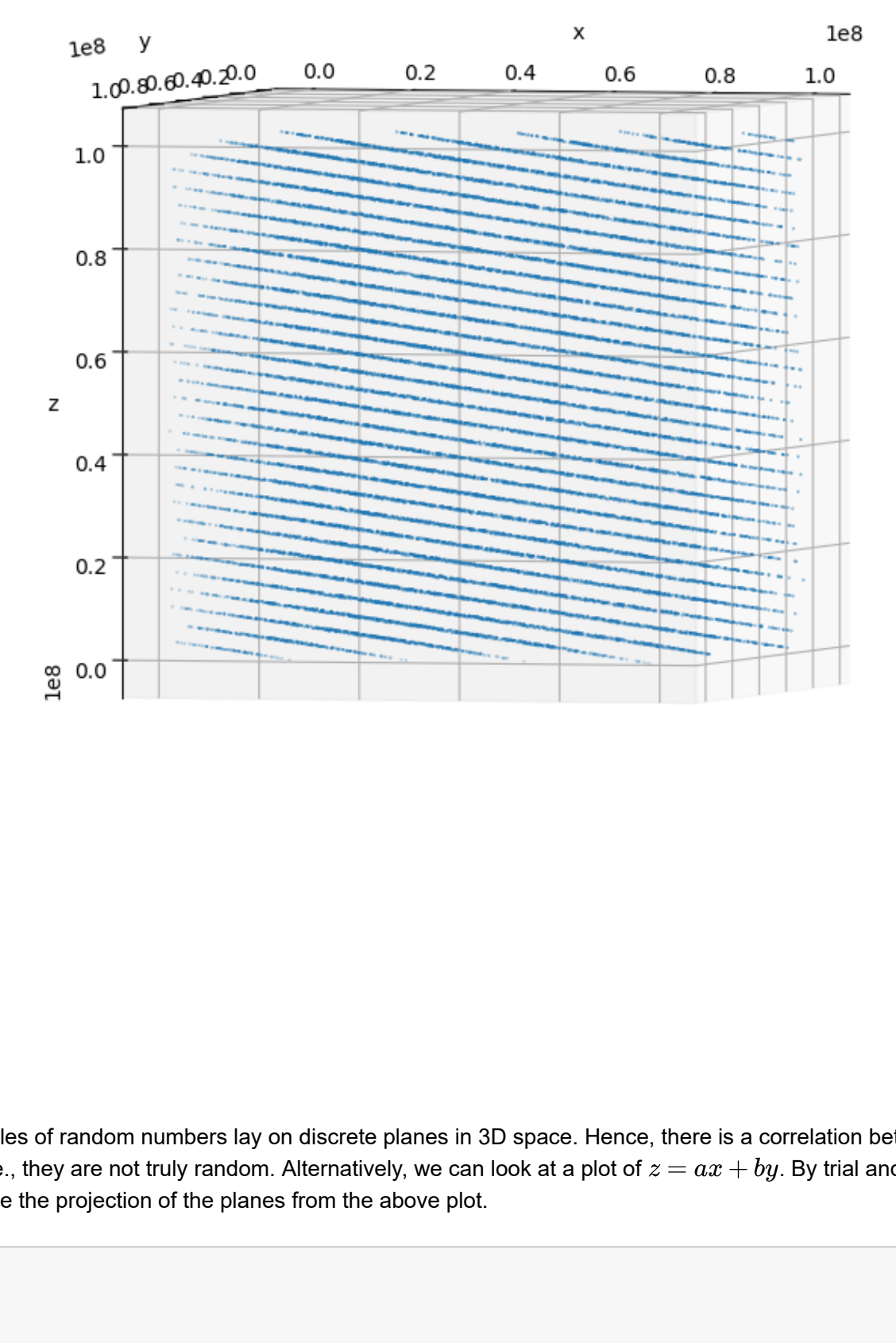
```
In [1]: import numpy as np
import matplotlib.pyplot as plt
%matplotlib notebook
from mpl_toolkits import mplot3d
```

1)

To look at whether or not the points generated by the standard C library are truly randomly distributed, we look at them grouped in triples (x, y, z). We can plot these values where each triple is represented by a point in 3D space. Were these numbers truly random, we should be able to see these points filling up the volume.

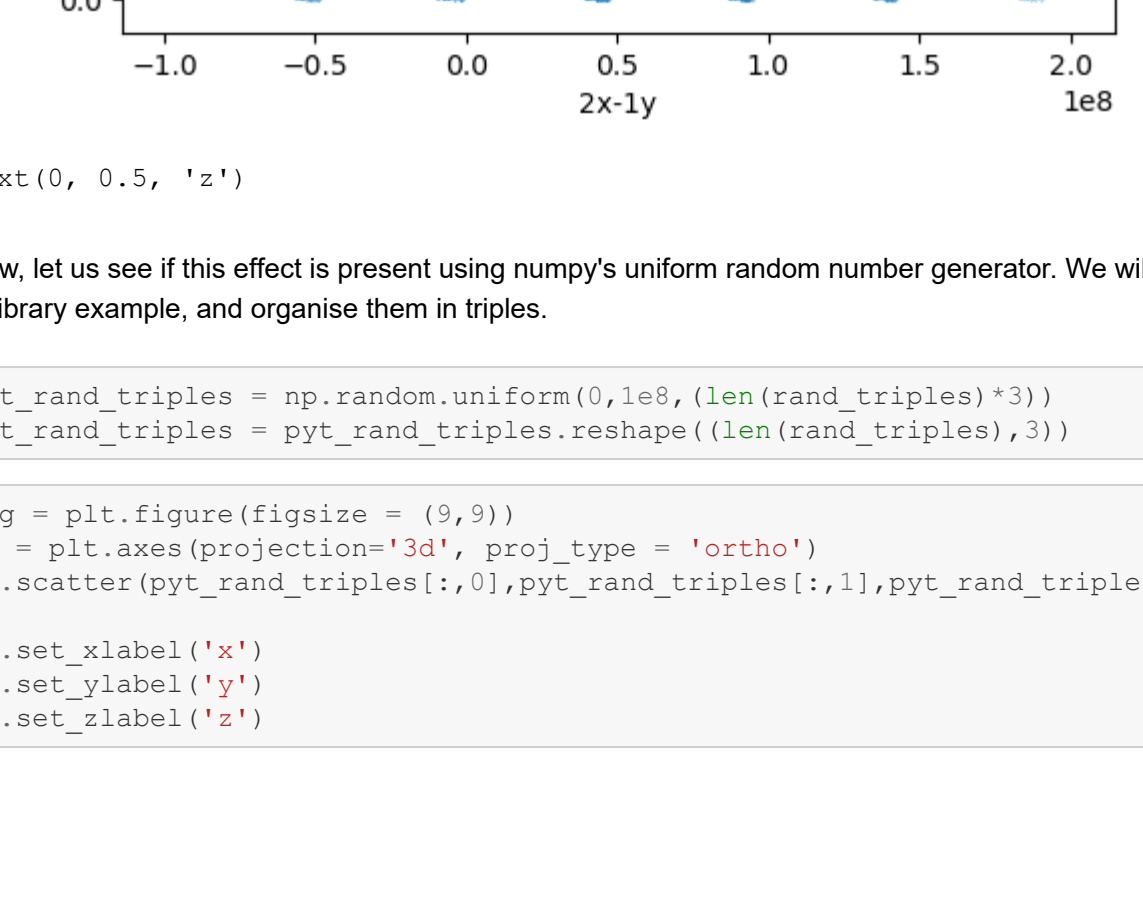
```
In [2]: rand_triples = np.loadtxt('rand_points.txt')
```

```
In [3]: fig = plt.figure(figsize = (9,9))
ax = plt.axes(projection='3d', proj_type = 'ortho')
ax.scatter(rand_triples[:,0],rand_triples[:,1],rand_triples[:,2],s = 0.1)
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')

ax.view_init(elev=-2., azim=-105)
```



As we can see, the triples of random numbers lay on discrete planes in 3D space. Hence, there is a correlation between subsequent generated numbers, i.e., they are not truly random. Alternatively, we can look at a plot of $z = ax + by$. By trial and error, we can get a and b values such that we see the projection of the planes from the above plot.

```
In [4]: a=2
b=-1

plt.figure()
plt.plot(a*rand_triples[:,0]+b*rand_triples[:,1],rand_triples[:,2],'.',markersize = 0.5)
plt.xlabel('{}x-{}y'.format(a,-b))
plt.ylabel('z')
```

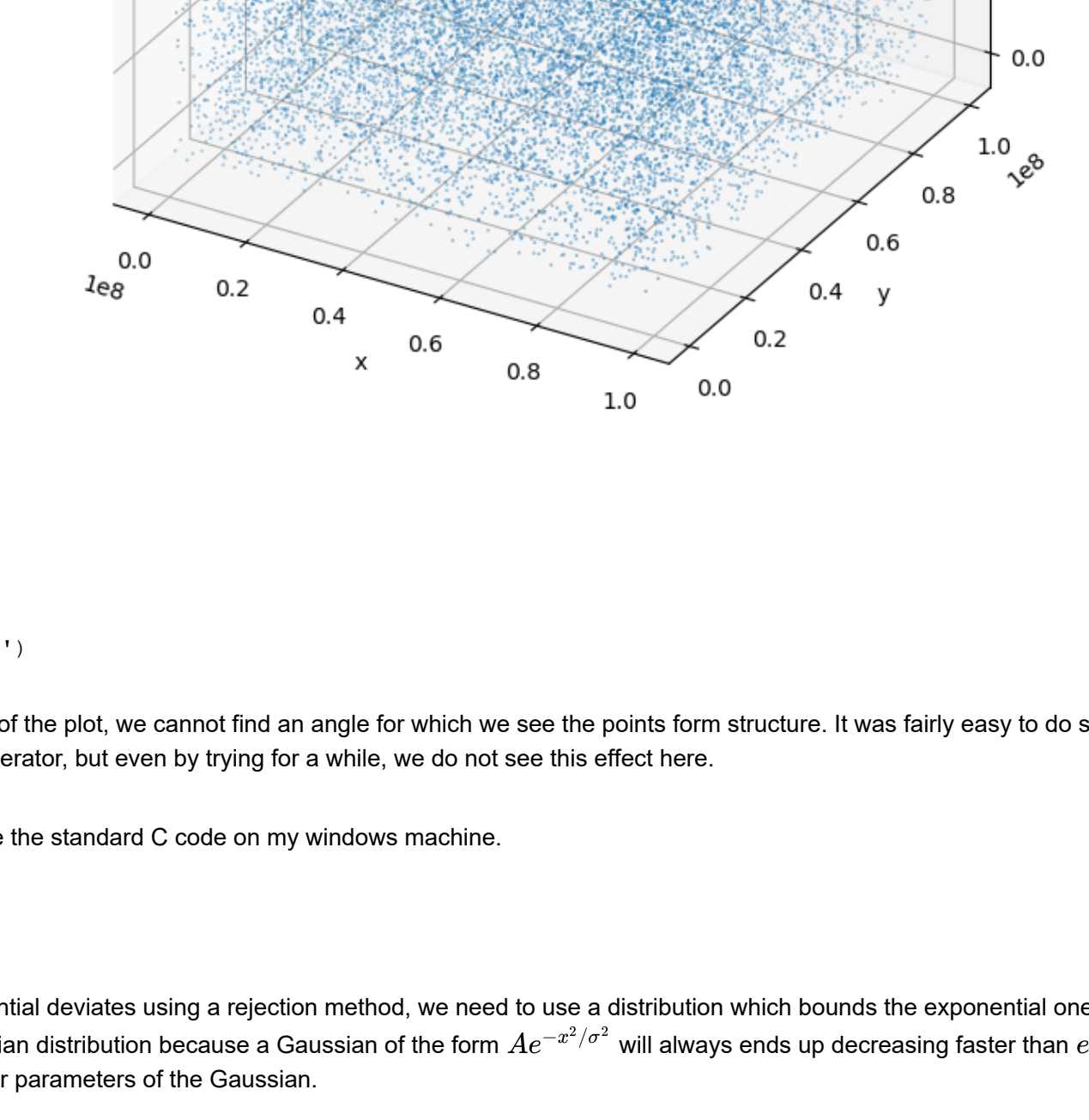



```
Out[4]: Text(0, 0.5, 'z')
```

Now, let us see if this effect is present using numpy's uniform random number generator. We will generate as many random numbers as the C library example, and organise them in triples.

```
In [5]: pyt_rand_triples = np.random.uniform(0,1e8,(len(rand_triples)*3))
pyt_rand_triples = pyt_rand_triples.reshape((len(rand_triples),3))
```

```
In [6]: fig = plt.figure(figsize = (9,9))
ax = plt.axes(projection='3d', proj_type = 'ortho')
ax.scatter(pyt_rand_triples[:,0],pyt_rand_triples[:,1],pyt_rand_triples[:,2],s = 0.1)

ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')
```




```
Out[6]: Text(0.5, 0, 'z')
```

By moving the view of the plot, we cannot find an angle for which we see the points form structure. It was fairly easy to do so for the C random number generator, but even by trying for a while, we do not see this effect here.

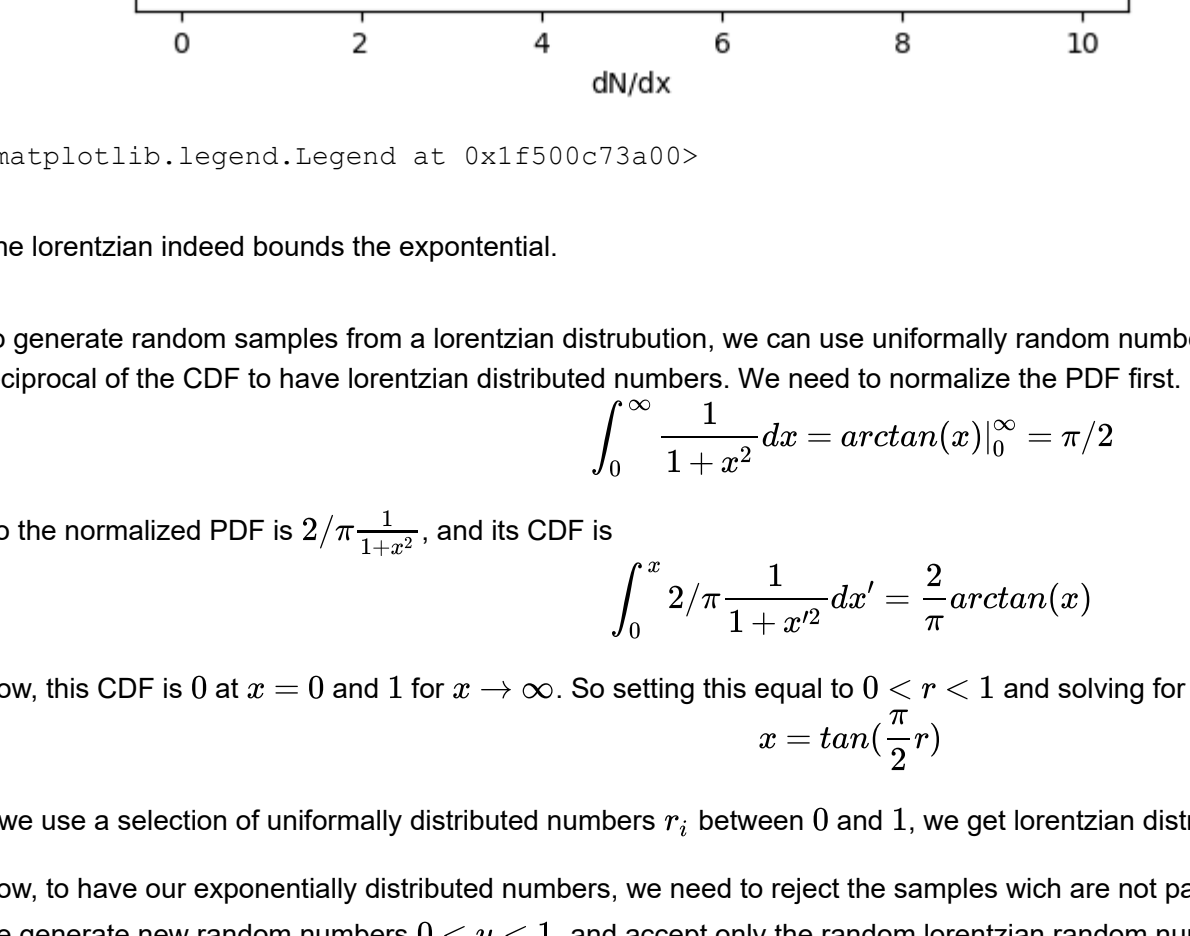
I was not able to use the standard C code on my windows machine.

2)

To generate exponential deviates using a rejection method, we need to use a distribution which bounds the exponential one for all x. We cannot use a Gaussian distribution because a Gaussian of the form Ae^{-x^2/σ^2} will always ends up decreasing faster than e^{-x} for some large x, whatever the other parameters of the Gaussian.

For a lorentzian distribution, this is not a problem, because the exponential will always decay faster. Hence, if we can draw random samples from a lorentzian distribution, we can use a rejection method to only keep samples that belong in the exponential distribution. Here, we will only generate deviates $0 < x < \infty$.

```
In [24]: def lorentzian(x):
    return (1+x**2)**-1

xmax = 10
x = np.linspace(0,xmax,1001)
plt.figure()
plt.plot(x,lorentzian(x),label = 'Lorentzian')
plt.plot(x,np.exp(-x),label = 'Exponential')
plt.xlabel('dN/dx')
plt.ylabel('PDF')
plt.legend()
```




```
Out[24]: <matplotlib.legend.Legend at 0x1f500c73a00>
```

The lorentzian indeed bounds the expontential.

To generate random samples from a lorentzian distrubution, we can use uniformally random numbers $0 < r < 1$ and plug those in the reciprocal of the CDF to have lorentzian distributed numbers. We need to normalize the PDF first.

$$\int_0^\infty \frac{1}{1+x^2}dx = arctan(x)|_0^\infty = \pi/2$$

So the normalized PDF is $2/\pi\frac{1}{1+x^2}$, and its CDF is

$$\int_0^x 2/\pi\frac{1}{1+x'^2}dx' = \frac{2}{\pi}arctan(x)$$

Now, this CDF is 0 at $x = 0$ and 1 for $x \rightarrow \infty$. So setting this equal to $0 < r < 1$ and solving for x yields

$$x = tan(\frac{\pi}{2}r)$$

if we use a selection of uniformally distributed numbers r_i between 0 and 1, we get lorentzian distributed x_i

Now, to have our exponentially distributed numbers, we need to reject the samples wich are not part of the exponential distribution. To do so, we generate new random numbers $0 < u < 1$, and accept only the random lorentzian random numbers which satisfy $u < \frac{e^{-x_i}}{1/(1+x_i^2)}$. To verify that everything works correctly, we expect to accept numbers with a rate that is equal to the ratio of the areas of the distributions, i.e.

$$acceptance\ rate = \frac{\int_0^\infty e^{-x}dx}{\int_0^\infty \frac{1}{1+x^2}dx} = 2/\pi$$

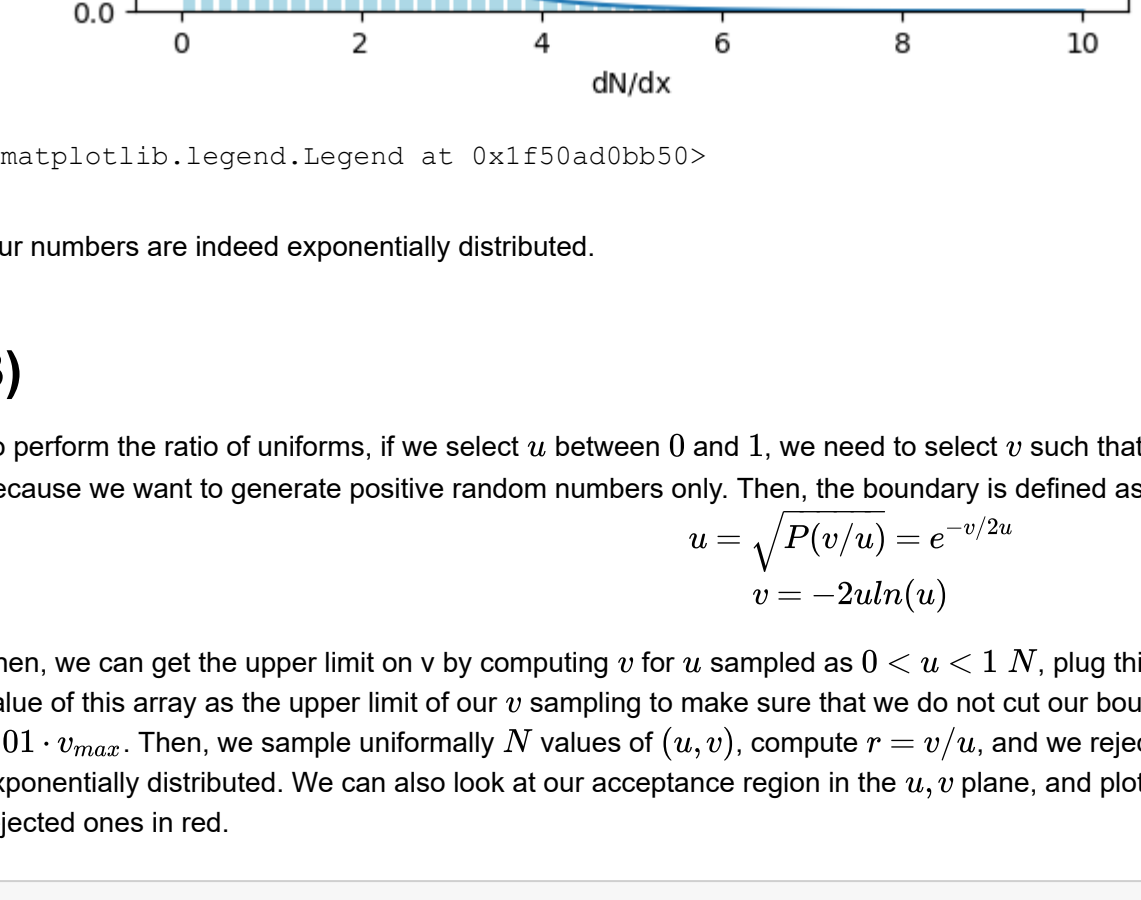
```
In [8]: def rand_exp(N):
    def lor_cdf_inv(r):
        return np.tan(np.pi/2*r)
    rs = np.random.rand(N)
    lor_rand = lor_cdf_inv(rs)
    acc = np.random.rand(N)<np.exp(-lor_rand)/lorentzian(lor_rand)
    print('Accepted ',np.mean(acc)*100,'% of samples')
    return lor_rand[acc]
```

```
In [19]: xmax = 10
a = rand_exp(100000)
print('Theoretical acceptance rate: ',2/np.pi*100,'%')
```

```
Accepted  63.708 % of samples
Theoretical acceptance rate:  63.66197723675814 %
```

Now, we plot our distribution by taking datapoints corresponding to the bins of the histograms:

```
In [216]: plt.figure()
ybins,bins= np.histogram(a,np.linspace(0,10,51),density = True)
binw = bins[1]-bins[0]
bins+=binw/2
plt.bar(bins[:-1],ybins,width = 0.15,label ='Rejection PDF',color = 'lightblue')
ytrue = np.exp(-x)
plt.plot(x,ytrue,label = 'Theoretical distribution')
plt.xlabel('dN/dx')
plt.ylabel('PDF')
plt.legend()
```




```
Out[216]: <matplotlib.legend.Legend at 0x1f50ad0bb50>
```

Our numbers are indeed exponentially distributed.

3)

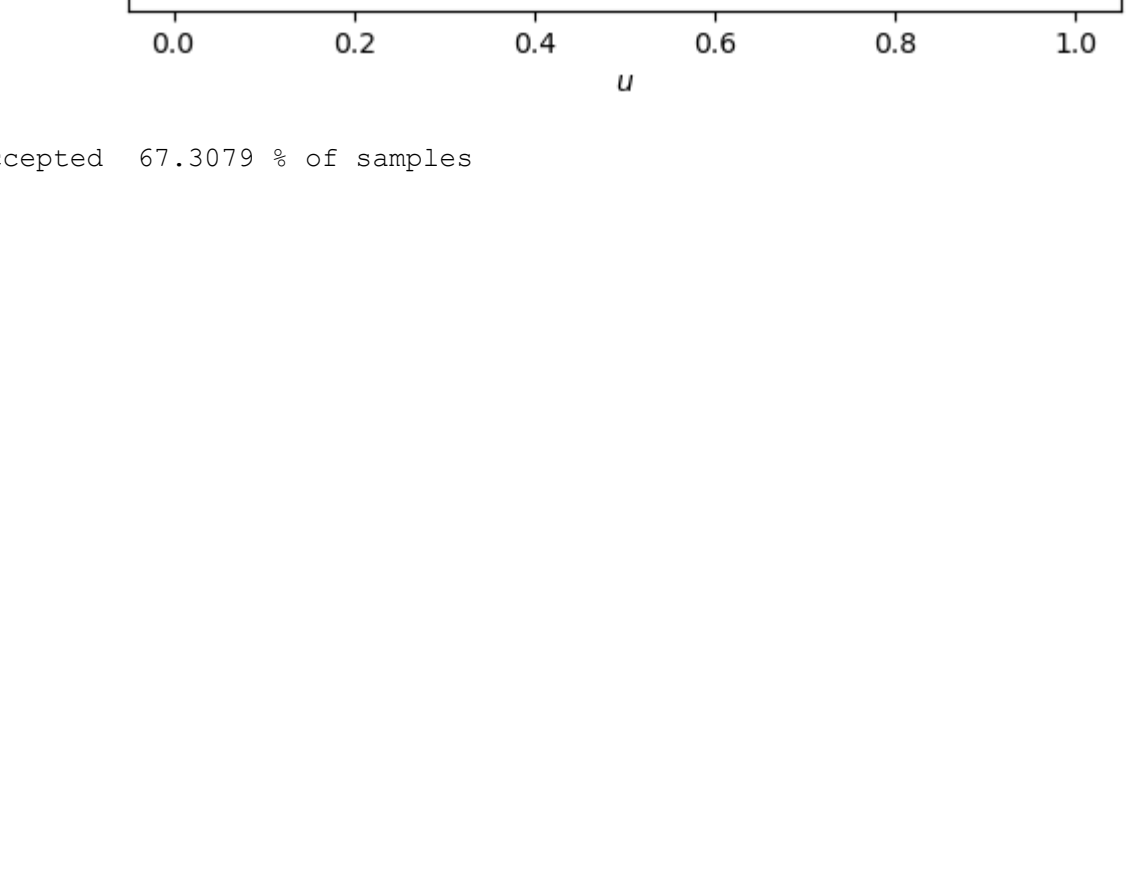
To perform the ratio of uniforms, if we select u between 0 and 1, we need to select v such that $0 < u < \sqrt{P(v/u)}$. We cannot allow $v < 0$ because we want to generate positive random numbers only. Then, the boundary is defined as

$$u = \sqrt{P(v/u)} = e^{-v/2u}$$
$$v = -2uln(u)$$

Then, we can get the upper limit on v by computing v for u sampled as $0 < u < 1$ N, plug this into our expression for v, and take the max value of this array as the upper limit of our v sampling to make sure that we do not cut our boundary by our sampling, we will sample to $1.01 \cdot v_{max}$. Then, we sample uniformally N values of (u, v), compute $r = v/u$, and we reject them if $u < e^{-r}$. Then, our r values are exponentially distributed. We can also look at our acceptance region in the u, v plane, and plot the accepted (u, v) doubles in green and rejected ones in red.

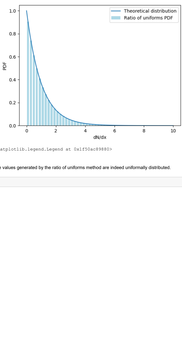
```
In [198]: def ROU_exp(N,plot = False):
    u = np.linspace(0,1,1001)
    u = u[1:]
    v = 2*u*-np.log(u)
    if plot:
        plt.figure()
        plt.plot(u,v,'k')
        plt.plot(u,[0]*len(u),'k')
    u = np.random.rand(N)
    v = np.random.uniform(0,v.max()*1.01,N)
    r = v/u
    acc = u**2<np.exp(-r)
    rej = ~acc
    if plot:
        plt.plot(u[acc],v[acc],'.',color = 'green',markersize = 0.01)
        plt.plot(u[rej],v[rej],'.',color = 'red',markersize = 0.01)
        plt.xlabel('$u$')
        plt.ylabel('$v$')
    print('Accepted ',np.mean(acc)*100,'% of samples')

    return r[acc]
```

```
In [199]: rand_ROU = ROU_exp(1000000,plot = True)
```




```
Accepted  67.3079 % of samples
```

In [213]:
```
plt.figure()
yhine,bine= np.histogram(rand_RCG,np.linspace(0,10,51),density = True)
bcne = bine[:-1]+bine[1]
bine=+bine/2
plt.bar(bine[:-1],yhine,width = 0.15,label ="Ratio of uniforms PDF",color = 'lightblue')
ytrue = 0.5*np.exp(-0.5*x)
plt.plot(x,ytrue,label = "Theoretical distribution")
plt.xlabel('dRUSs')
plt.ylabel('PDF')
plt.legend()
```



Out[213]: <matplotlib.legend.Legend at 0x1f50ac89880>

The values generated by the ratio of uniforms method are indeed uniformally distributed.

In [ ]: