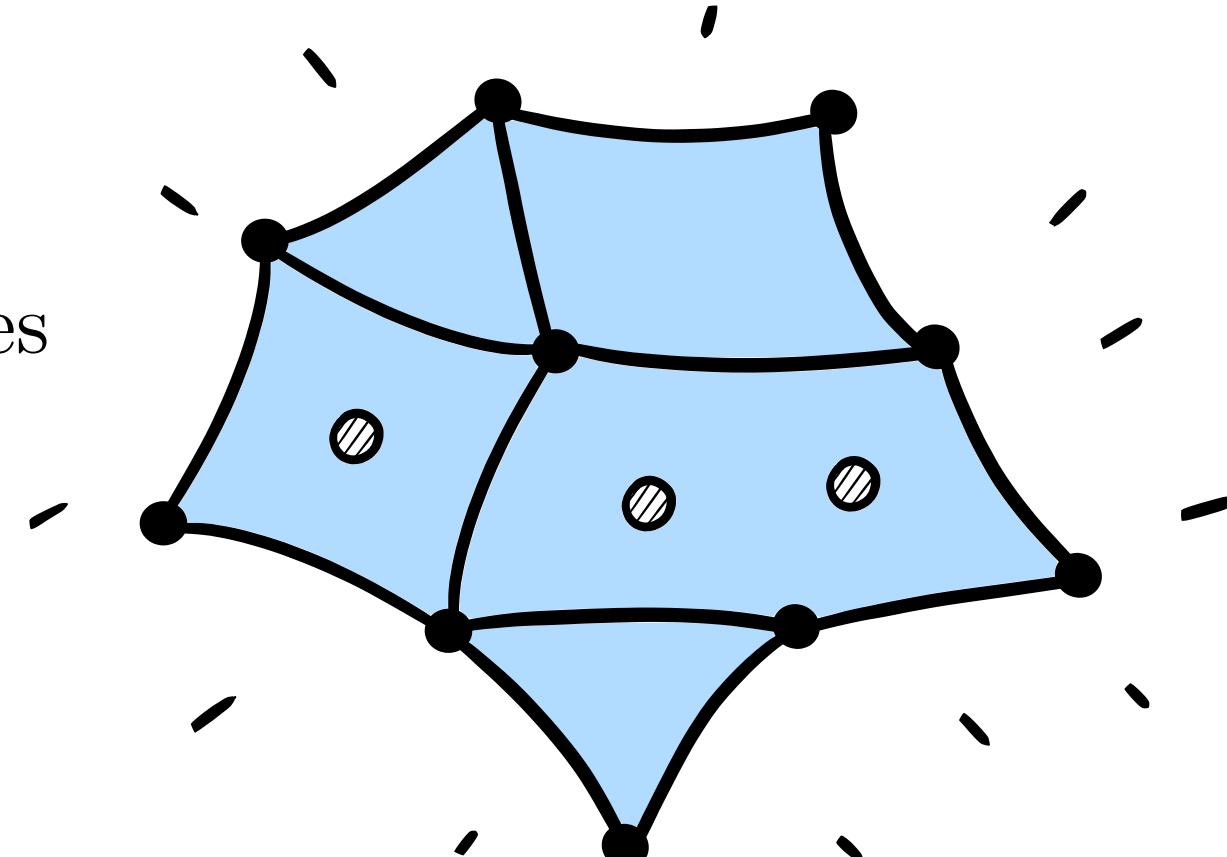
Input

take topological
polygons...



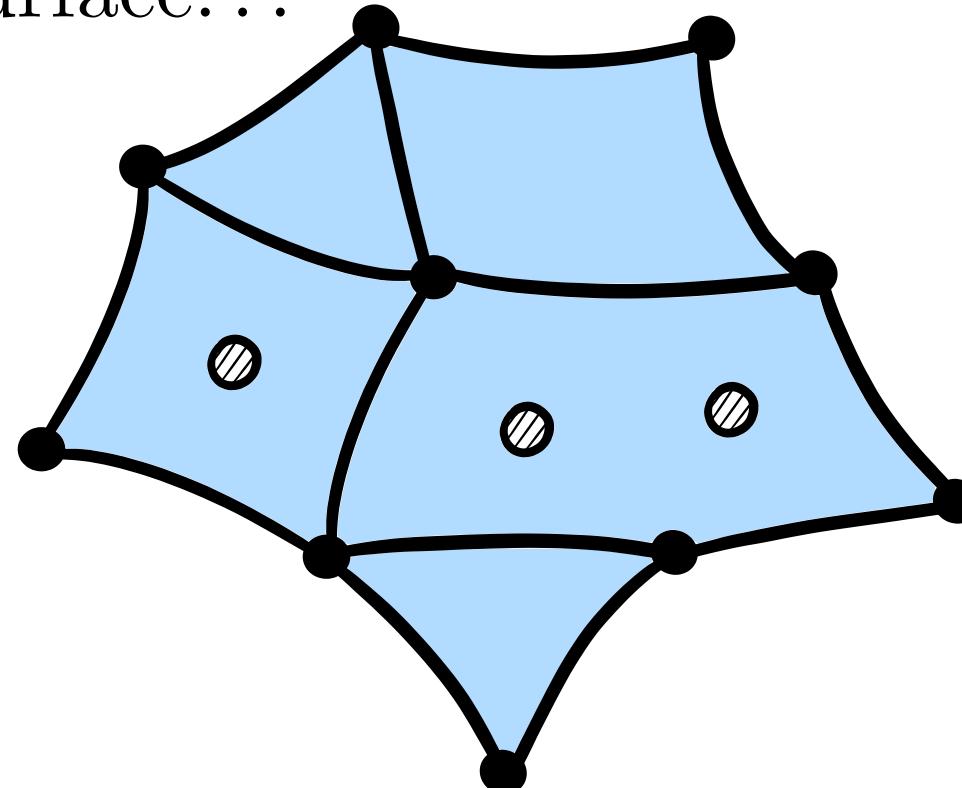
Input

glue edges



Input

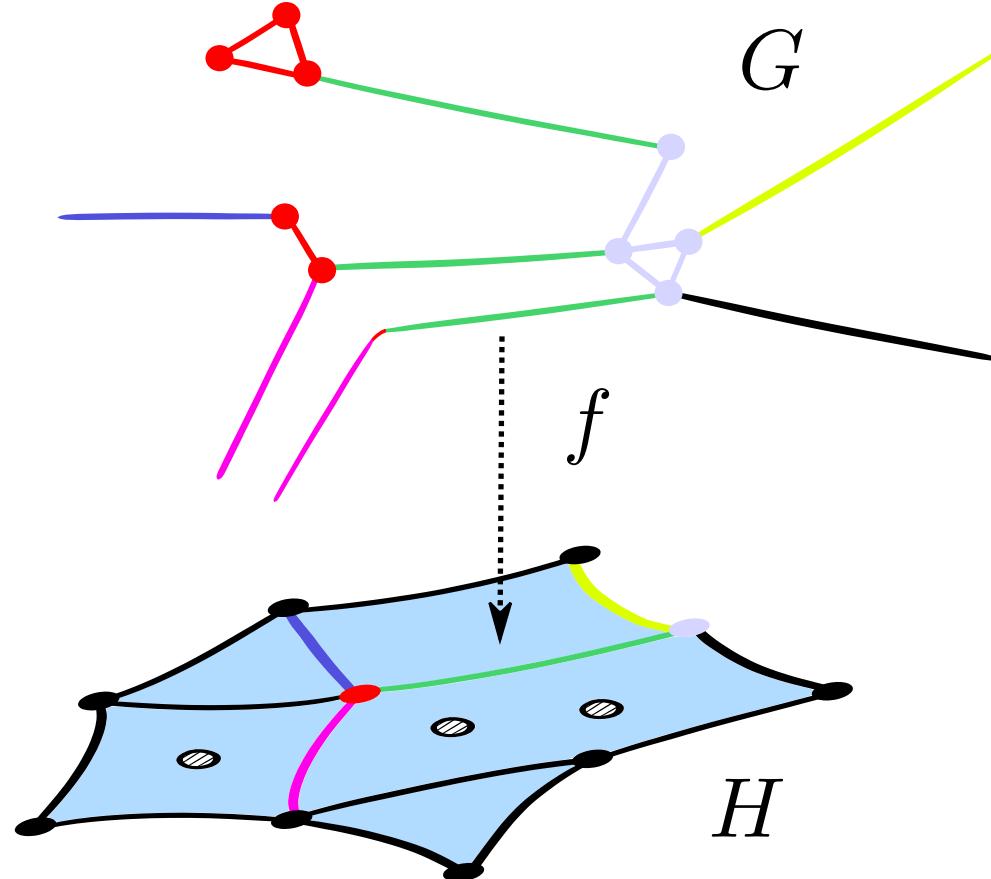
that encodes the surface . . .



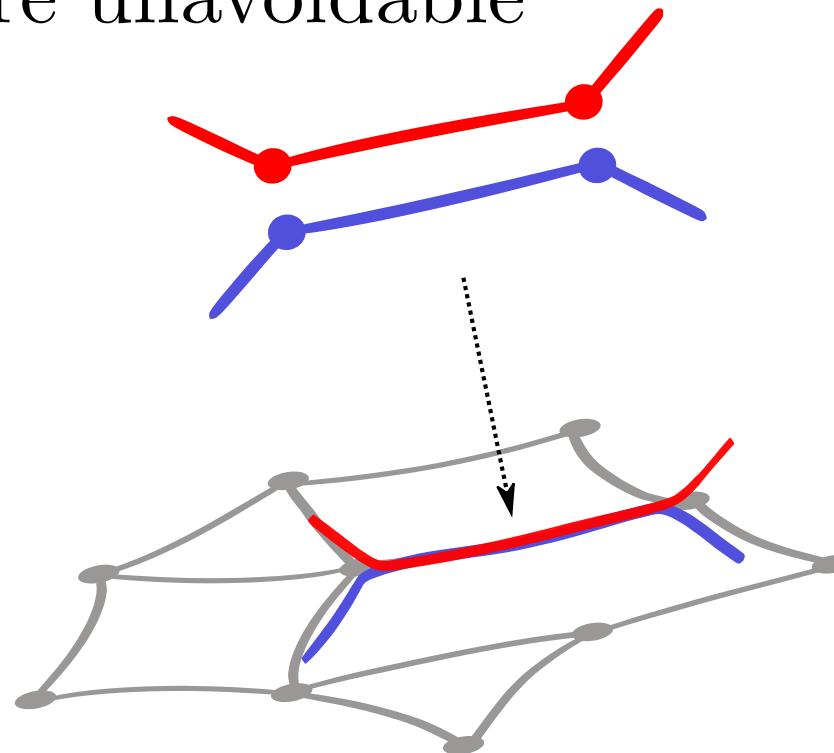
. . . and a graph H on it

Input

draw G in H

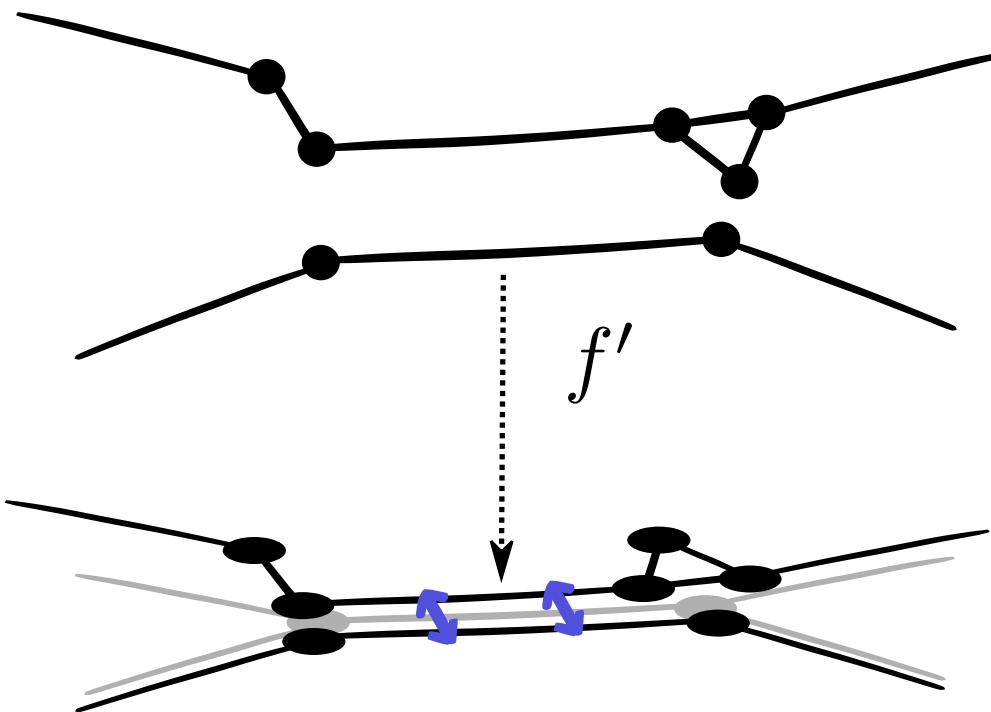


In this model overlaps
are unavoidable



Output

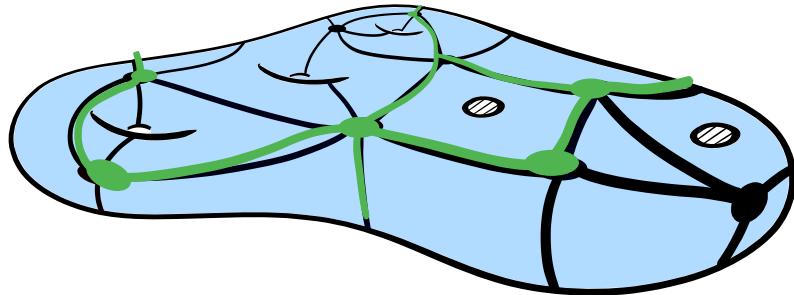
weak embedding: drawing f' that can be untangled by infinitesimal perturbation



Akitaya, Fulek, and
Tóth, 2019

algo to determine if
 f' is weak embedding,
and if so to perturb f'

Result



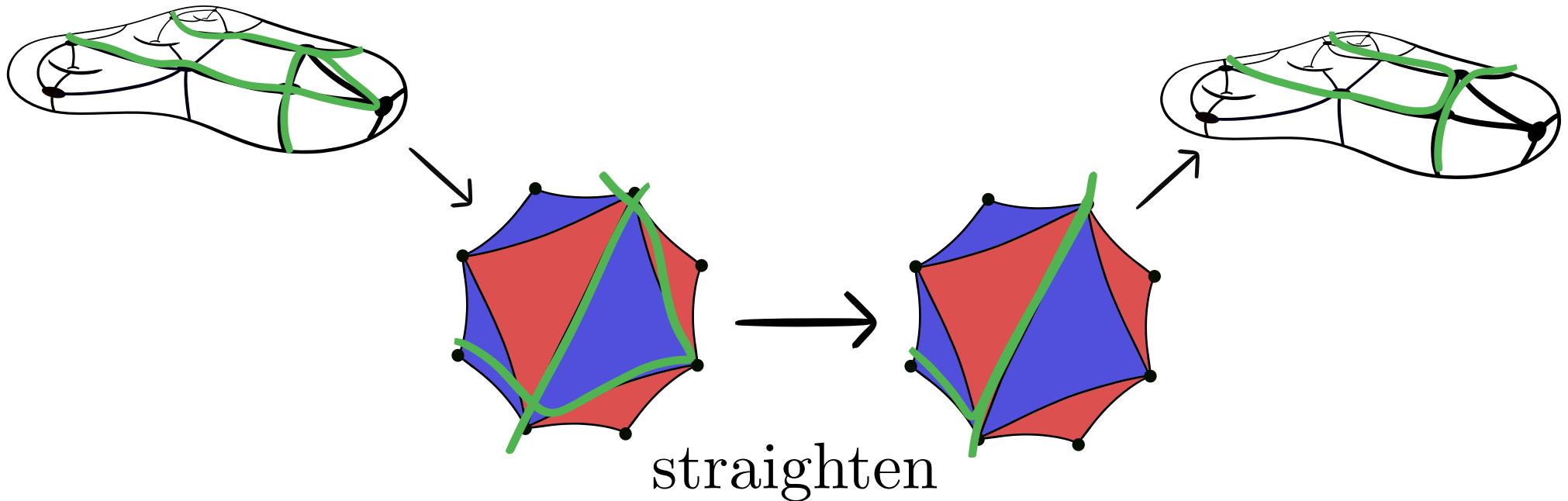
n : # times f uses an edge or vertex of H
 m : # vertices and edges of H
 s : genus + # holes

Colin de Verdière, Despré, D., 2023

We can decide if f can be untangled,
in $O(m + s^2 n \log(s n))$ time.
If so, we can compute a weak embedding homotopic
to f in additional $O(s^2 m n^2)$ time.

Algorithm overview

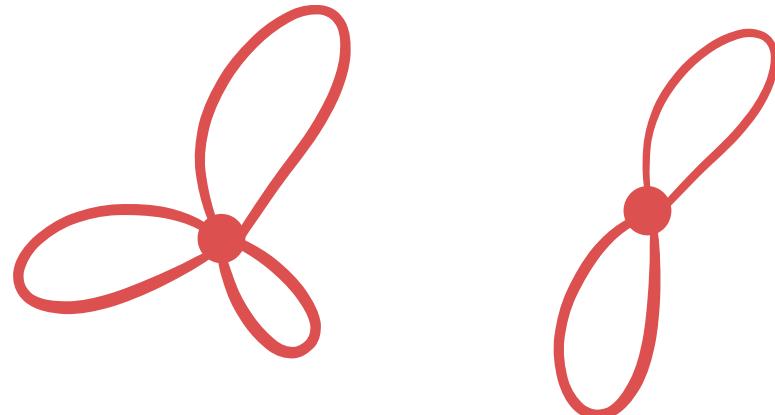
(assuming that the surface has no boundary)



Straightening a graph

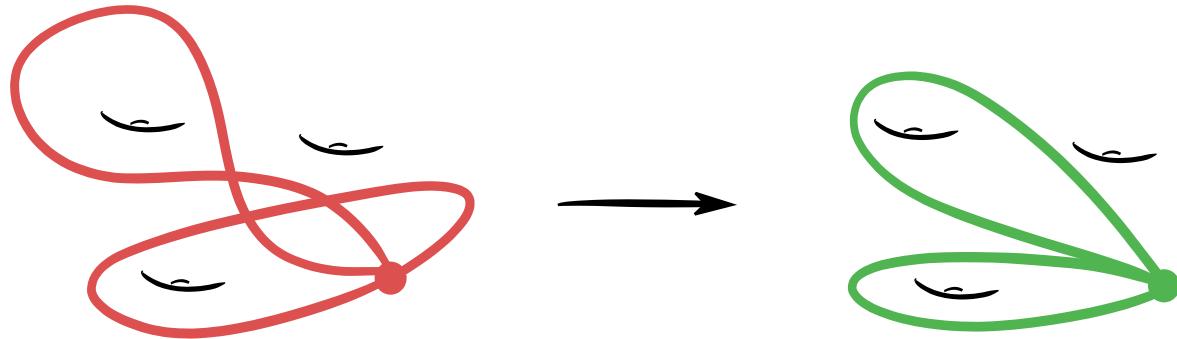
loop graph

Straightening a ~~graph~~



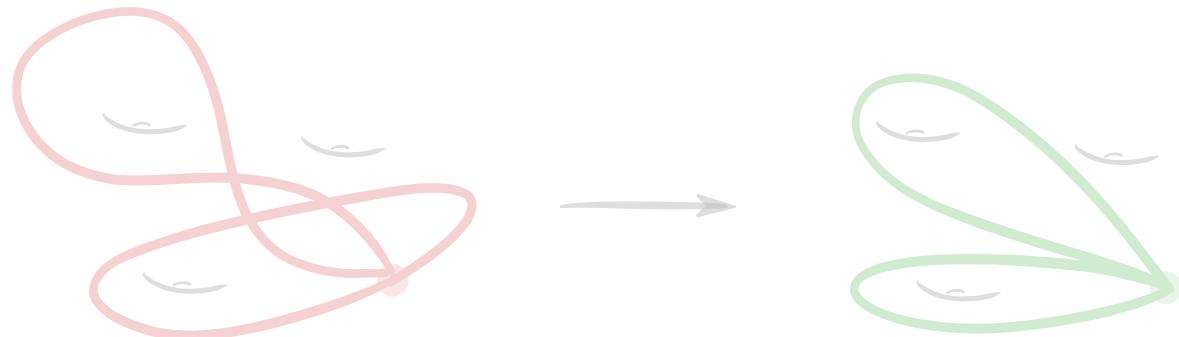
Straightening a loop graph

First attempt: reduce the loops, vertex fixed

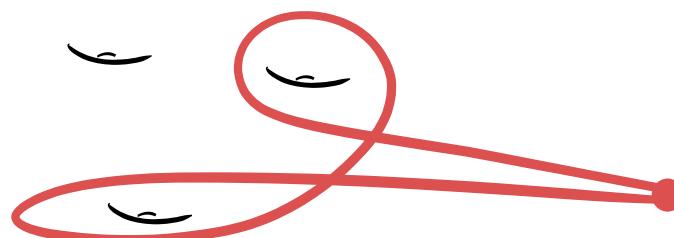


Straightening a loop graph

First attempt: reduce the loops, vertex fixed



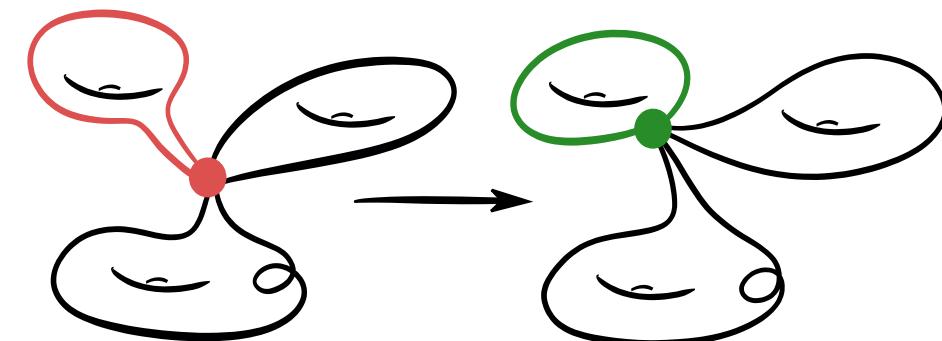
Problem:



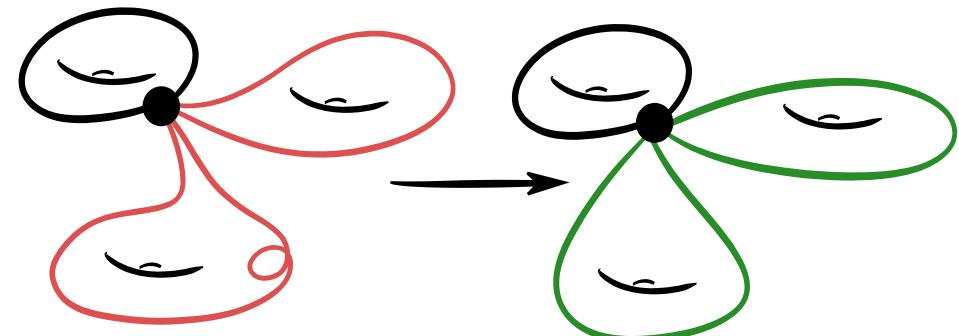
→ vertex must move

Straightening a loop graph

Solution:



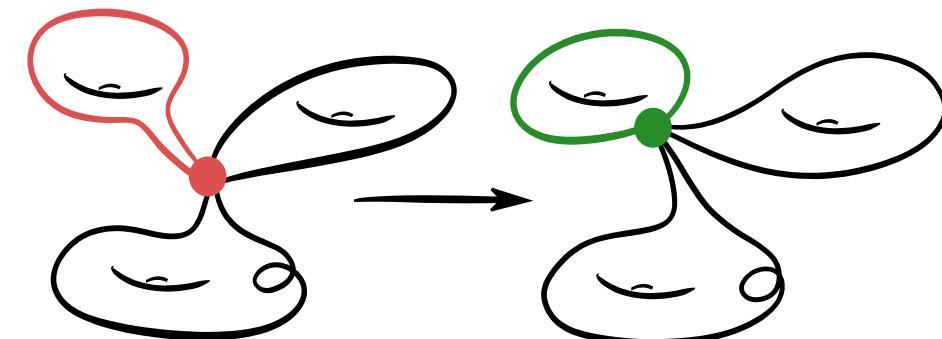
1. Reduce 1 loop cyclically



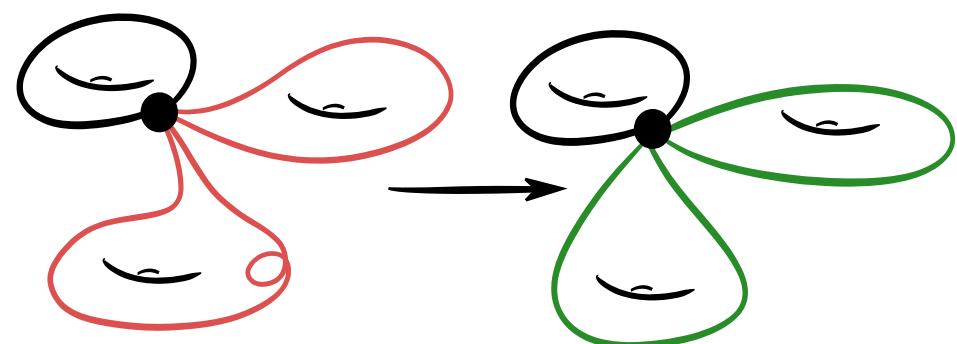
2. Reduce the other loops linearly

Straightening a loop graph

A straightened loop graph is a weak embedding
or cannot be untangled



1. Reduce 1 loop cyclically

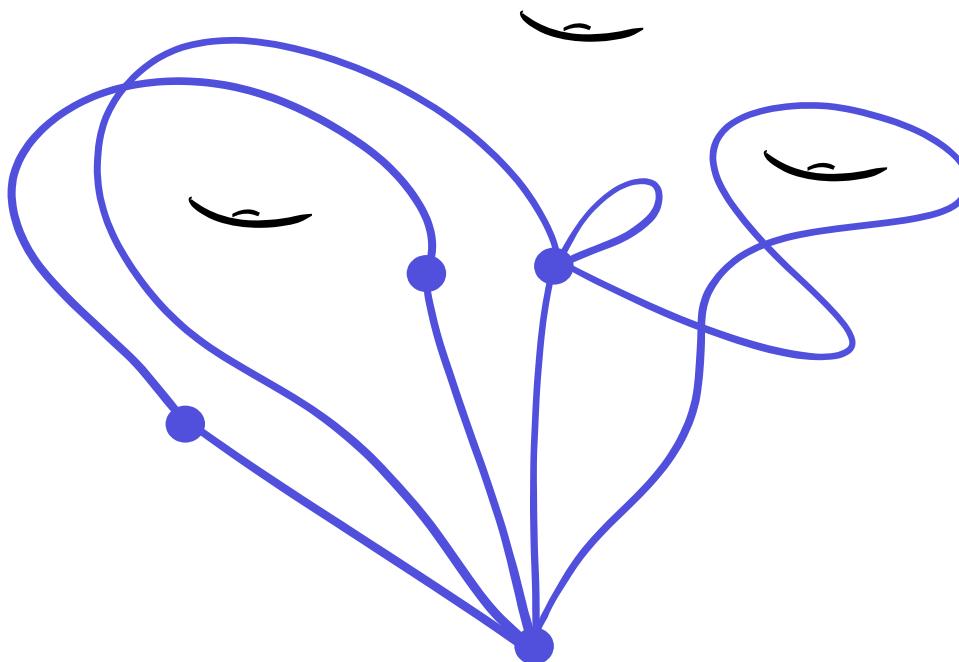


2. Reduce the other loops linearly

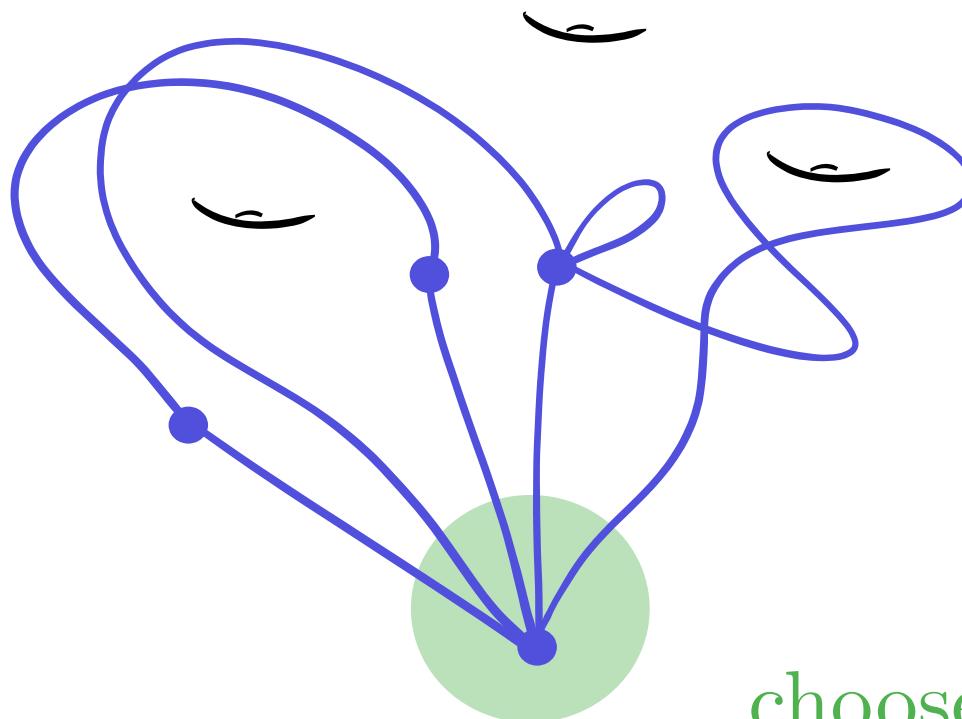
Straightening a graph

Straightening a graph

(connected, say)

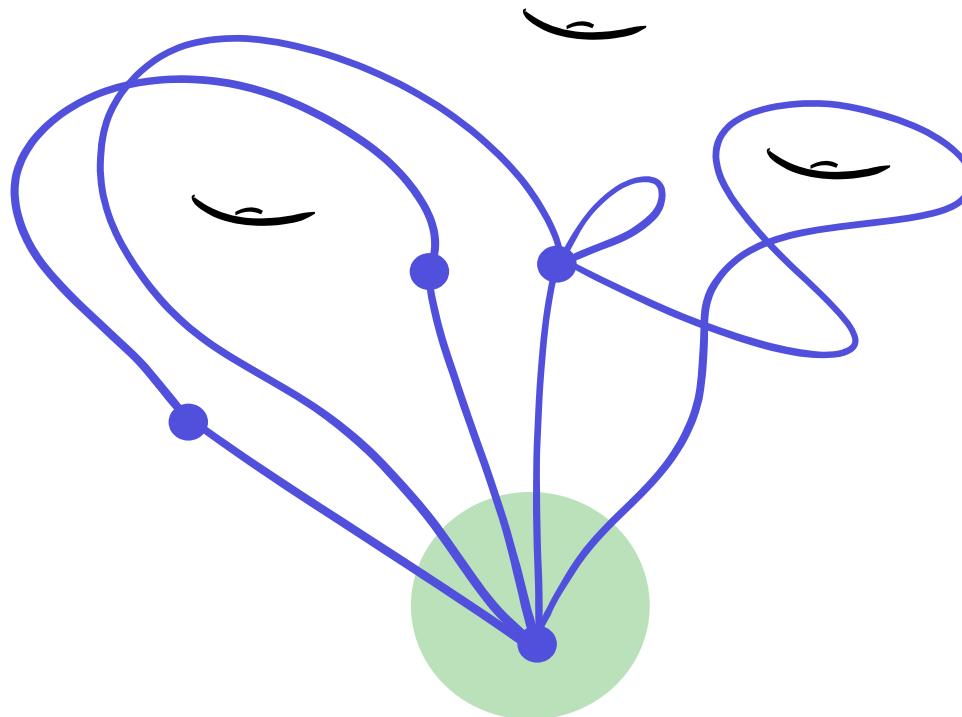


Straightening a graph

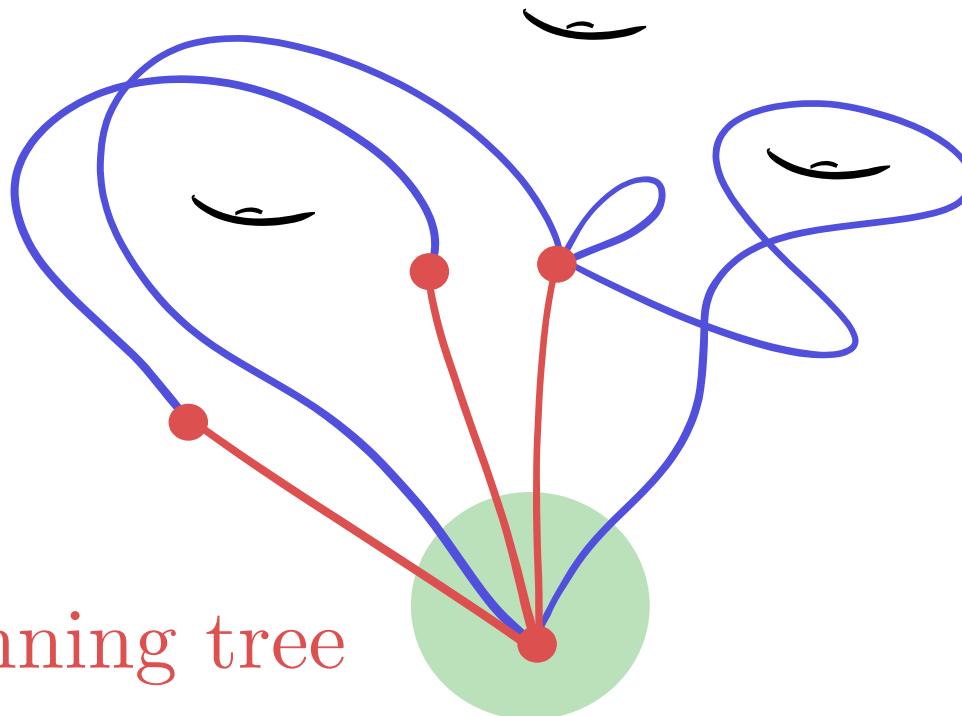


choose a base vertex

Straightening a graph

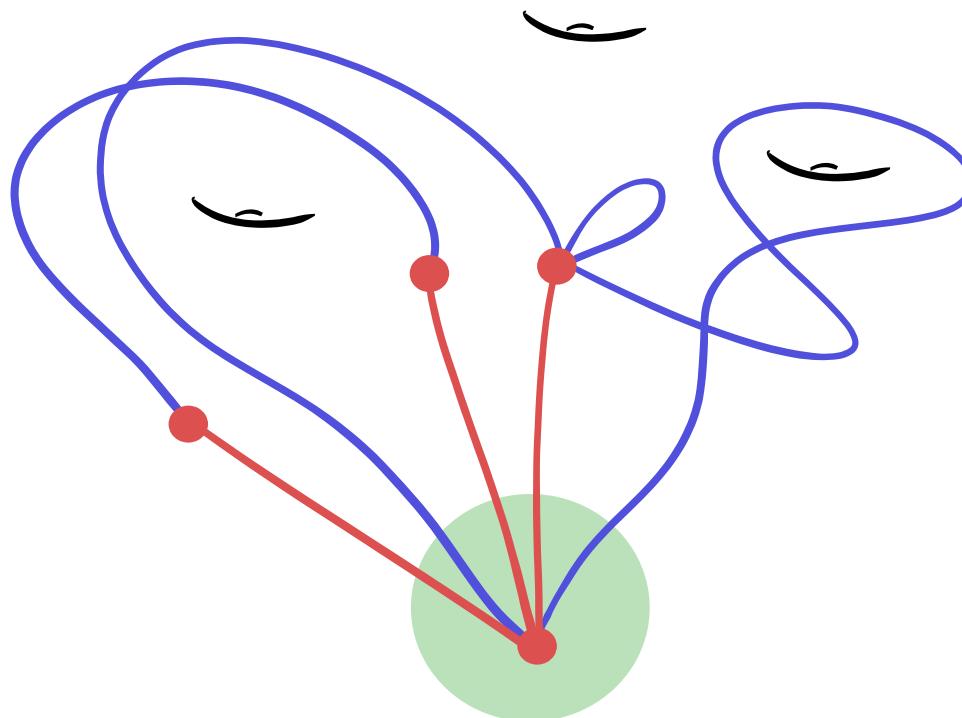


Straightening a graph



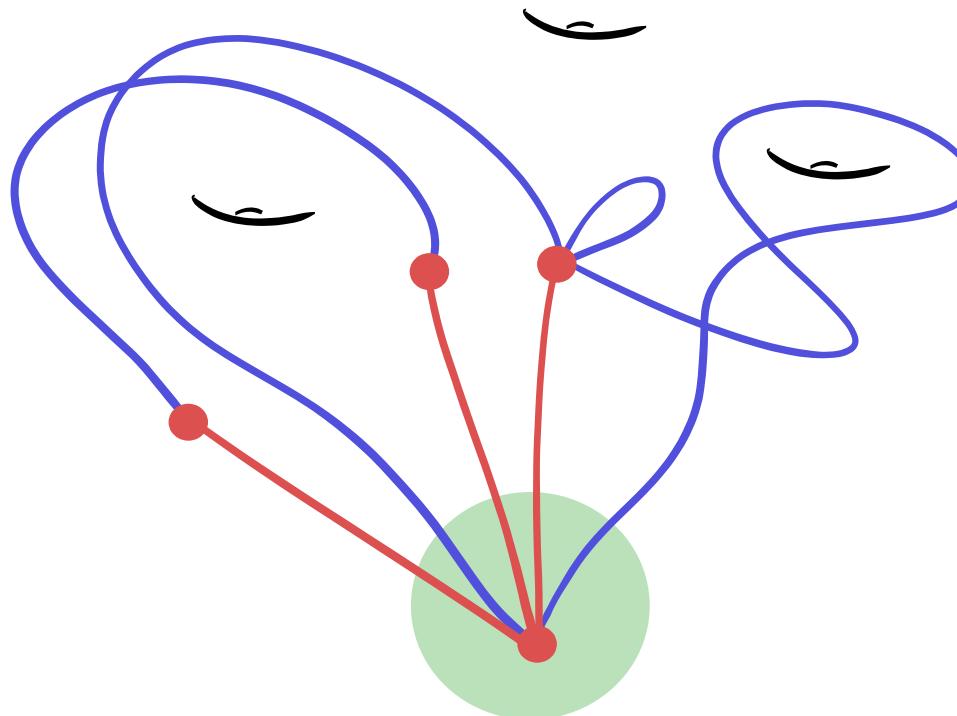
choose a spanning tree

Straightening a graph



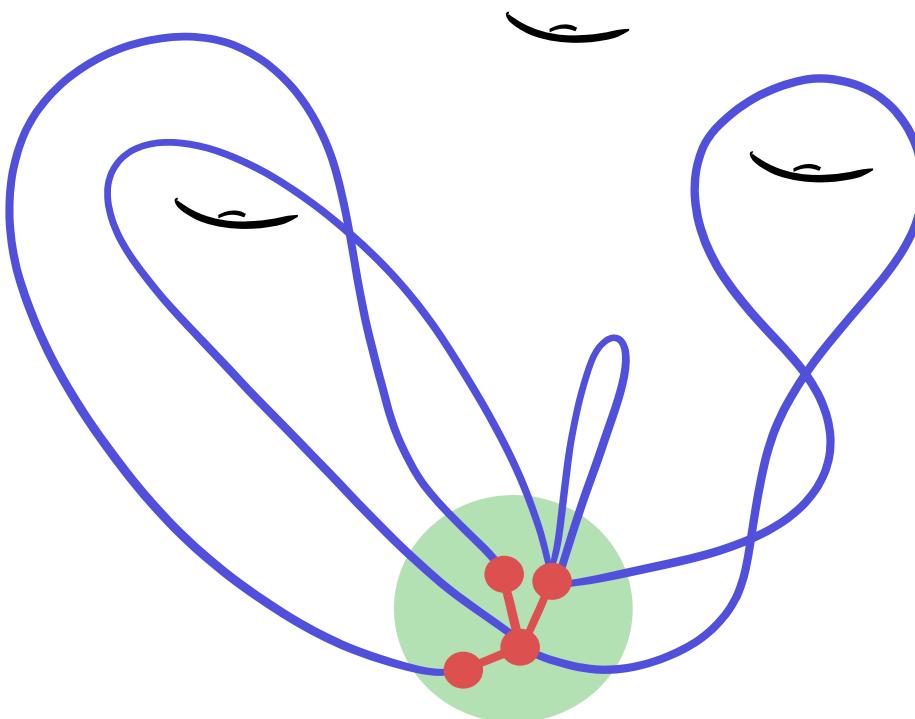
Straightening a graph

contract the spanning tree

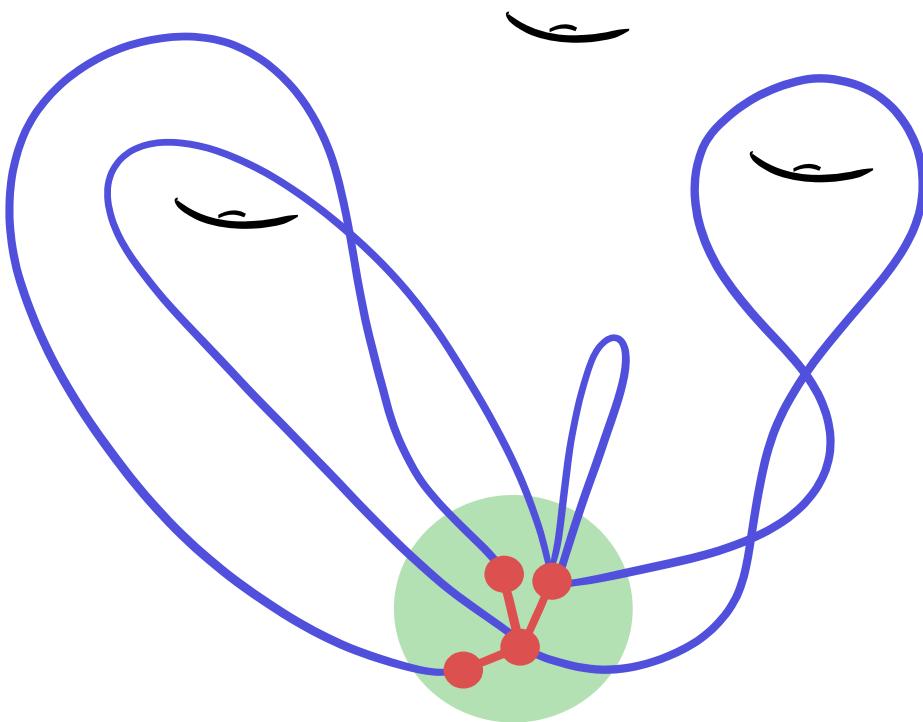


Straightening a graph

contract the spanning tree

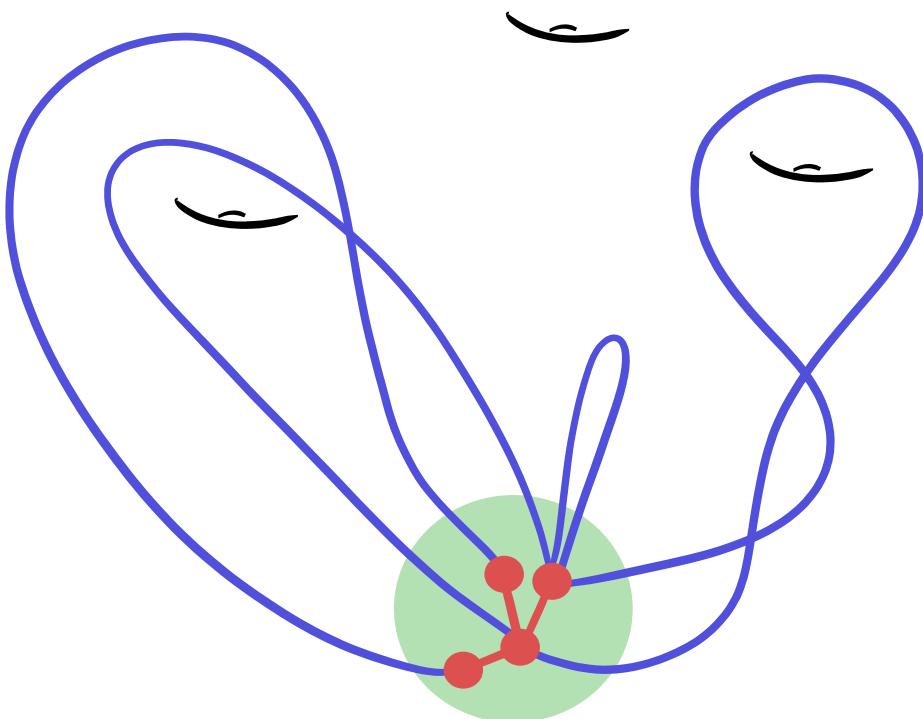


Straightening a graph



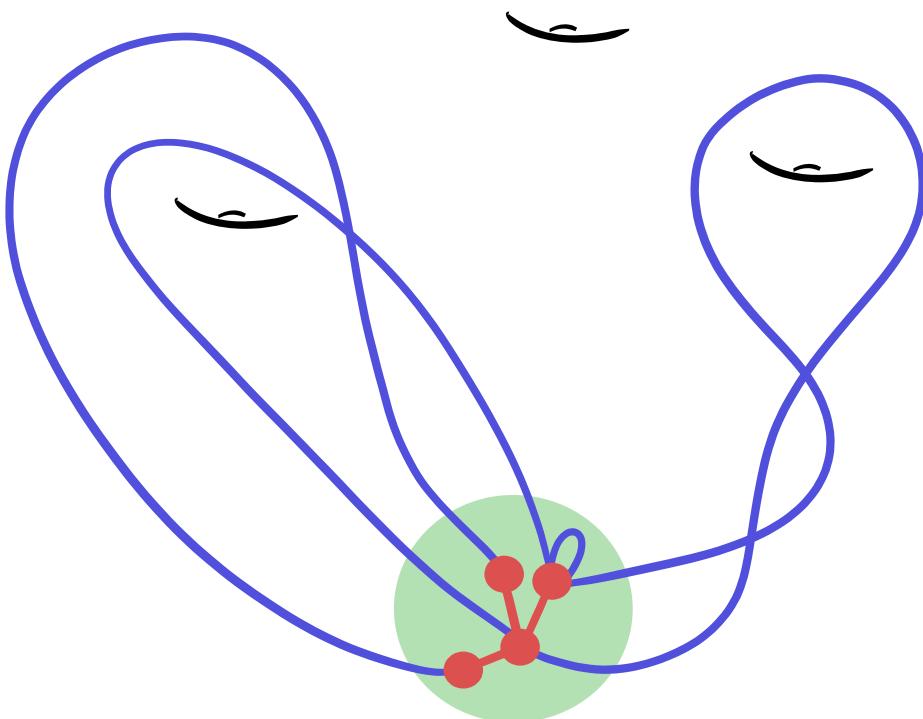
Straightening a graph

contract and bundle the other edges when possible



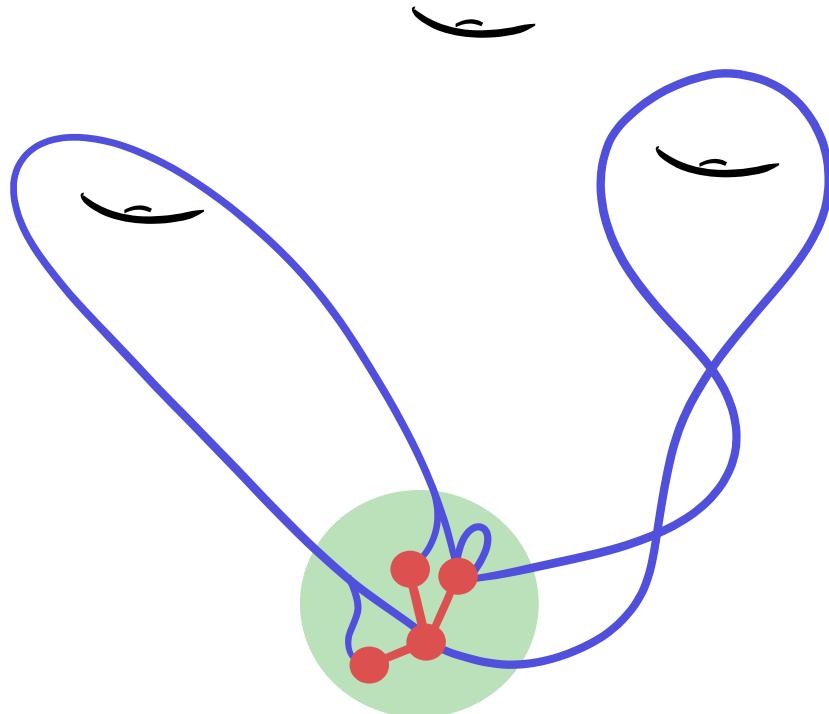
Straightening a graph

contract and bundle the other edges when possible

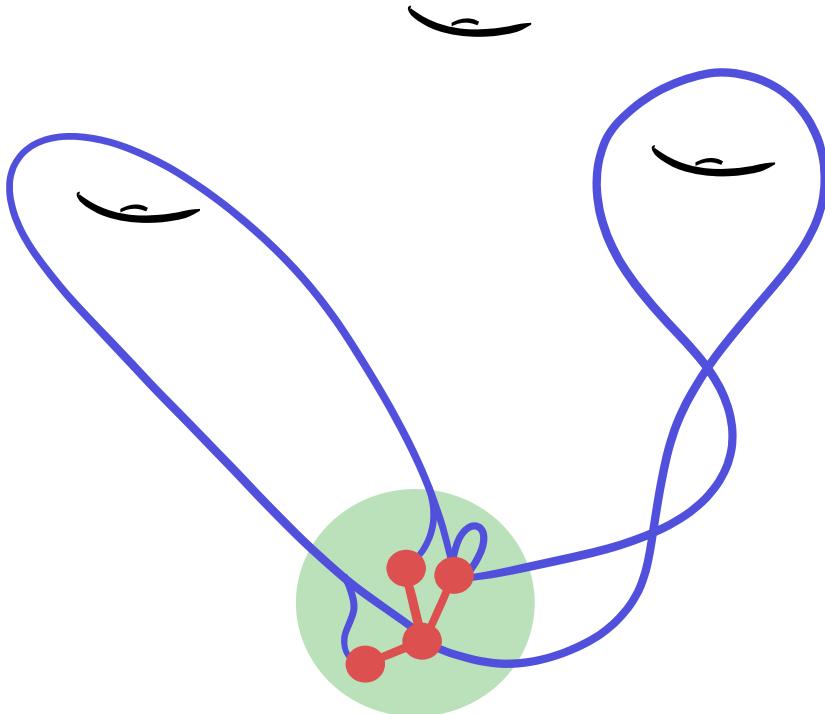


Straightening a graph

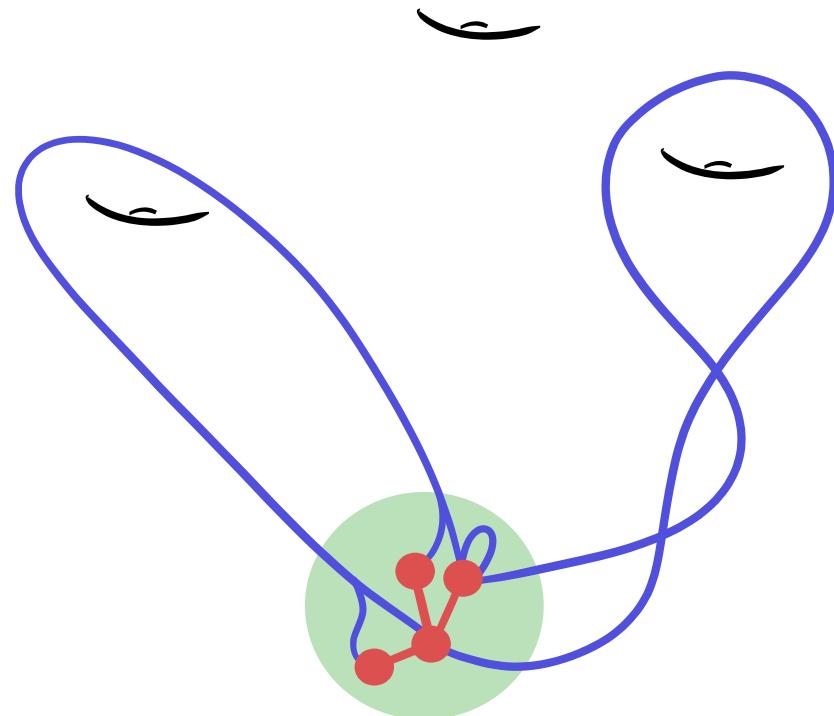
contract and bundle the other edges when possible



Straightening a graph

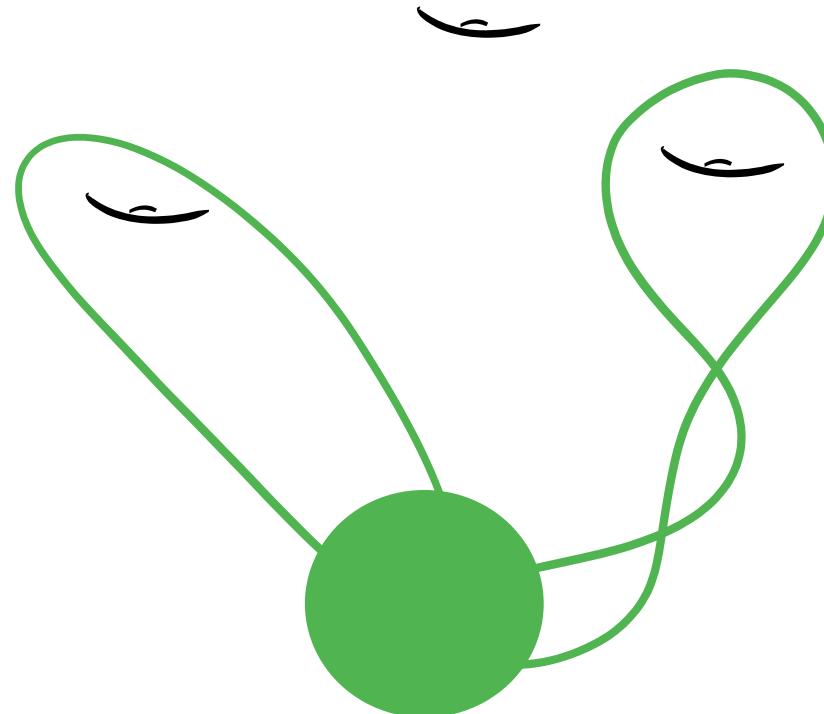


Straightening a graph



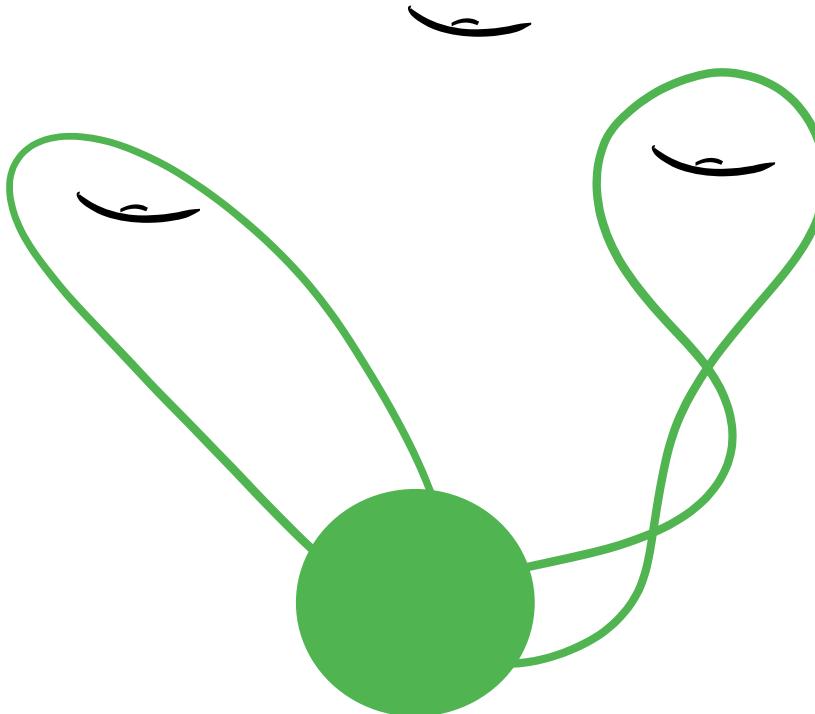
consider the associated loop graph

Straightening a graph



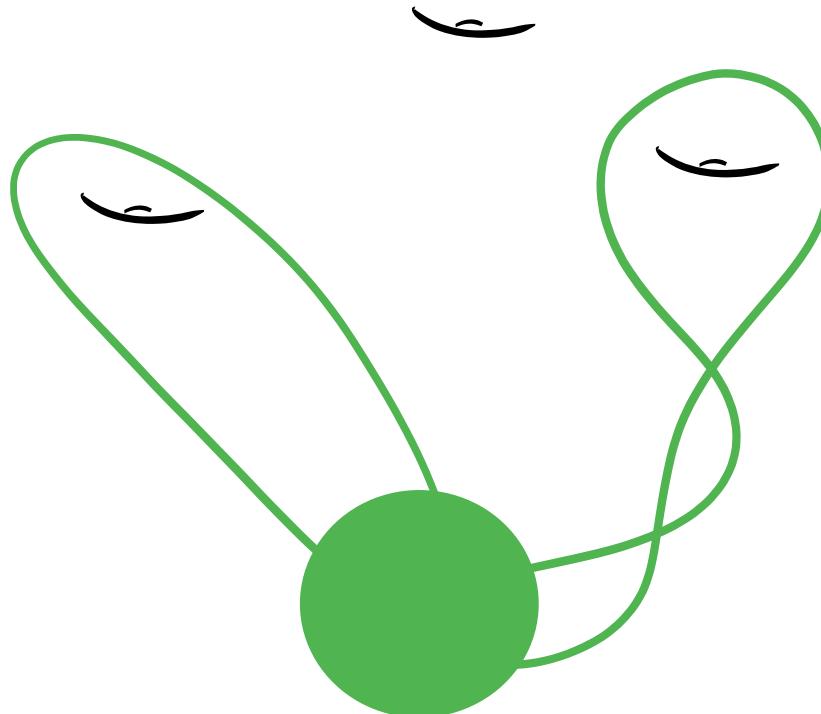
consider the associated loop graph

Straightening a graph



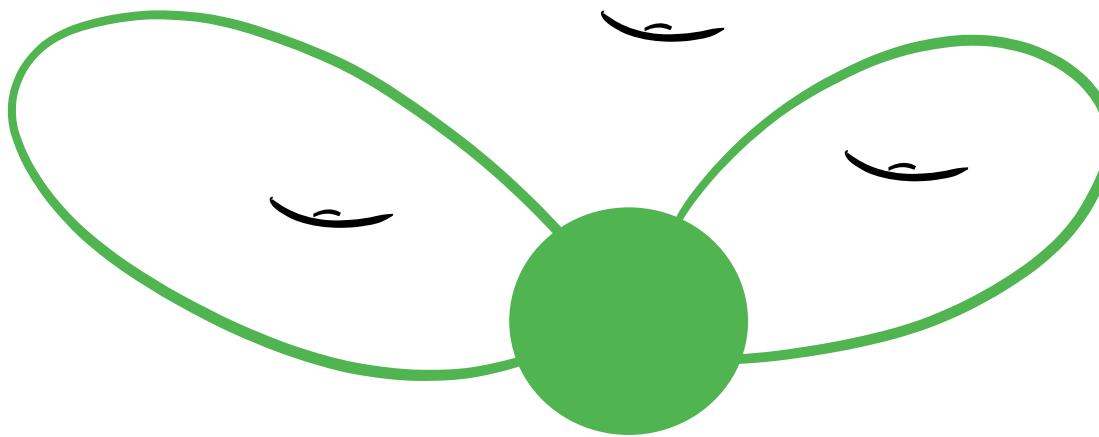
Straightening a graph

straighten the loop graph

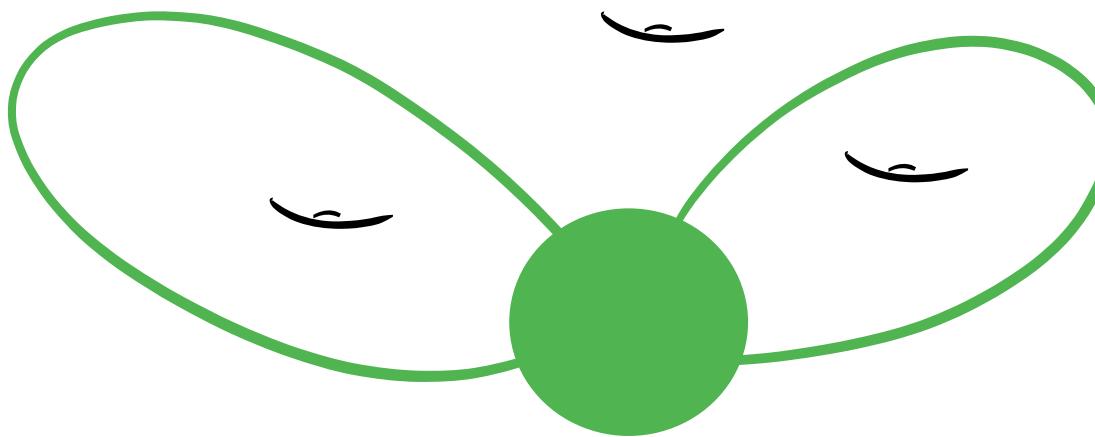


Straightening a graph

straighten the loop graph

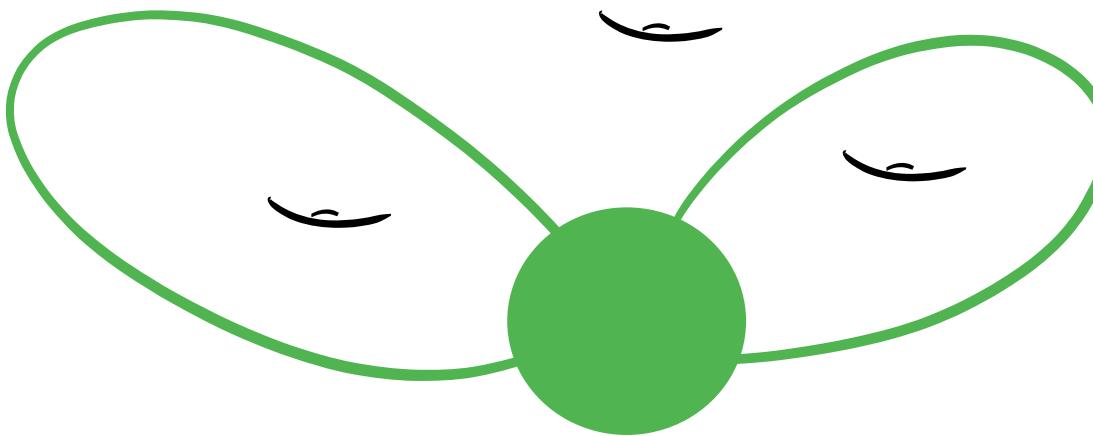


Straightening a graph



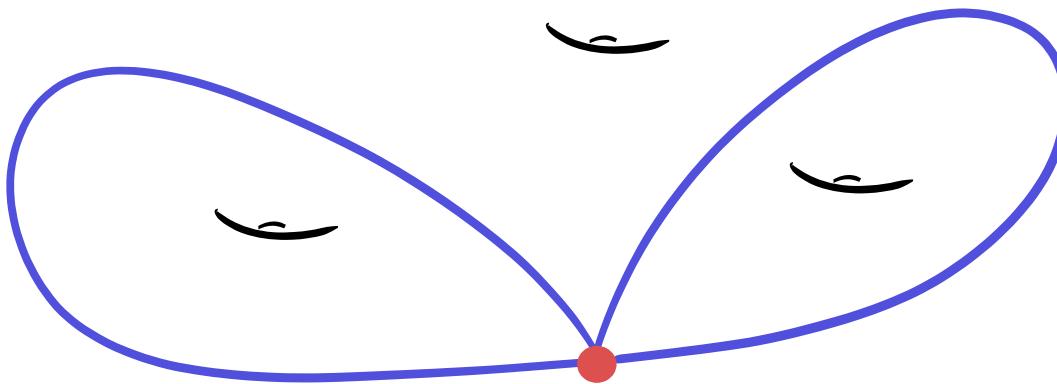
Straightening a graph

forget about the loop graph

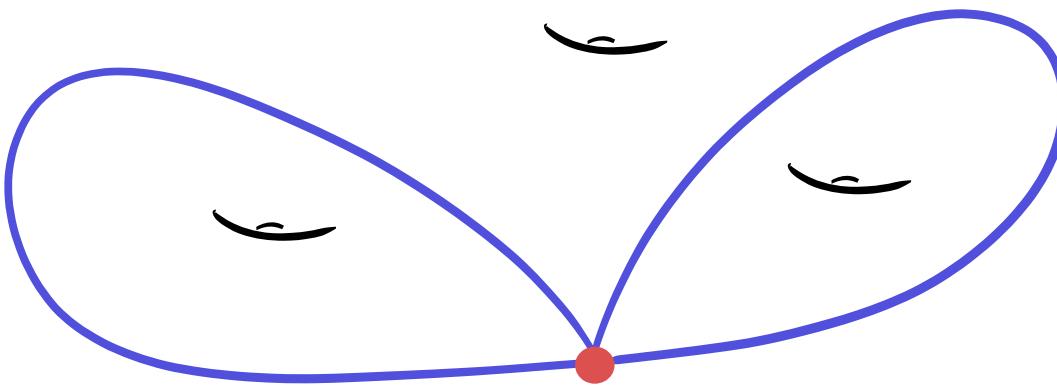


Straightening a graph

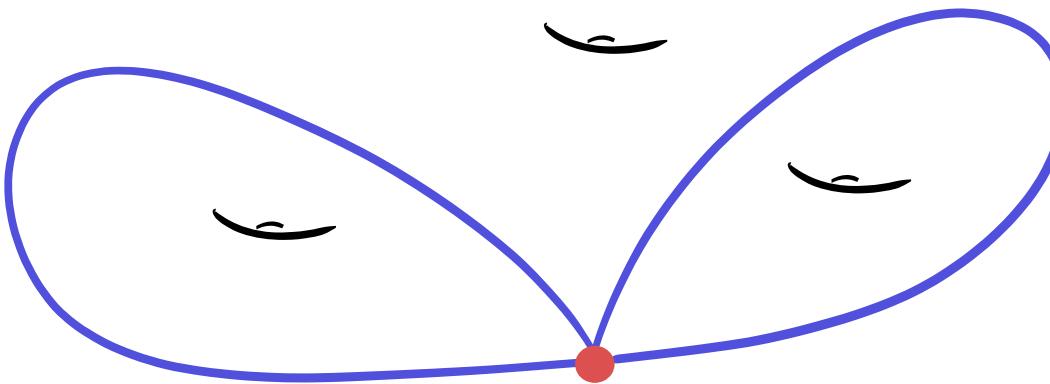
forget about the loop graph



Straightening a graph

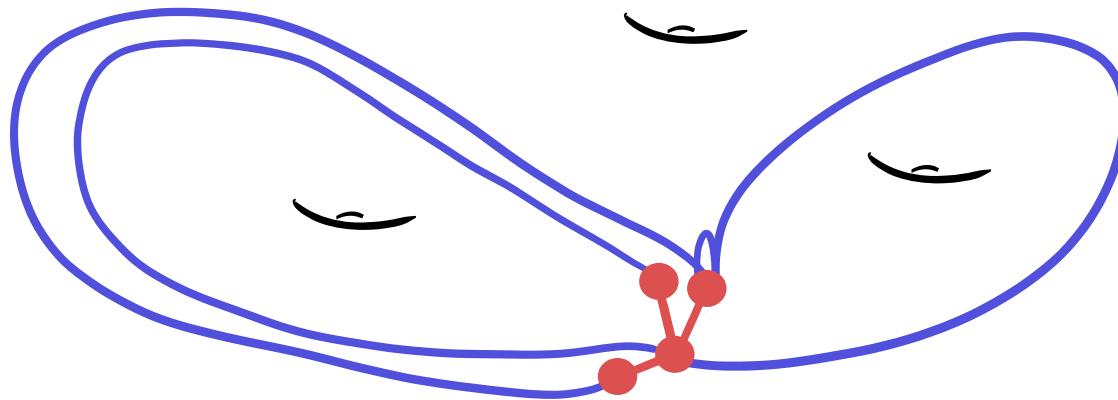


Straightening a graph



A straightened graph is a weak embedding
or cannot be untangled

Straightening a graph

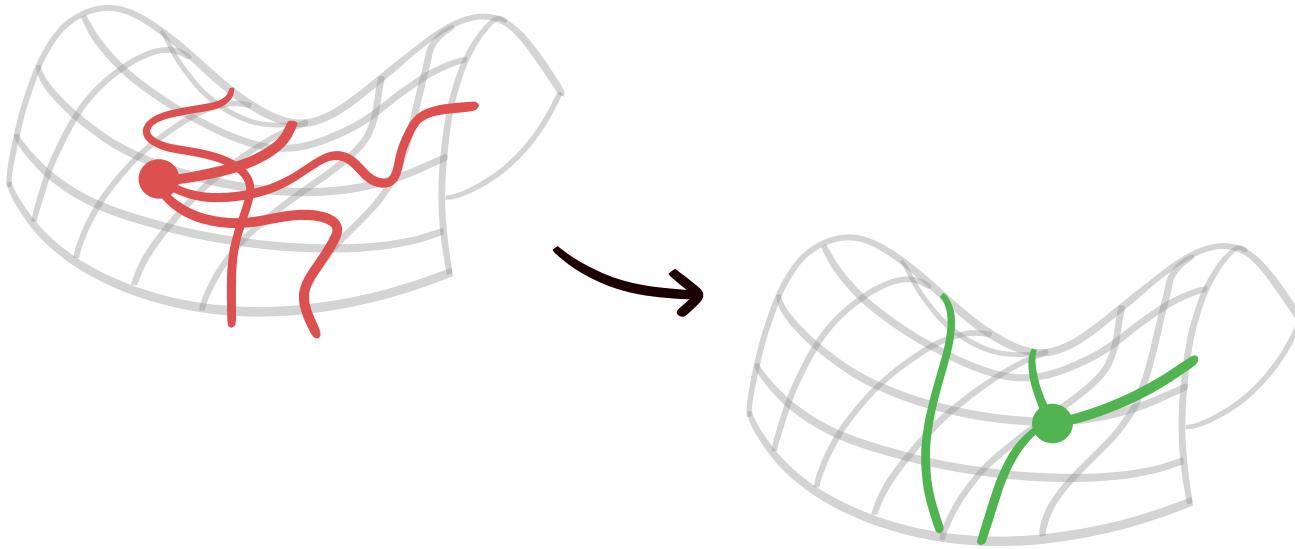


A straightened graph is a weak embedding
or cannot be untangled

(+ tricks and data structures for achieving
the announced time complexities)

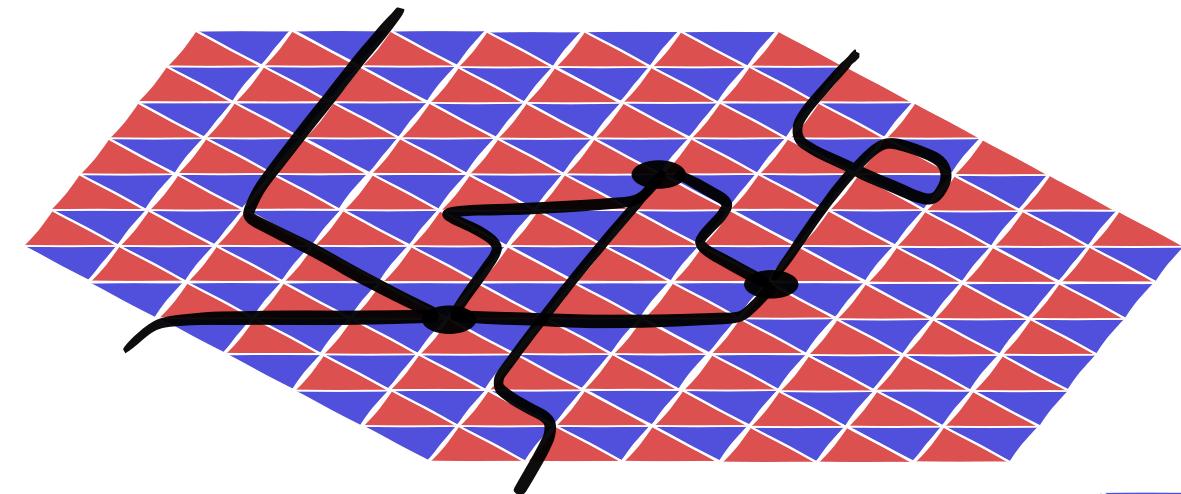
Discrete analogue of Tutte embeddings

Recall: Tutte embeddings

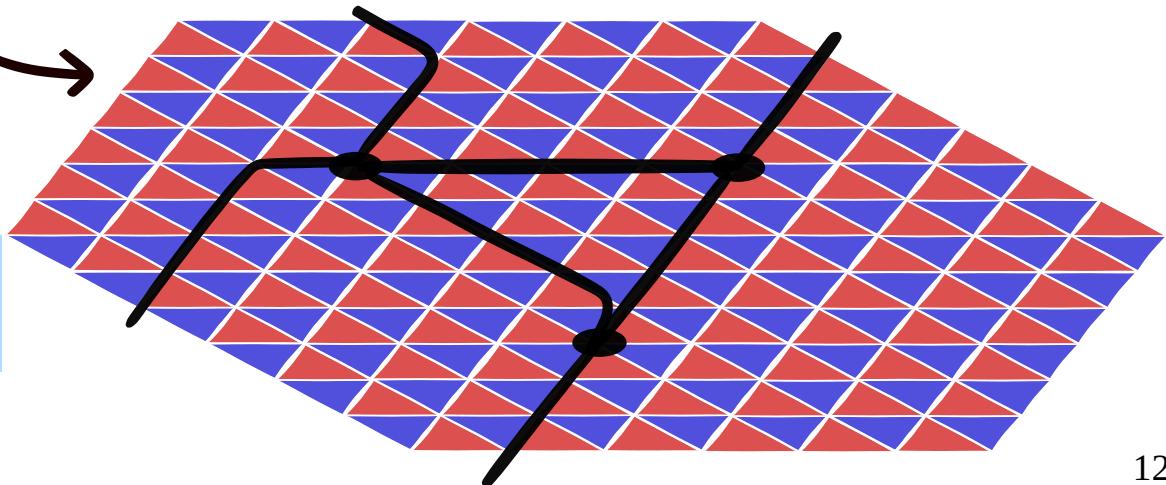


edges straight and
vertices “barycentric”

Harmonious drawings

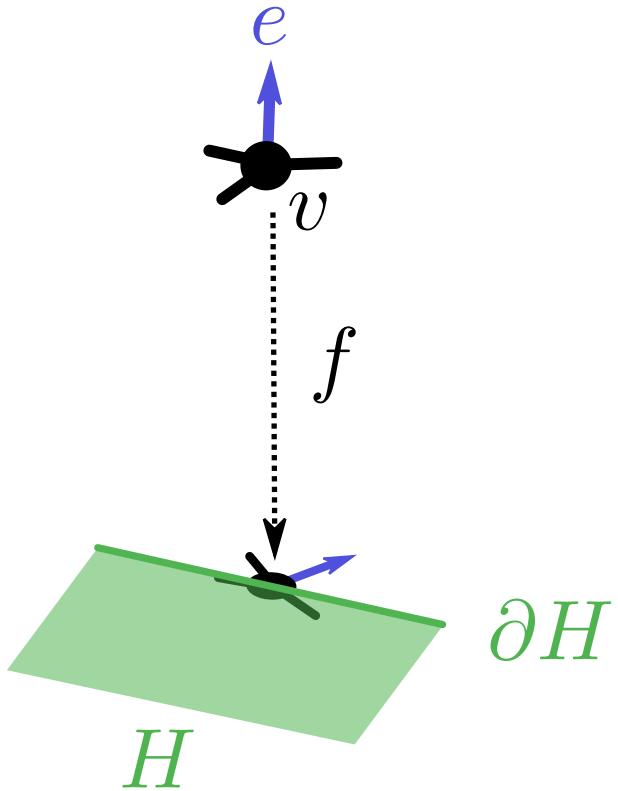


edges reduced and
vertices “barycentric”



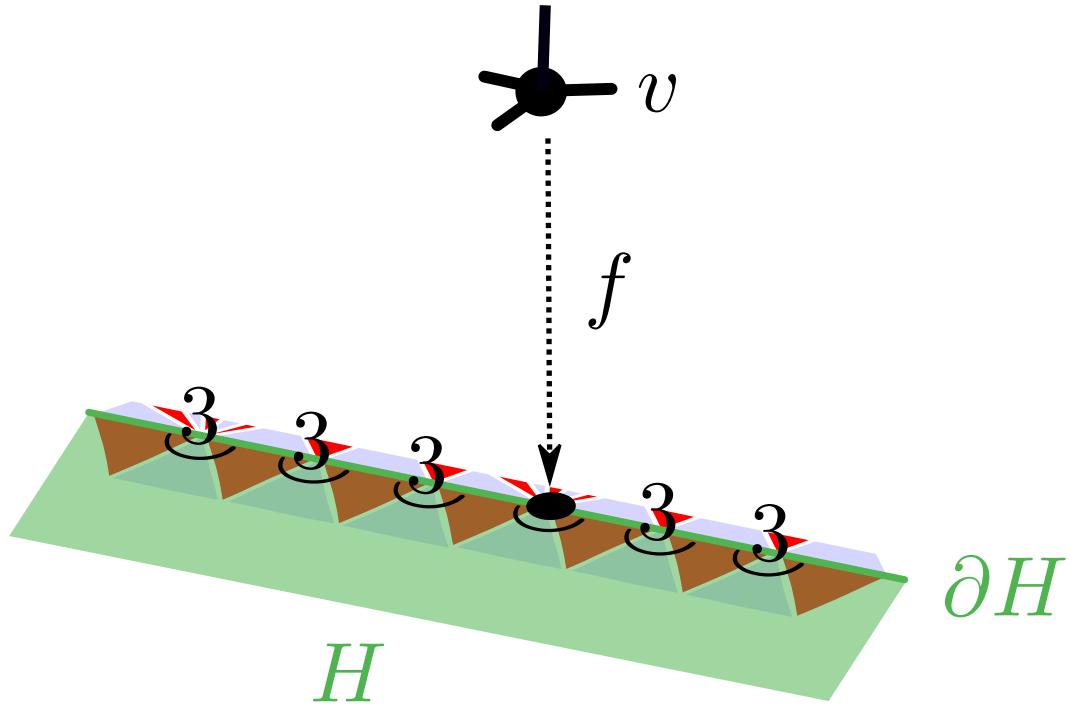
vertices “barycentric”

$$\begin{aligned} f(v) \in \partial H \\ \implies \exists e \quad f(e) \text{ escapes } H \end{aligned}$$



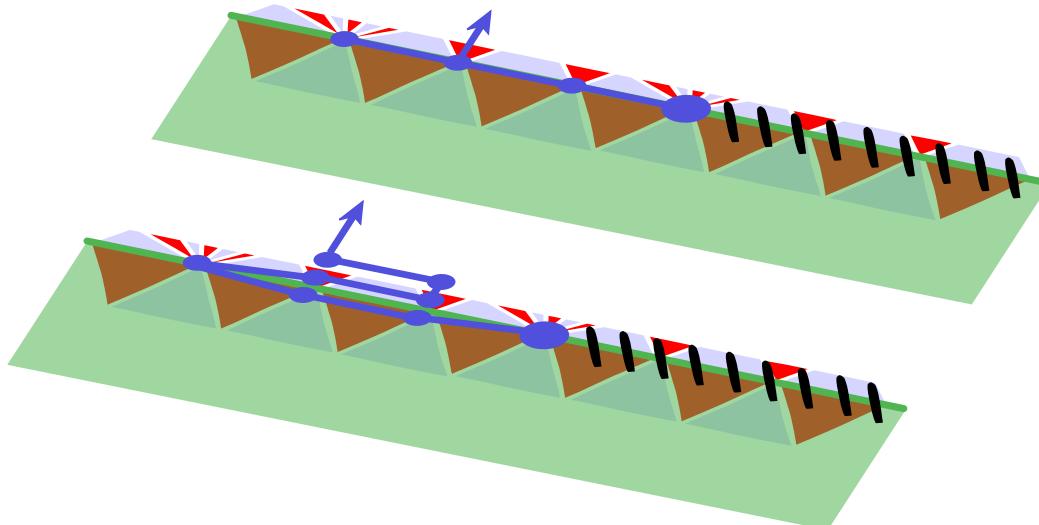
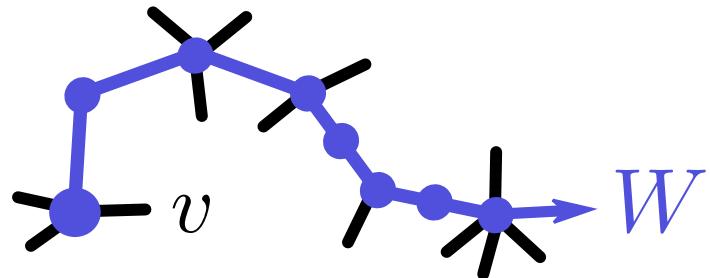
vertices “barycentric”

$$f(v) \in \partial H$$

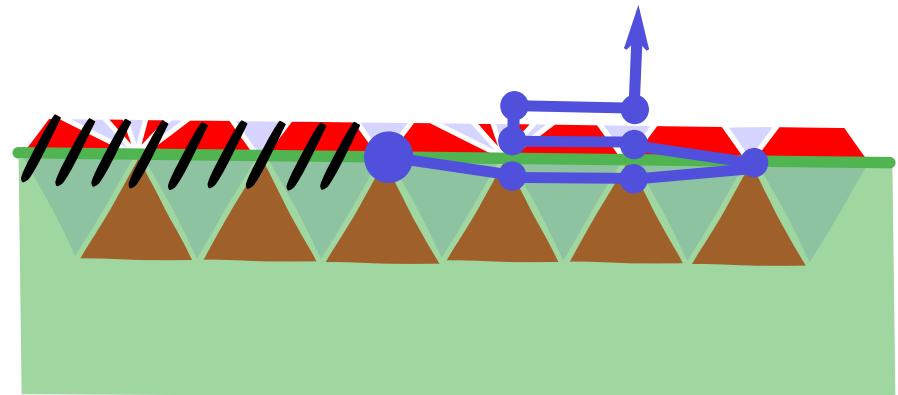
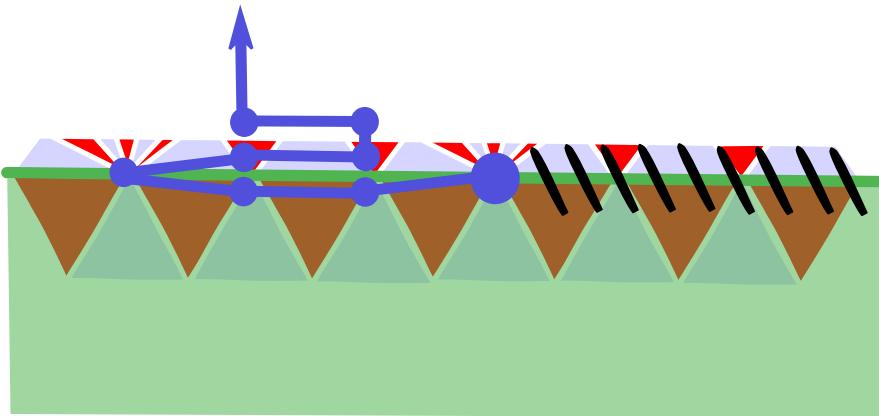


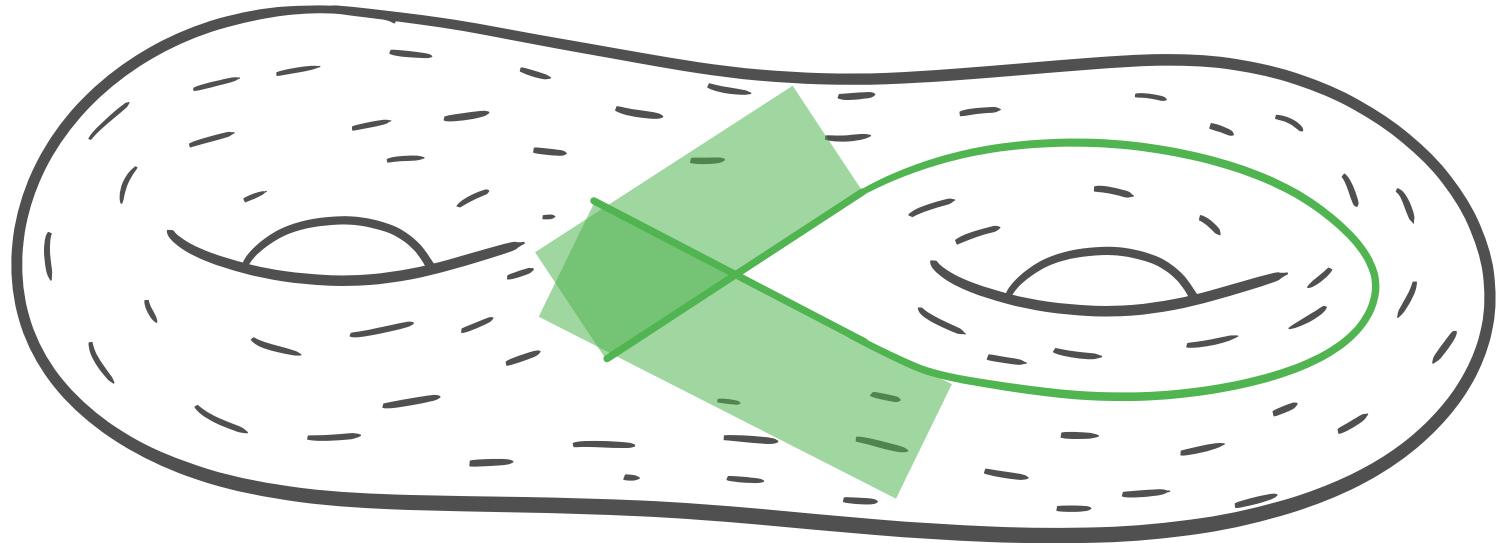
vertices “barycentric”

$$f(v) \in \partial H \\ \xrightarrow{} \\ \exists W f \circ W \text{“escapes”} H$$



where to escape depends on the coloring





this definition generalizes to surfaces

Results

graph G

reducing triangulation T with m edges

$f : G \rightarrow T^1$ of size n

Definition of *harmonious* drawings

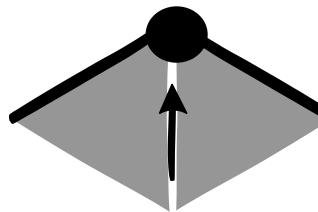
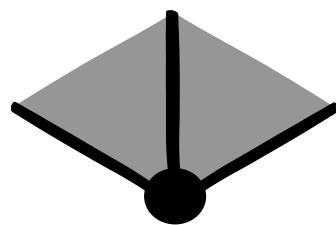
f harmonious and f can be untangled
 $\Rightarrow f$ weak embedding

Algo to make f harmonious in $O((m+n)^2 n^2)$ time,
without increasing any edge length

Harmonizing a drawing

- 1 we define monotonic moves to apply to f

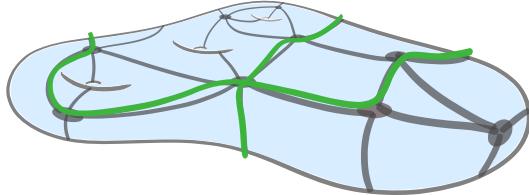
ex:



- 2 some moves do not seem to decrease any potential so we combine the moves carefully

Making curves cross
minimally

Related work

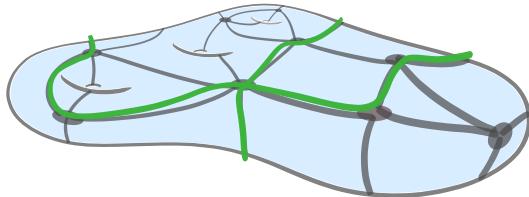


closed walks of total length n
on a graph of size m

Despré, Lazarus, 2019

- Put a single curve in minimal position in $O(m + n^4)$ time
- Compute the min. nb. of crossings in $O(m + n^2)$ time

Result



closed walks of total length n
on a graph of size m

simpler algos and proofs!

$$m^3n + mn \log(mn)$$

D., 2024

- Put a single curve in minimal position in $O(\cancel{m} + \cancel{n}^4)$ time
- Compute the min. nb. of crossings in $O(\cancel{m} + \cancel{n}^2)$ time

$$m^2 + mn \log(mn)$$

Untangling Graphs

Computing Delaunay Triangulations

Conclusion

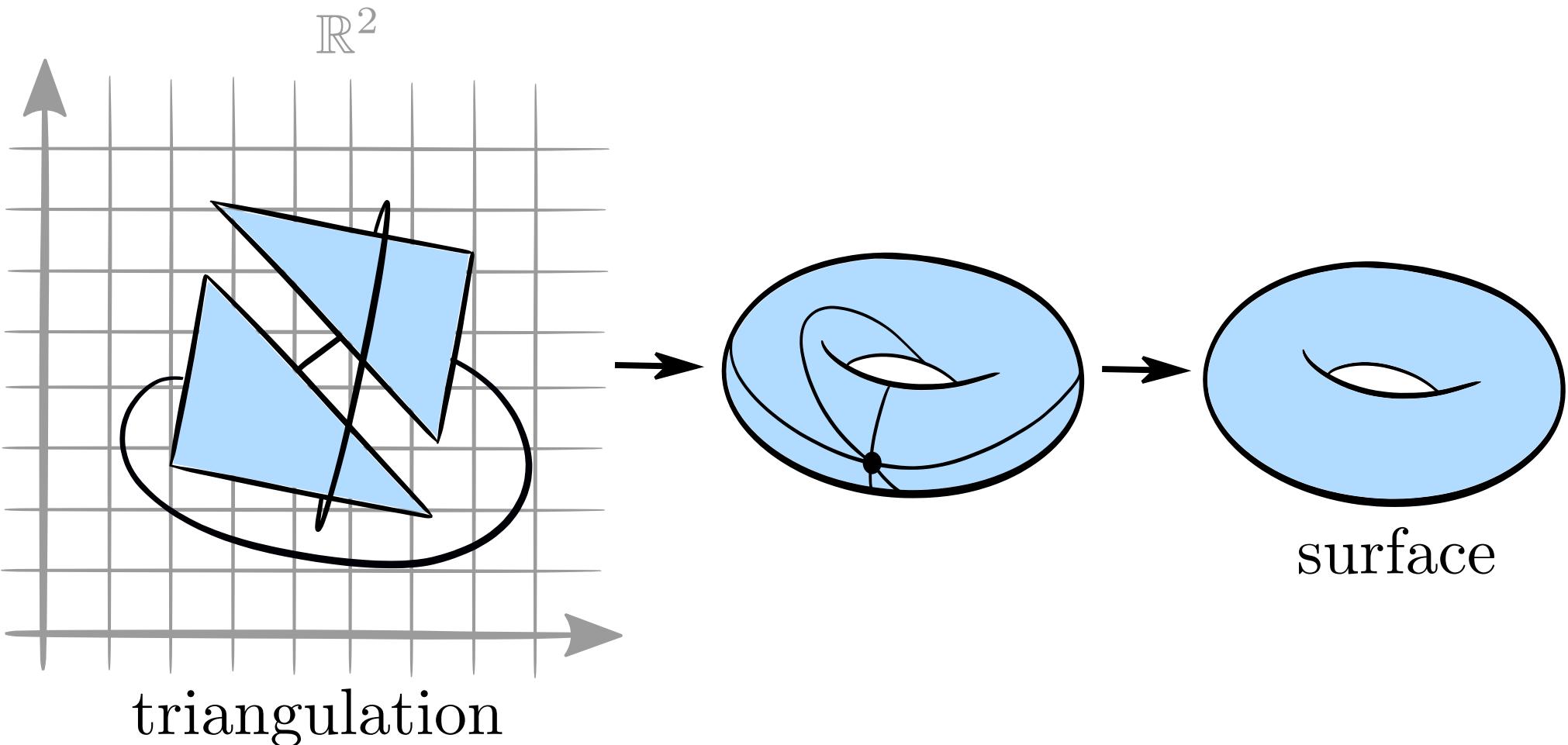
Untangling Graphs

Computing Delaunay Triangulations

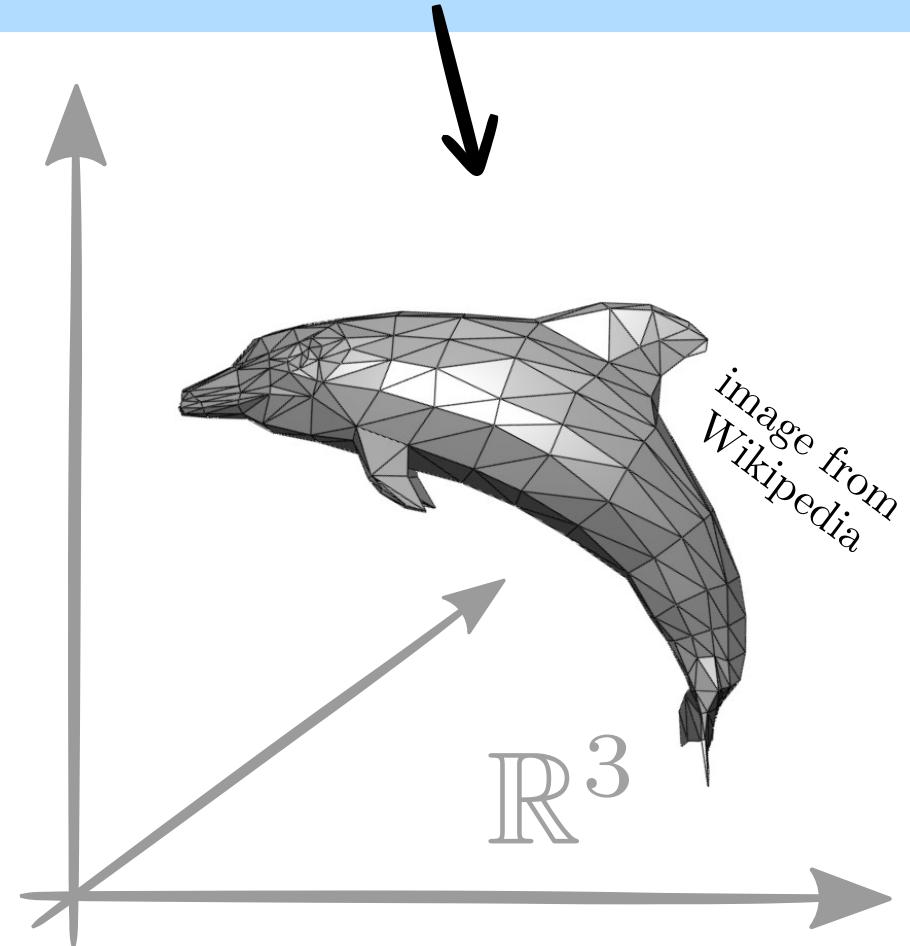
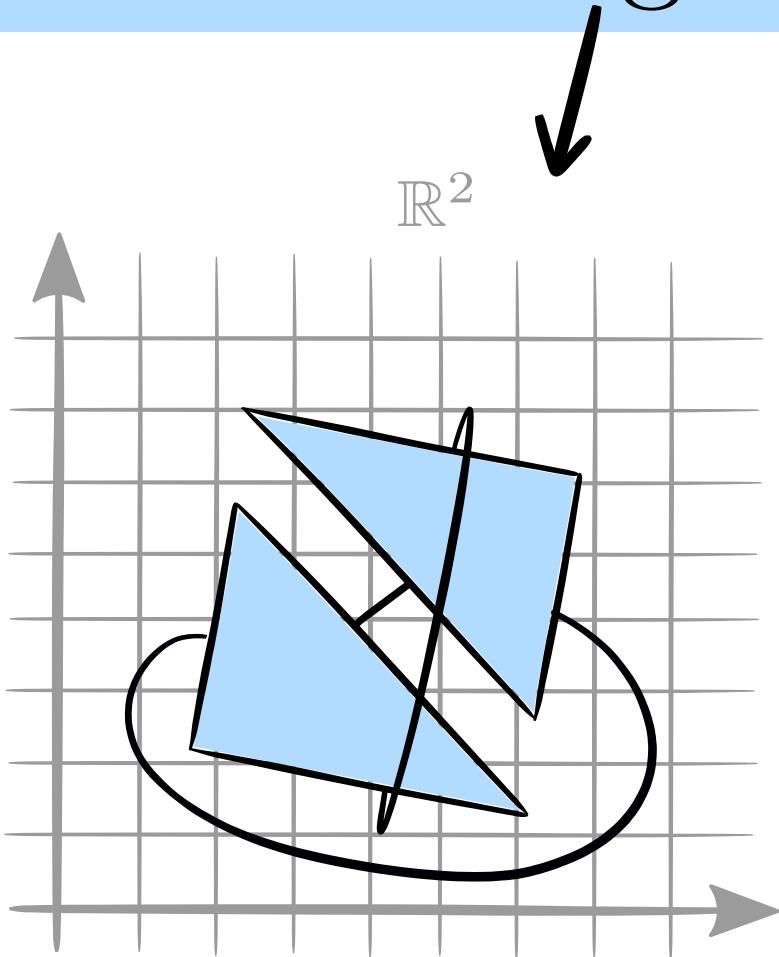
Conclusion

Triangulations of polyhedral surfaces

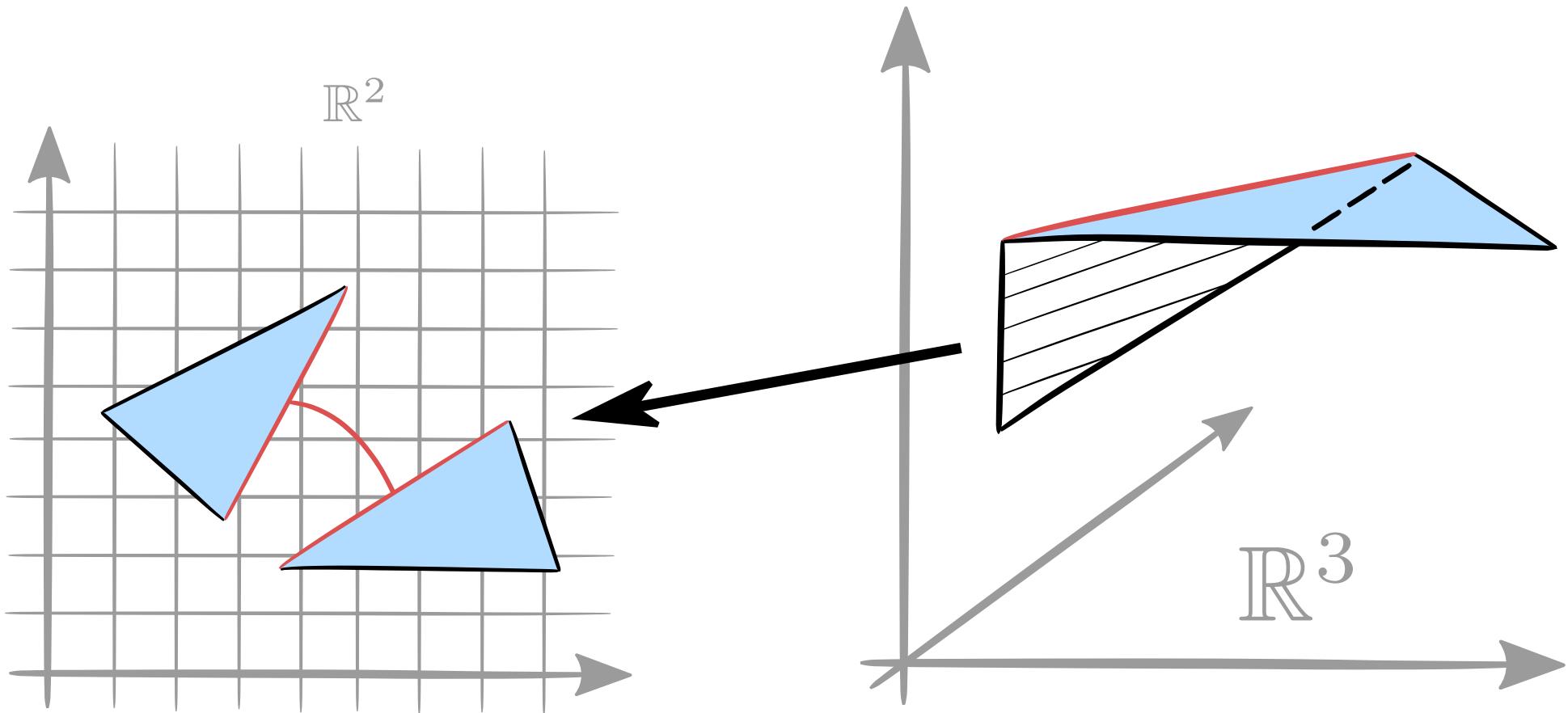
Triangulation of polyhedral surface



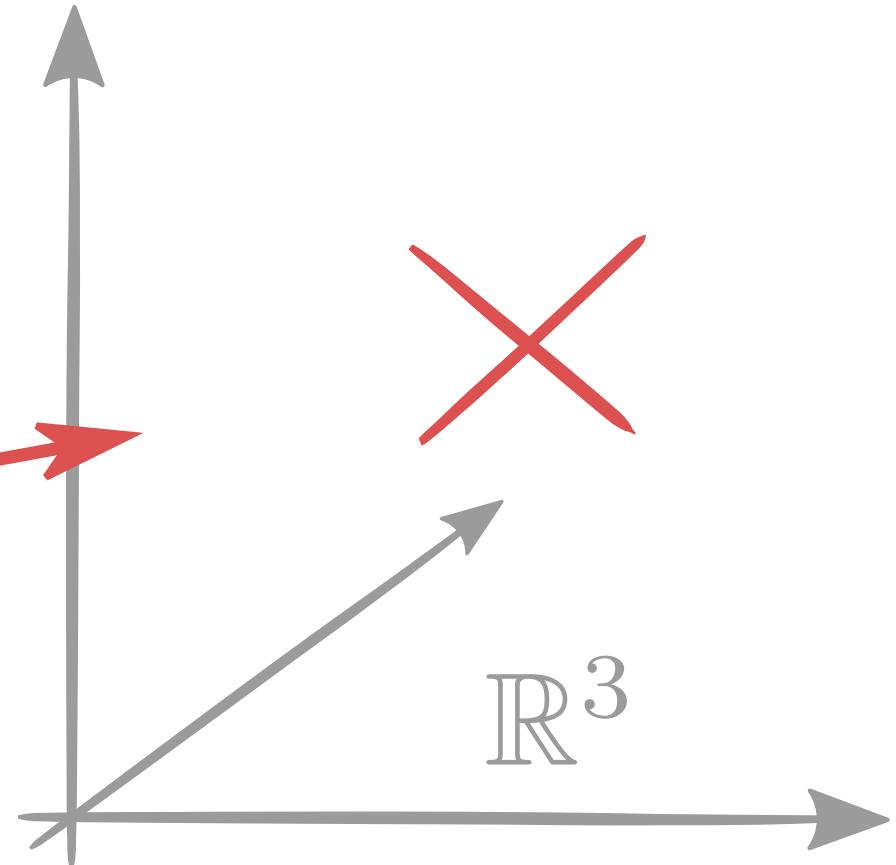
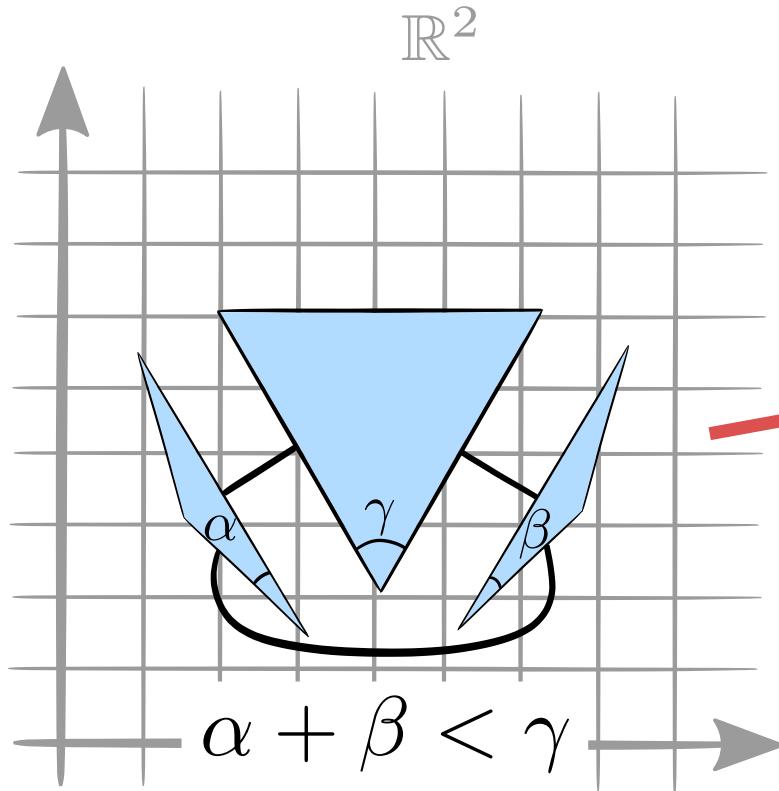
Triangulation vs. Mesh



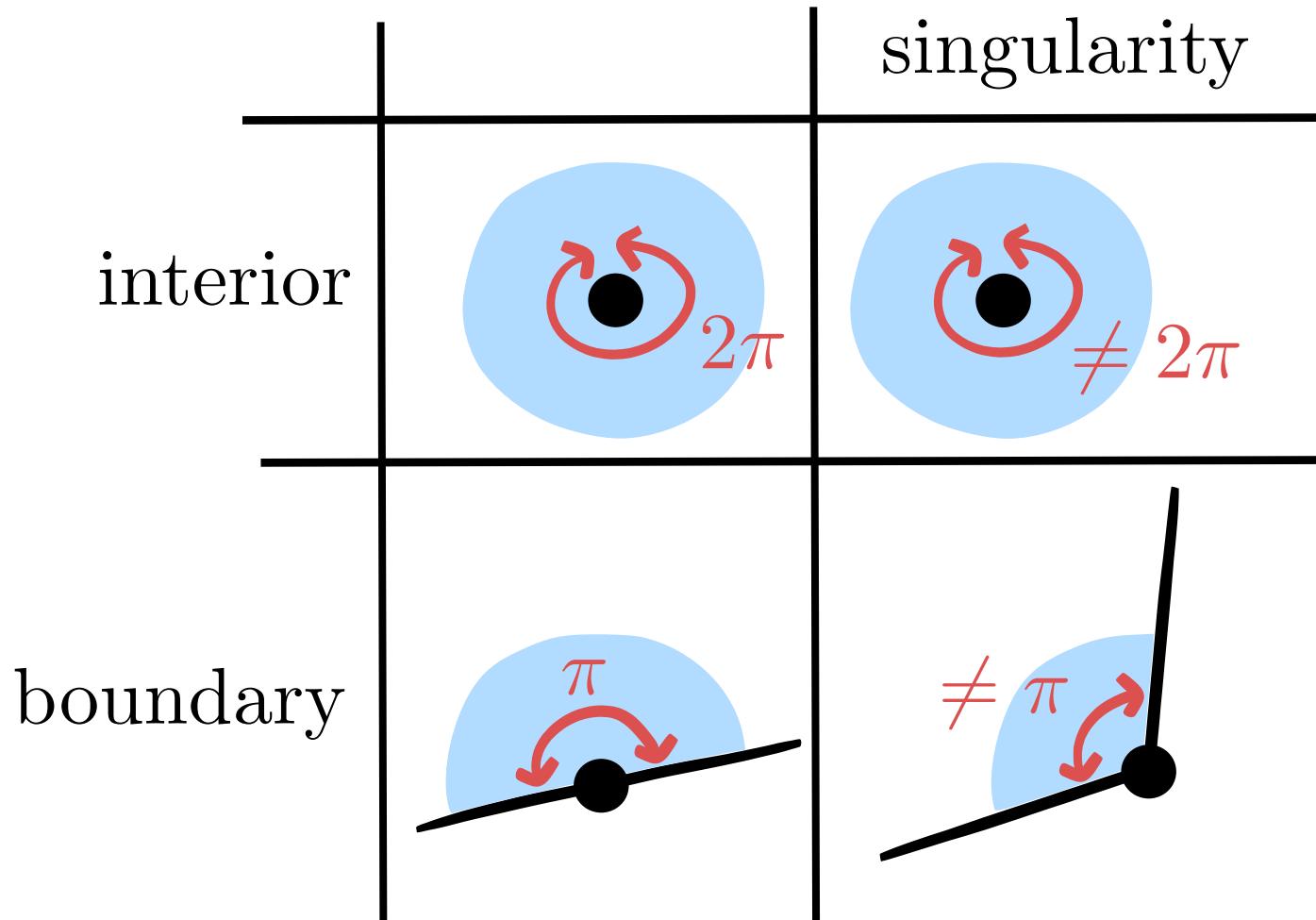
Every mesh gives a triangulation



Every mesh gives a triangulation
but converse is false!



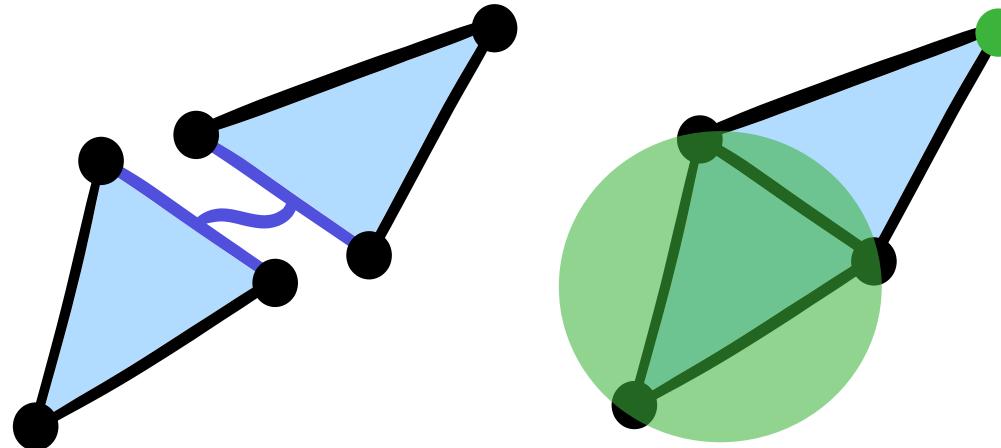
Types of points on the surface



Problem

Delaunay triangulation

triangulation in which
every edge is Delaunay



The Delaunay triangulation

Generically, every surface has a **unique**
Delaunay triangulation
with given vertex set*

*finite, non-empty, containing the singularities

Problem

Given triangulation T , and vertex set V^* ,
compute the Delaunay triangulation of the
surface of T whose vertex set is V

$*$ usually the vertex set of T or
the singularities of the surface of T

Motivations

- isometry testing
- shortest paths

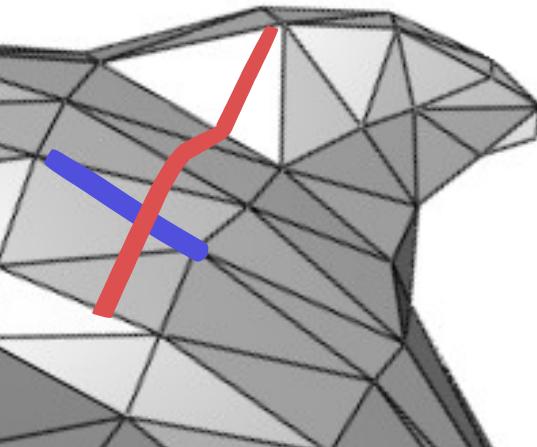
- shortest paths

Shortest paths on meshes

On a mesh M with n triangles...

a shortest path cannot cross an edge twice

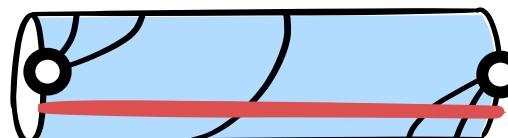
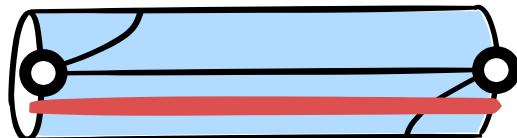
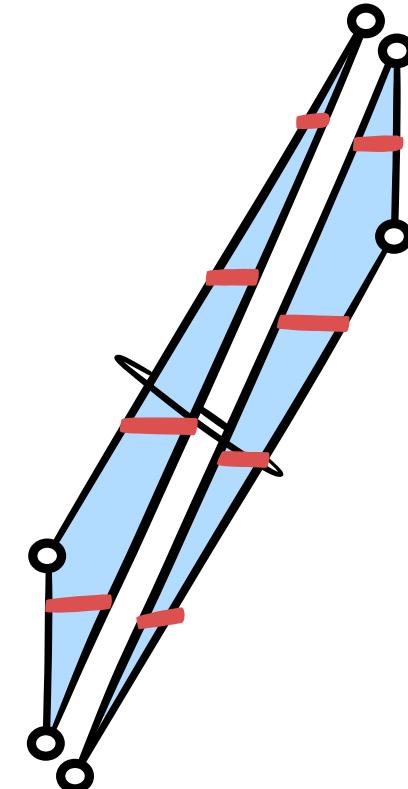
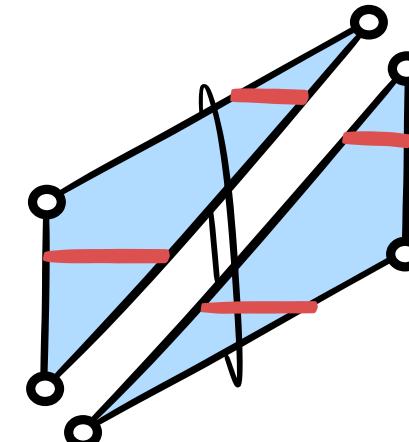
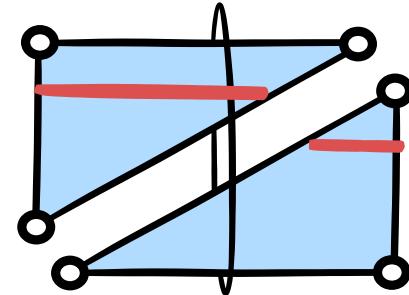
→ shortest path can be computed in $O(f(n))$ time



Mitchel, Mount, Papadimitriou, 1987

Shortest paths on triangulations

they can cross edges arbitrarily many times

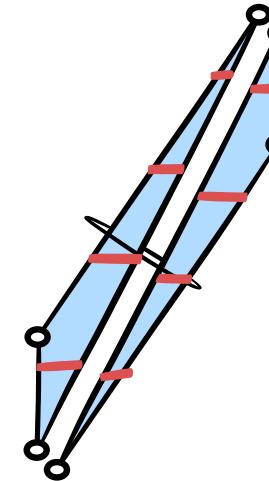


Erickson, 2006

Shortest paths on triangulations

Löffler, Ophelders, Staals, Silveira, 2023

happiness h : max number of times
a shortest path visits a triangle



→ shortest path can be computed in $O(f(n, h))$ time

Shortest paths on triangulations

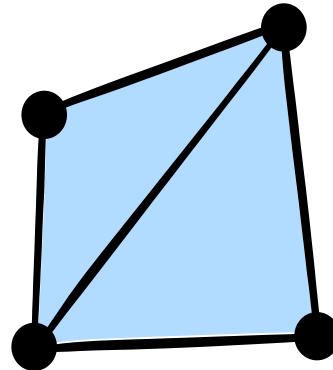
Löffler, Ophelders, Staals, Silveira, 2023

Delaunay triangulations have happiness $O(1)$

Existing algorithms

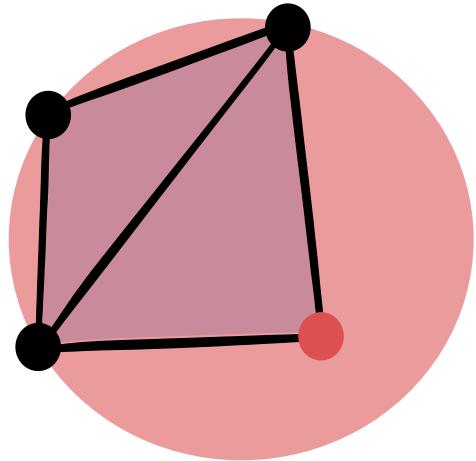
Delaunay flips algorithm

Delaunay flip:



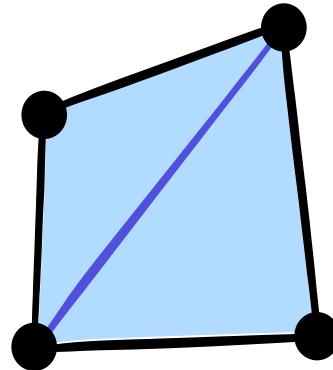
Delaunay flips algorithm

Delaunay flip:



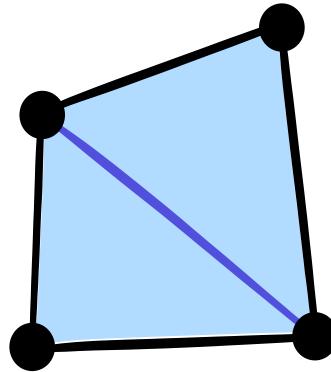
Delaunay flips algorithm

Delaunay flip:



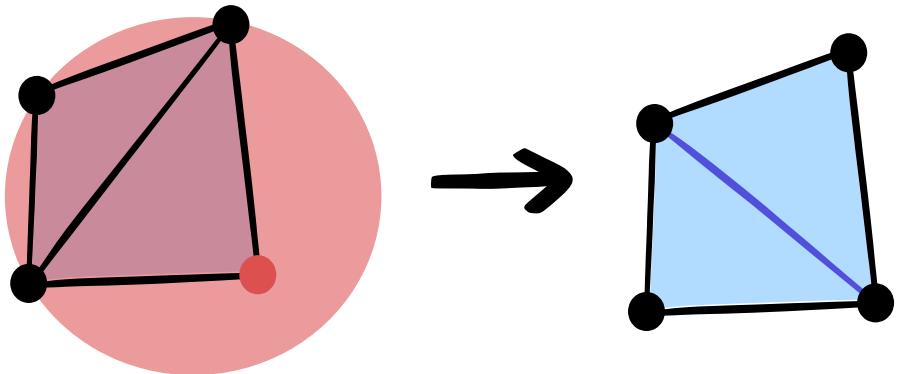
Delaunay flips algorithm

Delaunay flip:



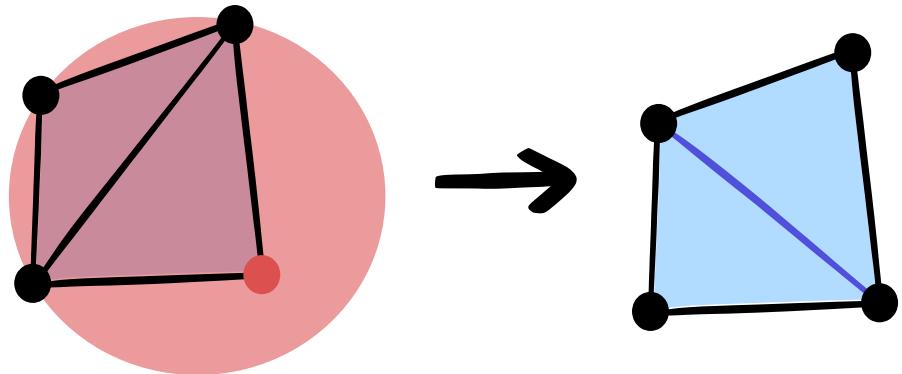
Delaunay flips algorithm

Delaunay flip:



Delaunay flips algorithm

Delaunay flip:

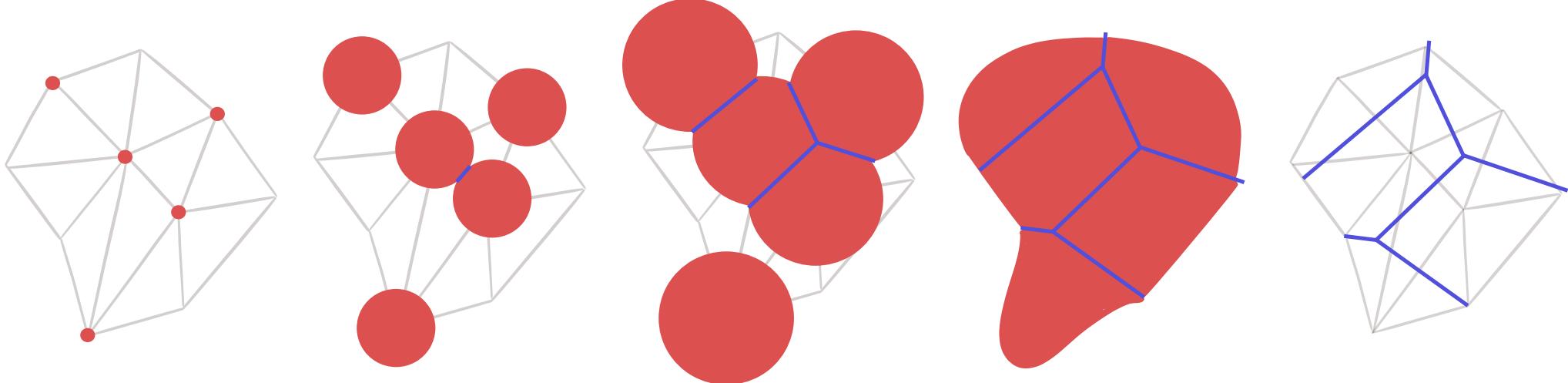


Algorithm:

apply Delaunay flips greedily as long as you can

Other method

compute the Voronoi diagram
by propagating waves

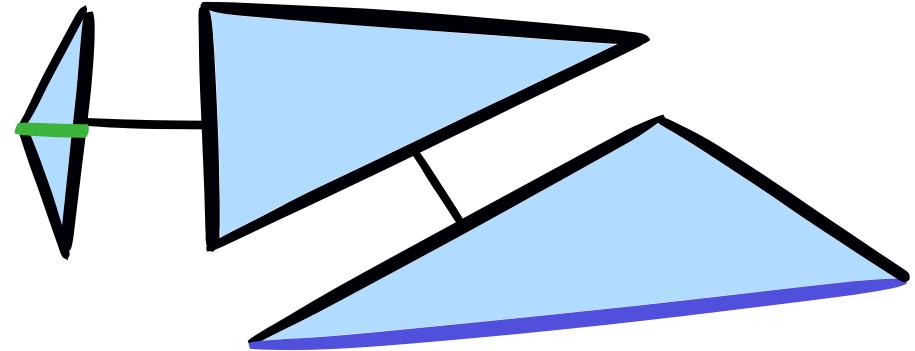


then derive Delaunay from it

Result

Result

aspect ratio =
 $\frac{\text{maximum side length}}{\text{minimum height}}$

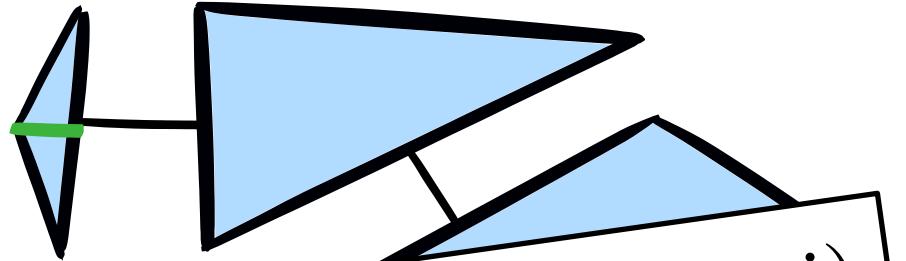


D. 2025

Given triangulation T of n triangles, of aspect ratio r , whose surface has no boundary, we can compute Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Result

aspect ratio =
maximum side length
minimum height

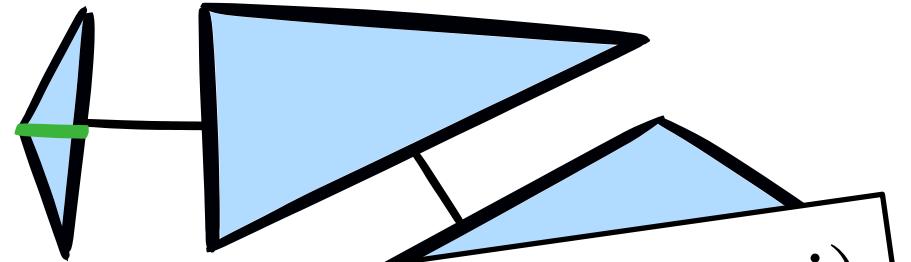


Previous algorithms (Delaunay flips and Voronoi)
achieved no better than $O(\text{Poly}(n, r))$

Given a surface with n vertices and m edges, of aspect ratio r ,
if the surface has no boundary, we can compute
Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Result

aspect ratio =
maximum side length
minimum height



D. So

G.

Previous algorithms (Delaunay flip and Voronoi)
achieved no better than $O(F)$

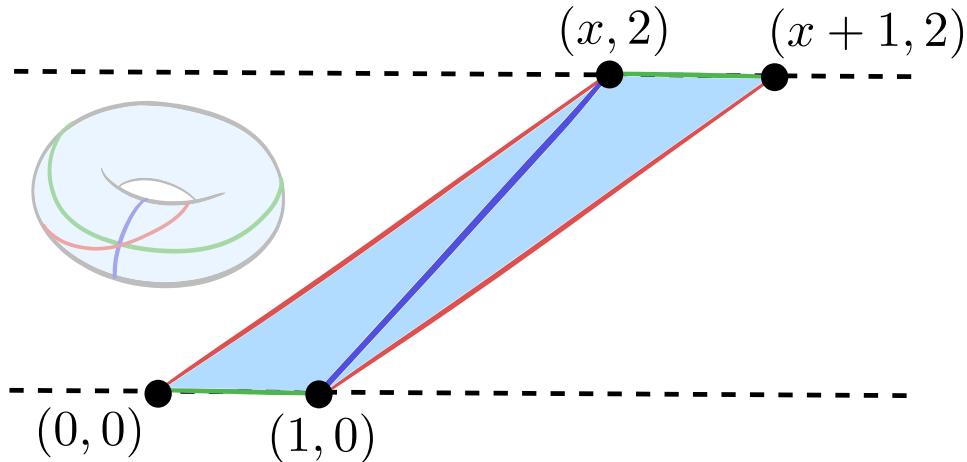
surface has no boundary, we can compute
Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Now backed by a
lower bound!

Lower bound

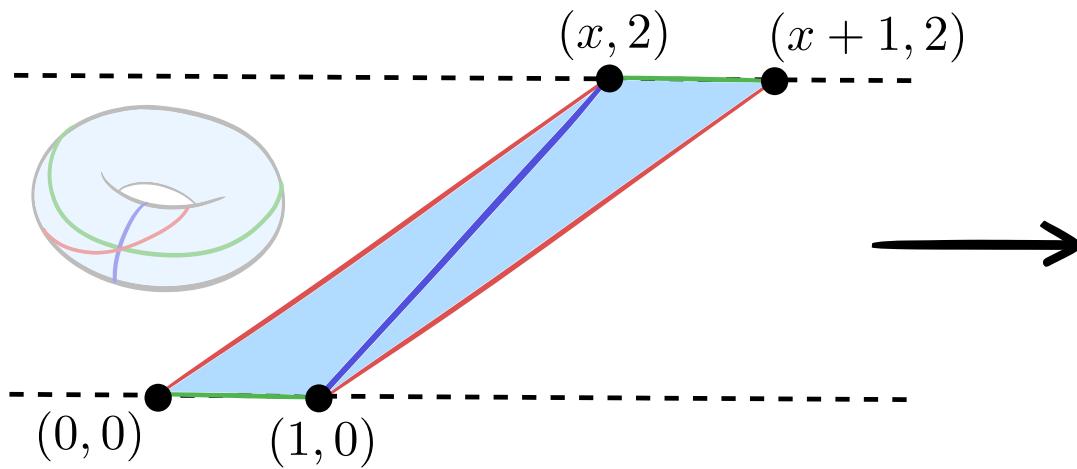
Lower bound

Input

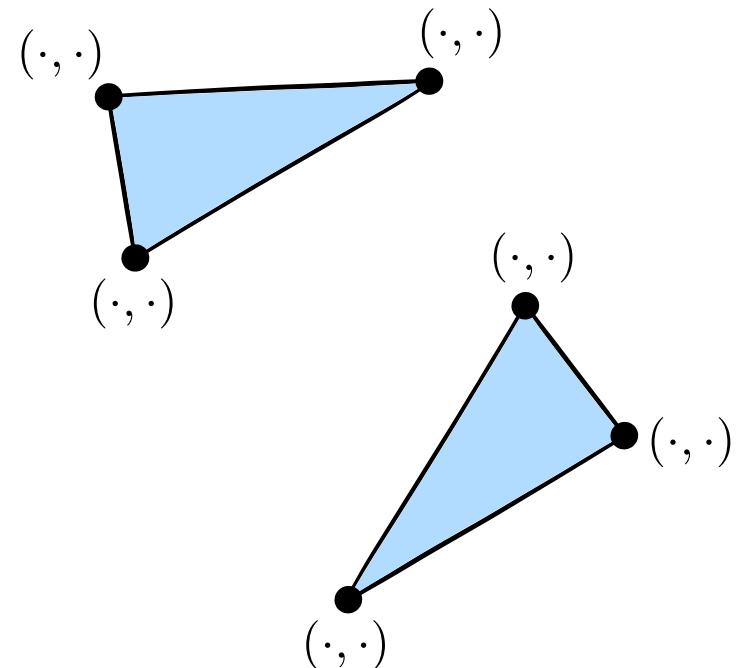


Lower bound

Input

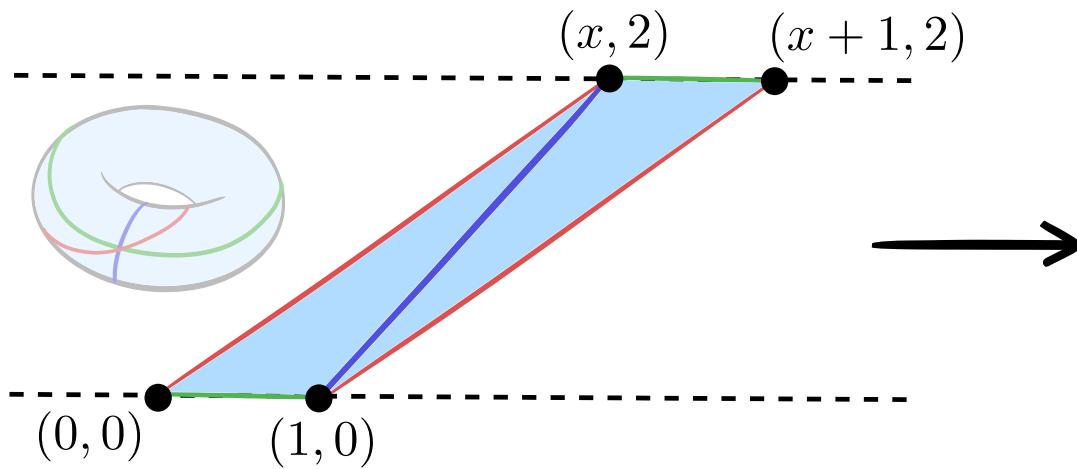


Output: Delaunay

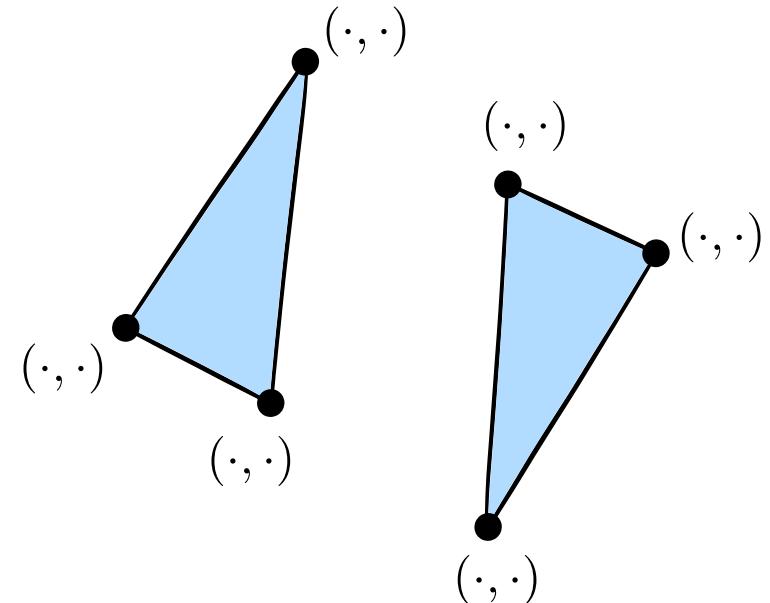


Lower bound

Input

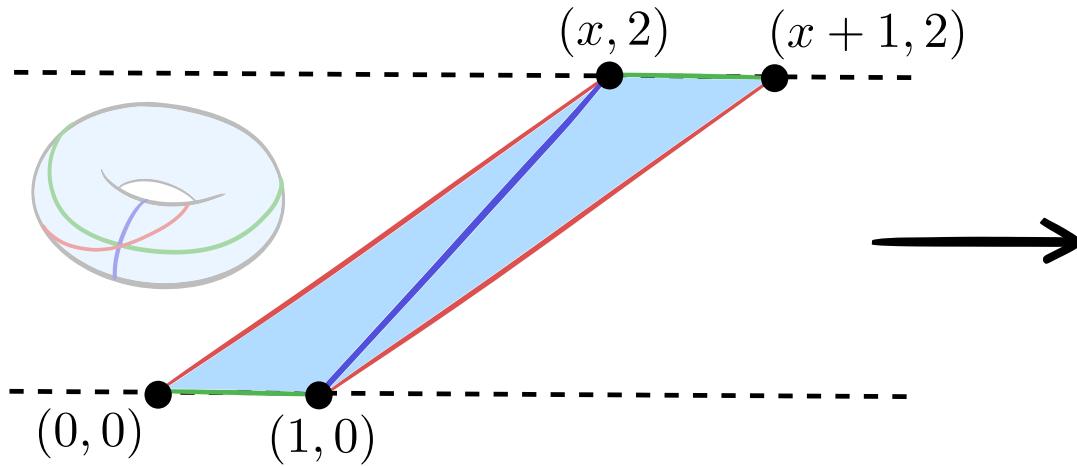


Output: Delaunay

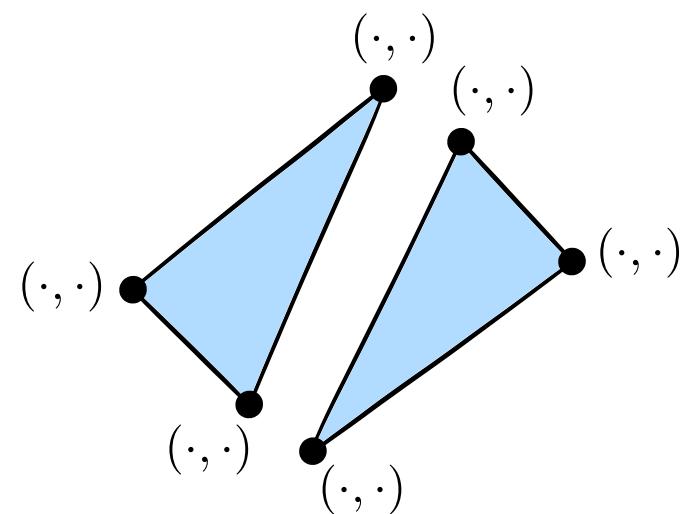


Lower bound

Input

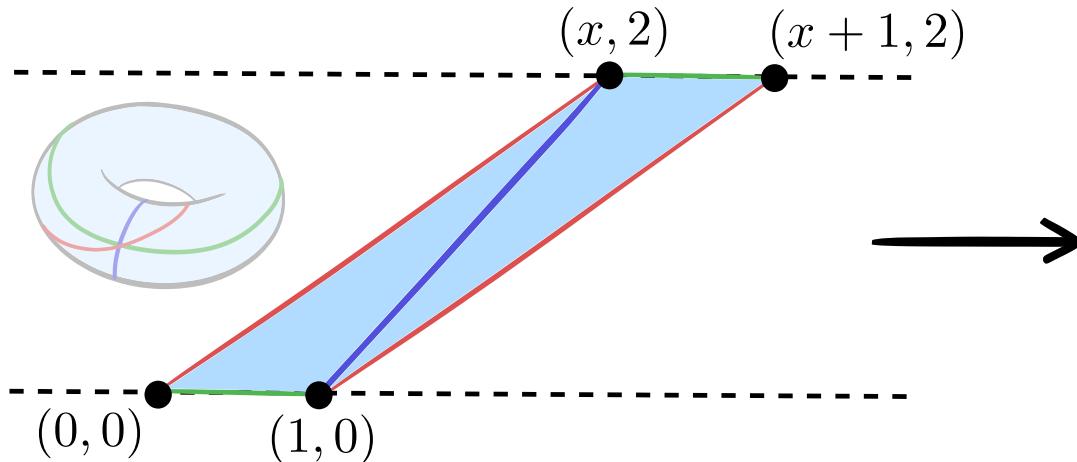


Output: Delaunay

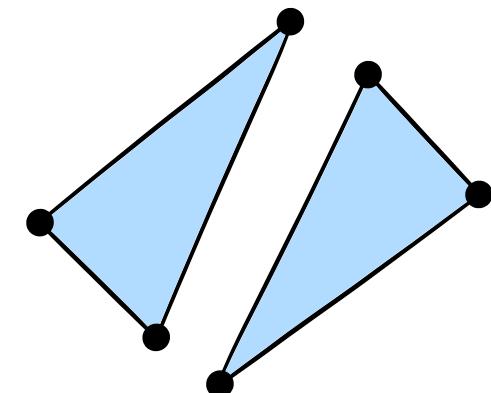


Lower bound

Input

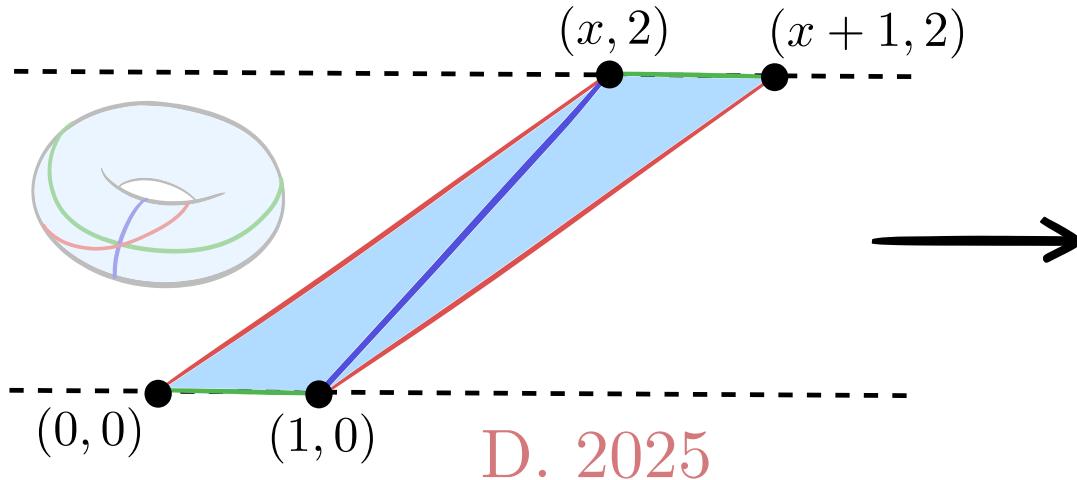


Output: Delaunay

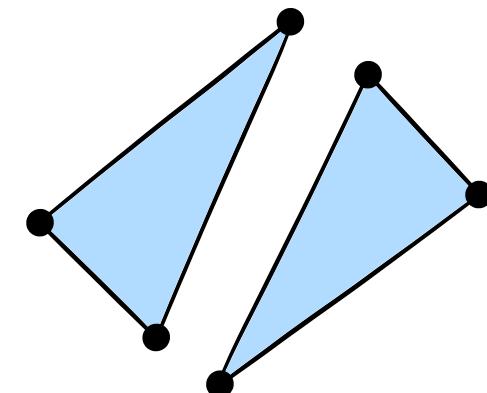


Lower bound

Input



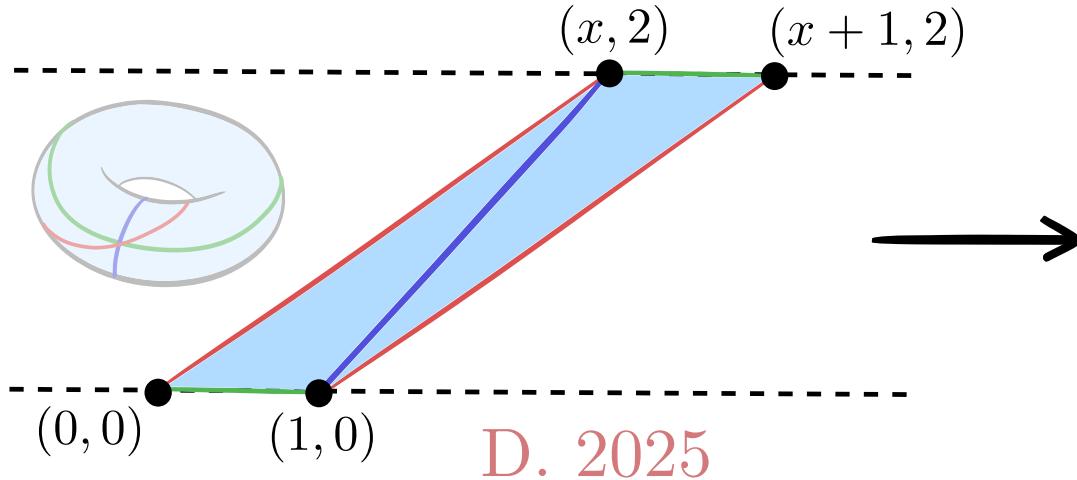
Output: Delaunay



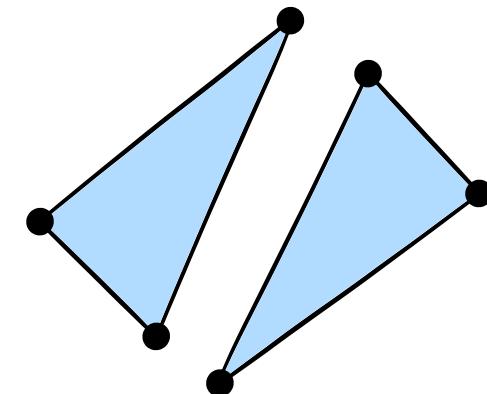
No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

Lower bound

Input



Output: Delaunay

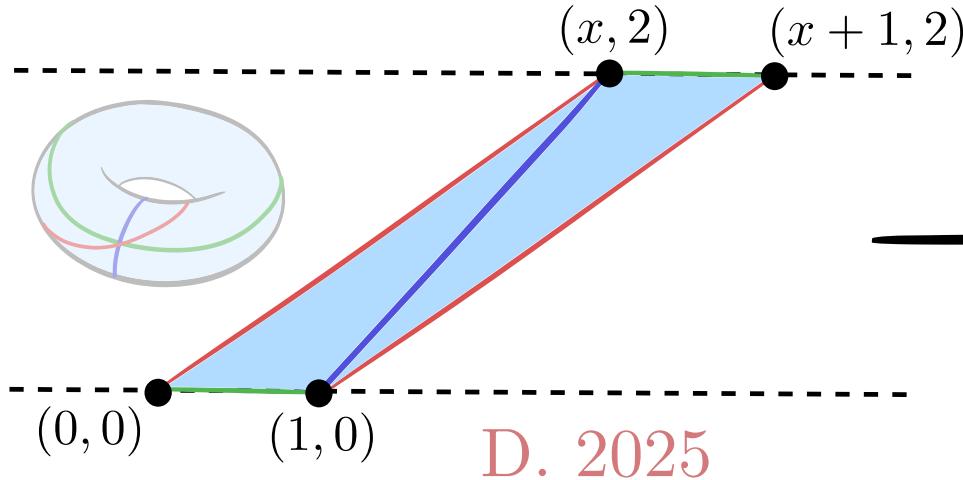


No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

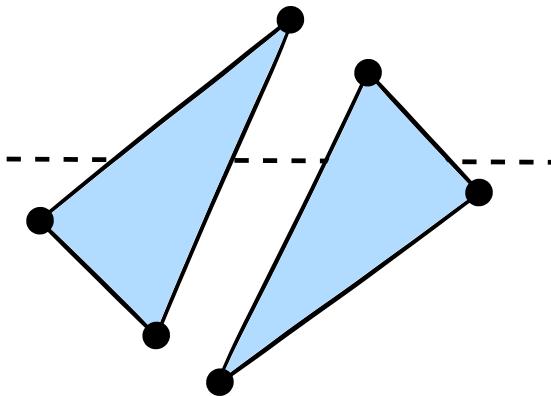
Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time

Lower bound

Input



Output: Delaunay

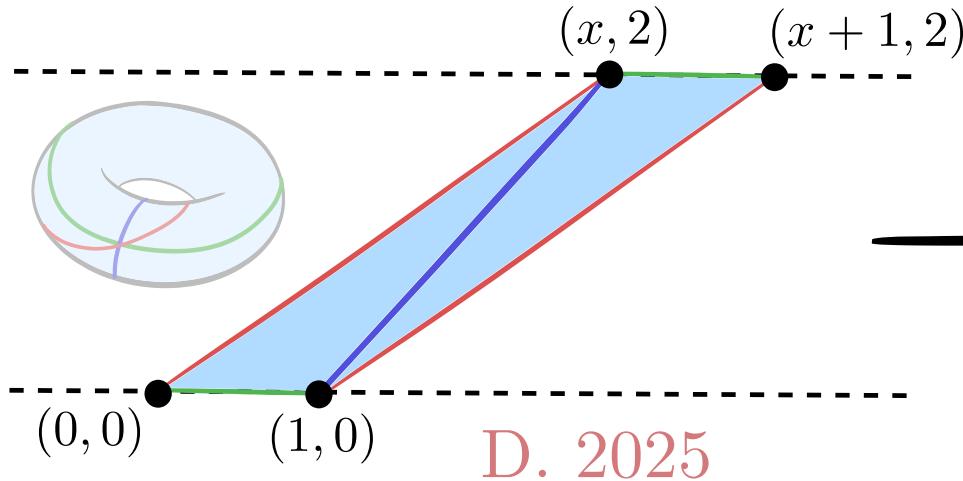


No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

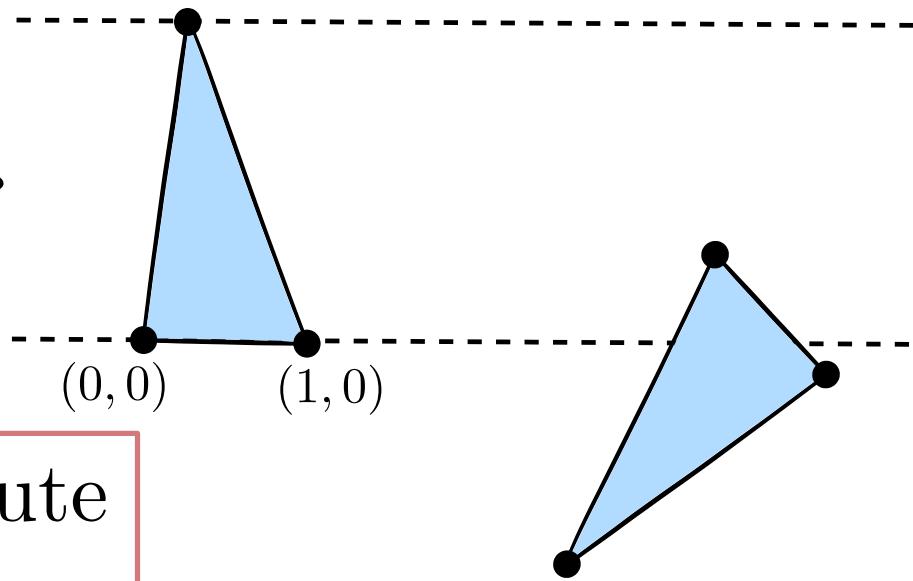
Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time

Lower bound

Input



Output: Delaunay

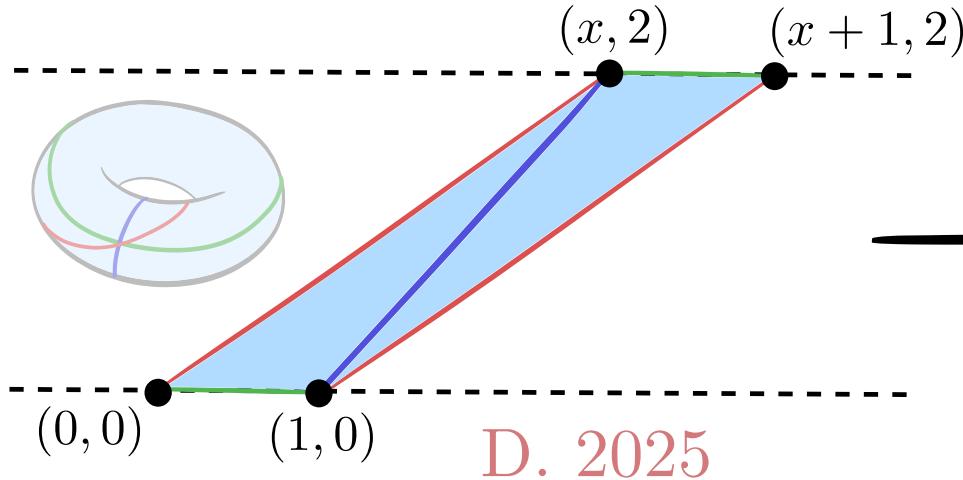


No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

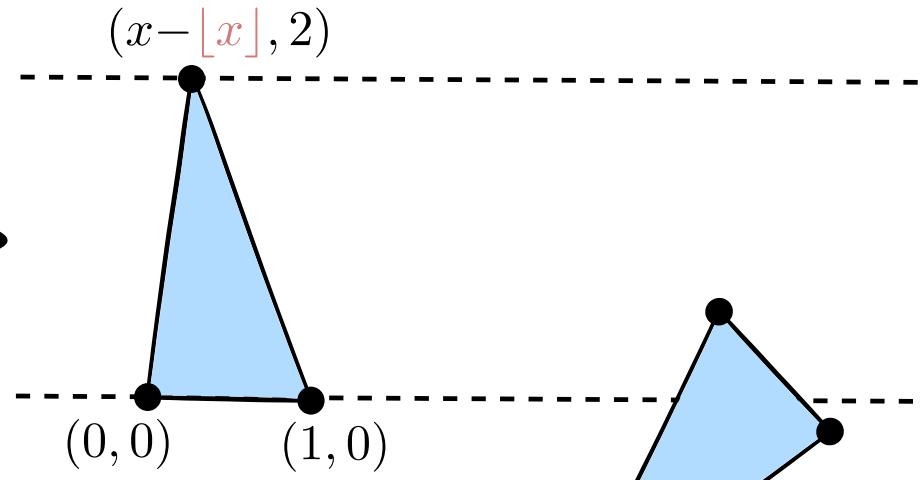
Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time

Lower bound

Input



Output: Delaunay



No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time

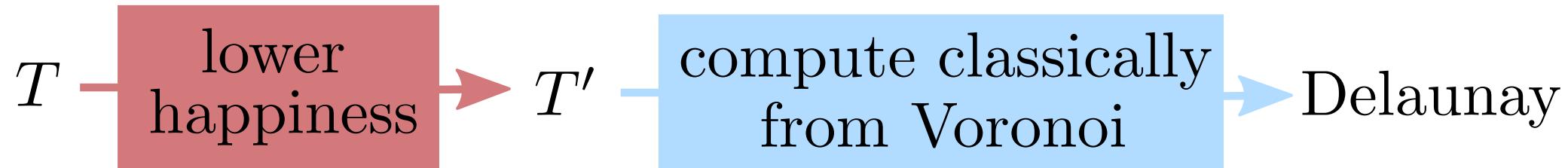
Algorithm overview

Algorithm



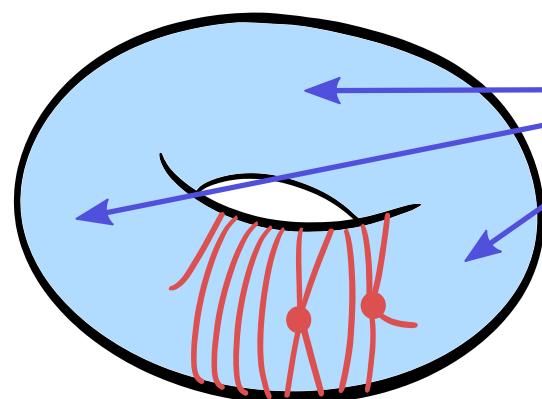
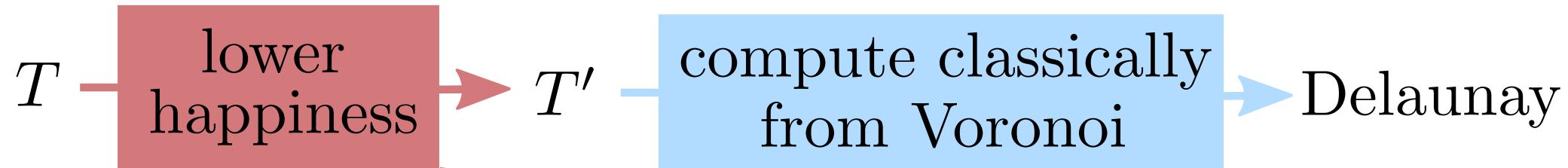
Algorithm

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Algorithm

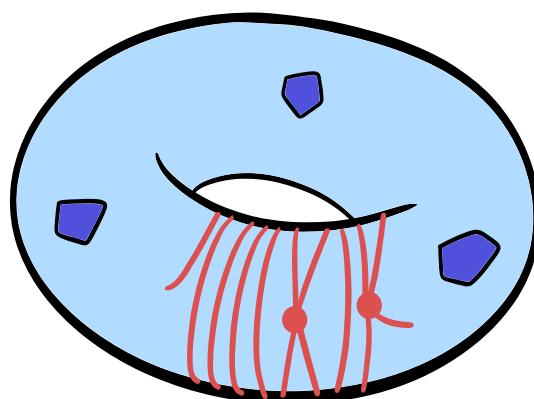
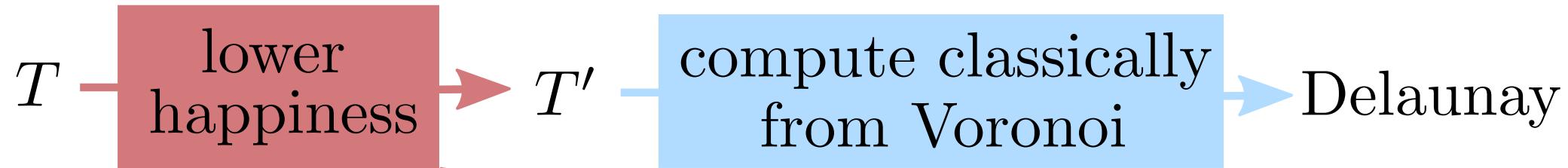
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consider the singularities

Algorithm

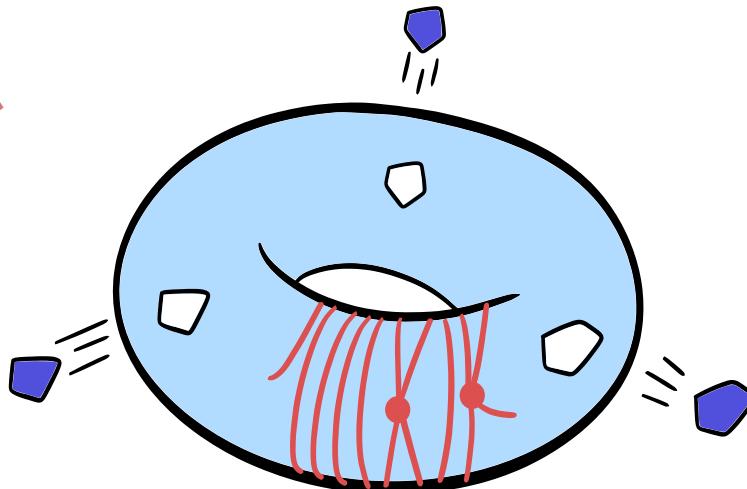
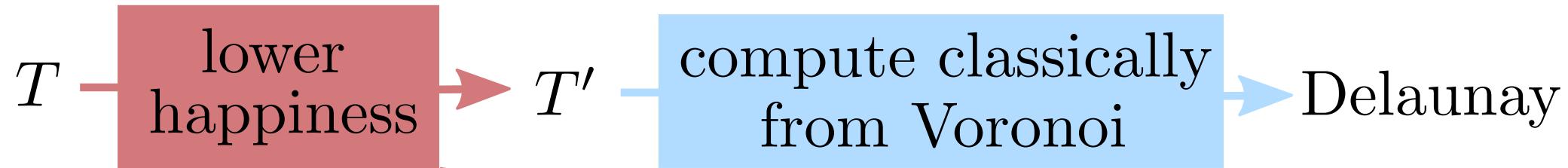
D. 2025



cut out caps around the singularities

Algorithm

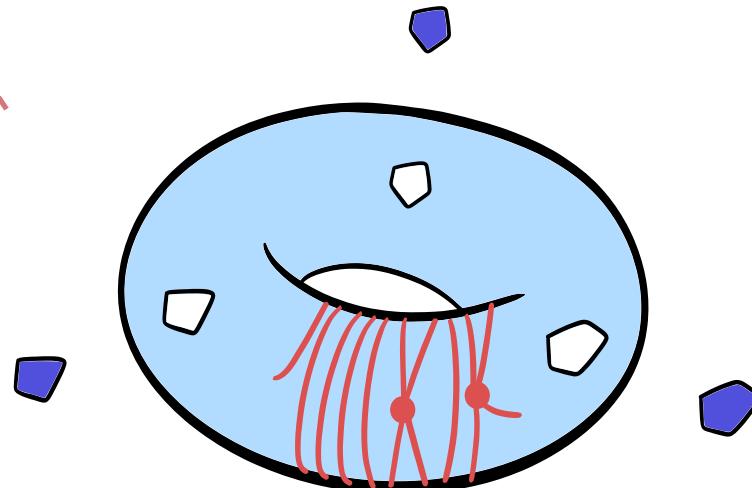
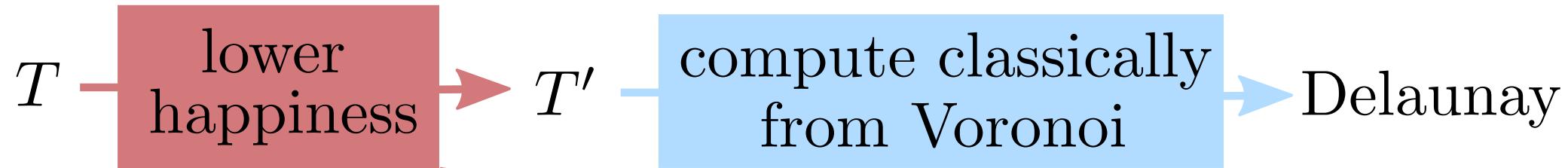
D. 2025



cut out caps around the singularities

Algorithm

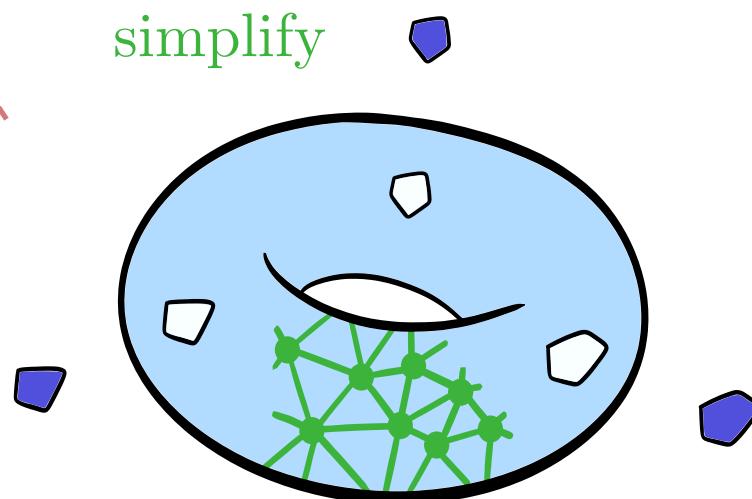
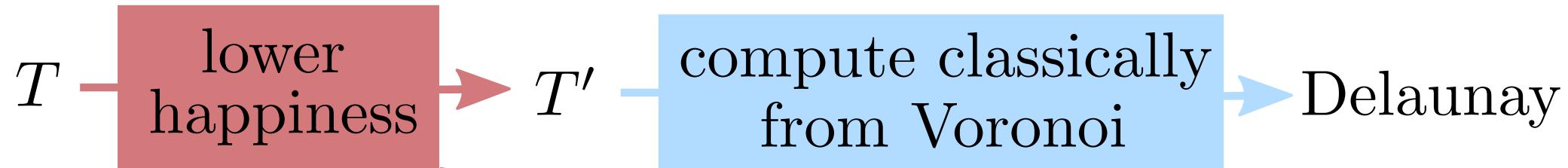
D. 2025



cut out caps around the singularities

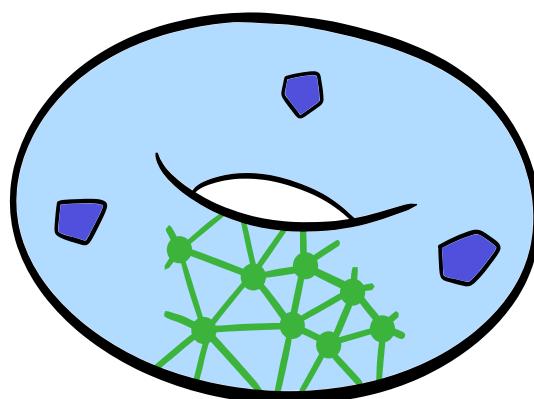
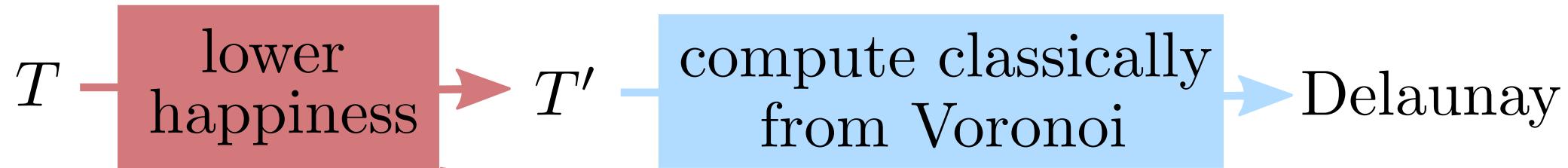
Algorithm

D. 2025



Algorithm

D. 2025



put the caps back

Simplification algorithm

Tuned combination of elementary operations, like

- inserting vertices in edges
- inserting edges in faces
- deleting vertices

repeated many times

some simplify the geometry,
others decrease # vertices

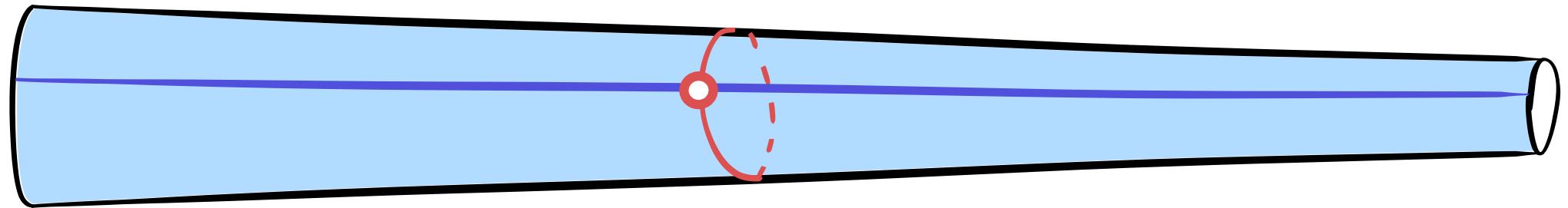
Analysis

Show that during execution:

1. # vertices stays bounded
2. Geometry gets simpler and simpler



Enclosure



Untangling Graphs

Computing Delaunay Triangulations

Conclusion

Untangling Graphs

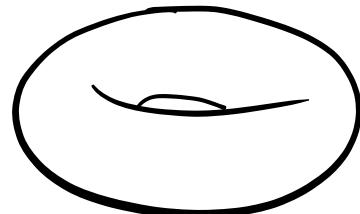
Computing Delaunay Triangulations

Conclusion

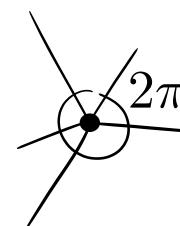
Other results

D., 2022-23

Upper bound for # Delaunay flips on flat tori,
tight up to constant factor



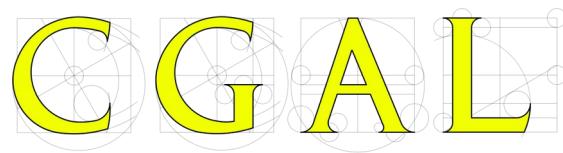
topological shape of a torus



2π around each vertex

Other results

Despré, D., Pouget, and Teillaud, 2025



package for computing with
hyperbolic surfaces

generation (genus 2 only)

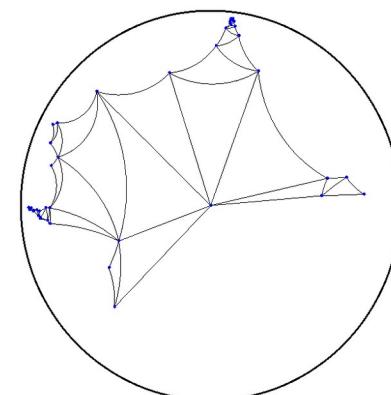
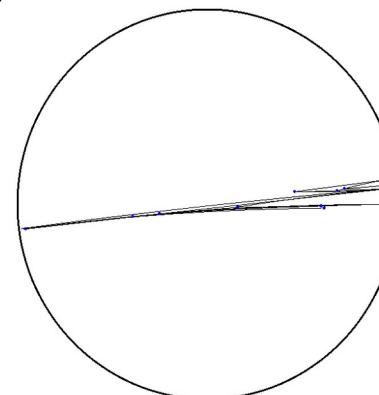


Triangulated hyperbolic
surface

Delaunay flip



visualization



Other results

Despré, D., Pouget, and Teillaud

Exact computations!
package for computing
hyperbolic surfaces

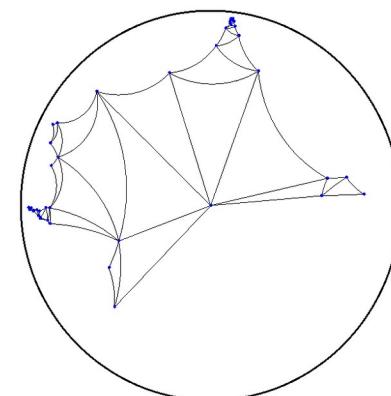
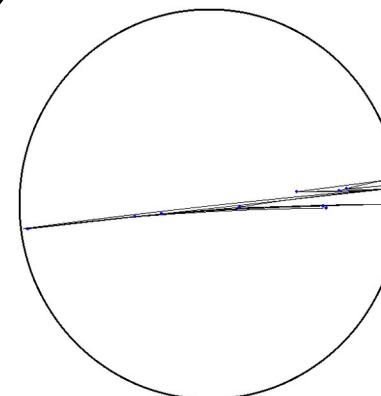
generation (genus 2 only)



Triangulated hyperbolic
surface

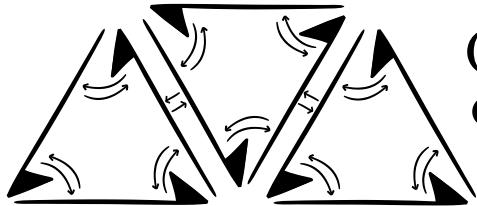
Delaunay flip

↓
visualization



Triangulated hyperbolic
surface

Combinatorics

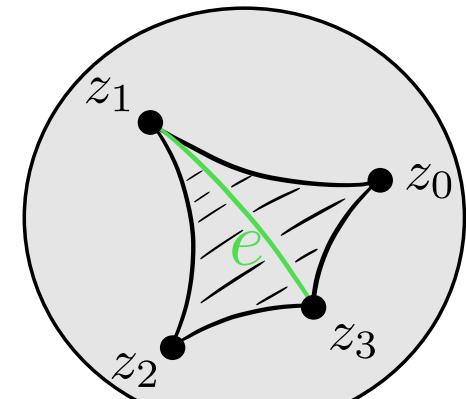


CGAL
combinatorial maps

Triangulated hyperbolic
surface

Geometry

- each edge is decorated with a complex number (cross ratio)



$$e \rightarrow \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

Possible continuations

- generation of hyperbolic surfaces of genus ≥ 3
- what is the complexity of Delaunay flips algo?
- certifying that a drawing cannot be untangled
- untangling by homotopy moves
- extension to non orientable surfaces
- what is the complexity of untangling?
- minimizing crossings of graphs by homotopy
- how unique reducing triangulations are?