Kermit-wave phenomena

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1 Modeling of Wave Phenomena

Wave phenomena play a central role in both physics and engineering, as they provide a fundamental mechanism for energy transfer without net transport of matter. Examples include acoustic propagation in air, stress waves in elastic solids, and surface oscillations in fluids. In the context of simulation, waves are also critical for modeling vibrational effects in robotics, resonances in mechanical systems, and visually plausible deformations in computer graphics.

In discretized systems composed of particles or dynamic units, wave-like behavior can be represented in multiple ways. Traditionally, partial differential equation (PDE) discretizations are employed, where local interactions approximate continuous differential operators [1, 2]. In contrast, this work also introduces a novel analytical formulation in dual quaternion space, designed for direct compatibility with rigid-body and pose-based representations [3, 4].

1.1 Scalar and Discrete Formulation

In a homogeneous isotropic medium, a linear harmonic wave is governed by the classical scalar wave equation [5]:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{1}$$

where $u(\mathbf{X}, t)$ is the displacement field, c the wave velocity, and ∇^2 the Laplacian operator.

In particle-based discretizations, the Laplacian is typically approximated through local neighborhood differences [6, 7]:

$$\nabla^2 u_i \approx \frac{1}{h^2} \sum_{j \in \mathcal{N}(i)} (u_j - u_i), \tag{2}$$

where $\mathcal{N}(i)$ denotes the neighborhood of particle i, and h is the characteristic spacing.

A localized impulse may be expressed as a Gaussian-modulated sinusoid [8]:

$$u_0(\mathbf{X}) = A \exp\left(-\frac{\|\mathbf{X} - \mathbf{X}_0\|^2}{\sigma^2} \right) \sin(k\|\mathbf{X} - \mathbf{X}_0\| + \phi)\,. \tag{3}$$

Explicit numerical integration must satisfy the Courant–Friedrichs–Lewy (CFL) condition [9]:

$$\Delta t < C\frac{h}{c}, \tag{4}$$

with C depending on the discretization scheme.

While this PDE-based approach is physically faithful, it requires careful spatial resolution and becomes computationally expensive for high-frequency or high-dimensional simulations.

1.2 Dual Quaternion Wave Field (Proposed)

To extend wave modeling beyond scalar displacement fields, we define a wave directly in dual quaternion space, enabling consistent perturbations of both translations and rotations in SE(3). Let $\hat{\xi} \in \mathbf{SE}(3)$ denote a unit dual twist (screw axis) specifying the local displacement/rotation direction. The resulting wave field is:

$$Q(\mathbf{X}, t) = Q_0(\mathbf{X}) \otimes \exp\big(\beta(\mathbf{X}, t)\,\hat{\xi}\big), \tag{5}$$

where $Q_0(\mathbf{X})$ is the undeformed pose field and exp the dual quaternion exponential.

The scalar modulation factor $\beta(\mathbf{X}, t)$ encodes the physical wave parameters:

$$\beta(\mathbf{X}, t) = \frac{A}{\rho} e^{-vt} \sin(k(\hat{n} \cdot \mathbf{X}) - \omega t + \phi)\,. \tag{6}$$

Here:

- A is the wave amplitude,
- ρ is the local density,
- v is viscosity/damping,
- $k = 2\pi/\lambda$ is the wave number,
- $\omega = 2\pi f$ is the angular frequency,
- ϕ is the initial phase,
- $\hat{n}$ is the propagation direction,
- $\hat{\xi}$ is the screw axis coupling rotation and translation.

This construction captures the physical behavior of the medium while lifting the oscillation into SE(3) via the exponential map.

1.2.1 Eulerian Incremental Alternative

For world-space propagation, an incremental update is:

$$Q_{t+\Delta t} = Q_t \otimes \exp\big(\Delta t \, \gamma(Q_t, t)\hat{\xi}\big), \tag{7}$$

with γ evaluated at the current pose.

Stability requires a CFL-like bound ensuring that wave propagation speed does not exceed the spatial discretization scale.

1.3 Discussion

This dual quaternion wave field differs fundamentally from Laplacian-based PDE methods. Instead of approximating differential operators through neighbor interactions, the field is expressed analytically and mapped directly onto the Lie group SE(3).

Advantages include:

- consistent translation/rotation coupling via $\hat{\xi}$,
- incorporation of physical material factors in β,
- compatibility with both Lagrangian and Eulerian interpretations,
- lightweight evaluation without PDE solvers,
- geometric coherence for robotics and animation tasks.

1.4 Related Work

Wave phenomena are traditionally modeled using PDEs in Euclidean space, such as the scalar and vector wave equations. Numerical approaches include finite difference (FDTD), finite element (FEM), and particle-based schemes including SPH and MPM. These methods approximate energy transport through local interactions but come with significant computational and stability demands.

Dual quaternions have become a standard tool for representing rigid-body motion in robotics and computer graphics [3]. While Lie-group integrators have been applied to dynamics on SO(3) and SE(3), prior work does not treat wave propagation in dual quaternion space. Procedural oscillation methods exist in graphics, but they operate in $\mathbf{R}^3$ without leveraging screw transformations or embedding physical parameters such as density or damping.

To our knowledge, no previous formulation defines a wave field directly in dual quaternion space. The proposed method offers a geometrically coherent, physically meaningful, and computationally lightweight alternative to PDE-based wave solvers for robotics and animation.

References

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