

# Algorithms for Topological and Metric Surfaces



Loïc Dubois

# Computational Geometry

Design algorithms for geometric problems

This thesis

Focus on surfaces

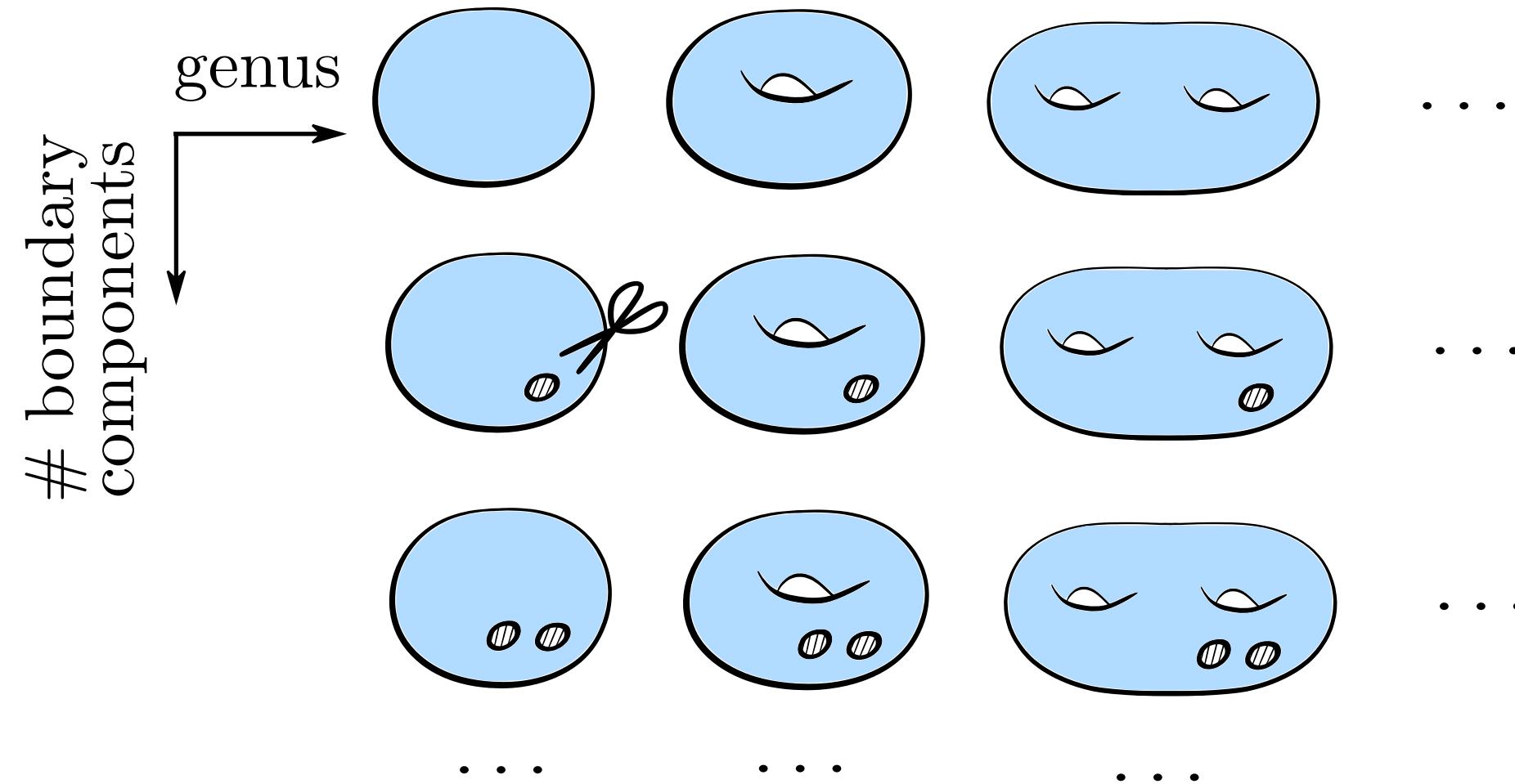


# Topology

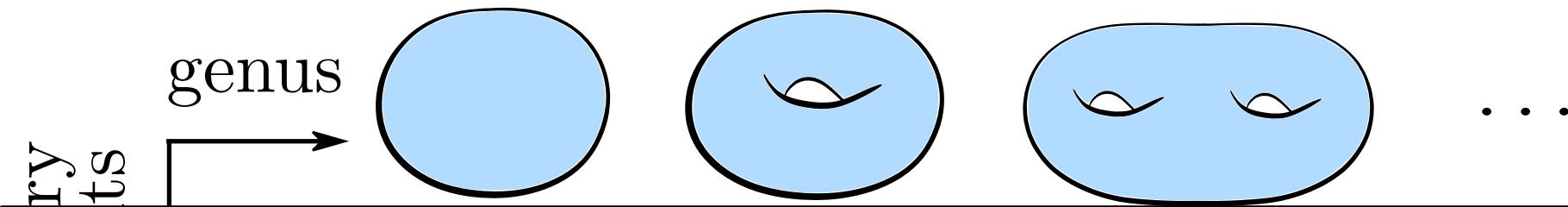


image by Crane and Segeiman

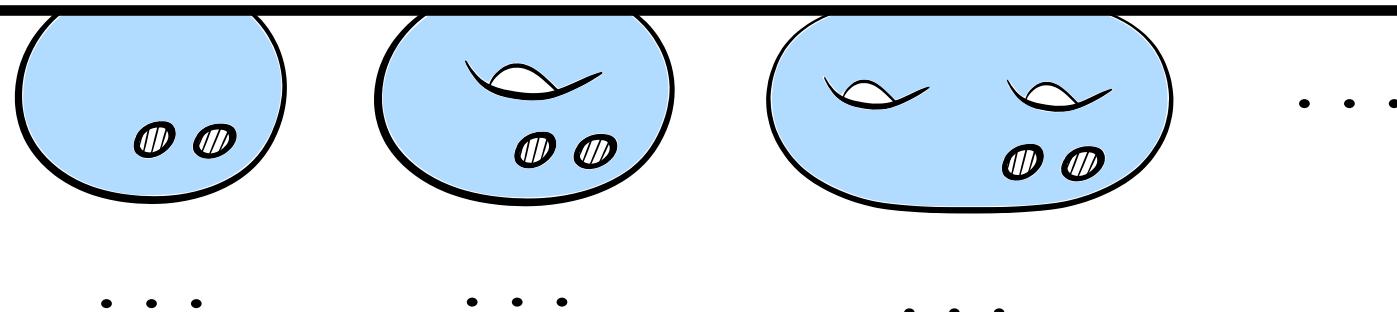
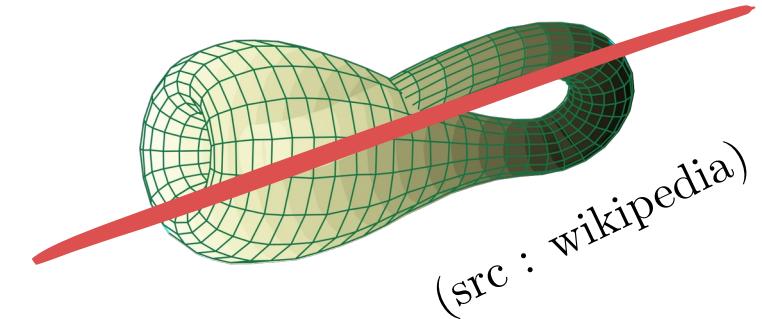
# Topological Surfaces



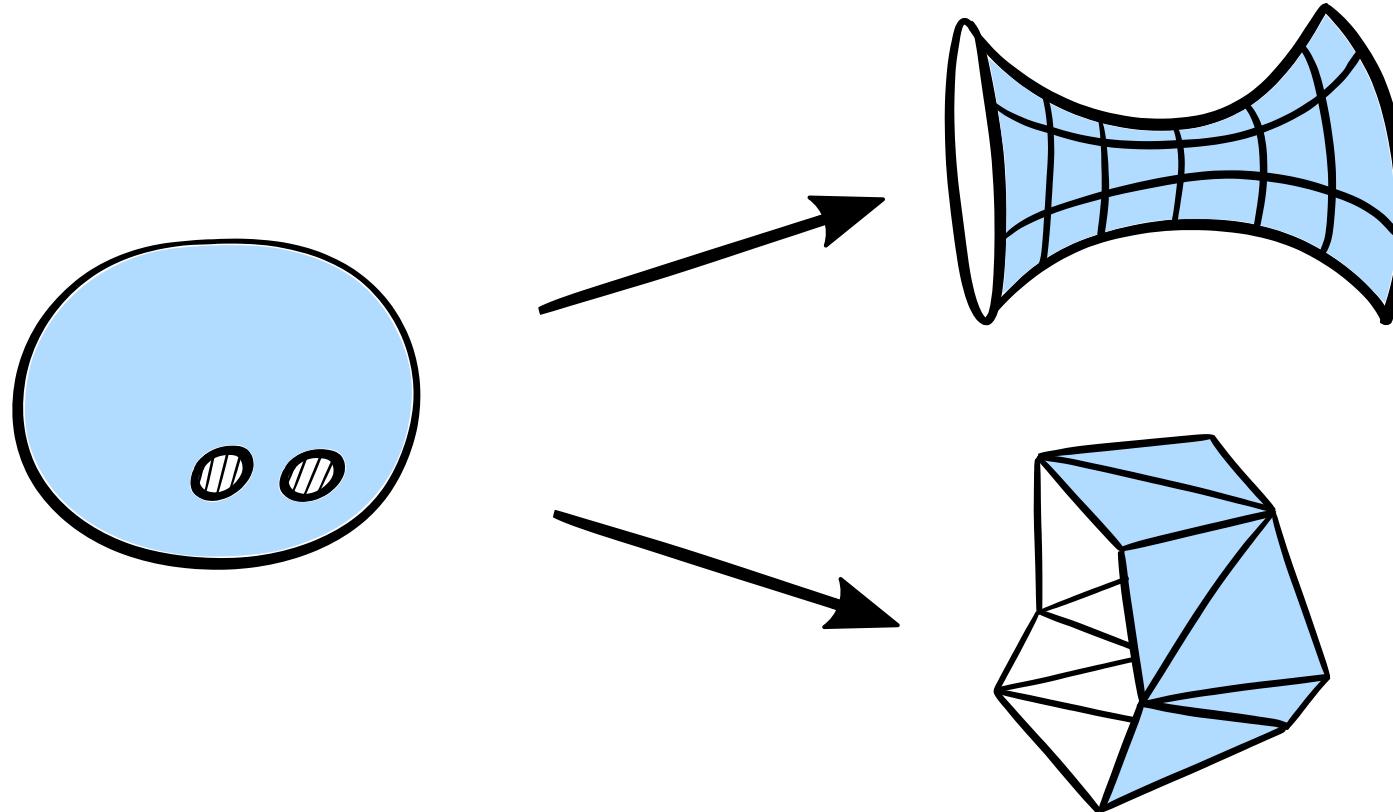
# Topological Surfaces



Only **orientable** surfaces today !



# Metrics on surfaces



# Untangling Graphs

# Computing Delaunay Triangulations

# Other works and conclusion

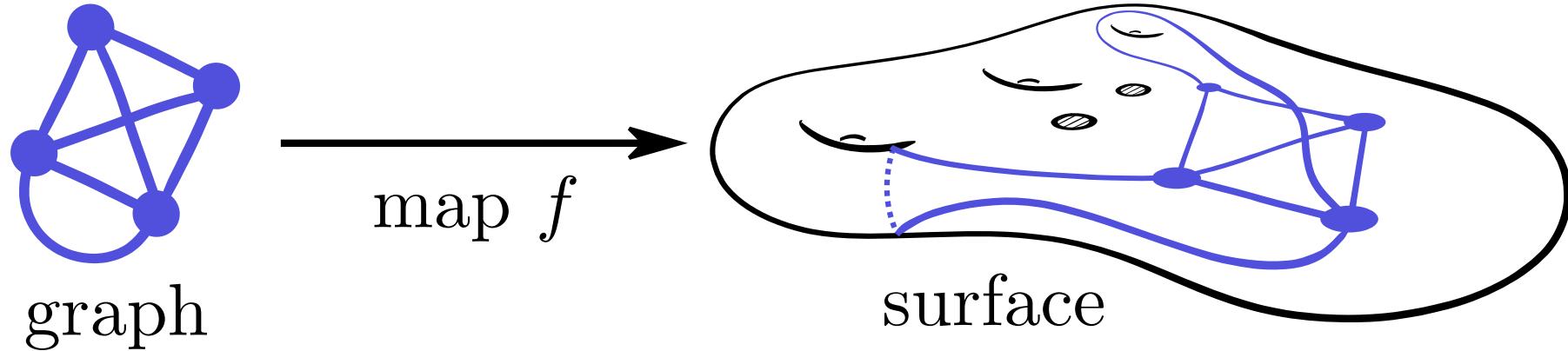
# Untangling Graphs

Computing Delaunay Triangulations

Other works and conclusion

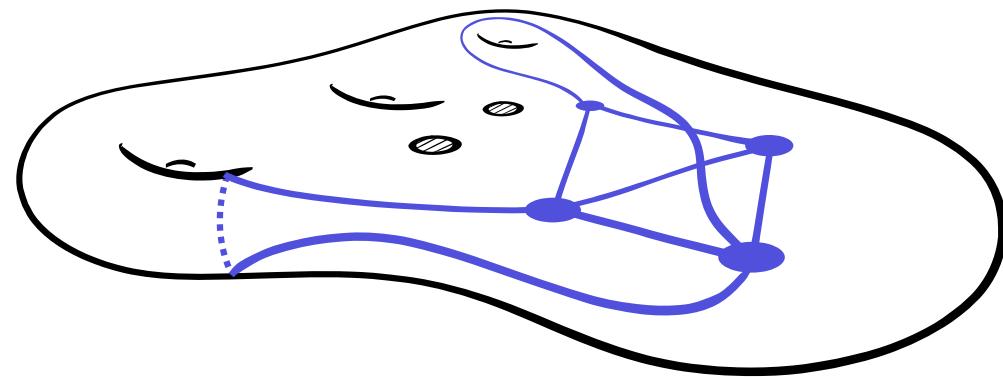
# Problem: untangling graphs

Input:

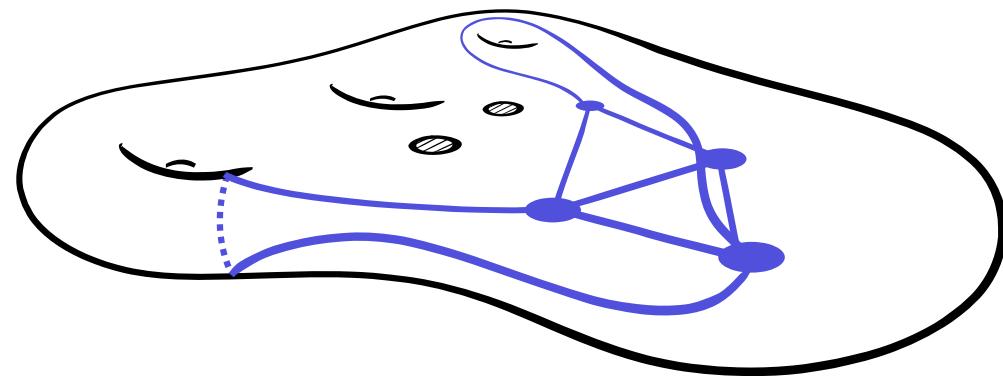


Goal: remove all crossings by deforming  $f$

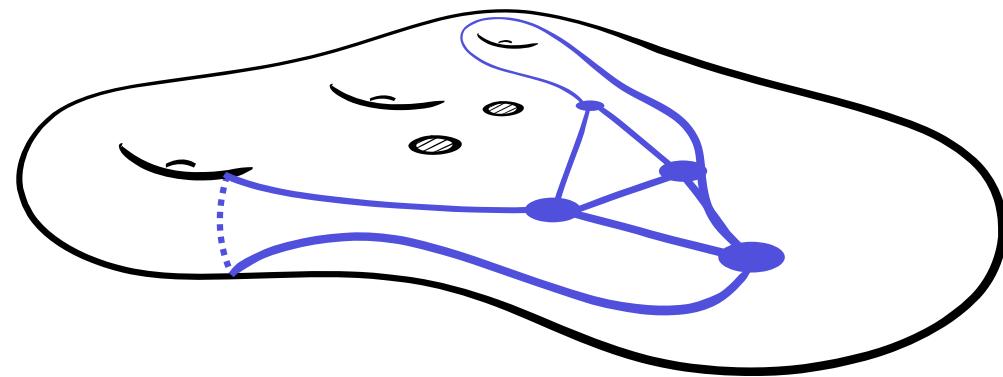
# Problem: untangling graphs



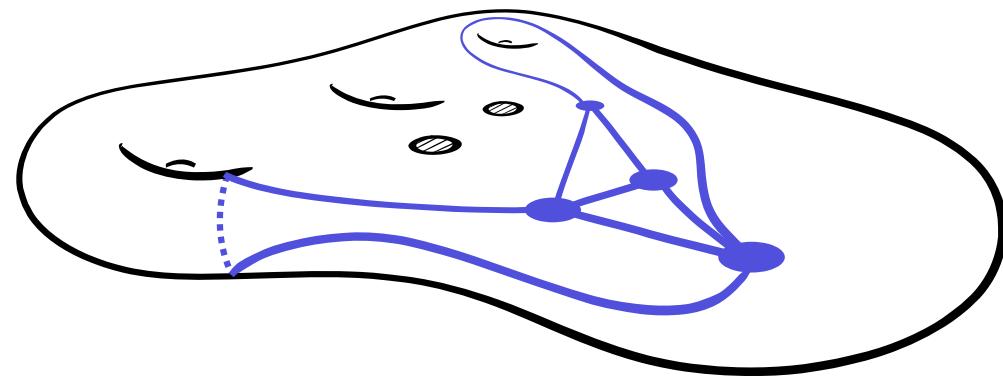
# Problem: untangling graphs



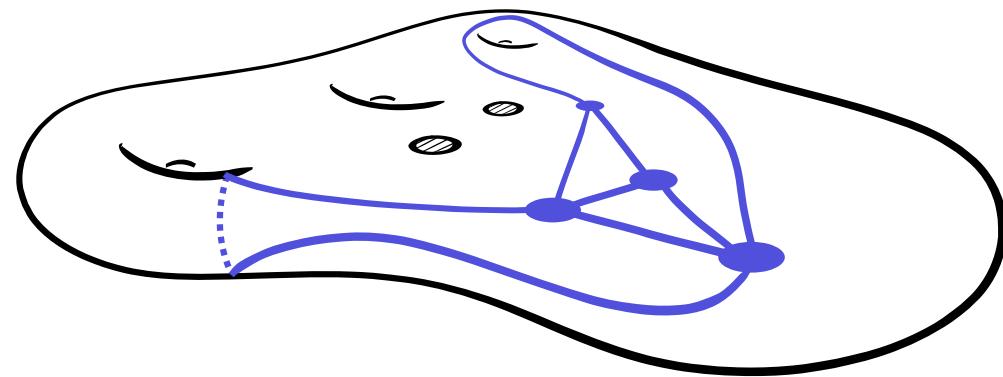
# Problem: untangling graphs



# Problem: untangling graphs



# Problem: untangling graphs

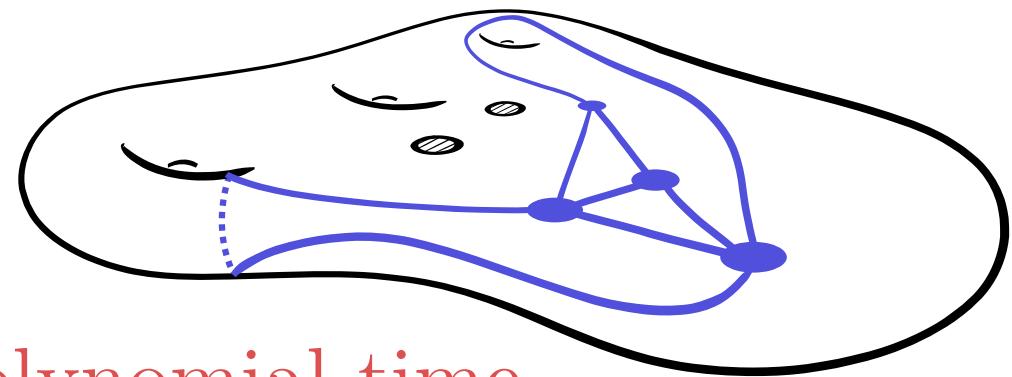


# Problem: untangling graphs



Output: Yes (+ untangled drawing) or No

# Problem: untangling graphs

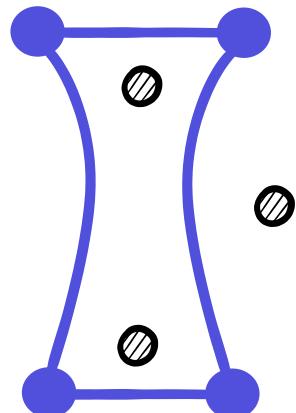


We obtain the first polynomial time  
algorithms for this problem

Output: Yes (+ untangled drawing) or No

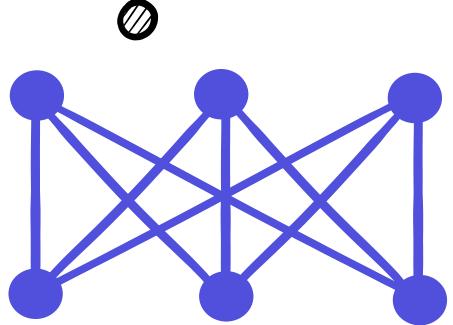


Yes:

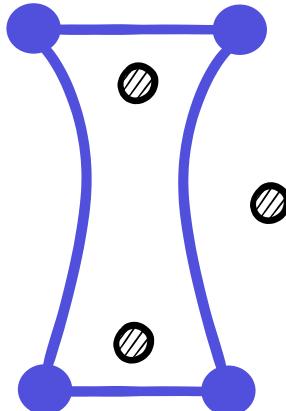


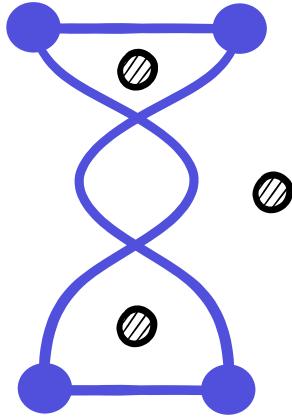


Yes:

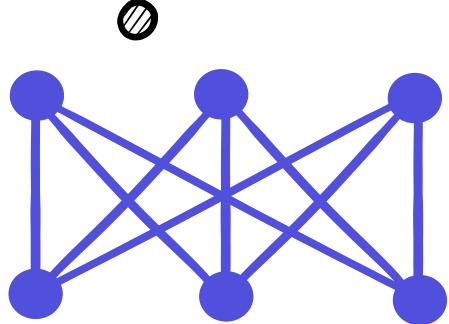


No





Yes:



No

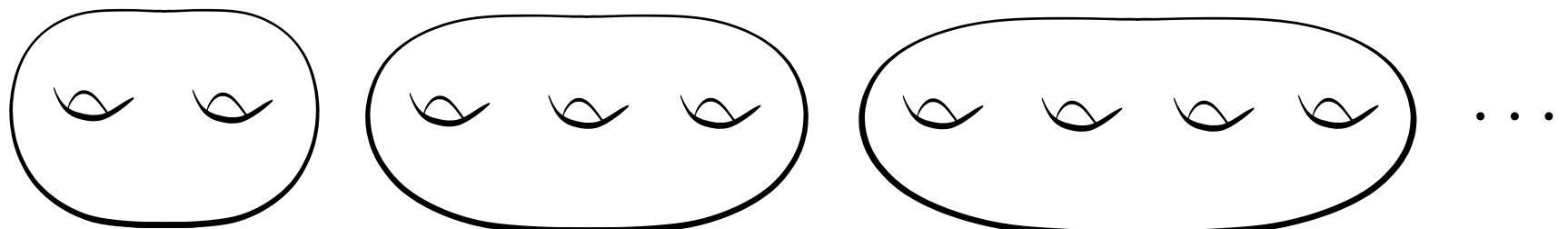


No



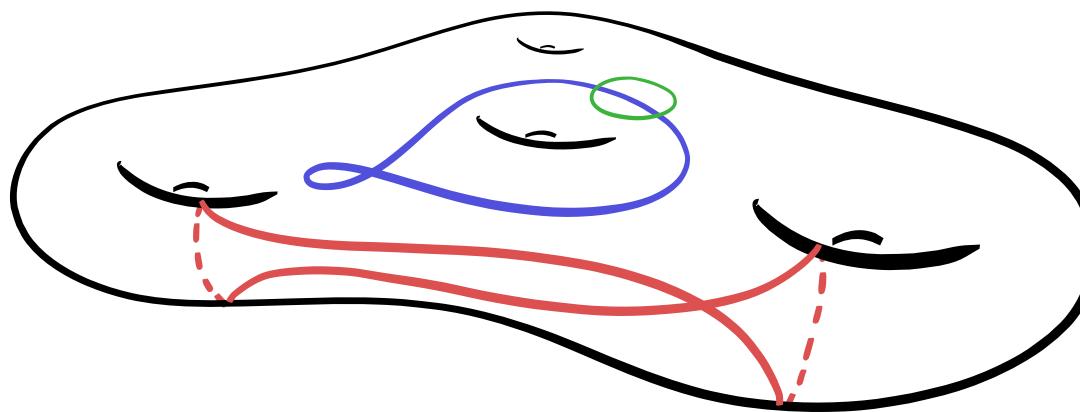
## Related works

We focus on surfaces without boundary  
of genus  $\geq 2$



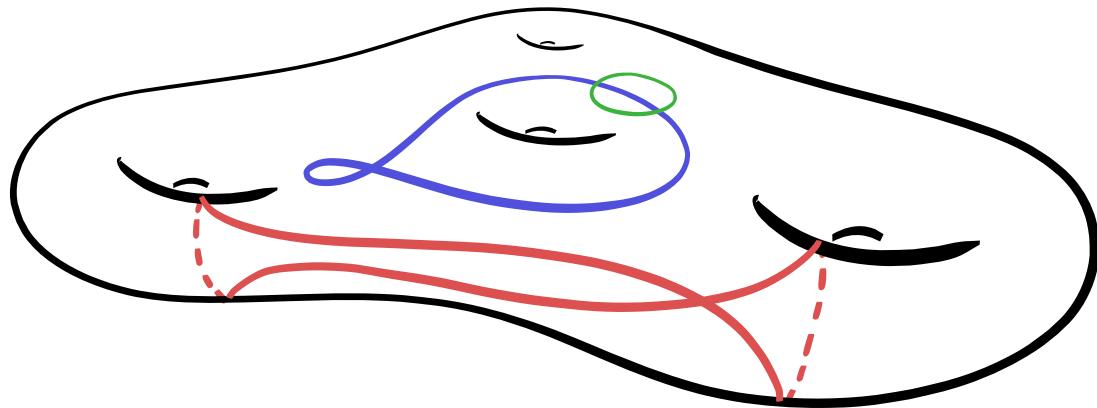
# Related problem: making curves cross minimally

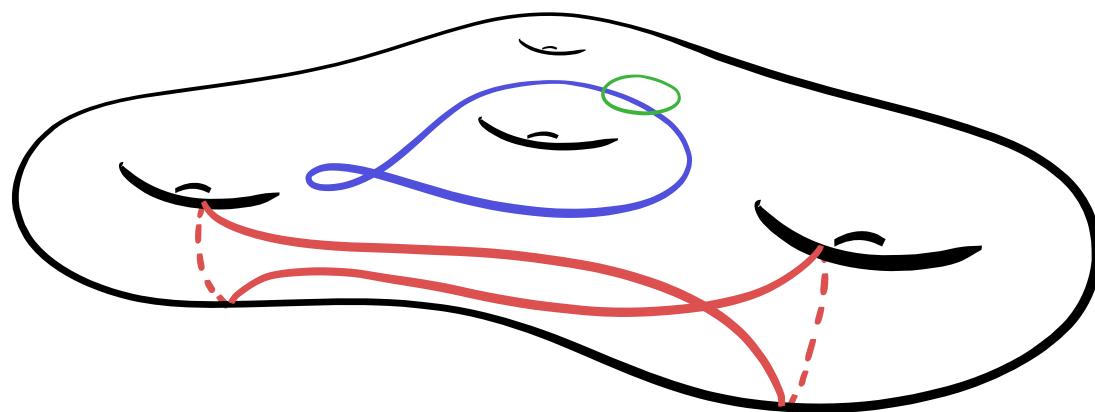
Input: closed curves on a surface

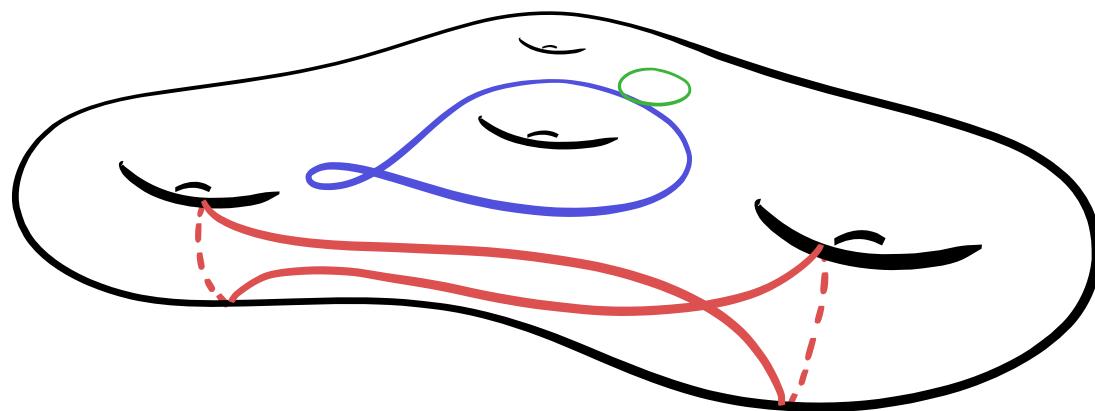


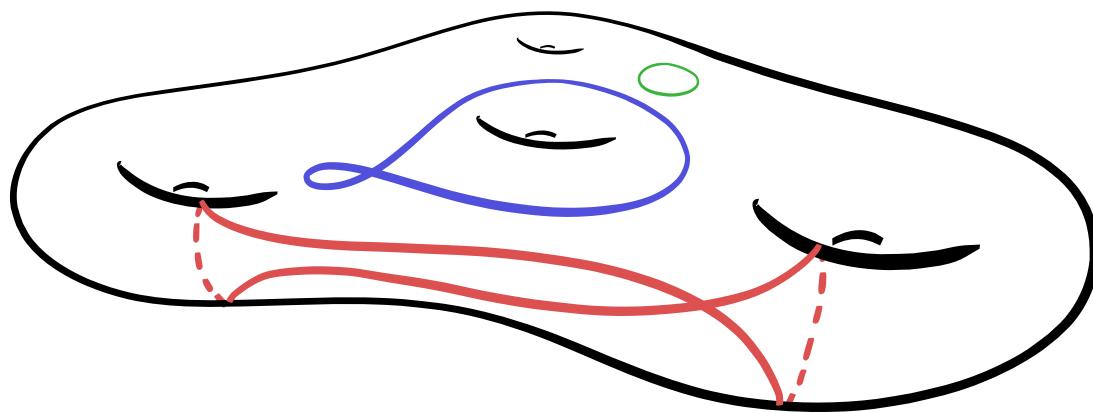
Goal:

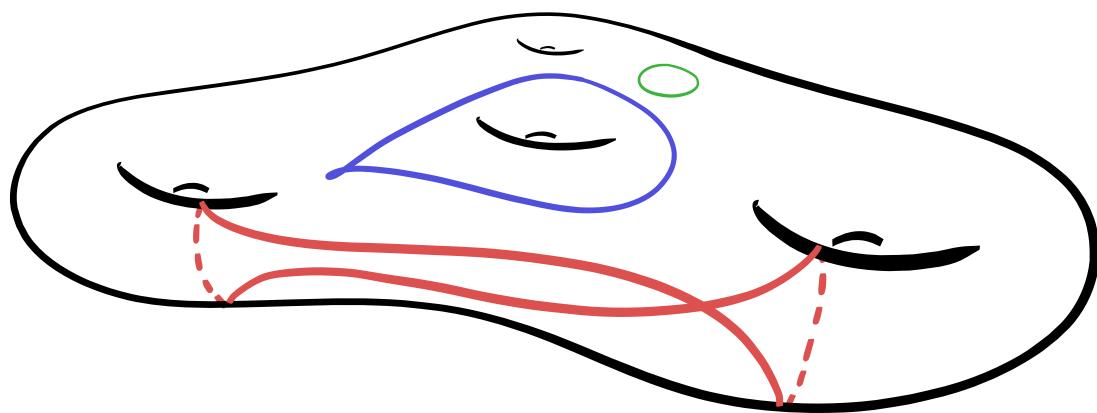
minimize the # crossings by deforming the curves

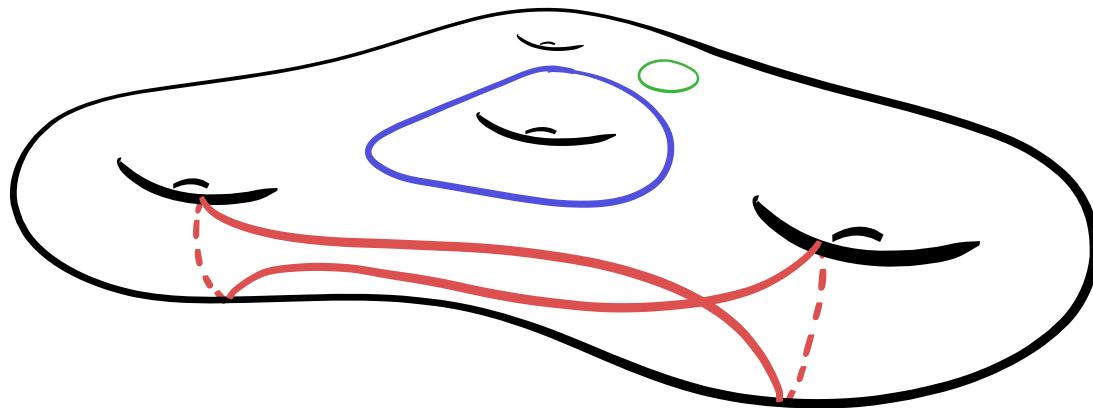


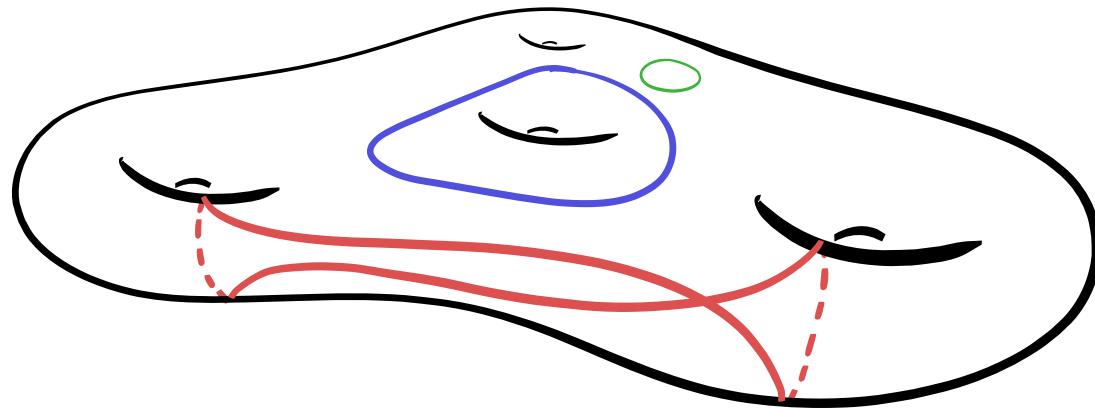












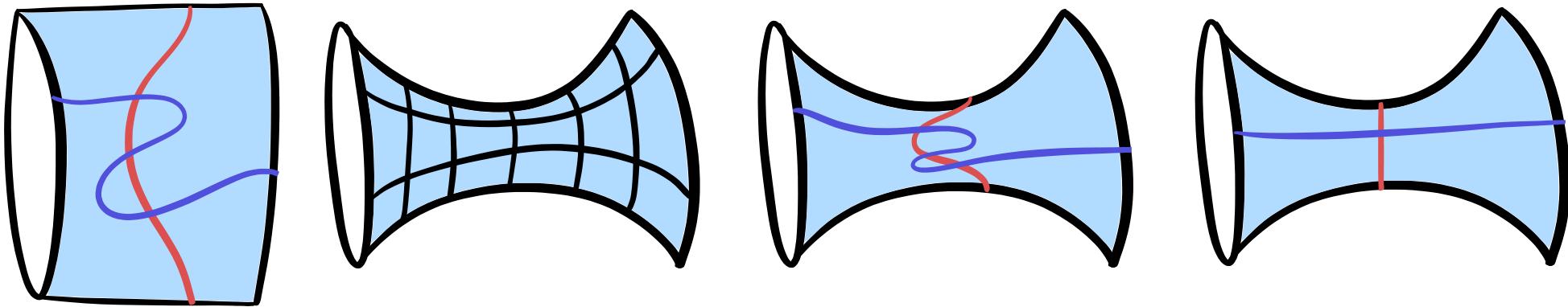
Output:  $\min \#$  of crossings (+ optimal curves)

# Many works related to making curves cross minimally!

- |                         |                               |
|-------------------------|-------------------------------|
| Poincaré, 1905          | de Graaf and Schrijver, 1987  |
| Dehn, 1911              | Dynnikov, 2002                |
| Dehn, 1912              | Paterson, 2002                |
| Reinhart, 1962          | Gonçalves et al., 2005        |
| Zieschang, 1965         | Schaefer et al., 2008         |
| Chillingworth, 1969     | Lazarus and Rivaud, 2012      |
| Zieschang, 1969         | Erickson and Whittlesey, 2013 |
| Chillingworth, 1971     | Arettines, 2015               |
| Turaev, 1979            | Chang et al., 2018            |
| Birman and Series, 1984 | Despré and Lazarus, 2019      |
| Cohen and Lustig, 1984  | Fulek and Tóth, 2020          |
| Hass and Scott, 1985    | Chang and de Mesmay, 2022     |
| Lustig, 1987            | Lackenby, 2024                |

# Method for making curves cross minimally

Poincaré, 1905



1. give special shape to surface
2. straighten the curves

# The special shape

negative curvature:

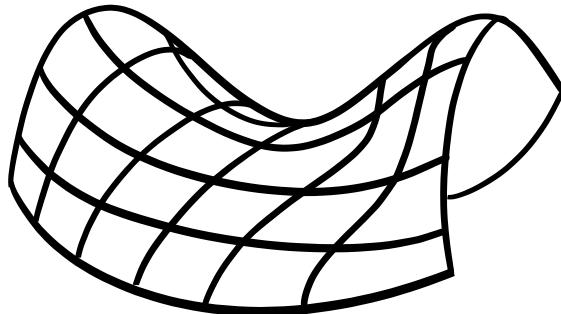


image by Susan Lombardo

almost all surfaces  
can be curved negatively

# The key property

On a negatively curved surface,

**straight** curves cross minimally

(does not hold  
on all surfaces)



# The key property

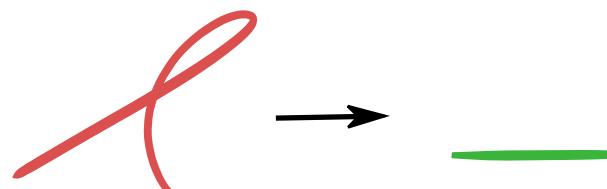
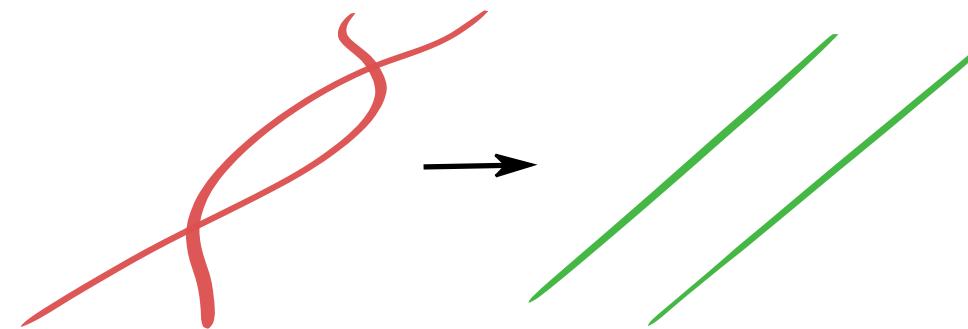
On a negatively curved surface,

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# The key property

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straight curves cross minimally



# The key property

On a negatively curved surface,

straight curves cross minimally

# The key property

On a negatively curved surface,

straight curves cross minimally



ever  
into

# The key property

On a negatively curved surface,

straight curves cross minimally



every path  
into a unit

# The key property

On a negatively curved surface,

straight curves cross minimally



every path can be  
into a unique straight curve

# The key property

On a negatively curved surface,

short curves cross minimally



every path can be deformed  
into a unique straight pa

# The key property

On a negatively curved surface,

cross minimally



every path can be deformed  
into a unique **straight** path

# The key property

On a negatively curved surface,

minimally



every path can be deformed  
into a unique **straight** path

# The key property

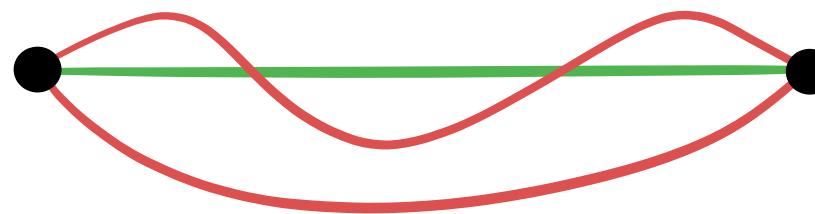
On a negatively curved surface,

every path can be deformed  
into a unique **straight** path

# The key property

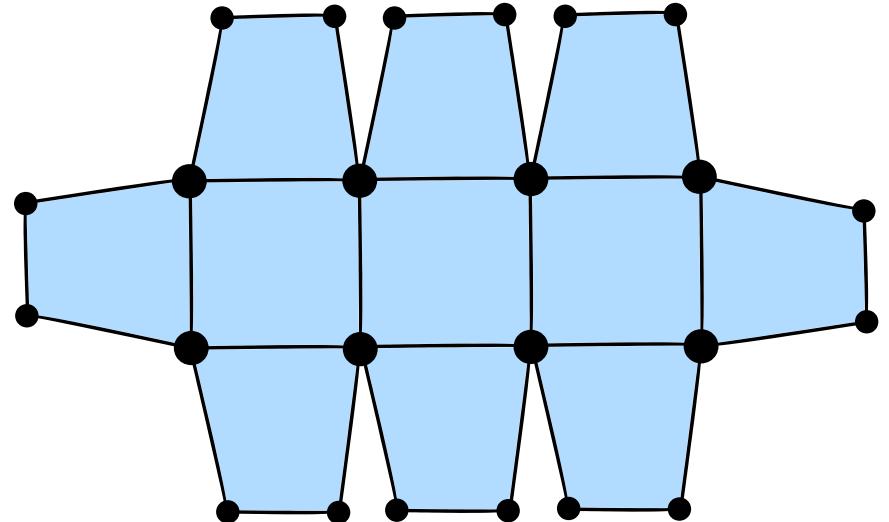
On a negatively curved surface,

every path can be deformed  
into a unique **straight** path

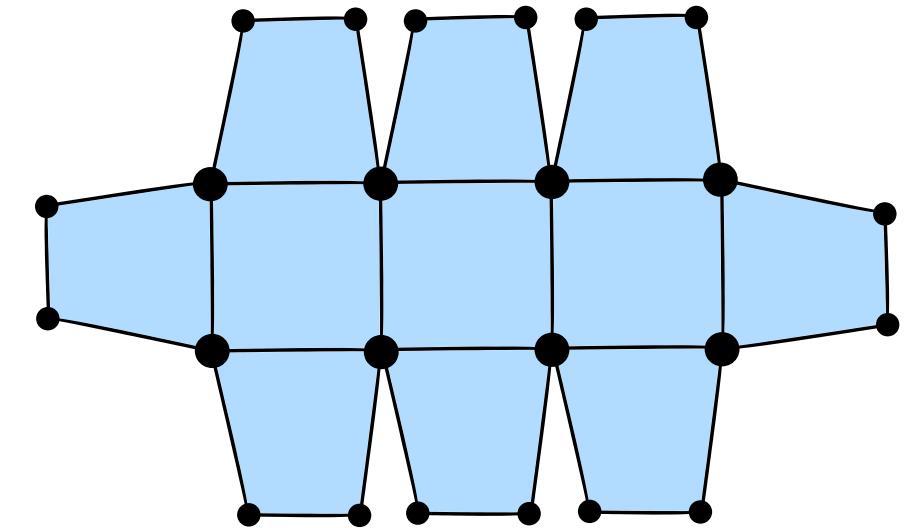


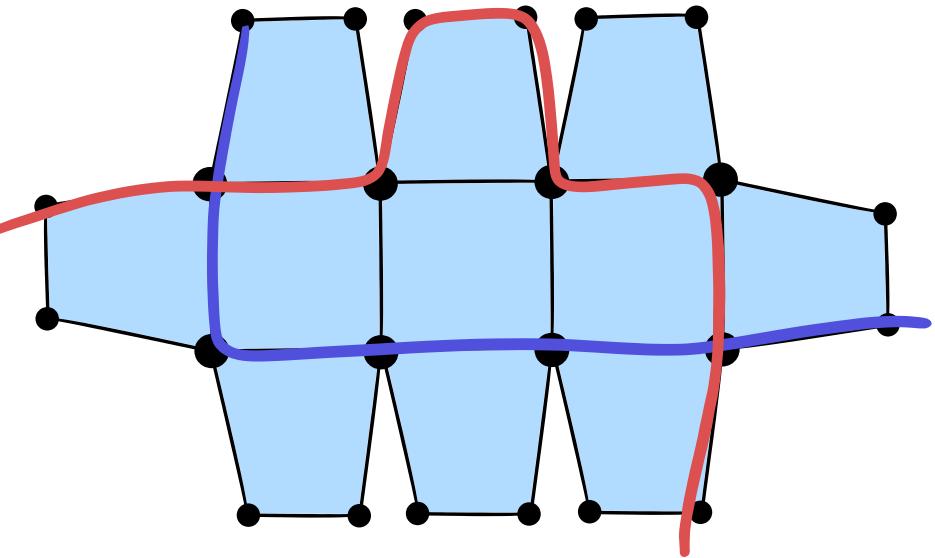
Discrete model of negatively  
curved surfaces?

Lazarus and Rivaud, 2012  
Erickson and Whittlesey, 2013

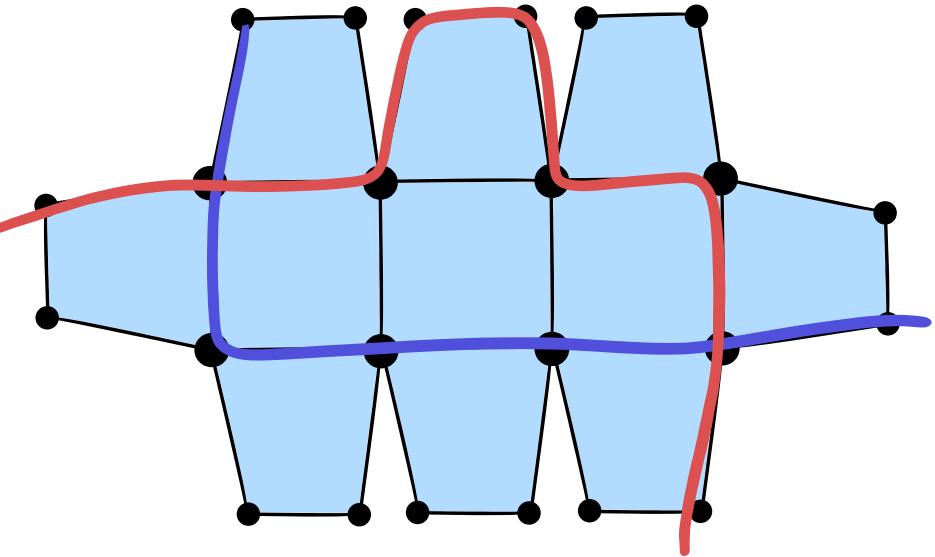


System of quads

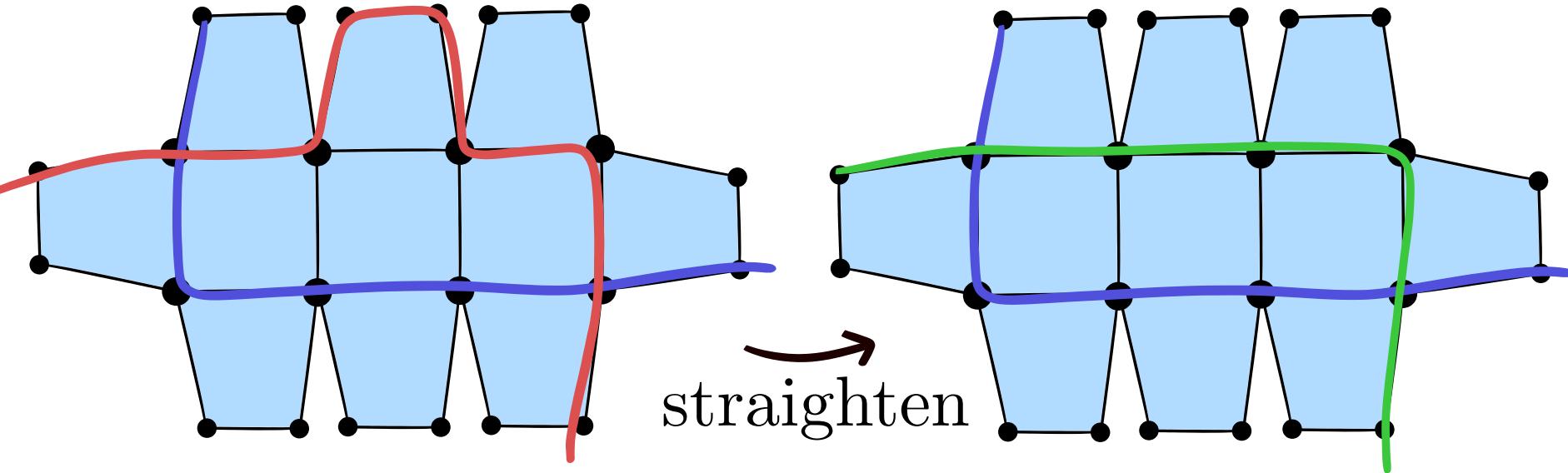




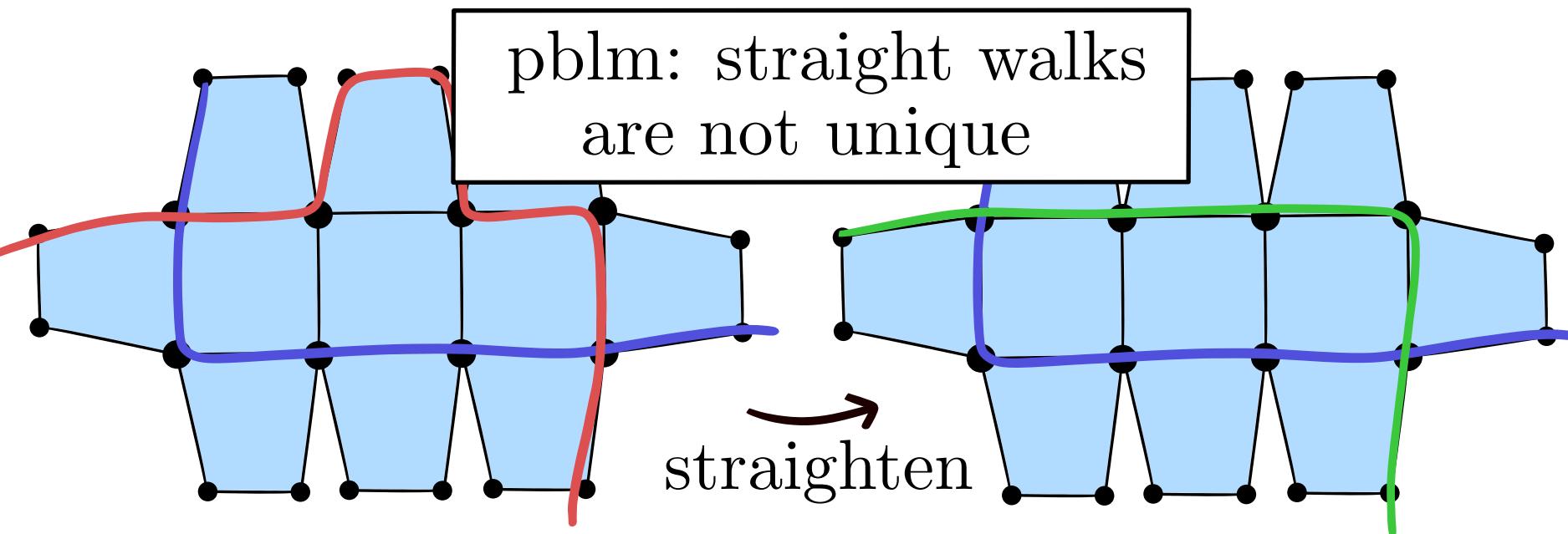
# Algo for making curves cross minimally



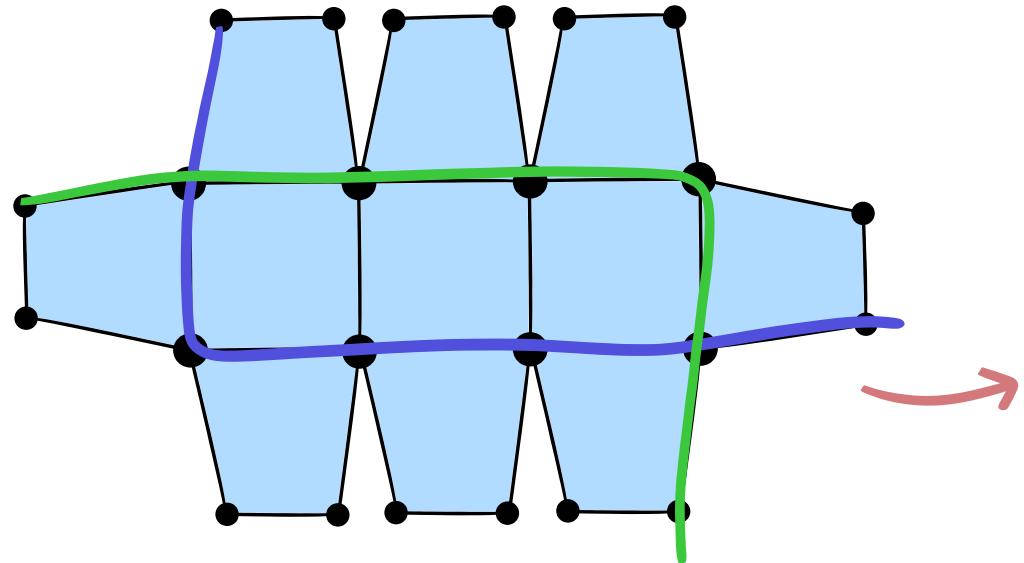
# Algo for making curves cross minimally



# Algo for making curves cross minimally



# Algo for making curves cross minimally



Despré and Lazarus, 2019

What about  
untangling graphs?

# Method for untangling graphs

Tutte, 1963

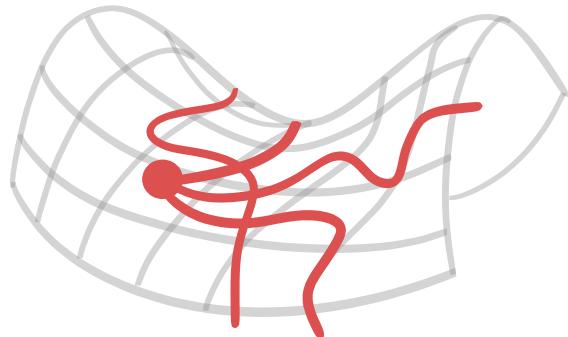
Y. Colin de Verdière, 1991



# Method for untangling graphs

Tutte, 1963

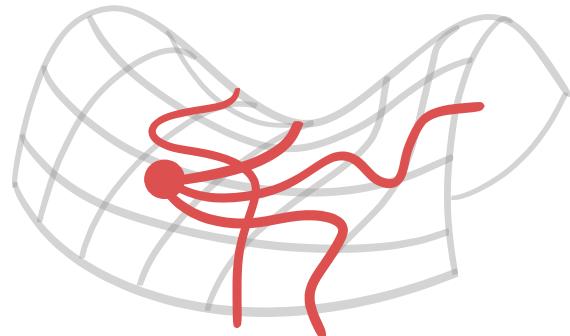
Y. Colin de Verdière, 1991



# Method for untangling graphs

Tutte, 1963

Y. Colin de Verdière, 1991



make edges straight

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Tutte, 1963

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make edges straight  
make vertices barycentric

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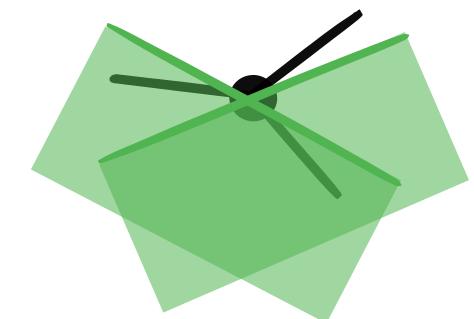
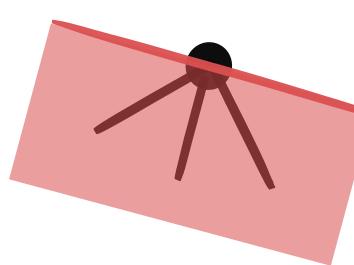
# Method for untangling graphs

Tutte, 1963

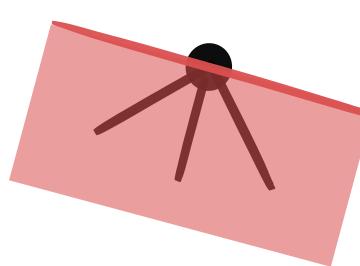


Y. Colin de Verdière, 1991

make edges straight  
make vertices barycentric

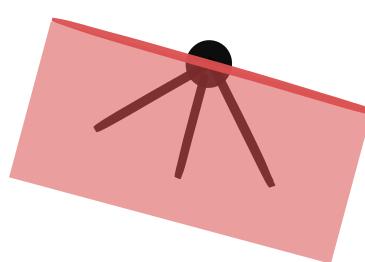


edges straight  
vertices barycentric



# Tutte embeddings

edges straight  
vertices barycentric



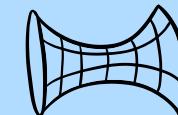
# Summary

Curves

Graphs

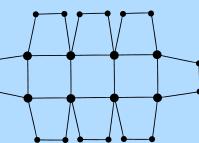
Method

negatively curved surface

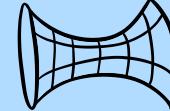


Algo

system of quads



# Our results

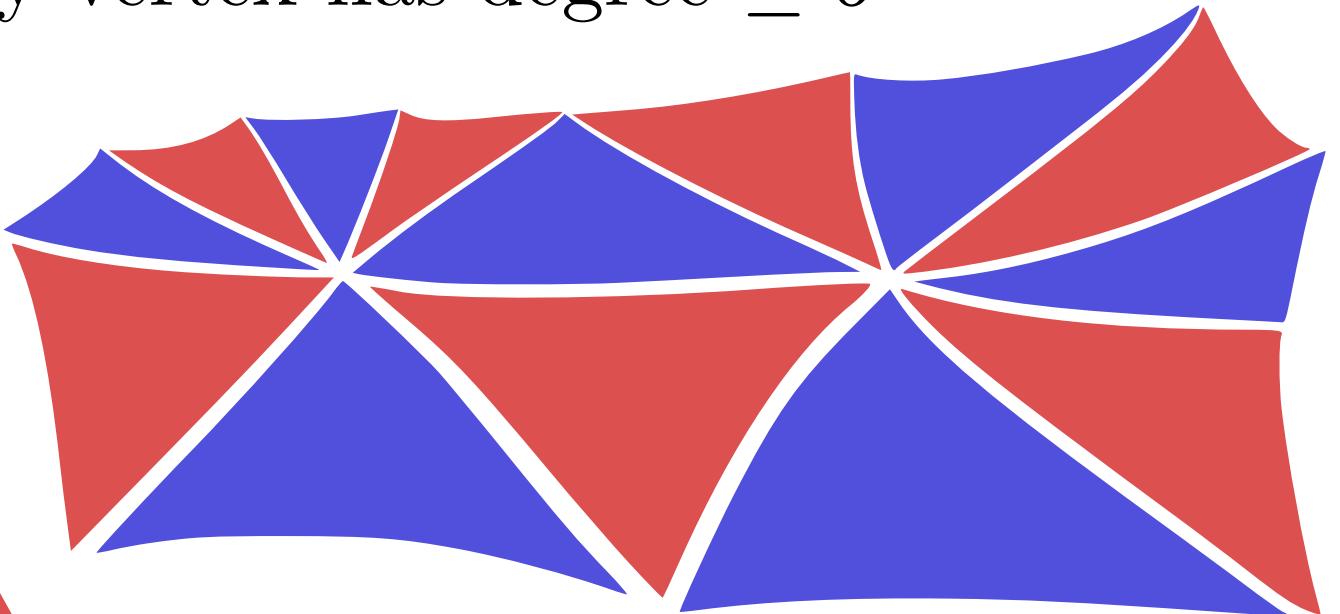
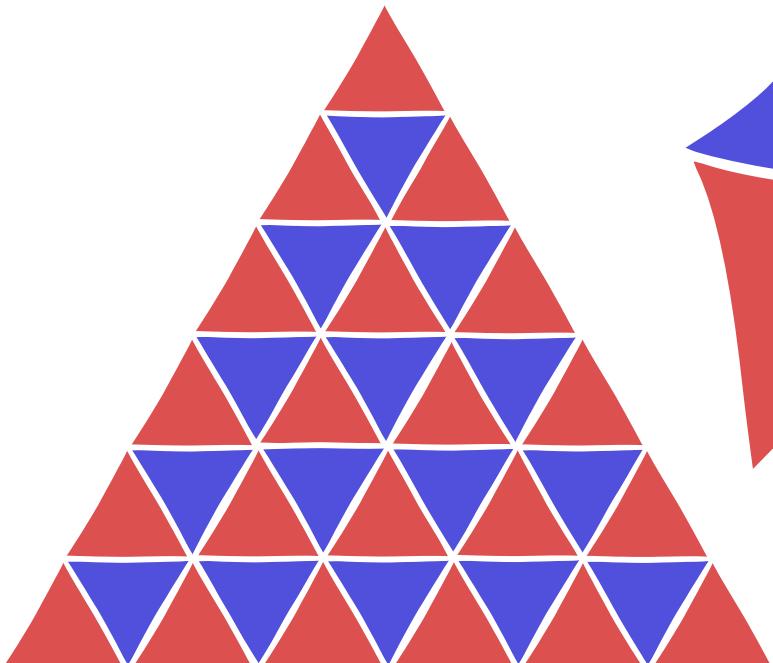
	Curves	Graphs
Method	negatively curved surface 	
Algo	reducing triangulations	
	improved algos for making curves cross minimally	first algos for untangling graphs
		Discrete analogue of Tutte embeddings

A new tool:

# Reducing triangulations

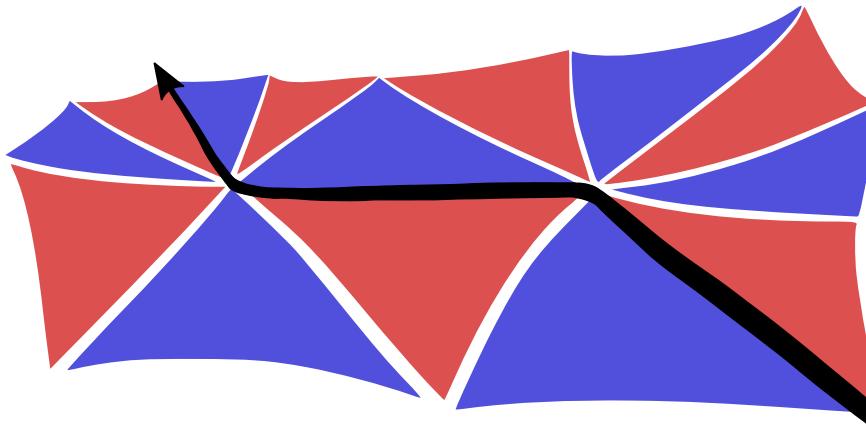
# Reducing triangulations

dual is bipartite and  
every vertex has degree  $\geq 6^*$

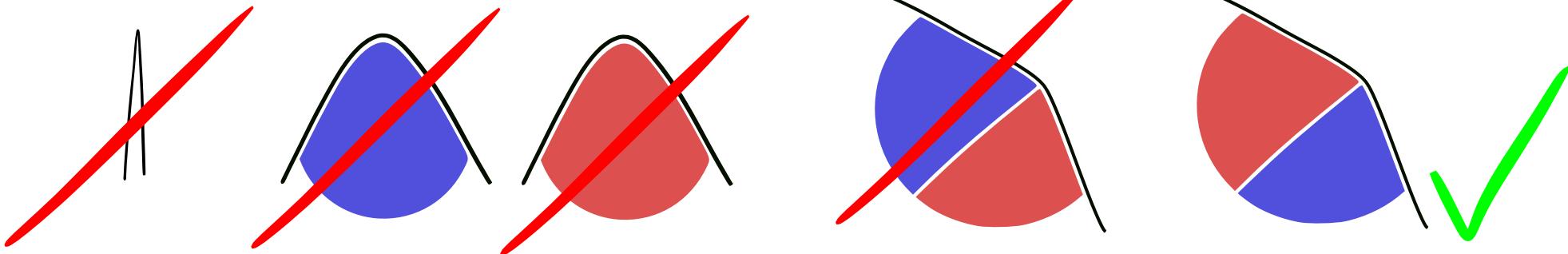


\*sometimes 8

# Reduced walks

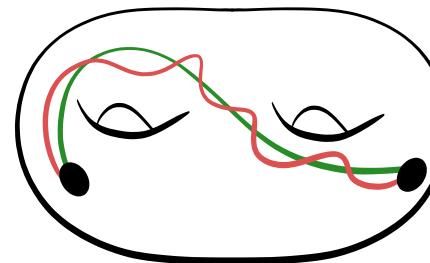
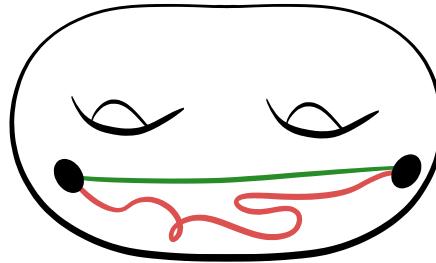


no bad turn



# Properties of reduced walks

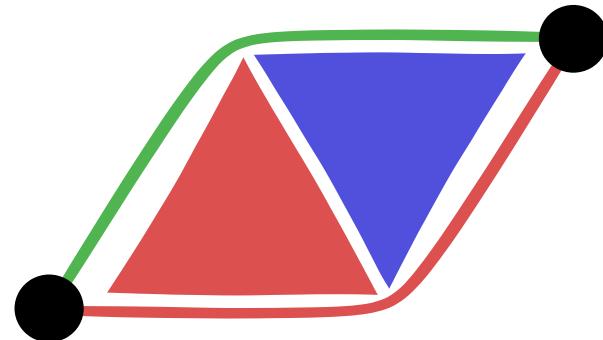
every walk can be deformed into a unique reduced walk, computable in linear time



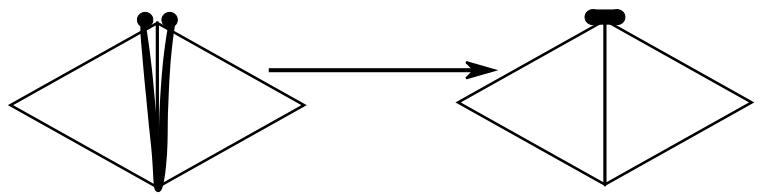
reduced walks are stable upon reversal and subwalk

## Purpose of the coloring

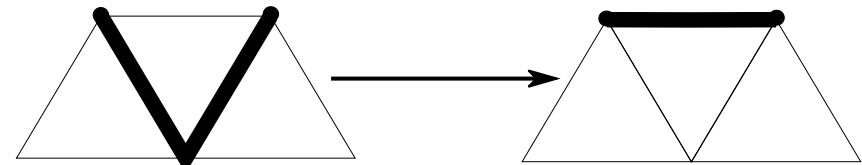
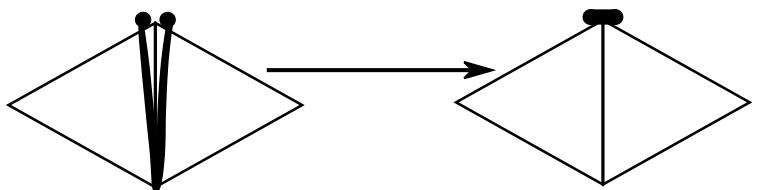
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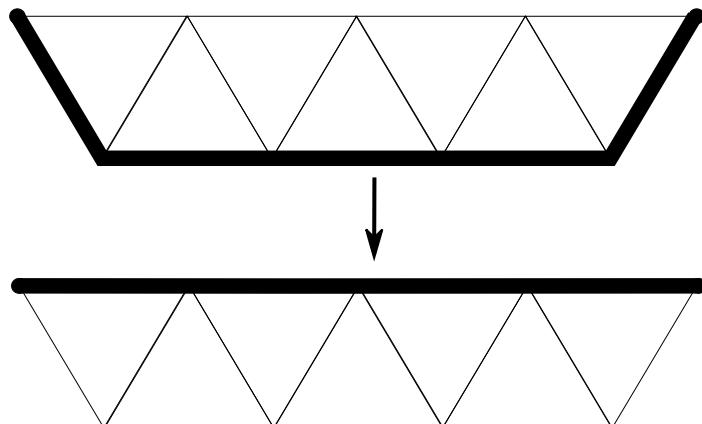
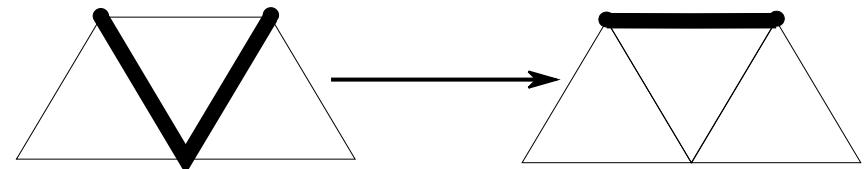
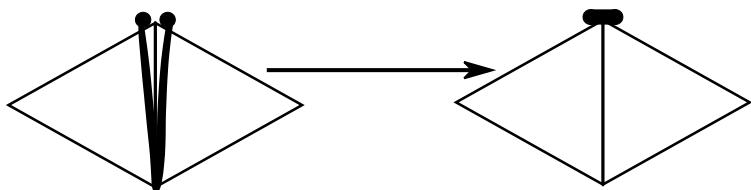
# Reducing a walk



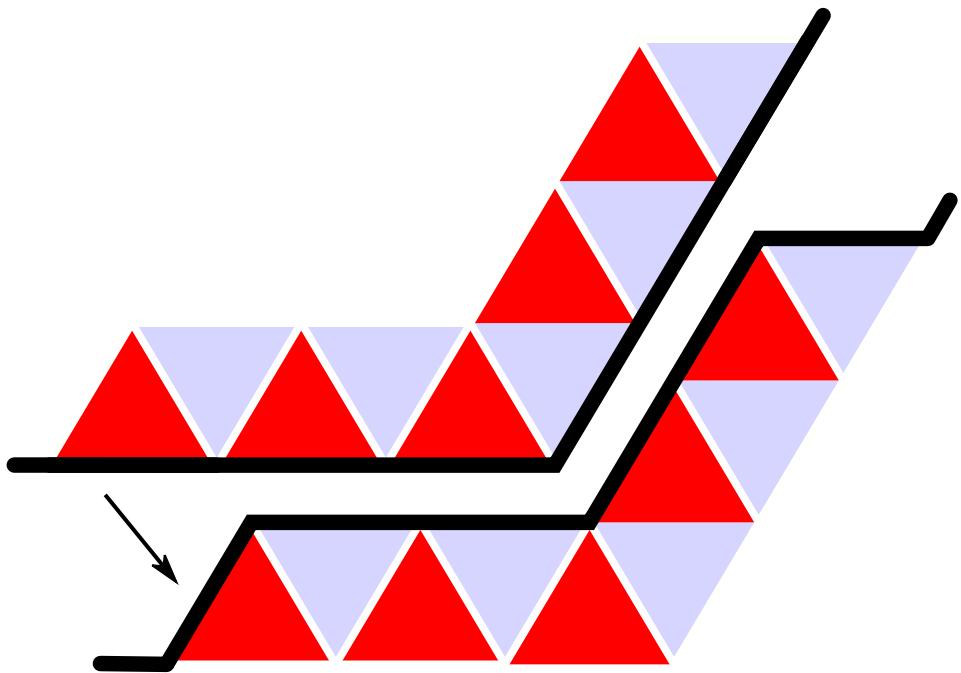
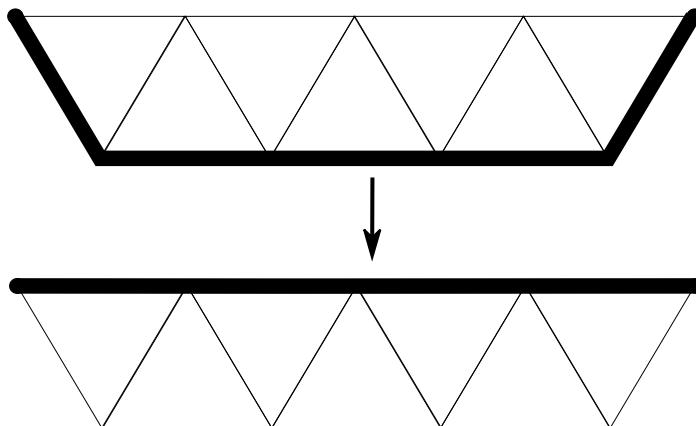
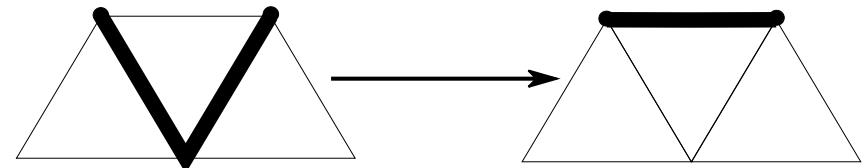
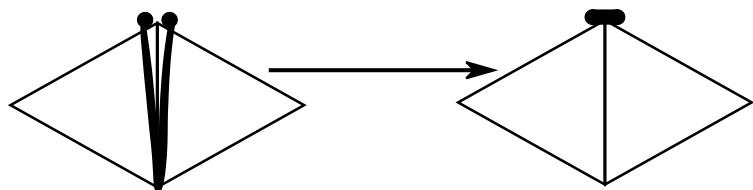
# Reducing a walk



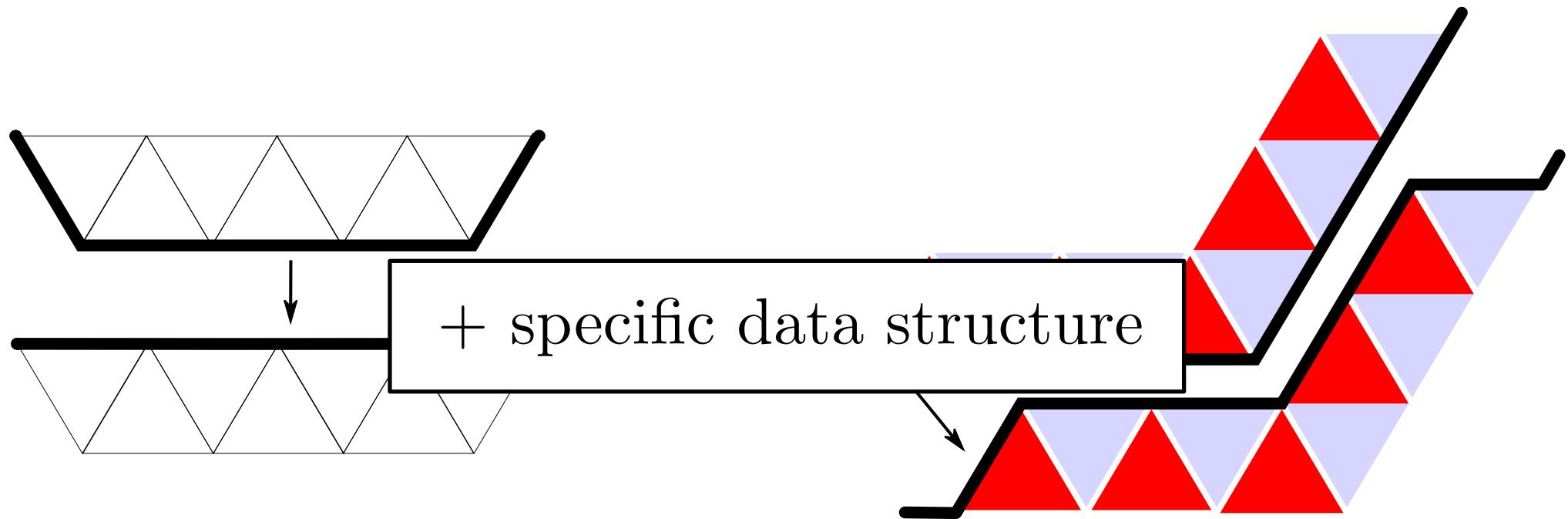
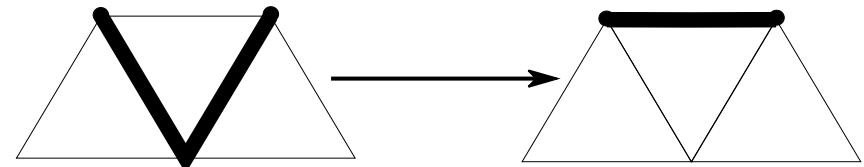
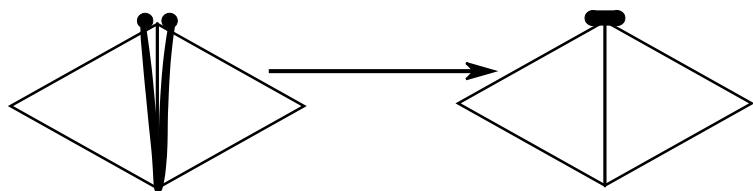
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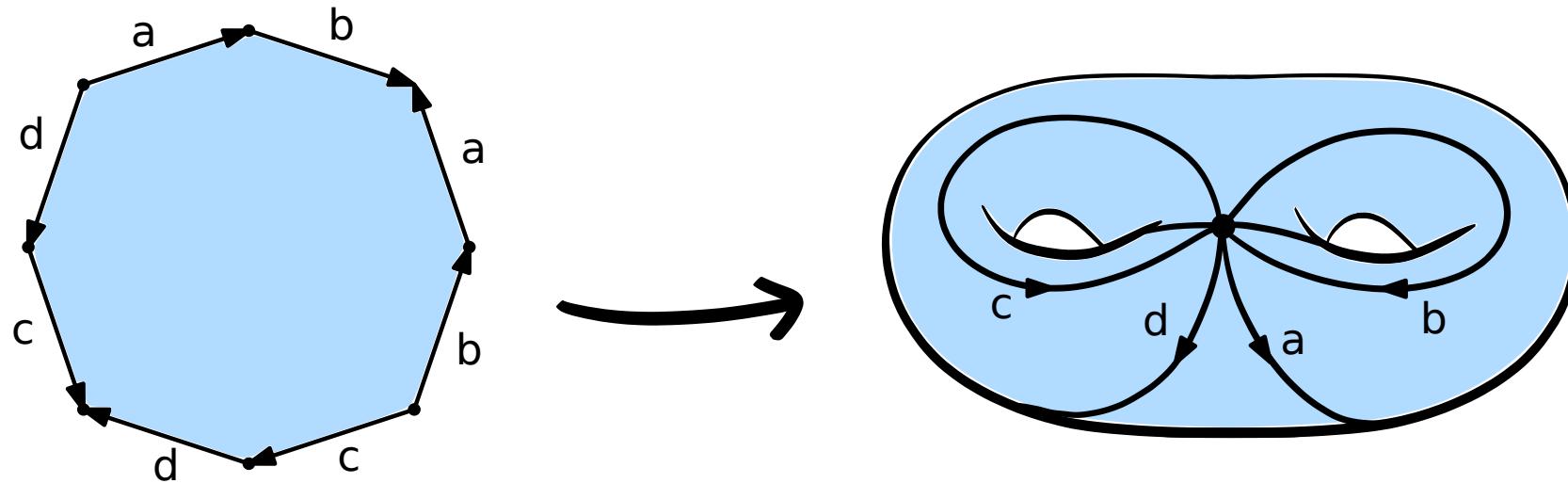
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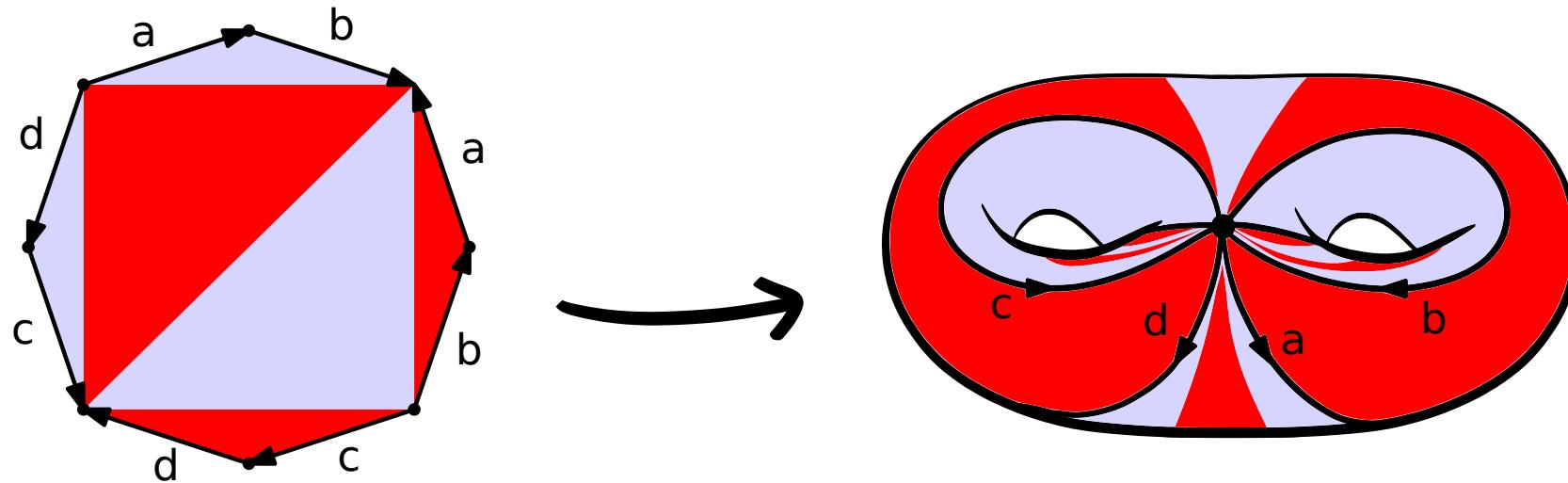
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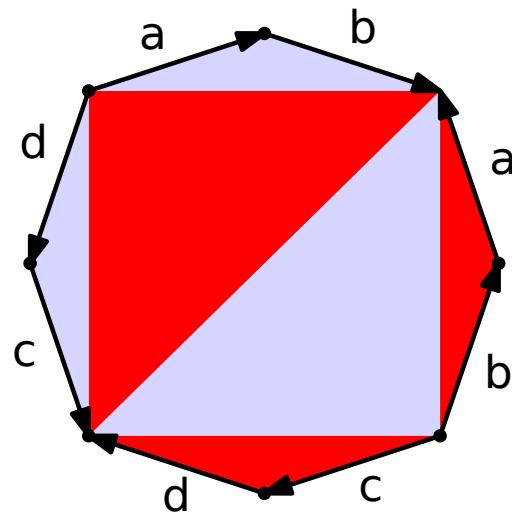
# Constructing reducing triangulations



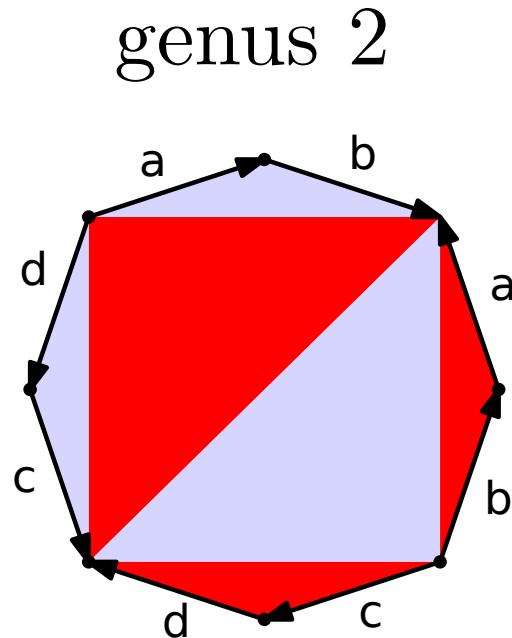
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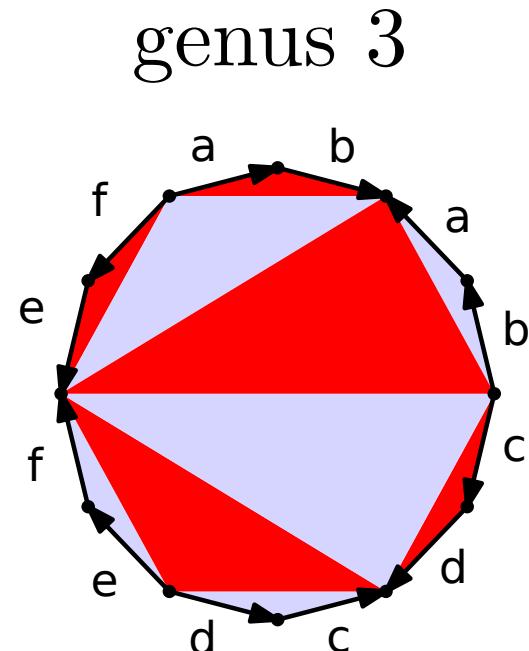
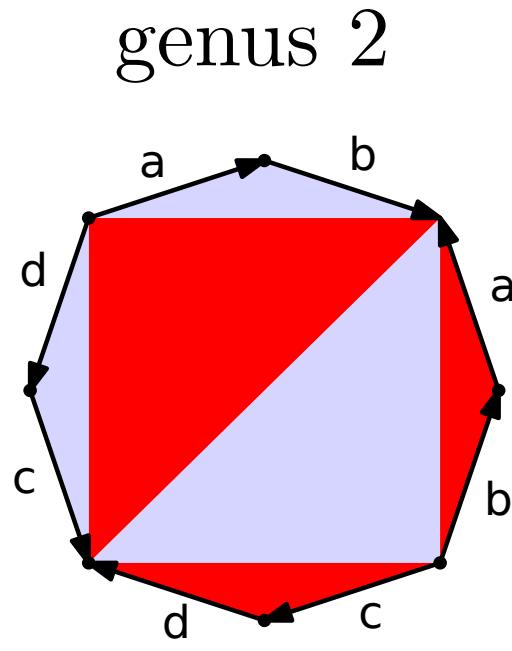
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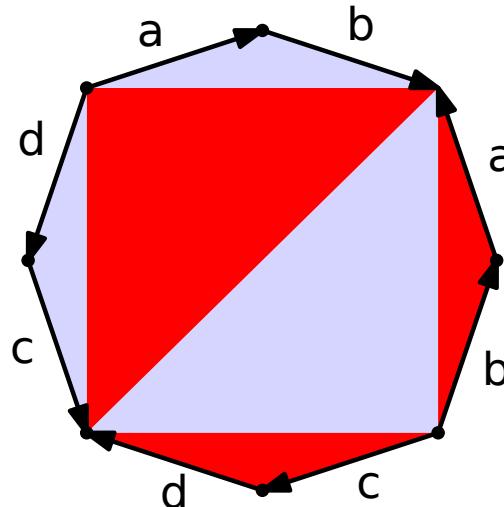


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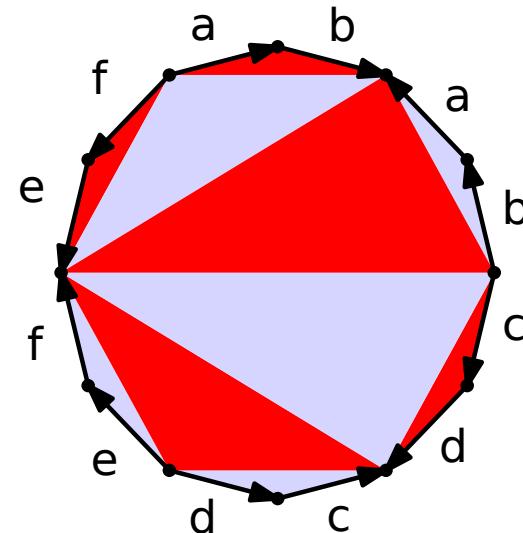


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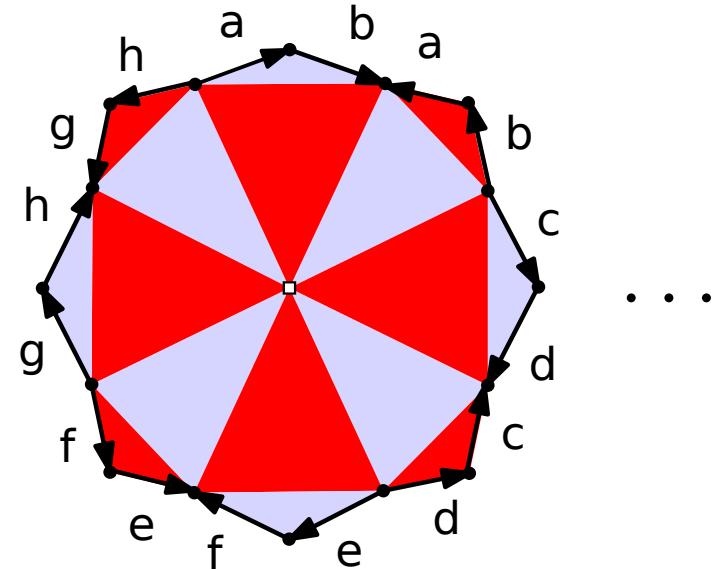
genus 2



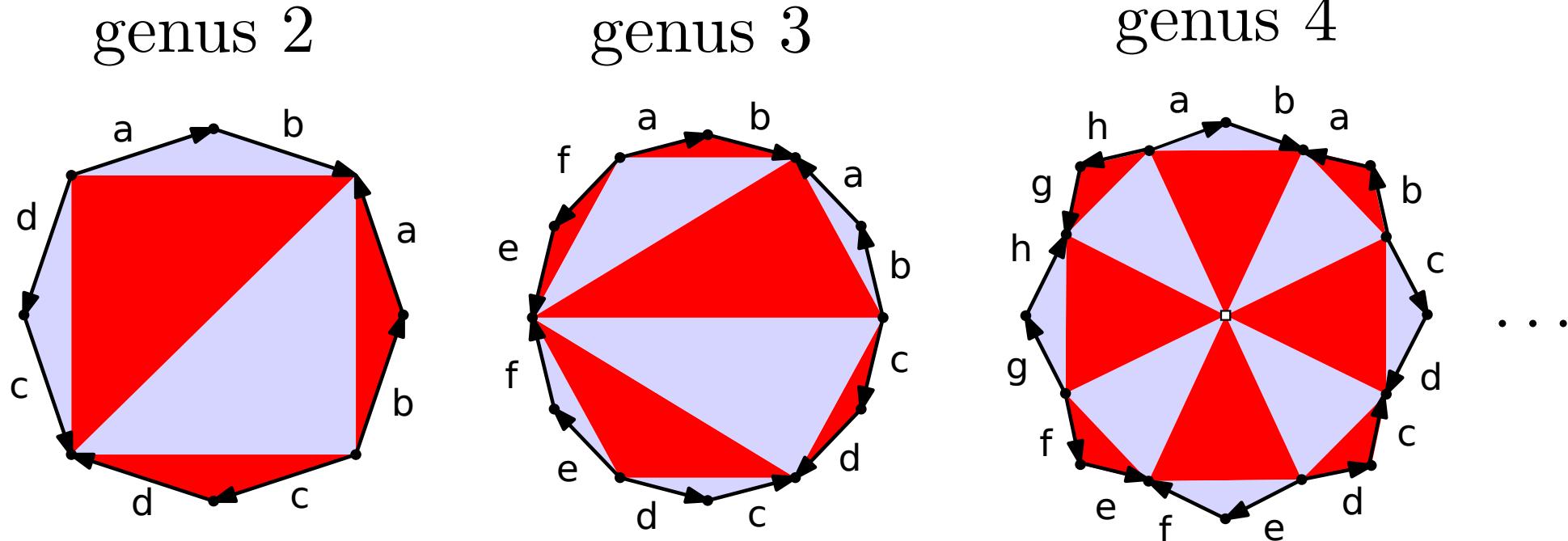
genus 3



genus 4



# Constructing reducing triangulations



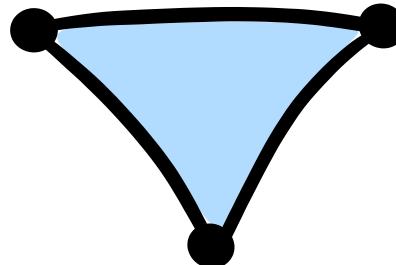
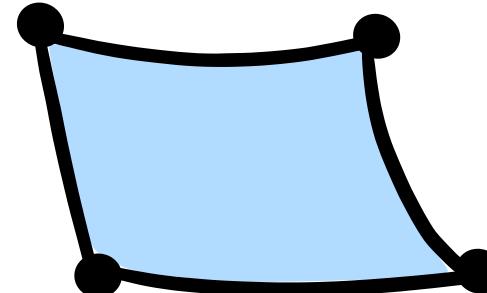
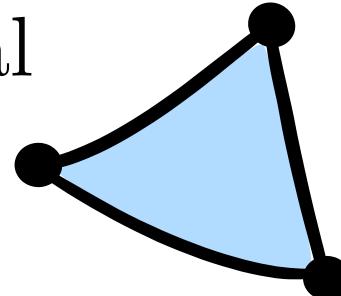
then subdivide at will:



Untangling graphs using  
reducing triangulations

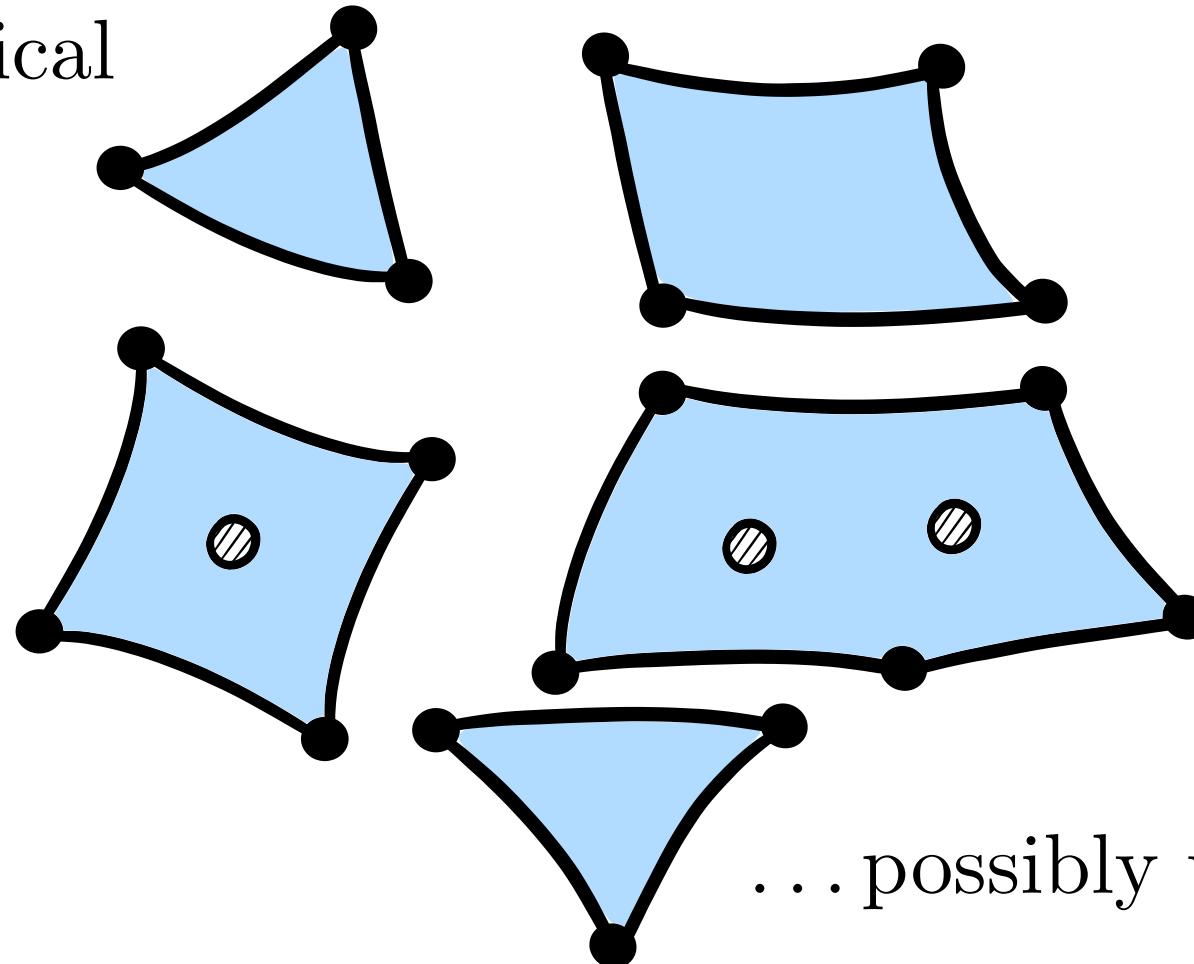
# Input

take topological  
polygons...



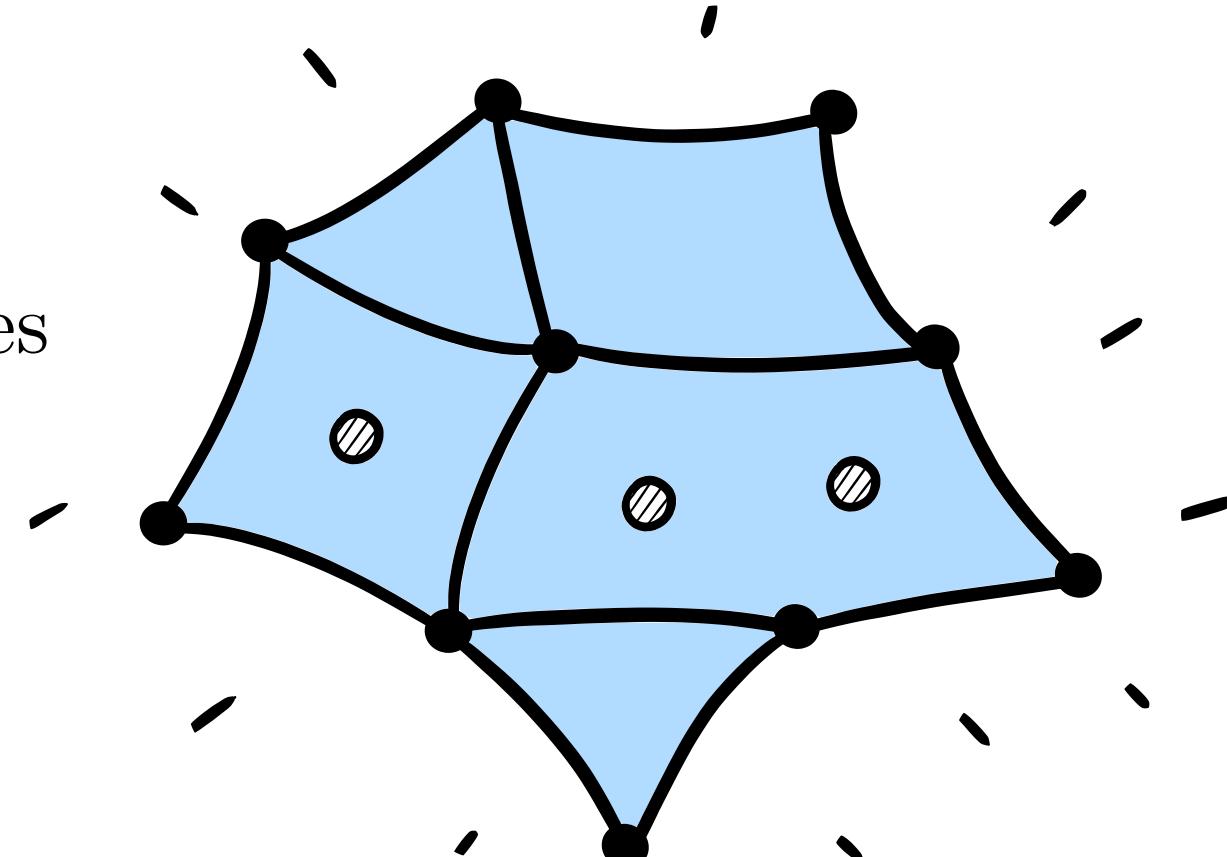
# Input

take topological  
polygons...



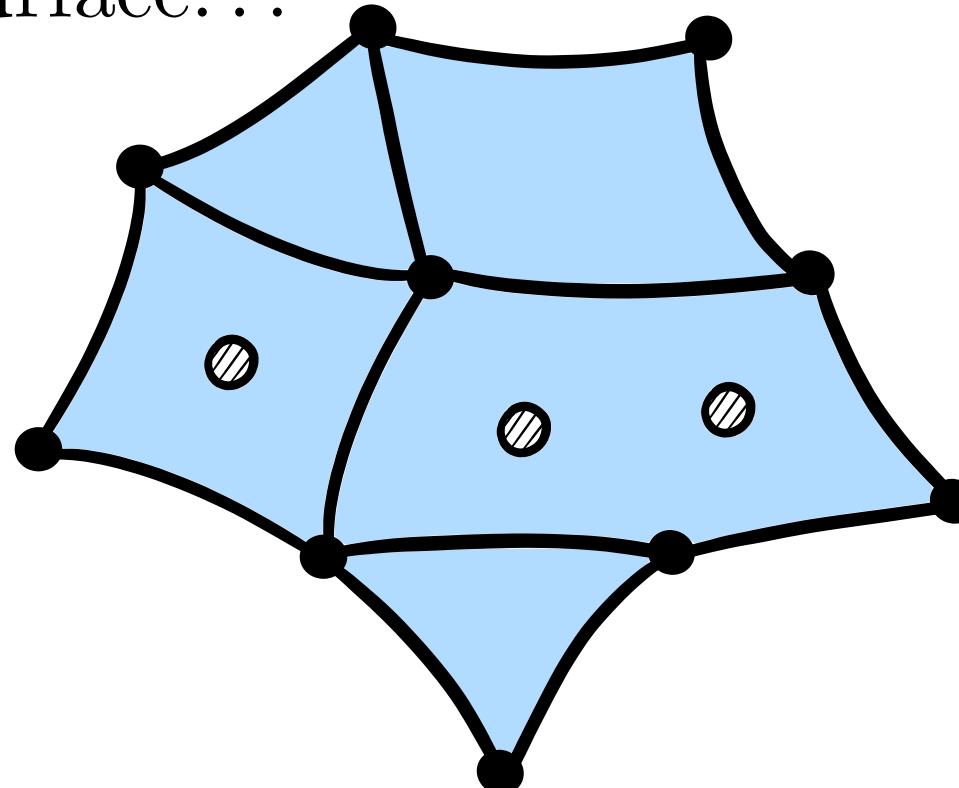
# Input

glue edges



# Input

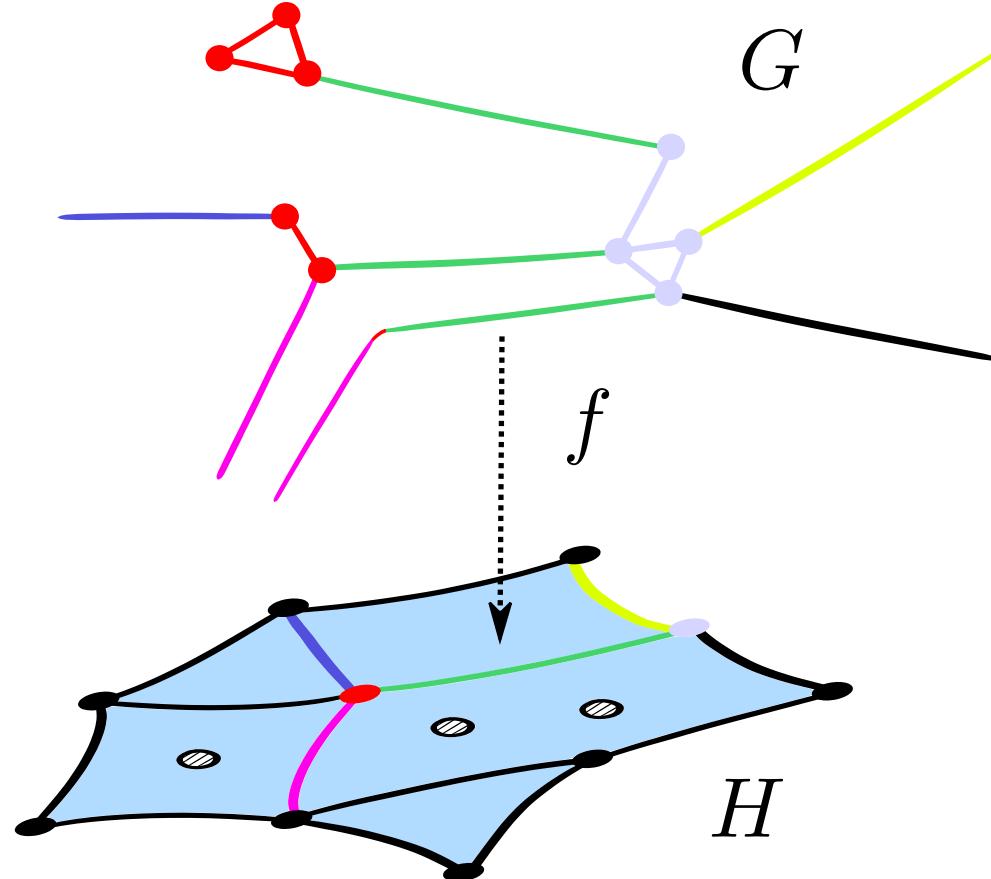
that encodes the surface . . .



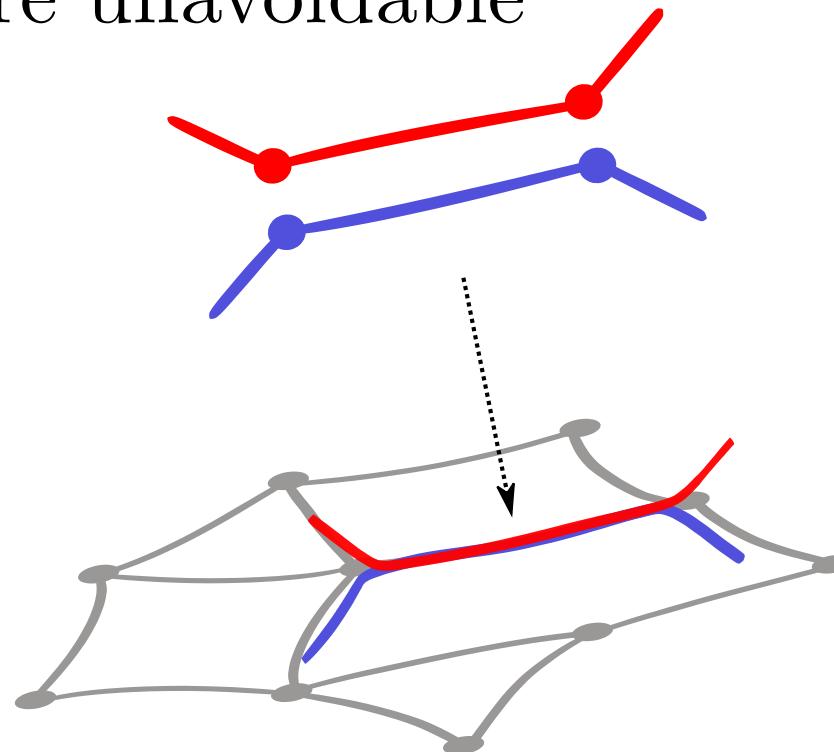
. . . and a graph  $H$  on it

# Input

draw  $G$  in  $H$

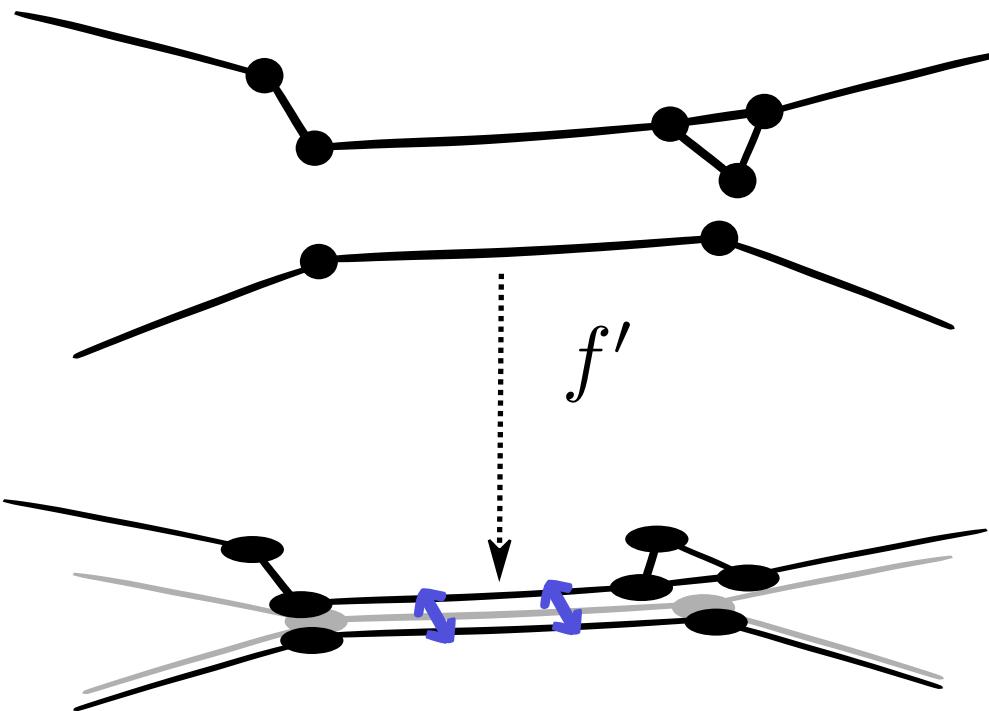


In this model overlaps  
are unavoidable



# Output

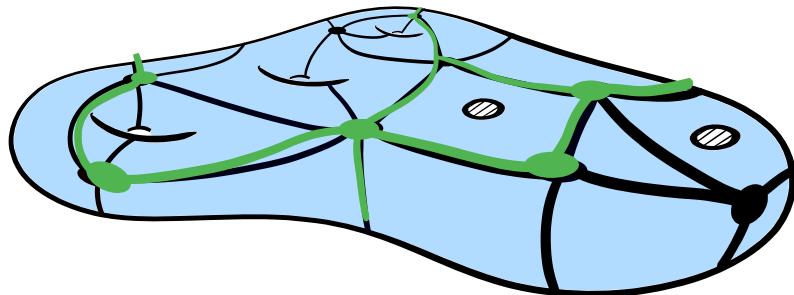
weak embedding: drawing  $f'$  that can be untangled by infinitesimal perturbation



Akitaya, Fulek, and  
Tóth, 2019

algo to determine if  
 $f'$  is weak embedding,  
and if so to perturb  $f'$

# Result



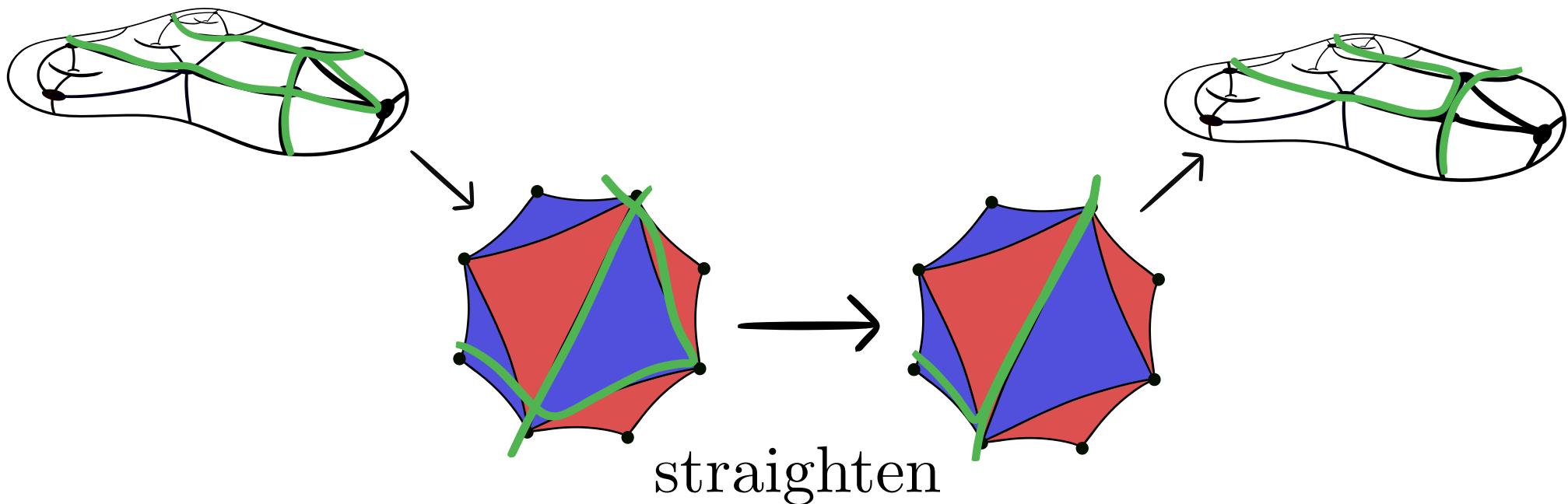
$n$  : # times  $f$  uses an edge or vertex of  $H$   
 $m$  : # vertices and edges of  $H$   
 $s$  : genus + # holes

Colin de Verdière, Despré, D., 2023

We can decide if  $f$  can be untangled,  
in  $O(m + s^2 n \log(s n))$  time.  
If so, we can compute a weak embedding homotopic  
to  $f$  in additional  $O(s^2 m n^2)$  time.

# Algorithm overview

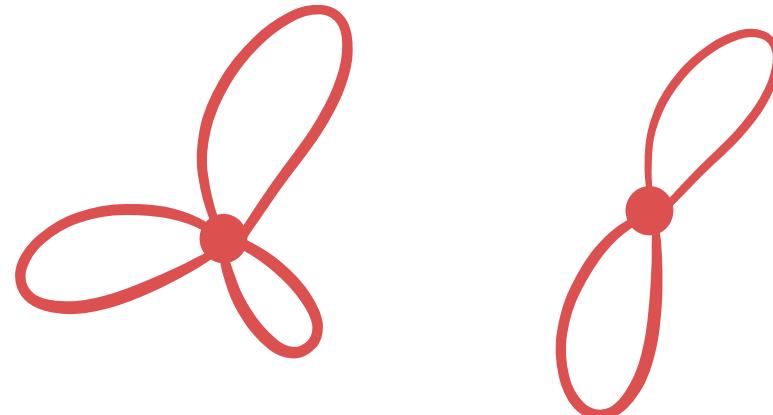
(assuming that the surface has no boundary)



# Straightening a graph

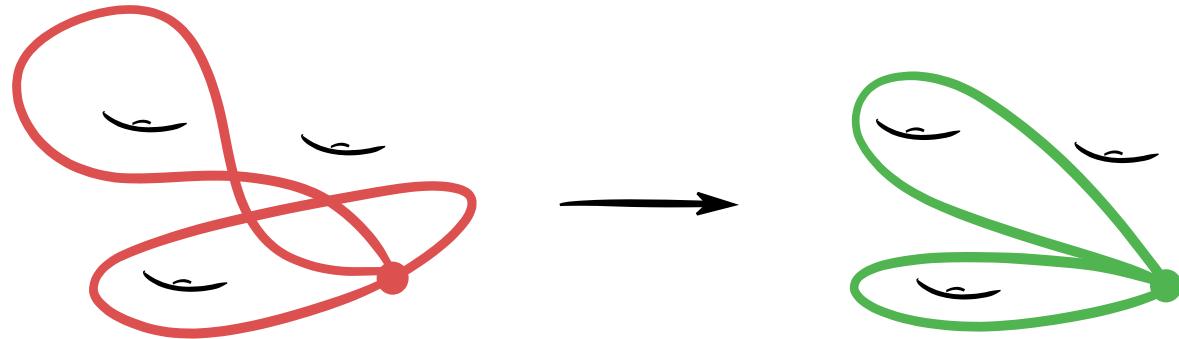
*loop graph*

# Straightening a ~~graph~~



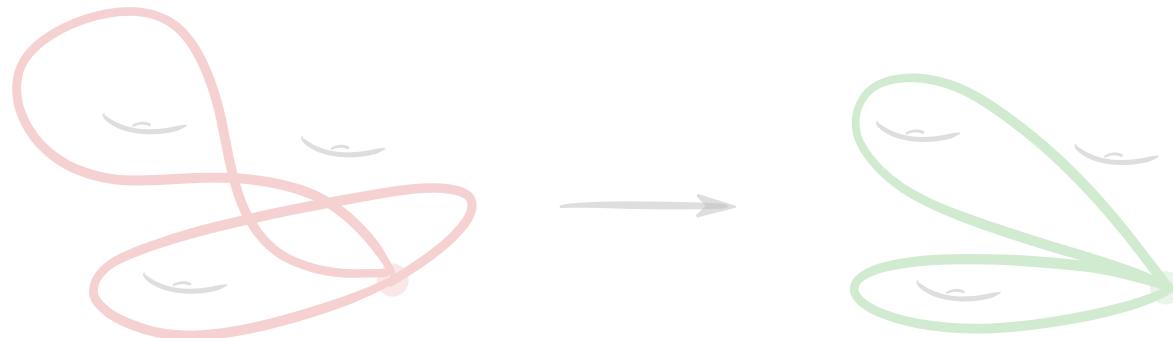
# Straightening a loop graph

First attempt: reduce the loops, vertex fixed

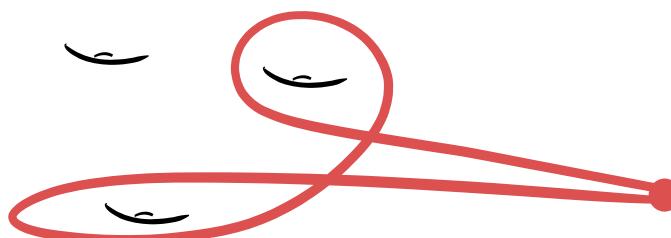


# Straightening a loop graph

First attempt: reduce the loops, vertex fixed



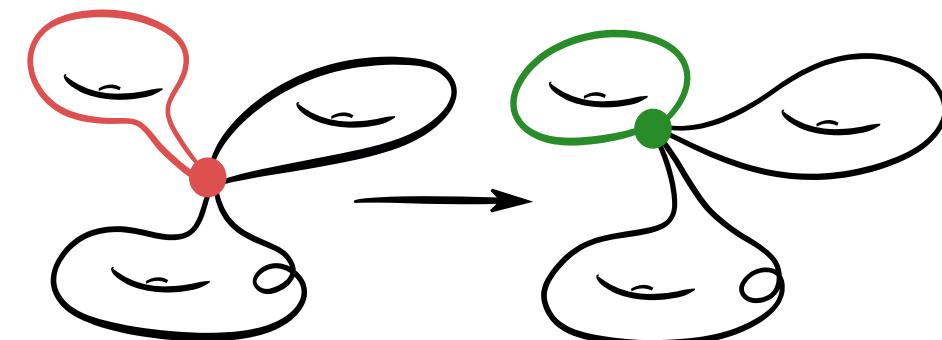
Problem:



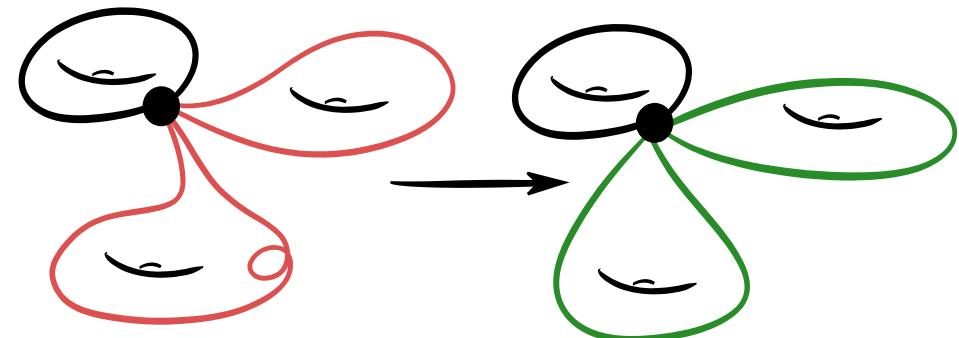
→ vertex must move

# Straightening a loop graph

Solution:



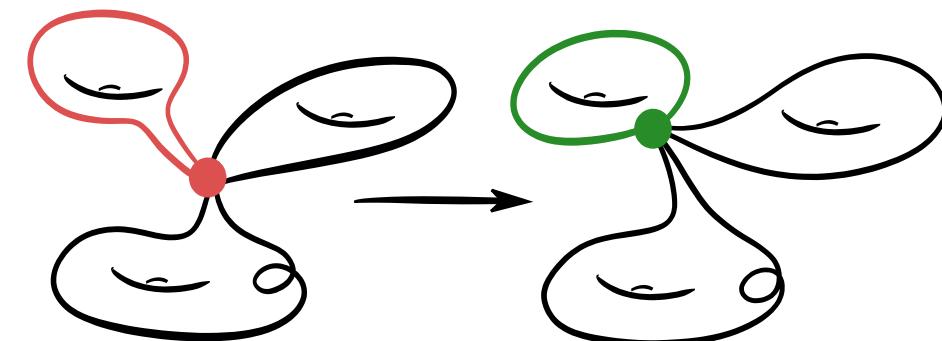
1. Reduce 1 loop cyclically



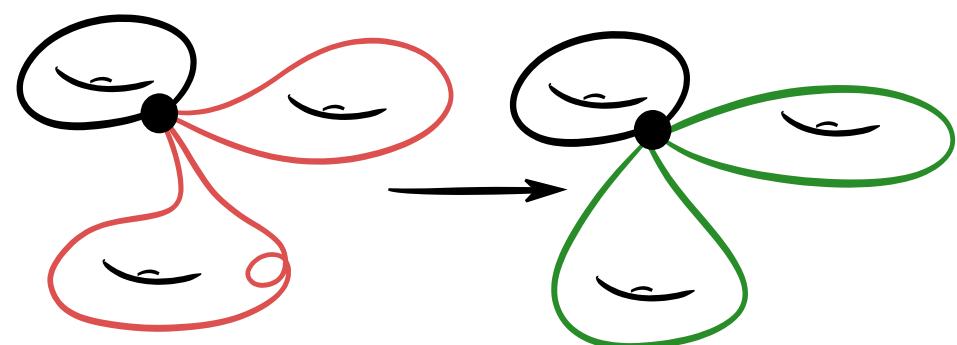
2. Reduce the other loops linearly

# Straightening a loop graph

A straightened loop graph is a weak embedding  
or cannot be untangled



1. Reduce 1 loop cyclically

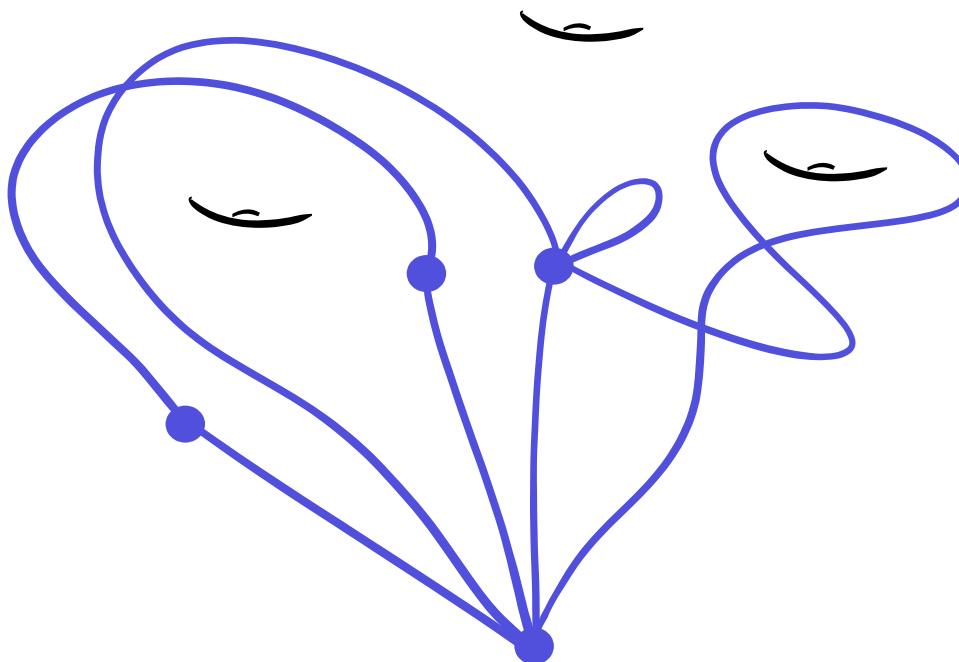


2. Reduce the other loops linearly

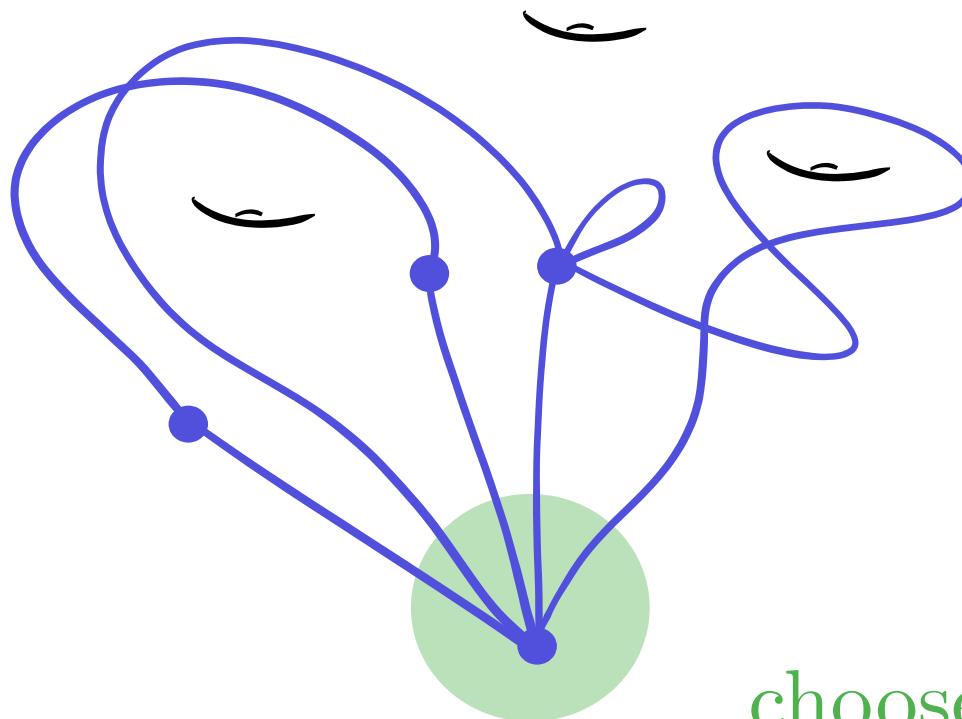
# Straightening a graph

# Straightening a graph

(connected, say)

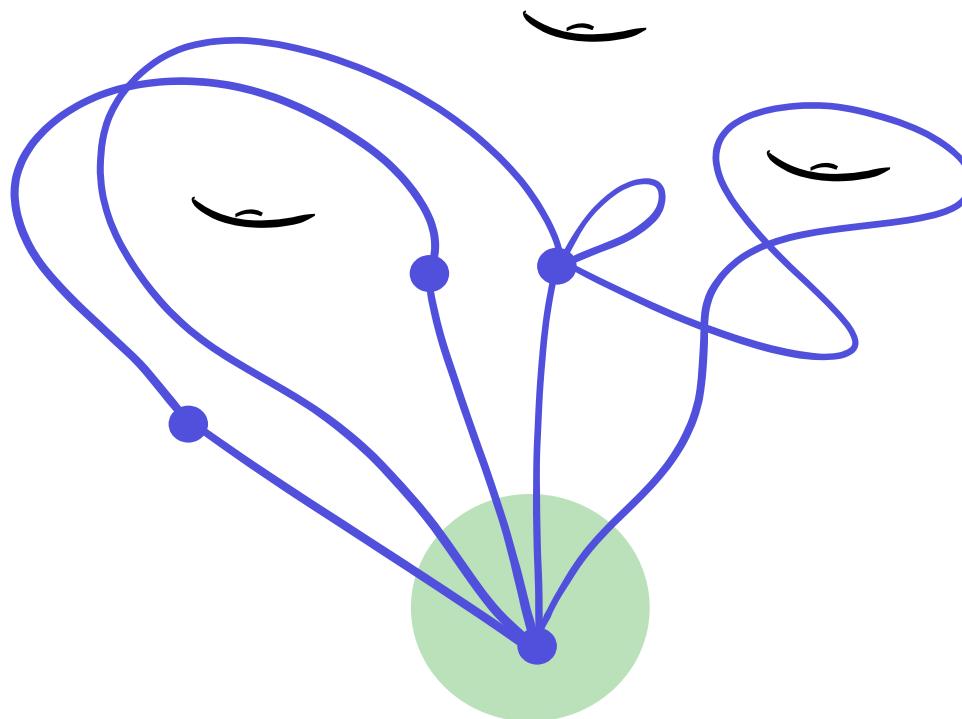


# Straightening a graph

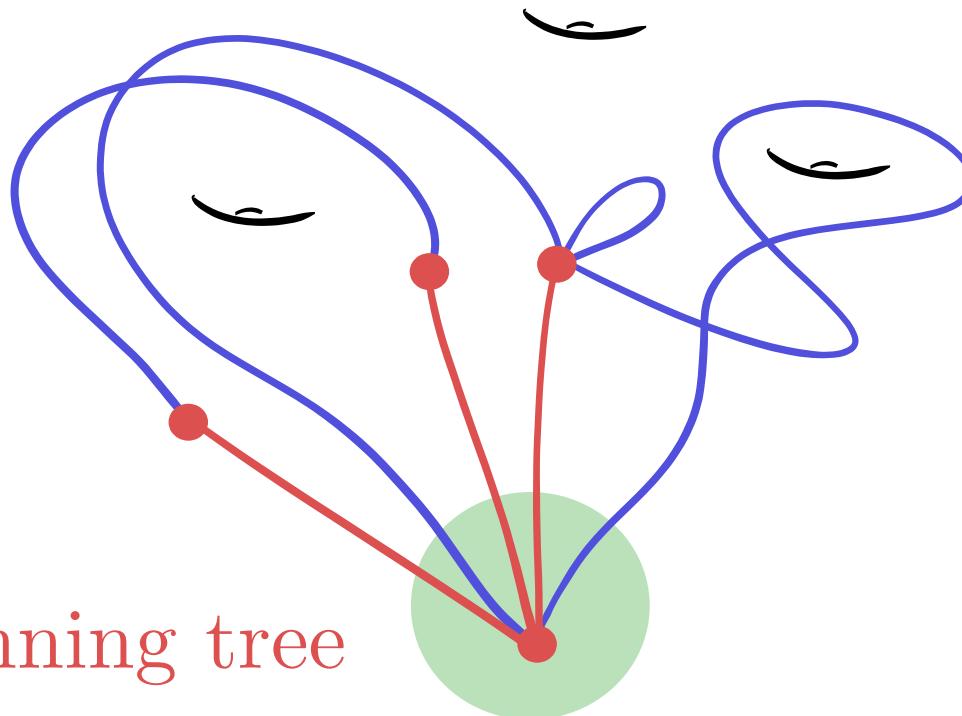


choose a base vertex

# Straightening a graph

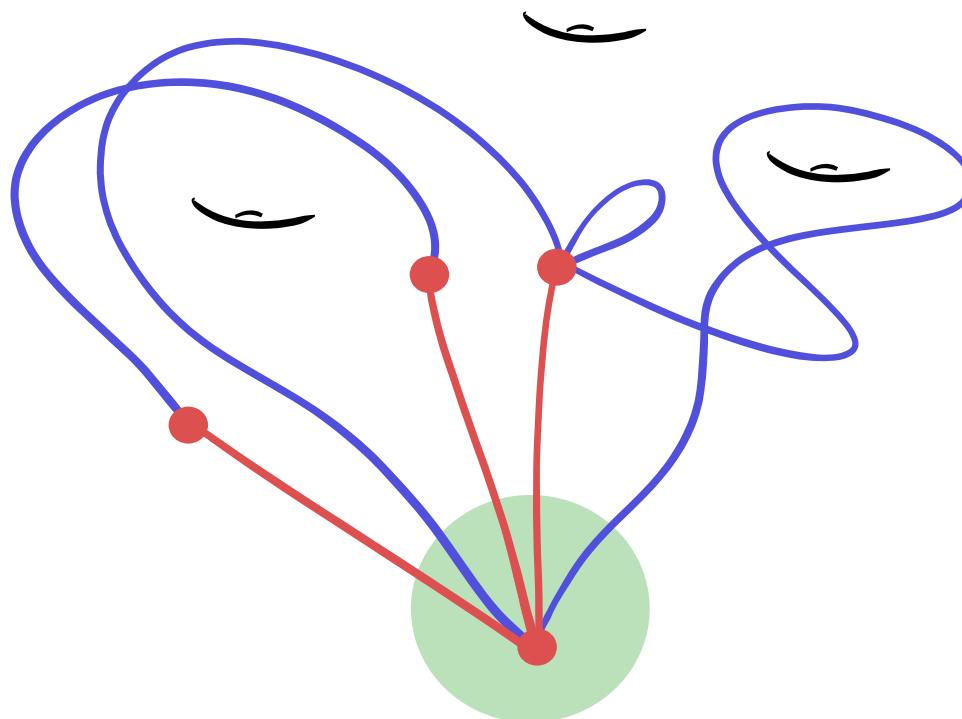


# Straightening a graph



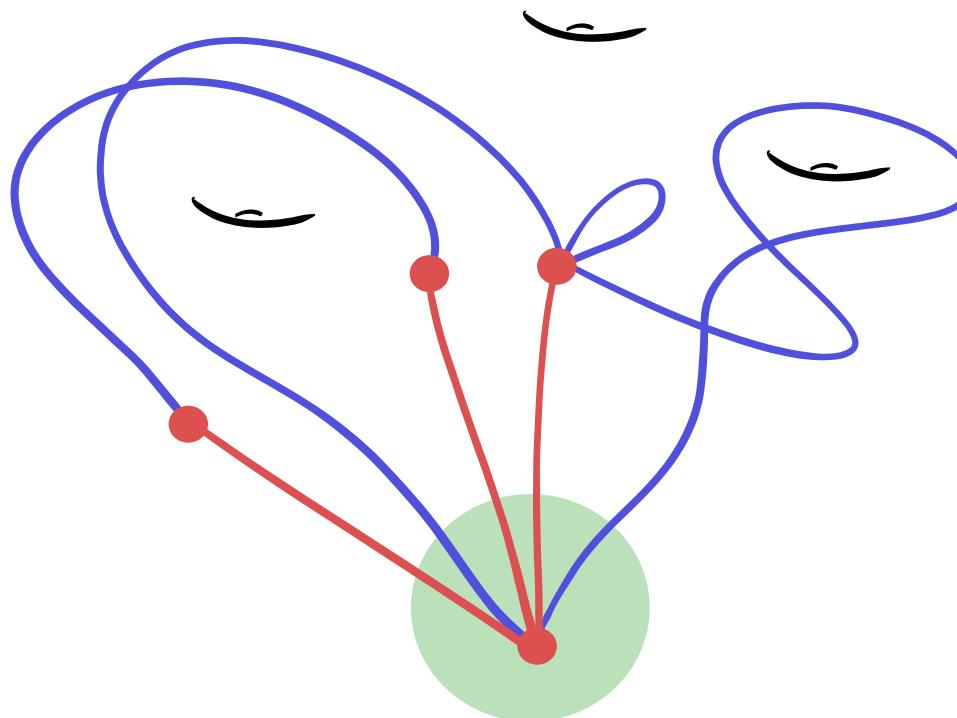
choose a spanning tree

# Straightening a graph



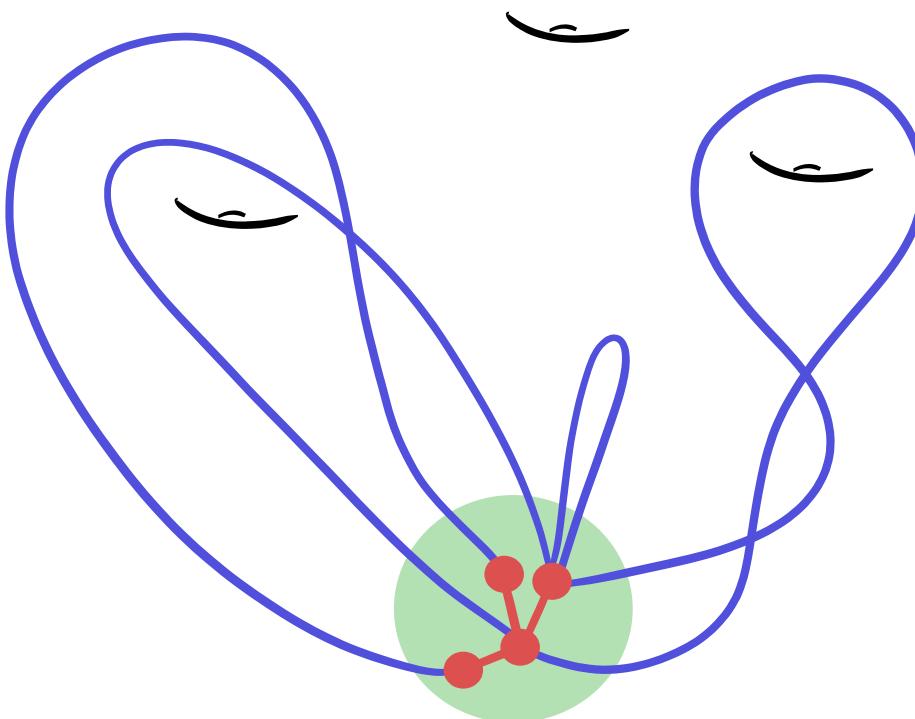
# Straightening a graph

contract the spanning tree

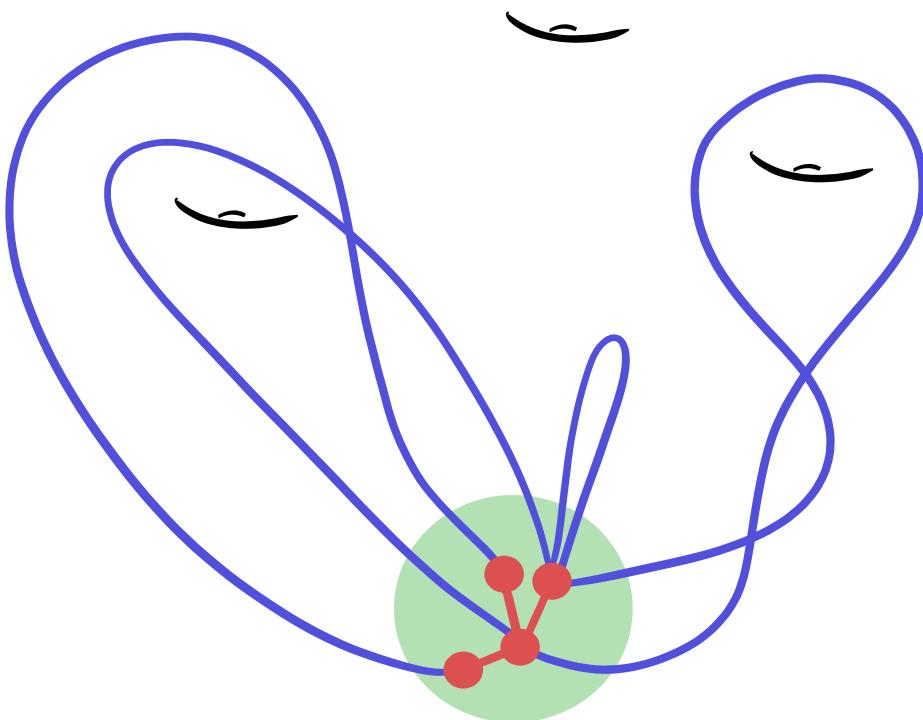


# Straightening a graph

contract the spanning tree

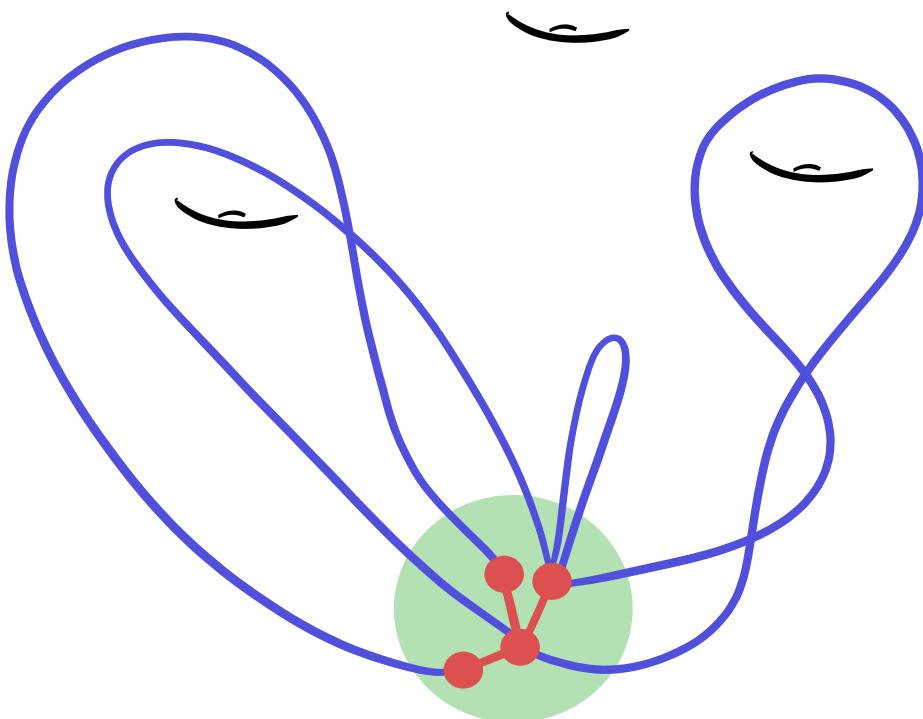


# Straightening a graph



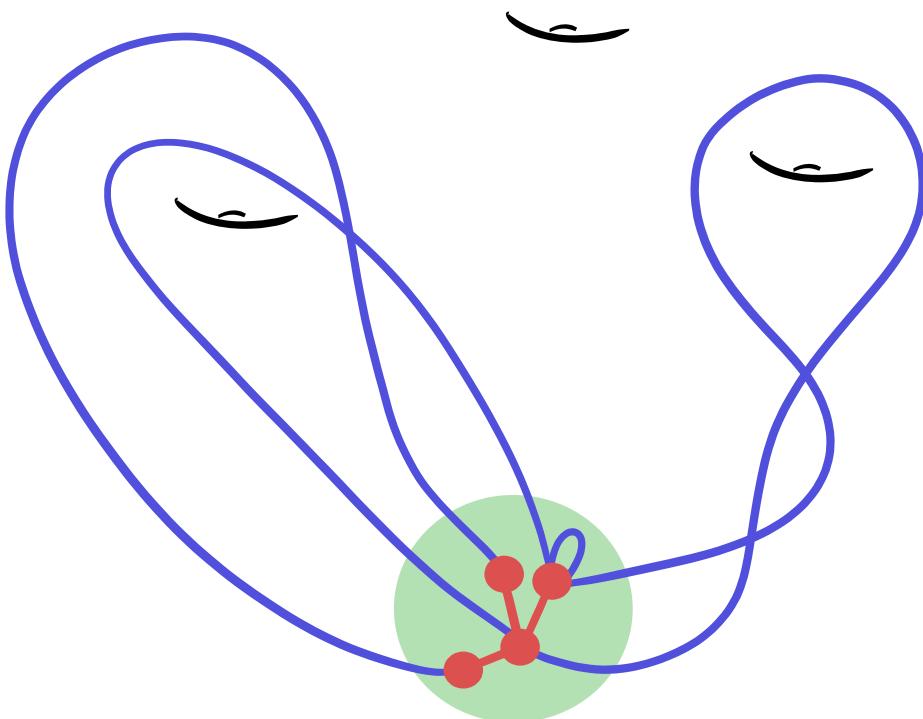
# Straightening a graph

contract and bundle the other edges when possible



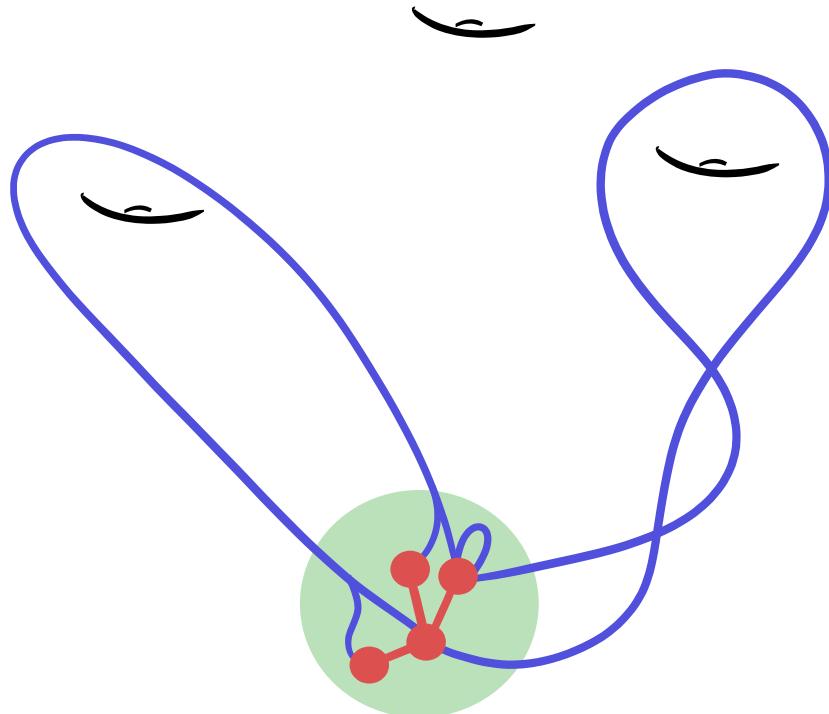
# Straightening a graph

contract and bundle the other edges when possible

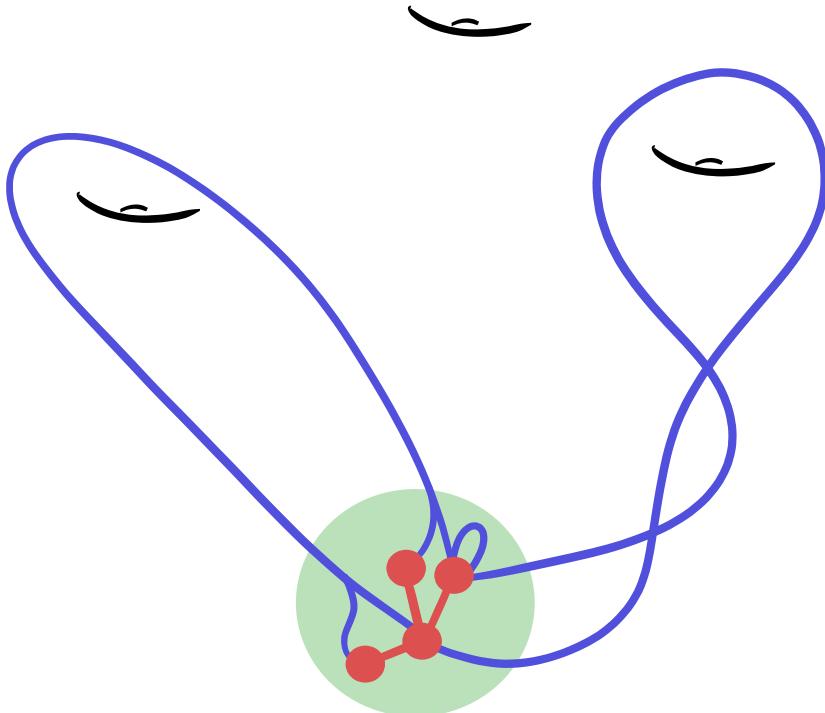


# Straightening a graph

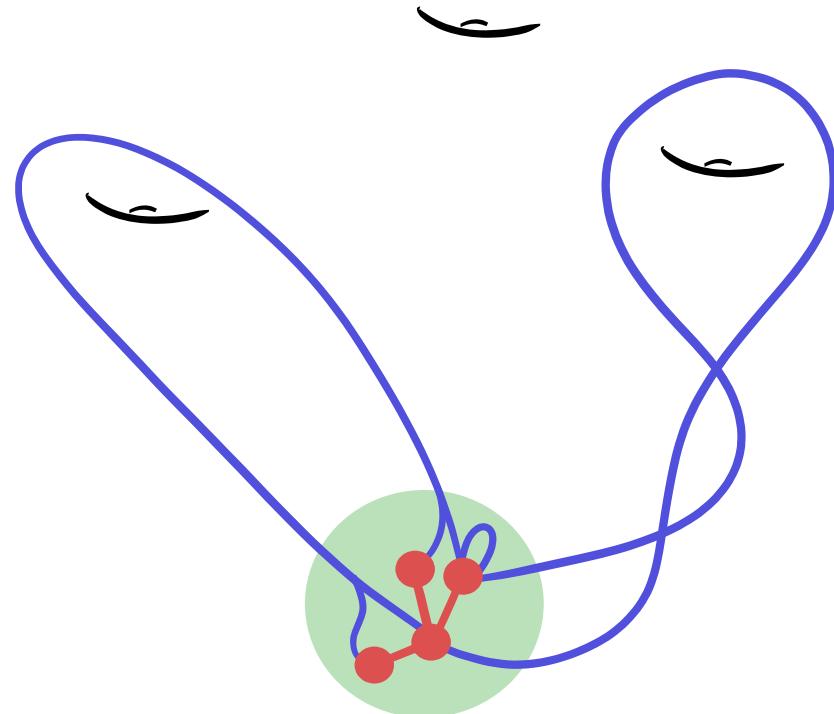
contract and bundle the other edges when possible



# Straightening a graph

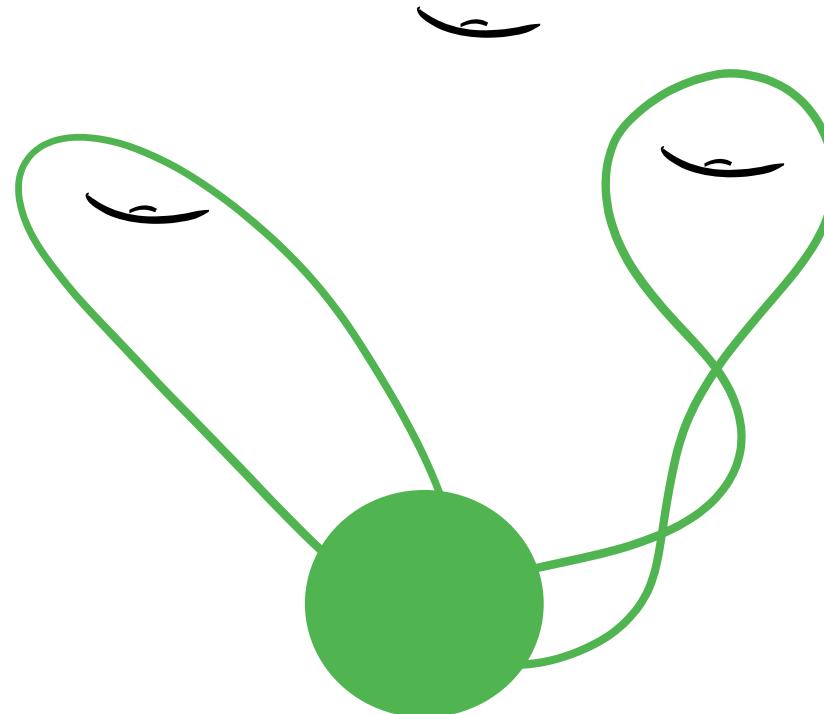


# Straightening a graph



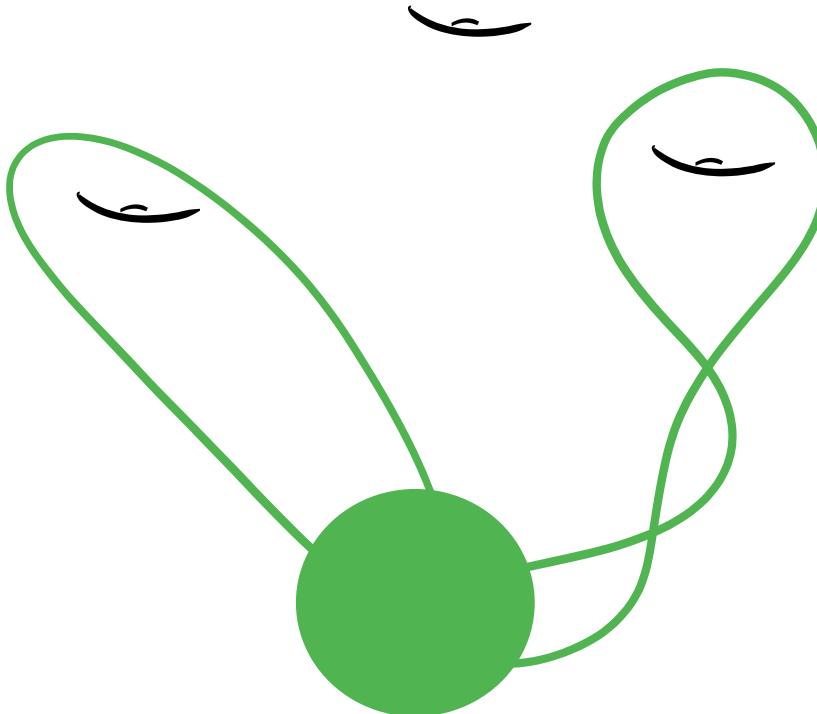
consider the associated loop graph

# Straightening a graph



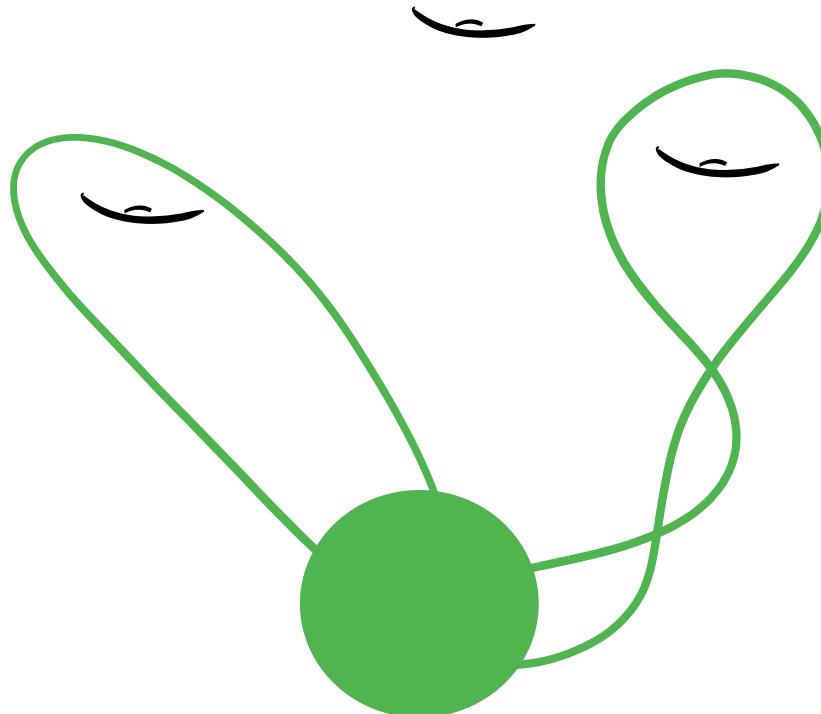
consider the associated loop graph

# Straightening a graph



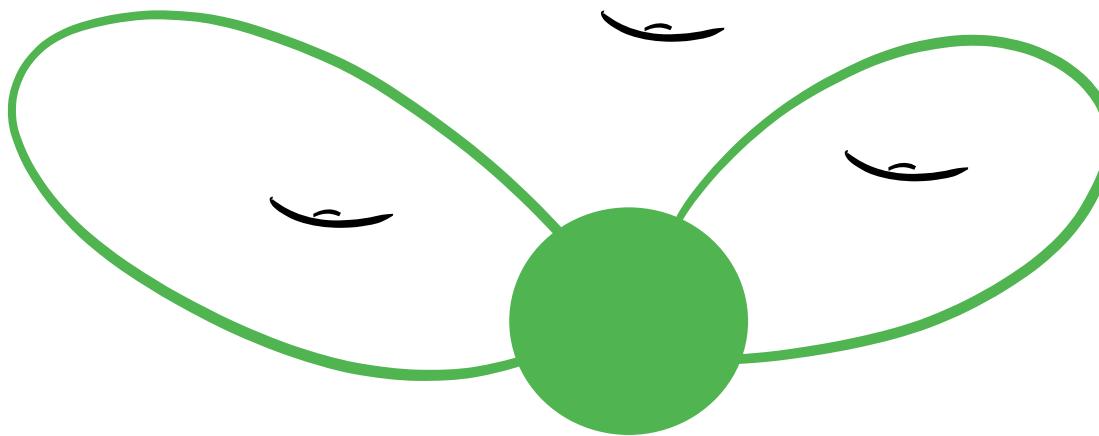
# Straightening a graph

straighten the loop graph

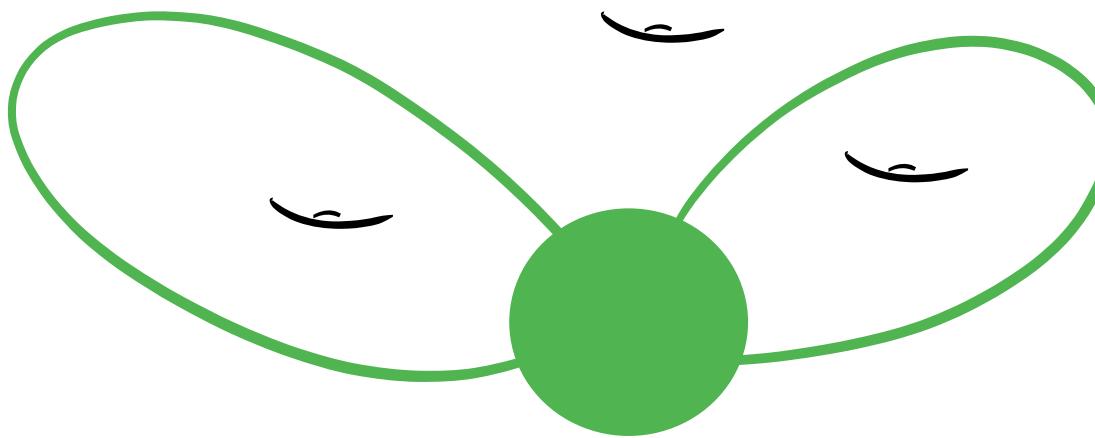


# Straightening a graph

straighten the loop graph

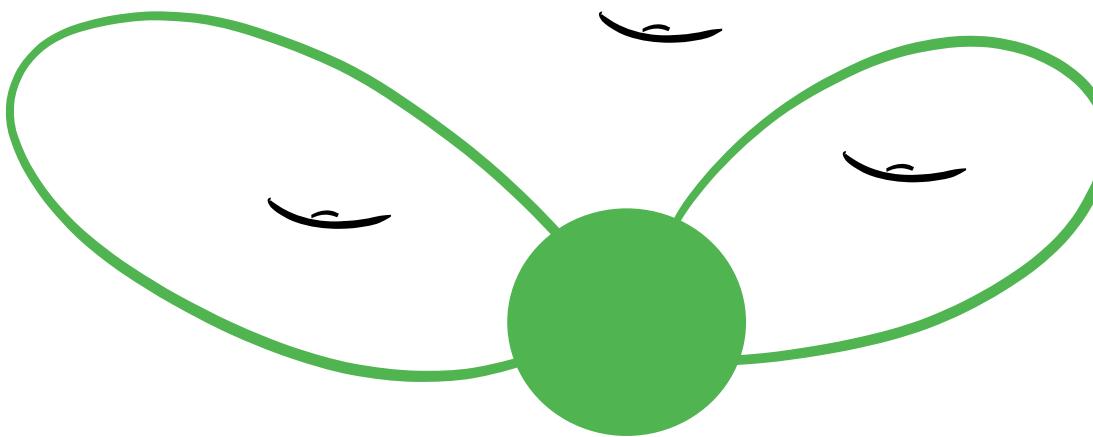


# Straightening a graph



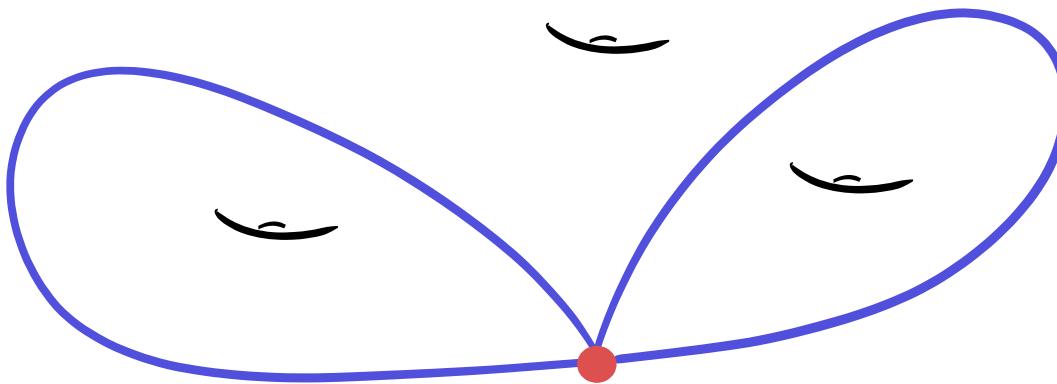
# Straightening a graph

forget about the loop graph

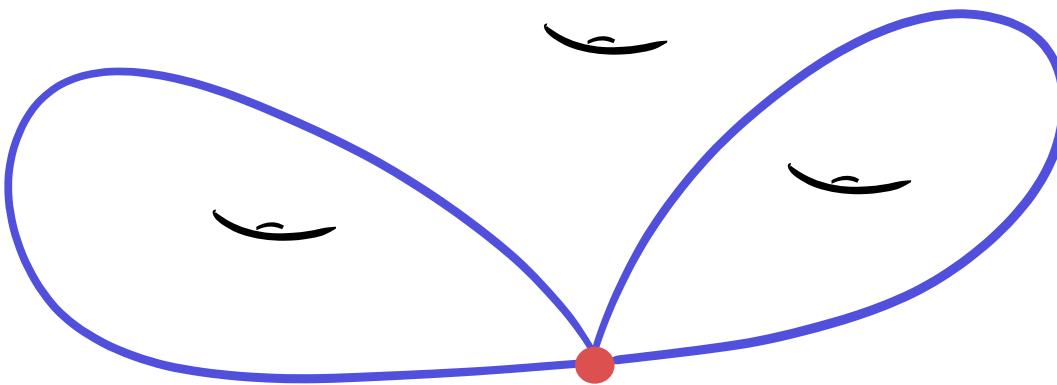


# Straightening a graph

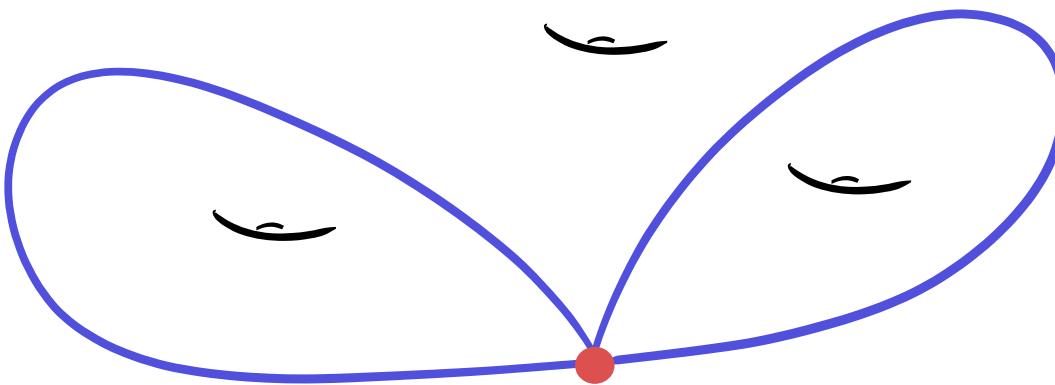
forget about the loop graph



# Straightening a graph

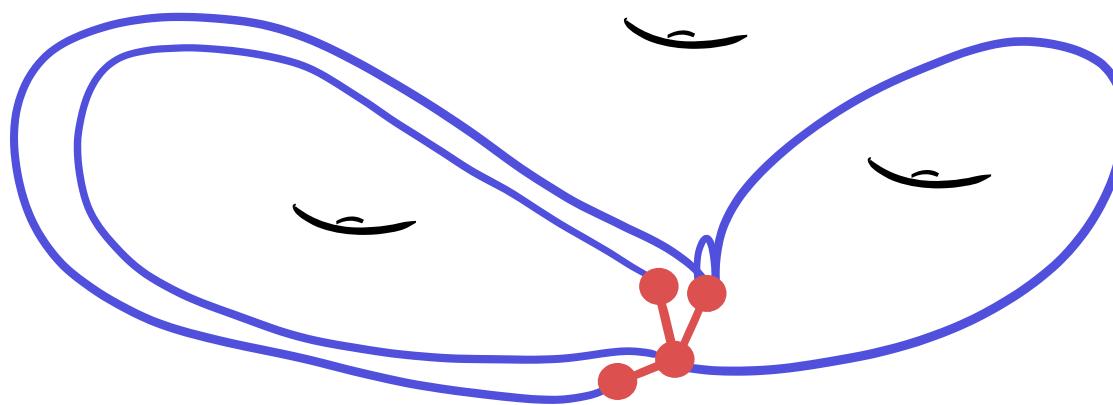


# Straightening a graph



A straightened graph is a weak embedding  
or cannot be untangled

# Straightening a graph

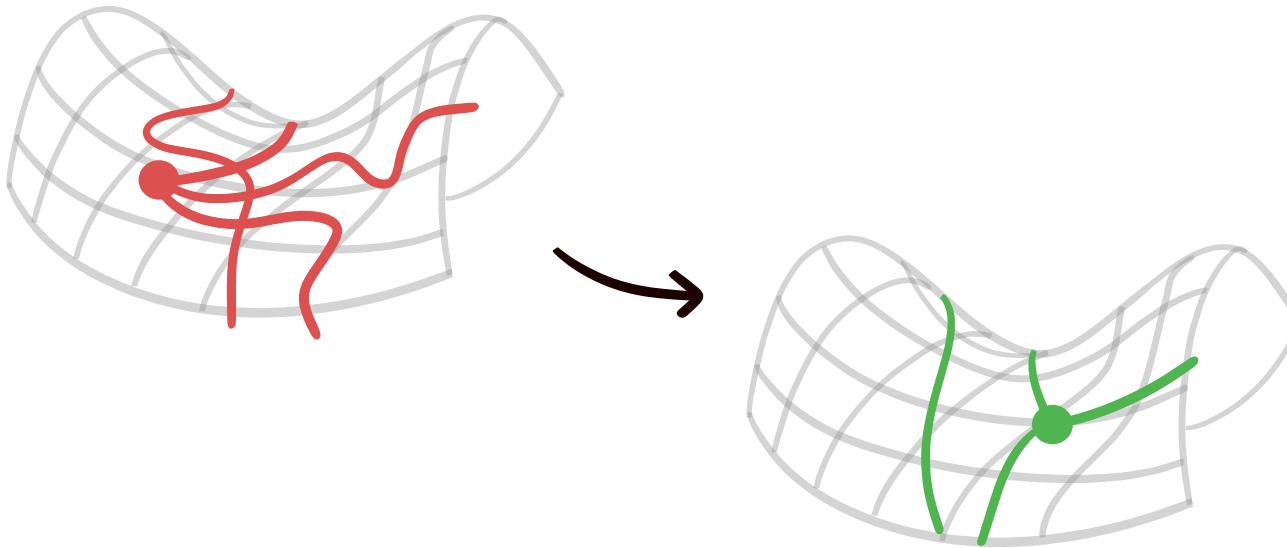


A straightened graph is a weak embedding  
or cannot be untangled

(+ tricks and data structures for achieving  
the announced time complexities)

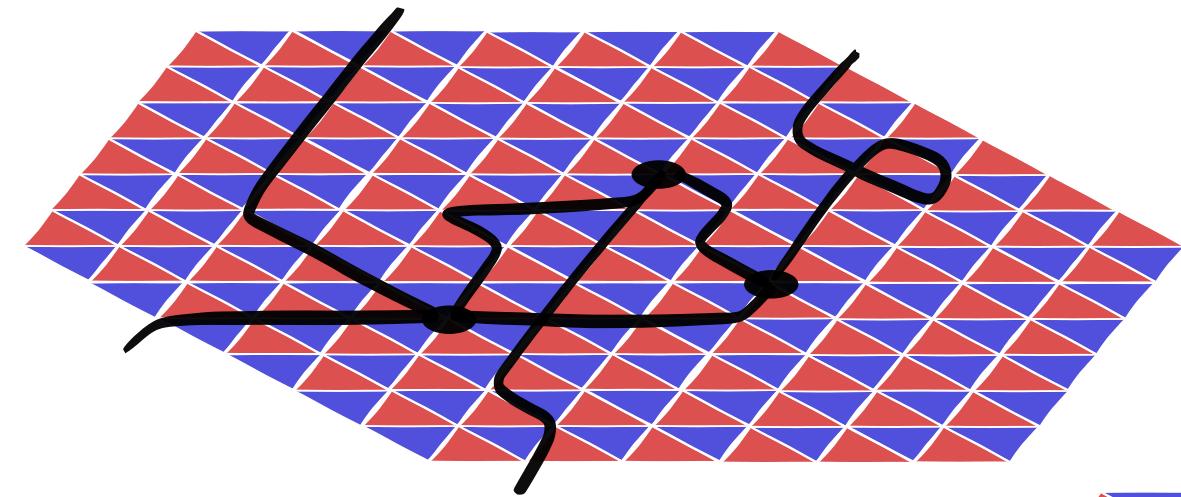
# Discrete analogue of Tutte embeddings

# Recall: Tutte embeddings

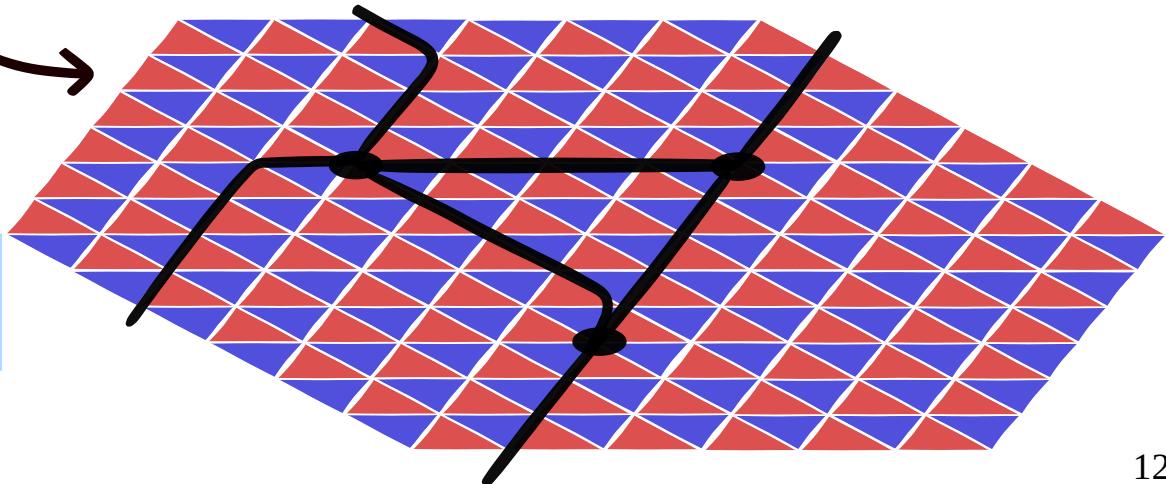


edges straight and  
vertices “barycentric”

# Harmonious drawings

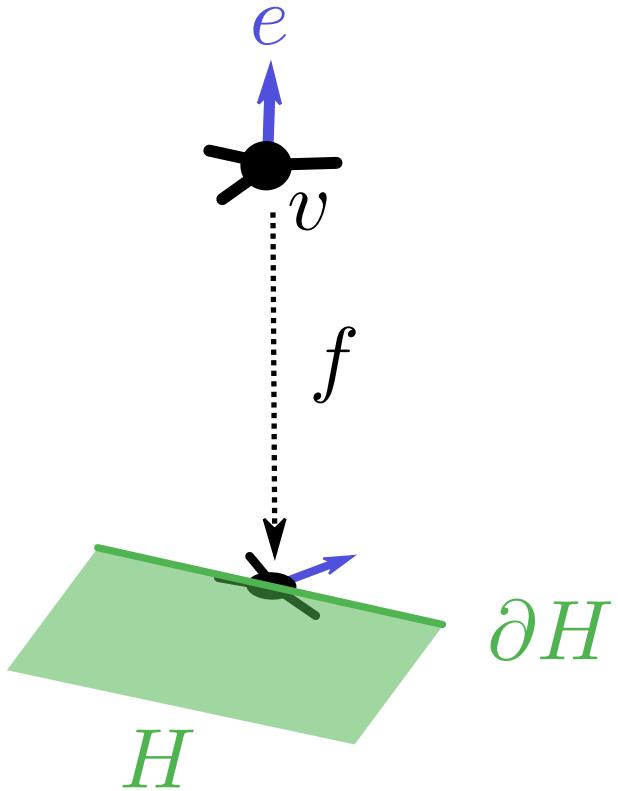


edges reduced and  
vertices “barycentric”



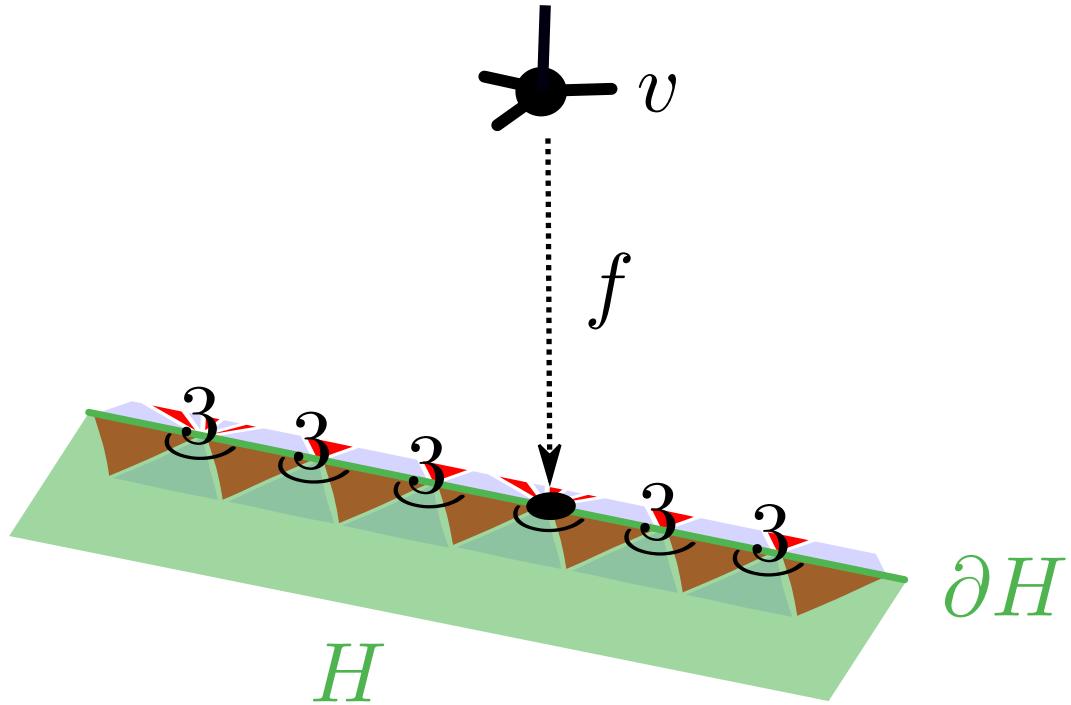
vertices “barycentric”

$$\begin{aligned} f(v) \in \partial H \\ \implies \exists e \quad f(e) \text{ escapes } H \end{aligned}$$



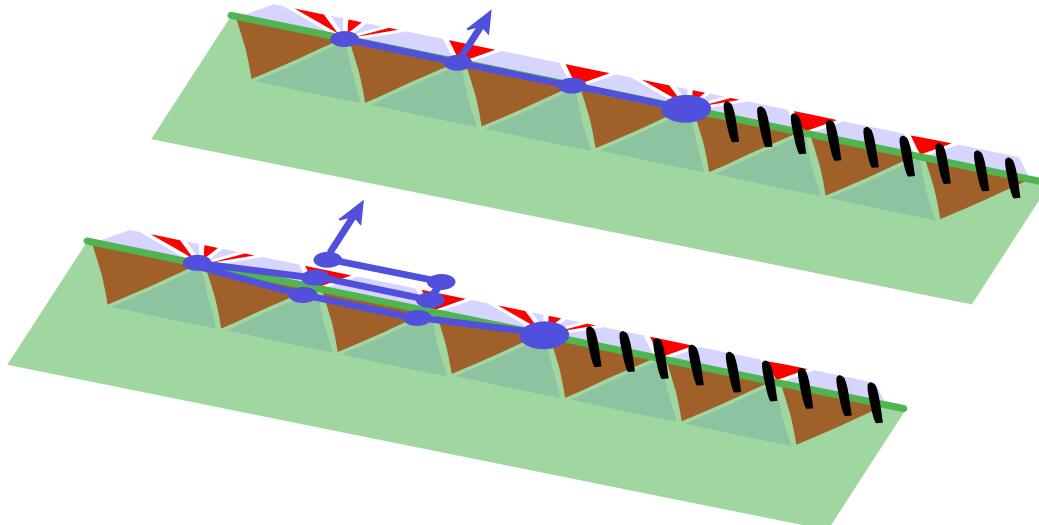
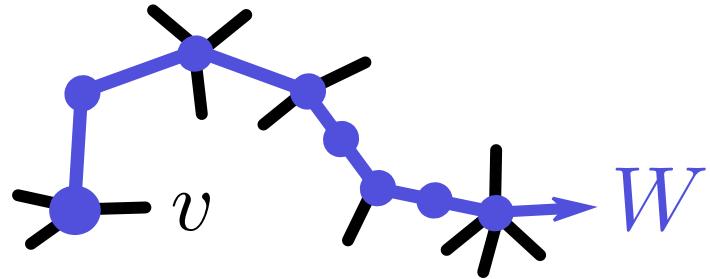
vertices “barycentric”

$$f(v) \in \partial H$$

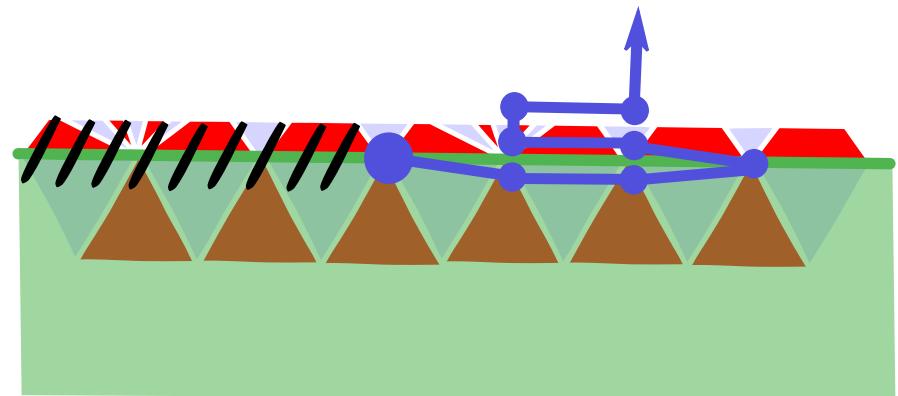
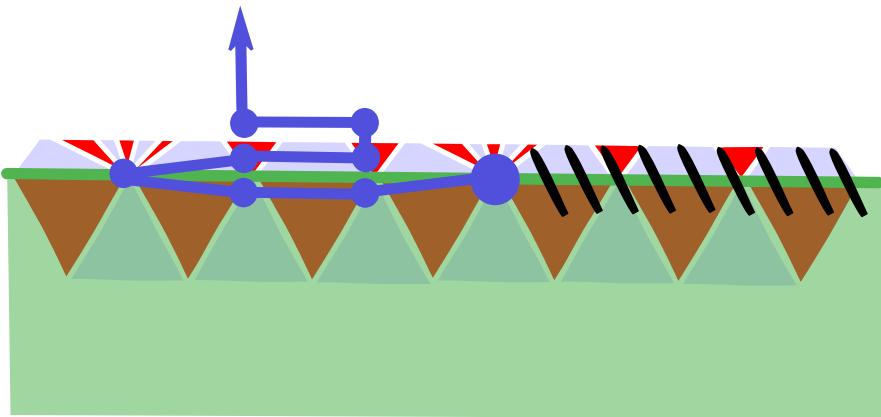


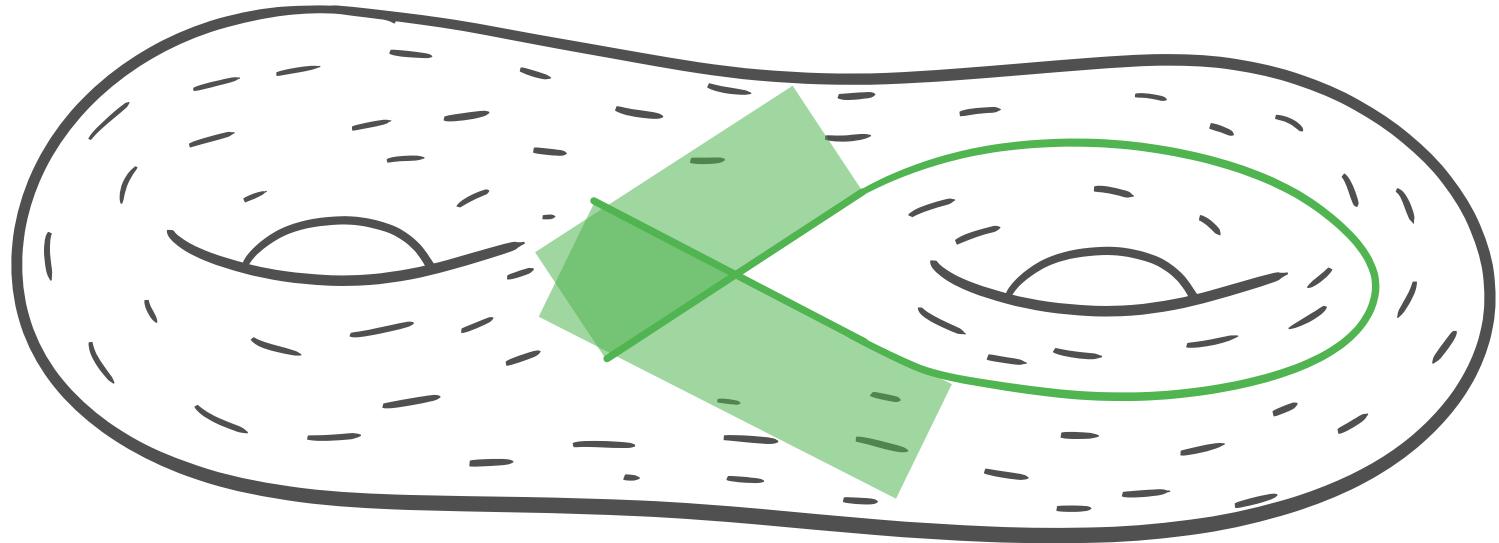
vertices “barycentric”

$$f(v) \in \partial H \\ \xrightarrow{} \\ \exists W f \circ W \text{“escapes”} H$$



where to escape depends on the coloring





this definition generalizes to surfaces

# Results

graph  $G$

reducing triangulation  $T$  with  $m$  edges

$f : G \rightarrow T^1$  of size  $n$

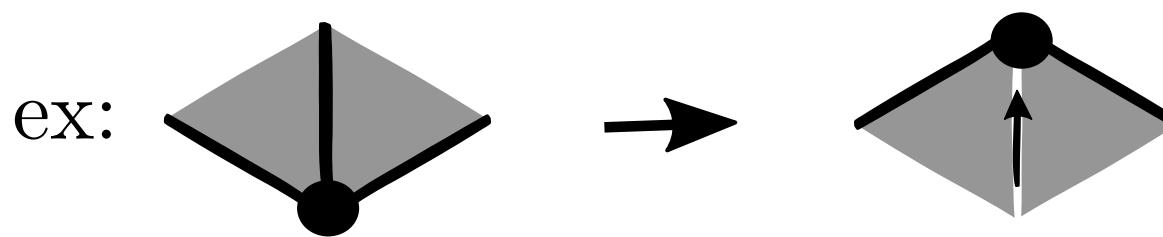
Definition of *harmonious* drawings

$f$  harmonious and  $f$  can be untangled  
 $\Rightarrow f$  weak embedding

Algo to make  $f$  harmonious in  $O((m+n)^2 n^2)$  time,  
without increasing any edge length

# Harmonizing a drawing

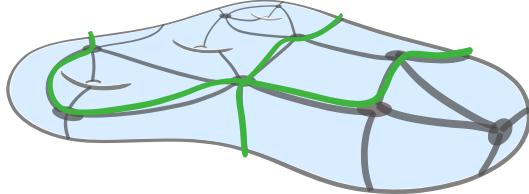
- 1 we define monotonic moves to apply to  $f$



- 2 some moves do not seem to decrease any potential so we combine the moves carefully

Making curves cross  
minimally

# Related work

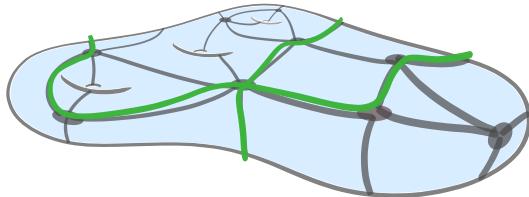


closed walks of total length  $n$   
on a graph of size  $m$

Despré, Lazarus, 2019

- Put a single curve in minimal position in  $O(m + n^4)$  time
- Compute the min. nb. of crossings in  $O(m + n^2)$  time

# Result



closed walks of total length  $n$   
on a graph of size  $m$

simpler algos and proofs!

$$m^3n + mn \log(mn)$$

D., 2024

- Put a single curve in minimal position in  $O(\cancel{m} + \cancel{n}^4)$  time
- Compute the min. nb. of crossings in  $O(\cancel{m} + \cancel{n}^2)$  time

$$m^2 + mn \log(mn)$$

# Untangling Graphs

Computing Delaunay Triangulations

Other works and conclusion

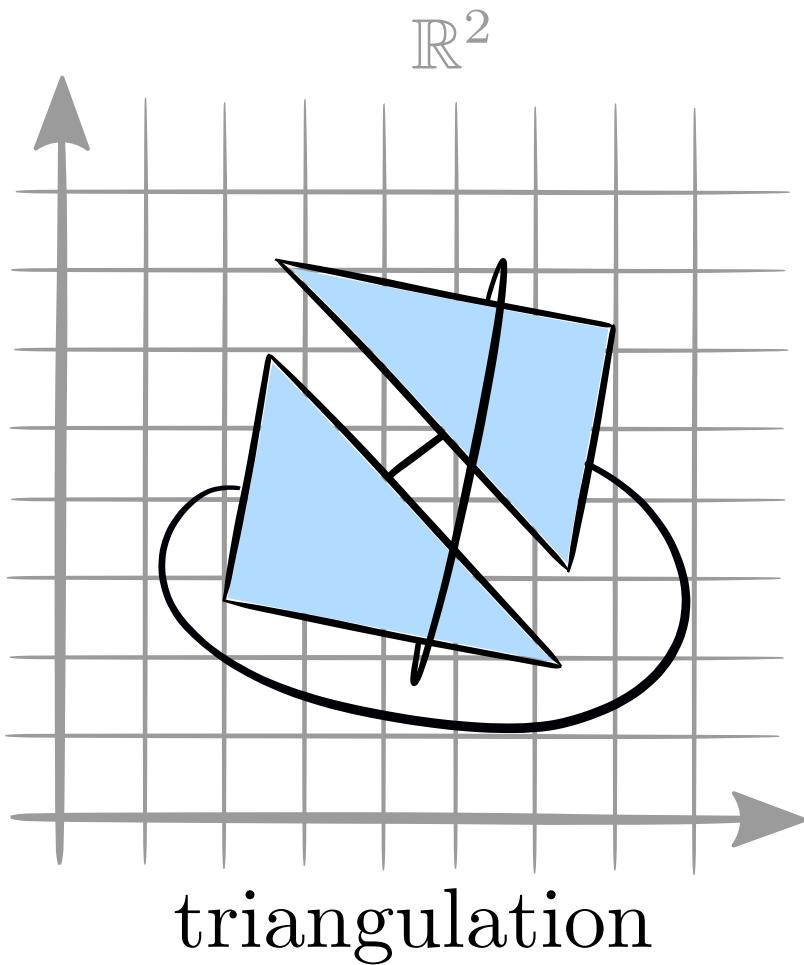
# Untangling Graphs

Computing Delaunay Triangulations

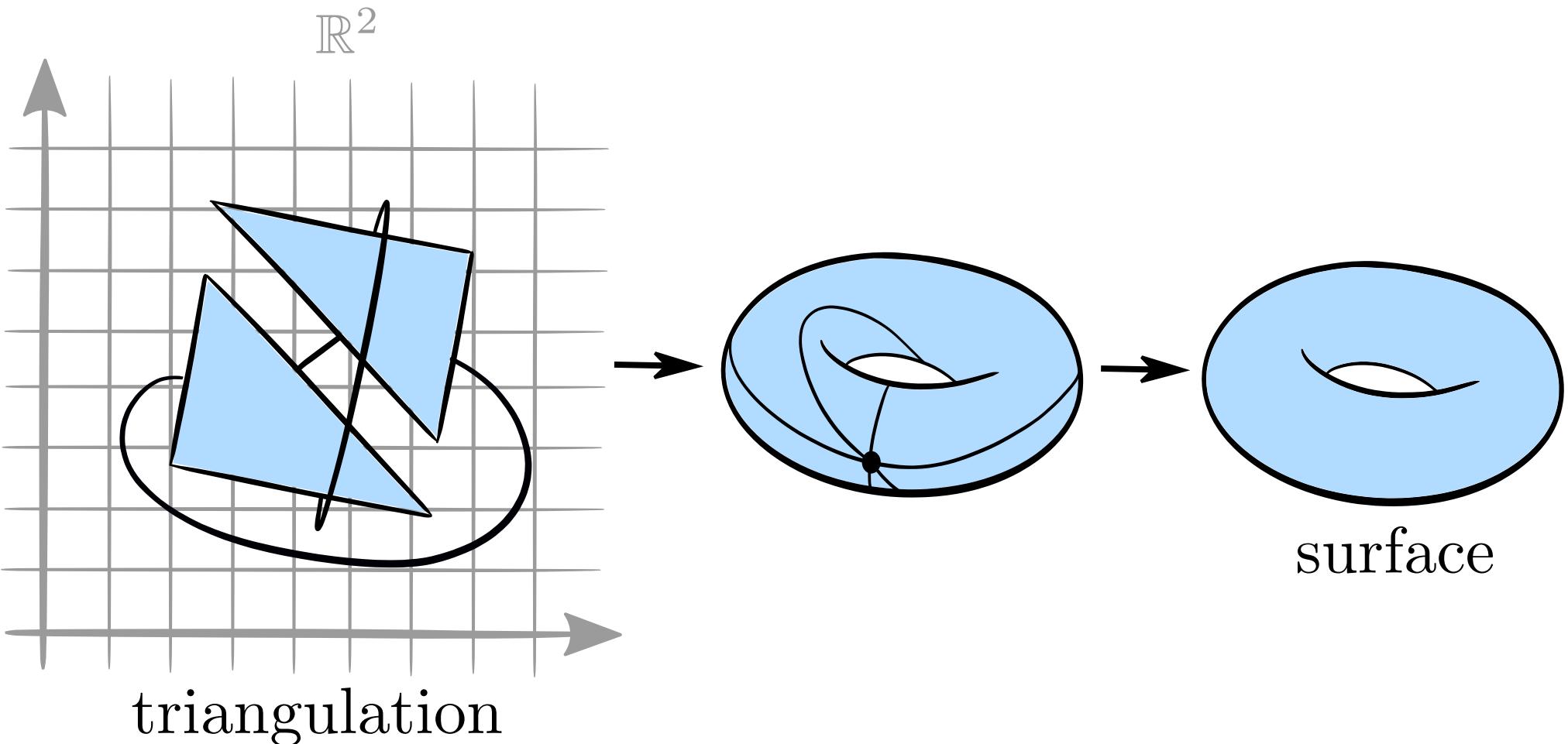
Other works and conclusion

# Triangulations of polyhedral surfaces

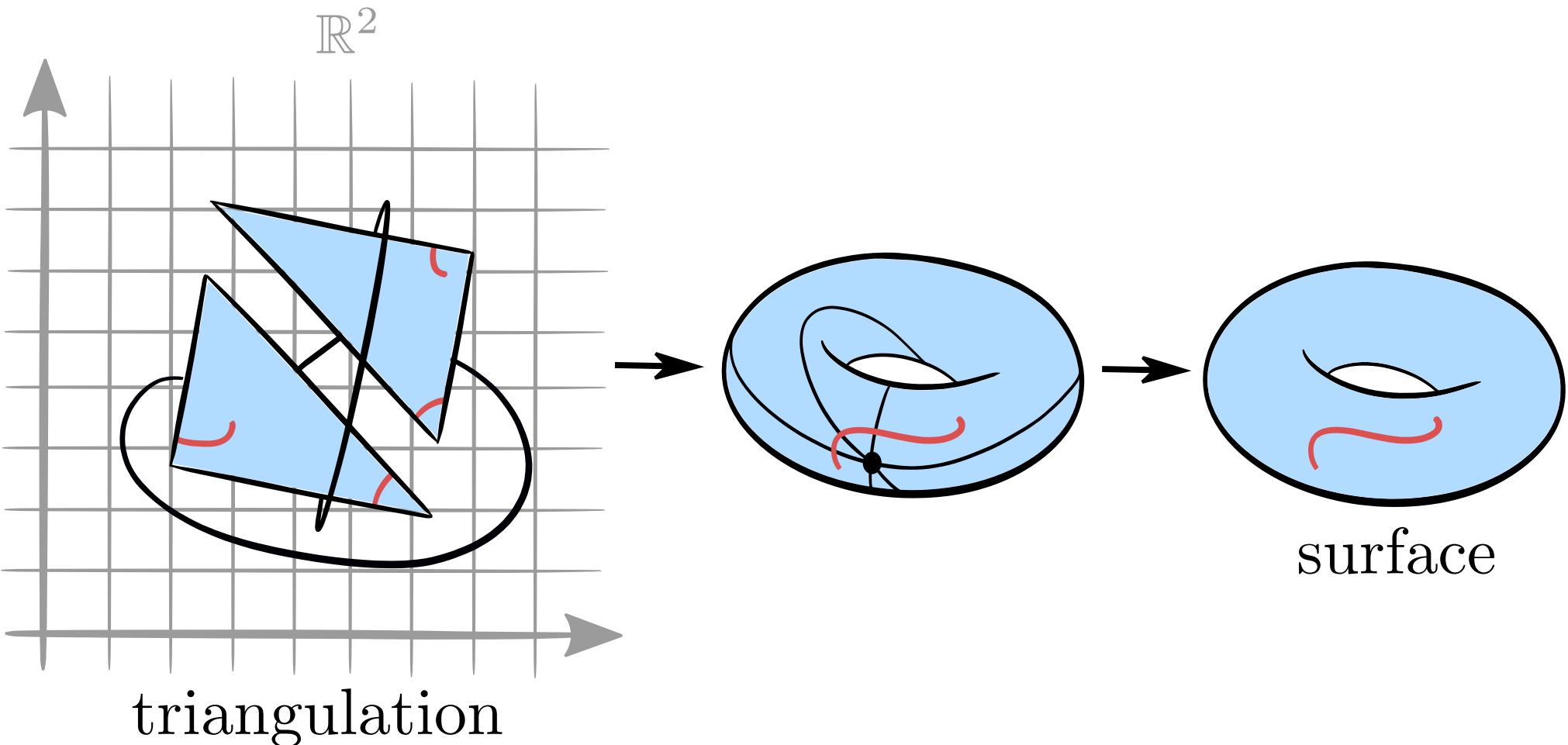
# Triangulation of polyhedral surface



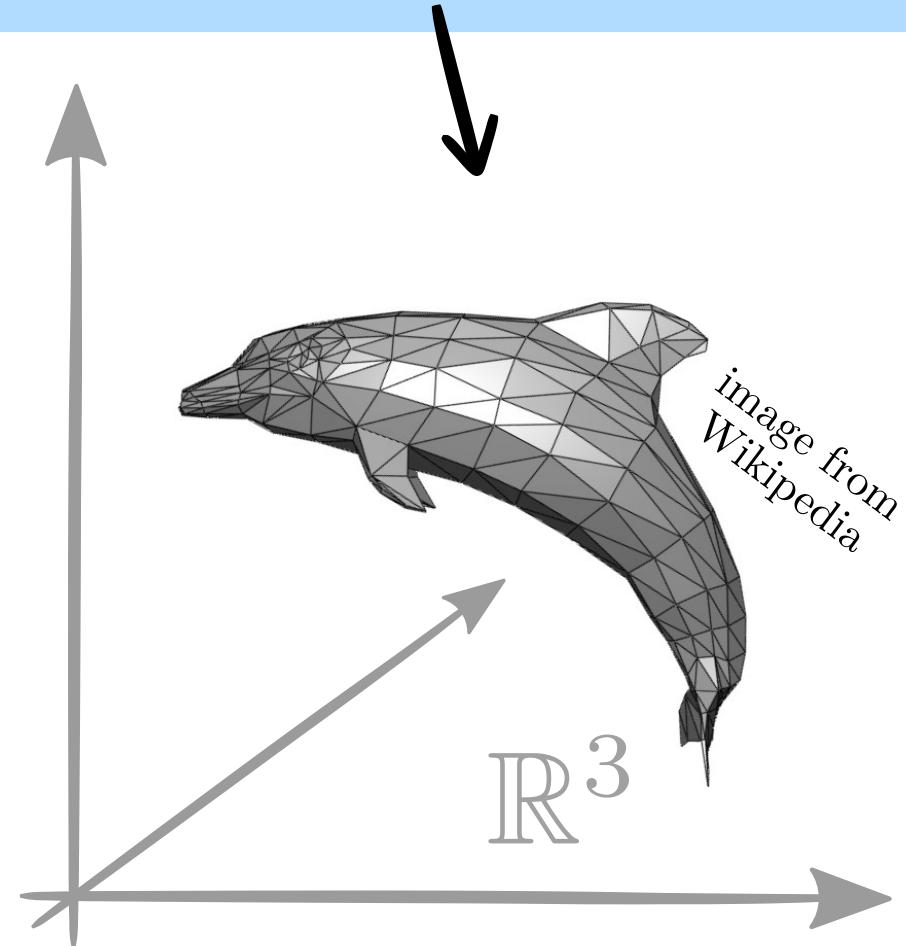
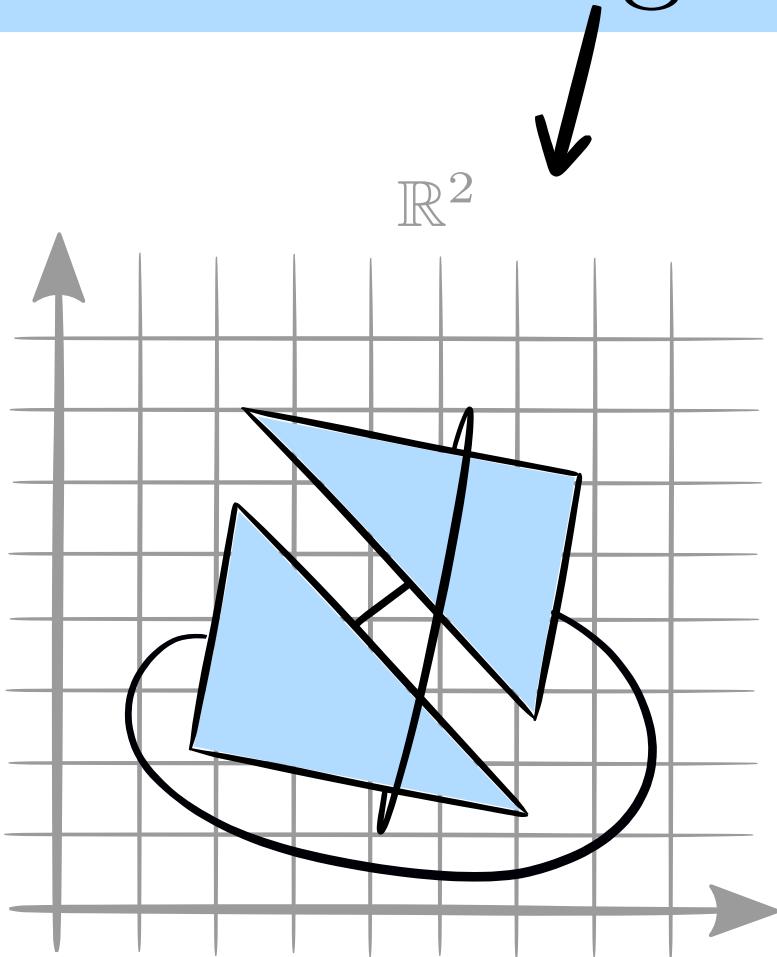
# Triangulation of polyhedral surface



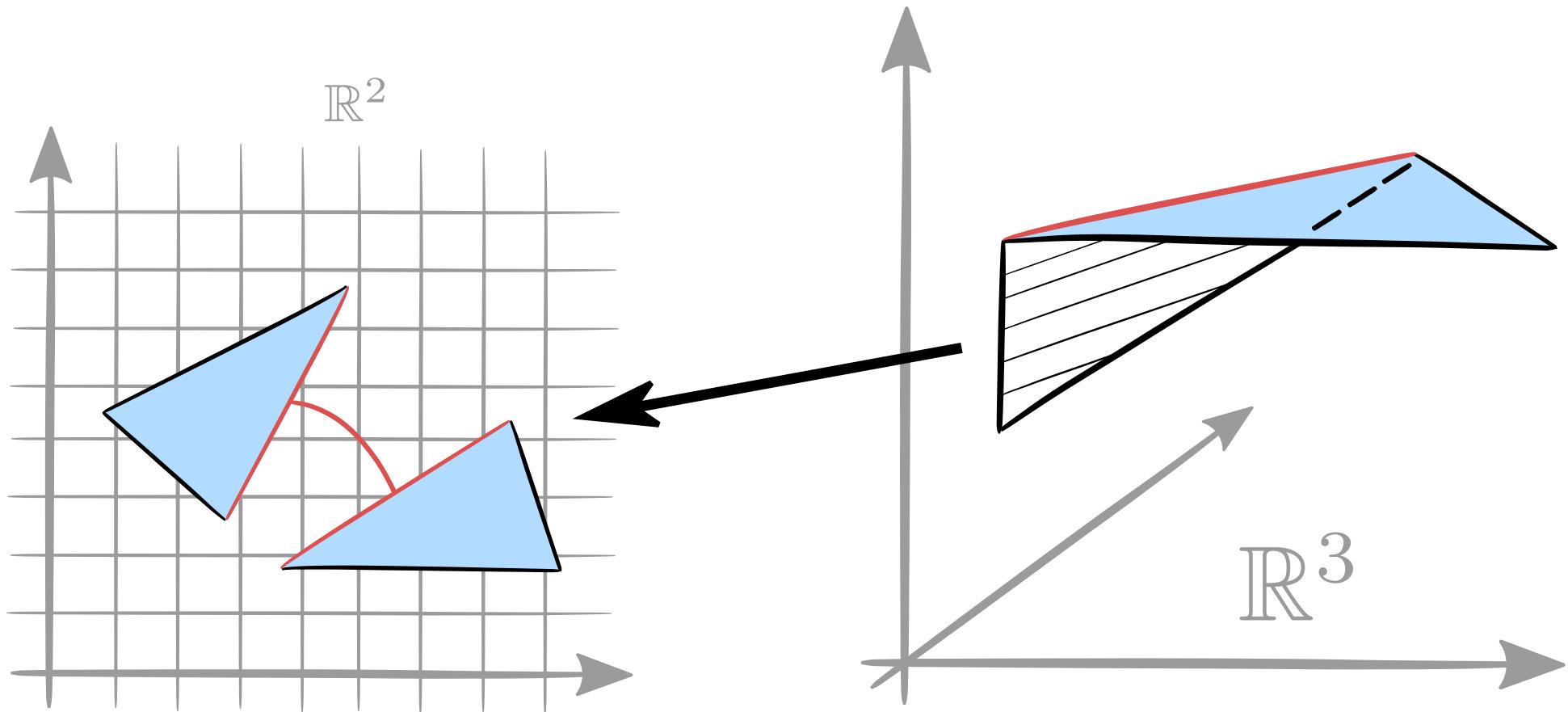
# Triangulation of polyhedral surface



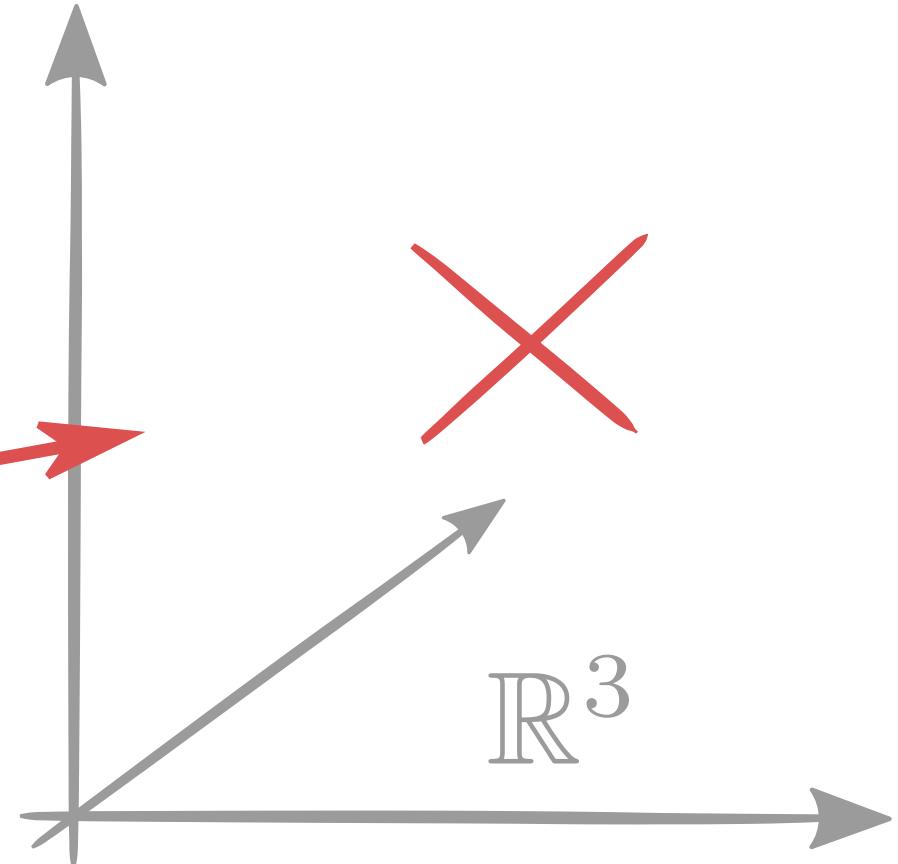
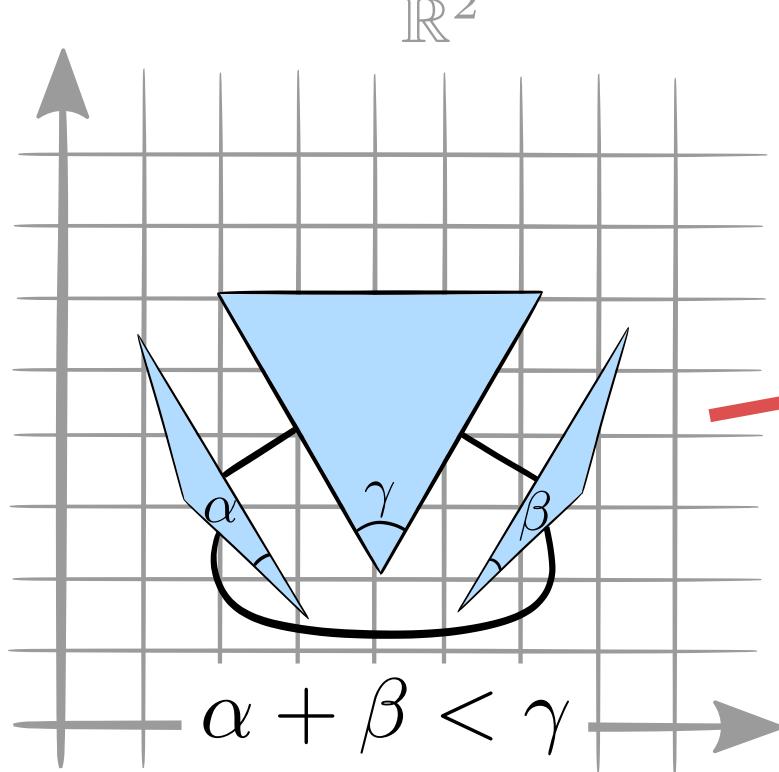
# Triangulation vs. Mesh



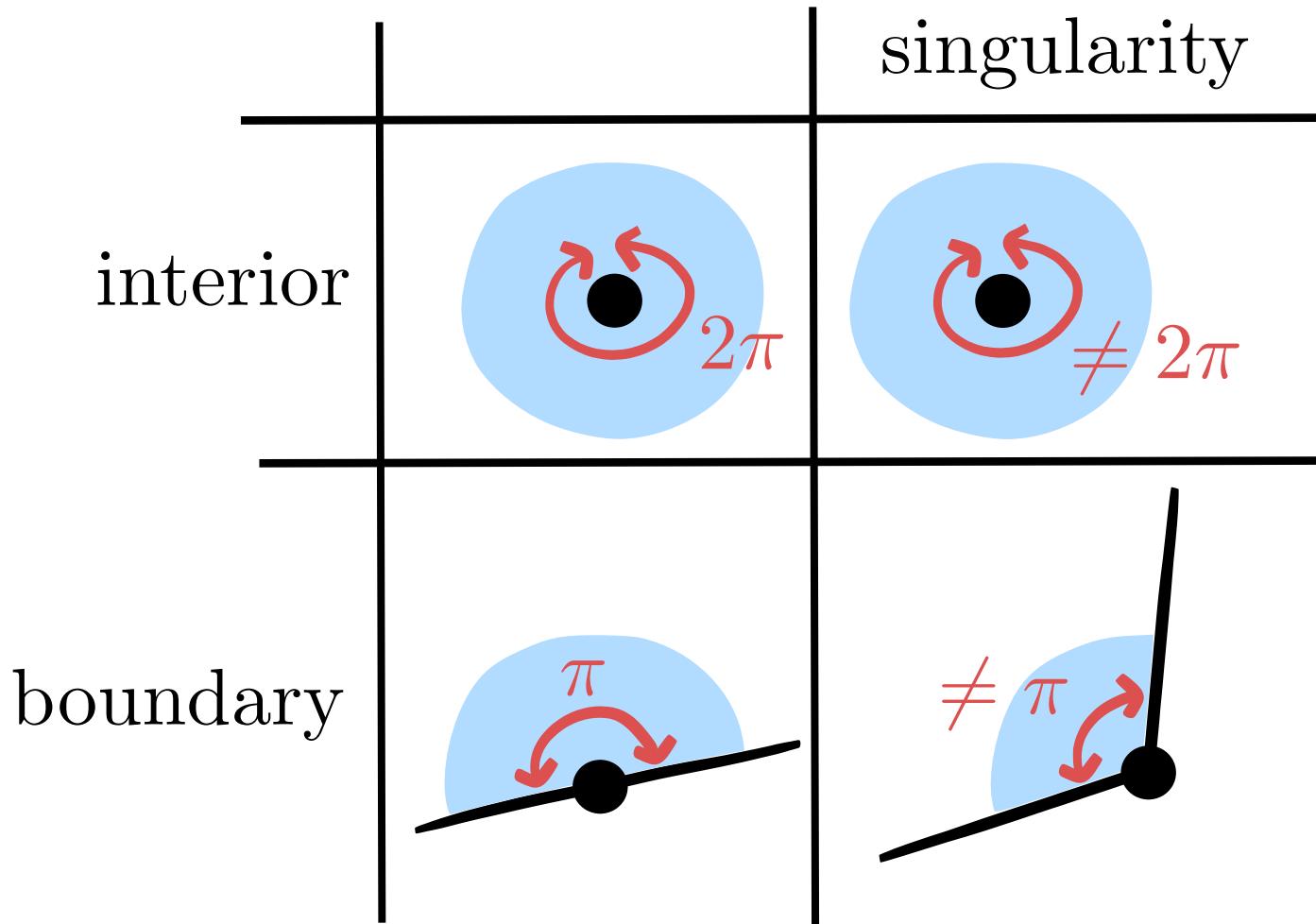
Every mesh gives a triangulation

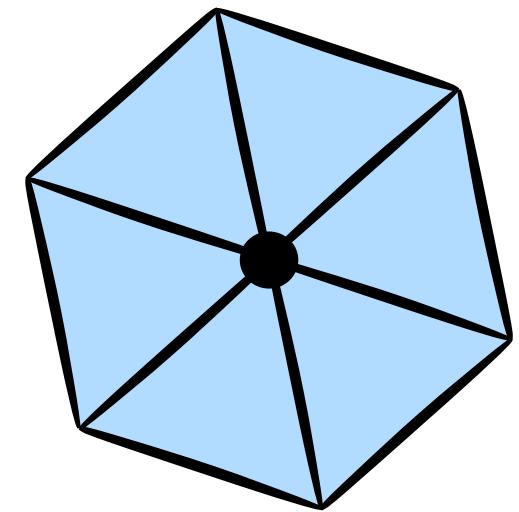
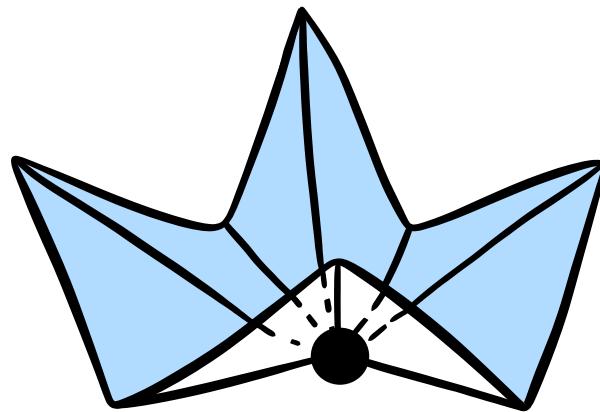
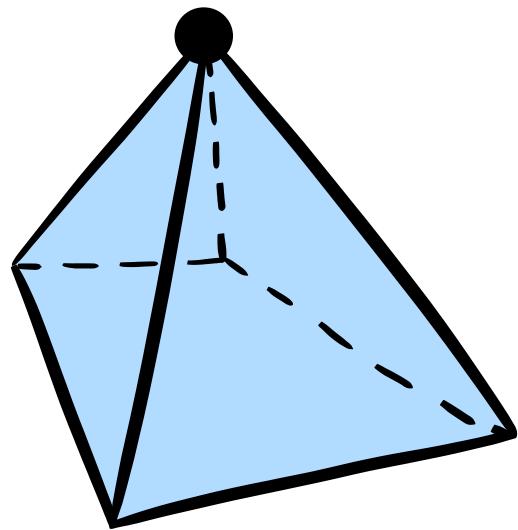


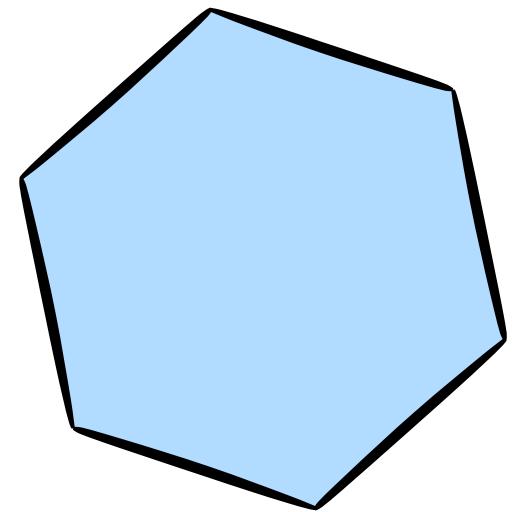
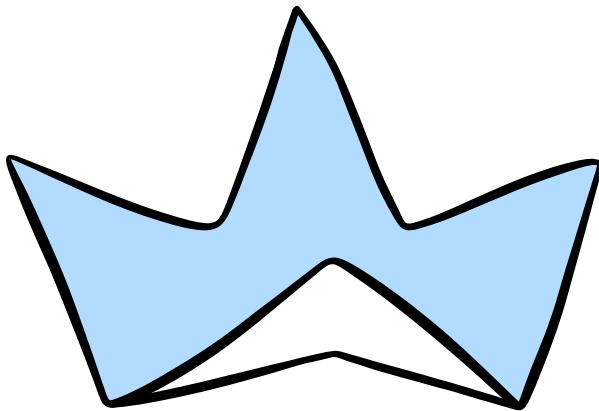
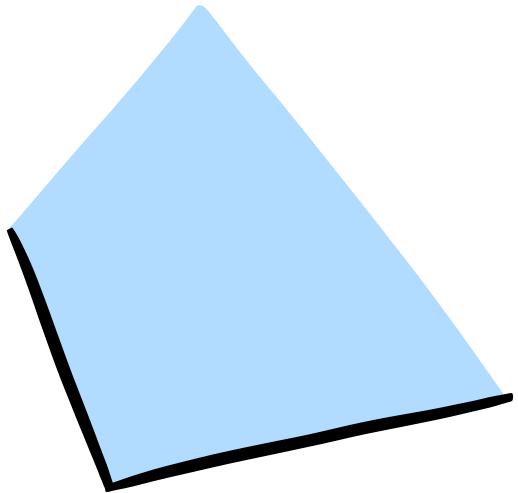
Every mesh gives a triangulation  
but converse is false!

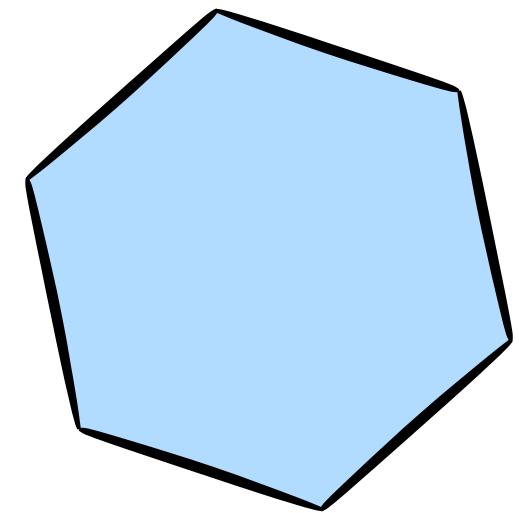
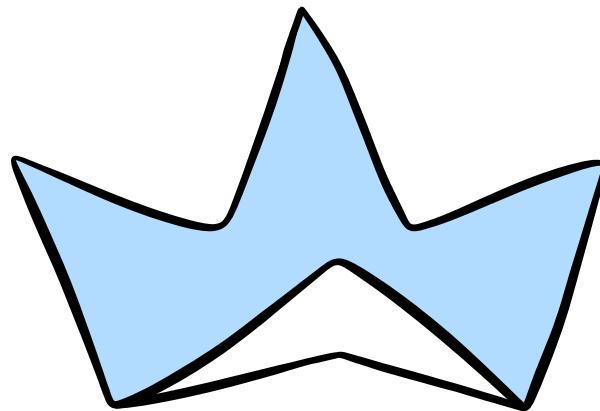
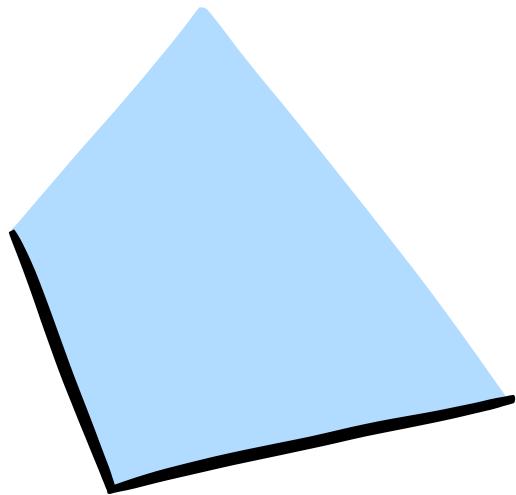


# Types of points on the surface

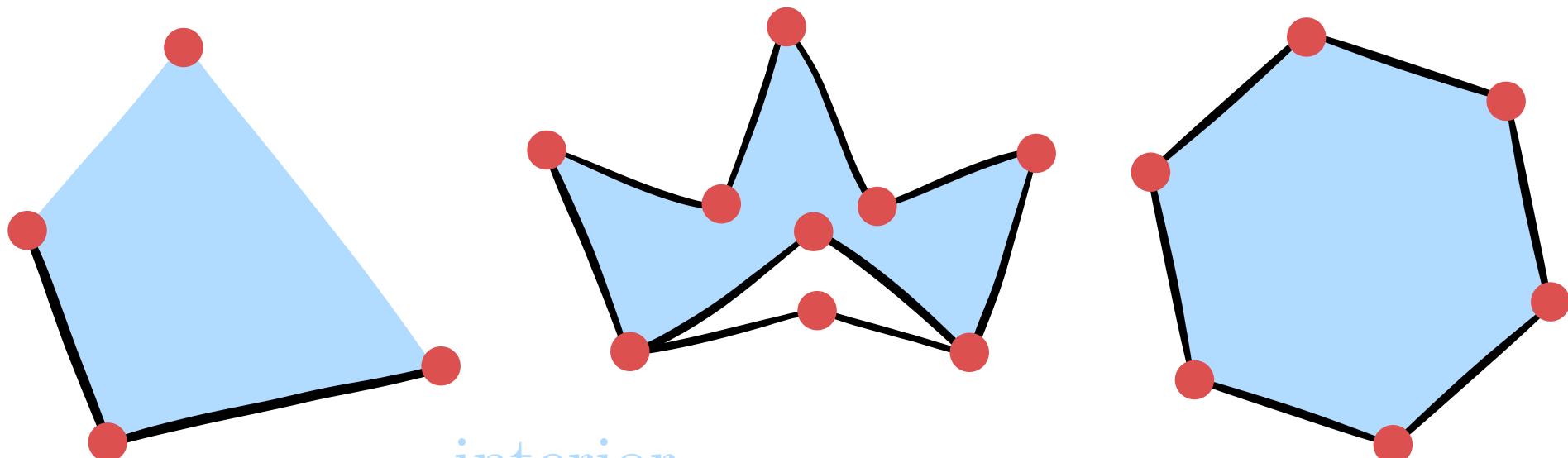








interior  
boundary



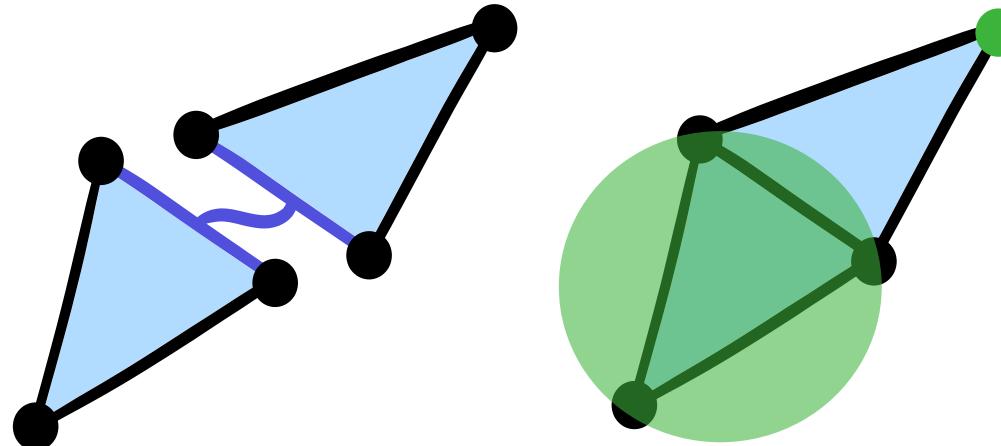
interior  
boundary

singularities

# Problem

# Delaunay triangulation

triangulation in which  
every edge is Delaunay



# The Delaunay triangulation

Generically, every surface has a **unique**  
Delaunay triangulation  
whose vertices are the singularities

# The Delaunay triangulation

Generically, every surface has a **unique**  
Delaunay triangulation  
whose vertices are the singularities

## Problem

Given triangulation  $T$ , compute  
“the” Delaunay triangulation  
of the surface of  $T$

# Motivations

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- isometry testing

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- isometry testing
- shortest paths

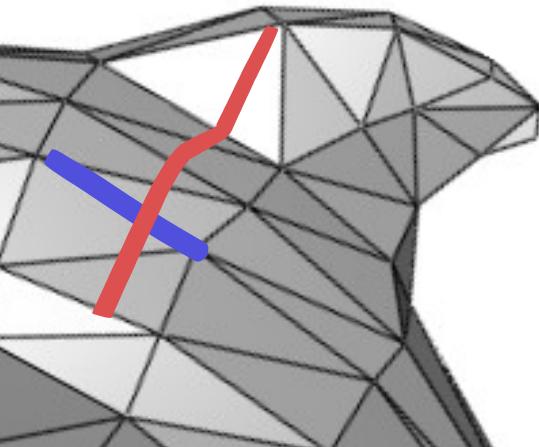
- shortest paths

# Shortest paths on meshes

On a mesh  $M$  with  $n$  triangles...

a shortest path cannot cross an edge twice

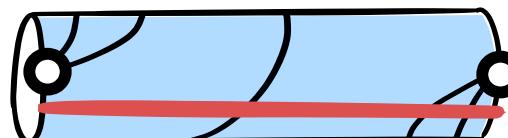
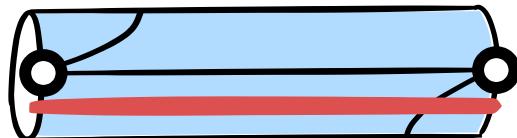
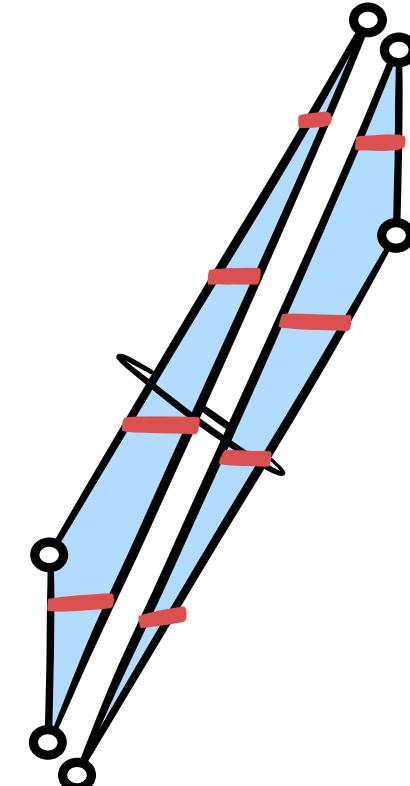
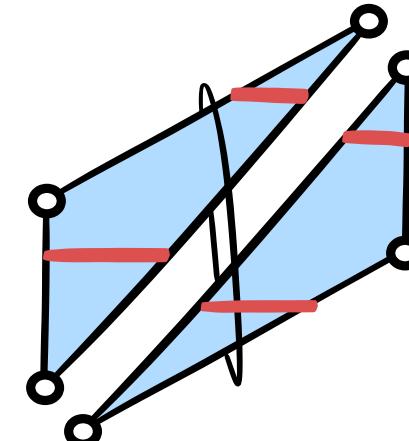
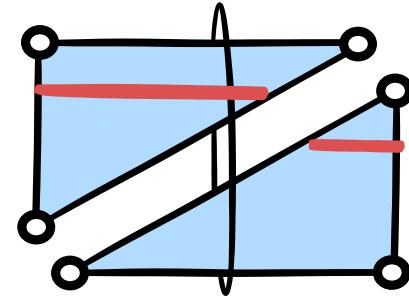
→ shortest path can be computed in  $O(f(n))$  time



Mitchel, Mount, Papadimitriou, 1987

# Shortest paths on triangulations

they can cross edges arbitrarily many times

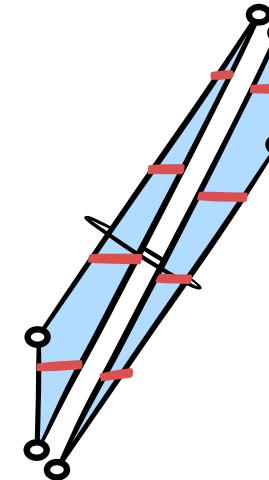


Erickson, 2006

# Shortest paths on triangulations

Löffler, Ophelders, Staals, Silveira, 2023

happiness  $h$ : max number of times  
a shortest path visits a triangle



→ shortest path can be computed in  $O(f(n, h))$  time

# Shortest paths on triangulations

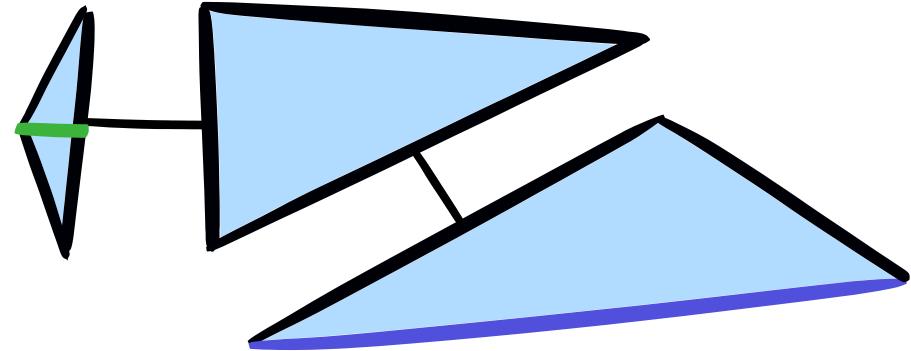
Löffler, Ophelders, Staals, Silveira, 2023

Delaunay triangulations have happiness  $O(1)$

# Result

# Result

aspect ratio =  
 $\frac{\text{maximum side length}}{\text{minimum height}}$

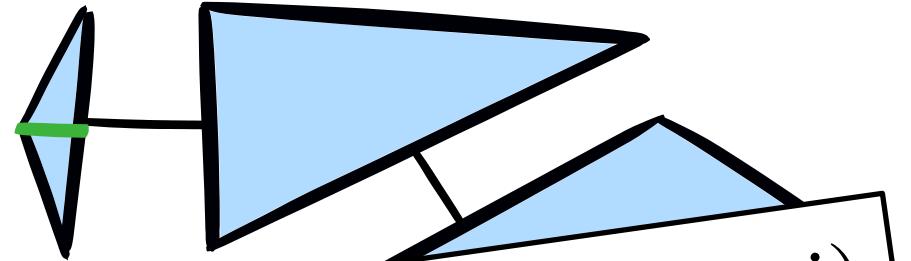


D. 2025

Given triangulation  $T$  of  $n$  triangles, of aspect ratio  $r$ , whose surface has no boundary, we can compute Delaunay in  $O(n^3 \log^2(n) \cdot \log^4(r))$  time

# Result

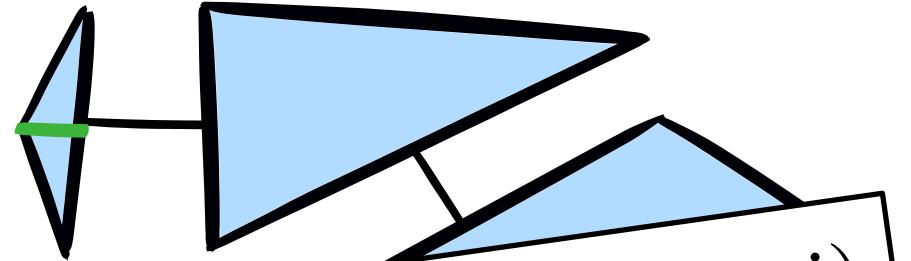
aspect ratio =  
maximum side length  
minimum height



D. S. G. Previous algorithms (Delaunay flips and Voronoi) achieved no better than  $O(\text{Poly}(n, r))$  time for triangulating a surface of aspect ratio  $r$ , of size  $n$ . If the surface has no boundary, we can compute Delaunay in  $O(n^3 \log^2(n) \cdot \log^4(r))$  time.

# Result

aspect ratio =  
maximum side length  
minimum height



D. So

G.

Previous algorithms (Delaunay flip and Voronoi)  
achieved no better than  $O(F)$

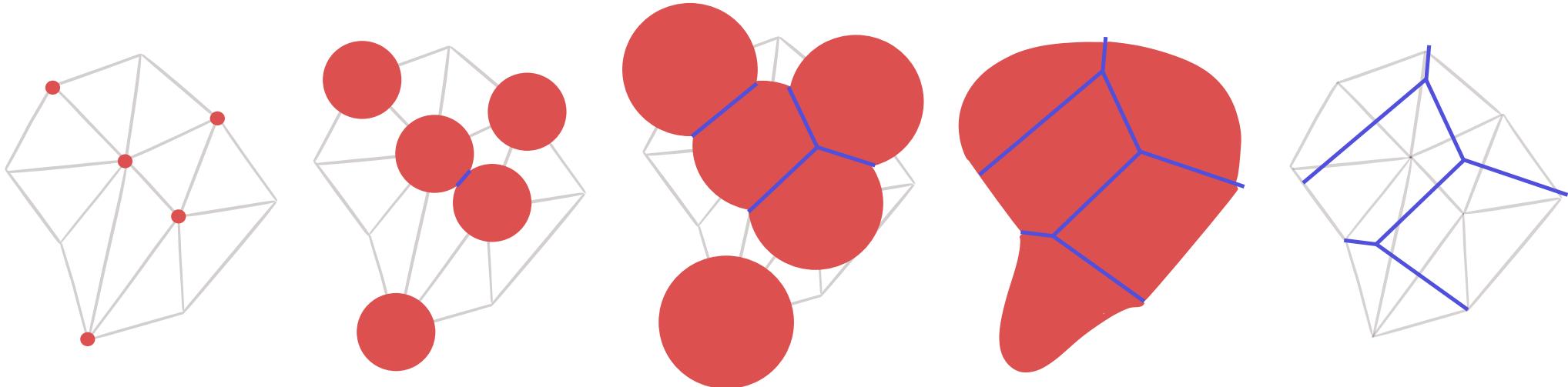
surface has no boundary, we can compute  
Delaunay in  $O(n^3 \log^2(n) \cdot \log^4(r))$  time

Now backed by a  
lower bound!

# Algorithm overview

# Classical method

compute the Voronoi diagram  
by propagating waves



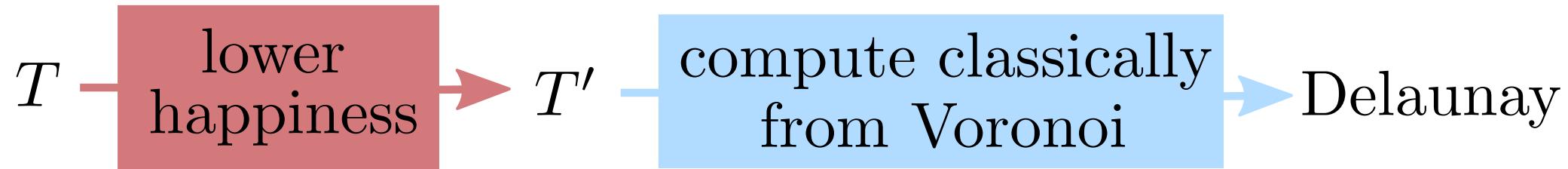
then derive Delaunay from it

# Algorithm



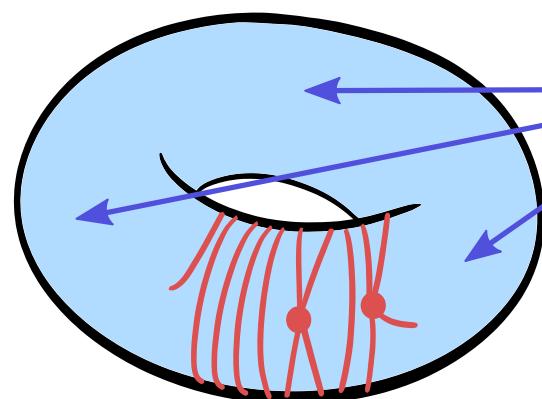
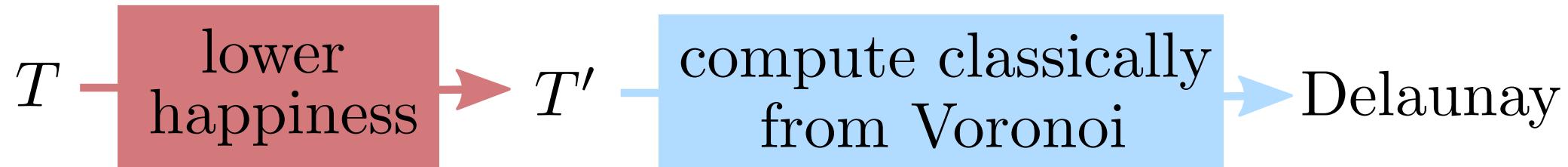
# Algorithm

D. 2025



# Algorithm

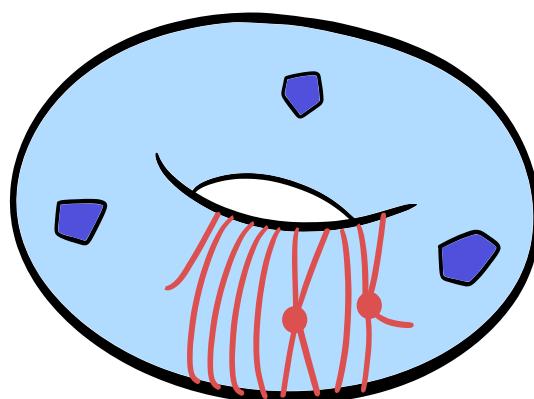
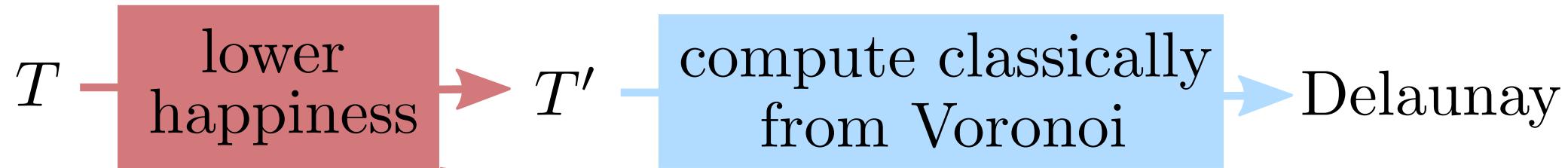
D. 2025



consider the singularities

# Algorithm

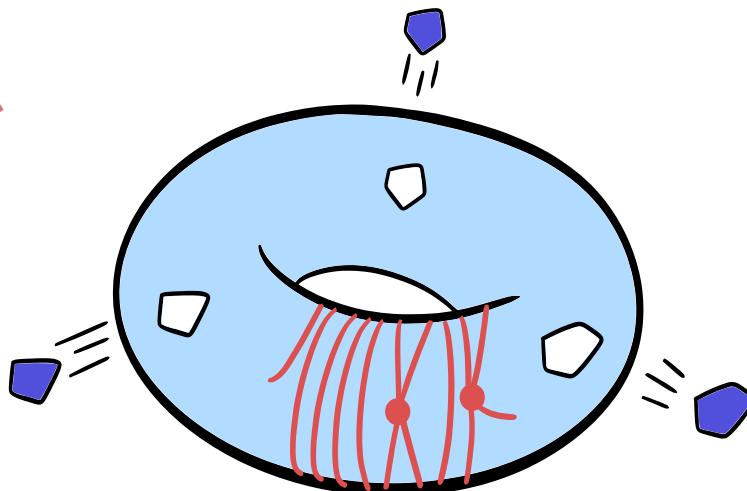
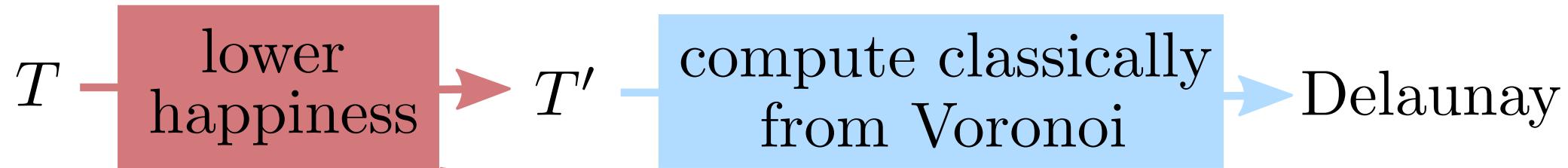
D. 2025



cut out caps around the singularities

# Algorithm

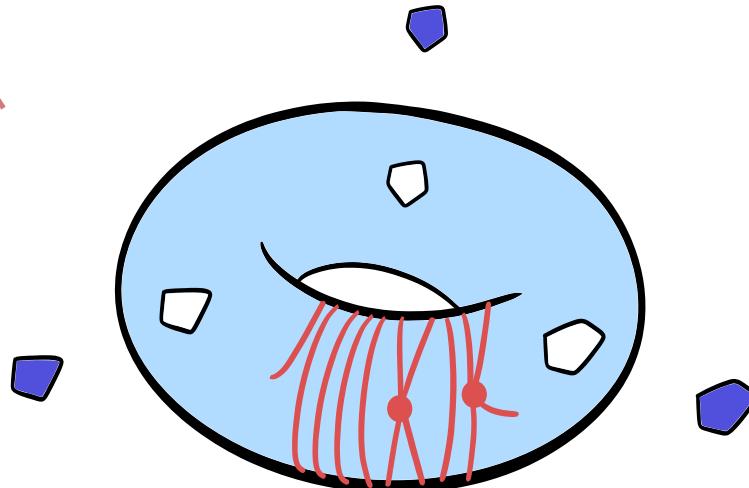
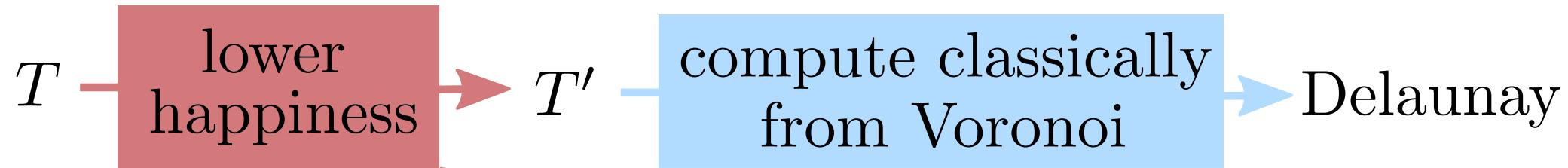
D. 2025



cut out caps around the singularities

# Algorithm

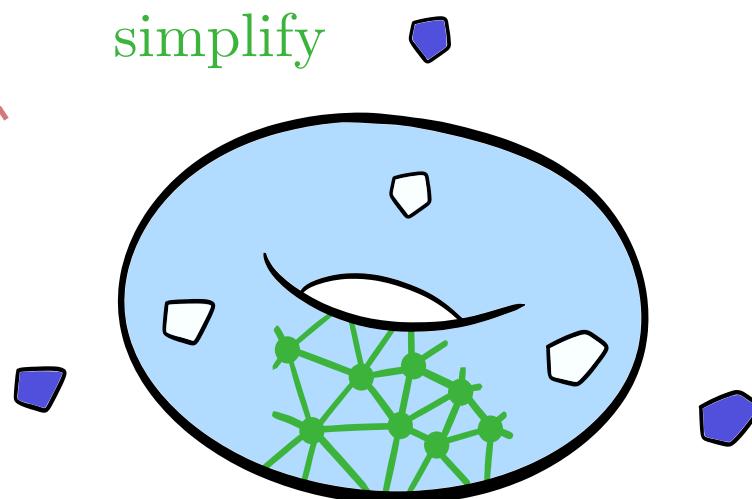
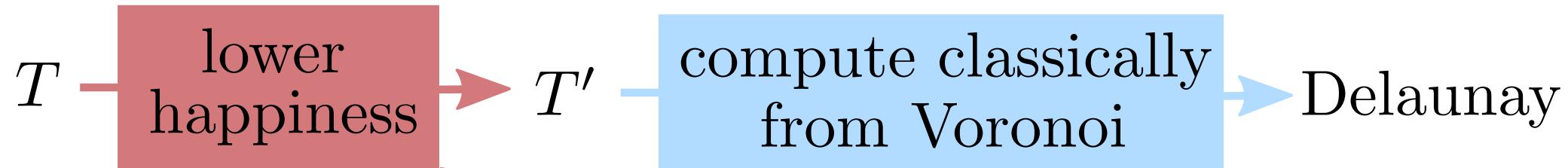
D. 2025



cut out caps around the singularities

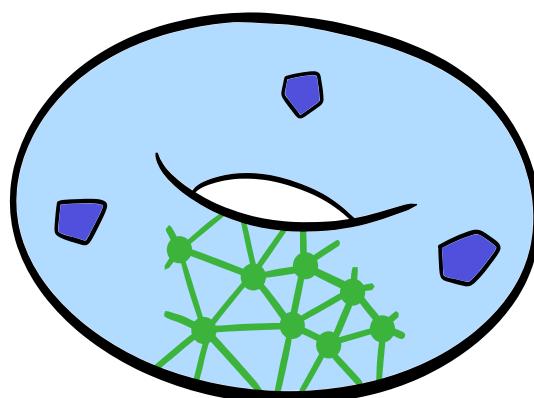
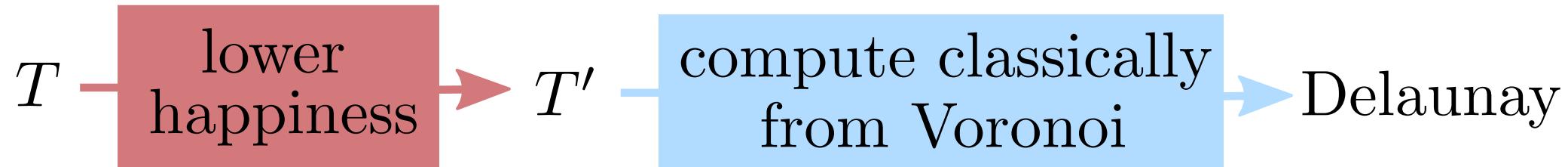
# Algorithm

D. 2025



# Algorithm

D. 2025



put the caps back

# Simplification algorithm

Tuned combination of elementary operations, like  
inserting vertices in edges  
inserting edges in faces  
deleting vertices  
repeated many times

some simplify the geometry,  
others decrease # vertices

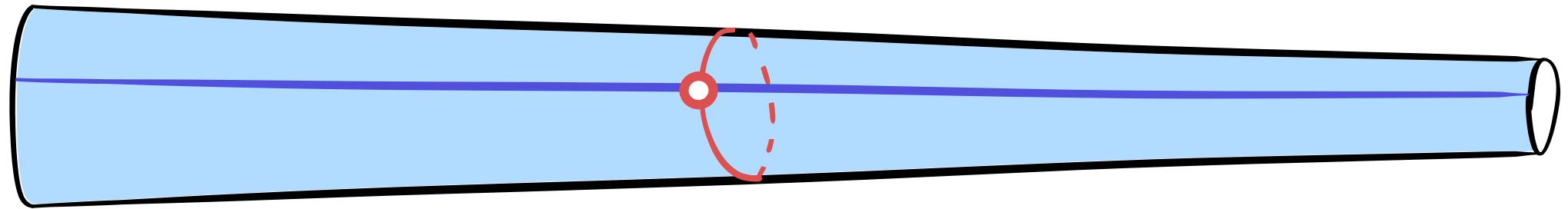
# Analysis

Show that during execution:

1. # vertices stays bounded
2. Geometry gets simpler and simpler



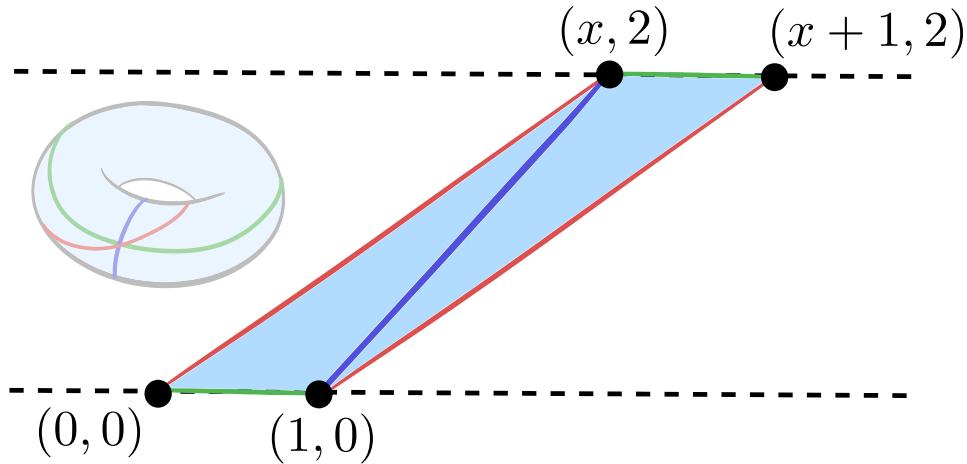
Enclosure



# Lower bound

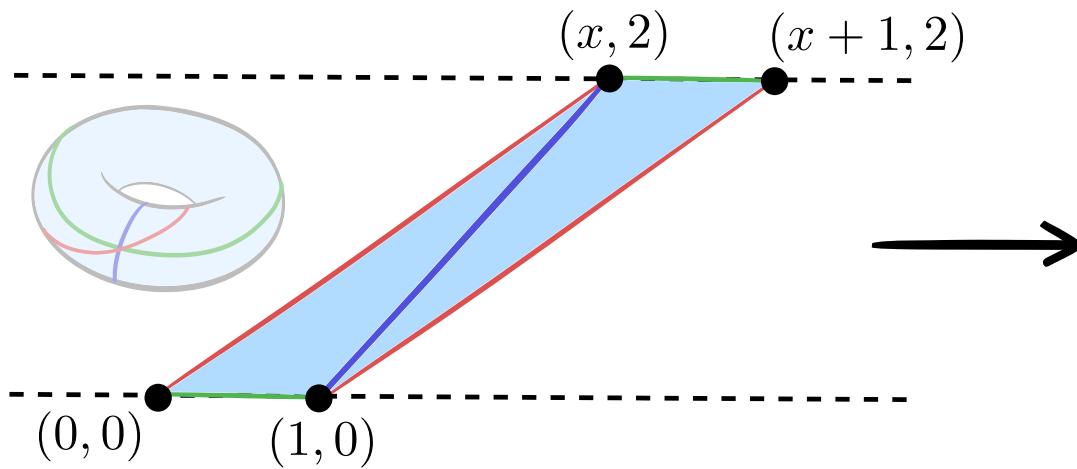
# Lower bound

Input

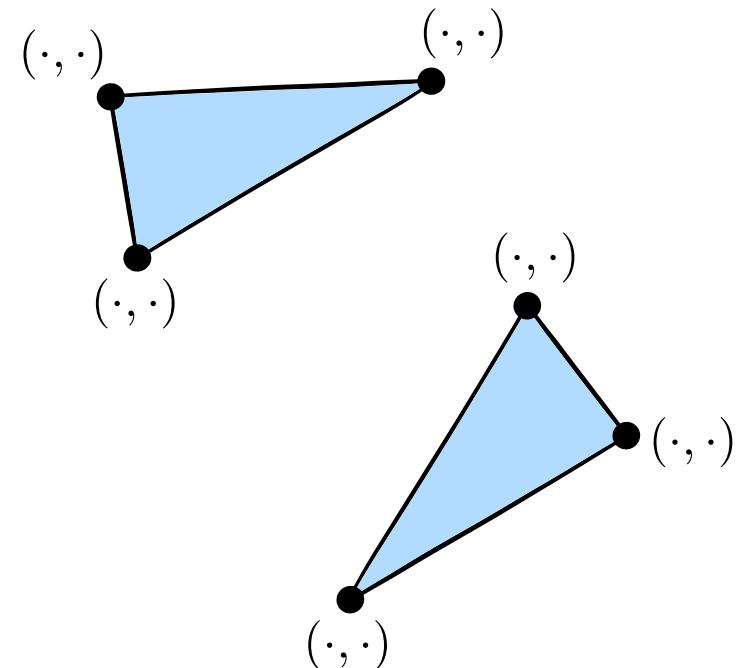


# Lower bound

Input

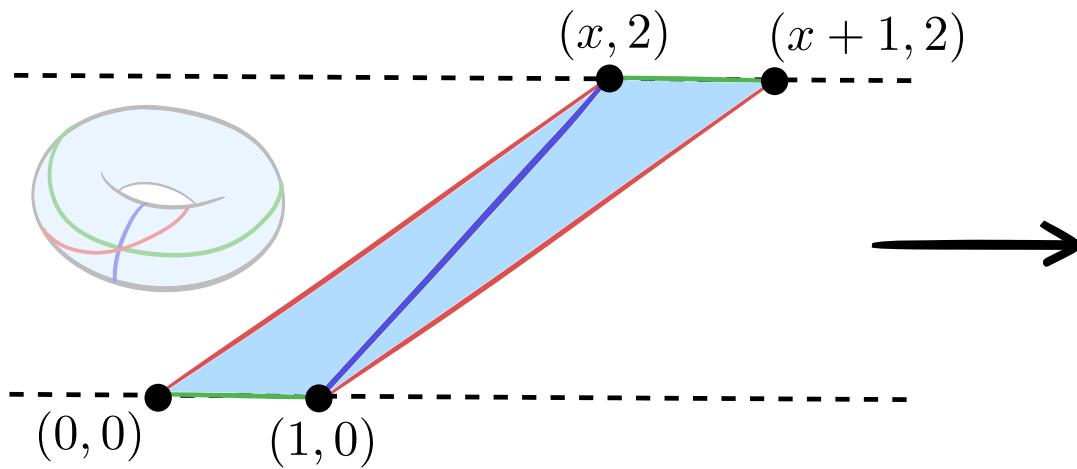


Output: Delaunay

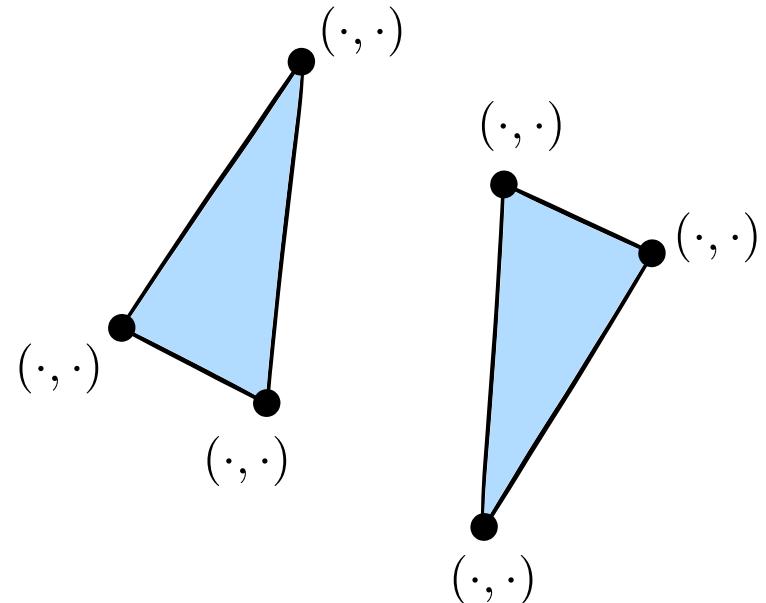


# Lower bound

Input

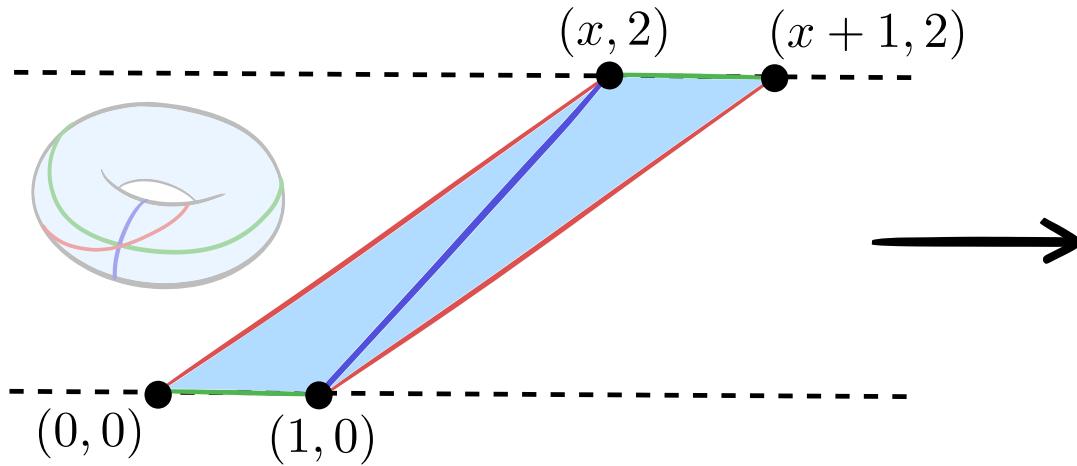


Output: Delaunay

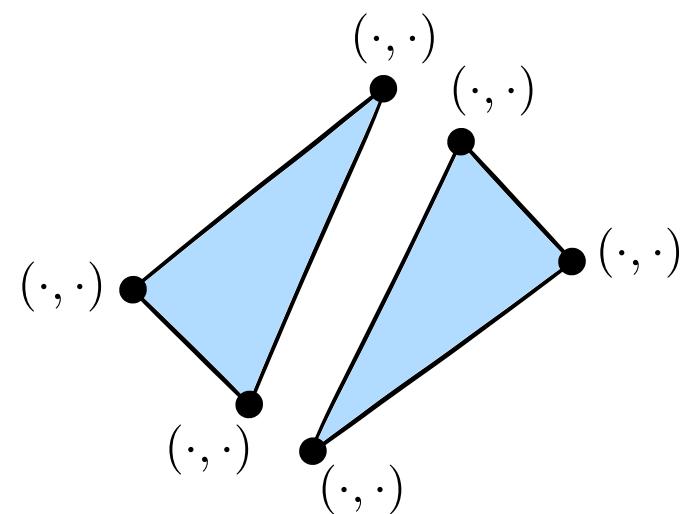


# Lower bound

Input

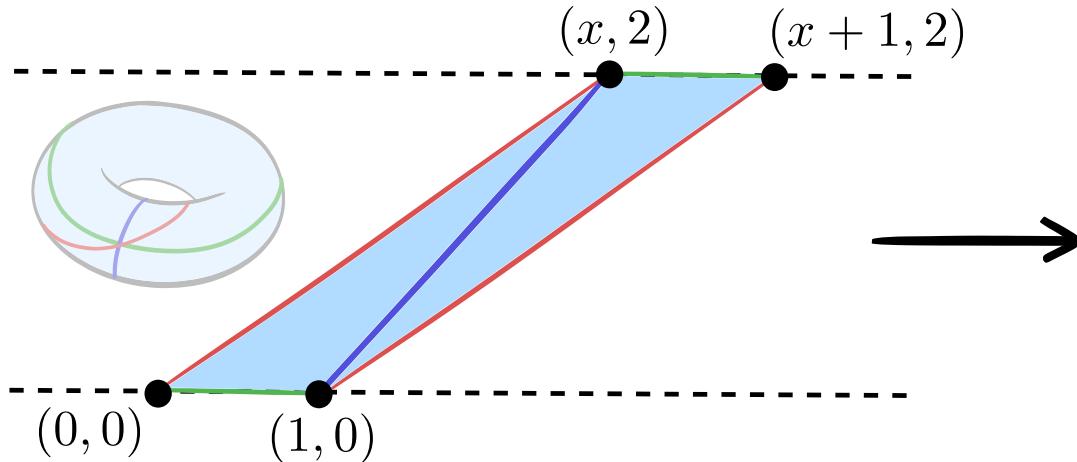


Output: Delaunay

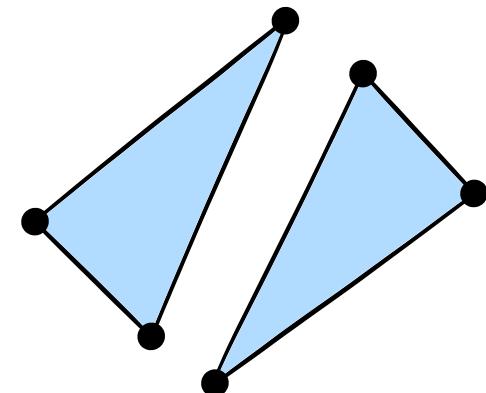


# Lower bound

Input

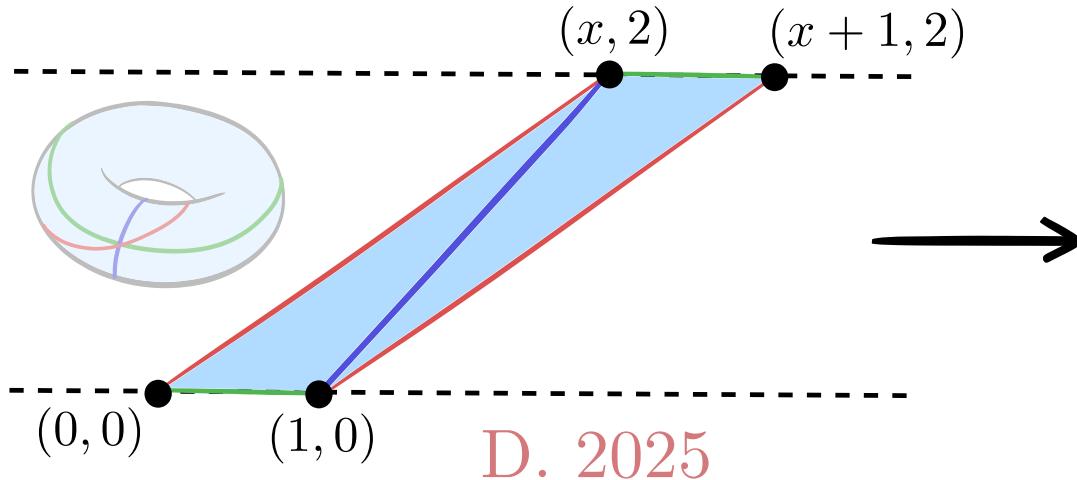


Output: Delaunay

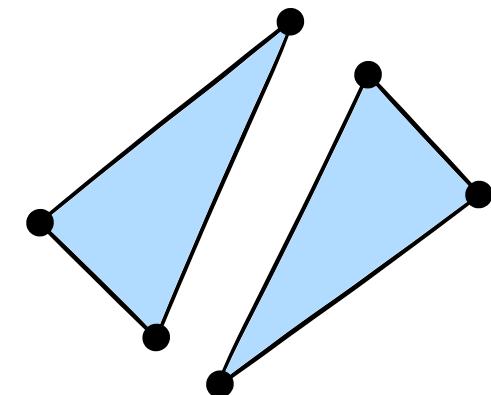


# Lower bound

Input



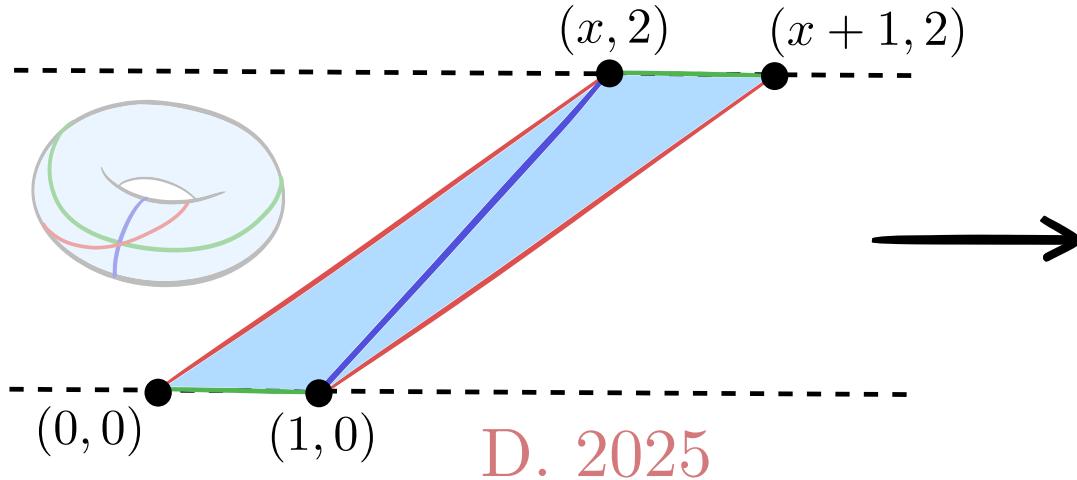
Output: Delaunay



No Real RAM algo can compute  
Delaunay from  $x$  in  $o(\log x)$  time

# Lower bound

Input



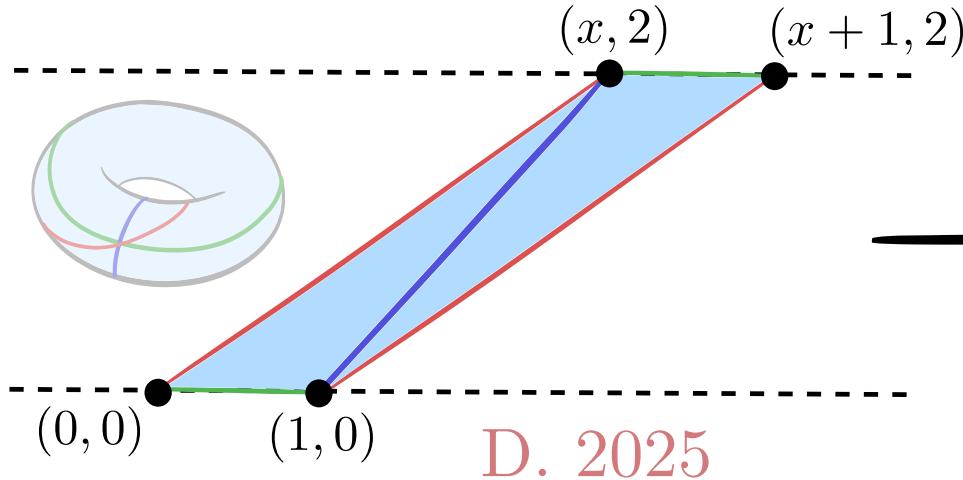
Output: Delaunay

No Real RAM algo can compute  
Delaunay from  $x$  in  $o(\log x)$  time

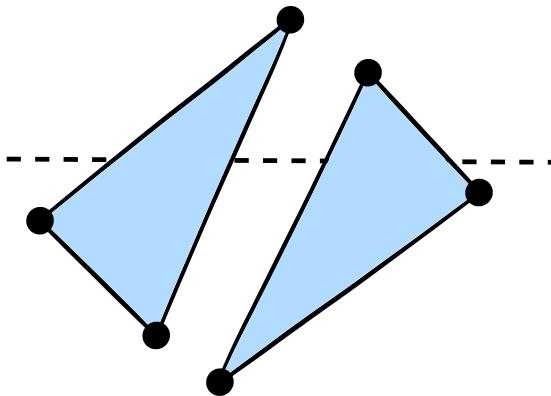
Otherwise we could compute  $\lfloor x \rfloor$  from  $x$  in  $o(\log x)$  time

# Lower bound

Input



Output: Delaunay

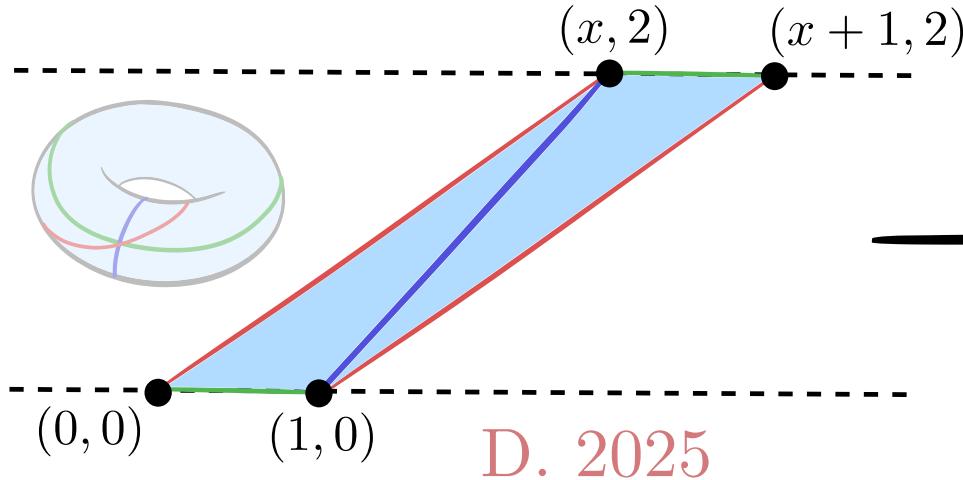


No Real RAM algo can compute  
Delaunay from  $x$  in  $o(\log x)$  time

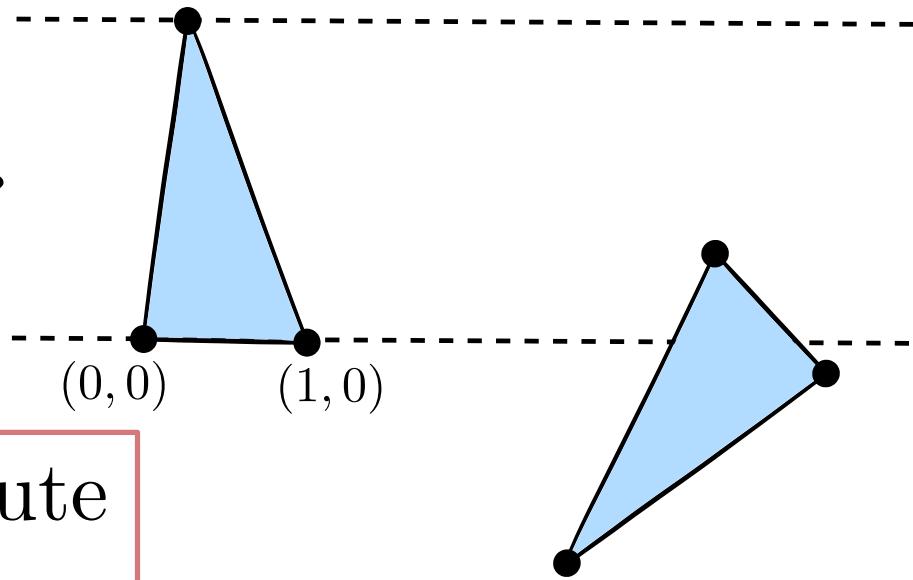
Otherwise we could compute  $\lfloor x \rfloor$  from  $x$  in  $o(\log x)$  time

# Lower bound

Input



Output: Delaunay

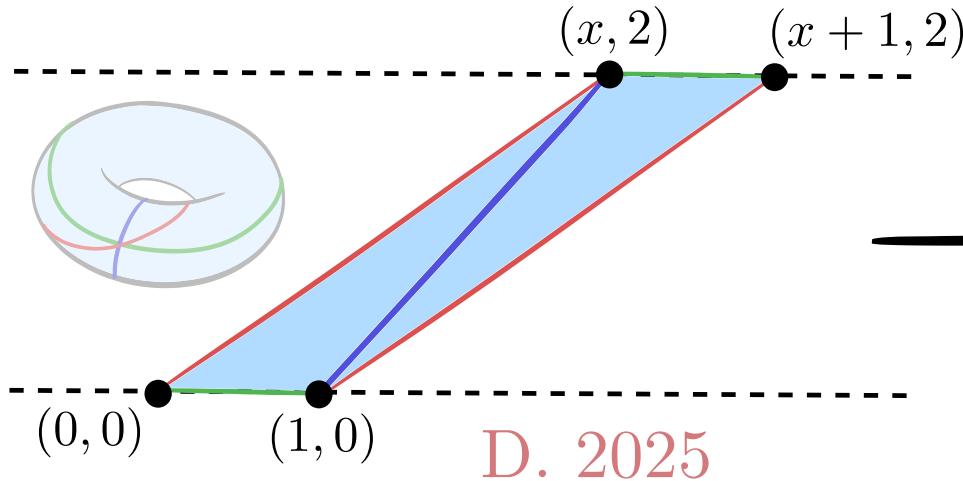


No Real RAM algo can compute  
Delaunay from  $x$  in  $o(\log x)$  time

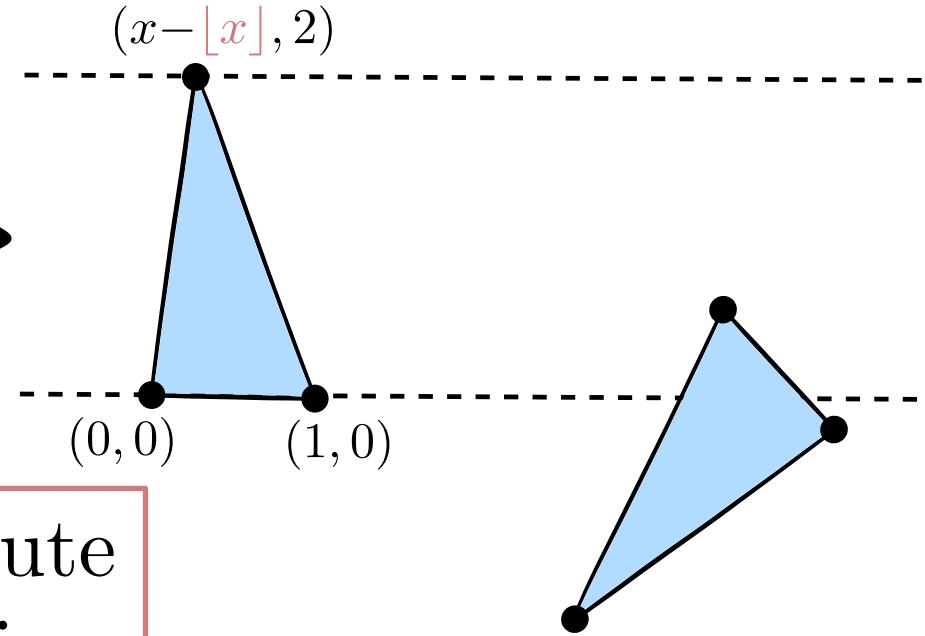
Otherwise we could compute  $\lfloor x \rfloor$  from  $x$  in  $o(\log x)$  time

# Lower bound

Input



Output: Delaunay



No Real RAM algo can compute  
Delaunay from  $x$  in  $o(\log x)$  time

Otherwise we could compute  $\lfloor x \rfloor$  from  $x$  in  $o(\log x)$  time

# Untangling Graphs

Computing Delaunay Triangulations

Other works and conclusion

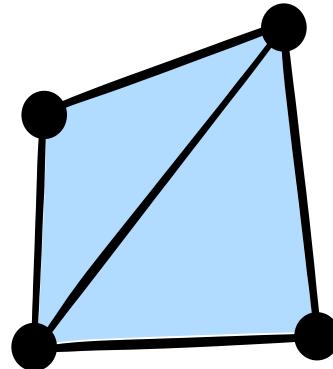
# Untangling Graphs

## Computing Delaunay Triangulations

Other works and conclusion

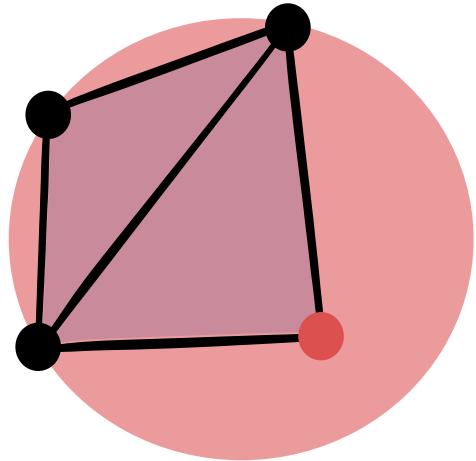
# Delaunay flips algorithm

Delaunay flip:



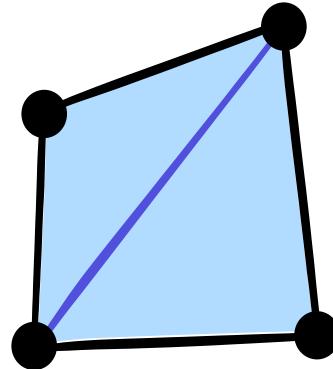
# Delaunay flips algorithm

Delaunay flip:



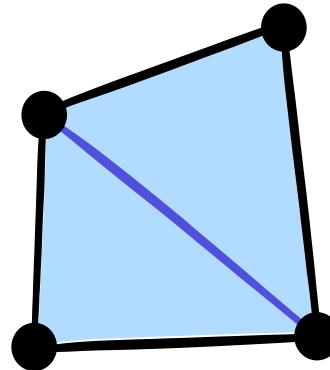
# Delaunay flips algorithm

Delaunay flip:



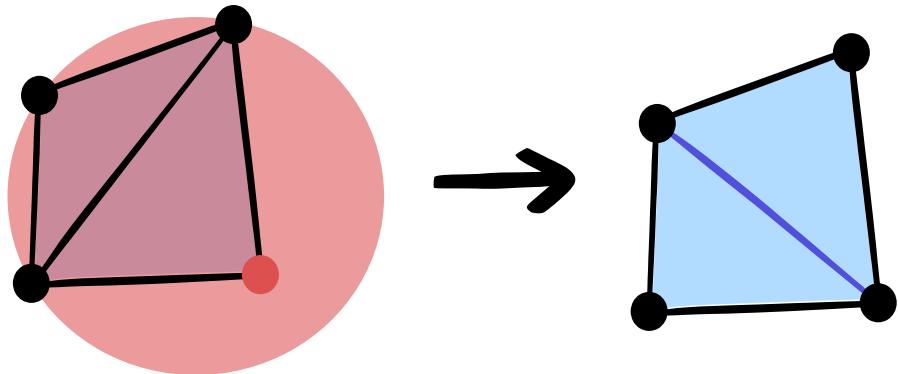
# Delaunay flips algorithm

Delaunay flip:



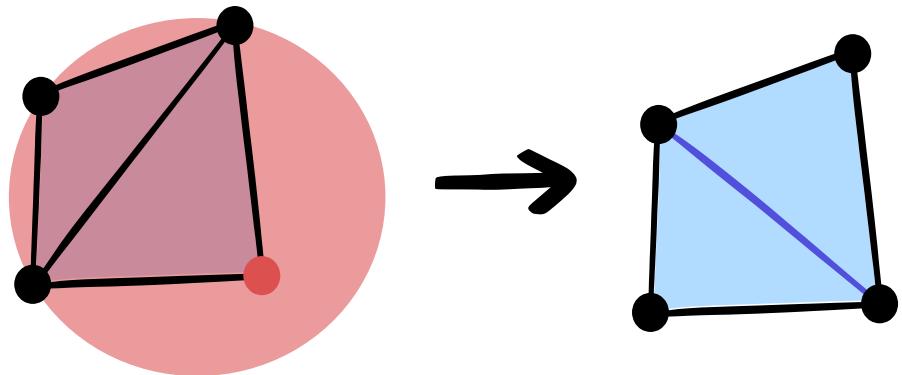
# Delaunay flips algorithm

Delaunay flip:



# Delaunay flips algorithm

Delaunay flip:

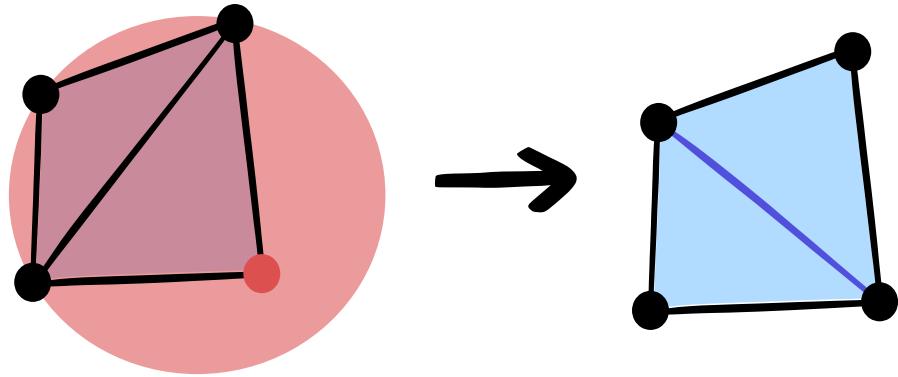


Algorithm:

apply Delaunay flips greedily as long as you can

# Delaunay flips algorithm

Delaunay flip:



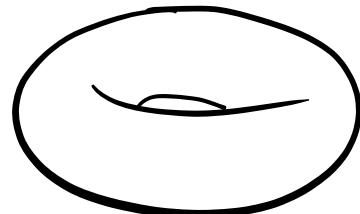
Algorithm:

apply Delaunay flips greedily as long as you can  
→ terminates and outputs a Delaunay triangulation

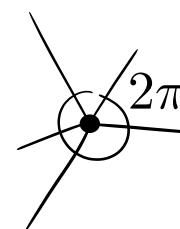
# Other results

D., 2022-23

Upper bound for # Delaunay flips on flat tori,  
tight up to constant factor



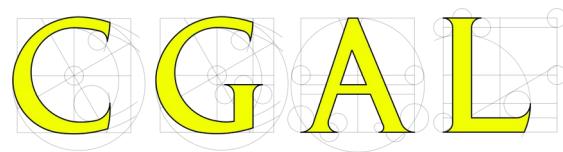
topological shape of a torus



$2\pi$  around each vertex

# Other results

Despré, D., Pouget, and Teillaud, 2025



package for computing with  
hyperbolic surfaces

generation (genus 2 only)

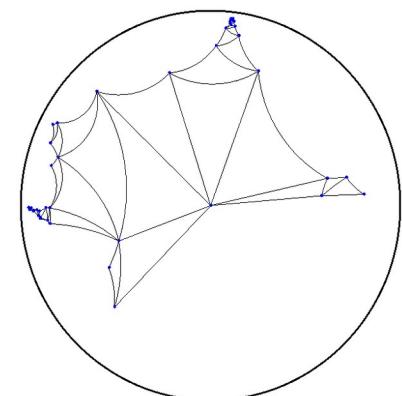
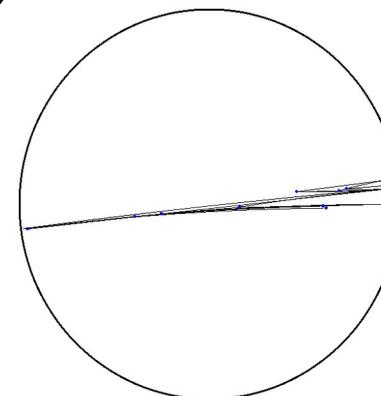


Triangulated hyperbolic  
surface

Delaunay flip



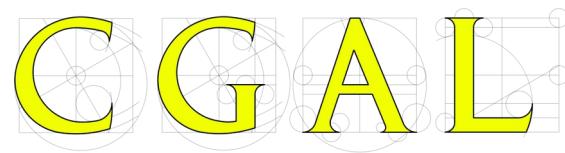
visualization



# Other results

Despré, D., Pouget, and Teillaud

*Exact computations!*  
package for computing  
hyperbolic surfaces



generation (genus 2 only)

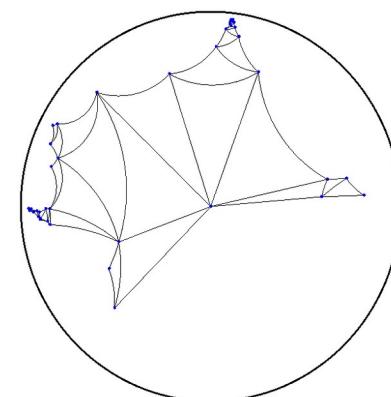
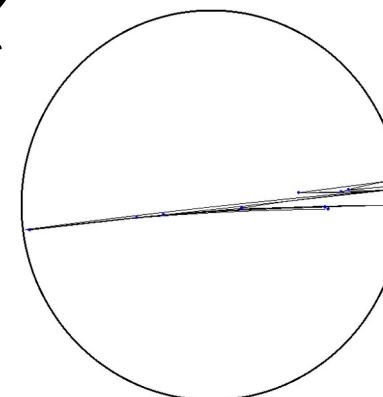


Triangulated hyperbolic  
surface

Delaunay flip

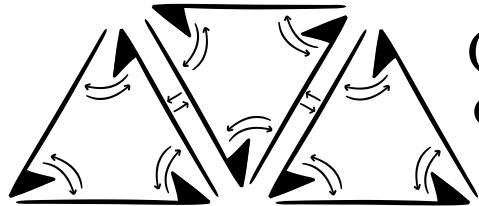


visualization



Triangulated hyperbolic  
surface

# Combinatorics

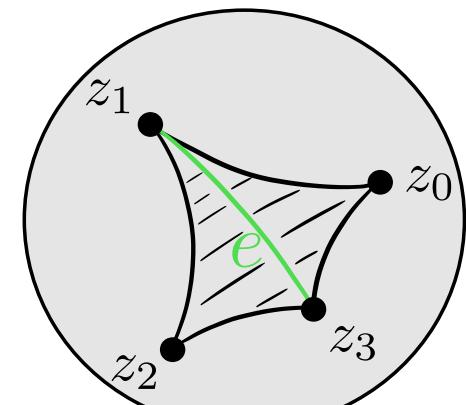


CGAL  
combinatorial maps

Triangulated hyperbolic  
surface

# Geometry

- each edge is decorated with a complex number (cross ratio)



$$e \rightarrow \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

# Possible continuations

- generation of hyperbolic surfaces of genus  $\geq 3$
- what is the complexity of Delaunay flips algo?
- certifying that a drawing cannot be untangled
- untangling by homotopy moves
- extension to non orientable surfaces
- what is the complexity of untangling?
- minimizing crossings of graphs by homotopy
- how unique reducing triangulations are?