

Algorithms for Topological and Metric Surfaces



Loïc Dubois

Computational Geometry

Design algorithms for geometric problems

This thesis

Focus on surfaces

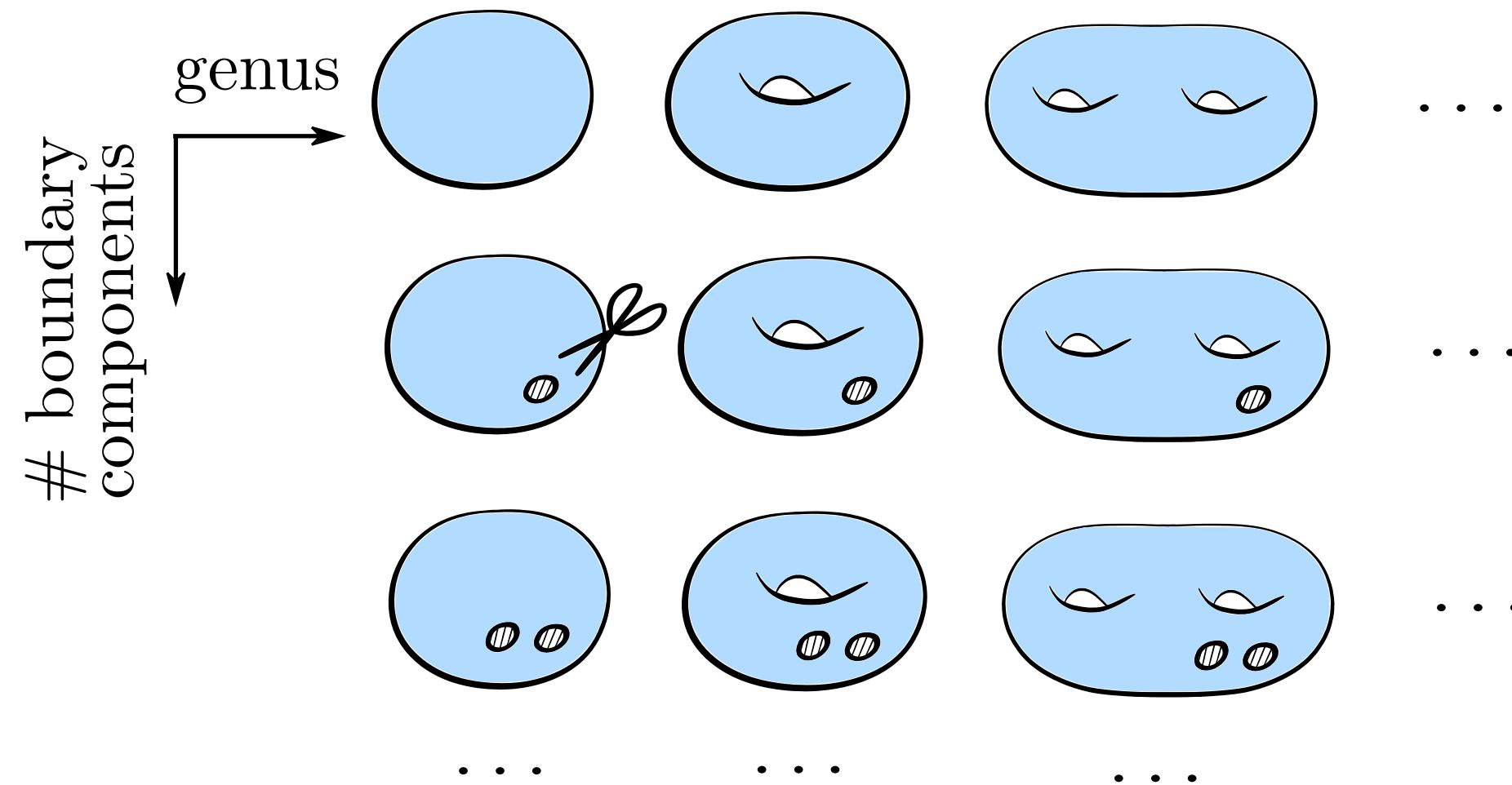


Topology

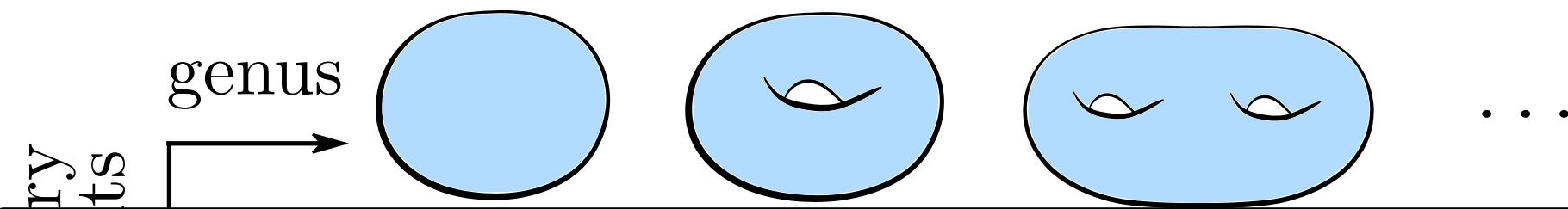


image by Crane and Segeiman

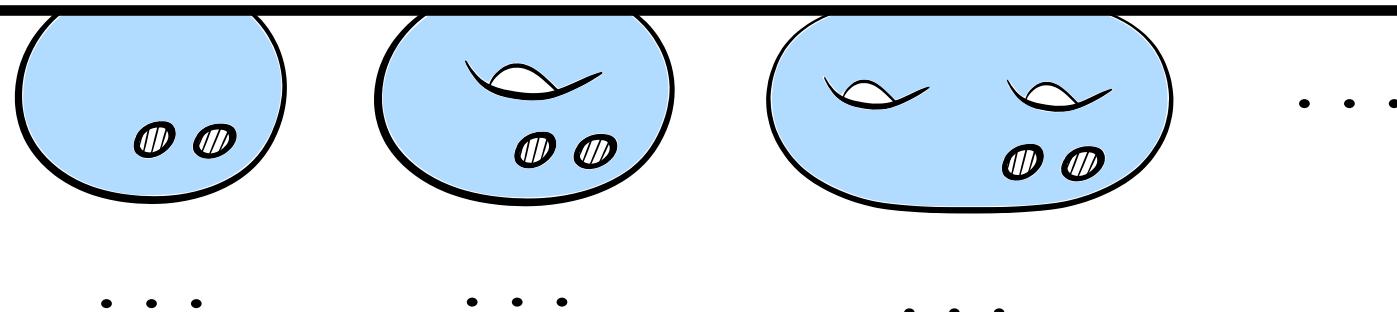
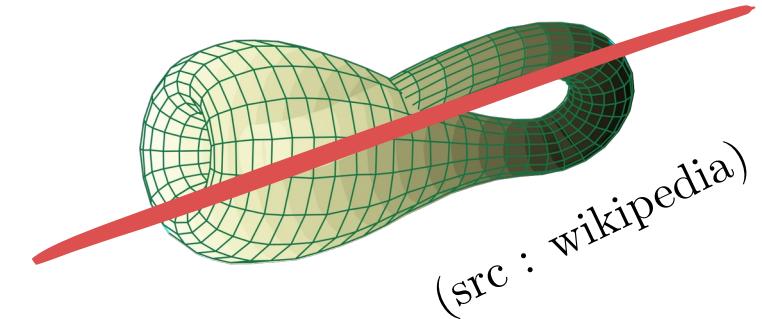
Topological Surfaces



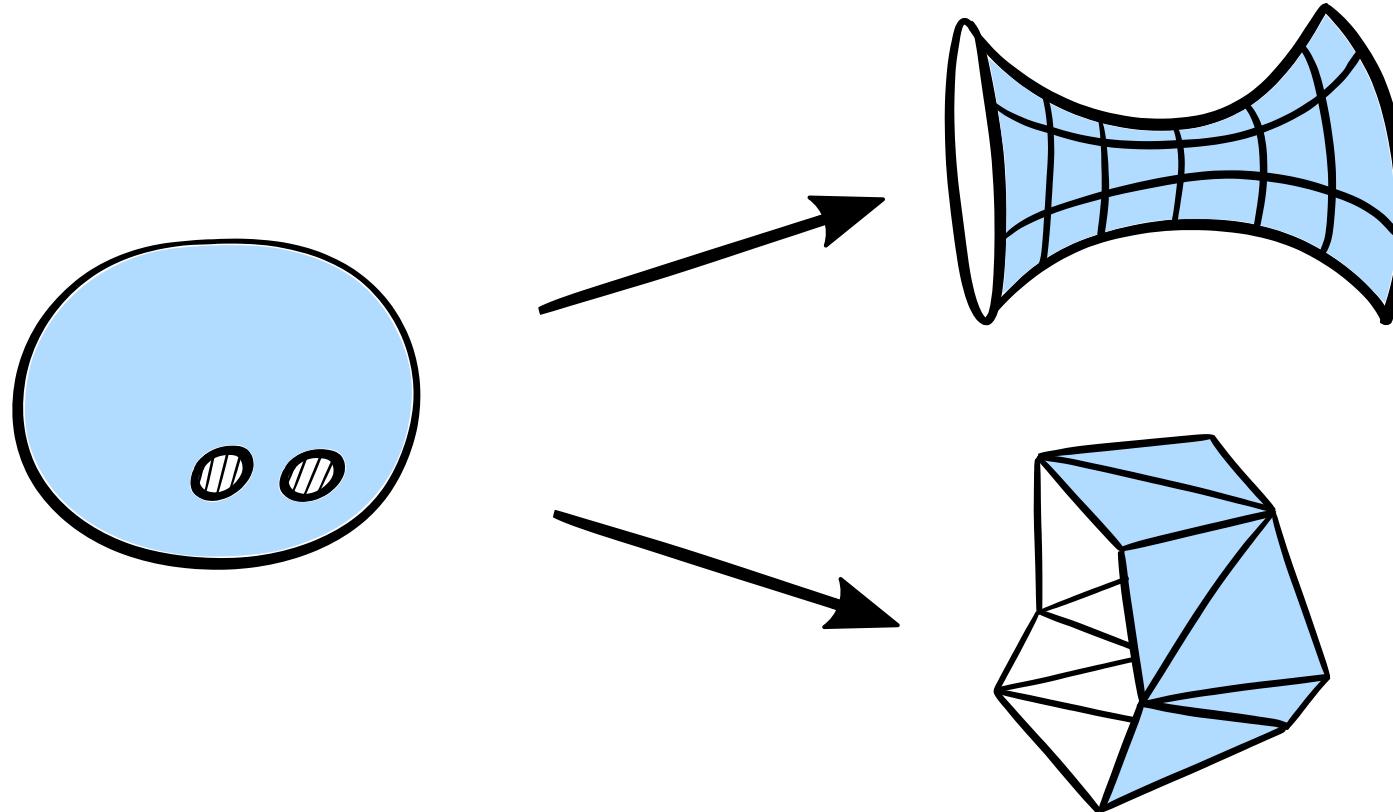
Topological Surfaces



Only **orientable** surfaces today !



Metrics on surfaces



Untangling Graphs

Computing Delaunay Triangulations

Other Results and Possible Continuations

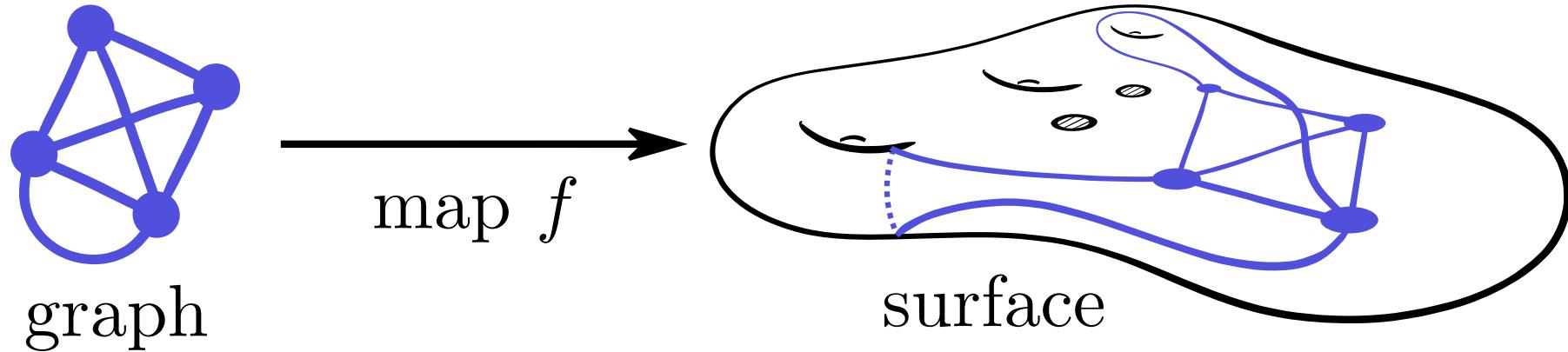
Untangling Graphs

Computing Delaunay Triangulations

Other Results
and Possible Continuations

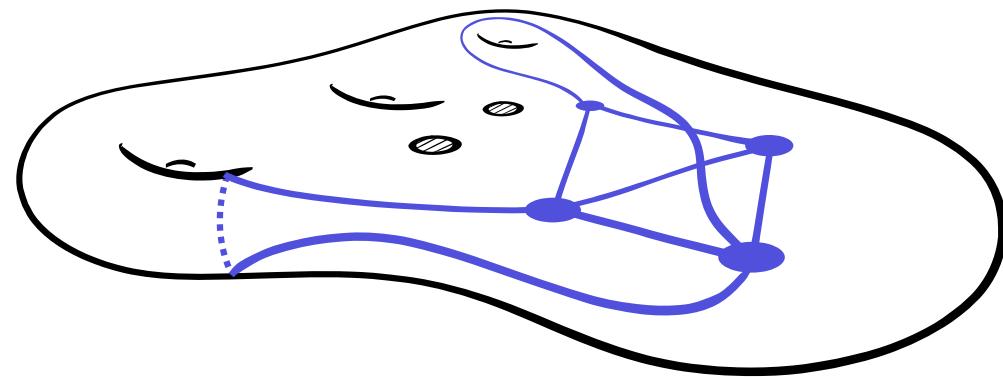
Problem: untangling graphs

Input:

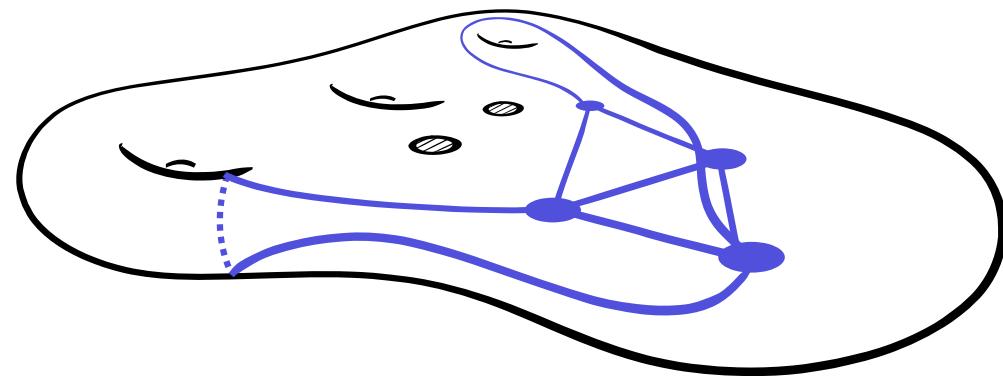


Goal: remove all crossings by deforming f

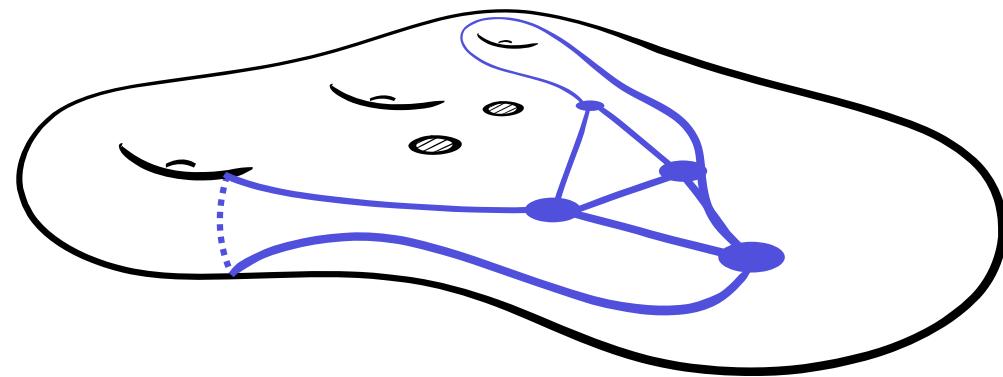
Problem: untangling graphs



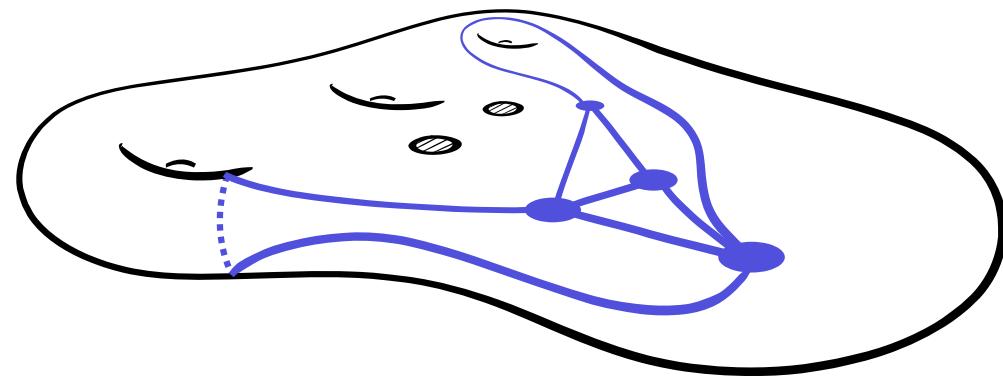
Problem: untangling graphs



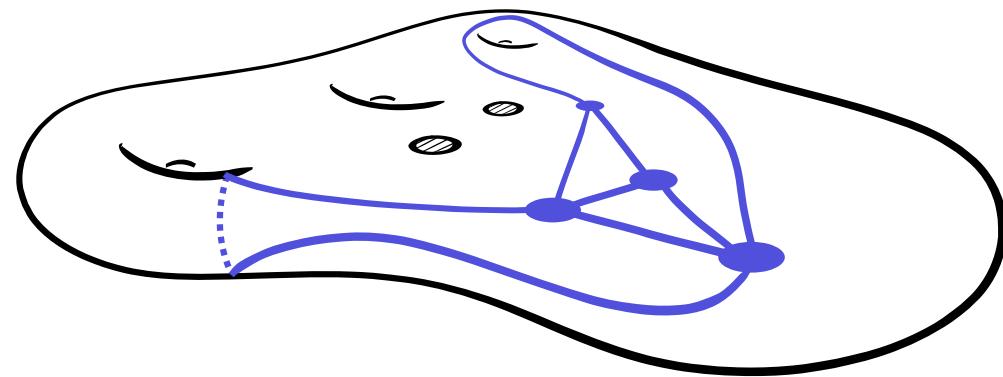
Problem: untangling graphs



Problem: untangling graphs



Problem: untangling graphs

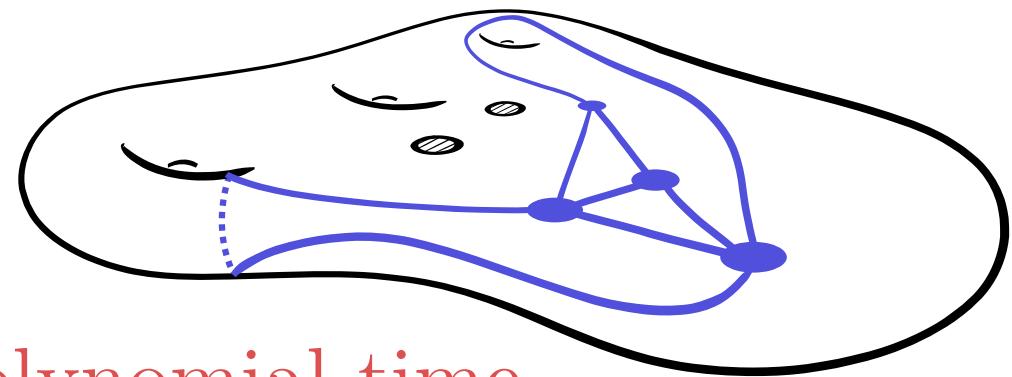


Problem: untangling graphs



Output: Yes (+ untangled drawing) or No

Problem: untangling graphs

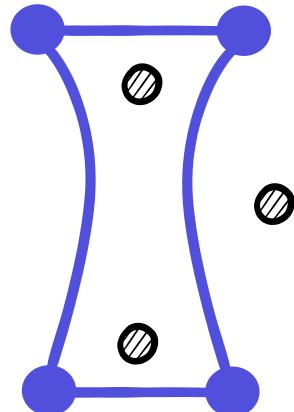


We obtain the first polynomial time
algorithms for this problem

Output: Yes (+ untangled drawing) or No

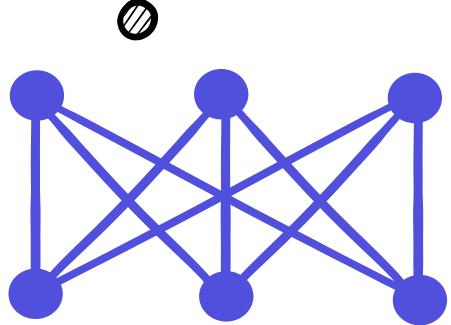


Yes:

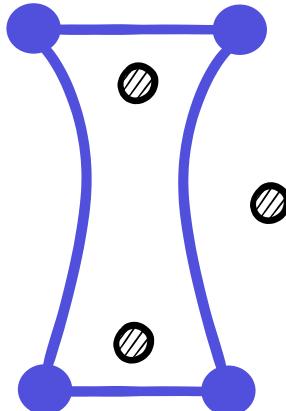


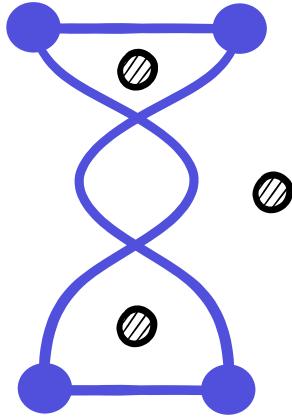


Yes:

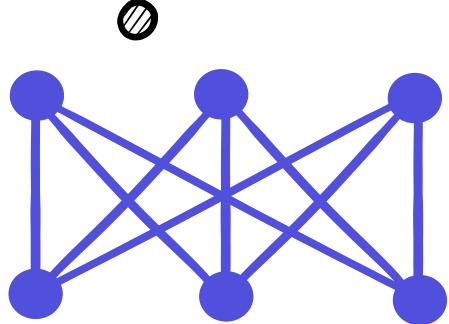


No





Yes:



No



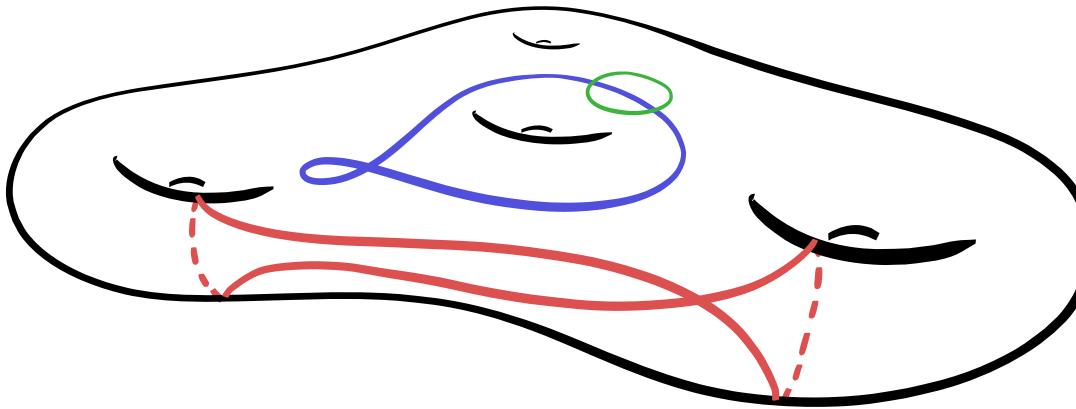
No



Related works

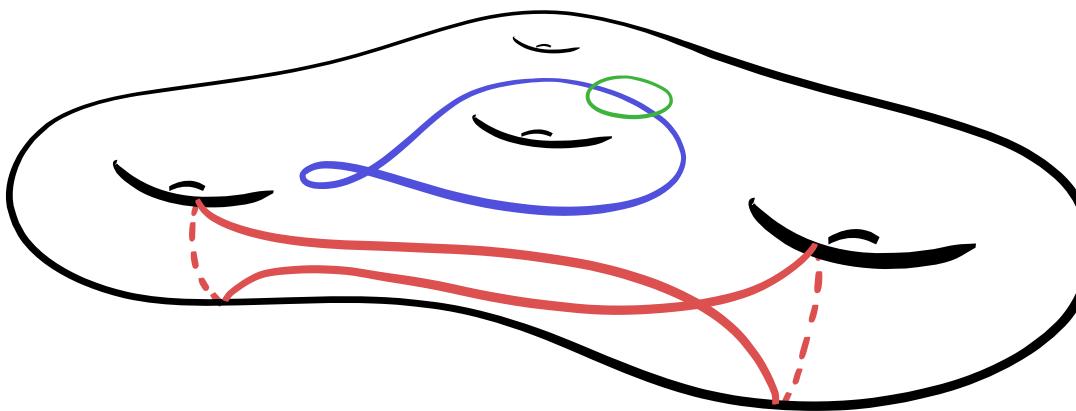
Related problem: untangling curves

Input: closed curves on surface

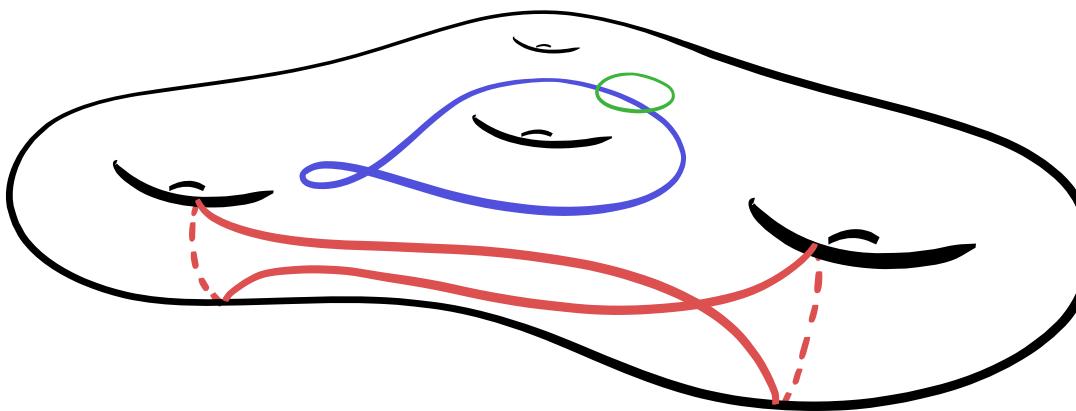


Goal: minimize the crossings by deforming the curves

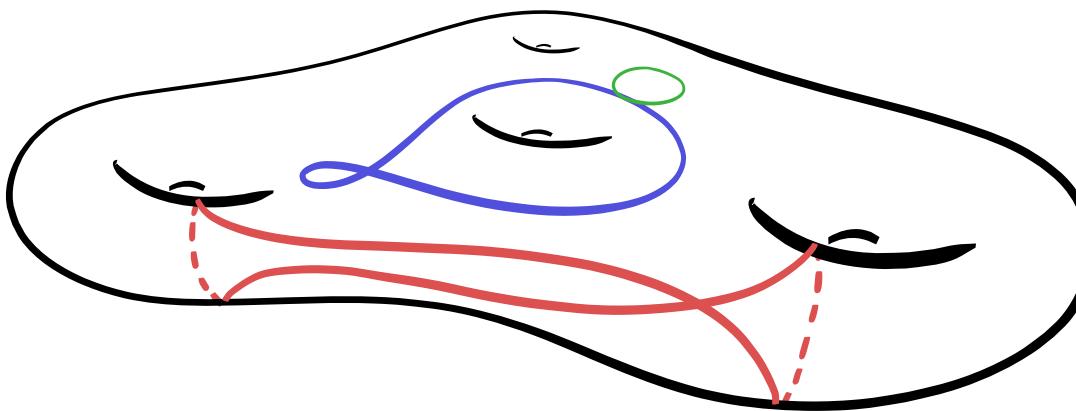
Related problem: untangling curves



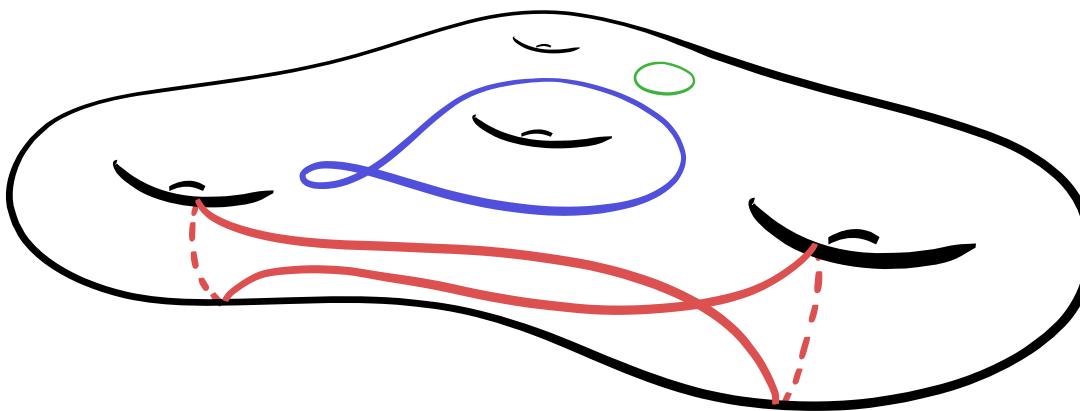
Related problem: untangling curves



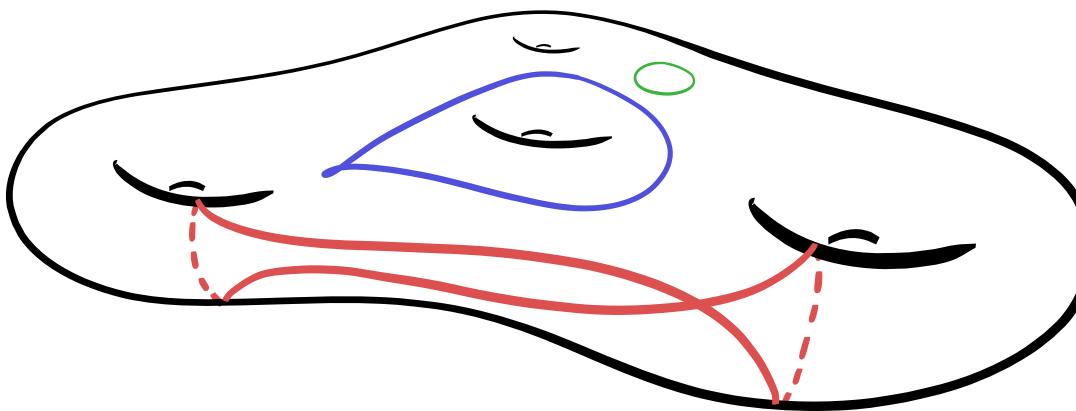
Related problem: untangling curves



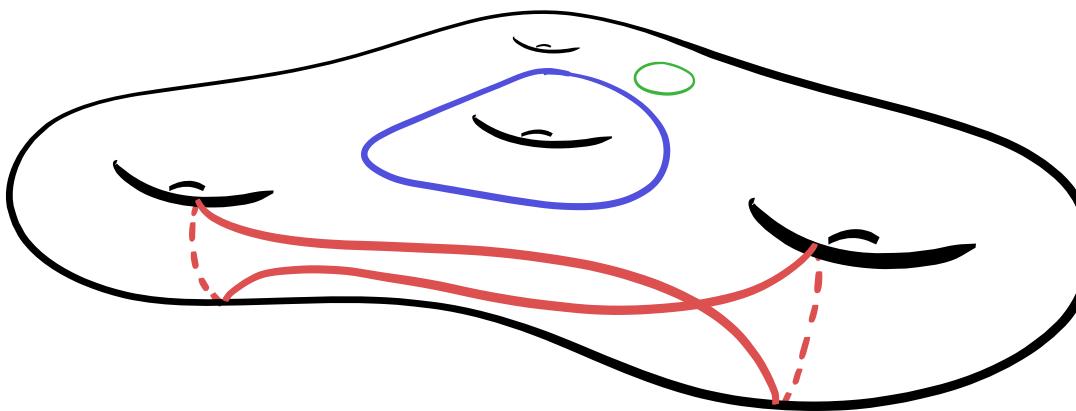
Related problem: untangling curves



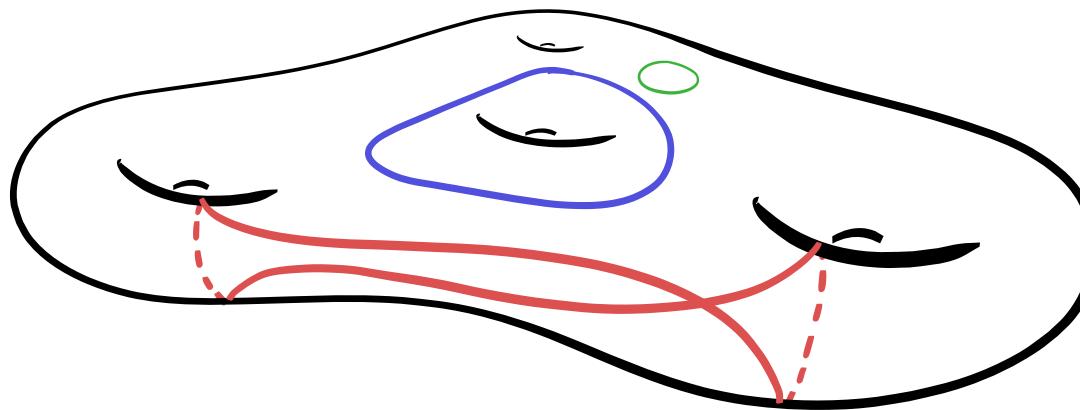
Related problem: untangling curves



Related problem: untangling curves



Related problem: untangling curves



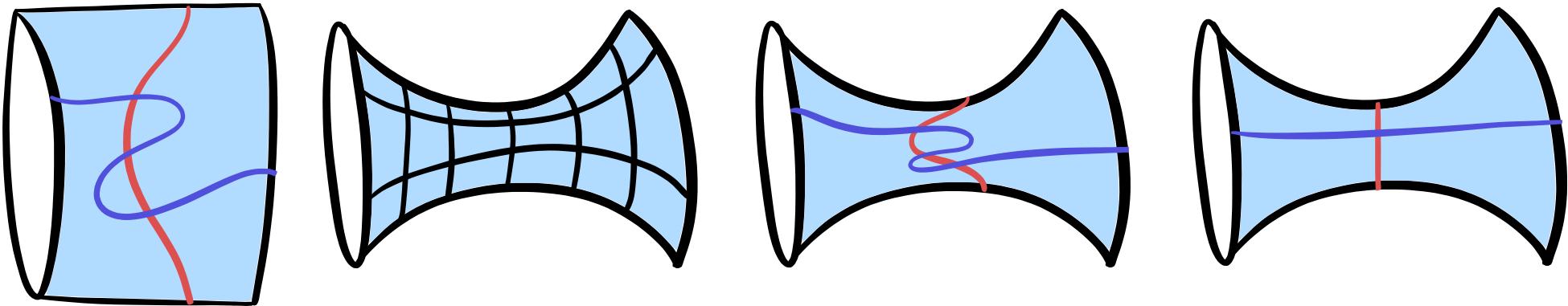
Output: $\min \#$ of crossings (+ optimal curves)

Many previous works!

- | | |
|-------------------------|-------------------------------|
| Poincaré, 1905 | de Graaf and Schrijver, 1987 |
| Dehn, 1911 | Dynnikov, 2002 |
| Dehn, 1912 | Paterson, 2002 |
| Reinhart, 1962 | Gonçalves et al., 2005 |
| Zieschang, 1965 | Schaefer et al., 2008 |
| Chillingworth, 1969 | Lazarus and Rivaud, 2012 |
| Zieschang, 1969 | Erickson and Whittlesey, 2013 |
| Chillingworth, 1971 | · · · |
| Turaev, 1979 | Arettines, 2015 |
| Birman and Series, 1984 | Chang et al., 2018 |
| Cohen and Lustig, 1984 | Despré and Lazarus, 2019 |
| Hass and Scott, 1985 | Fulek and Tóth, 2020 |
| Lustig, 1987 | Chang and de Mesmay, 2022 |
| | Lackenby, 2024 |

Method for untangling curves

Poincaré, 1905



1. give special shape to surface
2. straighten the curves

The special shape

negative curvature:

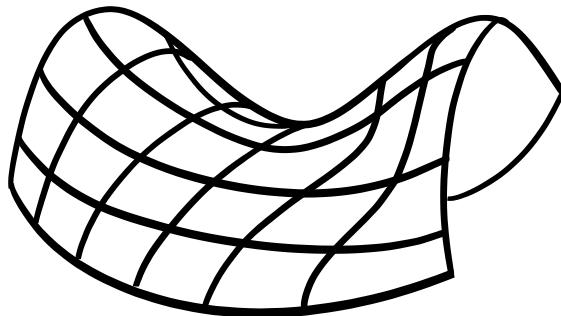


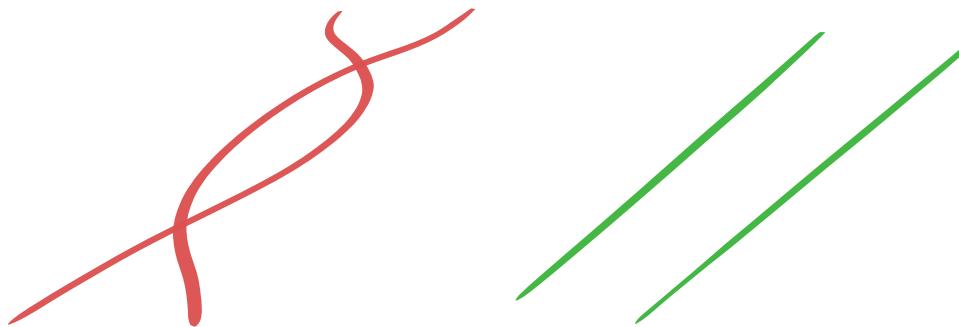
image by Susan Lombardo

almost all surfaces
can be curved negatively

The key property

On a negatively curved surface,

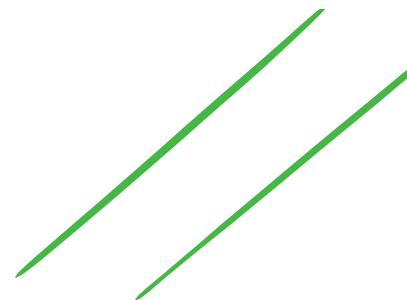
straight curves cross minimally



The key property

On a negatively curved surface,

straight curves cross minimally



(does not hold
on all surfaces)



The key property

On a negatively curved surface,

straight curves cross minimally



ever
into

The key property

On a negatively curved surface,

straight curves cross minimally



every path
into a unit

The key property

On a negatively curved surface,

straight curves cross minimally



every path can be
into a unique straight curve

The key property

On a negatively curved surface,

short curves cross minimally



every path can be deformed
into a unique straight pa

The key property

On a negatively curved surface,

cross minimally



every path can be deformed
into a unique **straight** path

The key property

On a negatively curved surface,

minimally



every path can be deformed
into a unique **straight** path

The key property

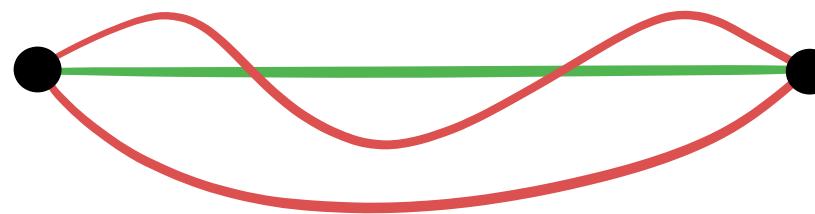
On a negatively curved surface,

every path can be deformed
into a unique **straight** path

The key property

On a negatively curved surface,

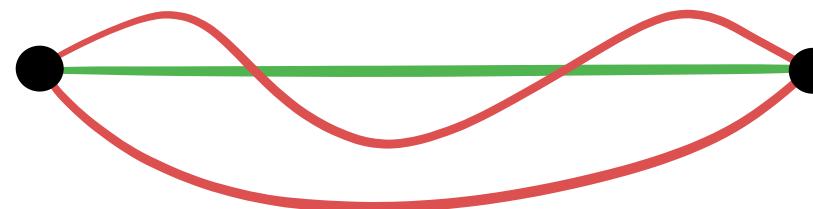
every path can be deformed
into a unique **straight** path



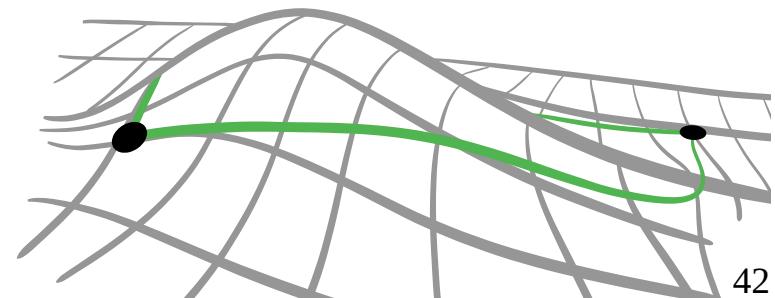
The key property

On a negatively curved surface,

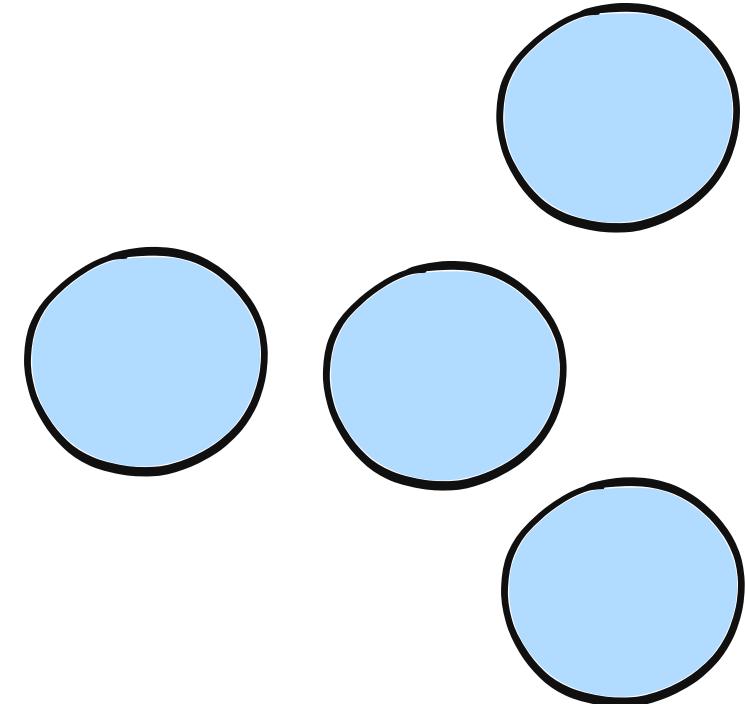
every path can be deformed
into a unique **straight** path



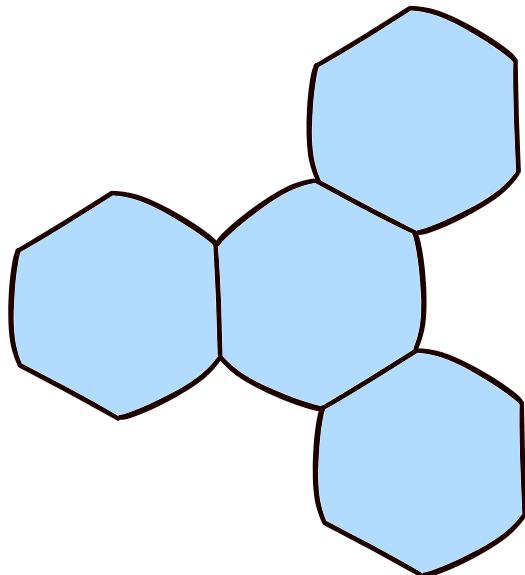
(does not hold
on all surfaces)



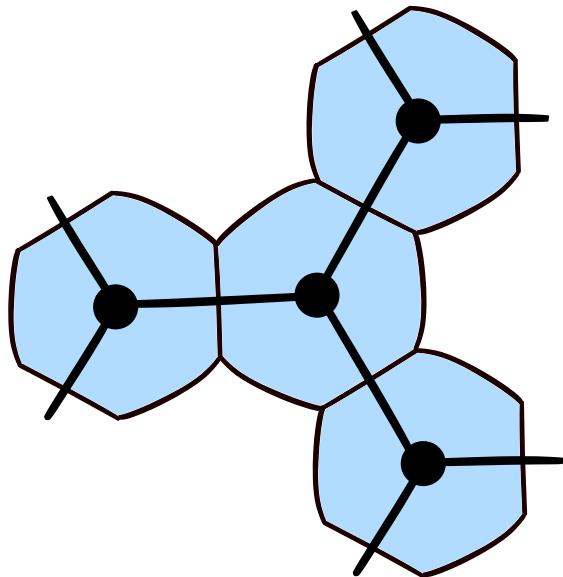
Algorithms for untangling curves



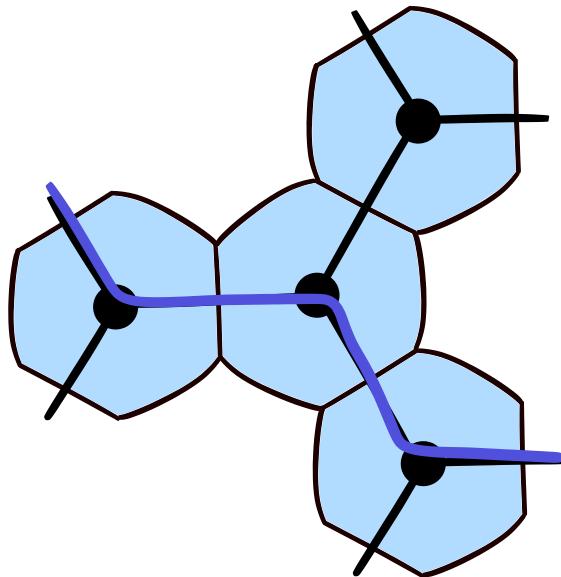
Algorithms for untangling curves



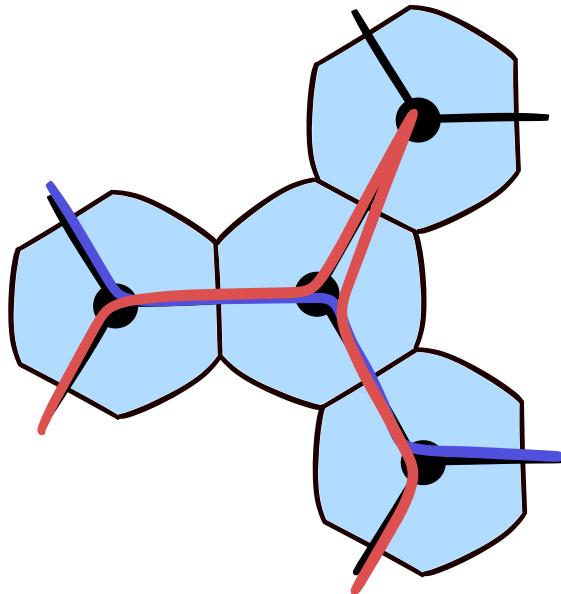
Algorithms for untangling curves



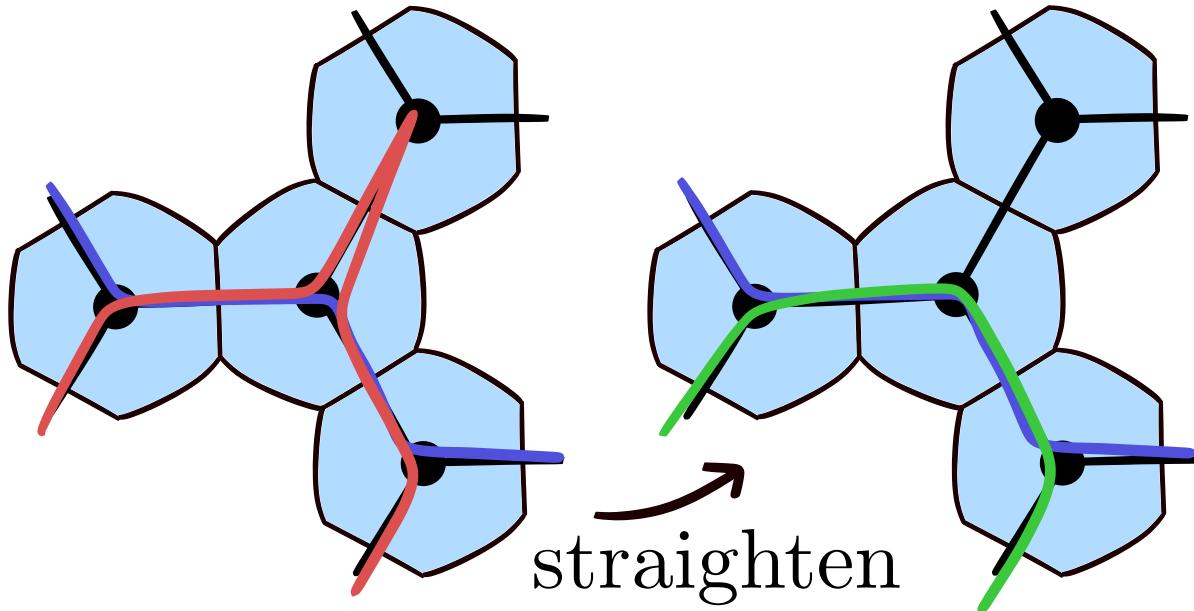
Algorithms for untangling curves



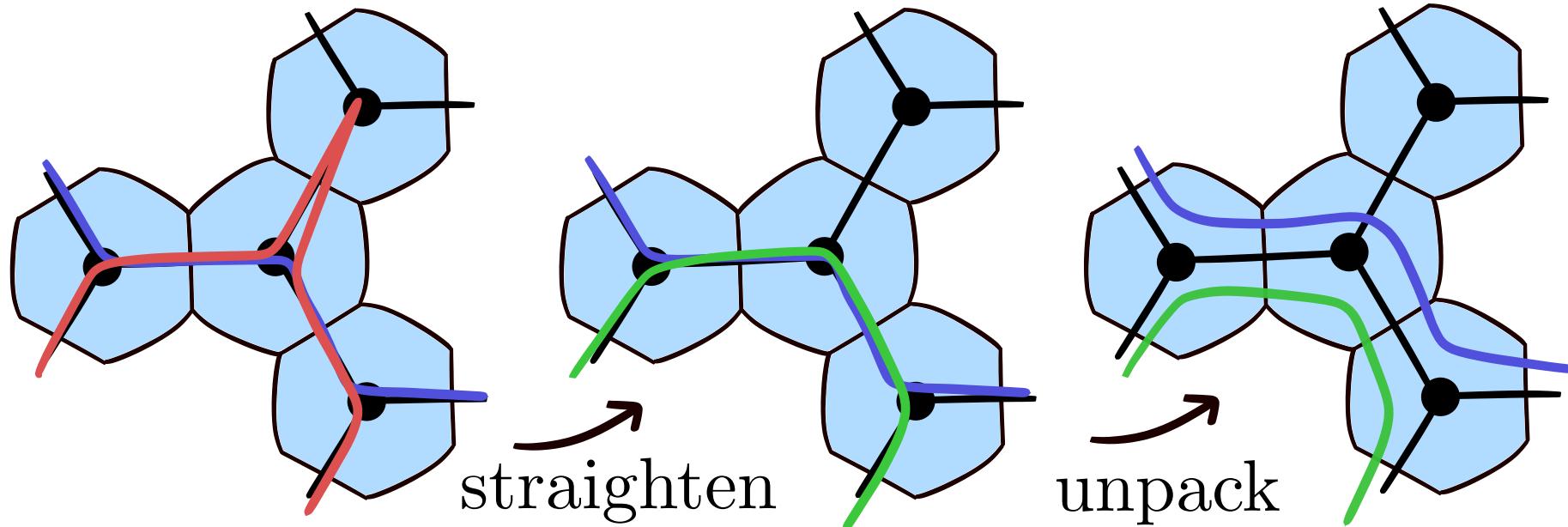
Algorithms for untangling curves



Algorithms for untangling curves

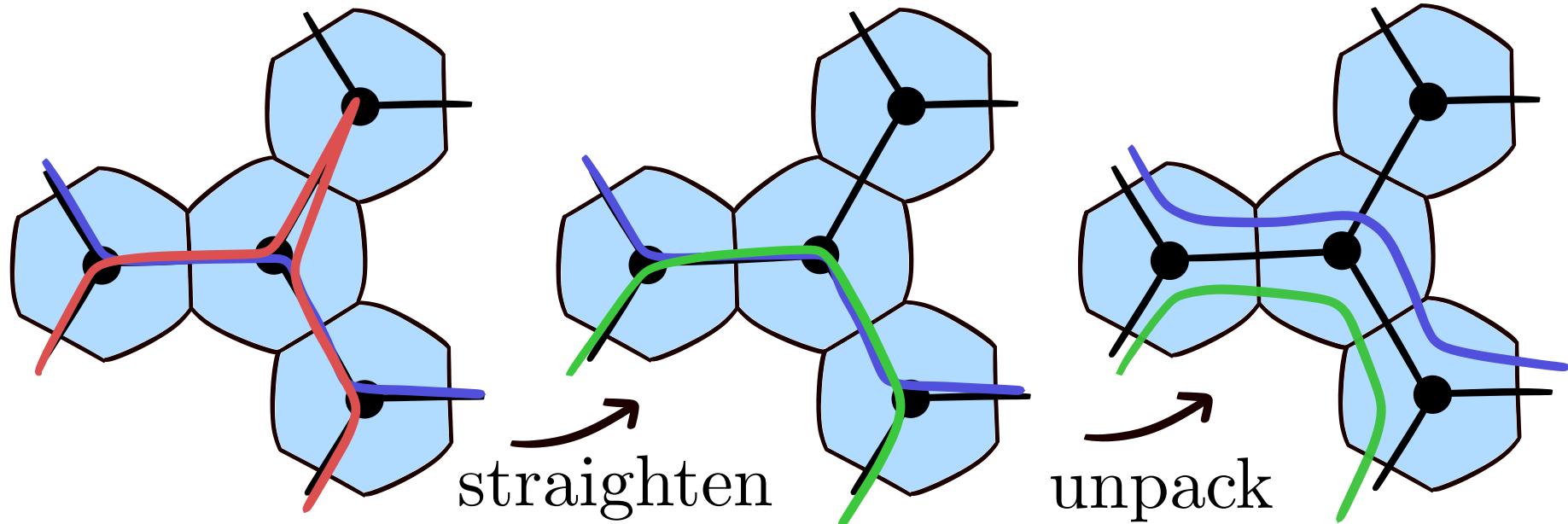


Algorithms for untangling curves



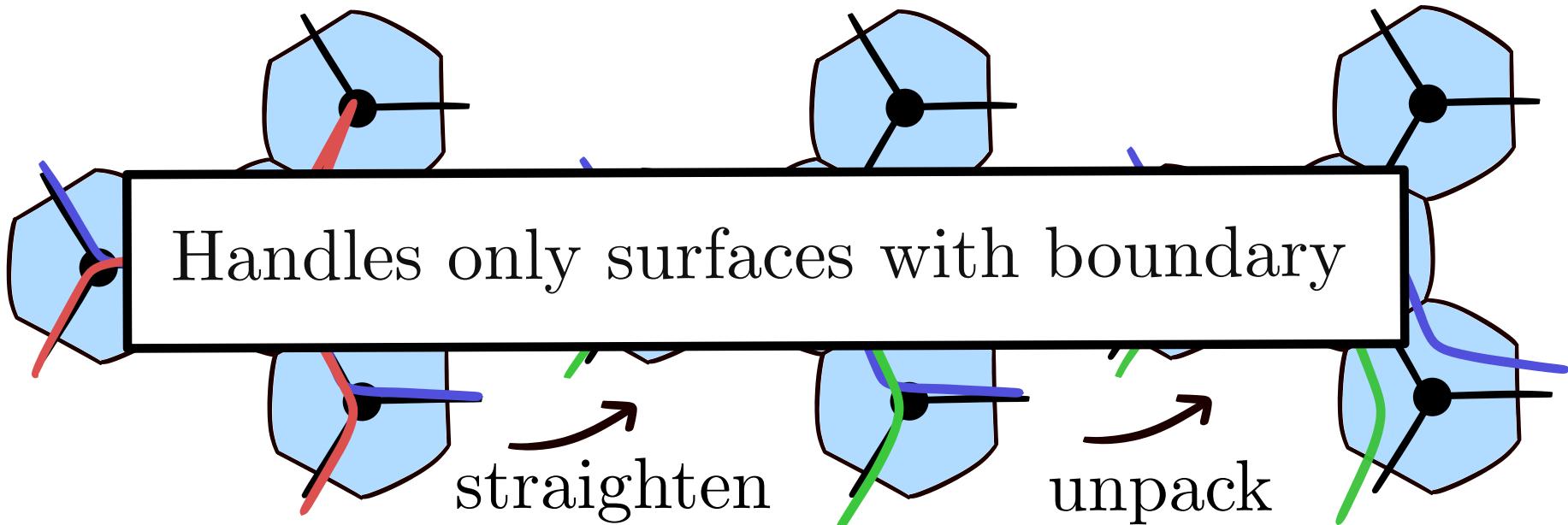
Fulek and Tóth, 2020

Algorithms for untangling curves



every walk can be deformed
into a unique straight walk

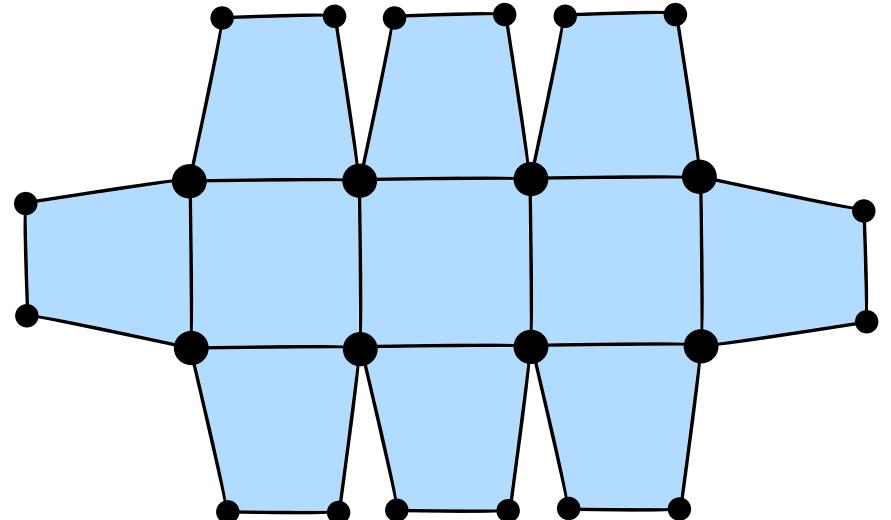
Algorithms for untangling curves



every walk can be deformed
into a unique straight walk

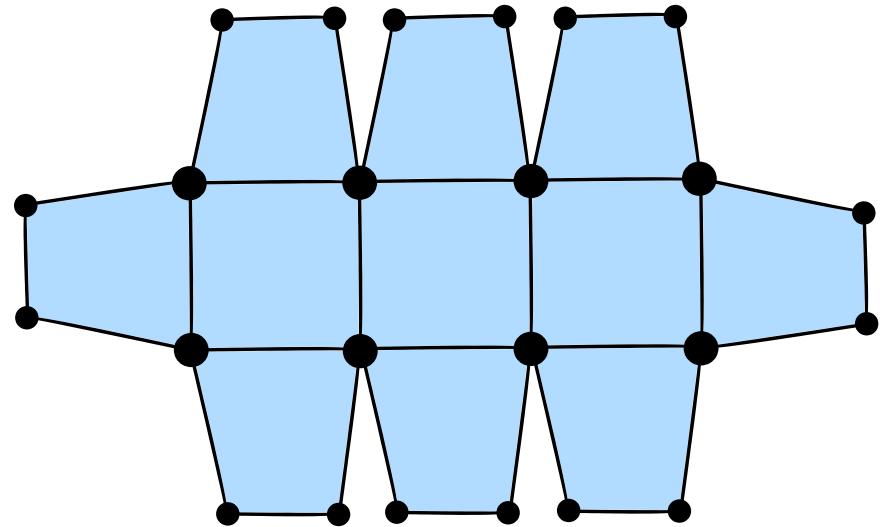
Algorithms for untangling curves

Lazarus and Rivaud, 2012
Erickson and Whittlesey, 2013

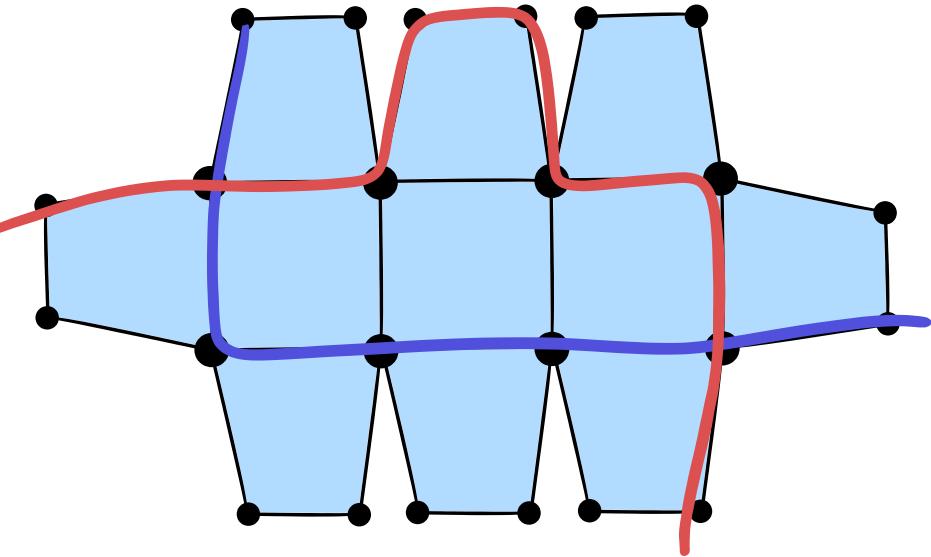


Model for surfaces
without boundary

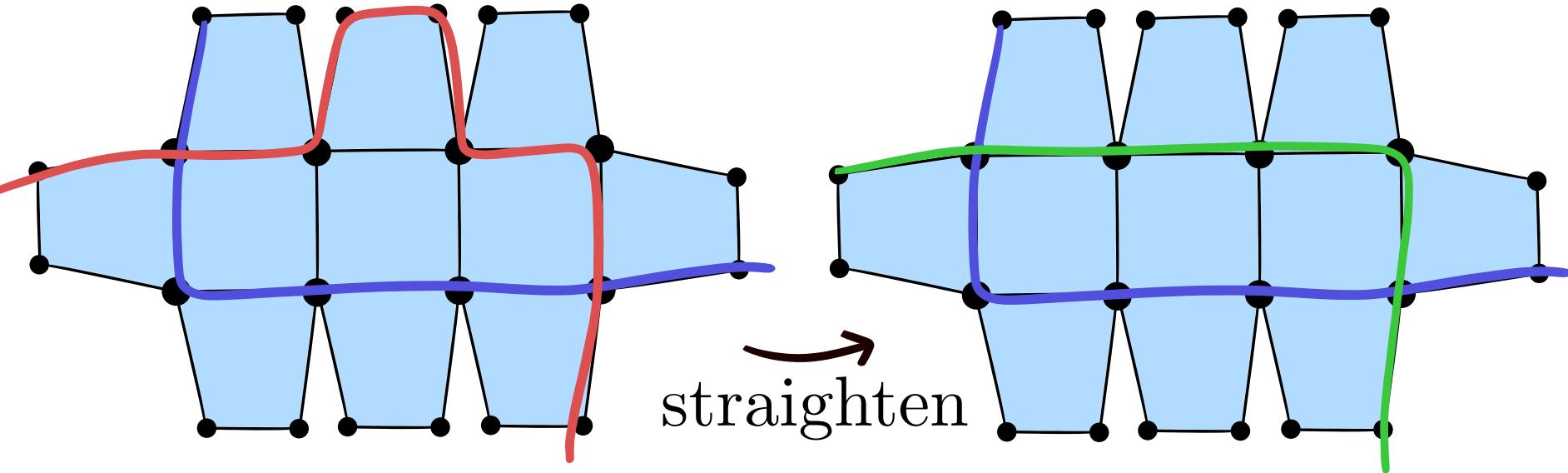
Algorithms for untangling curves



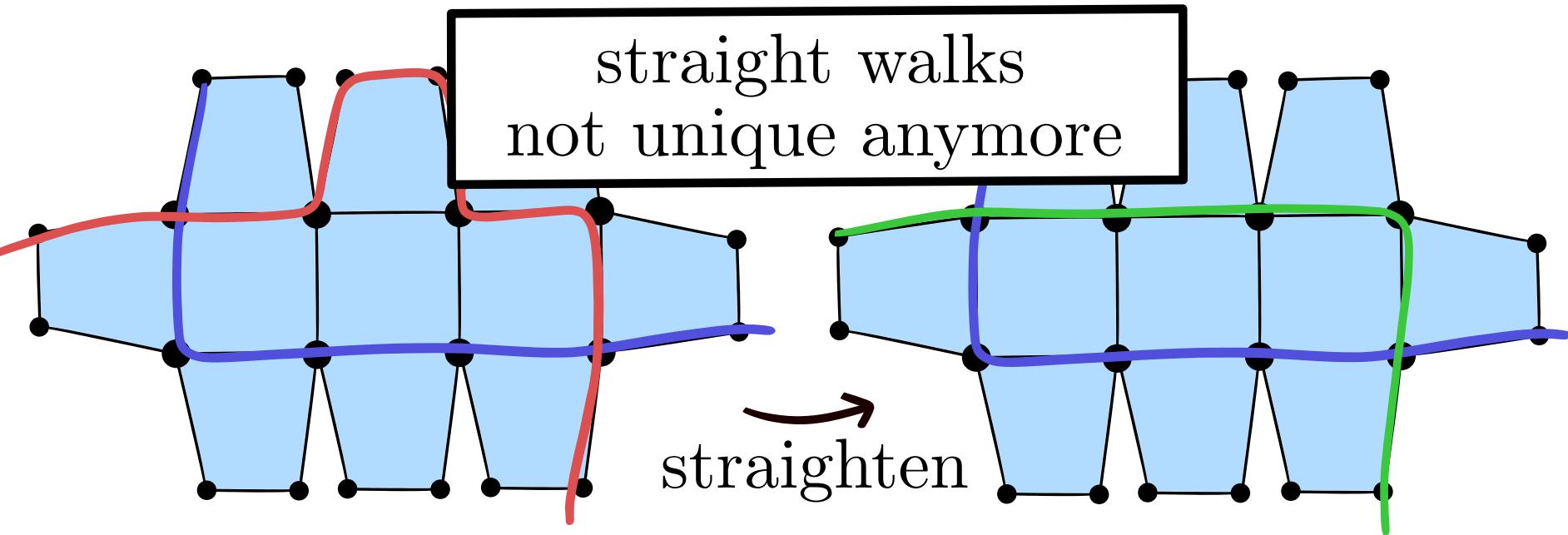
Algorithms for untangling curves



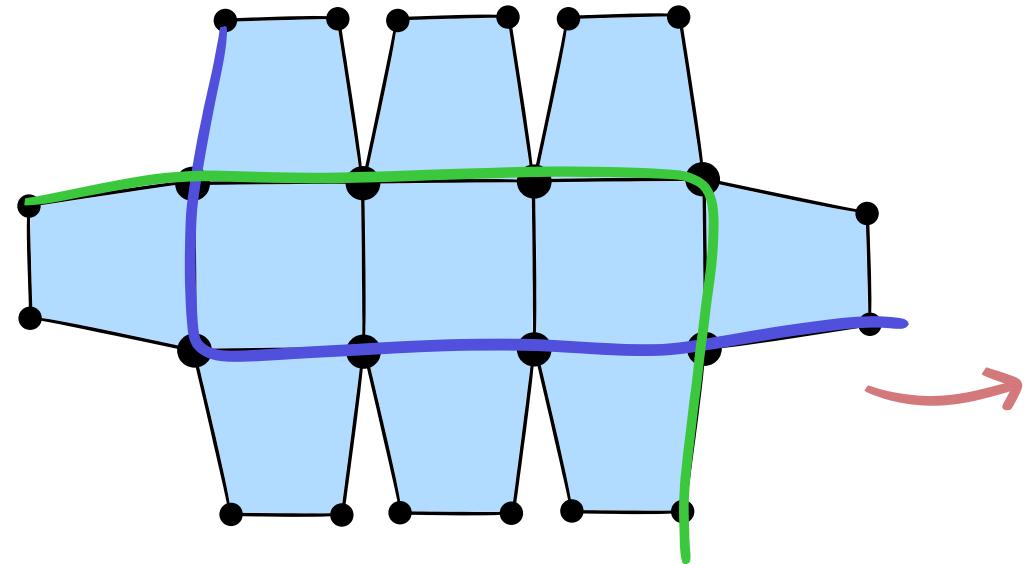
Algorithms for untangling curves



Algorithms for untangling curves



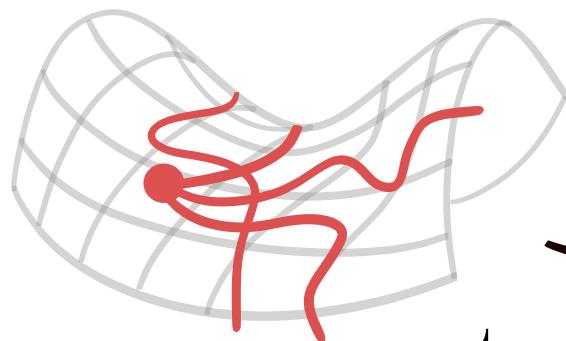
Algorithms for untangling curves



Despré and Lazarus, 2019

Method for untangling graphs

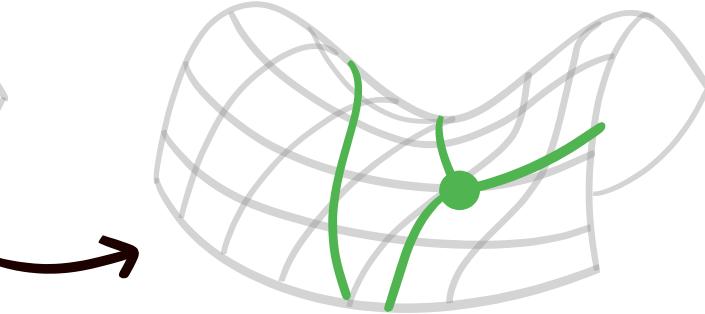
Tutte, 1963



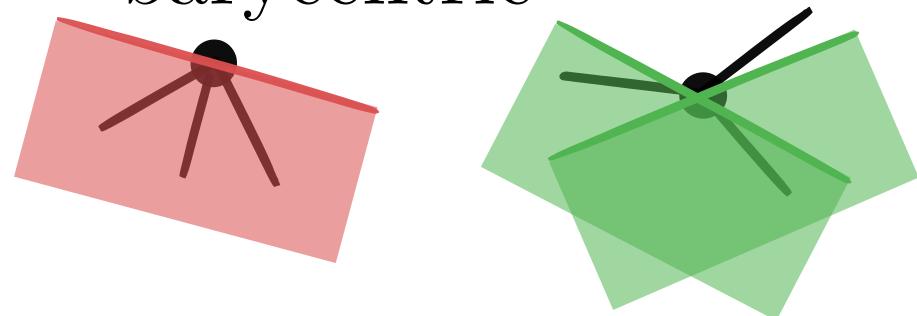
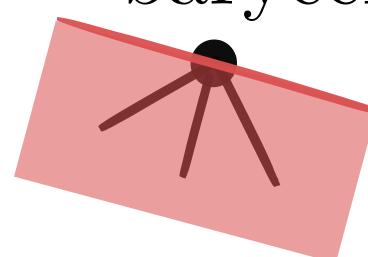
straighten
edges



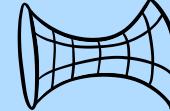
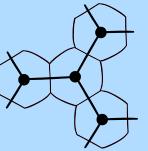
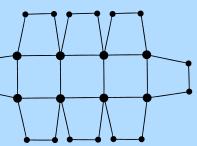
Y. Colin de Verdière, 1991



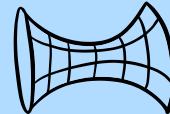
make vertices
barycentric



Straightening to untangle: summary

	Curves	Graphs
Method	negatively curved surface	
Algos	patch system 	system of quads 

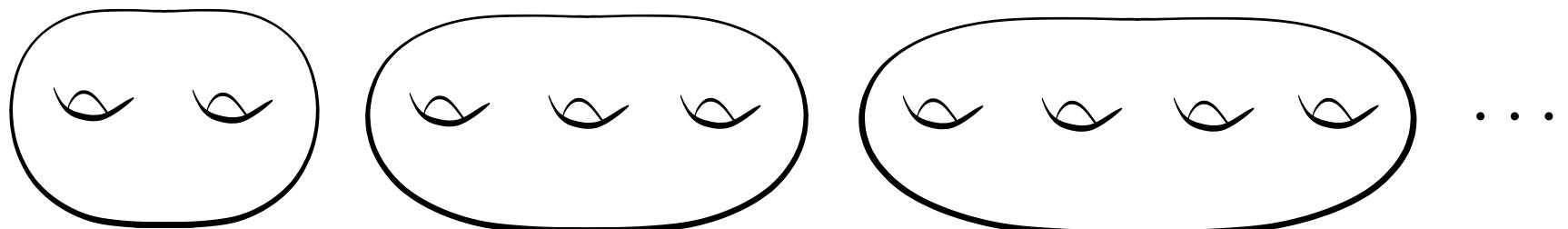
Our results

	Curves	Graphs
Method	negatively curved surface 	
Algos	reducing triangulations improved algos for untangling curves	first algos for untangling graphs discrete analogue of Tutte embeddings

A new tool:

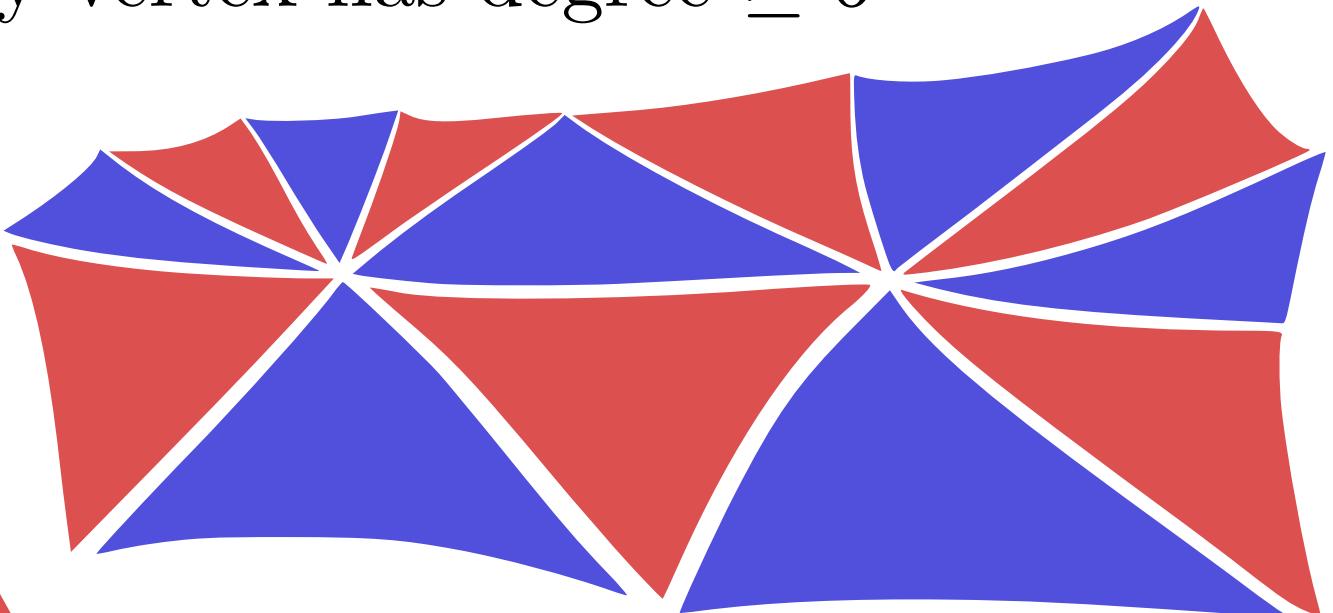
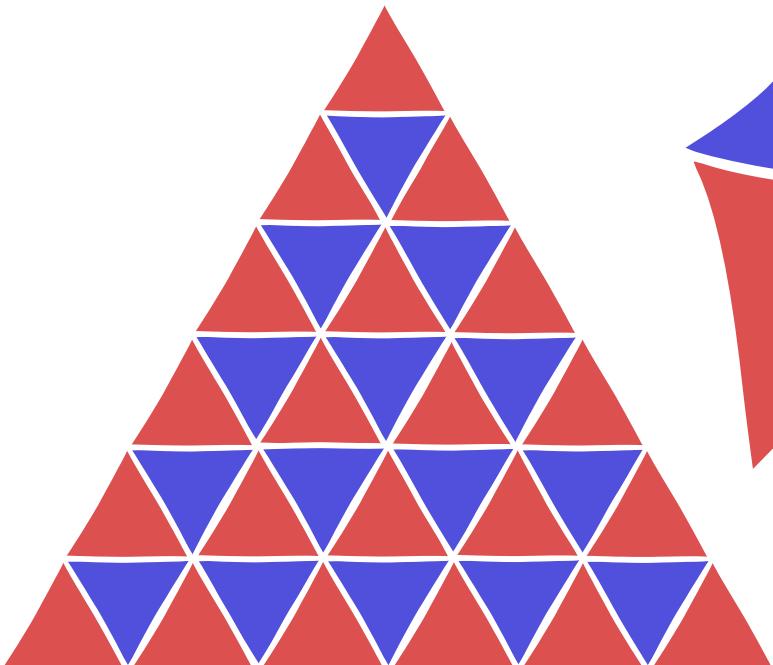
Reducing triangulations

We focus on surfaces without boundary
of genus ≥ 2

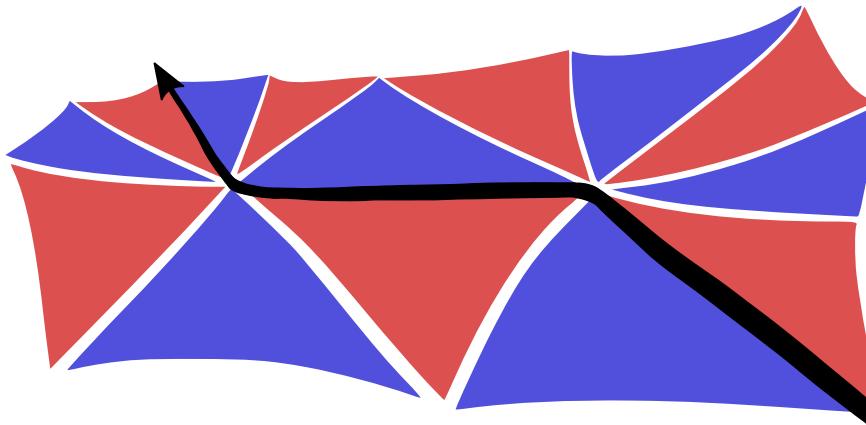


Reducing triangulations

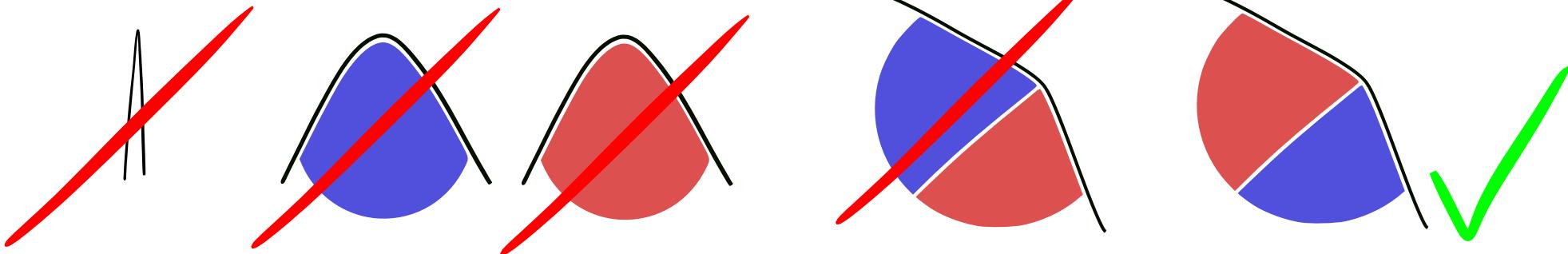
dual is bipartite and
every vertex has degree ≥ 6



Reduced walks

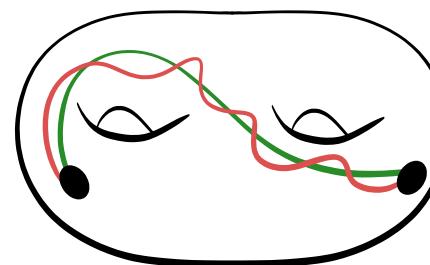
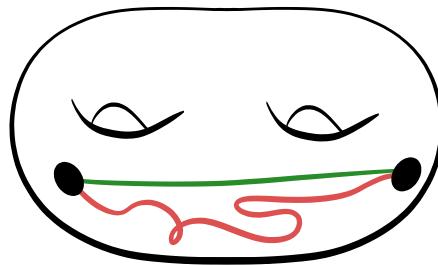


no bad turn



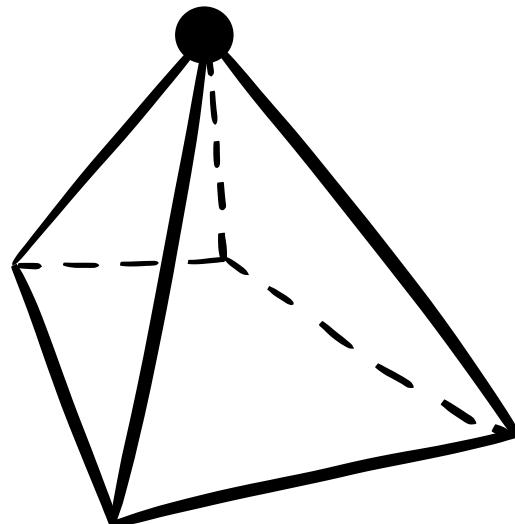
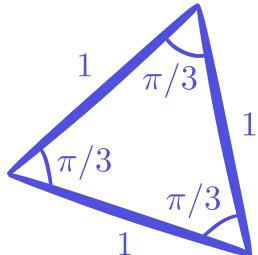
Properties of reduced walks

every walk can be deformed into a unique reduced walk, computable in linear time

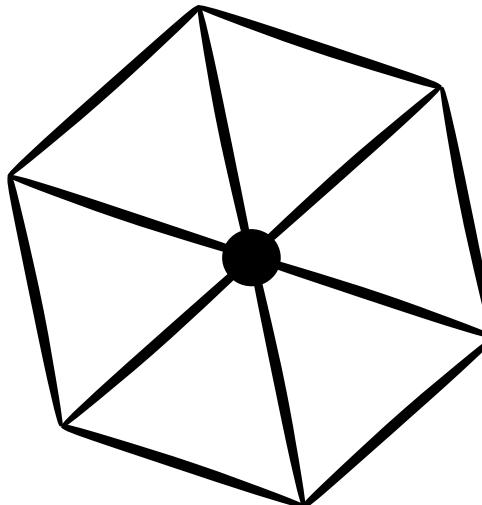


reduced walks are stable upon reversal and subwalk

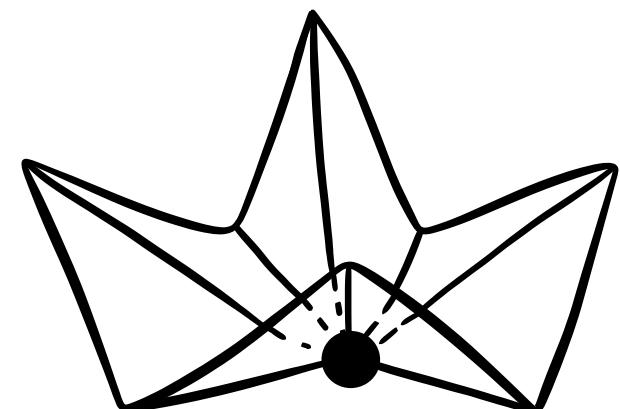
Analogy: vertex degree vs. curvature



degree < 6

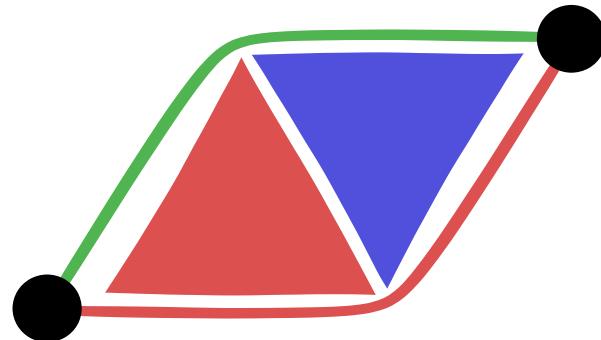


degree 6

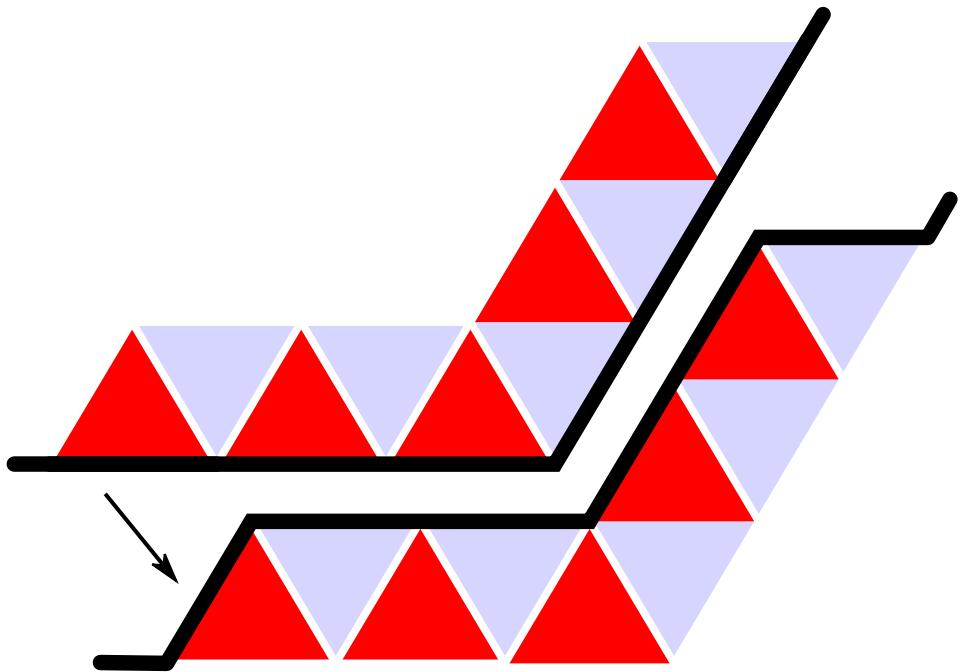
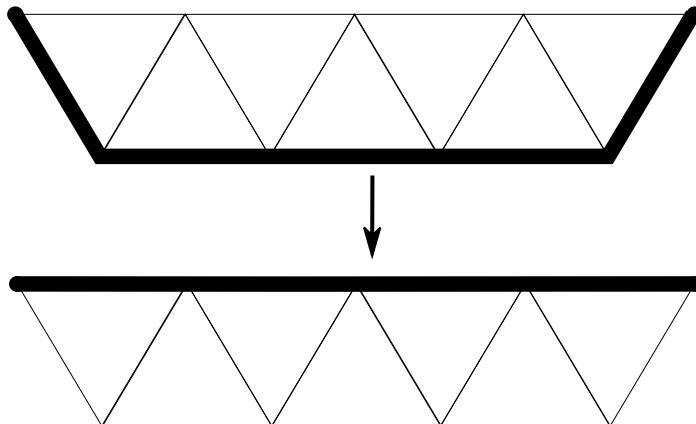
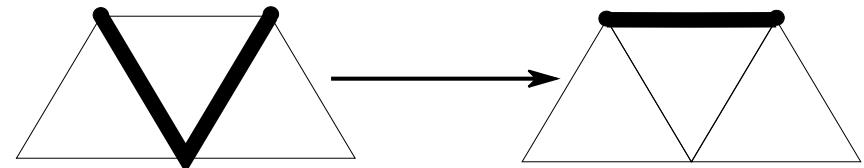
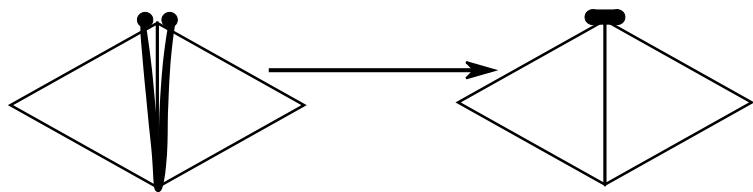


degree > 6

Purpose of the coloring



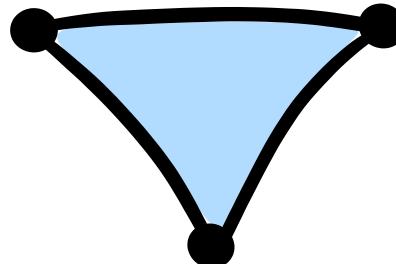
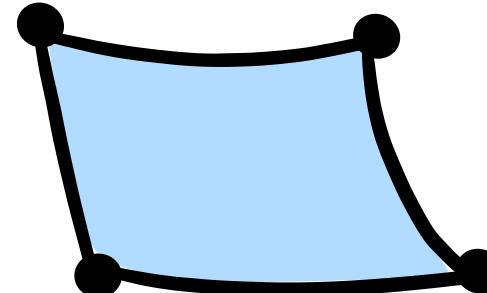
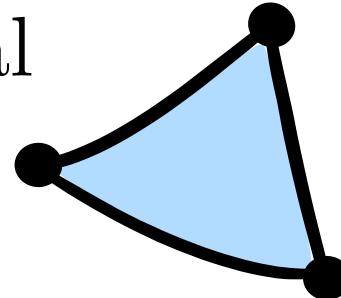
Reducing a walk



Untangling graphs using
reducing triangulations

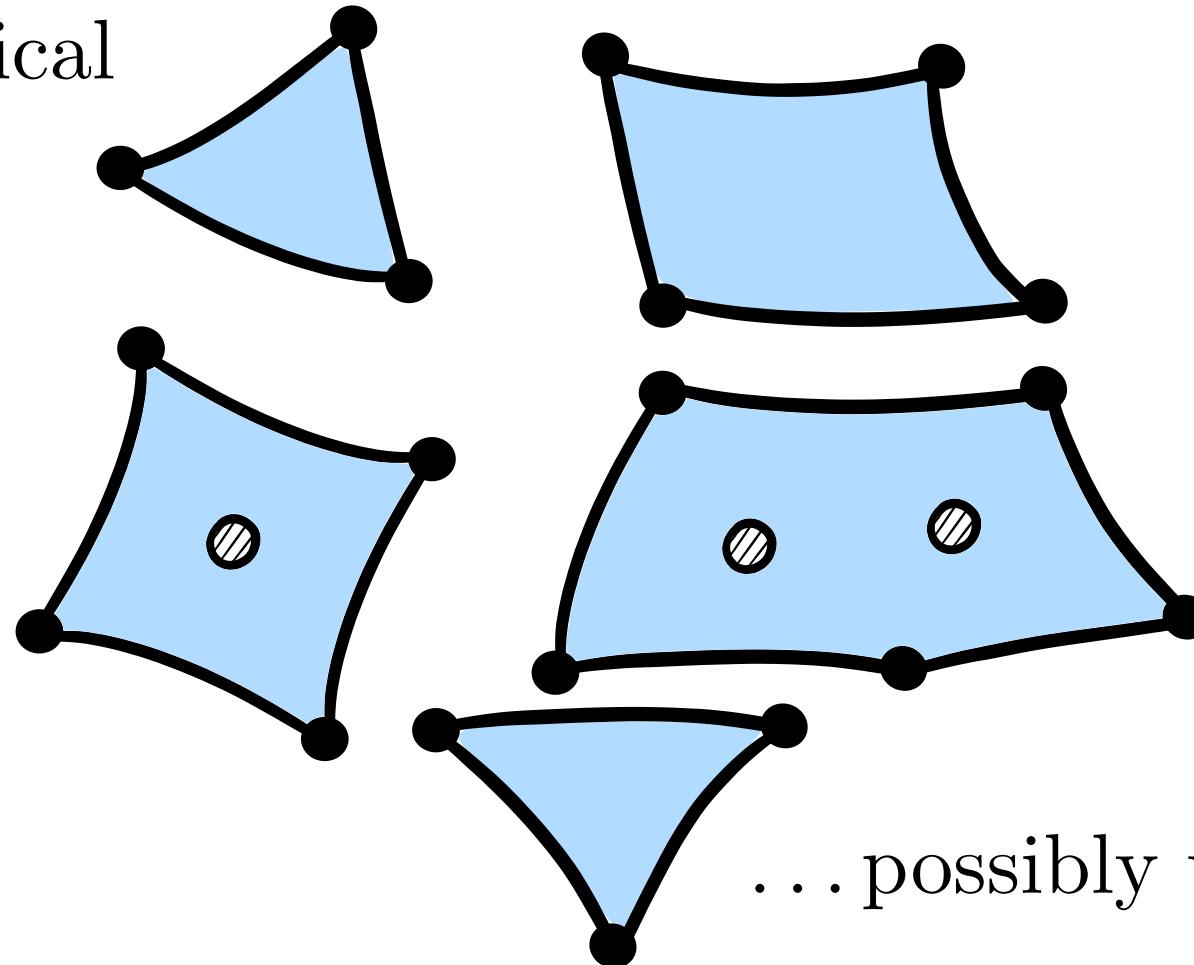
Input

take topological
polygons...



Input

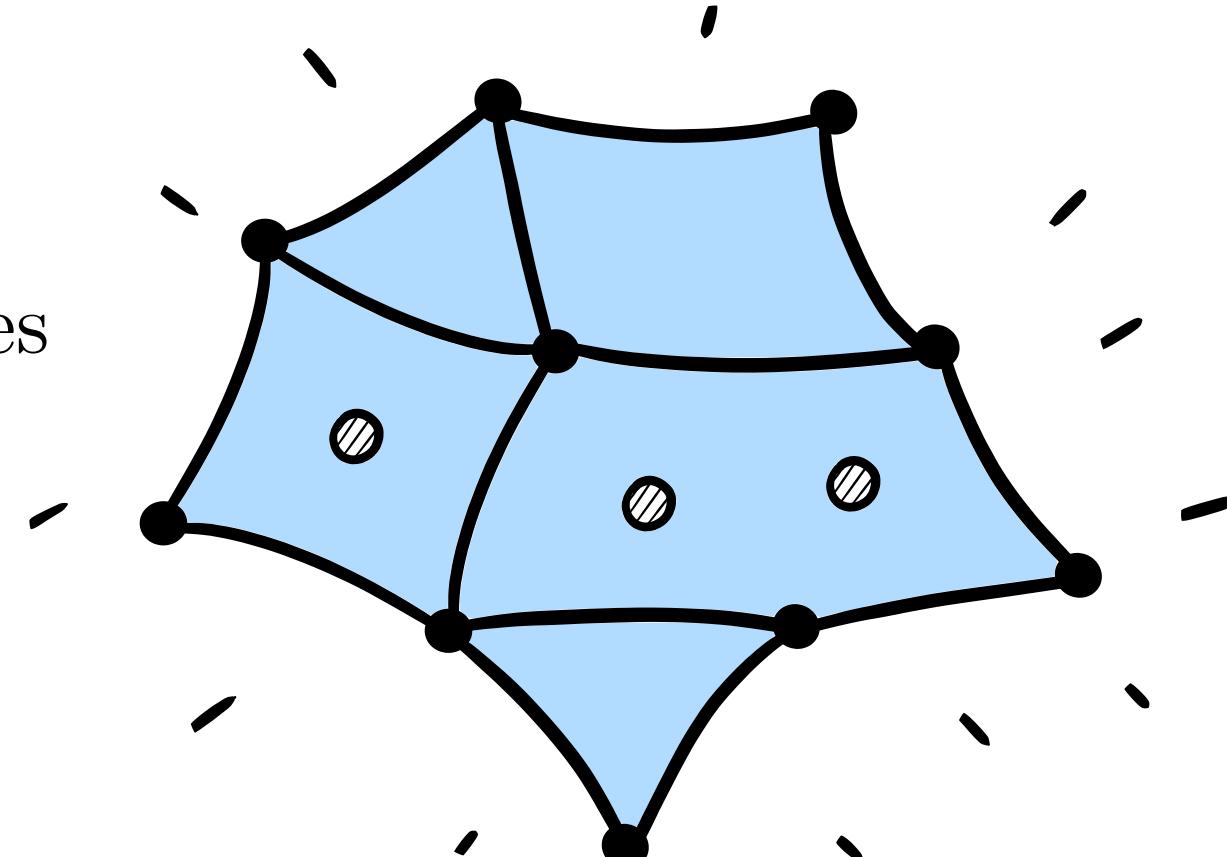
take topological
polygons...



... possibly with holes

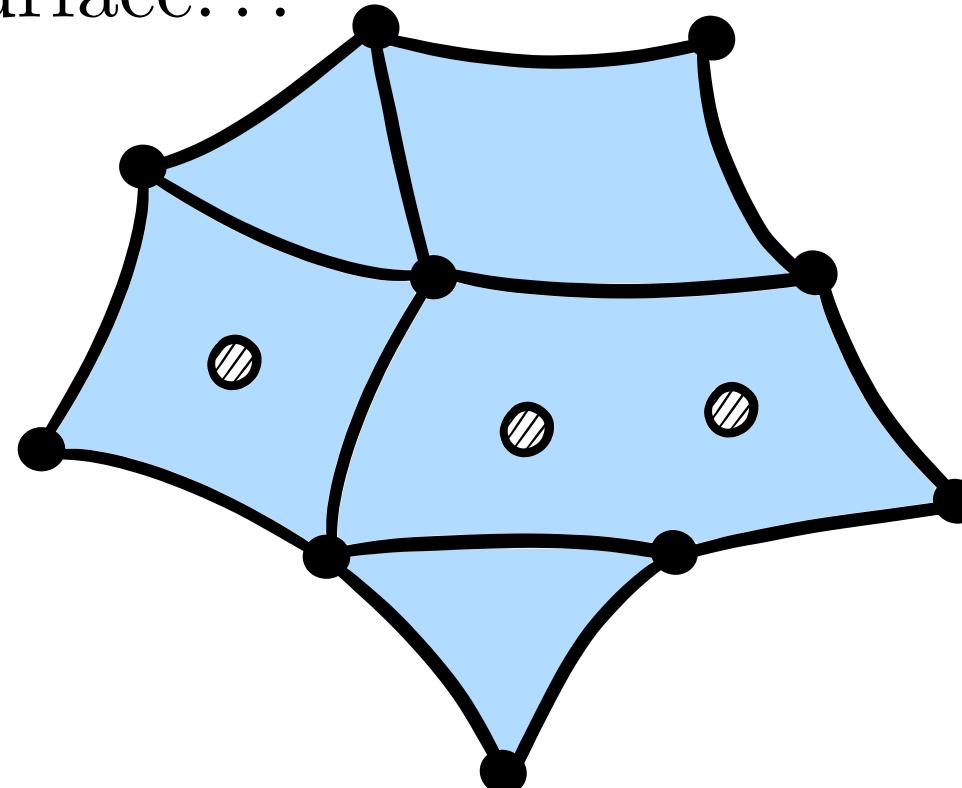
Input

glue edges



Input

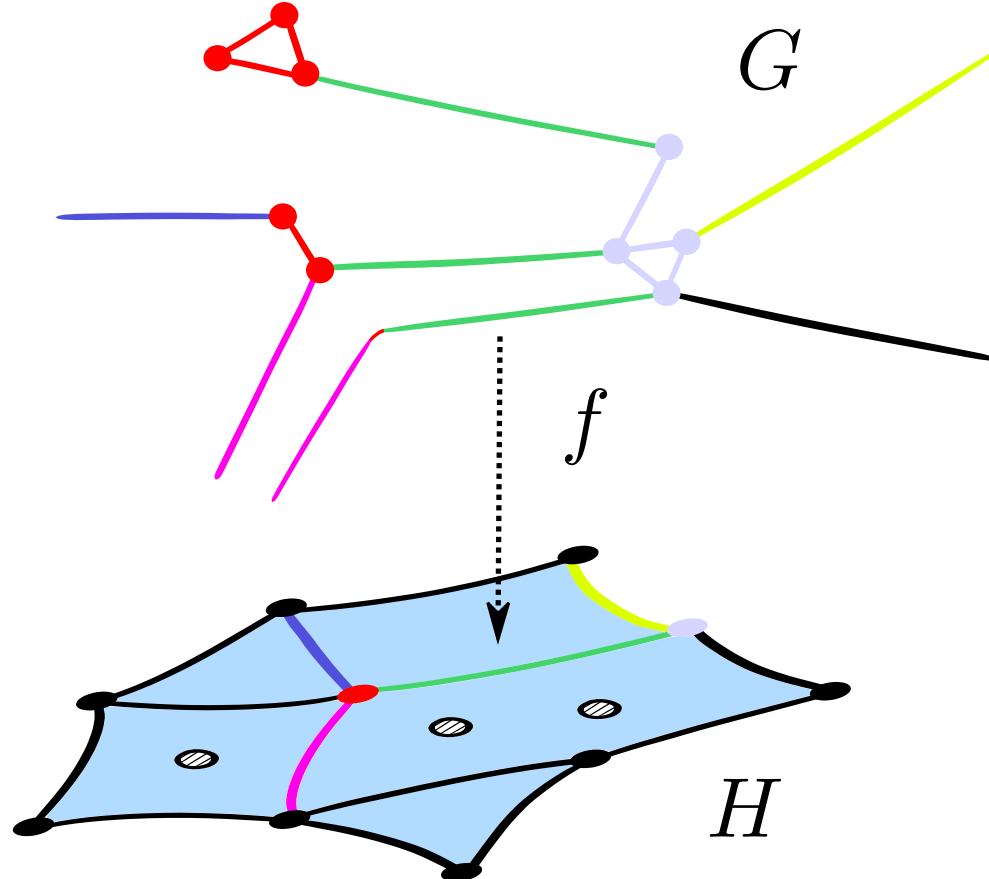
that encodes the surface . . .



. . . and a graph H on it

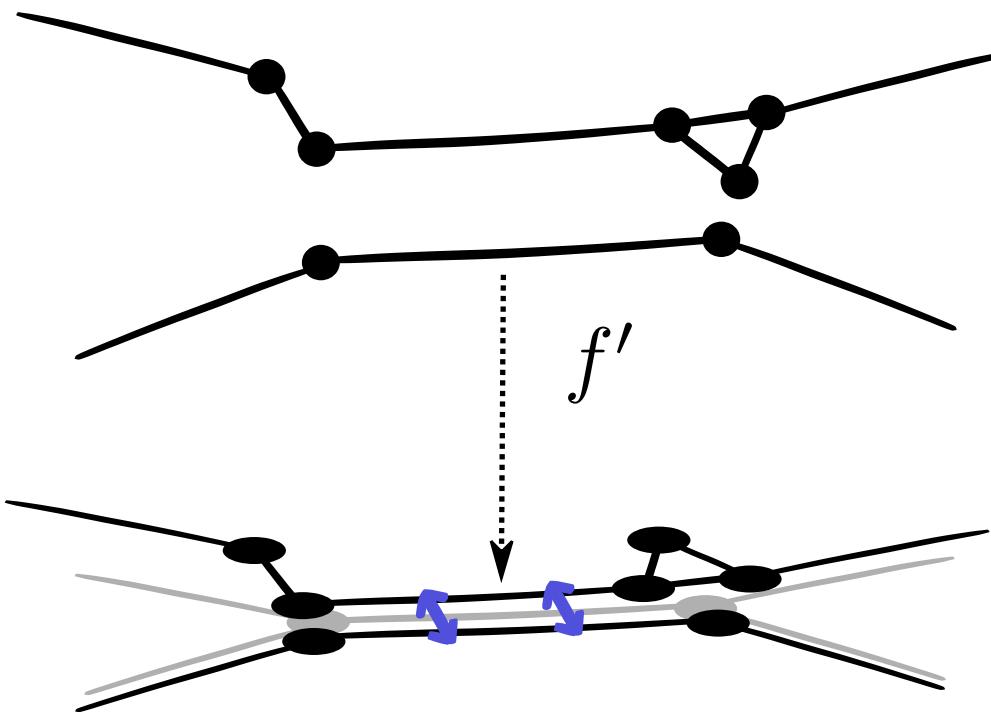
Input

draw G in H



Output

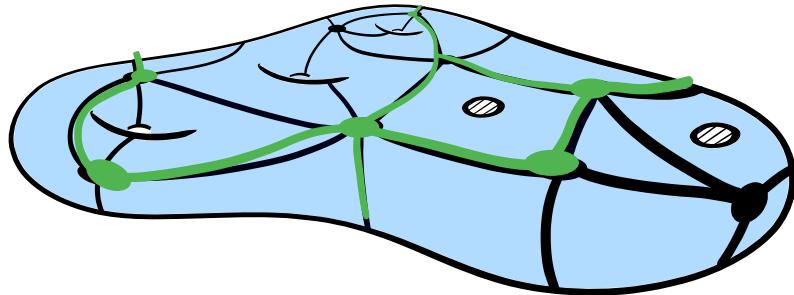
weak embedding: drawing f' that can be untangled by infinitesimal perturbation



Akitaya, Fulek, and
Tóth, 2019

algo to determine if
 f' is weak embedding,
and if so to perturb f'

Result

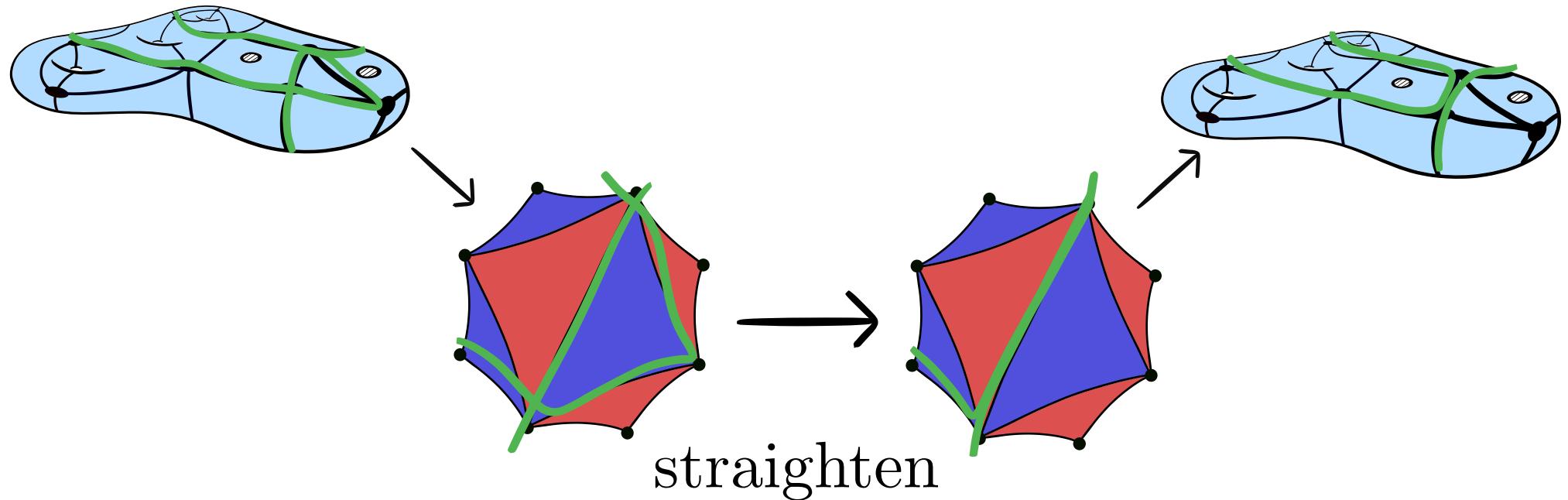


n : # times f uses an edge or vertex of H
 m : # vertices and edges of H
 s : genus + # holes

Colin de Verdière, Despré, D., 2023

We can decide if f can be untangled,
in $O(m + s^2 n \log(s n))$ time.
If so, we can compute a weak embedding homotopic
to f in additional $O(s^2 m n^2)$ time.

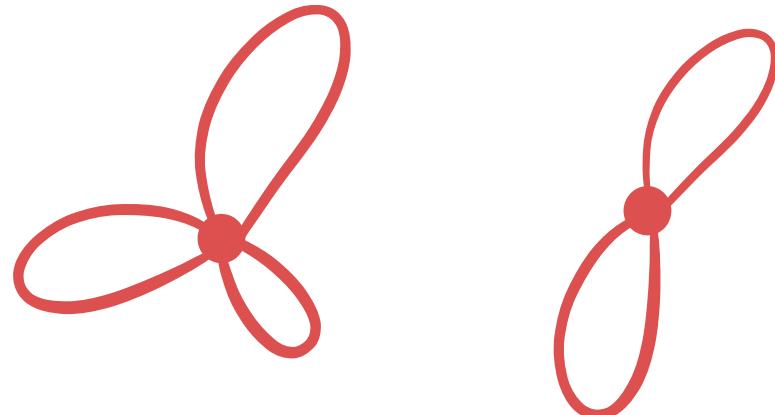
Algorithm overview



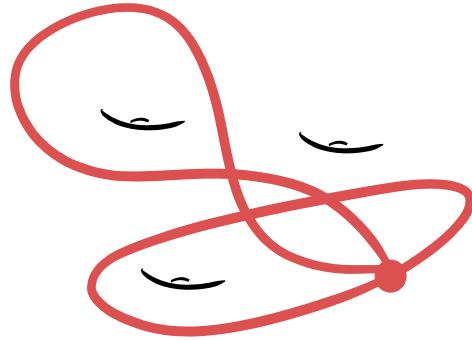
Straightening a graph

loop graph

Straightening a ~~graph~~

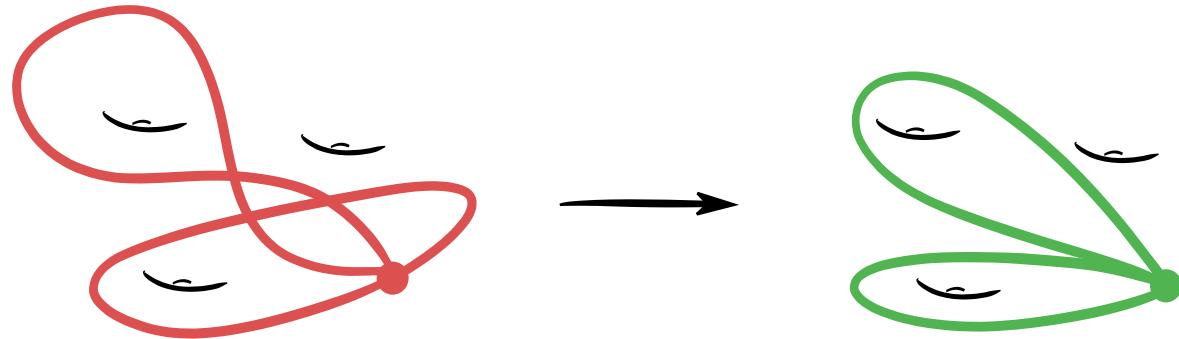


Straightening a loop graph



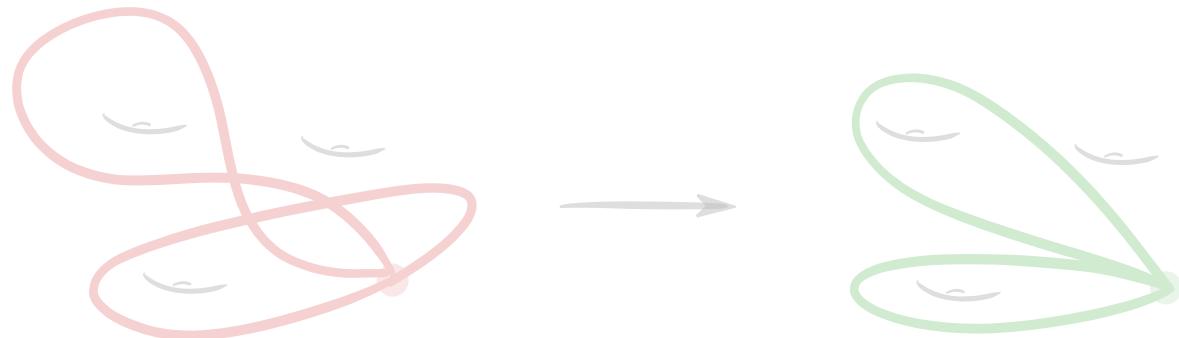
Straightening a loop graph

First attempt: reduce the loops, vertex fixed

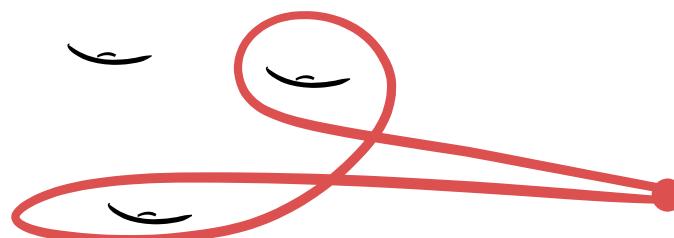


Straightening a loop graph

First attempt: reduce the loops, vertex fixed



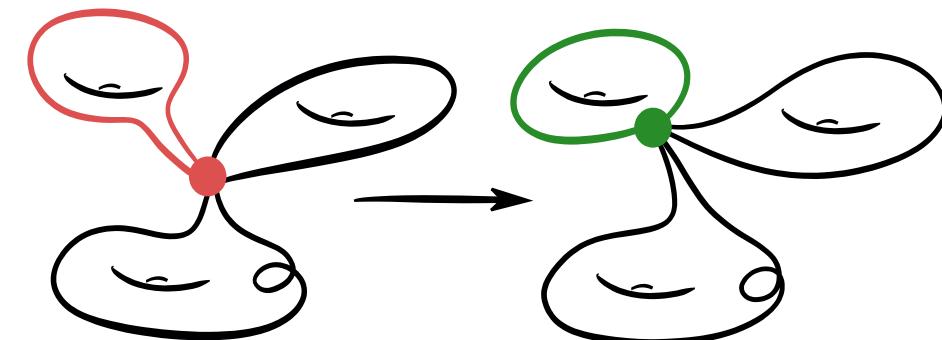
Problem:



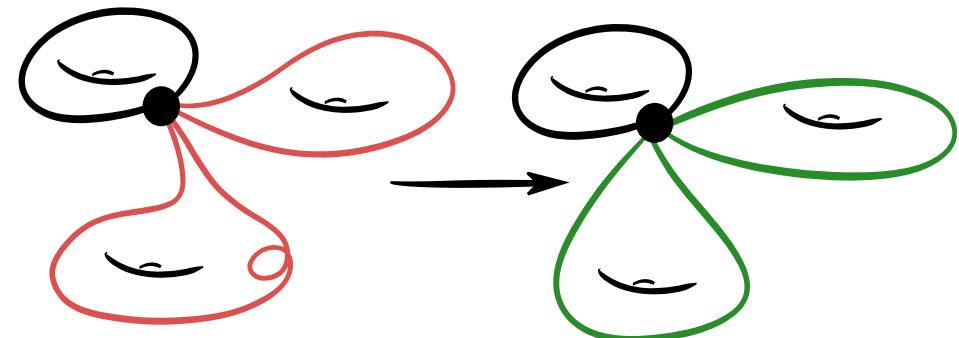
→ vertex must move

Straightening a loop graph

Solution:



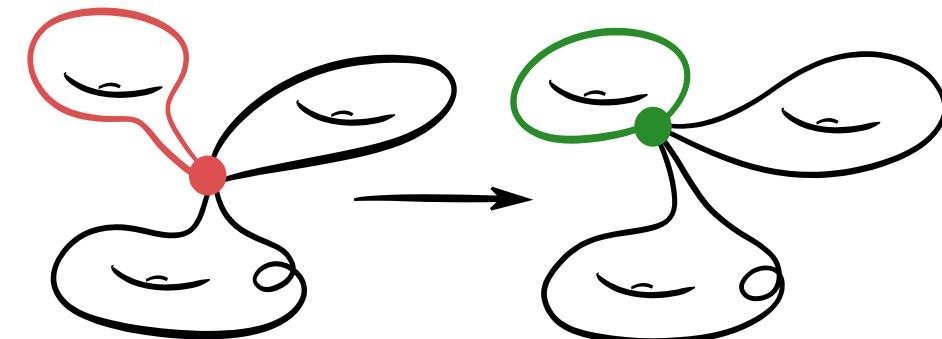
1. Reduce 1 loop cyclically



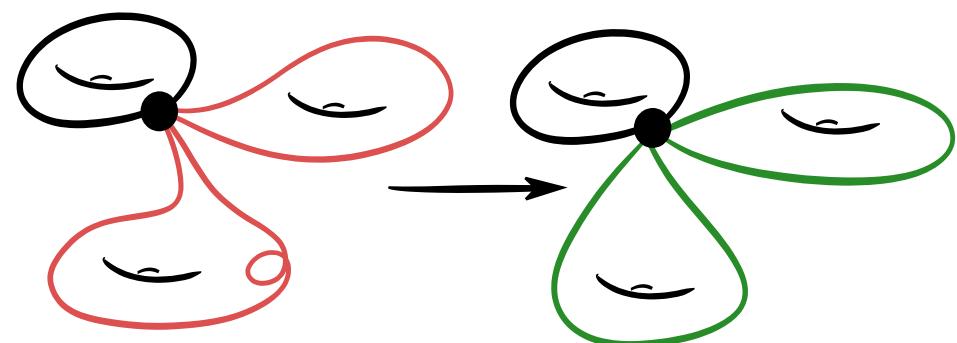
2. Reduce the other loops linearly

Straightening a loop graph

A straightened loop graph is a weak embedding
or cannot be untangled



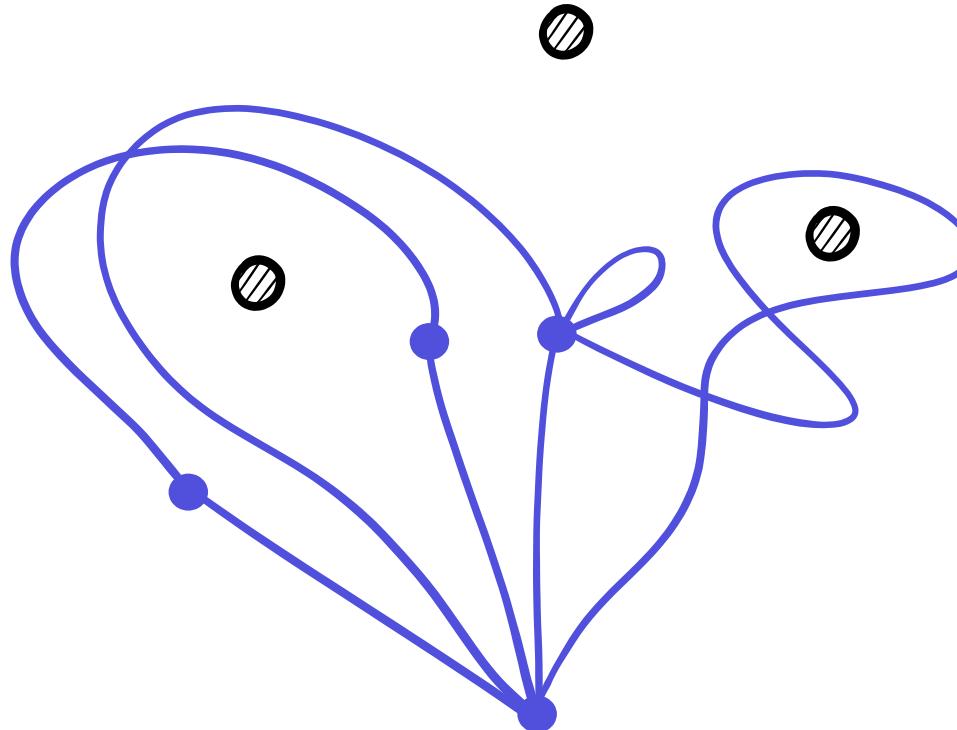
1. Reduce 1 loop cyclically



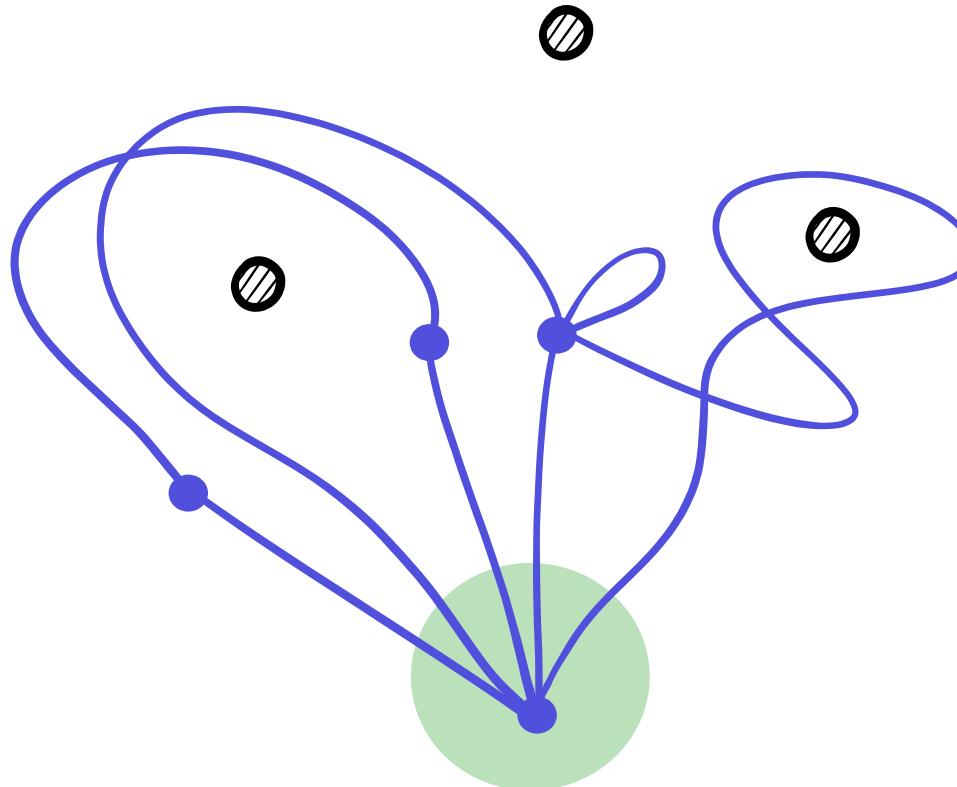
2. Reduce the other loops linearly

Straightening a graph

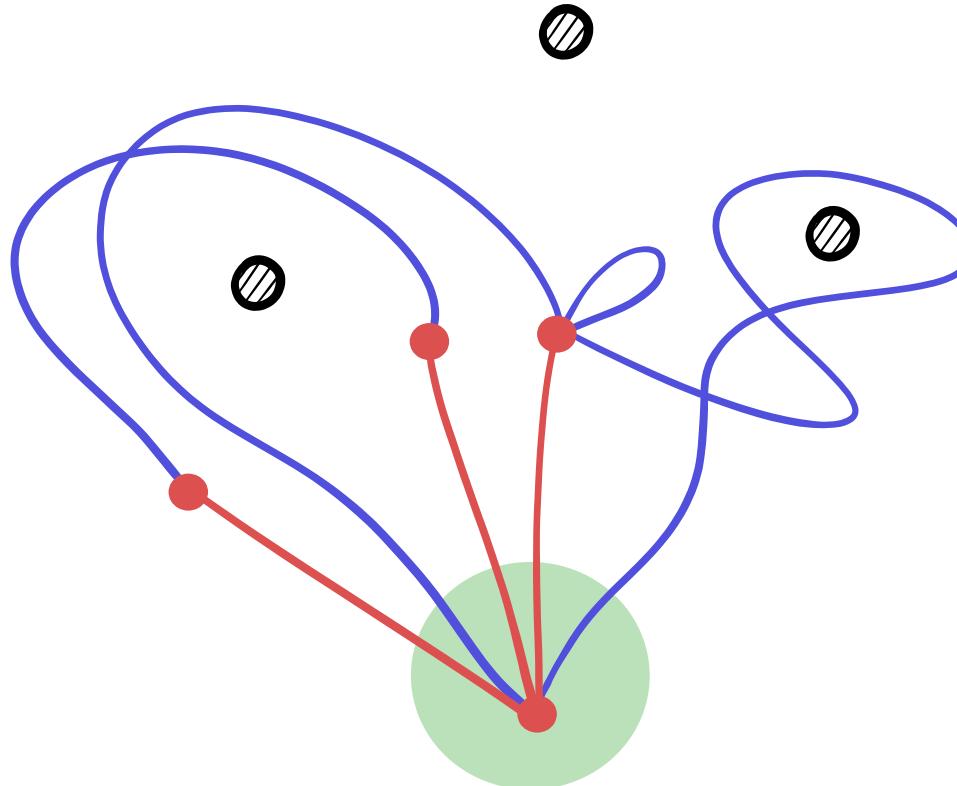
Straightening a graph



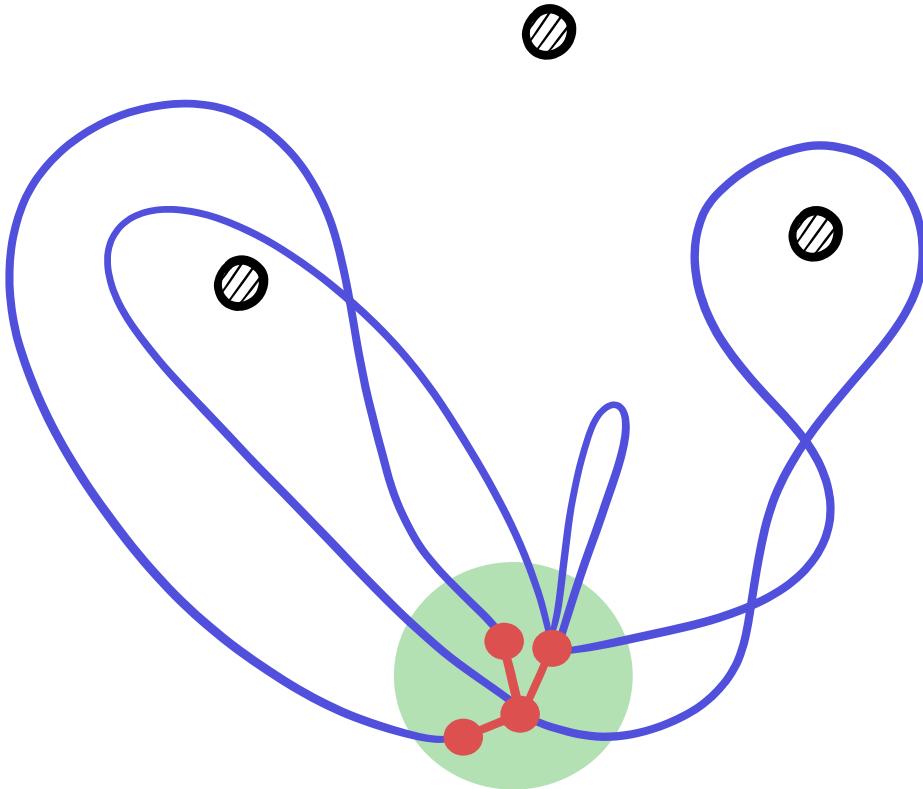
Straightening a graph



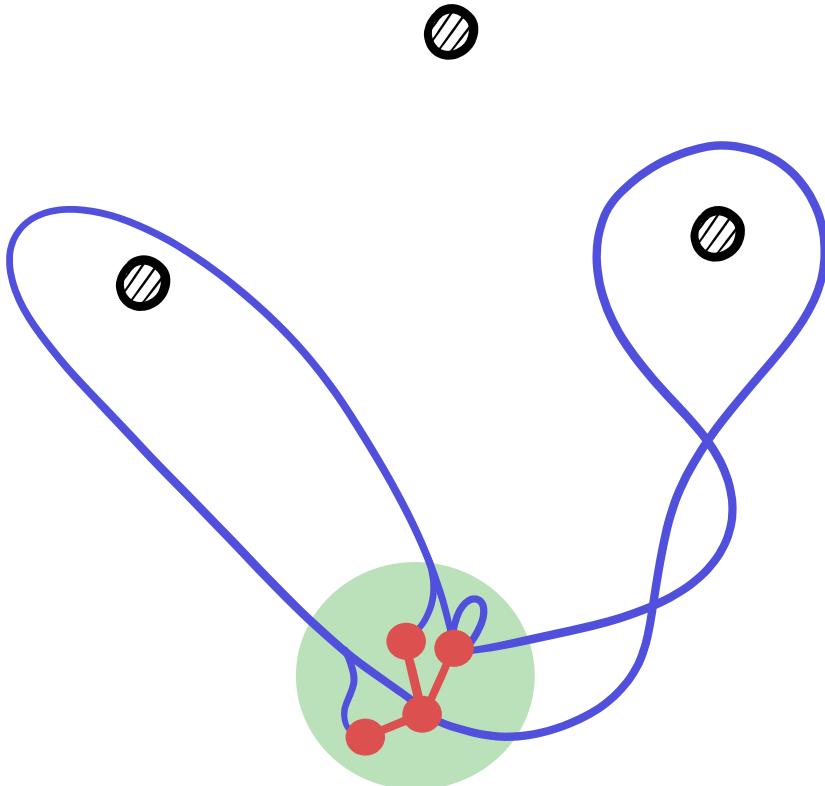
Straightening a graph



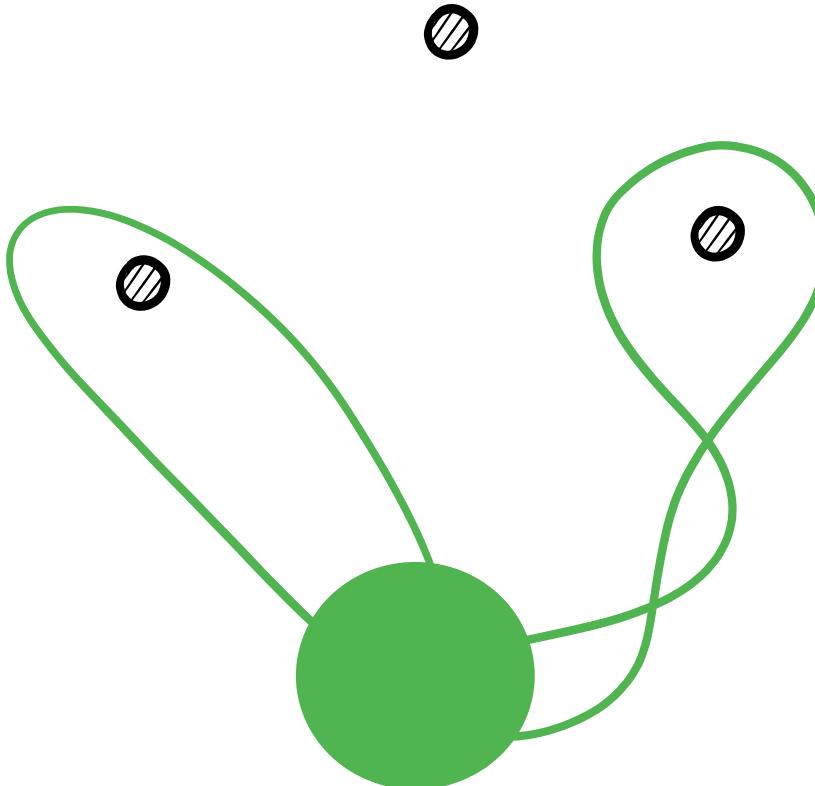
Straightening a graph



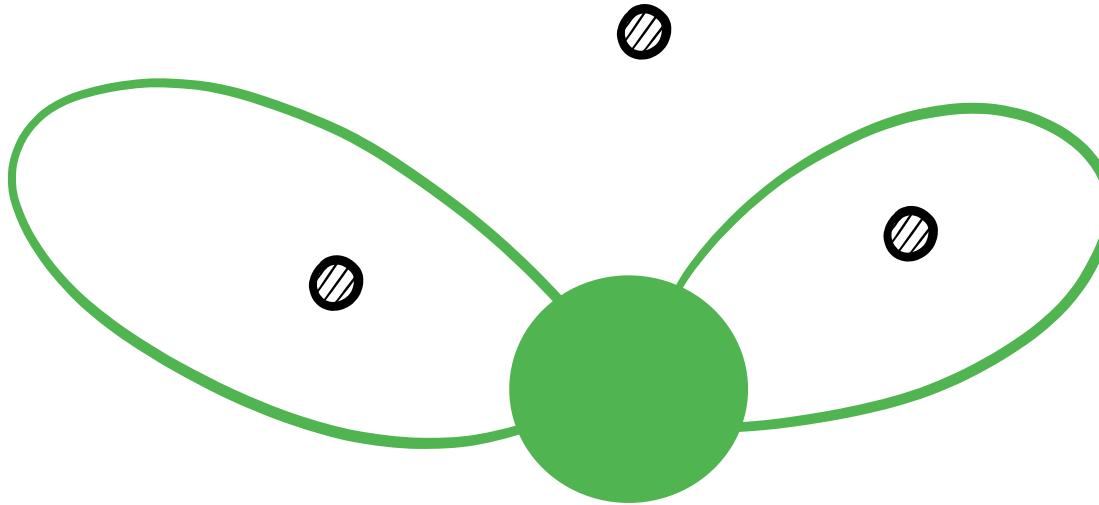
Straightening a graph



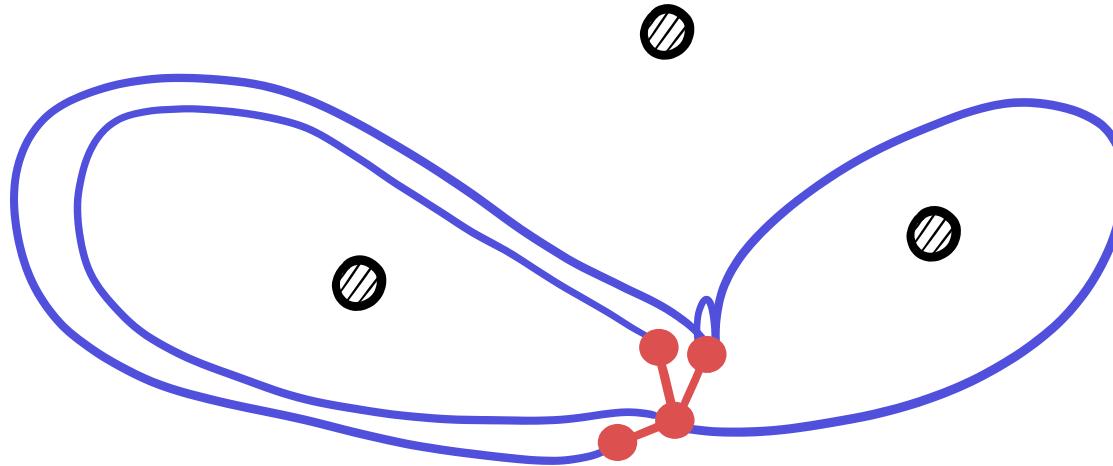
Straightening a graph



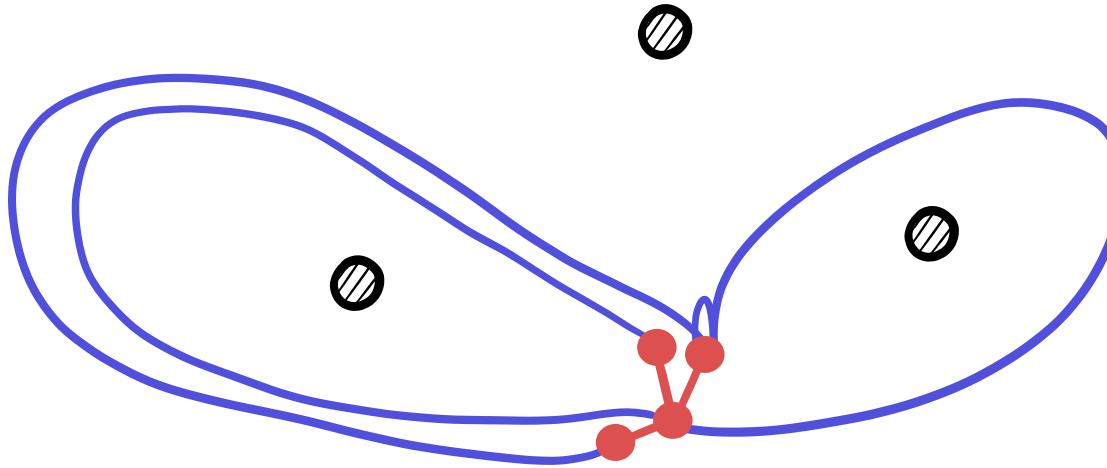
Straightening a graph



Straightening a graph



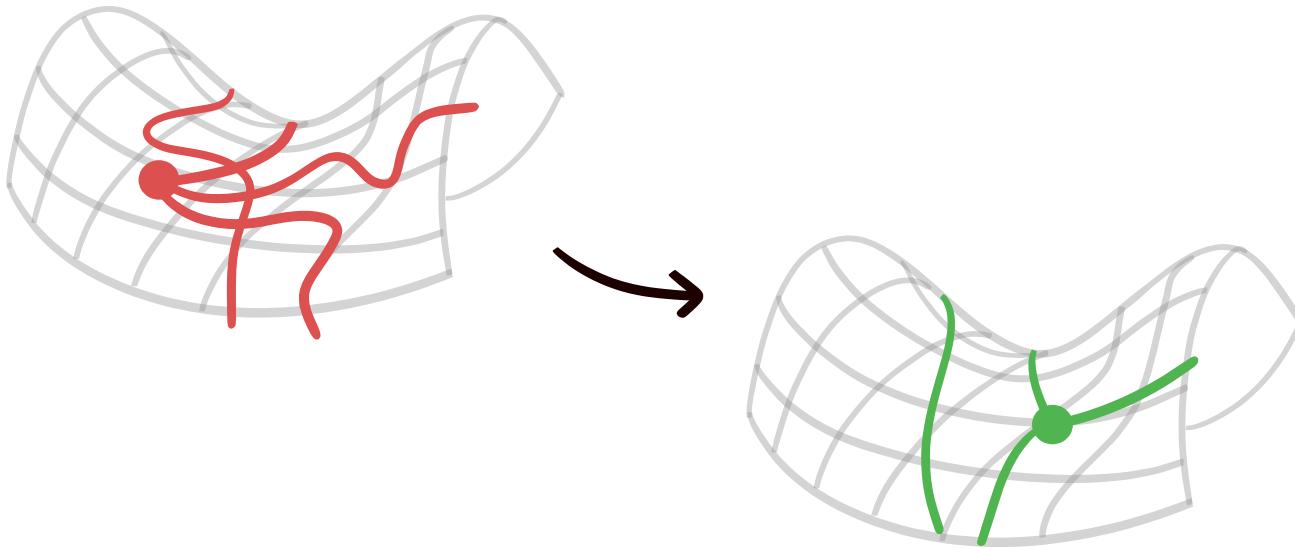
Straightening a graph



A straightened graph is a weak embedding
or cannot be untangled

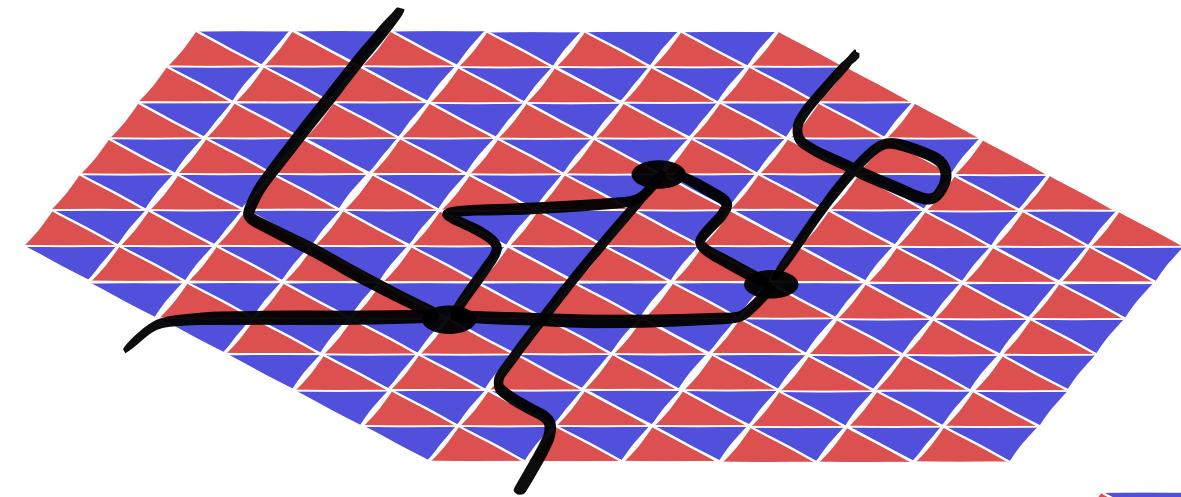
Discrete analogue of
Tutte embeddings

Recall: Tutte embeddings

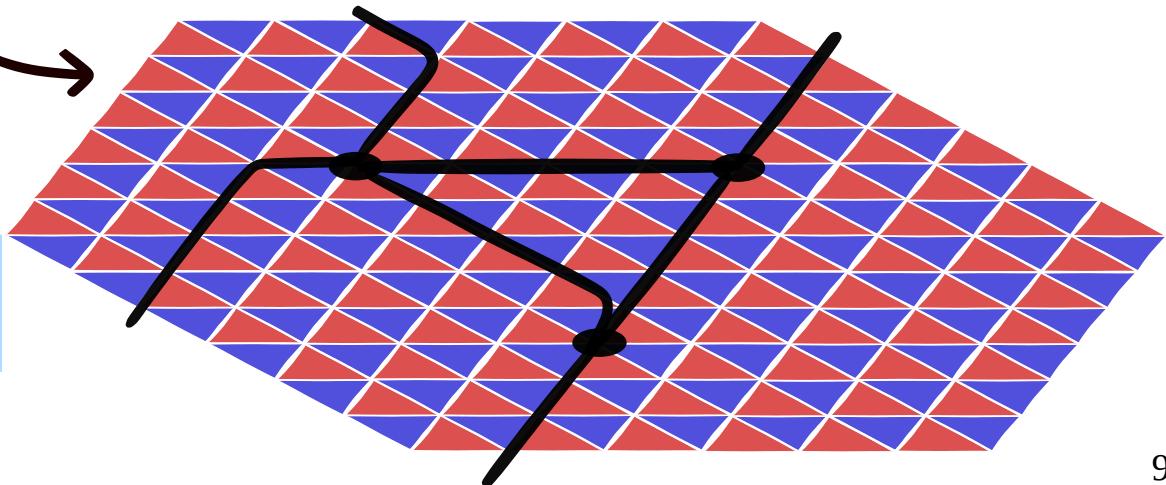


edges straight and
vertices “barycentric”

Harmonious drawings

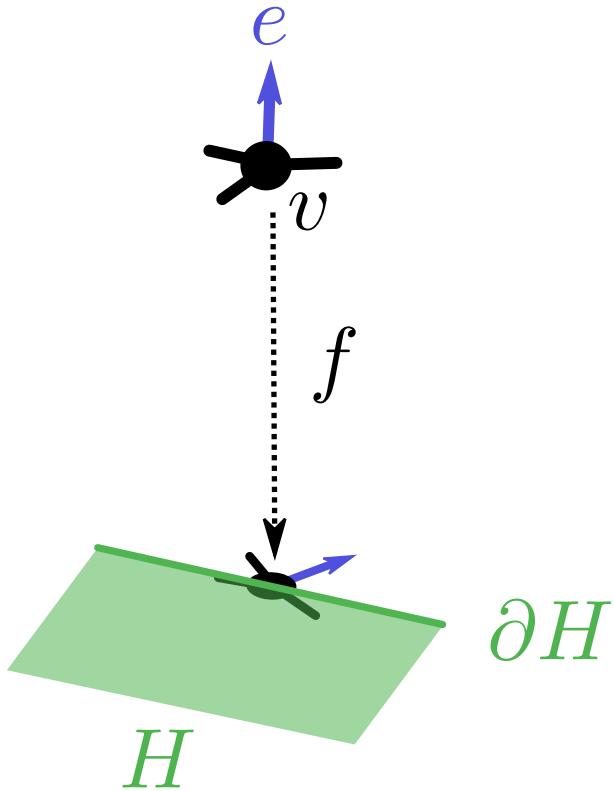


edges reduced and
vertices “barycentric”



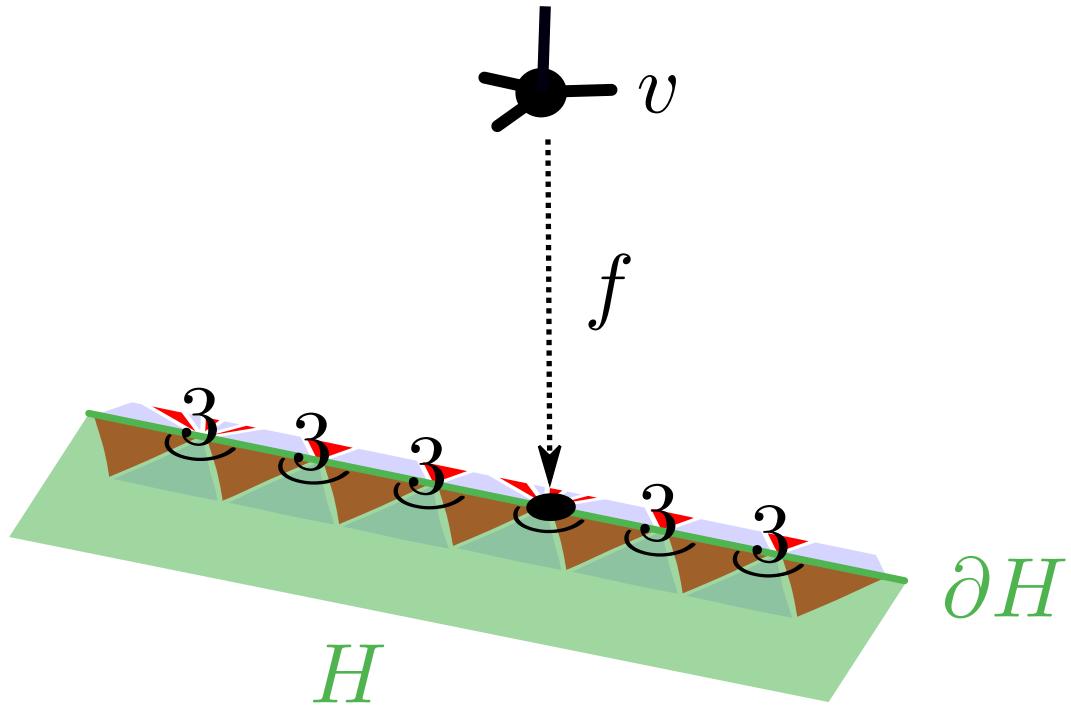
vertices “barycentric”

$$\begin{aligned} f(v) \in \partial H \\ \implies \exists e \quad f(e) \text{ escapes } H \end{aligned}$$



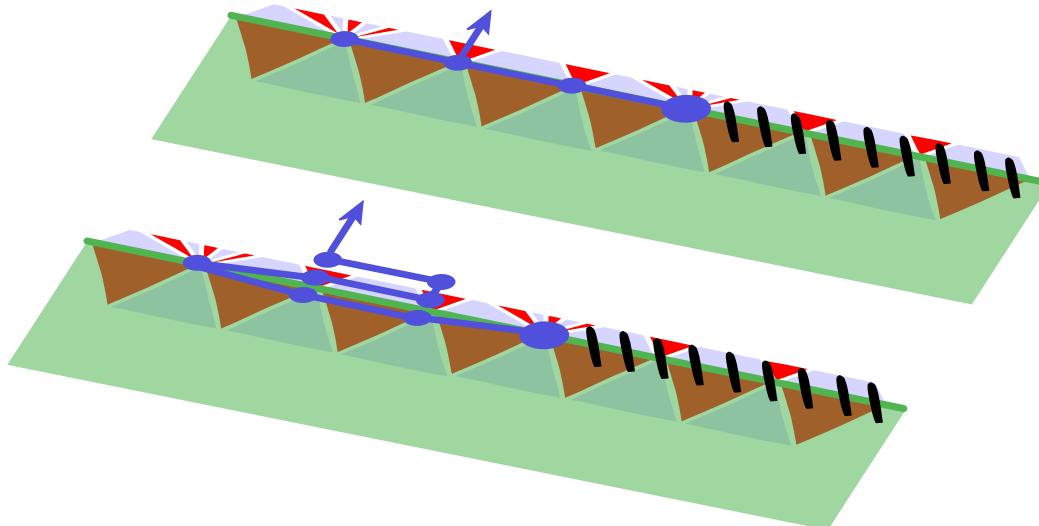
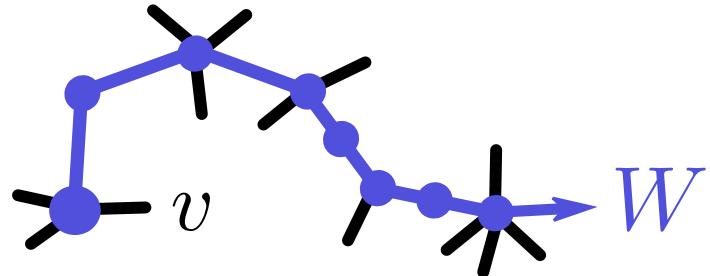
vertices “barycentric”

$$f(v) \in \partial H$$

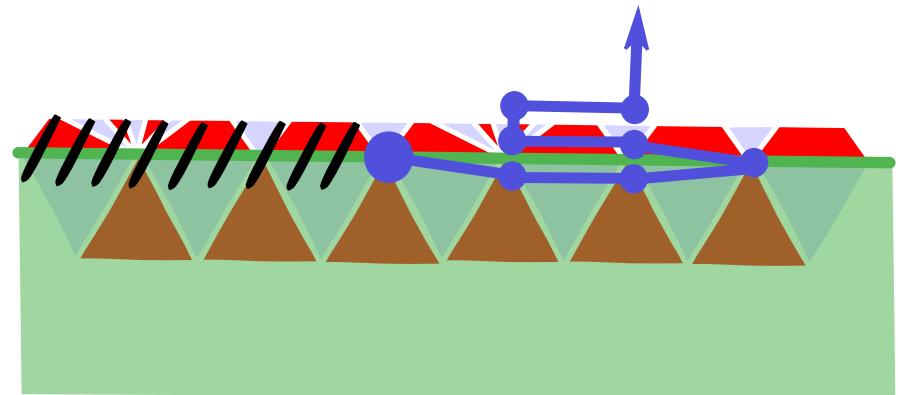
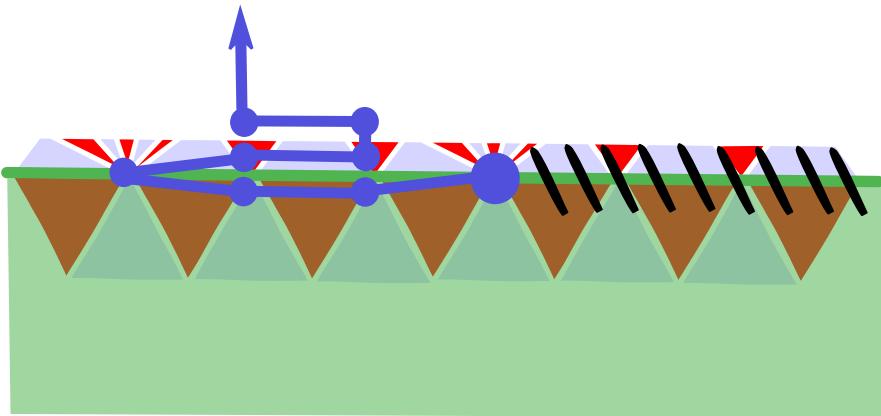


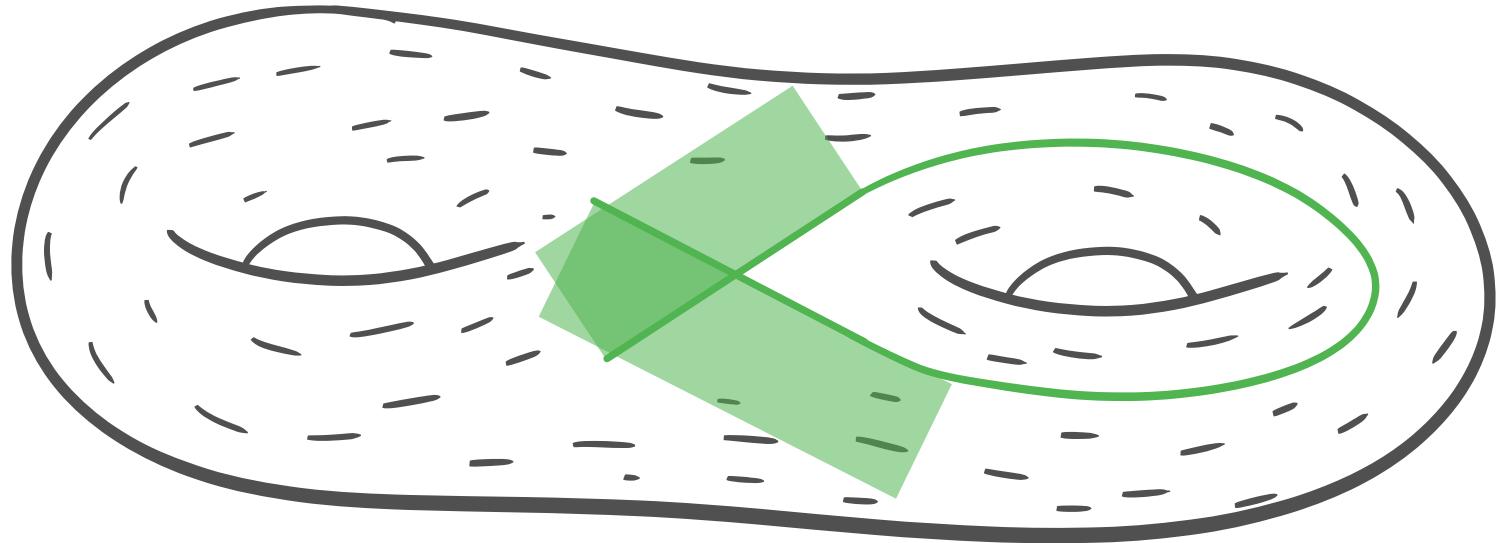
vertices “barycentric”

$$f(v) \in \partial H \\ \xrightarrow{} \\ \exists W f \circ W \text{“escapes”} H$$



where to escape depends on the coloring





this definition generalizes to surfaces

Results

graph G

reducing triangulation T with m edges

$f : G \rightarrow T^1$ of size n

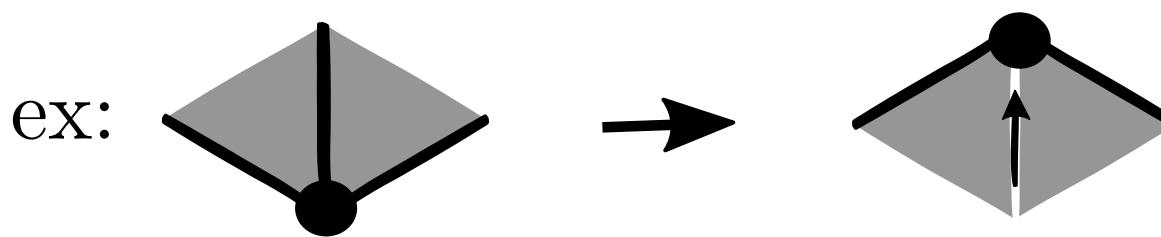
Definition of *harmonious* drawings

f harmonious and f can be untangled
 $\Rightarrow f$ weak embedding

Algo to make f harmonious in $O((m+n)^2 n^2)$ time,
without increasing any edge length

Harmonizing a drawing

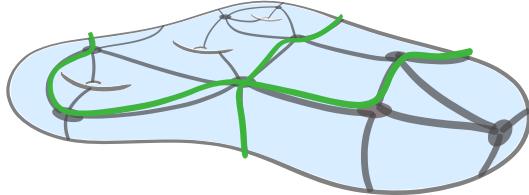
- 1 we define monotonic moves to apply to f



- 2 some moves do not seem to decrease any potential so we combine the moves carefully

Untangling curves

Related work

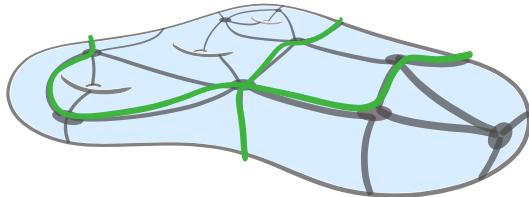


closed walks of total length n
on a graph of size m

Despré, Lazarus, 2019

- Put a single curve in minimal position in $O(m + n^4)$ time
- Compute the min. nb. of crossings in $O(m + n^2)$ time

Result



closed walks of total length n
on a graph of size m

simpler algos and proofs!

$$m^3n + mn \log(mn)$$

D., 2024

- Put a single curve in minimal position in $O(\cancel{m} + \cancel{n}^4)$ time
- Compute the min. nb. of crossings in $O(\cancel{m} + \cancel{n}^2)$ time

$$m^2 + mn \log(mn)$$

Untangling Graphs

Computing Delaunay Triangulations

Other Results
and Possible Continuations

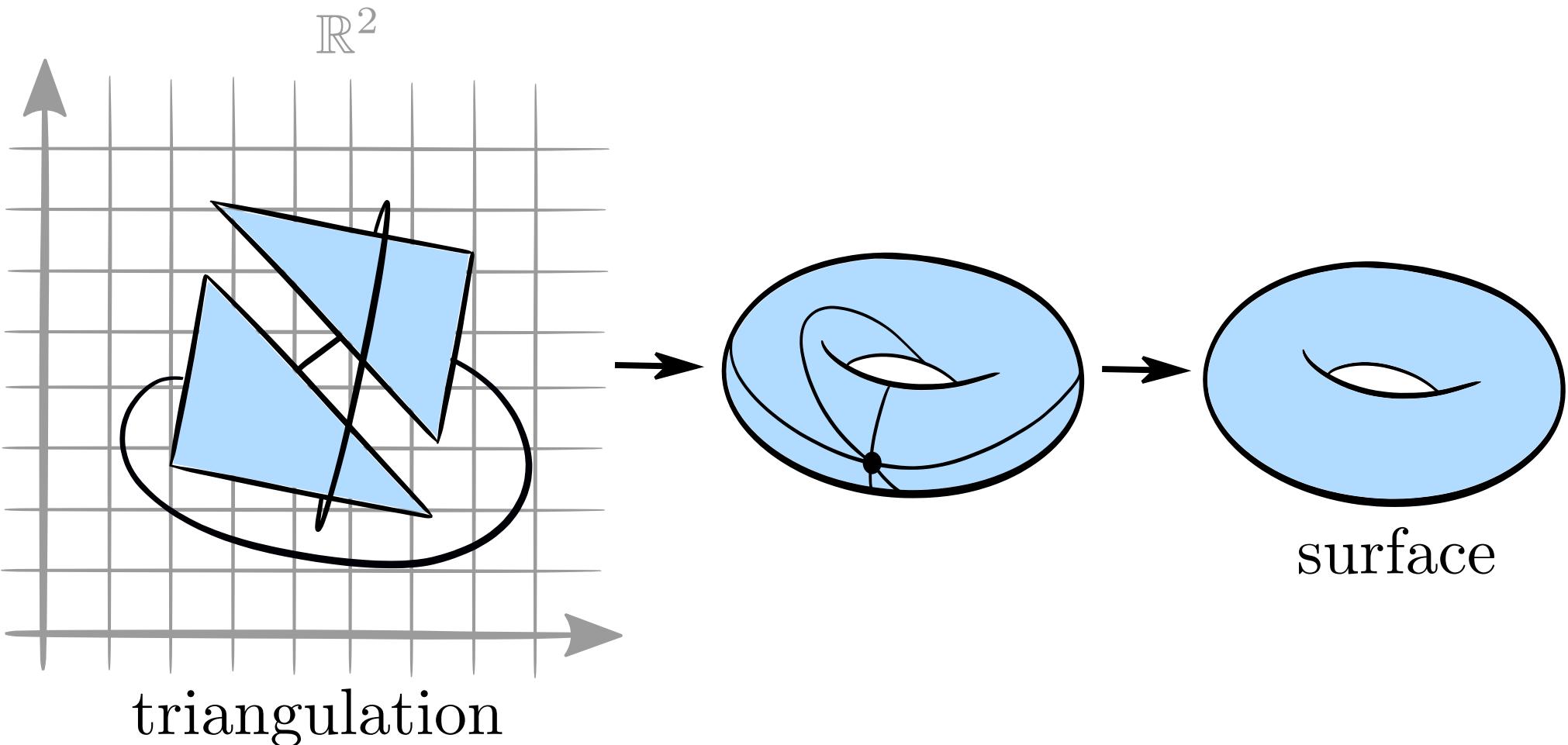
Untangling Graphs

Computing Delaunay Triangulations

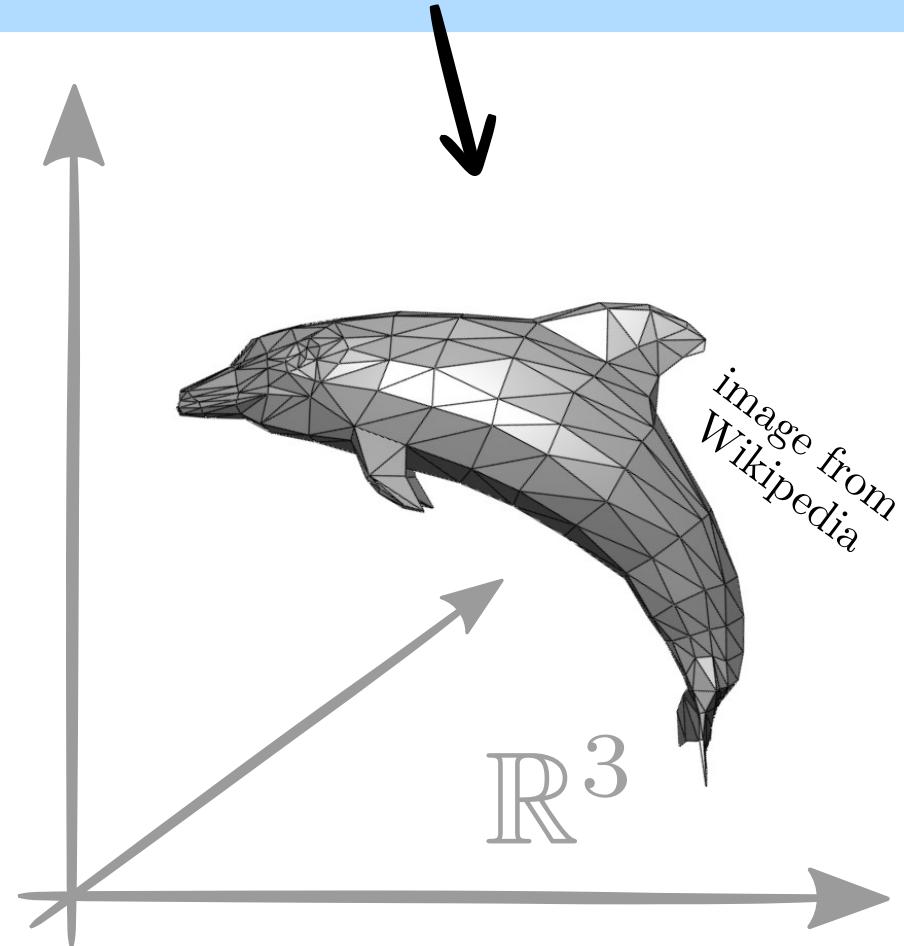
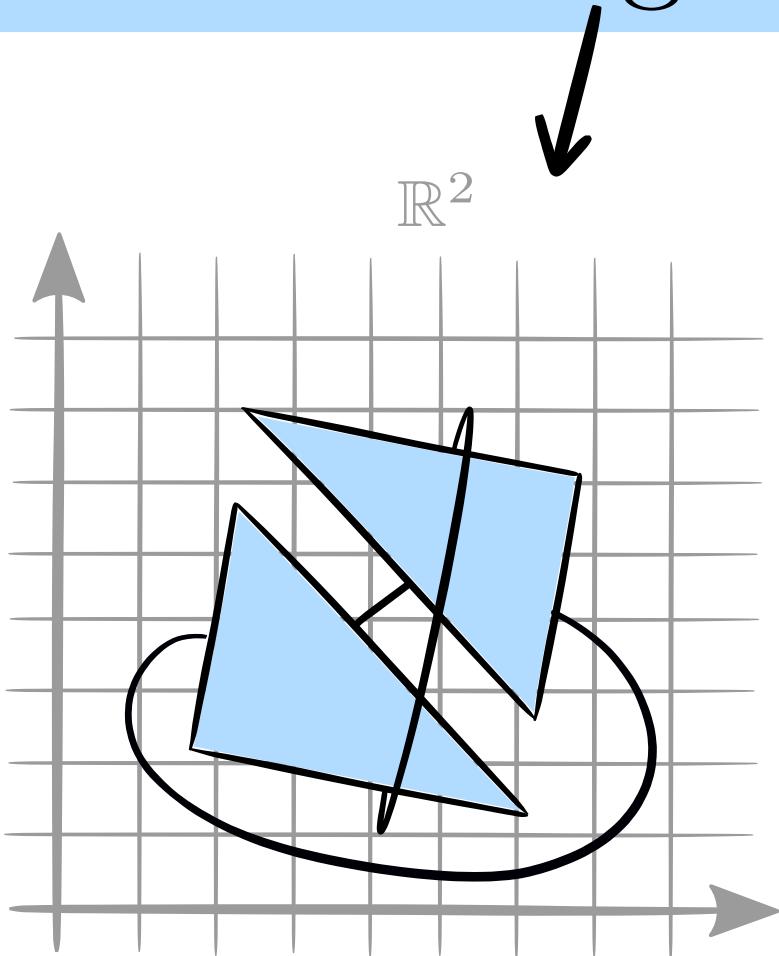
Other Results
and Possible Continuations

Triangulations of polyhedral surfaces

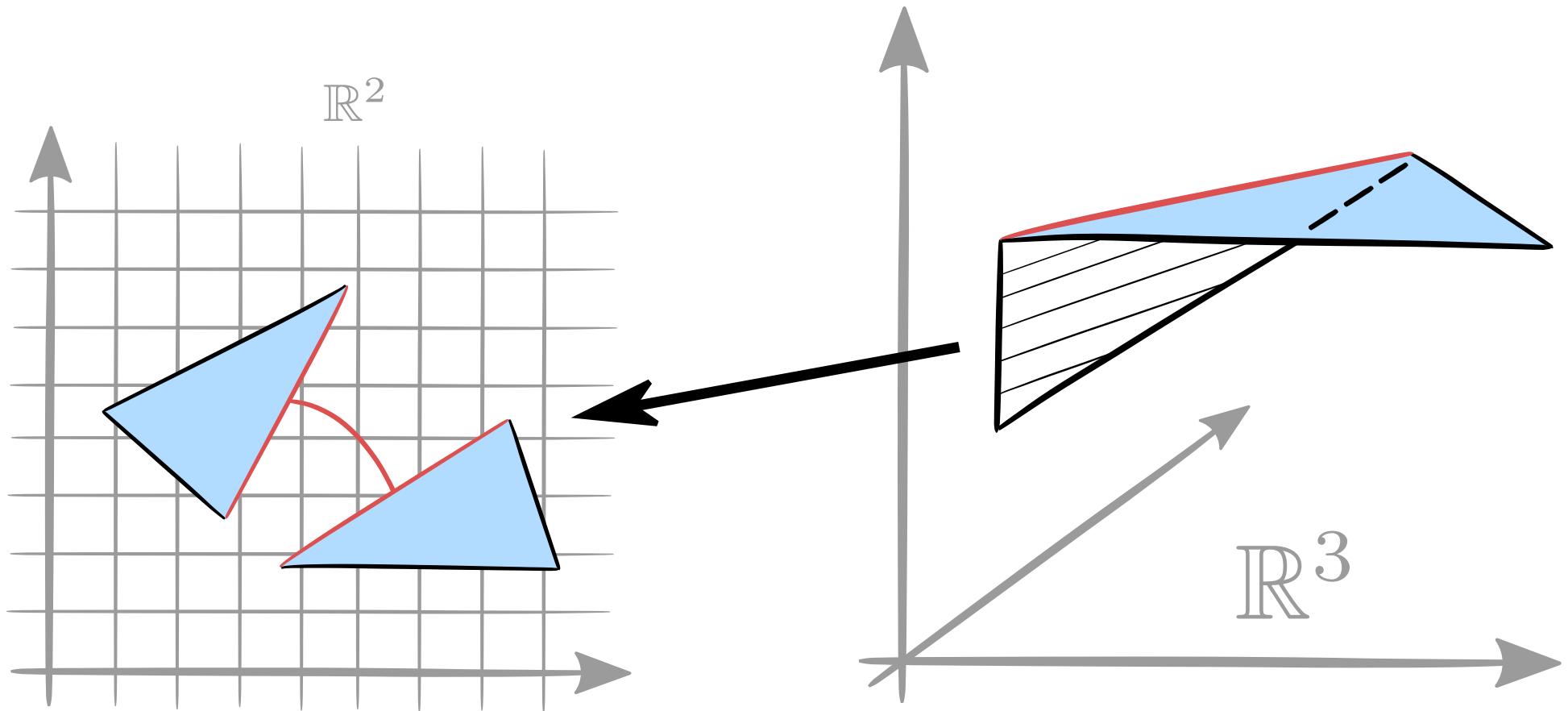
Triangulation of polyhedral surface



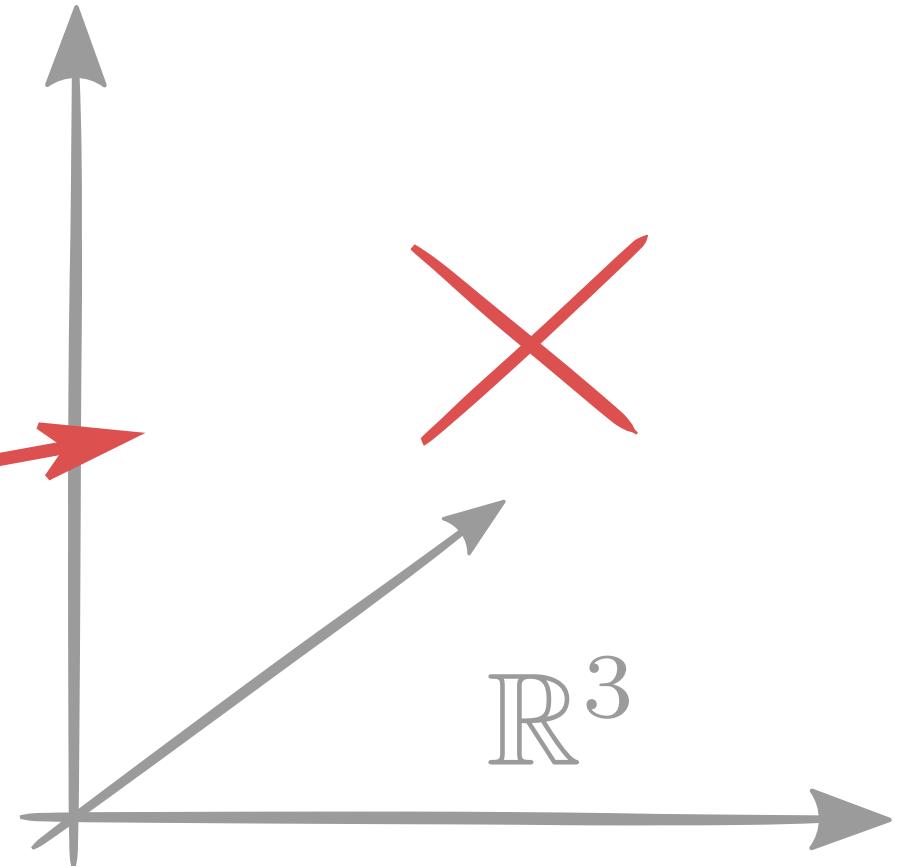
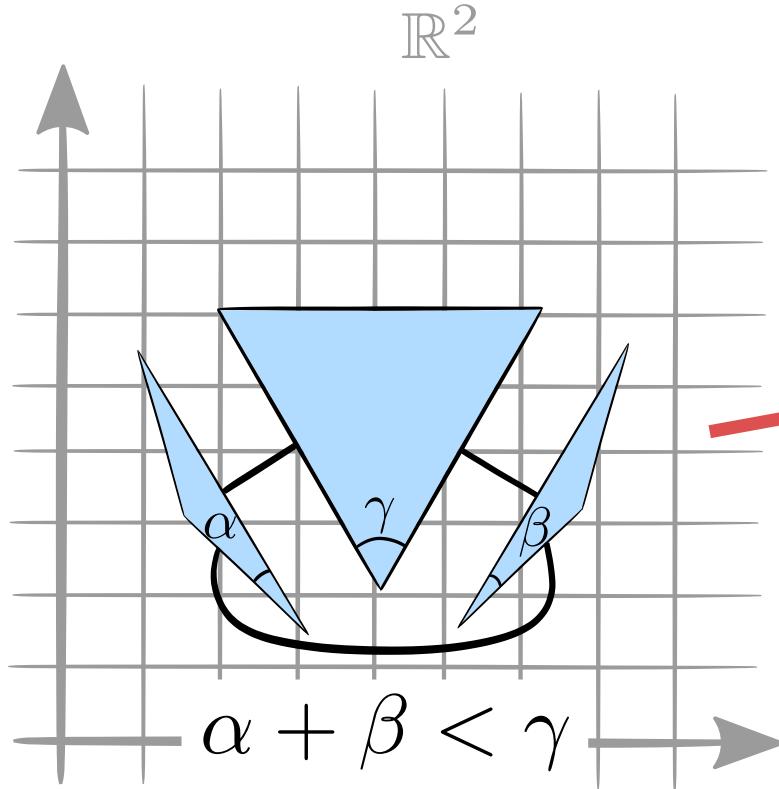
Triangulation vs. Mesh



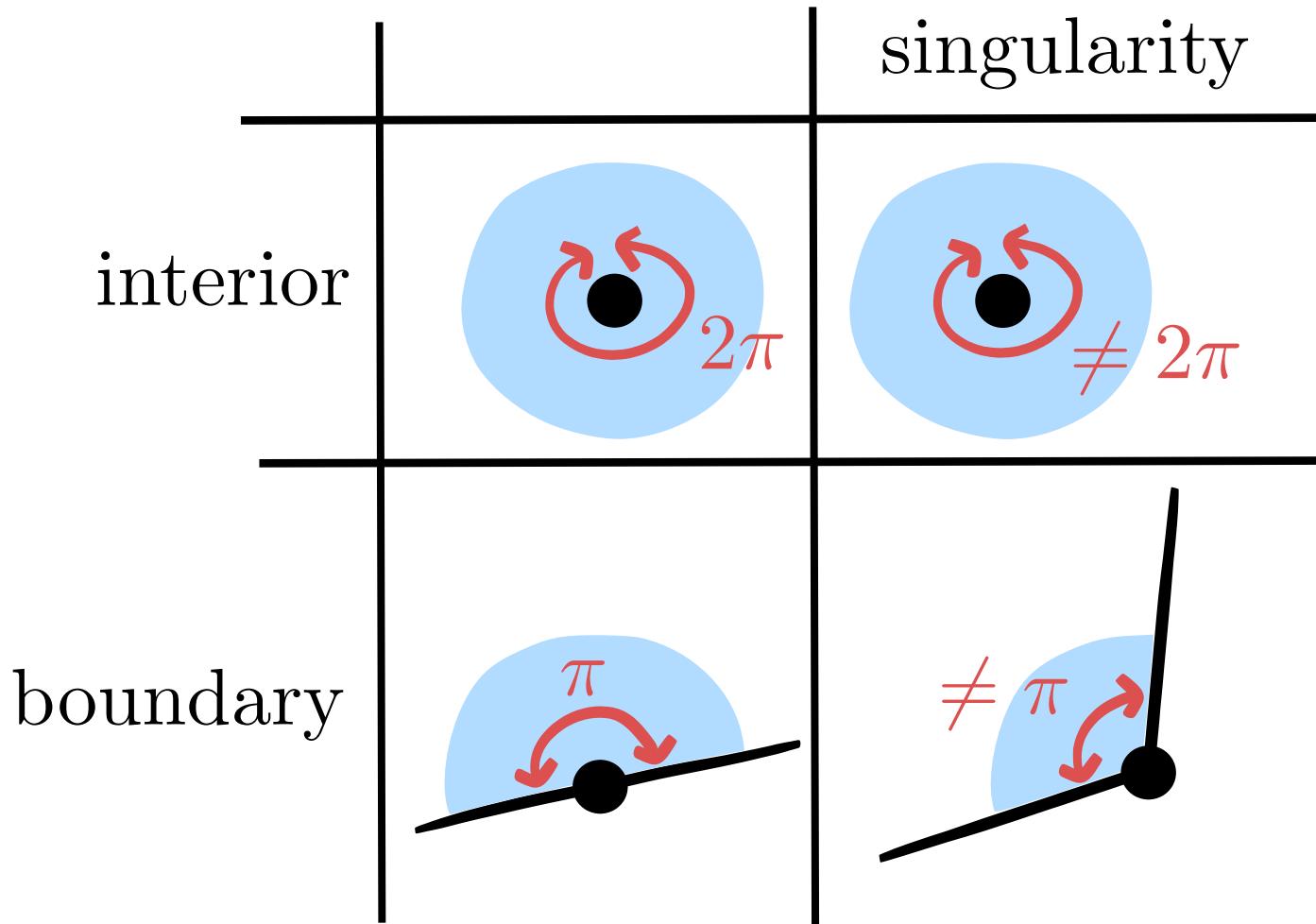
Every mesh gives a triangulation



Every mesh gives a triangulation
but converse is false!



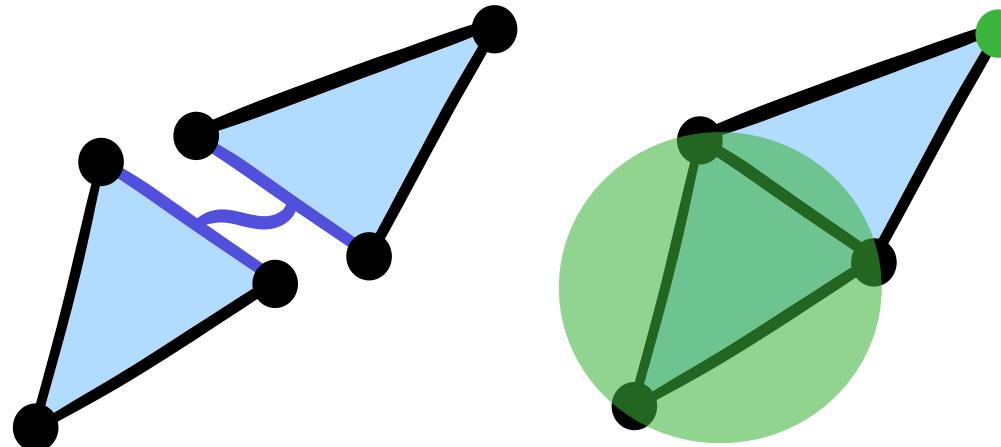
Types of points on the surface



Problem

Delaunay triangulation

triangulation in which
every edge is Delaunay



The Delaunay triangulation

Generically, every surface has a **unique**
Delaunay triangulation
with given vertex set*

*finite, non-empty, containing the singularities

Problem

Given triangulation T , and vertex set V^* ,
compute the Delaunay triangulation of the
surface of T whose vertex set is V

$*$ usually the vertex set of T or
the singularities of the surface of T

Motivations

- isometry testing
- shortest paths

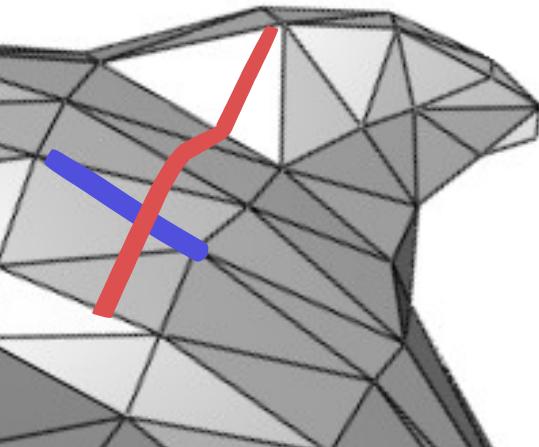
- shortest paths

Shortest paths on meshes

On a mesh M with n triangles...

a shortest path cannot cross an edge twice

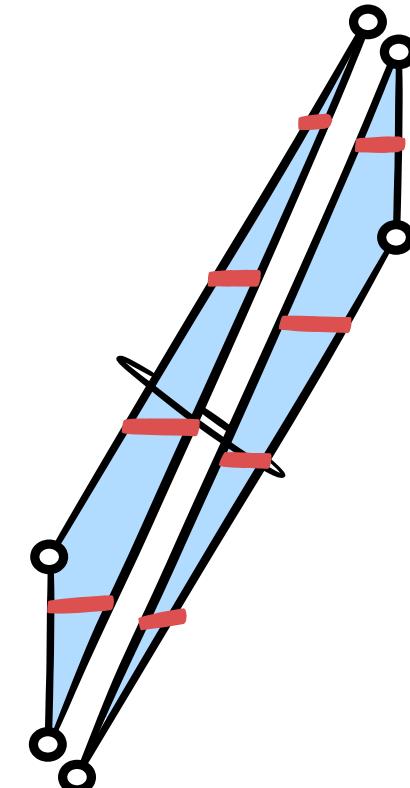
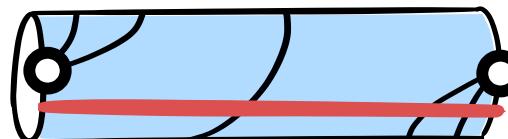
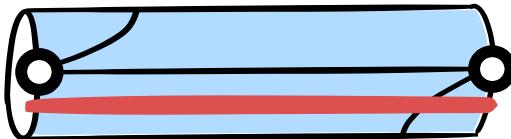
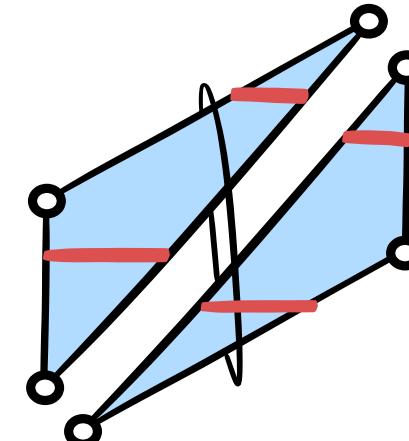
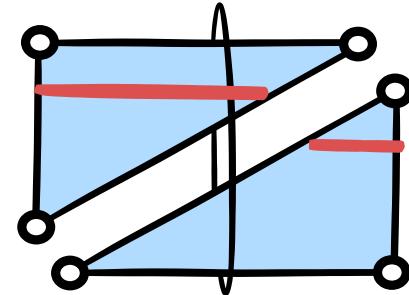
→ shortest path can be computed in $O(f(n))$ time



Mitchel, Mount, Papadimitriou, 1987

Shortest paths on triangulations

they can cross edges arbitrarily many times

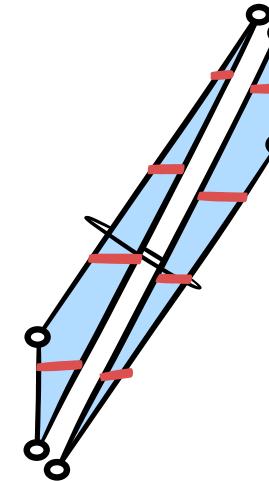


Erickson, 2006

Shortest paths on triangulations

Löffler, Ophelders, Staals, Silveira, 2023

happiness h : max number of times
a shortest path visits a triangle



→ shortest path can be computed in $O(f(n, h))$ time

Shortest paths on triangulations

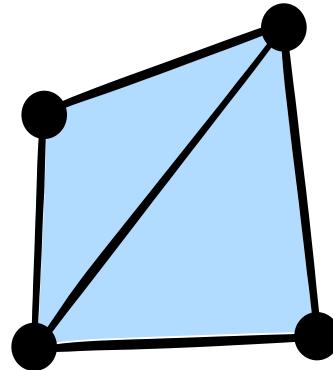
Löffler, Ophelders, Staals, Silveira, 2023

Delaunay triangulations have happiness $O(1)$

Existing algorithms

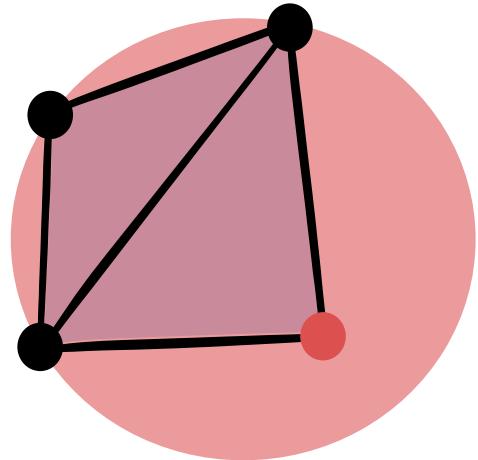
Delaunay flips algorithm

Delaunay flip:



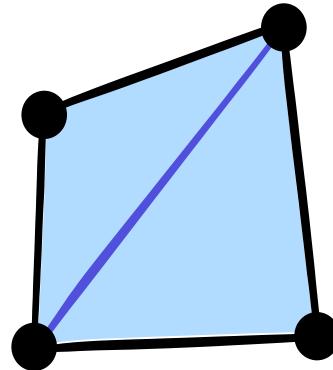
Delaunay flips algorithm

Delaunay flip:



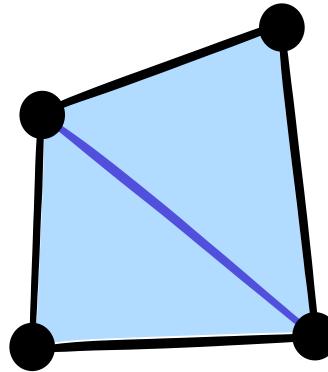
Delaunay flips algorithm

Delaunay flip:



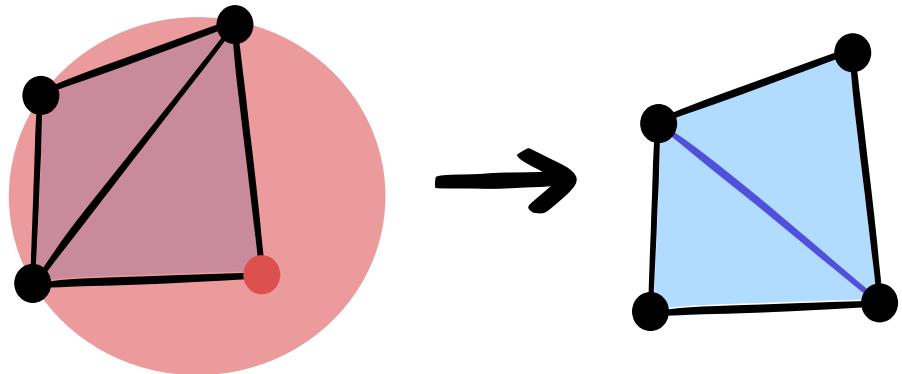
Delaunay flips algorithm

Delaunay flip:



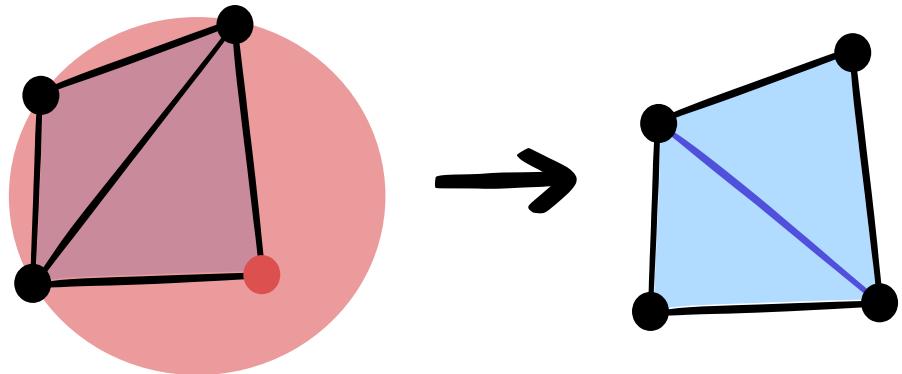
Delaunay flips algorithm

Delaunay flip:



Delaunay flips algorithm

Delaunay flip:

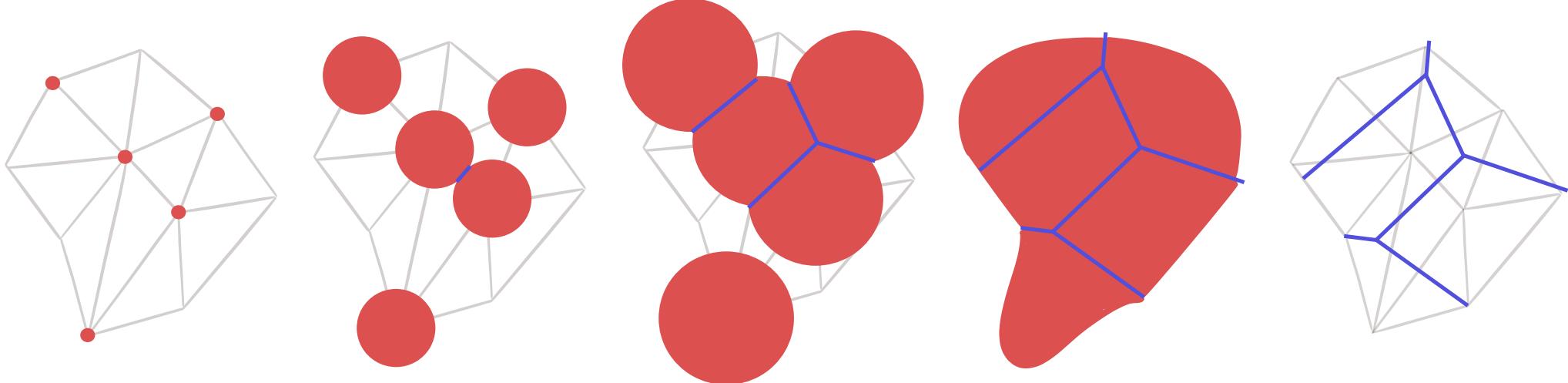


Algorithm:

apply Delaunay flips greedily as long as you can

Other method

compute the Voronoi diagram
by propagating waves

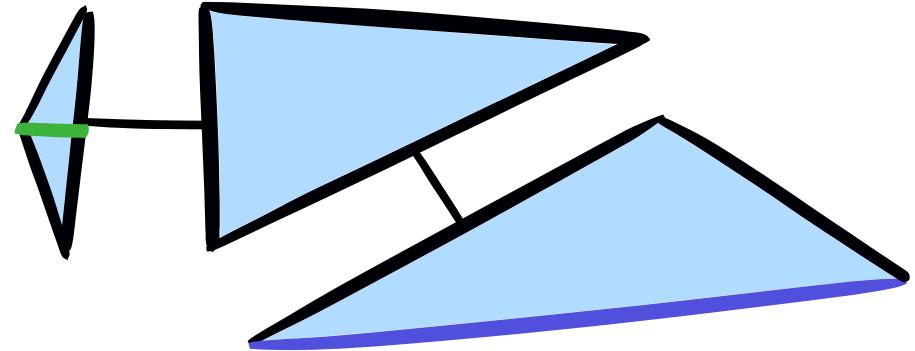


then derive Delaunay from it

Result

Result

aspect ratio =
 $\frac{\text{maximum side length}}{\text{minimum height}}$

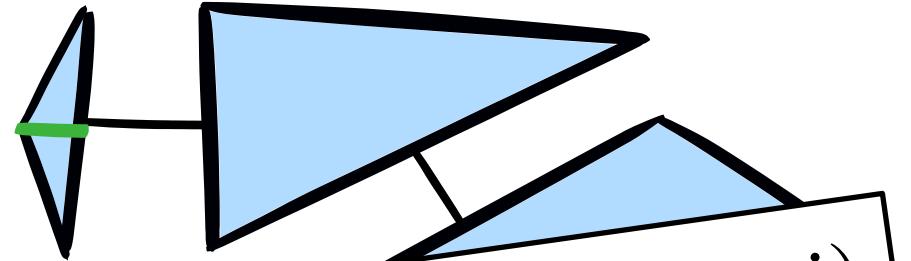


D. 2025

Given triangulation T of n triangles, of aspect ratio r , whose surface has no boundary, we can compute Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Result

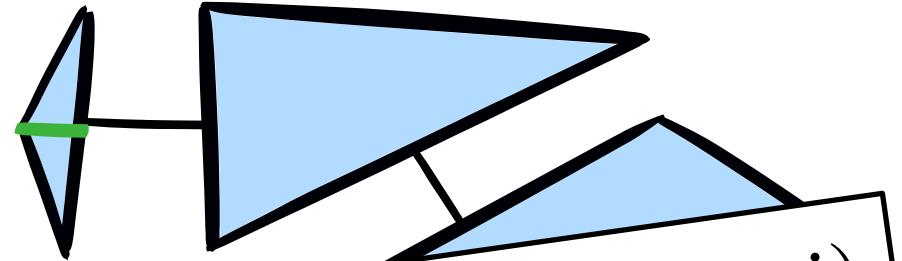
aspect ratio =
 $\frac{\text{maximum side length}}{\text{minimum height}}$



D. S. G. Previous algorithms (Delaunay flips and Voronoi) achieved $O(\text{Poly}(n, r))$ time for surfaces with boundaries. If the surface has no boundary, we can compute Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Result

aspect ratio =
maximum side length
minimum height



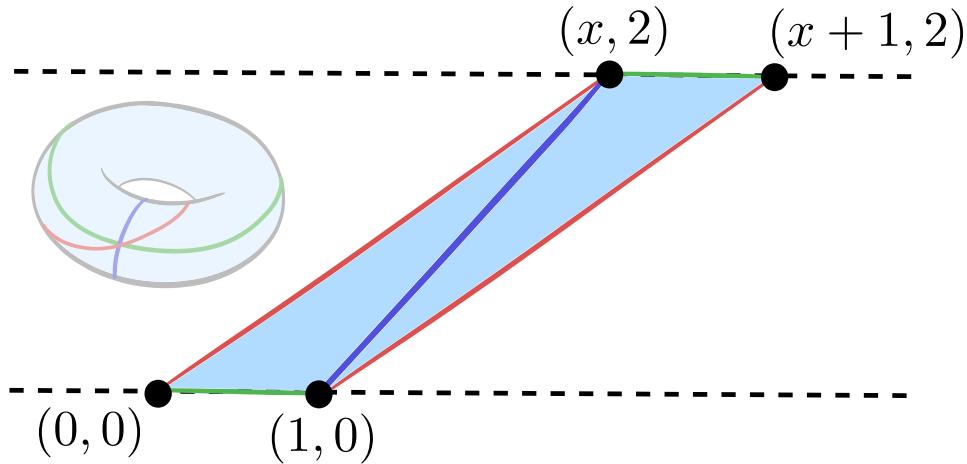
D. So
G. Previous algorithms (Delaunay flips and Voronoi)
achieved $O(\text{Poly}(n, r))$
surface has no boundary, we can compute
Delaunay in $O(n^3 \log^2(n) \cdot \log^4(r))$ time

Now backed by a lower bound!

Lower bound

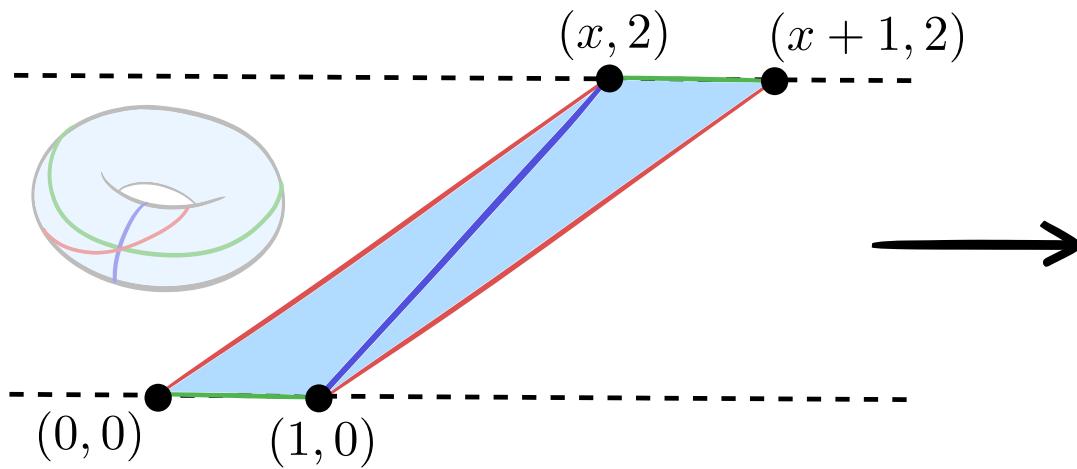
Lower bound

Input

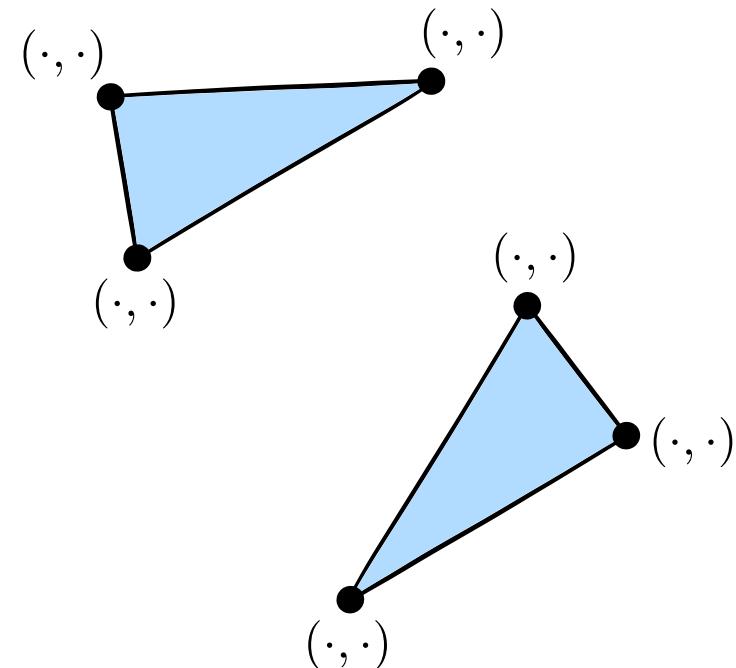


Lower bound

Input

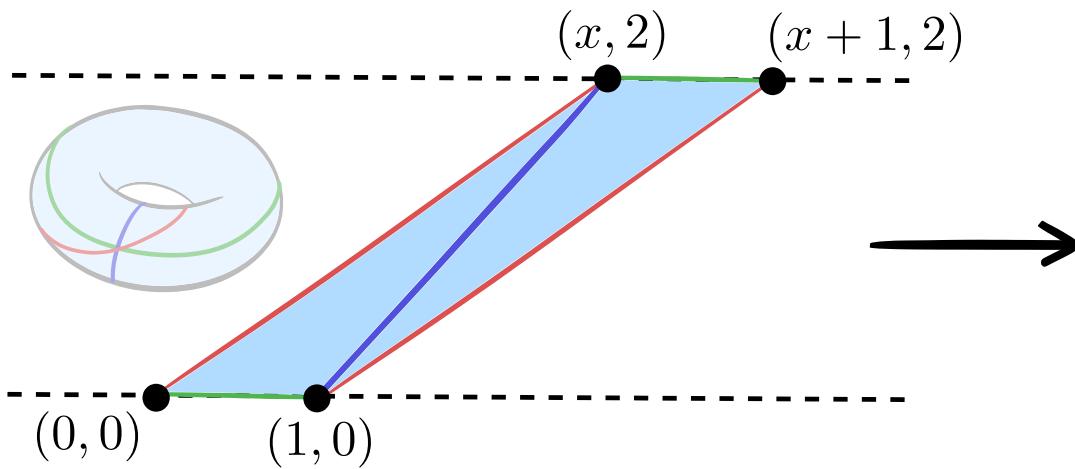


Output: Delaunay

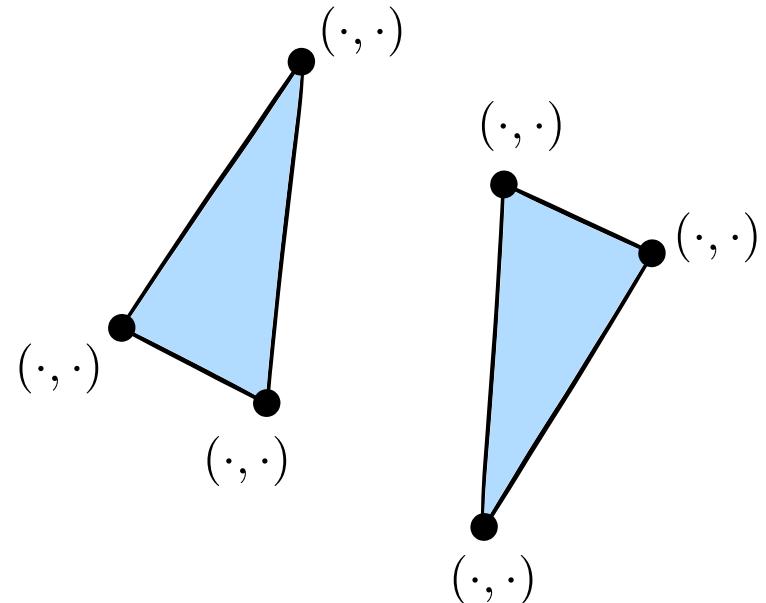


Lower bound

Input

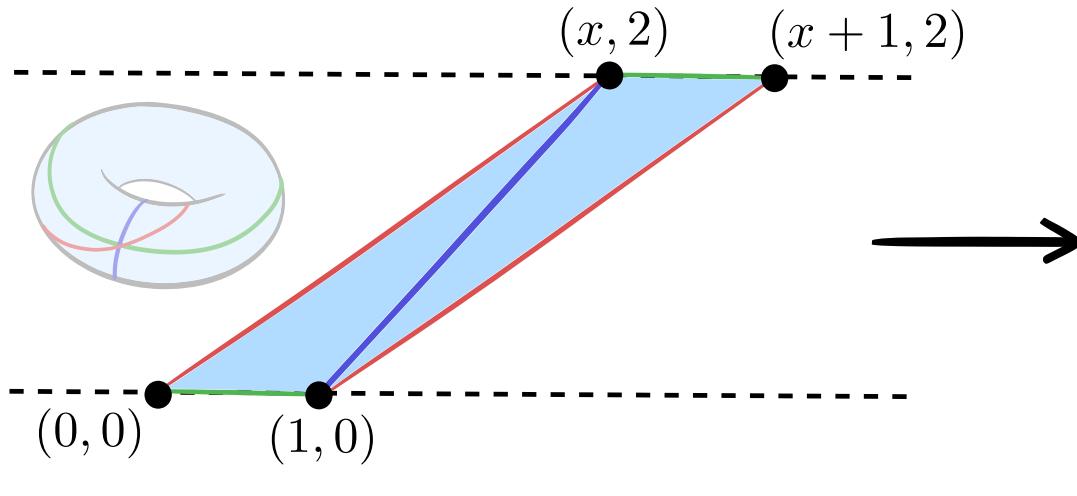


Output: Delaunay

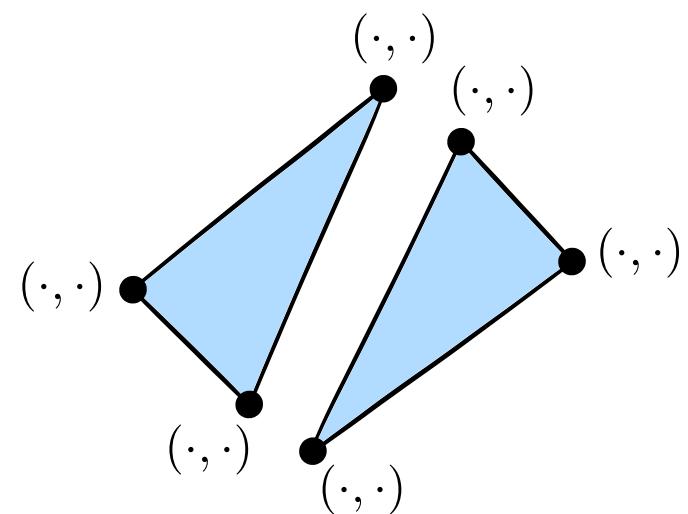


Lower bound

Input

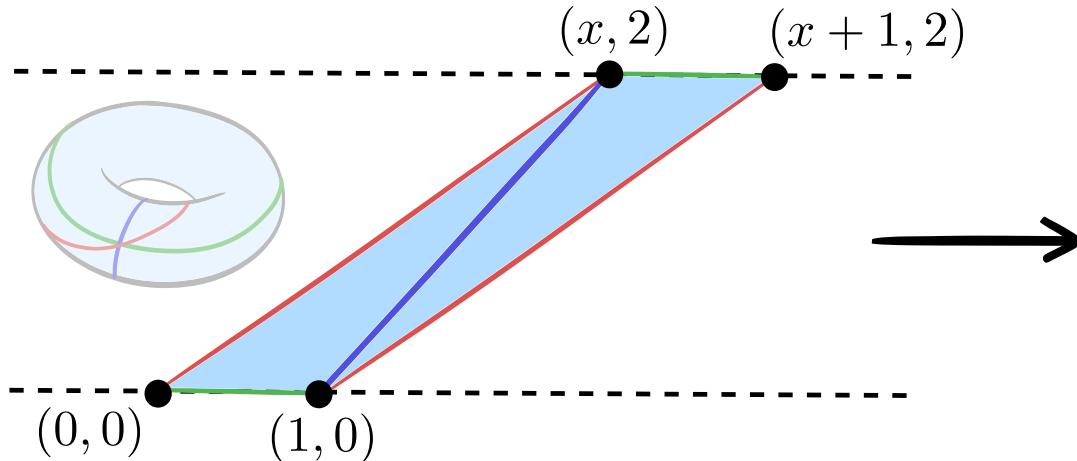


Output: Delaunay

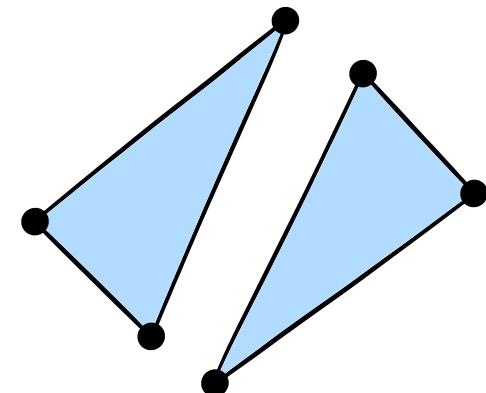


Lower bound

Input

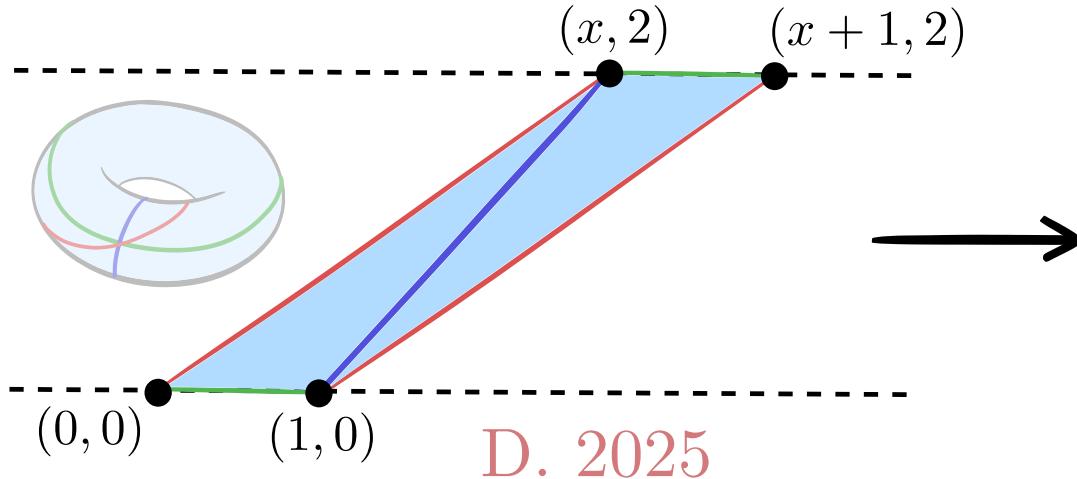


Output: Delaunay

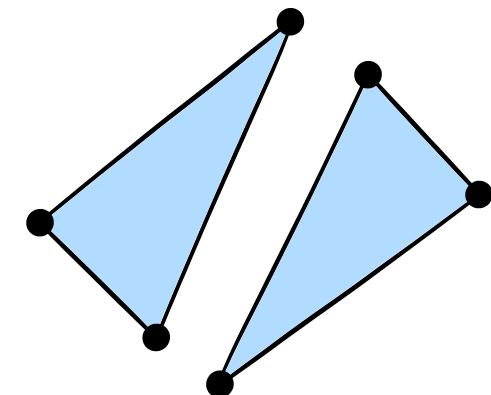


Lower bound

Input



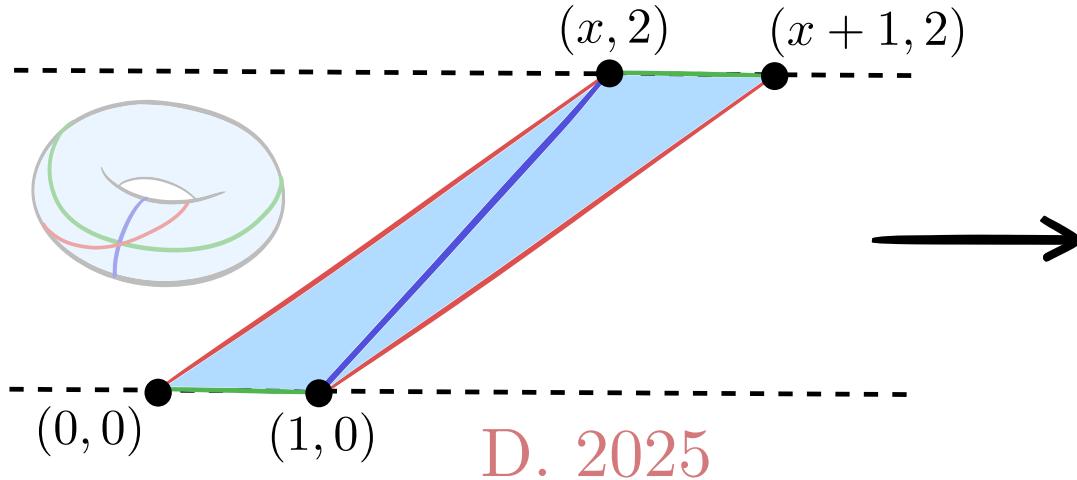
Output: Delaunay



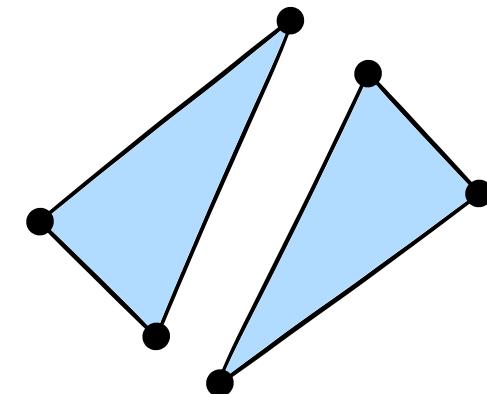
No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

Lower bound

Input



Output: Delaunay

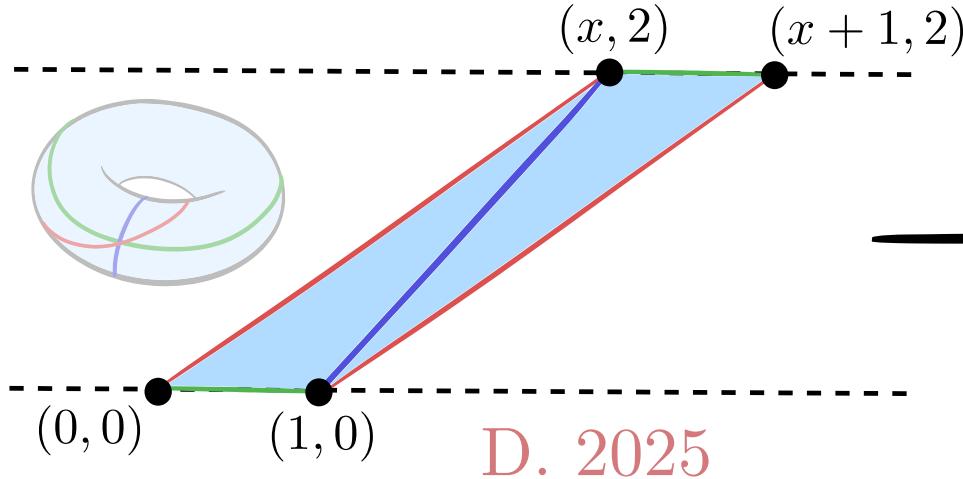


No Real RAM algo can compute
Delaunay from x in $o(\log x)$ time

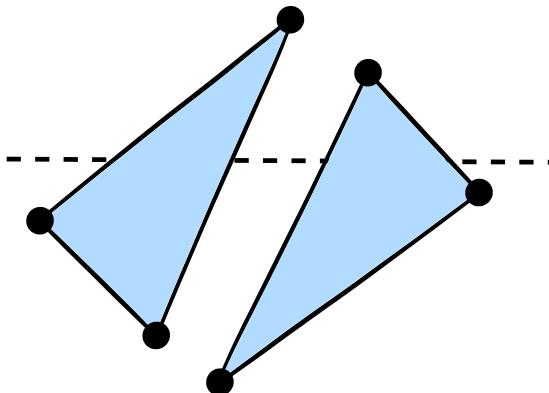
Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time

Lower bound

Input



Output: Delaunay

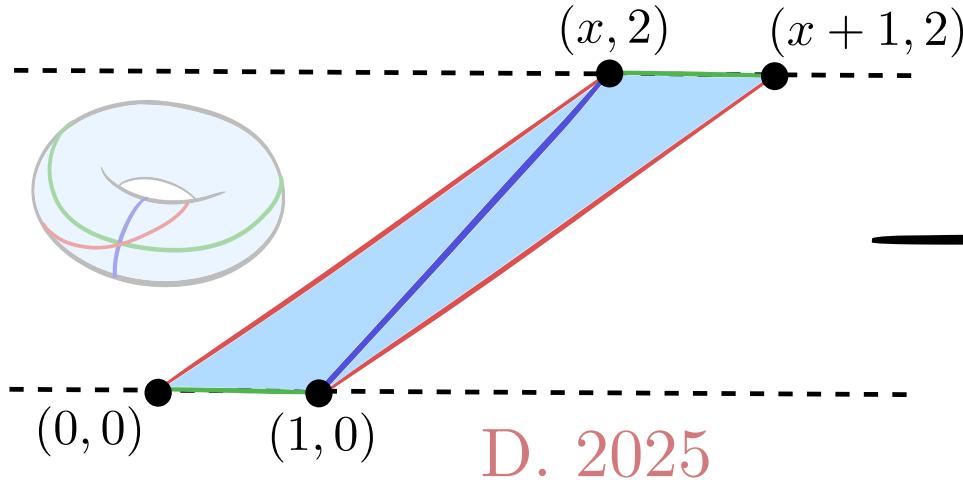


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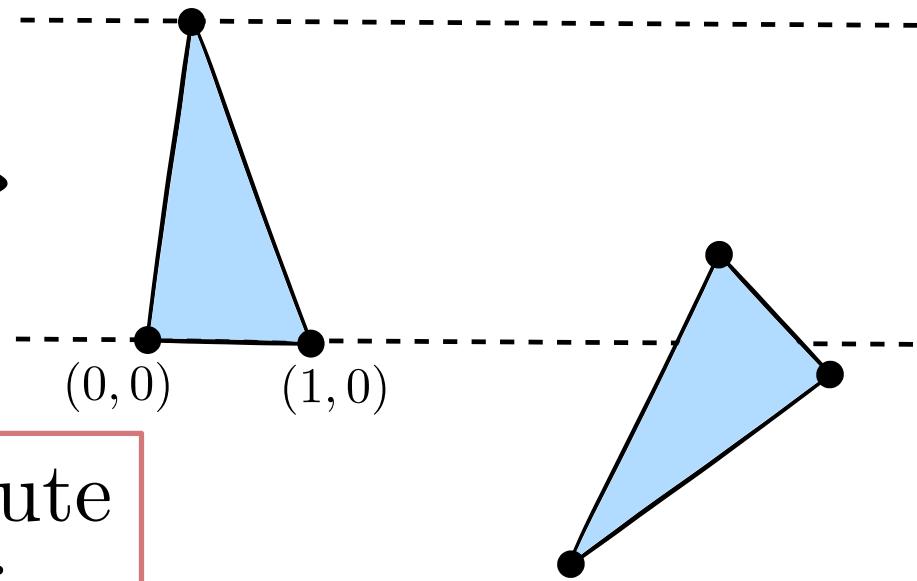
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Lower bound

Input



Output: Delaunay

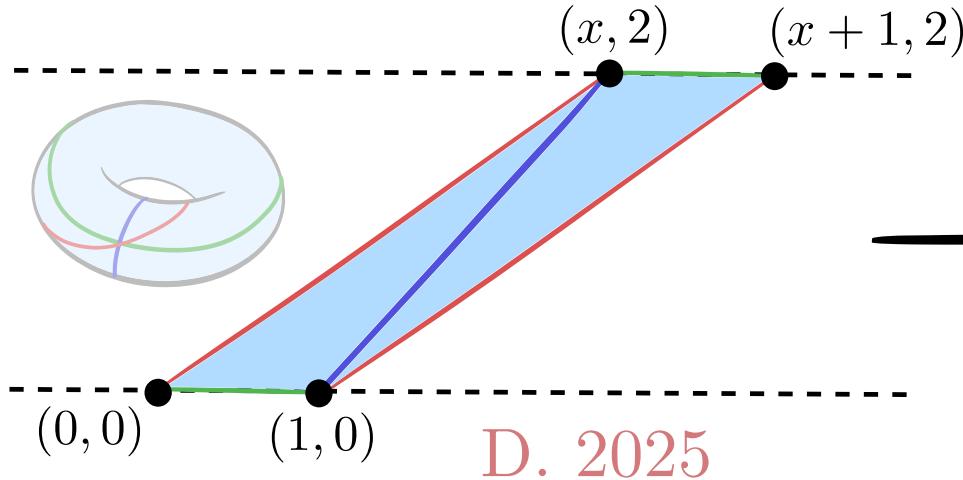


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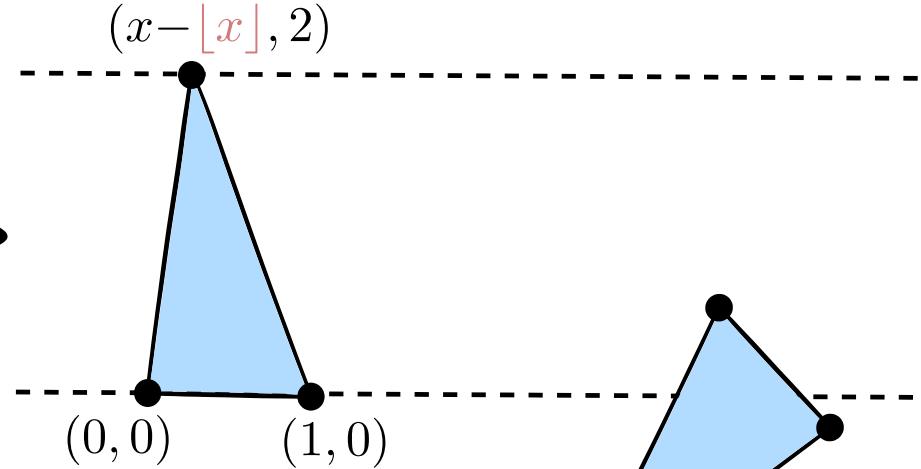
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Lower bound

Input



Output: Delaunay



No Real RAM algo can compute
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Otherwise we could compute $\lfloor x \rfloor$ from x in $o(\log x)$ time

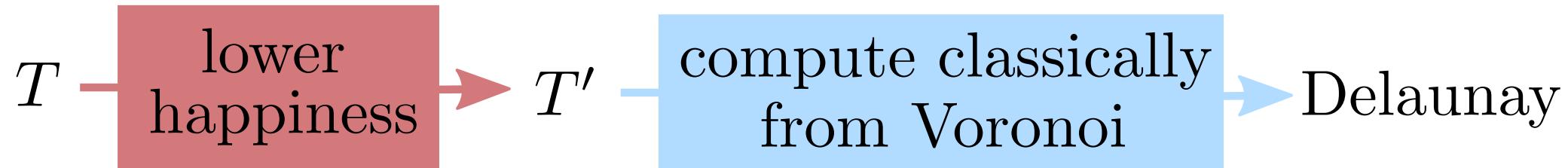
Algorithm overview

Algorithm



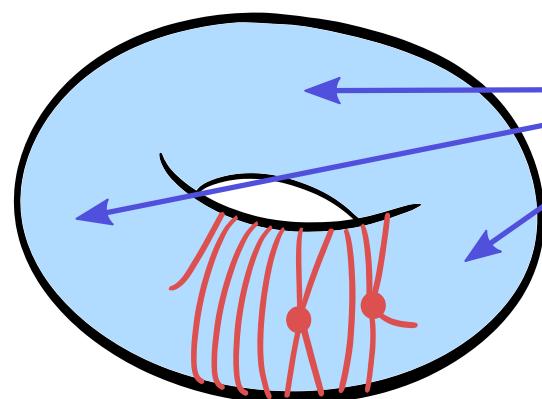
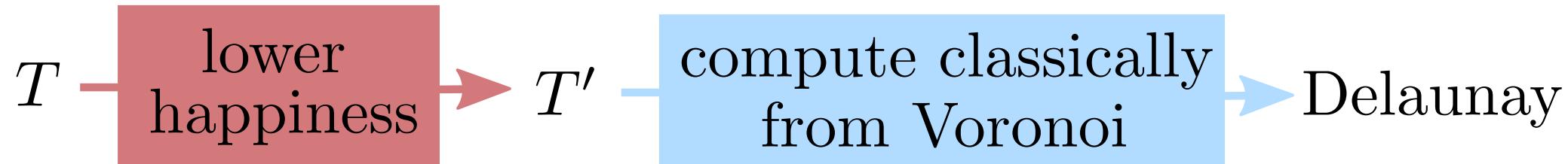
Algorithm

D. 2025



Algorithm

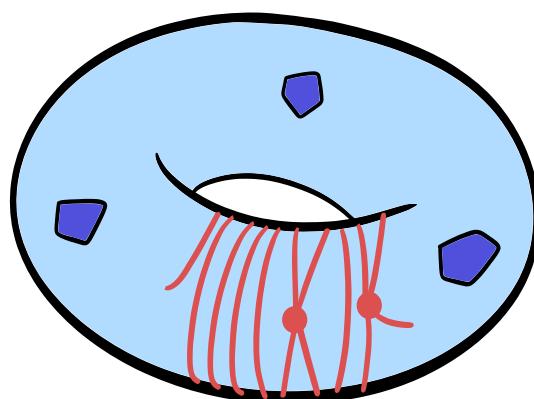
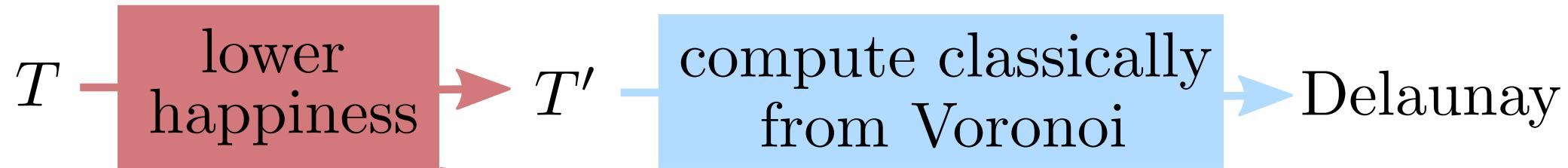
D. 2025



consider the singularities

Algorithm

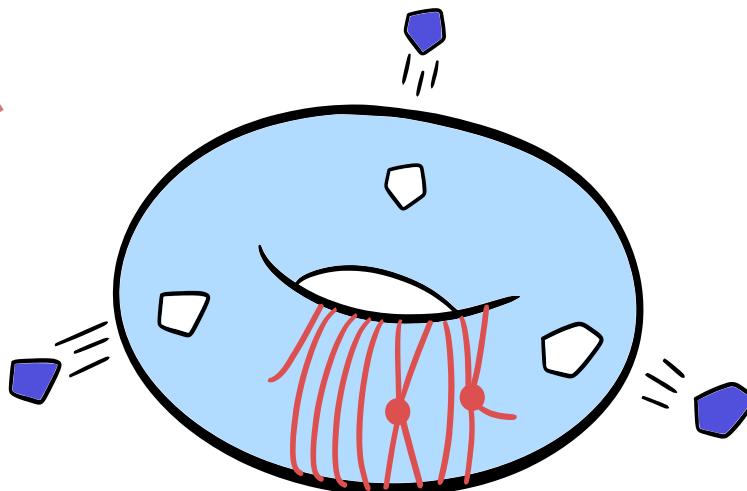
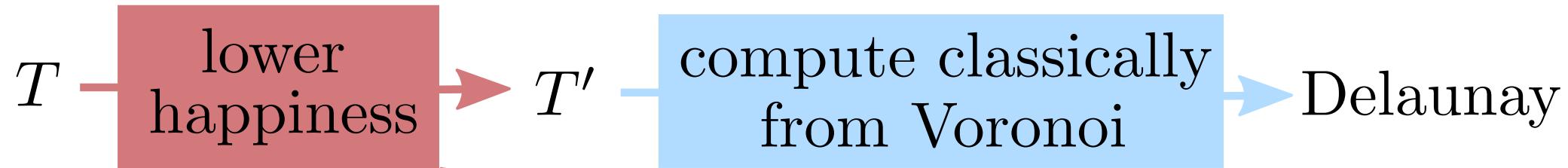
D. 2025



cut out caps around the singularities

Algorithm

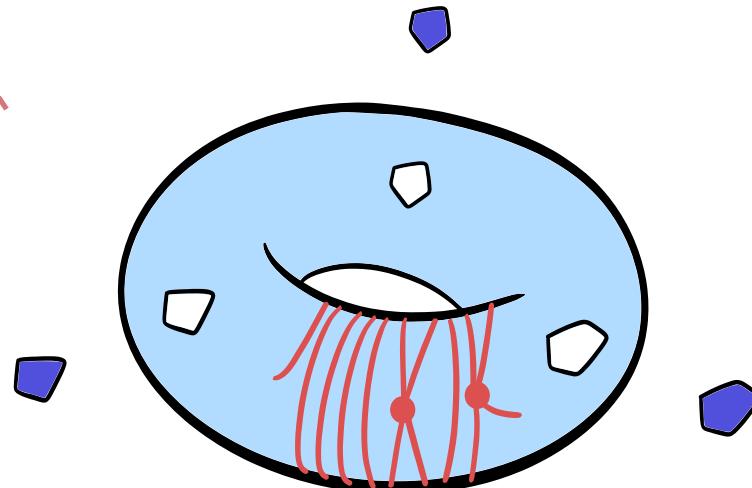
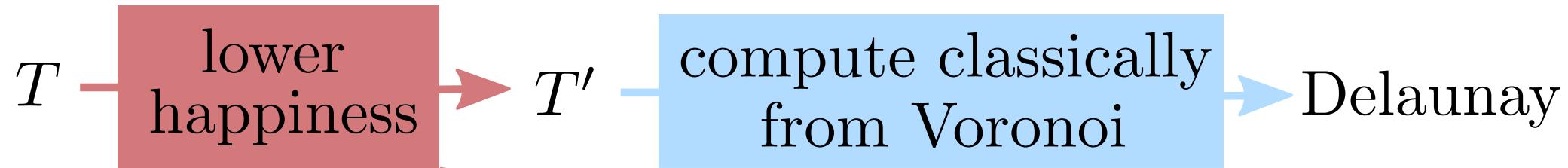
D. 2025



cut out caps around the singularities

Algorithm

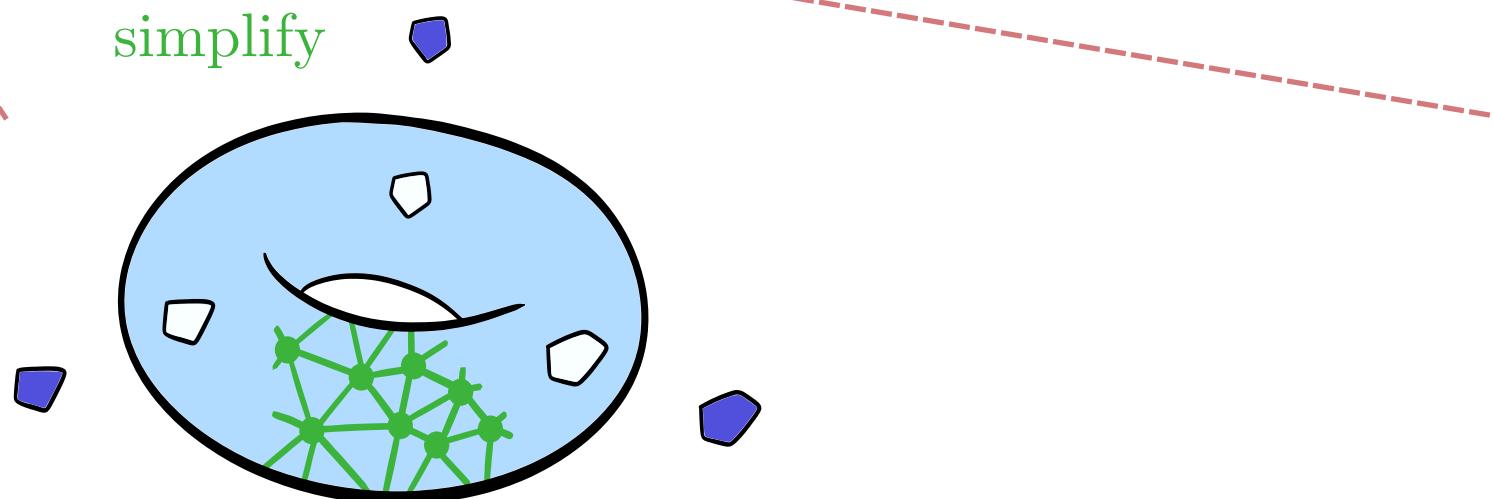
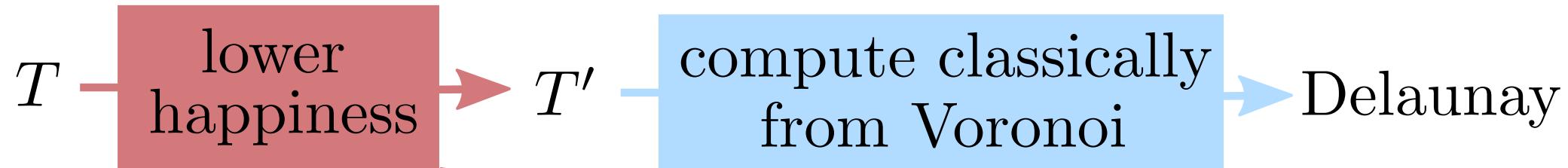
D. 2025



cut out caps around the singularities

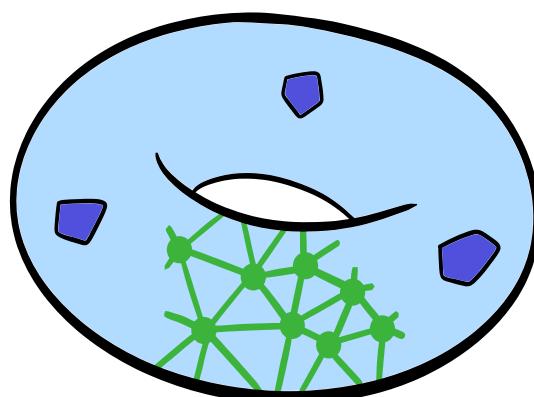
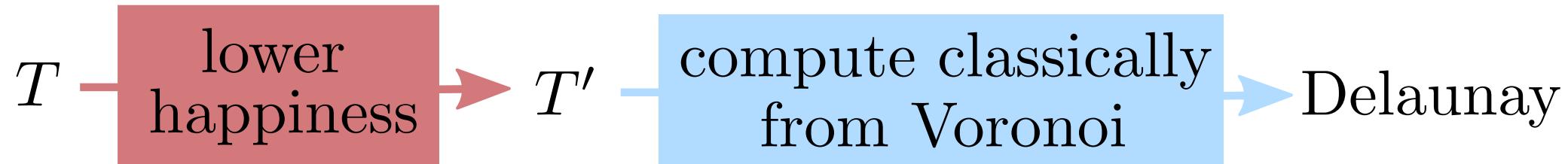
Algorithm

D. 2025



Algorithm

D. 2025



put the caps back

Simplification algorithm

Tuned combination of elementary operations, like

- inserting vertices in edges
- inserting edges in faces
- deleting vertices

repeated many times

some simplify the geometry,
others decrease # vertices

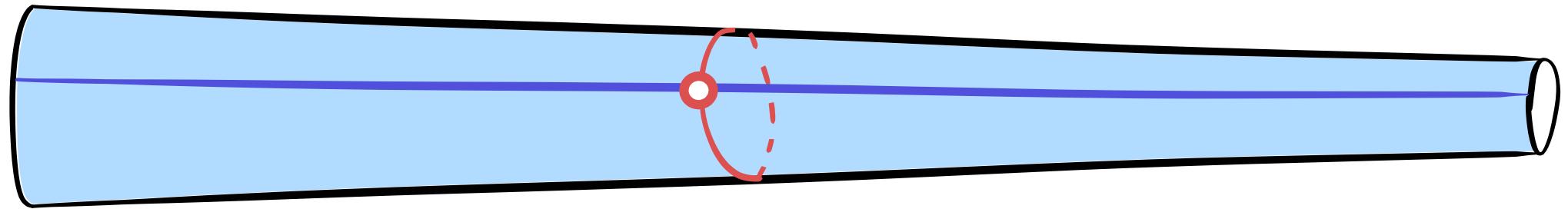
Analysis

Show that during execution:

1. # vertices stays bounded
2. Geometry gets simpler and simpler



Enclosure



Untangling Graphs

Computing Delaunay Triangulations

Other Results and Possible Continuations

Untangling Graphs

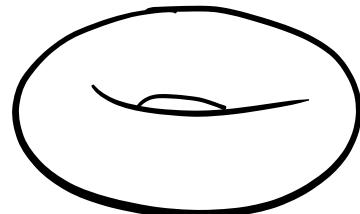
Computing Delaunay Triangulations

Other Results
and Possible Continuations

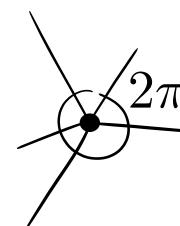
Other results

D., 2022-23

Upper bound for # Delaunay flips on flat tori,
tight up to constant factor



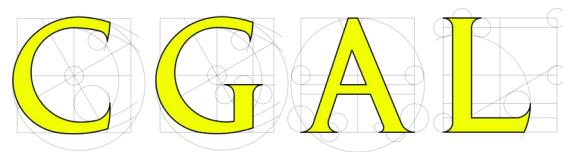
topological shape of a torus



2π around each vertex

Other results

Despré, D., Pouget, and Teillaud, 2025



package for computing with
hyperbolic surfaces

generation (genus 2 only)

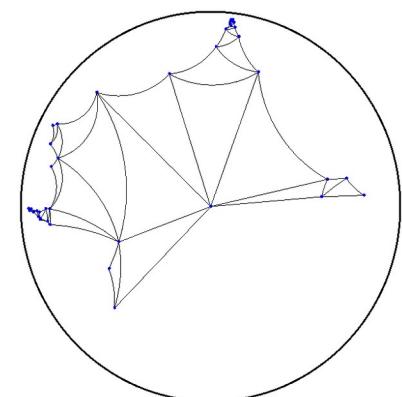
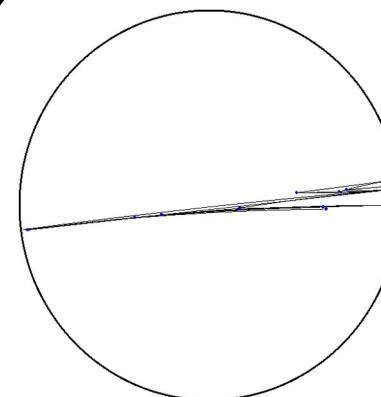


Triangulated hyperbolic
surface

Delaunay flip



visualization



Other results

Despré, D., Pouget, and Teillaud

Exact computations!
package for computing
hyperbolic surfaces

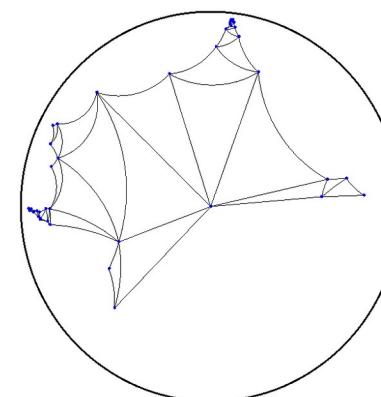
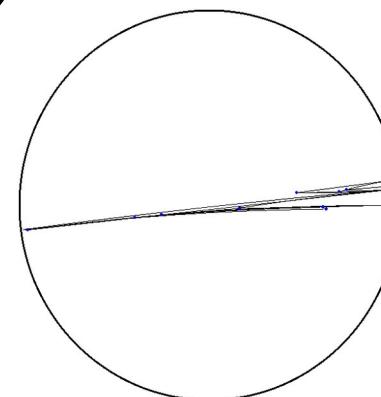
generation (genus 2 only)



Triangulated hyperbolic
surface

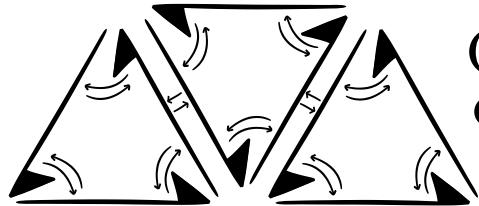
Delaunay flip

↓
visualization



Triangulated hyperbolic
surface

Combinatorics

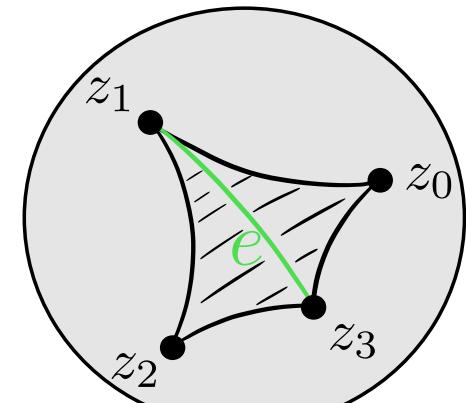


CGAL
combinatorial maps

Triangulated hyperbolic
surface

Geometry

- each edge is decorated with a complex number (cross ratio)



$$e \rightarrow \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

Possible continuations

- generation of hyperbolic surfaces of genus ≥ 3
- what is the complexity of Delaunay flips algo?
- certifying that a drawing cannot be untangled
- untangling by homotopy moves
- extension to non orientable surfaces
- what is the complexity of untangling?
- minimizing crossings of graphs by homotopy
- how unique reducing triangulations are?