forall x

THE MISSISSIPPI STATE EDITION

an introduction to formal logic

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Preface

I will begin by quoting E. J. Lemmon,

> It is not easy, and perhaps not even useful, to explain briefly what logic is. Like most subjects, it comprises many different kinds of problem and has no exact boundaries; at one end, it shades off into mathematics, at another, into philosophy. The best way to find out what logic is is to do some. (1965, p. 1)

Nonetheless, there are a couple of things that will be useful to know before you begin. First, formal logic is the study of a formal language. Unlike a natural language (such as English, Spanish, Mandarin, and so forth), in formal languages every part of the language—in particular, the content of the language and the rules for using the language—is precisely defined. Second, the study of formal logic focuses on certain relationships between sentences, namely, consistency and entailment.

The book is divided into four parts. Part I introduces the topic and basic concepts of logic in an informal way, without introducing a formal language. Parts II – IV cover the formal language *truth-functional logic* (TFL). (For reference, TFL also goes by other names: *propositional logic* or *sentential logic*.) In Part II, we begin with basic sentences. Basic sentences form more complex sentences with the connectives 'or', 'and', 'not', 'if ... then ...', and '... if and only if ...'. Once the connectives have been introduced, we investigate entailment in two ways: semantically, using the method of truth tables (in Part III) and proof-theoretically, using a system of formal derivations (in Part IV).

In the appendices you'll find a discussion of alternative notations for the language we discuss in this text and a "quick reference" listing most of the important rules and definitions. The central terms are listed in a glossary at the very end.

This book is based on a text originally written by P. D. Magnus and revised and expanded by Tim Button, J. Robert Loftis, Aaron Thomas-Bolduc, and Richard Zach. I have made additional revisions, taken out chapters that I do not need for teaching the Intro Logic course at Mississippi State University, and added instructions for using the logic software Carnap (http://carnap.io/). The resulting text is licensed under a Creative Commons Attribution-ShareAlike 4.0 license.

Incidentally, the title *forallx* (i.e., "for all x") is a reference to what is called *first-order logic*—although this version of the textbook does not, at least not right now, cover first-order logic. In any event, this is a symbolic expression in first-order logic: $\forall x(Kx \rightarrow Gx)$, and it is read, "for all x, if x is K, then x is G." Hence, the name of the textbook. (If, for instance, K stands for "is a king," and G stands for "is greedy," then $\forall x(Kx \rightarrow Gx)$ means "for all x, if x is a king, then x is greedy," or "everyone who is a king is greedy.")

Part I

Key notions of logic

Chapter 1

Arguments

Logic is the business of evaluating arguments; sorting the good from the bad. In everyday discourse, the word 'argument' generally refers to belligerent shouting matches. Logic, however, is not concerned with arguments in that sense of the word. An argument, as we will understand it, is something more like this:

It is raining heavily.
If you do not take an umbrella, you will get soaked.
∴ You should take an umbrella.

We here have a series of sentences. **The three dots on the third line of the argument are read 'therefore.'** They indicate that the final sentence expresses the *conclusion* of the argument. The two sentences before that are the *premises* of the argument. If you believe the premises, then the argument (perhaps) provides you with a reason to believe the conclusion.

This is the sort of thing that logicians are interested in. We will say that an argument is any collection of premises, together with a conclusion.

This Part discusses some basic logical notions that apply to arguments in a natural language like English. It is important to begin with a clear understanding of what arguments are and of what it means for an argument to be valid. Later we will translate arguments from English into a formal language. We want formal validity, as defined in the formal language, to have at least some of the important features of natural-language validity.

In the example just given, we used individual sentences to express both of the argument's premises, and we used a third sentence to express the argument's conclusion. Many arguments are expressed in this way, but a single sentence can contain a complete argument. Consider:

> I was wearing my sunglasses; so it must have been sunny.

This argument has one premise followed by a conclusion.

Many arguments start with premises, and end with a conclusion, but not all of them. The argument with which this section began might equally have been presented with the conclusion at the beginning, like so:

> You should take an umbrella. After all, it is raining heavily. And if you do not take an umbrella, you will get soaked.

Equally, it might have been presented with the conclusion in the middle:

> It is raining heavily. Accordingly, you should take an umbrella, given that if you do not take an umbrella, you will get soaked.

When approaching an argument, we want to know whether or not the conclusion follows from the premises. So the first thing to do is to separate out the conclusion from the premises. As a guide, these words are often used to indicate an argument's conclusion:

so, therefore, hence, thus, accordingly, consequently

By contrast, these expressions often indicate that we are dealing with a premise, rather than a conclusion:

since, because, given that

But in analysing an argument, there is no substitute for a good nose.

1.1 Sentences

To be perfectly general, we can define an ARGUMENT as a series of sentences. The sentences at the beginning of the series are premises. The final sentence in the series is the conclusion. If the premises are true and the argument is a good one, then you have a reason to accept the conclusion.

In logic, we are only interested in sentences that can figure as a premise or conclusion of an argument. So we will say that a SENTENCE is something that can be true or false.

You should not confuse the idea of a sentence that can be true or false with the difference between fact and opinion. Often, sentences in logic will express things that would count as facts— such as 'Kierkegaard was a hunchback' or 'Kierkegaard liked almonds.' They can also express things that you might think of as matters of opinion— such as, 'Almonds are tasty.'

Also, there are things that would count as 'sentences' in a linguistics or grammar course that we will not count as sentences in logic.

Questions In a grammar class, 'Are you sleepy yet?' would count as an interrogative sentence. Although you might be sleepy or you might be alert, the question itself is neither true nor false. For this reason, questions will not count as sentences in logic. Suppose you answer the question: 'I am not sleepy.' This is either true or false, and so it is a sentence in the logical sense. Generally, *questions* will not count as sentences, but *answers* will.

'What is this course about?' is not a sentence (in our sense). 'No one knows what this course is about' is a sentence.

Imperatives Commands are often phrased as imperatives like 'Wake up!', 'Sit up straight', and so on. In a grammar class, these would count as imperative sentences. Although it might be good for you to sit up straight or it might not, the command is neither true nor false. Note, however, that commands are not always phrased as imperatives. 'You will respect my authority' *is* either true or false— either you will or you will not— and so it counts as a sentence in the logical sense.

Exclamations 'Ouch!' is sometimes called an exclamatory sentence, but it is neither true nor false. We will treat 'Ouch, I hurt my toe!' as meaning the same thing as 'I hurt my toe.' The 'ouch' does not add anything that could be true or false.

Practice exercises

At the end of some chapters, there are exercises that review and explore the material covered in the chapter. There is no substitute for actually working through some problems, because learning logic is more about developing a way of thinking than it is about memorizing facts.

So here's the first exercise. Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

Chapter 2

Valid arguments

In §1, we gave a very permissive account of what an argument is. To see just how permissive it is, consider the following:

There is a bassoon-playing dragon in the *Cathedra Romana*.
∴ Salvador Dali was a poker player.

We have been given a premise and a conclusion. So we have an argument. Admittedly, it is a *terrible* argument, but it is still an argument.

2.1 Two ways that arguments can go wrong

It is worth pausing to ask what makes the argument so weak. In fact, there are two sources of weakness. First: the argument's (only) premise is obviously false. The Pope's throne is only ever occupied by a hat-wearing man. Second: the conclusion does not follow from the premise of the argument. Even if there were a bassoon-playing dragon in the Pope's throne, we would not be able to draw any conclusion about Dali's predilection for poker.

What about the main argument discussed in §1? The premises of this argument might well be false. It might be sunny outside; or it might be that you can avoid getting soaked without taking an umbrella. But even if both premises were true, it does not necessarily show you that you should take an umbrella. Perhaps you enjoy walking in the rain, and you would like to get soaked. So, even if both premises were true, the conclusion might nonetheless be false.

The general point is that, for any argument, there are two ways that it might go wrong:

- One or more of the premises might be false.
- The conclusion might not follow from the premises.

It is often important to determine whether or not the premises of an argument are true. However, that is normally a task best left to experts in the field: as it might be historians, scientists, or whomever. In our role as *logicians*, we are more concerned with arguments *in general*. So we are (usually) more concerned with the second way in which arguments can go wrong.

So: we are interested in whether or not a conclusion *follows from* some premises. Don't, though, say that the premises *infer* the conclusion. Entailment is a relation between premises and conclusions; inference is something we do. So if you want to mention inference when the conclusion follows from the premises, you could say that *one may infer* the conclusion from the premises.

2.2 Validity

As logicians, we want to be able to determine when the conclusion of an argument follows from the premises. One way to put this is as follows. We want to know whether, if all the premises were true, the conclusion would also have to be true. This motivates a definition:

> An argument is VALID if and only if it is impossible for all of the premises to be true and the conclusion false.

The crucial thing about a valid argument is that it is impossible for the premises to be true while the conclusion is false. Consider this example:

Oranges are either fruits or musical instruments.
Oranges are not fruits.
∴ Oranges are musical instruments.

The conclusion of this argument is ridiculous. Nevertheless, it follows from the premises. *If* both premises are true, *then* the conclusion just has to be true. So the argument is valid.

This highlights that valid arguments do not need to have true premises or true conclusions. Conversely, having true premises and a true conclusion is not enough to make an argument valid. Consider this example:

London is in England.
Beijing is in China.
∴ Paris is in France.

The premises and conclusion of this argument are, as a matter of fact, all true, but the argument is invalid. If Paris were to declare independence from the rest of France, then the conclusion would be false, even though both of the premises would remain true. Thus, it is *possible* for the premises of this argument to be true and the conclusion false. So the argument is invalid.

The important thing to remember is that validity is not about the actual truth or falsity of the sentences in the argument. It is about whether it is *possible* for all the premises to be true and the conclusion false. Nonetheless, we will say that an argument is SOUND if and only if it is both valid and all of its premises are true.

2.3 Inductive arguments

Many good arguments are invalid. Consider this one:

In January 1997, it rained in London.
In January 1998, it rained in London.
In January 1999, it rained in London.
In January 2000, it rained in London.
∴ It rains every January in London.

This argument generalises from observations about several cases to a conclusion about all cases. Such arguments are called INDUCTIVE arguments. The argument could be made stronger by adding additional premises before drawing the conclusion: In January 2001, it rained in London; In January 2002.... But, however many premises of this form we add, the argument will remain invalid. Even if it has rained in London in every January thus far, it remains *possible* that London will stay dry next January.

The point of all this is that inductive arguments—even good inductive arguments—are not (deductively) valid. They are not *watertight*. Unlikely though it might be, it is *possible* for their conclusion to be false, even when all of their premises are true. In this book, we will set aside (entirely) the question of what makes for a good inductive argument. Our interest is simply in sorting the (deductively) valid arguments from the invalid ones.

Practice exercises

A. Which of the following arguments are valid? Which are invalid?

1. Socrates is a man.
2. All men are carrots.

∴ Socrates is a carrot.

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.

∴ Abe Lincoln was born in Illinois.

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.

∴ Abe Lincoln will not die.

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.

∴ Abe Lincoln was from France.

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

∴ The world will not end today.

1. Joe is now 19 years old.
2. Joe is now 87 years old.

∴ Bob is now 20 years old.

B. Could there be:

1. A valid argument that has one false premise and one true premise?
2. A valid argument that has only false premises?
3. A valid argument with only false premises and a false conclusion?

4. An invalid argument that can be made valid by the addition of a new premise?
5. A valid argument that can be made invalid by the addition of a new premise?

In each case: if so, give an example; if not, explain why not.

Chapter 3

Other logical notions

The concept of a valid argument is central to logic. In this section, we will introduce some other important concepts that apply just to sentences, not to full arguments.

3.1 Joint possibility

Consider these two sentences:

B1. Jane's only brother is shorter than her.
B2. Jane's only brother is taller than her.

Logic alone cannot tell us which, if either, of these sentences is true. Yet we can say that *if* the first sentence (B1) is true, *then* the second sentence (B2) must be false. Similarly, if B2 is true, then B1 must be false. It is impossible that both sentences are true together. These sentences are inconsistent with each other; they cannot all be true at the same time. On the other hand, G1 and G2 can both be true at the same time.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

One of these sentences may be false and the other true, but it is *possible* that they are both true at the same time. This motivates the following definition:

> Sentences are JOINTLY POSSIBLE if and only if it is possible for them all to be true together.

So, G1 and G2 are *jointly possible* while B1 and B2 are *jointly impossible.*

We can investigate the joint possibility of any number of sentences. For example, let's add two more sentences to G1 and G2:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two extra-terrestrials at the wild animal park.
G4. Every giraffe at the wild animal park is an extra-terrestrial.

Together, G1 and G4 entail that there are at least four extra-terrestrials giraffes at the park. This conflicts with G3, which implies that there are no more than two extra-terrestrial giraffes there. So the sentences G1–G4 are jointly impossible. They cannot all be true together. (Note that the sentences G1, G3 and G4 are jointly impossible. But if sentences are already jointly impossible, adding an extra sentence to the mix will not make them jointly possible!)

3.2 Necessary truths, necessary falsehoods, and contingency

In assessing arguments for validity, we care about what would be true *if* the premises were true, but some sentences just *must* be true. Consider these sentences:

1. It is raining.
2. Either it is raining here, or it is not.
3. It is both raining here and not raining here.

In order to know if sentence 1 is true, you would need to look outside or check the weather channel. It might be true; it might be false. A sentence which is capable of being true and capable of being false (in different circumstances, of course) is called CONTINGENT.

Sentence 2 is different. You do not need to look outside to know that it is true. Regardless of what the weather is like, it is either raining or it is not. That is a NECESSARY TRUTH.

Equally, you do not need to check the weather to determine whether or not sentence 3 is true. It must be false, simply as a matter of logic. It might be raining here and not raining across town; it might be raining now but stop raining even as you finish this sentence; but it is impossible for it to be both raining and not raining in the same place and at the same time. So, whatever the world is like, it is not both raining here and not raining here. It is a NECESSARY FALSEHOOD.

Necessary equivalence

We can also ask about the logical relations *between* two sentences. For example:

> John went to the store after he washed the dishes.
> John washed the dishes before he went to the store.

These two sentences are both contingent, since John might not have gone to the store or washed dishes at all. Yet they must have the same truth-value. If either of the sentences is true, then they both are; if either of the sentences is false, then they both are. When two sentences necessarily have the same truth value, we say that they are NECESSARILY EQUIVALENT.

Summary of logical notions

- An argument is (deductively) VALID if it is impossible for the premises to be true and the conclusion false. It is INVALID otherwise.
- A NECESSARY TRUTH is a sentence that must be true; it could not possibly be false.
- A NECESSARY FALSEHOOD is a sentence that must be false; it could not possibly be true.
- A CONTINGENT SENTENCE is neither a necessary truth nor a necessary falsehood. It may be true but could have been false, or vice versa.
- Two sentences are NECESSARILY EQUIVALENT if they must have the same truth value.
- A collection of sentences is JOINTLY POSSIBLE if it is possible for all these sentences to be true together; it is JOINTLY IMPOSSIBLE otherwise.

Practice exercises

A. For each of the following: Is it a necessary truth, a necessary falsehood, or contingent?

1. Caesar crossed the Rubicon.
2. Someone once crossed the Rubicon.
3. No one has ever crossed the Rubicon.
4. If Caesar crossed the Rubicon, then someone has.
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
6. If anyone has ever crossed the Rubicon, it was Caesar.

B. For each of the following: Is it a necessary truth, a necessary falsehood, or contingent?

1. Elephants dissolve in water.
2. Wood is a light, durable substance useful for building things.
3. If wood were a good building material, it would be useful for building things.
4. I live in a three story building that is two stories tall.
5. If gerbils were mammals they would nurse their young.

C. Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.
   If you put an elephant in water, it will disintegrate.
2. All mammals dissolve in water.
   If you put an elephant in water, it will disintegrate.
3. George Bush was the 43rd president.
   Barack Obama is the 44th president.
4. Barack Obama is the 44th president.
   Barack Obama was president immediately after the 43rd president.
5. Elephants dissolve in water.
   All mammals dissolve in water.

D. Which of the following pairs of sentences are necessarily equivalent?

1. Thelonious Monk played piano.
   John Coltrane played tenor sax.

2. Thelonious Monk played gigs with John Coltrane.
   John Coltrane played gigs with Thelonious Monk.
3. All professional piano players have big hands.
   Piano player Bud Powell had big hands.
4. Bud Powell suffered from severe mental illness.
   All piano players suffer from severe mental illness.
5. John Coltrane was deeply religious.
   John Coltrane viewed music as an expression of spirituality.

E. Consider the following sentences:

G1 There are at least four giraffes at the wild animal park.

G2 There are exactly seven gorillas at the wild animal park.

G3 There are not more than two Martians at the wild animal park.

G4 Every giraffe at the wild animal park is a Martian.

Now consider each of the following collections of sentences. Which are jointly possible? Which are jointly impossible?

1. Sentences G2, G3, and G4
2. Sentences G1, G3, and G4
3. Sentences G1, G2, and G4
4. Sentences G1, G2, and G3

F. Consider the following sentences.

M1 All people are mortal.

M2 Socrates is a person.

M3 Socrates will never die.

M4 Socrates is mortal.

Which combinations of sentences are jointly possible? Mark each "possible" or "impossible."

1. Sentences M1, M2, and M3
2. Sentences M2, M3, and M4

3. Sentences M2 and M3
4. Sentences M1 and M4
5. Sentences M1, M2, M3, and M4

G. Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument that has one false premise and one true premise
2. A valid argument that has a false conclusion
3. A valid argument, the conclusion of which is a necessary falsehood
4. An invalid argument, the conclusion of which is a necessary truth
5. A necessary truth that is contingent
6. Two necessarily equivalent sentences, both of which are necessary truths
7. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
8. Two necessarily equivalent sentences that together are jointly impossible
9. A jointly possible collection of sentences that contains a necessary falsehood
10. A jointly impossible set of sentences that contains a necessary truth

H. Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument, whose premises are all necessary truths, and whose conclusion is contingent
2. A valid argument with true premises and a false conclusion
3. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
4. A jointly possible collection of sentences, all of which are contingent
5. A false necessary truth
6. A valid argument with false premises
7. A necessarily equivalent pair of sentences that are not jointly possible
8. A necessary truth that is also a necessary falsehood
9. A jointly possible collection of sentences that are all necessary falsehoods

Part II

Truth-functional logic

Chapter 4

First steps to symbolization

4.1 Validity in virtue of form

Consider this argument:

It is raining outside.
If it is raining outside, then Jenny is miserable.
∴ Jenny is miserable.

and another argument:

Jenny is an anarcho-syndicalist.
If Jenny is an anarcho-syndicalist, then John is a crypto-anarchist.
∴ John is a crypto-anarchist.

Both arguments are valid, and there is a straightforward sense in which we can say that they share a common structure. We might express the structure thus:

A
If A, then C
∴ C

This looks like an excellent argument *structure*. Indeed, surely any argument with this *structure* will be valid, and this is not the only good argument structure. Consider an argument like:

Jenny is either happy or sad.

Jenny is not happy.
∴ Jenny is sad.

Again, this is a valid argument. The structure here is something like:

A or B
not-A
∴ B

A superb structure! Here is another example:

It's not the case that Jim both studied hard and acted in lots of plays.
Jim studied hard
∴ Jim did not act in lots of plays.

This valid argument has a structure which we might represent thus:

not-(A and B)
A
∴ not-B

These examples illustrate an important idea, which we might describe as *validity in virtue of form*. The validity of the arguments just considered has nothing very much to do with the meanings of English expressions like 'Jenny is miserable', 'John is a crypto-anarchist', or 'Jim acted in lots of plays'. If it has to do with meanings at all, it is with the meanings of phrases like 'and', 'or', 'not,' and 'if..., then...'.

In Parts II–IV, we are going to develop a formal language which allows us to symbolize many arguments in such a way as to show that they are valid in virtue of their form. **That language will be *truth-functional logic*, or TFL.**

4.2 Validity for special reasons

There are plenty of arguments that are valid, but not for reasons relating to their form. Take an example:

Juanita is a vixen
∴ Juanita is a fox

It is impossible for the premise to be true and the conclusion false. So the argument is valid. However, the validity is not related to the form of the argument. Here is an invalid argument with the same form:

Juanita is a vixen
∴ Juanita is a cathedral

This might suggest that the validity of the first argument *is* keyed to the meaning of the words 'vixen' and 'fox'. But, whether or not that is right, it is not simply the *shape* of the argument that makes it valid. Equally, consider the argument:

The sculpture is green all over.
∴ The sculpture is not red all over.

Again, it seems impossible for the premise to be true and the conclusion false, for nothing can be both green all over and red all over. So the argument is valid, but here is an invalid argument with the same form:

The sculpture is green all over.
∴ The sculpture is not shiny all over.

The argument is invalid, since it is possible to be green all over and shiny all over. (One might paint their nails with an elegant shiny green varnish.) Plausibly, the validity of the first argument is keyed to the way that colours (or colour-words) interact, but, whether or not that is right, it is not simply the *shape* of the argument that makes it valid.

The important moral can be stated as follows. *At best, TFL will help us to understand arguments that are valid due to their form.*

4.3 Atomic sentences and symbolization

We started isolating the form of an argument, in §4.1 by replacing sentences and *subsentences* of sentences with individual letters. Thus, in the first example of this chapter, 'it is raining outside' is a subsentence of 'If it is raining outside, then Jenny is miserable', and we replaced that subsentence with 'A'.

Our artificial language, TFL, is built upon this idea. We start with some *atomic sentences*. These are the basic building blocks used to form more complex sentences. We will use uppercase Roman letters for atomic sentences of TFL.

There are only twenty-six letters of the alphabet, but there is no limit to the number of atomic sentences that we might want to consider. By adding subscripts to letters, we obtain new atomic sentences. Here, for instance, are five different atomic sentences of TFL:

$$A, P, P_1, P_2, A_{234}$$

We will use atomic sentences to represent, or *symbolize*, certain English sentences. To do this, we provide a SYMBOLIZATION KEY, such as the following:

A: It is raining outside
C: Jenny is miserable

In doing this, we are not fixing this symbolization *once and for all*. We are just saying that, for the time being, we will think of the atomic sentence of TFL, 'A', as symbolizing the English sentence 'It is raining outside', and the atomic sentence of TFL, 'C', as symbolizing the English sentence 'Jenny is miserable'. Later, when we are dealing with different sentences or different arguments, we can provide a new symbolization key; as it might be:

A: Jenny is an anarcho-syndicalist
C: John is a crypto-anarchist

It is important to understand that whatever structure an English sentence might have is lost when it is symbolized by an atomic sentence of TFL. From the point of view of TFL, an atomic sentence is just a letter. It can be used to build more complex sentences, but it cannot be taken apart.

Chapter 5

Connectives

In the previous chapter, we considered symbolizing fairly basic English sentences with atomic sentences of TFL. This leaves us wanting to deal with the English expressions 'and', 'or', 'not', and so forth. These are *connectives*—they can be used to form new sentences out of old ones. In TFL, we will make use of logical connectives to build complex sentences from atomic components. There are five logical connectives in TFL. This table summarises them, and they are explained throughout this section.

symbol	**what it is called**	**rough meaning**
$\neg$	negation	'It is not the case that...'
&	conjunction	'Both... and ...'
$\vee$	disjunction	'Either... or ...'
$\rightarrow$	conditional	'If ... then ...'
$\leftrightarrow$	biconditional	'... if and only if ...'

These are not the only connectives in English. Others are, for example, 'unless', 'neither ... nor ...', and 'because'. We will see that the first two can be expressed by the connectives we will discuss, while the last cannot. 'Because', in contrast to the others, is not *truth functional.*

5.1 Negation

Consider how we might symbolize these sentences:

1. Mary is in Barcelona.
2. It is not the case that Mary is in Barcelona.
3. Mary is not in Barcelona.

In order to symbolize sentence 1, we will need an atomic sentence. We might offer this symbolization key:

B: Mary is in Barcelona.

Since sentence 2 is obviously related to sentence 1, we will not want to symbolize it with a completely different sentence. Roughly, sentence 2 means something like 'It is not the case that B'. In order to symbolize this, we need a symbol for negation. We will use '$\neg$'. (Or, if you are typing, use '~'.) Now we can symbolize sentence 2 with '$\neg B$'.

Sentence 3 also contains the word 'not', and it is obviously equivalent to sentence 2. As such, we can also symbolize it with '$\neg B$'.

> A sentence can be symbolized as $\neg\mathsf{A}$ if it can be paraphrased in English as 'It is not the case that…'.

It will help to offer a few more examples:

4. The widget can be replaced.
5. The widget is irreplaceable.
6. The widget is not irreplaceable.

Let us use the following representation key:

R: The widget is replaceable

Sentence 4 can now be symbolized by 'R'. Moving on to sentence 5: saying the widget is irreplaceable means that it is not the case that the widget is replaceable. So even though sentence 5 does not contain the word 'not', we will symbolize it as follows: '$\neg R$'.

Sentence 6 can be paraphrased as 'It is not the case that the widget is irreplaceable.' Which can again be paraphrased as 'It is not the case that it is not the case that the widget is replaceable'. So we might symbolize this English sentence with the TFL sentence '$\neg\neg R$'.

But some care is needed when handling negations. Consider:

7. Jane is happy.
8. Jane is unhappy.

If we let the TFL-sentence 'H' symbolize 'Jane is happy', then we can symbolize sentence 7 as 'H'. However, it would be a mistake to symbolize sentence 8 with '$\neg H$'. If Jane is unhappy, then she is not happy; but sentence 8 does not mean the same thing as 'It is not the case that Jane is happy'. Jane might be neither happy nor unhappy; she might be in a state of blank indifference. In order to symbolize sentence 8, then, we would need a new atomic sentence of TFL.

5.2 Conjunction

Consider these sentences:

9. Adam is athletic.
10. Barbara is athletic.
11. Adam is athletic, and Barbara is also athletic.

We will need separate atomic sentences of TFL to symbolize sentences 9 and 10; perhaps

A: Adam is athletic.
B: Barbara is athletic.

Sentence 9 can now be symbolized as 'A', and sentence 10 can be symbolized as 'B'. Sentence 11 roughly says 'A and B'. We need another symbol, to deal with 'and'. We will use ' & '. Thus, we will symbolize it as '$(A \,\&\, B)$'. This connective is called CONJUNCTION. We also say that 'A' and 'B' are the two CONJUNCTS of the conjunction '$(A \,\&\, B)$'.

Notice that we make no attempt to symbolize the word 'also' in sentence 11. Words like 'both' and 'also' function to draw our attention to the fact that two things are being conjoined. Maybe they affect the emphasis of a sentence, but we will not (and cannot) symbolize such things in TFL.

Some more examples will bring out this point:

12. Barbara is athletic and energetic.
13. Barbara and Adam are both athletic.
14. Although Barbara is energetic, she is not athletic.

15. Adam is athletic, but Barbara is more athletic than him.

Sentence 12 is obviously a conjunction. The sentence says two things (about Barbara). In English, it is permissible to refer to Barbara only once. It *might* be tempting to think that we need to symbolize sentence 12 with something along the lines of 'B and energetic'. This would be a mistake. Once we symbolize part of a sentence as 'B', any further structure is lost, as 'B' is an atomic sentence of TFL. Conversely, 'energetic' is not an English sentence at all. What we are aiming for is something like 'B and Barbara is energetic'. So we need to add another sentence letter to the symbolization key. Let 'E' symbolize 'Barbara is energetic'. Now the entire sentence can be symbolized as '$(B \& E)$'.

Sentence 13 says one thing about two different subjects. It says of both Barbara and Adam that they are athletic, even though in English we use the word 'athletic' only once. The sentence can be paraphrased as 'Barbara is athletic, and Adam is athletic'. We can symbolize this in TFL as '$(B \& A)$', using the same symbolization key that we have been using.

Sentence 14 is slightly more complicated. The word 'although' sets up a contrast between the first part of the sentence and the second part. Nevertheless, the sentence tells us both that Barbara is energetic and that she is not athletic. In order to make each of the conjuncts an atomic sentence, we need to replace 'she' with 'Barbara'. So we can paraphrase sentence 14 as, '*Both* Barbara is energetic, *and* Barbara is not athletic'. The second conjunct contains a negation, so we paraphrase further: '*Both* Barbara is energetic *and it is not the case that* Barbara is athletic'. Now we can symbolize this with the TFL sentence '$(E \& \neg B)$'. Note that we have lost all sorts of nuance in this symbolization. There is a distinct difference in tone between sentence 14 and 'Both Barbara is energetic and it is not the case that Barbara is athletic'. TFL does not (and cannot) preserve these nuances.

Sentence 15 raises similar issues. There is a contrastive structure, but this is not something that TFL can deal with. So we can paraphrase the sentence as '*Both* Adam is athletic, *and* Barbara is more athletic than Adam'. (Notice that we once again replace the pronoun 'him' with 'Adam'.) How should we deal with the second conjunct? We already have the sentence letter 'A', which is being used to symbolize 'Adam is athletic', and the sentence 'B' which is being used to symbolize 'Barbara is athletic'; but neither of these concerns their relative athleticity. So, to symbolize the entire sentence, we need a new sentence letter. Let the TFL sentence 'R' symbolize the English sentence 'Barbara is more athletic than Adam'. Now we can symbolize sentence 15 by '(A & R)'.

> A sentence can be symbolized as (A & B) if it can be paraphrased in English as 'Both…, and…', or as '…, but …', or as 'although …, …'.

You might be wondering why we put brackets around the conjunctions. The reason for this is brought out by considering how negation might interact with conjunction. Consider:

16. It's not the case that you will get both soup and salad.
17. You will not get soup but you will get salad.

Sentence 16 can be paraphrased as 'It is not the case that: both you will get soup and you will get salad'. Using this symbolization key:

S_1: You will get soup.
S_2: You will get salad.

We would symbolize 'both you will get soup and you will get salad' as '(S_1 & S_2)'. To symbolize sentence 16, then, we simply negate the whole sentence, thus: '$\neg$(S_1 & S_2)'.

Sentence 17 is a conjunction: you *will not* get soup, and you *will* get salad. 'You will not get soup' is symbolized by '$\neg S_1$'. So to symbolize sentence 17 itself, we offer '($\neg S_1$ & S_2)'.

These English sentences are very different, and their symbolizations differ accordingly. In one of them, the entire conjunction is negated. In the other, just one conjunct is negated. Brackets help us to keep track of things like the *scope* of the negation.

5.3 Disjunction

Consider these sentences:

18. Either Fatima will play videogames, or she will watch movies.
19. Either Fatima or Omar will play videogames.

For these sentences we can use this symbolization key:

F: Fatima will play videogames.
O: Omar will play videogames.
M: Fatima will watch movies.

However, we will again need to introduce a new symbol. Sentence 18 is symbolized by '$(F \lor M)$'. The connective is called DISJUNCTION. We also say that 'F' and 'M' are the DISJUNCTS of the disjunction '$(F \lor M)$'.

Sentence 19 is only slightly more complicated. There are two subjects, but the English sentence only gives the verb once. However, we can paraphrase sentence 19 as 'Either Fatima will play videogames, or Omar will play videogames'. Now we can obviously symbolize it by '$(F \lor O)$' again.

> A sentence can be symbolized as $(\mathsf{A} \lor \mathsf{B})$ if it can be paraphrased in English as 'Either…, or….' Each of the disjuncts must be a sentence.

Sometimes in English, the word 'or' is used in a way that excludes the possibility that both disjuncts are true. This is called an EXCLUSIVE OR. An *exclusive or* is clearly intended when it says, on a restaurant menu, 'Entrees come with either soup or salad': you may have soup; you may have salad; but, if you want *both* soup *and* salad, then you have to pay extra.

At other times, the word 'or' allows for the possibility that both disjuncts might be true. This is probably the case with sentence 19, above. Fatima might play videogames alone, Omar might play videogames alone, or they might both play. Sentence 19 merely says that *at least* one of them plays videogames. This is called an INCLUSIVE OR. The TFL symbol '$\lor$' always symbolizes an *inclusive or*.

It might help to see negation interact with disjunction. Consider:

20. Either you will not have soup, or you will not have salad.
21. You will have neither soup nor salad.
22. You get either soup or salad, but not both.

Using the same symbolization key as before, sentence 20 can be paraphrased in this way: 'Either *it is not the case that* you get soup, or *it is not the case that* you get salad'. To symbolize this in TFL, we need both disjunction and negation. 'It is not the case that you get soup' is symbolized by '$\neg S_1$'. 'It is not the case that you get salad' is symbolized by '$\neg S_2$'. So sentence 20 itself is symbolized by '$(\neg S_1 \lor \neg S_2)$'.

Sentence 21 also requires negation. It can be paraphrased as, '*It is not the case that* either you get soup or you get salad'. Since this negates the entire disjunction, we symbolize sentence 21 with '$\neg(S_1 \vee S_2)$'.

Sentence 22 is an *exclusive or*. We can break the sentence into two parts. The first part says that you get one or the other. We symbolize this as '$(S_1 \vee S_2)$'. The second part says that you do not get both. We can paraphrase this as: 'It is not the case both that you get soup and that you get salad'. Using both negation and conjunction, we symbolize this with '$\neg(S_1 \,\&\, S_2)$'. Now we just need to put the two parts together. As we saw above, 'but' can usually be symbolized with ' & '. Sentence 22 can thus be symbolized as '$((S_1 \vee S_2) \,\&\, \neg(S_1 \,\&\, S_2))$'.

This last example shows something important. Although the TFL symbol '$\vee$' always symbolizes *inclusive or*, we can symbolize an *exclusive or* in TFL. We just have to use a few of our other symbols as well.

5.4 Conditional

Consider these sentences:

23. If Jean is in Paris, then Jean is in France.

Let's use the following symbolization key:

> P: Jean is in Paris.
> F: Jean is in France

Sentence 23 is roughly of this form: 'if P, then F'. We will use the symbol '$\rightarrow$' to symbolize this 'if..., then...' structure. So we symbolize sentence 23 by '$(P \rightarrow F)$'. The connective is called THE CONDITIONAL. Here, 'P' is called the ANTECEDENT of the conditional '$(P \rightarrow F)$', and 'F' is called the CONSEQUENT.

> A sentence can be symbolized as $\mathsf{A} \rightarrow \mathsf{B}$ if it can be paraphrased in English as 'If A, then B'.

Many English expressions can be represented using the conditional. These (and others) are all equivalent to 'If A, then B':

24. B if A. (Notice that the consequent is first here.)
25. Whenever A, B.

26. B provided that A.
27. Provided that A, B.
28. A only if B.

If you think about it, you'll see that all five of these sentences have the same meaning, and so they can all be symbolized by '$A \to B$'.

5.5 Biconditional

Consider these sentences:

29. Laika is a dog only if she is a mammal
30. Laika is a dog if she is a mammal
31. Laika is a dog if and only if she is a mammal

We will use the following symbolization key:

D: Laika is a dog
M: Laika is a mammal

Sentence 29, for reasons discussed above, can be symbolized by '$D \to M$'.

Sentence 30 is importantly different. It can be paraphrased as, 'If Laika is a mammal then Laika is a dog'. So it can be symbolized by '$M \to D$'.

Sentence 31 says something stronger than either 29 or 30. It can be paraphrased as 'Laika is a dog if Laika is a mammal, and Laika is a dog only if Laika is a mammal'. This is just the conjunction of sentences 29 and 30. So we can symbolize it as '$(D \to M) \& (M \to D)$'. We call this a BICONDITIONAL, because it entails the conditional in both directions.

We could treat every biconditional this way. So, just as we do not need a new TFL symbol to deal with *exclusive or*, we do not really need a new TFL symbol to deal with biconditionals. Because the biconditional occurs so often, however, we will use the symbol '$\leftrightarrow$' for it. We can then symbolize sentence 31 with the TFL sentence '$D \leftrightarrow M$'.

The expression 'if and only if' occurs a lot in philosophy, mathematics, and logic, and sometimes you will see it abbreviated 'iff'. (Although even when 'iff' is written, we still say 'if and only if.') So 'if' with only *one* 'f' is the English conditional. But 'iff' with *two* 'f's is the English biconditional.

> A sentence can be symbolized as $\mathsf{A} \leftrightarrow \mathsf{B}$ if it can be paraphrased in English as 'A iff B'—that is, as 'A if and only if B'.

A word of caution. Ordinary speakers of English often use 'if ..., then...' when they really mean to use something more like '... if and only if ...'. Perhaps your parents told you when you were a child: 'if you don't eat your greens, you won't get any dessert'. Suppose you ate your greens, but that your parents refused to give you any dessert, on the grounds that they were only committed to the *conditional* (roughly 'if you get dessert, then you will have eaten your greens'), rather than the biconditional (roughly, 'you get dessert iff you eat your greens'). Well, a tantrum would rightly ensue. So, be aware of this when interpreting people. And in your own writing, make sure you use *if and only if* if and only if you mean to use it.

5.6 Unless

We have now introduced all of the connectives of TFL. We can use them together to symbolize many kinds of sentences. An especially difficult case is when we use the English-language connective 'unless':

32. Unless you wear a jacket, you will catch a cold.
33. You will catch a cold unless you wear a jacket.

These two sentences are clearly equivalent. To symbolize them, we will use the symbolization key:

J: You will wear a jacket.
D: You will catch a cold.

Both sentences mean that if you do not wear a jacket, then you will catch a cold. With this in mind, we might symbolize them as '$\neg J \rightarrow D$'.

Equally, both sentences mean that if you do not catch a cold, then you must have worn a jacket. With this in mind, we might symbolize them as '$\neg D \rightarrow J$'.

Equally, both sentences mean that either you will wear a jacket or you will catch a cold. With this in mind, we might symbolize them as '$J \vee D$'.

All three are correct symbolizations. Indeed, in chapter 10 we will see that all three symbolizations are equivalent in TFL.

> If a sentence can be paraphrased as 'Unless A, B,' then it can be symbolized as '$A \lor B$'.

Again, though, there is a little complication. 'Unless' can be symbolized as a conditional; but as we said above, people often use the conditional (on its own) when they mean to use the biconditional. Equally, 'unless' can be symbolized as a disjunction; but there are two kinds of disjunction (exclusive and inclusive). So it will not surprise you to discover that ordinary speakers of English often use 'unless' to mean something more like the biconditional, or like exclusive disjunction. Suppose someone says: 'I will go running unless it rains'. They probably mean something like 'I will go running iff it does not rain' (i.e. the biconditional), or 'either I will go running or it will rain, but not both' (i.e. exclusive disjunction). Again: be aware of this when interpreting what other people have said, but be precise in your writing.

Practice exercises

A. Using the symbolization key given, symbolize each English sentence in TFL.

M: Those creatures are men in suits.
C: Those creatures are chimpanzees.
G: Those creatures are gorillas.

1. Those creatures are not men in suits.
2. Those creatures are men in suits, or they are not.
3. Those creatures are either gorillas or chimpanzees.
4. Those creatures are neither gorillas nor chimpanzees.
5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.
6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.

B. Using the symbolization key given, symbolize each English sentence in TFL.

A: Mister Ace was murdered.
B: The butler did it.
C: The cook did it.

D: The Duchess is lying.
E: Mister Edge was murdered.
F: The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.
2. If Mister Ace was murdered, then the cook did it.
3. If Mister Edge was murdered, then the cook did not do it.
4. Either the butler did it, or the Duchess is lying.
5. The cook did it only if the Duchess is lying.
6. If the murder weapon was a frying pan, then the culprit must have been the cook.
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
8. Mister Ace was murdered if and only if Mister Edge was not murdered.
9. The Duchess is lying, unless it was Mister Edge who was murdered.
10. If Mister Ace was murdered, he was done in with a frying pan.
11. Since the cook did it, the butler did not.
12. Of course the Duchess is lying!

C. Using the symbolization key given, symbolize each English sentence in TFL.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
2. If Ava is a firefighter, then she is satisfied with her career.
3. Ava is a firefighter, unless she is an electrician.
4. Harrison is an unsatisfied electrician.
5. Neither Ava nor Harrison is an electrician.
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
7. Harrison is satisfied only if he is a firefighter.
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.

9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
11. It cannot be that Harrison is both an electrician and a firefighter.
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.

D. Using the symbolization key given, symbolize each English-language sentence in TFL.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.
2. Neither Miles Davis nor John Coltrane played tuba.
3. John Coltrane did not play both tenor sax and tuba.
4. John Coltrane did not play tenor sax unless he also played soprano sax.
5. John Coltrane did not play tuba, but Miles Davis did.
6. Miles Davis played trumpet only if he also played tuba.
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
8. If John Coltrane played tuba then Miles Davis played neither trumpet nor tuba.
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.

E. Give a symbolization key and symbolize the following English sentences in TFL.

1. Alice and Bob are both spies.
2. If either Alice or Bob is a spy, then the code has been broken.
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
4. The German embassy will be in an uproar, unless someone has broken the code.

5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
6. Either Alice or Bob is a spy, but not both.

F. Give a symbolization key and symbolize the following English sentences in TFL.

1. If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.
2. Rafiki will talk about squashed bananas unless Simba is alive.
3. Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.
4. Scar will remain as king if and only if there is food to be found in the pridelands.
5. If Simba is alive, then Scar will not remain as king.

G. For each argument, write a symbolization key and symbolize all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.
2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.
3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

H. For each argument, write a symbolization key and symbolize the argument as well as possible in TFL. The part of the passage in italics is there to provide context for the argument, and doesn't need to be symbolized.

1. It is going to rain soon. I know because my leg is hurting, and my leg hurts if it's going to rain.
2. *Spider-man tries to figure out the bad guy's plan.* If Doctor Octopus gets the uranium, he will blackmail the city. I am certain of this because if Doctor Octopus gets the uranium, he can make a dirty bomb, and if he can make a dirty bomb, he will blackmail the city.

3. *A westerner tries to predict the policies of the Chinese government.* If the Chinese government cannot solve the water shortages in Beijing, they will have to move the capital. They don't want to move the capital. Therefore they must solve the water shortage. But the only way to solve the water shortage is to divert almost all the water from the Yangzi river northward. Therefore the Chinese government will go with the project to divert water from the south to the north.

I. We symbolized an *exclusive or* using '$\vee$', '$\&$', and '$\neg$'. How could you symbolize an *exclusive or* using only two connectives? Is there any way to symbolize an *exclusive or* using only one connective?

Chapter 6

Sentences of TFL

The sentence 'either apples are red, or berries are blue' is a sentence of English, and the sentence '$(A \vee B)$' is a sentence of TFL. Although we can identify sentences of English when we encounter them, we do not have a formal definition of 'sentence of English'. But in this chapter, we will offer a complete *definition* of what counts as a sentence of TFL. This is one respect in which a formal language like TFL is more precise than a natural language like English.

6.1 Expressions

We have seen that there are three kinds of symbols in TFL:

Atomic sentences	$A, B, C, \ldots, Z$
with subscripts, as needed	$A_1, B_1, Z_1, A_2, A_{25}, J_{375}, \ldots$
Connectives	$\neg, \ \&\ , \vee, \rightarrow, \leftrightarrow$
Brackets	(,)

We define an EXPRESSION OF TFL as any string of symbols of TFL. Take any of the symbols of TFL and write them down, in any order, and you have an expression of TFL.

6.2 Sentences

Of course, many expressions of TFL will be total gibberish. We want to know when an expression of TFL amounts to a *sentence*.

Obviously, individual atomic sentences like 'A' and 'G_{13}' should count as sentences. We can form further sentences out of these by using the various connectives. Using negation, we can get '$\neg A$' and '$\neg G_{13}$'. Using conjunction, we can get '$(A \& G_{13})$', '$(G_{13} \& A)$', '$(A \& A)$', and '$(G_{13} \& G_{13})$'. We could also apply negation repeatedly to get sentences like '$\neg\neg A$' or apply negation along with conjunction to get sentences like '$\neg(A \& G_{13})$' and '$\neg(G_{13} \& \neg G_{13})$'. The possible combinations are endless, even starting with just these two sentence letters, and there are infinitely many sentence letters. So there is no point in trying to list all the sentences one by one.

Instead, we will describe the process by which sentences can be *constructed*. Consider negation: Given any sentence A of TFL, $\neg\mathsf{A}$ is a sentence of TFL. (Notice that A and A are different fonts. This is intentional, and we will discuss the reason for the difference in §7.3.) We can say similar things for each of the other connectives. For instance, if A and B are sentences of TFL, then $(\mathsf{A} \& \mathsf{B})$ is a sentence of TFL. Providing clauses like this for all of the connectives, we arrive at the following formal definition for a SENTENCE OF TFL:

1. Every atomic sentence is a sentence.
2. If A is a sentence, then $\neg\mathsf{A}$ is a sentence.
3. If A and B are sentences, then $(\mathsf{A} \& \mathsf{B})$ is a sentence.
4. If A and B are sentences, then $(\mathsf{A} \lor \mathsf{B})$ is a sentence.
5. If A and B are sentences, then $(\mathsf{A} \to \mathsf{B})$ is a sentence.
6. If A and B are sentences, then $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a sentence.
7. Nothing else is a sentence.

Definitions like this are called *recursive*. Recursive definitions begin with some specifiable base elements, and then present ways to generate indefinitely many more elements by compounding together previously established ones. To give you a better idea of what a recursive definition is, we can give a recursive definition of the idea of *an ancestor of mine.* We specify a base clause.

- My parents are ancestors of mine.

and then offer further clauses like:

- If x is an ancestor of mine, then x's parents are ancestors of mine.
- Nothing else is an ancestor of mine.

Using this definition, we can easily check to see whether someone is my ancestor: just check whether she is the parent of the parent of... one of my parents. And the same is true for our recursive definition of sentences of TFL. Just as the recursive definition allows complex sentences to be built up from simpler parts, the definition allows us to decompose sentences into their simpler parts. Once we get down to atomic sentences, then we know we are ok.

Let's consider some examples.

Suppose we want to know whether or not '$\neg\neg\neg D$' is a sentence of TFL. Looking at the second clause of the definition, we know that '$\neg\neg\neg D$' is a sentence *if* '$\neg\neg D$' is a sentence. So now we need to ask whether or not '$\neg\neg D$' is a sentence. Again looking at the second clause of the definition, '$\neg\neg D$' is a sentence *if* '$\neg D$' is. So, '$\neg D$' is a sentence *if* 'D' is a sentence. Now 'D' is an atomic sentence of TFL, so we know that 'D' is a sentence by the first clause of the definition. So for a compound sentence like '$\neg\neg\neg D$', we must apply the definition repeatedly. Eventually we arrive at the atomic sentences from which the sentence is built up.

Next, consider the example '$\neg(P \,\&\, \neg(\neg Q \vee R))$'. Looking at the second clause of the definition, this is a sentence if '$(P \,\&\, \neg(\neg Q \vee R))$' is, and this is a sentence if *both* 'P' *and* '$\neg(\neg Q \vee R)$' are sentences. The former is an atomic sentence, and the latter is a sentence if '$(\neg Q \vee R)$' is a sentence. It is. Looking at the fourth clause of the definition, this is a sentence if both '$\neg Q$' and 'R' are sentences, and both are! Ultimately, every sentence is constructed nicely out of atomic sentences.

The recursive structure of sentences in TFL will also be important when we consider the circumstances under which a particular sentence would be true or false. The sentence '$\neg\neg\neg D$' is true if and only if the sentence '$\neg\neg D$' is false, and so on through the structure of the sentence, until we arrive at the atomic components. We will return to this point in Part III.

6.3 The main logical operator

When we are dealing with a *sentence* other than an atomic sentence, we can see that there must be some sentential connective that was introduced *last*, when constructing the sentence. We call that connective the MAIN LOGICAL OPERATOR of the sentence. In the simplest case, when there is only one connective—for instance, as there is here: $(P \,\&\, Q)$—the main logical operator is the one connective, the '&'. If we add to this sentence, for instance like this: $((P \,\&\, Q) \vee R)$, then the main logical operator becomes '$\vee$'. This is not only the operator that was added last, but it also dictates, at the most general level, what kind of sentence this is: it's a disjunction. (And one of the disjuncts happens to be a conjunction.)

Adding to this sentence again, we get $(((P \,\&\, Q) \vee R) \rightarrow T)$. Now the main logical operator is '$\rightarrow$', and the sentence is a conditional. And finally, if we negate the sentence: $\neg(((P \,\&\, Q) \vee R) \rightarrow T)$, '$\neg$' becomes the main logical operator. Some more examples:

1. $((P \,\&\, R) \rightarrow (\neg Q \,\&\, S))$ The main logical operator is the '$\rightarrow$'.
2. $(((T \rightarrow P) \,\&\, R) \vee (S \leftrightarrow Q))$ The main logical operator is the '$\vee$'.
3. $\neg\neg\neg D$ The main logical operator is the first '$\neg$' sign.
4. $(P \,\&\, \neg(\neg Q \vee R))$ The main logical operator is '&'.
5. $((\neg E \vee F) \rightarrow \neg G)$ The main logical operator is '$\rightarrow$'.

When the sentence includes the outermost brackets, you can find the main logical operator for a sentence by using the following method:

(1) If the first symbol in the sentence is '$\neg$', then that is the main logical operator.

(2) Otherwise, start counting the brackets by following one of these two procedures.

  (2a) Start from the left, and for each open-bracket, i.e. '(', add 1; for each closing-bracket, i.e. ')', subtract 1. When your count is at exactly 1, the next operator you come to (*apart* from a '$\neg$') is the main logical operator.

  (2b) If starting at the left-side of the sentence doesn't seem to work, follow the same procedure, but begin at the far right (and work left).

As we will discuss in the next section (§6.4), it is, in some cases, acceptable to omit the outermost brackets. For instance, although it is not strictly allowable according to the rules given in §6.2, we can write '$(P \,\&\, R) \to Q$' instead of '$((P \,\&\, R) \to Q)$'. When the outermost brackets are omitted, '$\neg$' won't be the main logical operator, and so (1) will not apply. In other words, if '$\neg$' is the main logical operator, then the outermost brackets have to be used. But remember, when '$\neg$' is the main logical operator—as it is in this example: $\neg(((P \,\&\, Q) \lor R) \to T)$—the '$\neg$' will be outside the outermost brackets. When the outermost brackes are omitted, (2a) and (2b) can be used, but stop when your count gets to zero instead of 1.

Finally, let's give a formal definition of the *scope* of a negation (which was mentioned in §5.2). The scope of a '$\neg$' is the subsentence for which '$\neg$' is the main logical operator. Consider a sentence like:

$$(P \,\&\, (\neg(R \,\&\, B) \leftrightarrow Q))$$

which was constructed by conjoining 'P' with '$(\neg(R \,\&\, B) \leftrightarrow Q)$'. This last sentence was constructed by placing a biconditional between '$\neg(R \,\&\, B)$' and 'Q'. The former of these sentences—a subsentence of our original sentence—is a sentence for which '$\neg$' is the main logical operator. So the scope of the negation is just '$\neg(R \,\&\, B)$'. The same holds for every connective, and so we have the following definition.

> The SCOPE of a connective (in a sentence) is the subsentence for which that connective is the main logical operator.

6.4 Bracketing conventions

Strictly speaking, the brackets in '$(Q \,\&\, R)$' are required. One reason for this is because the rules for forming sentences in TFL state that two atomic sentences connected by a connective are enclosed in brackets. (See section 6.2.) Another reason is that we might use '$(Q \,\&\, R)$' as a subsentence in a more complicated sentence. For example, we might want to negate '$(Q \,\&\, R)$', obtaining '$\neg(Q \,\&\, R)$'. If we just had '$Q \,\&\, R$' without the brackets and put a negation in front of it, we would have '$\neg Q \,\&\, R$'. But '$\neg Q \,\&\, R$'—or '$(\neg Q \,\&\, R)$'—is very different than '$\neg(Q \,\&\, R)$'.

That said, there are some convenient conventions that we can use as long as we are careful. First, we allow ourselves to omit the *outermost* brackets of a sentence. Thus, we allow ourselves to write '$Q \,\&\, R$' instead of the sentence

'$(Q \,\&\, R)$'. We must remember, however, to put brackets around it when we want to embed the sentence into a more complicated sentence.

Second, it can be a bit painful to stare at long sentences with many nested pairs of brackets. To make things a bit easier on the eyes, we will allow ourselves to use square brackets, '[' and ']', instead of rounded ones. So there is no logical difference between '$(P \lor Q)$' and '$[P \lor Q]$', for example.

Combining these two conventions, we can rewrite the unwieldy sentence

$$(((H \to I) \lor (I \to H)) \,\&\, (J \lor K))$$

rather more clearly as follows:

$$\big[(H \to I) \lor (I \to H)\big] \,\&\, (J \lor K)$$

The scope of each connective is now much easier to pick out.

Practice exercises

A. For each of the following: (a) Is it a sentence of TFL, strictly speaking? (b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

1. (A)
2. $J_{374} \lor \neg J_{374}$
3. $\neg\neg\neg\neg F$
4. $\neg \,\&\, S$
5. $(G \,\&\, \neg G)$
6. $(A \to (A \,\&\, \neg F)) \lor (D \leftrightarrow E)$
7. $[(Z \leftrightarrow S) \to W] \,\&\, [J \lor X]$
8. $(F \leftrightarrow \neg D \to J) \lor (C \,\&\, D)$

B. Are there any sentences of TFL that contain no atomic sentences? Explain your answer.

C. What is the scope of each connective in the sentence

$$\big[(H \to I) \lor (I \to H)\big] \,\&\, (J \lor K)$$

Chapter 7

Use and mention

In this Part, we have talked a lot *about* sentences. So we should pause to explain an important, and very general, point.

7.1 Quotation conventions

Consider these two sentences:

- Justin Trudeau is the Prime Minister.
- The expression 'Justin Trudeau' is composed of two uppercase letters and eleven lowercase letters

When we want to talk about the Prime Minister, we *use* his name. When we want to talk about the Prime Minister's name, we *mention* that name, which we do by putting it in quotation marks.

There is a general point here. When we want to talk about things in the world, we just *use* words. When we want to talk about words, we typically have to *mention* those words. We need to indicate that we are mentioning them, rather than using them. To do this, some convention is needed. We can put them in quotation marks, or display them centrally in the page (say). So this sentence:

- 'Justin Trudeau' is the Prime Minister.

says that some *expression* is the Prime Minister. That's false. The *man* is the Prime Minister; his *name* isn't. Conversely, this sentence:

- Justin Trudeau is composed of two uppercase letters and eleven lowercase letters.

also says something false: Justin Trudeau is a man, made of flesh rather than letters. One final example:

- " 'Justin Trudeau' " is the name of 'Justin Trudeau'.

On the left-hand-side, here, we have the name of a name. On the right hand side, we have a name. Perhaps this kind of sentence only occurs in logic textbooks, but it is true nonetheless.

Those are just general rules for quotation, and you should observe them carefully in all your work! To be clear, the quotation-marks here do not indicate indirect speech. They indicate that you are moving from talking about an object, to talking about the name of that object.

7.2 Object language and metalanguage

These general quotation conventions are of particular importance for us. After all, we are describing a formal language here, TFL, and so we are often *mentioning* expressions from TFL.

When we talk about a language, the language that we are talking about is called the OBJECT LANGUAGE. The language that we use to talk about the object language is called the METALANGUAGE.

For the most part, the object language in this chapter has been the formal language that we have been developing: TFL. The metalanguage is English. Not conversational English exactly, but English supplemented with some additional vocabulary which helps us to get along.

Now, we have used italic uppercase letters for atomic sentences of TFL:

$$A, B, C, Z, A_1, B_4, A_{25}, J_{375}, \ldots$$

These are sentences of the object language (TFL). They are not sentences of English. So we must not say, for example:

- D is an atomic sentence of TFL.

Obviously, we are trying to come out with an English sentence that says something about the object language (TFL), but 'D' is a sentence of TFL, and not part of English. So the preceding is gibberish, just like:

- Schnee ist weiß is a German sentence.

What we surely meant to say, in this case, is:

- 'Schnee ist weiß' is a German sentence.

Equally, what we meant to say above is just:

- 'D' is an atomic sentence of TFL.

The general point is that, whenever we want to talk in English about some specific expression of TFL, we need to indicate that we are *mentioning* the expression, rather than *using* it. We can either deploy quotation marks, or we can adopt some similar convention, such as placing it centrally in the page.

7.3 Metavariables

However, we do not just want to talk about *specific* expressions of TFL. We also want to be able to talk about *any arbitrary* sentence of TFL. Indeed, we had to do this in §6.2, when we presented the recursive definition of a sentence of TFL. We used uppercase script letters to do this, namely:

$$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$$

These symbols do not belong to TFL. Rather, they are part of our (augmented) metalanguage that we use to talk about *any* expression of TFL. To repeat the second clause of the recursive definition of a sentence of TFL, we said:

2. If A is a sentence, then $\neg\mathsf{A}$ is a sentence.

This talks about *arbitrary* sentences. If we had instead offered:

- If 'A' is a sentence, then '$\neg A$' is a sentence.

this would not have allowed us to determine whether '$\neg B$' is a sentence. To emphasize, then:

> 'A' is a symbol (called a METAVARIABLE) in augmented English, which we use to talk about any TFL expression. 'A' is a particular atomic sentence of TFL.

But this last example raises a further complication for our quotation conventions. We have not included any quotation marks in the second clause of our recursive definition. Should we have done so?

The problem is that the expression on the right-hand-side of this rule is not a sentence of English, since it contains '$\neg$'. So we might try to write:

2′. If A is a sentence, then '$\neg\mathsf{A}$' is a sentence.

But this is no good: '$\neg\mathsf{A}$' is not a TFL sentence, since 'A' is a symbol of (augmented) English rather than a symbol of TFL.

What we really want to say is something like this:

2″. If A is a sentence, then the result of concatenating the symbol '$\neg$' with the sentence A is a sentence.

This is impeccable, but rather long-winded. But we can avoid long-windedness by creating our own conventions. We can perfectly well stipulate that an expression like '$\neg\mathsf{A}$' should simply be read *directly* in terms of rules for concatenation. So, *officially*, the metalanguage expression '$\neg\mathsf{A}$' simply abbreviates:

the result of concatenating the symbol '$\neg$' with the sentence A

and similarly, for expressions like '$(\mathsf{A} \,\&\, \mathsf{B})$', '$(\mathsf{A} \lor \mathsf{B})$', etc.

7.4 Quotation conventions for arguments

One of our main purposes for using TFL is to study arguments, and that will be our concern in Parts III and IV. In English, the premises of an argument are often expressed by individual sentences, and the conclusion by a further sentence. Since we can symbolize English sentences, we can symbolize English arguments using TFL. Thus we might ask whether the argument whose premises are the TFL sentences 'A' and '$A \to C$', and whose conclusion is the TFL sentence 'C', is valid. However, it is quite a mouthful to write that every time. So instead we will introduce another bit of abbreviation. This:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \therefore \mathsf{C}$$

abbreviates:

the argument with premises $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ and conclusion C

To avoid unnecessary clutter, we will not regard this as requiring quotation marks around it. (Note, then, that ‘∴’ is a symbol of our augmented *metalanguage*, and not a new symbol of TFL.)

Part III

Truth tables

Chapter 8

Characteristic truth tables

Any non-atomic sentence of TFL is composed of atomic sentences with sentential connectives. The truth value of the compound sentence depends on the truth value of the atomic sentences that comprise it. To know the truth value of '$(D \& E)$', for instance, you need to know the truth value of 'D' and the truth value of 'E'. You also, however, need to know the rule for when a conjunction is true and when it is false.

We introduced five connectives in chapter 5, and now we need to explain when sentences using each connective are true and false. For convenience, we will abbreviate 'True' with 'T' and 'False' with 'F'. (But just to be clear, the two truth values are *true* and *false*; the truth values are not letters!)

Conjunction. For any sentences A and B, the conjunction $(\mathsf{A} \& \mathsf{B})$ is true if and only if both A and B are true. If one or both of A and B are false, then the sentence $(\mathsf{A} \& \mathsf{B})$ is false. We can summarize this in the characteristic truth table for conjunction:

A	B	$\mathsf{A} \& \mathsf{B}$
T	T	T
T	F	F
F	T	F
F	F	F

That's a truth table. At the top (above the horizontal line) we have, on the right, the sentence whose truth and falsity we are investigating. On the left, are all of the atomic sentences that appear in the sentence on the right.

Below the horizontal line on the left are different combinations of *true* and *false* for the atomic sentences. On the first line, there is a 'T' below A and a 'T' below B. So, this line represents the situation where A is true and B is true. What does that mean for the sentence $(\mathsf{A} \,\&\, \mathsf{B})$? Of course, you know. But you can look on the right side of the truth table, on line 1, and see that, in this situation, $(\mathsf{A} \,\&\, \mathsf{B})$ is true. Likewise, on line 2, there is a 'T' below A and an 'F' below B. Checking the right side, then, we see that when A is true and B is false, $(\mathsf{A} \,\&\, \mathsf{B})$ is false. That's the basic idea, although there is more explanation and some examples in the next chapter.

Note that conjunction is *symmetrical*. The truth value for $\mathsf{A} \,\&\, \mathsf{B}$ is always the same as the truth value for $\mathsf{B} \,\&\, \mathsf{A}$.

Negation. For any sentence A: If A is true, then $\neg\mathsf{A}$ is false. If A is false, then $\neg\mathsf{A}$ is true. We can summarize this in the *characteristic truth table* for negation:

A	$\neg\mathsf{A}$
T	F
F	T

Disjunction. Recall that '$\vee$' always represents inclusive-or. So, for any sentences A and B, the disjunction $(\mathsf{A} \vee \mathsf{B})$ is true when either A or B or both are true. The only instance when $(\mathsf{A} \vee \mathsf{B})$ is false is when both A and B are false. We can summarize this in the characteristic truth table for disjunction:

A	B	$\mathsf{A} \vee \mathsf{B}$
T	T	T
T	F	T
F	T	T
F	F	F

This is a good time to explain another point. We are, in this chapter, simply defining the characteristic truth table for each connective. We have reasons for defining them these ways, and there is a consensus that these are the best definitions. But, in the end, these are the correct truth tables for each connective because these are the ways that we have chosen to set them. Conceivably, we could say that $(A \vee B)$ is false when both A and B are false *and* when both A and B are true. That would agree with the way that we, at least some of the time, use *or* in English. But that's not what we've chosen to do, and so the way that $(A \vee B)$ is defined in the truth table above is going to apply from this point forward (and similarly for all of the other connectives).

Conditional. The conditional is interesting and, for some, philosophically contentious. You may know that one way to think about the conditional is as rule: if the antecedent happens, then the consequent has to happen. So, for instance, take the conditional 'if I am at the grocery store, I am getting bread'. This sentence is obviously true when (1) I go to the grocery store and get bread. Conversely, this sentence is false when (2) I go to the grocery store, but don't buy bread. (If that happens, the rule has been broken.) Those two scenarios are represented by lines 1 and 2 in the characteristic truth table for the conditional.

For the other two scenarios, we have to concentrate a bit. The sentence is also true when (3) I'm not at the grocery store, but I'm getting bread somewhere else. In this case, the rule *if I am at the grocery store, then I am getting bread* hasn't been broken; it just doesn't really apply. So, when the antecedent is false and the consequent is true, the conditional is true. That's represented by line 3 of the characteristic truth table for the conditional. Similarly, when (4) I am not at the grocery store, and I am not getting bread, the rule hasn't been broken. It just hasn't been invoked at all. So even though the antecedent didn't happen and the consequent didn't happen, the sentence is still true. (Again, it's still true that *if I am at the grocery store, then I am getting bread*, but since, let's say, I'm home taking a nap, it's false that 'I am at the grocery store' and it's false that 'I am buying bread'.)

We summarize all of this with the characteristic truth table for the conditional:

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Hopefully, you understood the explanation just given for each of the four scenarios. The conditional is philosophically contentious, however, because every conditional is not as simple and straightforward as 'if I am at the grocery store, then I am getting bread'. Take a conditional where the antecedent is always false: 'if the queen of England is on the moon, then Mississippi State University is in Starkville.' This isn't much of a rule, but it is a conditional. And since the antecedent is false and the consequent is true, the sentence is true. Even stranger, consider this conditional: 'if the queen of England is on the moon, then pigs can fly.' Now, the antecedent is false and the consequent is false, but, as is shown in line 4 of the truth table, the sentence is true.

Sometimes the truth values for the antecedent, the consequent, and the whole conditional make sense (as in our first example) and sometimes they seem odd. That has generated philosophical debate, but it actually does not present a problem for us. The conditional is very precisely defined by the characteristic truth table for the conditional—as are sentences using all of the connectives. We, then, simply use that definition, and we don't have to make any decisions about whether a particular conditional should really be true or false.

Also, unlike the conjunction and the disjunction, the conditional is *asymmetrical.* You cannot swap the antecedent and consequent without changing the meaning of the sentence, because $\mathsf{A} \to \mathsf{B}$ has a very different truth table from $\mathsf{B} \to \mathsf{A}$.

Biconditional. Since a biconditional is to be the same as the conjunction of a conditional running in each direction, the truth table for the biconditional is this:

A	B	$\mathsf{A} \leftrightarrow \mathsf{B}$
T	T	T
T	F	F
F	T	F
F	F	T

Unsurprisingly, the biconditional is symmetrical.

Chapter 9

Complete truth tables

In chapter 8, we examined the characteristic truth table for each connective. Those truth tables define the truth values for each connective. For example, $(\mathsf{A} \vee \mathsf{B})$ is false when A and B are both false, otherwise it is true; and so on for the other connectives. Now that we have those definitions, we can investigate when other, more complex sentences are true and false—for instance, ones like '$(H \& I) \rightarrow H$' and '$(M \& (N \vee P))$', which we will go through in this chapter. Once we understand how to create truth tables, we can investigate other properties of sentences, which we will do in chapters 10 and 11.

Before we begin, we will define VALUATION.

> A VALUATION is any assignment of truth values to particular atomic sentences of TFL.

So, **each row of a truth table represents a possible valuation**. **The entire truth table represents all possible valuations**. Thus, the truth table provides us with a way of finding the truth values of complex sentences on each possible valuation—that is, for every combination of 'true' and 'false' for every atomic sentence. This is easiest to explain by example.

9.1 An example

Consider the sentence '$(H \& I) \rightarrow H$', which contains three atomic sentences, although only two different ones. We set up the truth table for this sentence by putting H and I on the left side of the vertical line and '$(H \,\&\, I) \rightarrow H$' on the right. (Although H appears twice in '$(H \,\&\, I) \rightarrow H$', we only need one H on the left.) Below the H and I on the left side, we put every combination of 'T' and 'F'. That is, on one line we want to have T and T; on another we want to have T and F; on another F and T; and on another F and F.

Since we have two atomic sentences on the left, there are four combinations of true and false. For consistency, the Ts and Fs should always be listed this way: (a) in the first column (the one next to the vertical line), they alternate T, F, T, F; (b) in the second column, they alternate in pairs, T, T, F, F; and (c) if there are more atomic sentences than just H and I, then more columns and more rows are needed, but the pattern remains the same.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T					
T	F					
F	T					
F	F					

Once the left side of the truth table is filled in, we copy (**just copy**) the truth values for the atomic sentences to the right-side of the truth table. For the H, that gives us:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T				T
T	F	T				T
F	T	F				F
F	F	F				F

And then we add the truth values for I:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T		T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

Now consider the subsentence '$(H \,\&\, I)$'. This is a conjunction, $(\mathsf{A} \,\&\, \mathsf{B})$, and our next step is to determine the truth values for just this subsentence. The characteristic truth table for conjunction gives the truth conditions for *any* sentence of the form (A & B), whatever A and B might be. It represents the point that a conjunction is true only when both conjuncts are true. In this case, our conjuncts are 'H' and 'I'. They are both true on (and only on) the first line of the truth table. Accordingly, we put 'T' on the first line below the ' & ' and 'F' on each of the other lines.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	**T**	T		T
T	F	T	**F**	F		T
F	T	F	**F**	T		F
F	F	F	**F**	F		F

Now, that we have the truth values for $(H \,\&\, I)$ the sentence that we are dealing with is a conditional. '$(H \,\&\, I)$' is the antecedent of this conditional and 'H' is the consequent. To simplify the table before our next step, we will just show the truth values for the antecedent and the consequent.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T		T			T
T	F		F			T
F	T		F			F
F	F		F			F

To complete the truth table, we must determine the truth values for this conditional. On the first row, '$(H \,\&\, I)$' is true and the consequent, 'H', is true. Since a conditional is true when the antecedent is true and the consequent is true, we write a 'T' in the first row underneath the conditional symbol. On the second row, '$(H \,\&\, I)$' is false and 'H' is true. Since a conditional is true whenever the antecedent is false (check p. 50), we write a 'T' in the second row underneath the conditional symbol. We continue for the last two rows and get this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T		T		**T**	T
T	F		F		**T**	T
F	T		F		**T**	F
F	F		F		**T**	F

The conditional is the main logical operator of the sentence, so the column of 'T's underneath the conditional tells us that the sentence '$(H \,\&\, I) \rightarrow H$' is true regardless of the truth values of 'H' and 'I'. Those atomic sentences can be true or false in any combination, and the full sentence, '$(H \& I) \rightarrow H$', still comes out true. Since we have considered all four possible assignments of truth and falsity to 'H' and 'I', we can say that '$(H \& I) \rightarrow H$' is true on every *valuation*.

In this example, we have not repeated all of the entries in every column in every successive table. When actually writing truth tables on paper, however, it is impractical to erase whole columns or rewrite the whole table for every step. Although it is more crowded, a complete truth table looks like this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Most of the columns underneath the sentence are only there for bookkeeping purposes. The column that matters most is the column underneath the *main logical operator* for the sentence, since this tells you the truth value of the entire sentence. We have emphasized this, by putting this column in bold. When you work through truth tables yourself, you should similarly emphasize it (perhaps by underlining or circling).

9.2 Building complete truth tables

A COMPLETE TRUTH TABLE has a line for every possible assignment of True and False to the relevant atomic sentences. Each line represents a *valuation*, and a complete truth table has a line for all the different valuations.

The size of the complete truth table depends on the number of different atomic sentences in the table. A sentence that contains only one atomic sentence requires only two rows, as in the characteristic truth table for negation. This is true even if the same letter is repeated many times, as in the sentence '$[(C \leftrightarrow C) \rightarrow C] \& \neg(C \rightarrow C)$'. The complete truth table requires only two lines because there are only two possibilities: 'C' can be true or it can be false. The truth table for this sentence looks like this:

C	$[(C$	$\leftrightarrow$	$C)$	$\rightarrow$	$C]$	$\&$	$\neg$	$(C$	$\rightarrow$	$C)$
T	T	T	T	T	T	**F**	F	T	T	T
F	F	T	F	F	F	**F**	F	F	T	F

Looking at the column underneath the main logical operator, we see that the sentence is false on both rows of the table; i.e., the sentence is false regardless of whether 'C' is true or false. It is false on every valuation.

A sentence that contains two atomic sentences requires four lines for a complete truth table, as in the characteristic truth tables for our binary connectives, and as in the complete truth table for '$(H \,\&\, I) \rightarrow H$'.

A sentence that contains three atomic sentences requires eight lines, as shown in the table right below. Notice that the Ts and Fs in the columns below N and P (on the left side) follow the same pattern as the example in the previous section. The column under the P, meanwhile, has four Ts and then four Fs.

M	N	P	M	&	$(N$	$\vee$	$P)$
T	T	T	T	**T**	T	T	T
T	T	F	T	**T**	T	T	F
T	F	T	T	**T**	F	T	T
T	F	F	T	**F**	F	F	F
F	T	T	F	**F**	T	T	T
F	T	F	F	**F**	T	T	F
F	F	T	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F

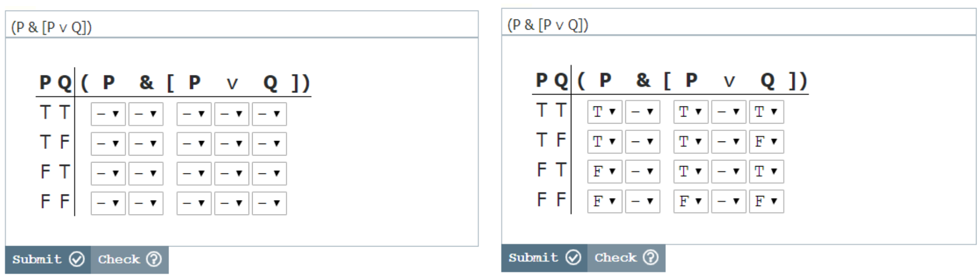
From this table, we know that the sentence '$M \,\&\, (N \vee P)$' can be true or false, depending on the truth values of 'M', 'N', and 'P'.

A complete truth table for a sentence that contains four different atomic sentences requires 16 lines. Five letters, 32 lines. Six letters, 64 lines. And so on. To be perfectly general: If a complete truth table has n different atomic sentences, then it must have 2^n lines.

To fill in the columns of a complete truth table, begin with the right-most atomic sentence (on the left side of the table) and alternate between 'T' and 'F'. In the next column to the left, write two 'T's, write two 'F's, and repeat. For the third atomic sentence, write four 'T's followed by four 'F's. This yields an eight line truth table like the one above. For a 16 line truth table, the next column of atomic sentences should have eight 'T's followed by eight 'F's. For a 32 line table, the next column would have 16 'T's followed by 16 'F's, and so on.

9.3 Building truth tables in Carnap

You should practice making truth tables on paper, but you also need to make them using the software package Carnap (https://carnap.io/). Using Carnap is pretty straightforward, and it's made easier because the left side of the truth table is completed for you. (See figure 9.1a.) On the right side, below each atomic sentence and connective, you have the option of selecting a 'T' or an 'F'. As we do anytime we are making a truth table, we begin by filling in the truth values for each atomic sentence (figure 9.1b).

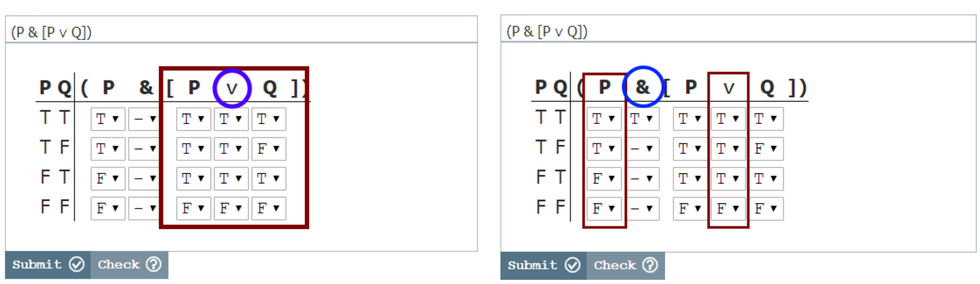


(a) *(b)*

Figure 9.1

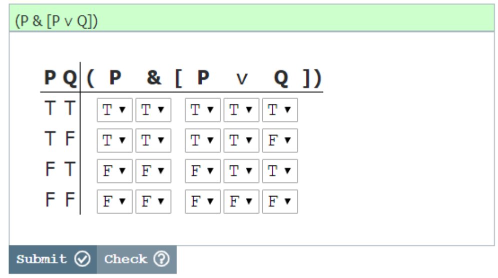
Next, in this example, we fill in the truth values for $(P \vee Q)$. Those values go below the '$\vee$', and, to determine what they are, we just look at the truth values for the P and Q that are inside the box in figure 9.2a. (Note that we cannot put truth values below the ' & ' until we have done $(P \vee Q)$.)

Once we have determined the truth values for $(P \vee Q)$, we can complete the



(a) *(b)*

Figure 9.2



P	Q	(P	&	[P	v	Q])
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	F	F	T	T
F	F	F	F	F	F	F

Figure 9.3

truth table by filling in the values below the & . To do this, we must look at the values for the P at the beginning of the sentence and the values for $(P \vee Q)$. Those are marked in figure 9.2b.

We're done. Except sometimes we're not because it is easy to make a mistake when filling in a truth table. Select 'Check' below the truth table, and Carnap will tell you "Success!" or "Something's not quite right." If something is not quite right, then you have to check every truth value until you find the mistake. Then select 'Check' again. Once Carnap confirms that the truth table is correct, select 'Submit'.

9.4 More about brackets

Consider these two sentences:

$$((A \,\&\, B) \,\&\, C)$$
$$(A \,\&\, (B \,\&\, C))$$

These are truth functionally equivalent. Consequently, it will never make any difference from the perspective of truth value—which is all that TFL cares about—which of the two sentences we assert (or deny). But even though the order of the brackets does not matter to the truth of the sentence, we should not just drop them. The expression

$$A \,\&\, B \,\&\, C$$

is ambiguous between the two sentences above. The same observation holds for disjunctions. The following sentences are logically equivalent:

$$((A \vee B) \vee C)$$
$$(A \vee (B \vee C))$$

But we should not simply write:

$$A \vee B \vee C$$

In fact, it is a specific fact about the characteristic truth table of $\vee$ and $\&$ that guarantees that any two conjunctions (or disjunctions) of the same sentences are truth functionally equivalent, however you place the brackets. *But be careful.* These two sentences have *different* truth tables:

$$((A \rightarrow B) \rightarrow C)$$
$$(A \rightarrow (B \rightarrow C))$$

So if we were to write:

$$A \rightarrow B \rightarrow C$$

it would be dangerously ambiguous. So we must not do the same with conditionals. Equally, these sentences have different truth tables:

$$((A \vee B) \,\&\, C)$$
$$(A \vee (B \,\&\, C))$$

So if we were to write:

$$A \vee B \,\&\, C$$

it would be dangerously ambiguous. *Never write this.* The moral is never drop brackets, unless there is no possibility of ambiguity.

Practice exercises

A. Make complete truth tables for each of the following:

1. $A \rightarrow A$
2. $C \rightarrow \neg C$
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
4. $(A \rightarrow B) \vee (B \rightarrow A)$
5. $(A \,\&\, B) \rightarrow (B \vee A)$
6. $\neg(A \vee B) \leftrightarrow (\neg A \,\&\, \neg B)$

7. $\big[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)\big] \,\&\, C$
8. $[(A \,\&\, B) \,\&\, C] \rightarrow B$
9. $\neg\big[(C \vee A) \vee B\big]$

B. Check all the claims made in §9.4; that is, show that:

1. '$((A \,\&\, B) \,\&\, C)$' and '$(A \,\&\, (B \,\&\, C))$' have the same truth table
2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table
3. '$((A \vee B) \,\&\, C)$' and '$(A \vee (B \,\&\, C))$' do not have the same truth table
4. '$((A \rightarrow B) \rightarrow C)$' and '$(A \rightarrow (B \rightarrow C))$' do not have the same truth table

Also, check whether:

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table

C. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1. $\neg(S \leftrightarrow (P \rightarrow S))$
2. $\neg[(X \,\&\, Y) \vee (X \vee Y)]$
3. $(A \rightarrow B) \leftrightarrow (\neg B \leftrightarrow \neg A)$
4. $[C \leftrightarrow (D \vee E)] \,\&\, \neg C$
5. $\neg(G \,\&\, (B \,\&\, H)) \leftrightarrow (G \vee (B \vee H))$

D. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1. $(D \,\&\, \neg D) \rightarrow G$
2. $(\neg P \vee \neg M) \leftrightarrow M$
3. $\neg\neg(\neg A \,\&\, \neg B)$
4. $[(D \,\&\, R) \rightarrow I] \rightarrow \neg(D \vee R)$
5. $\neg[(D \leftrightarrow O) \leftrightarrow A] \rightarrow (\neg D \,\&\, O)$

Chapter 10

Semantic concepts

In the previous section, we introduced the idea of a valuation and showed how to determine the truth value of any TFL sentence, on any valuation, using a truth table. In this section, we will introduce some related ideas, and show how to use truth tables to test whether or not they apply.

10.1 Tautologies and contradictions

In §3.2, we explained *necessary truth*, *necessary falsity*, and *contingency*. The first two have surrogates in TFL. We will start with a surrogate for necessary truth.

> A is a TAUTOLOGY if and only if it is true on every valuation.

We can determine whether a sentence is a tautology using a truth table. If the sentence is true on every line of a complete truth table (that is, if there are only 'T's under the main connective), then it is true on every valuation. And if it is true on every valuation, it is a tautology. The example in §9.1, '$(H \,\&\, I) \to H$', for instance, is a tautology.

This is only, though, a *surrogate* for necessary truth. There are some necessary truths that we cannot adequately symbolize in TFL. An example is '$2 + 2 = 4$'. This *must* be true, but if we try to symbolize it in TFL, the best we can offer is an atomic sentence, and no atomic sentence is a tautology. Still, if we can adequately symbolize some English sentence using a TFL sentence which is a tautology, then that English sentence expresses a necessary truth.

We have a similar surrogate for necessary falsity:

> A is a CONTRADICTION if and only if it is false on every valuation.

We can determine whether a sentence is a contradiction just by using truth tables. If the sentence is false on every line of a complete truth table, then it is false on every valuation, so it is a contradiction. The example in §9.2, '$[(C \leftrightarrow C) \rightarrow C] \,\&\, \neg(C \rightarrow C)$', is a contradiction.

In §3.2, we defined CONTINGENT as "a sentence that is capable of being true and capable of being false (in different circumstances, of course)." A truth table, then, provides us with those different circumstances. A sentence that is true on some lines (or even just on one line) and false on the others is contingent. Or, we can also say: any sentence that is neither a tautology nor a contradiction is contingent. $\neg(P \vee Q)$, for instance, is contingent.

P	Q	$\neg\,(P \vee Q)$
T	T	**F** T T T
T	F	**F** T T F
F	T	**F** F T T
F	F	**T** F F F

10.2 Equivalence

When we have two sentences, three possible logical relations can exist between the sentences. (Actually, there are more than three, but we'll focus on three.) The first is EQUIVALENCE.

> A and B are EQUIVALENT if and only if, for every valuation, their truth values agree, i.e. if there is no valuation in which they have opposite truth values.
> (Equivalently, if $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology, then A and B are EQUIVALENT.)

We have already made use of this notion, in effect, in §9.4; the point was that '$(A \,\&\, B) \,\&\, C$' and '$A \,\&\, (B \,\&\, C)$' are logically equivalent. Again, it is easy to test for equivalence using truth tables. Consider the sentences '$\neg(P \vee Q)$' and '$\neg P \,\&\, \neg Q$'. Are they logically equivalent? To find out, we construct a truth table containing both sentences.

P	Q	$\neg(P \vee Q)$	$\neg P \;\&\; \neg Q$
T	T	**F** T T T	F T **F** F T
T	F	**F** T T F	F T **F** T F
F	T	**F** F T T	T F **F** F T
F	F	**T** F F F	T F **T** T F

Look at the columns for the main logical operators; negation for the first sentence, conjunction for the second. On the first three rows, both are false. On the final row, both are true. Since they match on every row, the two sentences are logically equivalent.

10.3 Consistency

In §3.1, we said that sentences are jointly possible if and only if it is possible for all of them to be true at once. We have a surrogate for this notion too.

> A and B are JOINTLY CONSISTENT if and only if there is some valuation that makes them both true.
> (Equivalently, if there is at least one valuation that makes (A & B) true, then A and B are JOINTLY CONSISTENT.)

This was one of the examples in §3.1:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

These are jointly possible because it is possible for them both to be true at the same time. It takes nothing away from their joint possibility that they can also be false at the same time or one can be false while the other is true. Applying that same observation to *jointly consistent*, all we need is one or more lines where both sentences are true. $(P \vee Q)$ and $(P \;\&\; \neg Q)$ have one line where they are both true, and so they are jointly consistent:

P	Q	$P \vee Q$	$P \;\&\; \neg Q$
T	T	T **T** T	T **F** F T
T	F	T **T** F	T **T** T F
F	T	F **T** T	F **F** F T
F	F	F **F** F	F **F** T F

Conversely, sentences are JOINTLY INCONSISTENT if there is no valuation that makes them all true. If we think about this definition for a moment, we see that there are three ways that two sentences can be jointly inconsistent.

(1) One each line, the truth value for one sentence is 'T' and the truth value for the other sentence is 'F'. For instance, the truth values for $P \vee Q$ and $\neg P$ & $\neg Q$ never match. On each line, one is true and the other is false. Hence, for this relationship between two sentences, all of these criteria are satisfied: $\neg(\mathsf{A}$ & $\mathsf{B})$ is a tautology; $\neg(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology; and $(\mathsf{A} \vee \mathsf{B})$ is a tautology.

(2) When the truth value for one sentence is 'T', then the truth value for the other sentence is 'F', but both sentences can be false at the same time. For example, $\neg(\neg P \vee Q)$ and $(\neg P$ & $\neg Q)$ are never both true on the same line, but they are false on the same line. For sentences that are jointly inconsistent in this way, only this criterion is satisfied: $\neg(\mathsf{A}$ & $\mathsf{B})$ is a tautology.

(3) Both sentences are false on every line. For example, the truth values for $\neg P$ & P and $\neg Q$ & Q are always the same. On each line, both sentences are false. So, for sentences that are jointly inconsistent in this way, both of these criteria are satisfied: $\neg(\mathsf{A}$ & $\mathsf{B})$ is a tautology and $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology. (And the latter, recall, means that these sentences are equivalent.)

P	Q	$P \vee Q$	$\neg P$ & $\neg Q$
T	T	T **T** T	F T **F** F T
T	F	T **T** F	F T **F** T F
F	T	F **T** T	T F **F** F T
F	F	F **F** F	T F **T** T F

P	Q	$\neg(\neg P \vee Q)$	$\neg P$ & $\neg Q$
T	T	**F** F T T T	F T **F** F T
T	F	**T** F T F F	F T **F** T F
F	T	**F** T F T T	T F **F** F T
F	F	**F** T F T F	T F **T** T F

P	Q	$\neg P$ & P	$\neg Q$ & Q
T	T	F T **F** T	F T **F** T
T	F	F T **F** T	T F **F** F
F	T	T F **F** F	F T **F** T
F	F	T F **F** F	T F **F** F

Chapter 11

Entailment & validity

11.1 Entailment

The following idea is closely related to that of joint consistency:

> The sentences $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ ENTAIL the sentence C if there is no valuation of the atomic sentences that makes all of $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ true and C false.

Again, it is easy to test this with a truth table. Let us check whether '$\neg L \rightarrow (J \vee L)$' and '$\neg L$' entail '$J$', we simply need to check whether there is any valuation which makes both '$\neg L \rightarrow (J \vee L)$' and '$\neg L$' true whilst making '$J$' false. So we use a truth table:

J	L	$\neg\, L \rightarrow (J \vee L)$	$\neg\, L$	J
T	T	F T **T** T T T	**F** T	**T**
T	F	T F **T** T T F	**T** F	**T**
F	T	F T **T** F T T	**F** T	**F**
F	F	T F **F** F F F	**T** F	**F**

The only row on which both'$\neg L \rightarrow (J \vee L)$' and '$\neg L$' are true is the second row, and that is a row on which 'J' is also true. So '$\neg L \rightarrow (J \vee L)$' and '$\neg L$' entail '$J$'.

We now make an important observation:

> If $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ entail C, then $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \therefore \mathsf{C}$ is valid.

And, just to remind ourselves,

> An argument is VALID if and only if it is impossible for all of the premises to be true and the conclusion false.

Here's why entailment equals validity. If $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ entail C, then there is no valuation which makes all of $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ true whilst making C false. This means that it is *logically impossible* for $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ all to be true whilst C is false. But this is just what it takes for an argument, with premises $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ and conclusion C, to be valid!

In short, we have a way to test for the validity of English arguments. First, we symbolize them in TFL, as having premises $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$, and conclusion C. Then we test for entailment using truth tables.

11.2 Validity

When using a truth table to determine if an argument is valid, the premise or premises are listed first on the right side of the truth table. (In the example in §11.1, '$\neg L \rightarrow (J \vee L)$' and '$\neg L$' are the premises.) The conclusion is then put last. (J is the conclusion of that argument.) Once the truth table is completed, we check for *bad* lines.

- Any line where all of the premises are true and the conclusion is true **is a good line**.
- Any line where all of the premises are true and the conclusion is false **is a bad line**.
- Any line where the conclusion is true is **not** a bad line.
- Any line where at least one premise is false is **not** a bad line.

Let's look at the truth table for an argument with one small (but significant) change: $\neg L \rightarrow (J \vee L), \neg L \therefore \neg J$. The premises are the same, but now the conclusion is $\neg J$ instead of J. Here is the truth table:

J	L	$\neg L \rightarrow (J \vee L)$	$\neg L$	$\neg J$
T	T	F T **T** T T T	**F** T	**F**
T	F	T F **T** T T F	**T** F	**F**
F	T	F T **T** F T T	**F** T	**T**
F	F	T F **F** F F F	**T** F	**T**

The truth values for the premises are the same, and the truth values for the conclusion have, on each line, flipped from T to F or vice versa. Now, when we evaluate each line, what do we fine? As before, on lines 1, 3, and 4, one of the premises is false, and so they are not bad lines. On line 2, the premises are true and the conclusion is false. Bad line! That means that $\neg L \rightarrow (J \vee L)$ and $\neg L$ do not entail $\neg J$ and the argument '$\neg L \rightarrow (J \vee L), \neg L \therefore \neg J$' is not valid.

11.3 The double-turnstile

We are going to use the notion of entailment rather a lot in this course. It will help us, then, to introduce a symbol that abbreviates it. Rather than saying that the TFL sentences $\mathsf{A}_1, \mathsf{A}_2, \ldots$ and A_n together entail C, we will abbreviate this by:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vDash \mathsf{C}$$

The symbol '$\vDash$' is known as *the double-turnstile*, since it looks like a turnstile with two horizontal beams.

Let me be clear. '$\vDash$' is not a symbol of TFL. Rather, it is a symbol of our metalanguage, augmented English (recall the difference between object language and metalanguage from §7.2). So the metalanguage sentence:

- $P, P \rightarrow Q \vDash Q$

is just an abbreviation for the English sentence:

- The TFL sentences 'P' and '$P \rightarrow Q$' entail 'Q'

Note that there is no limit on the number of TFL sentences that can be mentioned before the symbol '$\vDash$'. Indeed, we can even consider the limiting case:

$$\vDash \mathsf{C}$$

This says that there is no set of sentences and no valuation for those sentences that makes C false—for example, $\vDash \neg(R \mathbin{\&} \neg R)$ is one such sentence. Otherwise put, every valuation makes C true, which means that C is a tautology. Equally:

$$\mathsf{A} \vDash$$

says that A is a contradiction—for example, $(R \mathbin{\&} \neg R) \vDash$.

11.4 Checking for validity in Carnap

When checking whether an argument is valid in Carnap, the truth table will look like the one shown in figure 11.1. The premise, $(P \mathbin{\&} R)$, is separated from the conclusion, R, by a turnstile ('$\vdash$'). (This symbol represents a different concept than the double turnstile ('$\vDash$'), and chapter 16 explains how they are different. Nonetheless, for both, the premises precede the $\vdash$ or $\vDash$ and the conclusion follows it. And Carnap uses only the turnstile.) When an argument contains more than one premise, the premises will be separated by a comma—for example, like this: $(P \leftrightarrow (Q \lor R)), P \vdash (\neg Q \rightarrow R)$.

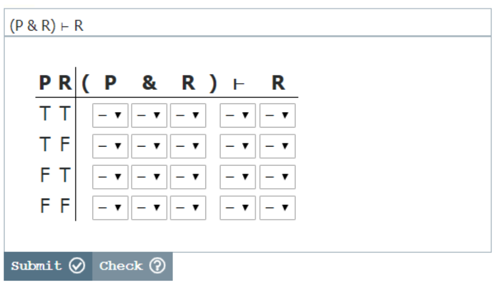


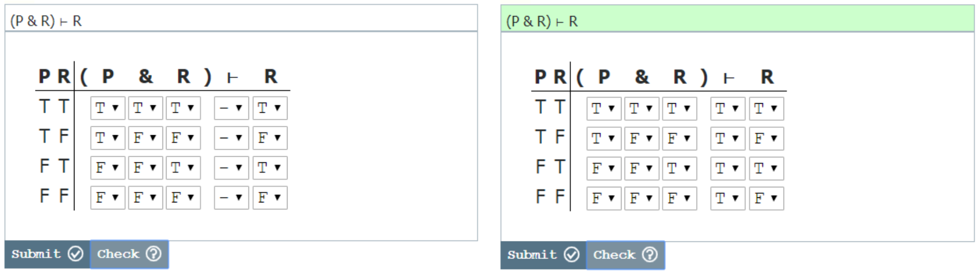
Figure 11.1

The truth values for each atomic sentence and the connectives are filled in normally. But that will leave one column under the '$\vdash$' as shown in figure 11.2a. That column will also be filled in with 'T's and 'F's, but there 'T' and 'F' don't really mean 'true' and 'false'. 'T' means that the line is a good line. 'F' means that the line is a bad line. (See §11.2 for the definition of valid and when lines are good and when they are bad.) If all of the lines are good, then the argument is valid. If one or more of the lines is bad, the argument is invalid.

In this example, all of the lines are good, and so we have all 'T's below the '$\vdash$' (figure 11.2b). To reiterate, each line is a good line and the argument is valid even though, on the third line, the premise is false and the conclusion is true, and on the second and fourth lines, the premise is false and the conclusion is false. The only scenario that makes a line a bad line is when the premise (or premises) are true and the conclusion is false.

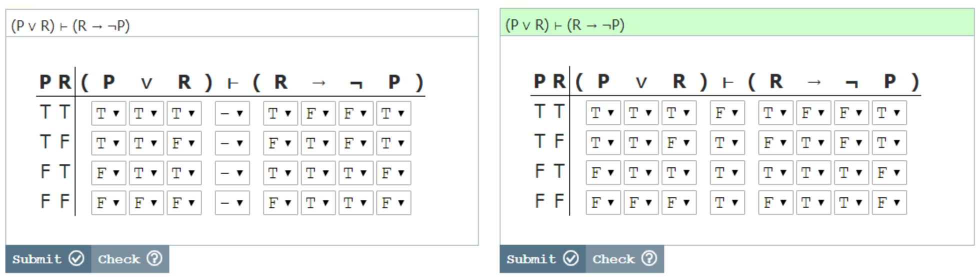
Consider the argument in figure 11.3a: $P \lor R \vdash R \rightarrow \neg P$. Here we do have a line where the premise is true and the conclusion is false. That's a bad line, and so in the column under the '$\vdash$', we put an 'F' there. Even though all of the other lines are fine, because we have one bad line, the argument is not valid.

Once the truth table is completed and has been checked, submit it.



(a) *(b)*

Figure 11.2



(a) *(b)*

Figure 11.3

11.5 '⊨' versus '→'

We now want to compare and contrast '⊨' and '→'.

Observe: $\mathsf{A} \vDash \mathsf{C}$ if and only if there is no valuation of the atomic sentences that makes A true and C false.

Observe: $\mathsf{A} \rightarrow \mathsf{C}$ is a tautology if and only if there is no valuation of the atomic sentences that makes $\mathsf{A} \rightarrow \mathsf{C}$ false. Since a conditional is true except when its antecedent is true and its consequent false, $\mathsf{A} \rightarrow \mathsf{C}$ is a tautology iff there is no valuation that makes A true and C false.

Combining these two observations, we see that $\mathsf{A} \rightarrow \mathsf{C}$ is a tautology iff $\mathsf{A} \vDash \mathsf{C}$. But there is a really, really important difference between '⊨' and '→':

> '→' is a sentential connective of TFL.
> '⊨' is a symbol of augmented English.

Indeed, when '$\rightarrow$' is flanked with two TFL sentences, the result is a longer TFL sentence. By contrast, when we use '$\vDash$', we form a metalinguistic sentence that *mentions* the surrounding TFL sentences.

11.6 The limits of these tests

We have reached an important milestone: a test for the validity of arguments! However, we should not get carried away just yet. It is important to understand the *limits* of our achievement. We will illustrate these limits with three examples.

First, consider the argument:

1. Daisy has four legs. Therefore, Daisy has more than two legs.

To symbolize this argument in TFL, we would have to use two different atomic sentences—perhaps 'F' and 'T'—for the premise and the conclusion respectively. Now, it is obvious that 'F' does not entail 'T'. The English argument surely seems valid, though!

Second, consider the sentence:

2. John is neither bald nor not-bald.

To symbolize this sentence in TFL, we would offer something like '$\neg J \;\&\; \neg\neg J$'. This a contradiction (check this with a truth-table), but sentence 2 does not seem like a contradiction; for we might have happily gone on to add "John is on the borderline of baldness"!

Third, consider the following sentence:

3. It's not the case that, if God exists, he answers malevolent prayers.

Symbolizing this in TFL, we would offer something like '$\neg(G \rightarrow M)$'. Now, '$\neg(G \rightarrow M)$' entails 'G' (again, check this with a truth table). So if we symbolize sentence 3 in TFL, it seems to entail that God exists. But that's strange: surely even an atheist can accept sentence 3, without contradicting herself!

One lesson of this is that the symbolization of 3 as '$\neg(G \rightarrow M)$' shows that 3 does not express what we intend. Perhaps we should rephrase it as

3. If God exists, he does not answer malevolent prayers.

and symbolize 3 as '$G \to \neg M$'. Now, if atheists are right, and there is no God, then 'G' is false and so '$G \to \neg M$' is true, and the puzzle disappears. However, if 'G' is false, then '$G \to M$' (i.e., 'If God exists, he answers malevolent prayers') is *also* true!

In different ways, these four examples highlight some of the limits of working with a language like TFL that can *only* handle truth-functional connectives. Moreover, these limits give rise to some interesting questions in philosophical logic. The case of John's baldness (or otherwise) raises the general question of what logic we should use when dealing with *vague* discourse. The case of the atheist raises the question of how to deal with the (so-called) *paradoxes of the material conditional*. Part of the purpose of this course is to equip you with the tools to explore these questions of *philosophical logic*. But we have to walk before we can run; and so we have to become proficient using TFL, before we can adequately discuss its limits and consider alternatives.

Practice exercises

A. Revisit your answers to the exercises in part A of chapter 9, and determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

B. Use truth tables to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \to A$, $\neg A \to \neg A$, $A \;\&\; A$, $A \vee A$
2. $A \vee B$, $A \to C$, $B \to C$
3. $B \;\&\; (C \vee A)$, $A \to B$, $\neg(B \vee C)$
4. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$

C. Use truth tables to determine whether each argument is valid or invalid.

1. $A \to A \mathrel{\therefore} A$
2. $A \to (A \;\&\; \neg A) \mathrel{\therefore} \neg A$
3. $A \vee (B \to A) \mathrel{\therefore} \neg A \to \neg B$
4. $A \vee B, B \vee C, \neg A \mathrel{\therefore} B \;\&\; C$
5. $(B \;\&\; A) \to C, (C \;\&\; A) \to B \mathrel{\therefore} (C \;\&\; B) \to A$

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence, using a complete truth table.

1. $\neg B \;\&\; B$
2. $\neg D \vee D$
3. $(A \;\&\; B) \vee (B \;\&\; A)$
4. $\neg[A \rightarrow (B \rightarrow A)]$
5. $A \leftrightarrow [A \rightarrow (B \;\&\; \neg B)]$
6. $[(A \;\&\; B) \leftrightarrow B] \rightarrow (A \rightarrow B)$

E. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write "equivalent." Otherwise write, "Not equivalent."

1. A and $\neg A$
2. $A \;\&\; \neg A$ and $\neg B \leftrightarrow B$
3. $[(A \vee B) \vee C]$ and $[A \vee (B \vee C)]$
4. $A \vee (B \;\&\; C)$ and $(A \vee B) \;\&\; (A \vee C)$
5. $[A \;\&\; (A \vee B)] \rightarrow B$ and $A \rightarrow B$

F. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are equivalent, write "equivalent." Otherwise write, "not equivalent."

1. $A \rightarrow A$ and $A \leftrightarrow A$
2. $\neg(A \rightarrow B)$ and $\neg A \rightarrow \neg B$
3. $A \vee B$ and $\neg A \rightarrow B$
4. $(A \rightarrow B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$
5. $A \leftrightarrow (B \leftrightarrow C)$ and $A \;\&\; (B \;\&\; C)$

G. Determine whether each collection of sentences is jointly consistent or jointly inconsistent using a complete truth table.

1. $A \;\&\; \neg B$, $\neg(A \rightarrow B)$, $B \rightarrow A$
2. $A \vee B$, $A \rightarrow \neg A$, $B \rightarrow \neg B$
3. $\neg(\neg A \vee B)$, $A \rightarrow \neg C$, $A \rightarrow (B \rightarrow C)$
4. $A \rightarrow B$, $A \;\&\; \neg B$

5. $A \rightarrow (B \rightarrow C)$, $(A \rightarrow B) \rightarrow C$, $A \rightarrow C$

H. Determine whether each collection of sentences is jointly consistent or jointly inconsistent, using a complete truth table.

1. $\neg B$, $A \rightarrow B$, A
2. $\neg(A \vee B)$, $A \leftrightarrow B$, $B \rightarrow A$
3. $A \vee B$, $\neg B$, $\neg B \rightarrow \neg A$
4. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \rightarrow B$
5. $(A \vee B) \vee C$, $\neg A \vee \neg B$, $\neg C \vee \neg B$

I. Determine whether each argument is valid or invalid, using a complete truth table.

1. $A \rightarrow B$, $B \therefore A$
2. $A \leftrightarrow B$, $B \leftrightarrow C \therefore A \leftrightarrow C$
3. $A \rightarrow B$, $A \rightarrow C \therefore B \rightarrow C$
4. $A \rightarrow B$, $B \rightarrow A \therefore A \leftrightarrow B$

J. Determine whether each argument is valid or invalid, using a complete truth table.

1. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \therefore A$
2. $A \vee B$, $B \vee C$, $\neg B \therefore A \,\&\, C$
3. $A \rightarrow B$, $\neg A \therefore \neg B$
4. A, $B \therefore \neg(A \rightarrow \neg B)$
5. $\neg(A \,\&\, B)$, $A \vee B$, $A \leftrightarrow B \therefore C$

K. Answer each of the questions below and justify your answer.

1. Suppose that A and B are logically equivalent. What can you say about $\mathsf{A} \leftrightarrow \mathsf{B}$?
2. Suppose that $(\mathsf{A} \,\&\, \mathsf{B}) \rightarrow \mathsf{C}$ is neither a tautology nor a contradiction. What can you say about whether $\mathsf{A}, \mathsf{B} \therefore \mathsf{C}$ is valid?
3. Suppose that A, B and C are jointly inconsistent. What can you say about $(\mathsf{A} \,\&\, \mathsf{B} \,\&\, \mathsf{C})$?
4. Suppose that A is a contradiction. What can you say about whether $\mathsf{A}, \mathsf{B} \vDash \mathsf{C}$?

5. Suppose that C is a tautology. What can you say about whether $A, B \vDash C$?
6. Suppose that A and B are logically equivalent. What can you say about $(A \lor B)$?
7. Suppose that A and B are *not* logically equivalent. What can you say about $(A \lor B)$?

L. Consider the following principle:

- Suppose A and B are logically equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

Is this principle correct? Explain your answer.

Chapter 12

Truth table shortcuts

With practice, you will quickly become adept at filling out truth tables. In this section, we want to give you some permissible shortcuts to help you along the way.

12.1 Working through truth tables

You will quickly find that you do not need to copy the truth value of each atomic sentence, but can simply refer back to them. So you can speed things up by writing:

P	Q	$(P \vee Q) \leftrightarrow \neg P$
T	T	T **F** F
T	F	T **F** F
F	T	T **T** T
F	F	F **F** T

You also know for sure that a disjunction is true whenever one of the disjuncts is true. So if you find a true disjunct, there is no need to work out the truth values of the other disjuncts. Thus you might offer:

P	Q	$(\neg P \vee \neg Q) \vee \neg P$
T	T	F F F **F** F
T	F	F T T **T** F
F	T	**T** T
F	F	**T** T

Equally, you know for sure that a conjunction is false whenever one of the conjuncts is false. So if you find a false conjunct, there is no need to work out the truth value of the other conjunct. Thus you might offer:

P	Q	$\neg$	$(P$	$\&$	$\neg$	$Q)$	$\&$	$\neg$	P
T	T						**F**	F	
T	F						**F**	F	
F	T	T		F			**T**	T	
F	F	T		F			**T**	T	

A similar short cut is available for conditionals. You immediately know that a conditional is true if either its consequent is true, or its antecedent is false. Thus you might present:

P	Q	$((P$	$\rightarrow$	$Q)$	$\rightarrow$	$P)$	$\rightarrow$	P
T	T						**T**	
T	F						**T**	
F	T		T		F		**T**	
F	F		T		F		**T**	

So '$((P \rightarrow Q) \rightarrow P) \rightarrow P$' is a tautology. In fact, it is an instance of *Peirce's Law*, named after Charles Sanders Peirce.

12.2 Testing for validity and entailment

As we said in §11.2, when we use truth tables to test for validity or entailment, we are checking for *bad* lines: lines where the premises are all true and the conclusion is false. Consequently,

- Any line where the conclusion is true is not a bad line.
- Any line where some premise is false is not a bad line.

Since *all* we are doing is looking for bad lines, if we find a line where the conclusion is true, we do not need to evaluate anything else on that line. That line definitely isn't bad. Likewise, if we find a line where some premise is false, we do not need to evaluate anything else on that line.

With this in mind, consider how we might test the following for validity:

$$\neg L \rightarrow (J \vee L), \neg L \therefore J$$

The *first* thing we should do is evaluate the conclusion. If we find that the conclusion is *true* on some line, then that is not a bad line, and so we can simply ignore the rest of the line. At our first stage, we are left with something like this:

J	L	$\neg L \rightarrow (J \vee L)$	$\neg L$	J
T	T			T
T	F			T
F	T	?	?	F
F	F	?	?	F

The blank spaces under $\neg L \rightarrow (J \vee L)$ and $\neg L$ indicate that we are not going to bother doing any more investigation (since the line is not bad). The question-marks indicate that we need to keep investigating. On those lines, it is possible that the premises are true and the conclusion is false.

The easiest premise to evaluate is the second ($\neg L$), so we do that next:

J	L	$\neg L \rightarrow (J \vee L)$	$\neg L$	J
T	T			T
T	F			T
F	T		F	F
F	F	?	T	F

Now we see that we no longer need to consider the third line: it will not be a bad line, because (at least) one of the premises is false on that line, $\neg L$. Finally, we complete the fourth line:

J	L	$\neg L \rightarrow (J \vee L)$	$\neg L$	J
T	T			T
T	F			T
F	T		F	F
F	F	T **F** F	T	F

Since the fourth line tells us that—for those valuations of J and L—the first premise is false, the truth table has no bad lines. Hence, the argument is valid: any valuation for which all the premises are true is a valuation for which the conclusion is true.

It might be worth illustrating the tactic again. Let us check whether the following argument is valid

$$A \vee B, \neg(A \,\&\, C), \neg(B \,\&\, \neg D) \therefore (\neg C \vee D)$$

At the first stage, we determine the truth value of the conclusion. Since this is a disjunction, it is true whenever either disjunct is true, so we can speed things along a bit. We can then ignore every line apart from the few lines where the conclusion is false. (Notice that the negation in the conclusion is determined for just those lines were D is false.)

A	B	C	D	$A \vee B$	$\neg(A \,\&\, C)$	$\neg(B \,\&\, \neg D)$	$(\neg C$	$\vee$	$D)$
T	T	T	T					**T**	
T	T	T	F	?	?	?	F	**F**	
T	T	F	T					**T**	
T	T	F	F				T	**T**	
T	F	T	T					**T**	
T	F	T	F	?	?	?	F	**F**	
T	F	F	T					**T**	
T	F	F	F				T	**T**	
F	T	T	T					**T**	
F	T	T	F	?	?	?	F	**F**	
F	T	F	T					**T**	
F	T	F	F				T	**T**	
F	F	T	T					**T**	
F	F	T	F	?	?	?	F	**F**	
F	F	F	T					**T**	
F	F	F	F				T	**T**	

We must now evaluate the premises. The first premise is the simplest, so we start there. Of the four lines where the conclusion is false, there are three where $A \vee B$ is true. So the truth values for the next premise have to be determined for those three lines. (The second premise is simpler to evaluate than the third, so it's next.) Knowing the truth value for $\neg(A \,\&\, C)$ leaves us with one line where the first two premise are true. A little bit more work tells us that the third premise is false on that line. There is no line where the premises are true and the conclusion is false! The argument is valid.

A	B	C	D	$A \vee B$	$\neg(A$	$\&\ C)$	$\neg(B$	$\&$	$\neg D)$	$(\neg C$	$\vee D)$
T	T	T	T								**T**
T	T	T	F	**T**	**F**	T				F	**F**
T	T	F	T								**T**
T	T	F	F							T	**T**
T	F	T	T								**T**
T	F	T	F	**T**	**F**	T				F	**F**
T	F	F	T								**T**
T	F	F	F							T	**T**
F	T	T	T								**T**
F	T	T	F	**T**	**T**	F	**F**	T	T	F	**F**
F	T	F	T								**T**
F	T	F	F							T	**T**
F	F	T	T								**T**
F	F	T	F	**F**						F	**F**
F	F	F	T								**T**
F	F	F	F							T	**T**

If we had used no shortcuts, we would have had to write 256 'T's or 'F's on this table. Using shortcuts, we only had to write 37. We have saved ourselves a *lot* of work.

We have been discussing shortcuts in testing for logically validity, but exactly the same shortcuts can be used in testing for entailment. By employing a similar notion of *bad* lines, you can save yourself a huge amount of work.

Practice exercises

A. Using shortcuts, determine whether each sentence is a tautology, a contradiction, or neither.

1. $\neg B \,\&\, B$
2. $\neg D \vee D$
3. $(A \,\&\, B) \vee (B \,\&\, A)$
4. $\neg[A \rightarrow (B \rightarrow A)]$
5. $A \leftrightarrow [A \rightarrow (B \,\&\, \neg B)]$
6. $\neg(A \,\&\, B) \leftrightarrow A$
7. $A \rightarrow (B \vee C)$

8. $(A \,\&\, \neg A) \rightarrow (B \vee C)$
9. $(B \,\&\, D) \leftrightarrow [A \leftrightarrow (A \vee C)]$

Chapter 13

Partial truth tables

Sometimes, we do not need to know what happens on every line of a truth table. Sometimes, just a line or two will do.

Tautology. In order to show that a sentence is a tautology, we need to show that it is true on every valuation. That is to say, we need to know that it comes out true on every line of the truth table. So we need a complete truth table.

To show that a sentence is *not* a tautology, however, we only need one line: a line on which the sentence is false. Therefore, in order to show that some sentence is not a tautology, it is enough to provide a single valuation—a single line of the truth table—which makes the sentence false.

Suppose that we want to show that the sentence '$(U \& T) \rightarrow (S \& W)$' is *not* a tautology. We set up a PARTIAL TRUTH TABLE:

S	T	U	W	$(U$	$\&$	$T)$	$\rightarrow$	$(S$	$\&$	$W)$
							F			

We have only left space for one line, rather than 16, since we are only looking for one line on which the sentence is false. For just that reason, we have filled in 'F' for the entire sentence.

The main logical operator of the sentence is a conditional. In order for the conditional to be false, the antecedent must be true and the consequent must be false. So we fill these in on the table:

S	T	U	W	$(U$	$\&$	$T)$	$\rightarrow$	$(S$	$\&$	$W)$
					T		**F**		F	

In order for the '$(U \ \& \ T)$' to be true, both 'U' and 'T' must be true.

S	T	U	W	$(U$	&	$T)$	$\rightarrow$	$(S$	&	$W)$
	T	T		T	T	T	**F**		F	

Now we just need to make '$(S \ \& \ W)$' false. To do this, we need to make at least one of 'S' and 'W' false. We can make both 'S' and 'W' false if we want. All that matters is that the whole sentence turns out false on this line. Making an arbitrary decision, we finish the table in this way:

S	T	U	W	$(U$	&	$T)$	$\rightarrow$	$(S$	&	$W)$
F	T	T	F	T	T	T	**F**	F	F	F

We now have a partial truth table, which shows that '$(U \ \& \ T) \rightarrow (S \ \& \ W)$' is not a tautology. Put otherwise, we have shown that there is a valuation which makes '$(U \ \& \ T) \rightarrow (S \ \& \ W)$' false, namely, the valuation which makes 'S' false, 'T' true, 'U' true and 'W' false.

Contradiction. Showing that something is a contradiction requires a complete truth table: we need to show that there is no valuation which makes the sentence true; that is, we need to show that the sentence is false on every line of the truth table.

However, to show that something is *not* a contradiction, all we need to do is find a valuation which makes the sentence true, and a single line of a truth table will suffice. We can illustrate this with the same example.

S	T	U	W	$(U$	&	$T)$	$\rightarrow$	$(S$	&	$W)$
							T			

To make the sentence true, it will suffice to ensure that the antecedent is false. Since the antecedent is a conjunction, we can just make one of them false. For no particular reason, we choose to make 'U' false; and then we can assign whatever truth value we like to the other atomic sentences.

S	T	U	W	$(U$	&	$T)$	$\rightarrow$	$(S$	&	$W)$
F	T	F	F	F	F	T	**T**	F	F	F

Truth functional equivalence. To show that two sentences are logically equivalent, we must show that the sentences have the same truth value on every valuation. So this requires a complete truth table.

To show that two sentences are *not* logically equivalent, we only need to show that there is a valuation on which they have different truth values. So this requires only a one-line partial truth table: make the table so that one sentence is true and the other false.

Consistency. To show that some sentences are jointly consistent, we must show that there is a valuation which makes all of the sentences true,so this requires only a partial truth table with a single line.

To show that some sentences are jointly inconsistent, we must show that there is no valuation which makes all of the sentence true. So this requires a complete truth table: You must show that on every row of the table at least one of the sentences is false.

Validity. To show that an argument is valid, we must show that there is no valuation which makes all of the premises true and the conclusion false. So this requires a complete truth table. (Likewise for entailment.)

To show that argument is *invalid*, we must show that there is a valuation which makes all of the premises true and the conclusion false. So this requires only a one-line partial truth table on which all of the premises are true and the conclusion is false. (Likewise for a failure of entailment.)

This table summarises what is required:

	Yes	**No**
tautology?	complete	one-line partial
contradiction?	complete	one-line partial
equivalent?	complete	one-line partial
consistent?	one-line partial	complete
valid?	complete	one-line partial
entailment?	complete	one-line partial

Practice exercises

A. Use complete or partial truth tables (as appropriate) to determine whether these pairs of sentences are logically equivalent:

1. A, $\neg A$
2. A, $A \vee A$
3. $A \rightarrow A$, $A \leftrightarrow A$
4. $A \vee \neg B$, $A \rightarrow B$
5. $A \,\&\, \neg A$, $\neg B \leftrightarrow B$
6. $\neg(A \,\&\, B)$, $\neg A \vee \neg B$
7. $\neg(A \rightarrow B)$, $\neg A \rightarrow \neg B$
8. $(A \rightarrow B)$, $(\neg B \rightarrow \neg A)$

B. Use complete or partial truth tables (as appropriate) to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \,\&\, B$, $C \rightarrow \neg B$, C
2. $A \rightarrow B$, $B \rightarrow C$, A, $\neg C$
3. $A \vee B$, $B \vee C$, $C \rightarrow \neg A$
4. A, B, C, $\neg D$, $\neg E$, F
5. $A \,\&\, (B \vee C)$, $\neg(A \,\&\, C)$, $\neg(B \,\&\, C)$
6. $A \rightarrow B$, $B \rightarrow C$, $\neg(A \rightarrow C)$

C. Use complete or partial truth tables (as appropriate) to determine whether each argument is valid or invalid:

1. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \therefore A$
2. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$
3. $A \rightarrow B, B \therefore A$
4. $A \vee B, B \vee C, \neg B \therefore A \,\&\, C$
5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $A \rightarrow \neg A$
2. $A \rightarrow (A \,\&\, (A \vee B))$

3. $(A \rightarrow B) \leftrightarrow (B \rightarrow A)$
4. $A \rightarrow \neg(A \,\&\, (A \vee B))$
5. $\neg B \rightarrow [(\neg A \,\&\, A) \vee B]$
6. $\neg(A \vee B) \leftrightarrow (\neg A \,\&\, \neg B)$
7. $[(A \,\&\, B) \,\&\, C] \rightarrow B$
8. $\neg\big[(C \vee A) \vee B\big]$
9. $\big[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)\big] \,\&\, C$
10. $(A \,\&\, B)] \rightarrow [(A \,\&\, C) \vee (B \,\&\, D)]$

E. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $\neg(A \vee A)$
2. $(A \rightarrow B) \vee (B \rightarrow A)$
3. $[(A \rightarrow B) \rightarrow A] \rightarrow A$
4. $\neg[(A \rightarrow B) \vee (B \rightarrow A)]$
5. $(A \,\&\, B) \vee (A \vee B)$
6. $\neg(A \,\&\, B) \leftrightarrow A$
7. $A \rightarrow (B \vee C)$
8. $(A \,\&\, \neg A) \rightarrow (B \vee C)$
9. $(B \,\&\, D) \leftrightarrow [A \leftrightarrow (A \vee C)]$
10. $\neg[(A \rightarrow B) \vee (C \rightarrow D)]$

F. Determine whether each the following pairs of sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write "equivalent." Otherwise write, "not equivalent."

1. A and $A \vee A$
2. A and $A \,\&\, A$
3. $A \vee \neg B$ and $A \rightarrow B$
4. $(A \rightarrow B)$ and $(\neg B \rightarrow \neg A)$
5. $\neg(A \,\&\, B)$ and $\neg A \vee \neg B$
6. $((U \rightarrow (X \vee X)) \vee U)$ and $\neg(X \,\&\, (X \,\&\, U))$
7. $((C \,\&\, (N \leftrightarrow C)) \leftrightarrow C)$ and $(\neg\neg\neg N \rightarrow C)$

8. $[(A \vee B) \,\&\, C]$ and $[A \vee (B \,\&\, C)]$
9. $((L \,\&\, C) \,\&\, I)$ and $L \vee C$

G. Determine whether each collection of sentences is jointly consistent or jointly inconsistent. Justify your answer with a complete or partial truth table where appropriate.

1. $A \to A$, $\neg A \to \neg A$, $A \,\&\, A$, $A \vee A$
2. $A \to \neg A$, $\neg A \to A$
3. $A \vee B$, $A \to C$, $B \to C$
4. $A \vee B$, $A \to C$, $B \to C$, $\neg C$
5. $B \,\&\, (C \vee A)$, $A \to B$, $\neg(B \vee C)$
6. $(A \leftrightarrow B) \to B$, $B \to \neg(A \leftrightarrow B)$, $A \vee B$
7. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$
8. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \to B$
9. $A \leftrightarrow B$, $A \to C$, $B \to D$, $\neg(C \vee D)$
10. $\neg(A \,\&\, \neg B)$, $B \to \neg A$, $\neg B$

H. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1. $A \to (A \,\&\, \neg A) \therefore \neg A$
2. $A \vee B$, $A \to B$, $B \to A \therefore A \leftrightarrow B$
3. $A \vee (B \to A) \therefore \neg A \to \neg B$
4. $A \vee B$, $A \to B$, $B \to A \therefore A \,\&\, B$
5. $(B \,\&\, A) \to C$, $(C \,\&\, A) \to B \therefore (C \,\&\, B) \to A$
6. $\neg(\neg A \vee \neg B)$, $A \to \neg C \therefore A \to (B \to C)$
7. $A \,\&\, (B \to C)$, $\neg C \,\&\, (\neg B \to \neg A) \therefore C \,\&\, \neg C$
8. $A \,\&\, B$, $\neg A \to \neg C$, $B \to \neg D \therefore A \vee B$
9. $A \to B \therefore (A \,\&\, B) \vee (\neg A \,\&\, \neg B)$
10. $\neg A \to B, \neg B \to C, \neg C \to A \therefore \neg A \to (\neg B \vee \neg C)$

I. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$
2. $A \vee B$, $B \vee C$, $\neg A \therefore B \,\&\, C$
3. $A \to C$, $E \to (D \vee B)$, $B \to \neg D \therefore (A \vee C) \vee (B \to (E \,\&\, D))$
4. $A \vee B$, $C \to A$, $C \to B \therefore A \to (B \to C)$

5. $A \rightarrow B, \neg B \vee A \therefore A \leftrightarrow B$

Part IV

Natural deduction for TFL

Chapter 14

The very idea of natural deduction

In §2.2, we said that an argument is valid if and only if it is impossible to make all of the premises true and the conclusion false.

In the case of TFL, this led us to develop truth tables. Each line of a complete truth table corresponds to a valuation. So, given an argument in TFL, we have a very direct way to assess whether it is possible to make all of the premises true and the conclusion false: just investigate the truth table.

However, truth tables do not necessarily give us much *insight*. Consider this argument:

$$(P \,\&\, Q) \vee R, \neg R \therefore Q$$

This is a valid argument, and you can confirm that it is by constructing a four-line truth table. But we might want to know *why* it is valid—that is, why (or how) the conclusion follows from the premises.

One aim of a *natural deduction system* is to show that particular arguments are valid and why they are valid. That is to say, the system allows us to make explicit the reasoning process that get us from the premises to the conclusion. We begin with very basic rules of inference. These rules can be combined, and with just a small number of them, we hope to be able to explicate all of the valid arguments that can be represented in TFL.

This is a different way of thinking about arguments. With truth tables, we directly consider different scenarios where the atomic sentences are true or false and see what that means for the premises and conclusion. With natural deduction systems, we manipulate the sentences in accordance with rules that we have set down. This gives us a better insight—or at least, a different insight—into how arguments work.

The move to natural deduction might be motivated by more than the search for insight. It might also be motivated by necessity. Take, for instance, this argument:

$$A \mathbin{\&} B \mathrel{\therefore} (A \lor C) \mathbin{\&} (B \lor D)$$

To test this argument for validity with a truth table, you need 16 lines. If you do it correctly, then you will see that there is no line on which all the premises are true and on which the conclusion is false. So you will know that the argument is valid. (But, as just mentioned, there is a sense in which you will not know *why* the argument is valid.) On the other hand, using our natural deduction system, you can demonstrate that this argument is valid in six lines. (And after reading §15.2 and §15.3, you'll be able to do it easily.)

When an argument contains more letters, it gets even more difficult to use truth tables (since the number of lines needed is 2^n where n = the number of letters). In principle, we can set a computer to grind through truth tables and report back when it is finished. But, in practice, complicated arguments in TFL can become *intractable* if we use truth tables.

◊ ◊ ◊

The modern development of natural deduction dates from simultaneous and unrelated papers by Gerhard Gentzen and Stanisław Jaśkowski (1934). However, the natural deduction system that we shall consider is based largely around work by Frederic Fitch (first published in 1952).

CHAPTER 15

BASIC RULES FOR TFL

15.1 Proofs

The purpose of a *proof system* is to show that particular arguments are valid in a way that allows us to understand the reasoning involved in the argument. We begin with seven basic rules. These rules can then be combined to demonstrate each step that must be taken to get from the premises to the conclusion.

A PROOF is a sequence of sentences. The first sentences of the sequence are assumptions; these are the premises of the argument. Every sentence later in the sequence follows from earlier sentences by one of the rules. The final sentence of the sequence is the conclusion of the argument.

As an illustration, consider:

$$\neg(A \vee B) \therefore \neg A \,\&\, \neg B$$

We will start a proof by writing the premise:

$$\begin{array}{r|ll} 1 & \underline{\neg(A \vee B)} & \text{:PR} \end{array}$$

Notice that we have numbered the premise at the beginning of the line, since we will want to refer back to it. Indeed, every line on a proof is numbered, so that we can refer back to it. We have also indicated that this is a premise by putting 'PR' at the end of the line. And finally, we have drawn a line underneath the premise. Everything written above the line is an *initial assumption* (i.e., a premise). Everything written below the line will either be something that follows from that assumption, or it will be some new assumption. We are hoping to conclude that '$\neg A \,\&\, \neg B$'; so we are hoping ultimately to conclude our proof with

n	$\neg A \,\&\, \neg B$

for some number n. It doesn't matter what line number we end on, but we would obviously prefer a short proof to a long one.

Similarly, suppose we wanted to consider:

$$A \vee B, \neg(A \,\&\, C), \neg(B \,\&\, \neg D) \therefore \neg C \vee D$$

The argument has three premises, so we start by writing them all down, numbering each line, and drawing a line under the final premise:

1	$A \vee B$	:PR
2	$\neg(A \,\&\, C)$	:PR
3	$\neg(B \,\&\, \neg D)$	:PR

We are hoping to conclude with this line:

n	$\neg C \vee D$

and so all that remains to do is to explain each of the rules that we can use to get from the premises to the conclusion.

To construct proofs, we will develop what is called a NATURAL DEDUCTION system. In the natural deduction system, there are two rules for each logical operator: an INTRODUCTION rule, which allow us to prove a new sentence that has the logical operator as the main connective, and an ELIMINATION rule, which allows us to extract a subsentence from a sentence with the logical operator as the main connective.

All of the rules introduced in this chapter are summarized starting on p. 135.

15.2 Conjunction

Suppose we want to show that Sarah is both a nurse and a mother. One obvious way to do this would be as follows: first we show that Sarah is a nurse; then we show that Sarah is a mother; then we put these two demonstrations together to obtain the conjunction.

Our natural deduction system will capture this thought straightforwardly. In the example given, we might adopt the following symbolization key:

N: Sarah is a nurse
M: Sarah is a mother

Perhaps we are working through a proof, and we have obtained 'N' on line 8 and 'M' on line 15. Then on any subsequent line—say, line 17—we can get 'N & M' by using the & I rule:

8	N	...
15	M	...
17	N & M	& I 8, 15

Note that every line of our proof must either be an assumption, or it must be justified by some rule. We cite ' & I 8, 15' here to indicate that the line is obtained by the CONJUNCTION INTRODUCTION RULE (& I) applied to lines 8 and 15. We could equally well obtain:

8	N	
15	M	
17	M & N	& I 15, 8

with the citation reversed, to reflect the order of the conjuncts. More generally, here is our conjunction introduction rule:

m	A	
n	B	
	A & B	& I m, n

To be clear, the statement of the rule is *schematic*. It is not itself a proof. 'A' and 'B' are not sentences of TFL. Rather, they are symbols in the metalanguage, which we use when we want to talk about any sentence of TFL (see §7). Similarly, 'm' and 'n' are not a numerals that will appear on any actual proof. Rather, they are symbols in the metalanguage, which we use when we want to talk about any line number of any proof. In an actual proof, the lines are numbered '1', '2', '3', and so forth, but when we define the rule, we use variables to emphasize that the rule may be applied at any point. The rule requires only that we have both conjuncts available to us somewhere in the proof. They can be separated from one another, and they can appear in any order.

The rule is called 'conjunction *introduction*' because it introduces the symbol ' & ' into our proof where it may have been absent. Correspondingly, we have a rule that *eliminates* that symbol. Suppose you have shown that Sarah is both a nurse and a mother. You are entitled to conclude that Sarah is a nurse. Equally, you are entitled to conclude that Sarah is a mother. Putting this together, we obtain our CONJUNCTION ELIMINATION RULE (which is actually two similar rules):

m	A & B	
	A	& E m

and equally:

m	A & B	
	B	& E m

The point is simply that, when you have a conjunction on some line of a proof, you can obtain either of the conjuncts by & E. (But one point is worth emphasizing: you can only apply this rule when conjunction is the main logical operator. So you cannot infer 'D' straight from '$C \vee (D \,\&\, E)$'!)

Even with just this one rule, we can start to see how our formal proof system works. Consider:

$$(A \vee B) \,\&\, [(E \vee F) \,\&\, (G \rightarrow H)]$$
$$\therefore\ [(A \vee B) \,\&\, (G \rightarrow H)]$$

The main logical operator in both the premise and conclusion of this argument is ' & '.

In order to provide a proof, we begin by writing down the premise, which is our assumption. We draw a line below this: everything after this line must follow from our assumptions by the application of our rules of inference. So the beginning of the proof looks like this:

1	$(A \vee B)\ \&\ [(E \vee F)\ \&\ (G \to H)]$	PR

From the premise, we can eliminate the main connective only using & E. Using & E twice gives us this:

1	$(A \vee B)\ \&\ [(E \vee F)\ \&\ (G \to H)]$	PR
2	$(A \vee B)$	& E 1
3	$[(E \vee F)\ \&\ (G \to H)]$	& E 1

Now that $[(E \vee F)\ \&\ (G \to H)]$ is on its own line, we can use & E again to get $(G \to H)$

1	$(A \vee B)\ \&\ [(E \vee F)\ \&\ (G \to H)]$	PR
2	$(A \vee B)$	& E 1
3	$[(E \vee F)\ \&\ (G \to H)]$	& E 1
4	$(G \to H)$	& E 3

Our final step requires & I to get the conclusion, $[(A \vee B)\ \&\ (G \to H)]$.

1	$(A \vee B)\ \&\ [(E \vee F)\ \&\ (G \to H)]$	PR
2	$(A \vee B)$	& E 1
3	$[(E \vee F)\ \&\ (G \to H)]$	& E 1
4	$(G \to H)$	& E 3
5	$[(A \vee B)\ \&\ (G \to H)]$	& I 2, 4

And we're done. Notice that there is nothing in this representation of the proof to indicate that the last line is the conclusion. It's only because we began with

$$\begin{array}{ll} & (A \vee B)\ \&\ [(E \vee F)\ \&\ (G \to H)] \\ \therefore & [(A \vee B)\ \&\ (G \to H)] \end{array}$$

that we know that we arrived at the conclusion that we wanted.

That was a very simple proof, but it shows how we can combine the rules to complete a proof. (As a side note, investigating this argument with a truth table would have required 64 lines; our formal proof required only five lines.)

15.3 Disjunction

Suppose that Sarah is a nurse. Then 'Sarah is a nurse or a runner' is true. After all, to say that 'Sarah is a nurse or a runner' is to say something weaker than to say that 'Sarah is a nurse'.

Let's emphasize this point. Suppose Sarah is a nurse. It follows that Sarah is *either* a nurse *or* a pineapple. Equally, it follows that *either* Sarah is a nurse *or* the Queen of England is on the moon. These are strange inferences to draw from 'Sarah is a nurse', but there is nothing logically wrong with them. (They may violate some implicit conversational norms, but they don't violate the truth conditions for *or*. Just check the characteristic truth table for the disjunction.)

Knowing all this, we have the DISJUNCTION INTRODUCTION RULE (which, again, is two similar rules):

m	A	
	$\mathsf{A} \vee \mathsf{B}$	$\vee$I m

m	A	
	$\mathsf{B} \vee \mathsf{A}$	$\vee$I m

B can be *any* sentence whatsoever. The only line that we need is the one containing A. Hence, the following is a perfectly acceptable proof:

1	M	PR
2	$M \vee [(A \leftrightarrow B) \rightarrow (C \,\&\, D)]$	$\vee$I 1

The DISJUNCTION ELIMINATION RULE, on the other hand, requires citing more than one line. Consider, what can you conclude from 'Amy is a chef or a rock climber'? You cannot conclude that Amy is a chef. 'Amy is a chef or a rock climber' might be true because 'Amy is a chef' is true, but it might be true because only 'Amy is a rock climber' is true. Or, since this is the inclusive-or, it's also possible that 'Amy is a chef' and 'Amy is a rock climber' are both true. The problem is that we don't know anything except that Amy is a chef or a rock climber.

To make an inference from 'Amy is a chef or a rock climber', we also need the denial of one of the disjunctions—for instance, 'Amy is not a chef'. If we know that 'Amy is a chef or a rock climber' and that 'Amy is not a chef', then we can safely conclude that 'Amy is a rock climber'. That is an application of the disjunction elimination rule (∨E), which is written this way:

$$\begin{array}{l|ll} m & \mathsf{A} \lor \mathsf{B} & \\ n & \neg\mathsf{B} & \\ & \mathsf{A} & \lor\text{E } m, n \end{array}$$

$$\begin{array}{l|ll} m & \mathsf{A} \lor \mathsf{B} & \\ n & \neg\mathsf{A} & \\ & \mathsf{B} & \lor\text{E } m, n \end{array}$$

15.4 Conditional elimination

For the conditional, we will begin with the elimination rule since it is a little bit simpler than the conditional introduction rule. Consider the following argument:

If Jane is smart, then she is fast.
Jane is smart.
∴ Jane is fast.

In this argument—which is valid—we have a conditional and then, on a separate line, the antecedent of that conditional ('Jane is smart'). That allows us to infer the antecedent ('Jane is fast'), and when we do so, we are using the conditional elimination rule (→E).

m	$\mathsf{A} \to \mathsf{B}$	
n	A	
	B	$\to$E m, n

This rule is also sometimes called *modus ponens*. When we use the rule, the conditional and the antecedent can be separated from one another, and they can appear in any order. However, in the citation for $\to$E, we always cite the conditional first, followed by the antecedent.

15.5 Conditional introduction

The CONDITIONAL INTRODUCTION RULE is a little bit more complicated than the conditional elimination rule, but, with some thought, it's easily understood. We'll start with this short argument:

1. The green van is next to the building.
2. Therefore, if today is Tuesday, then the green van is next to the building *and* today is Tuesday.

Maybe you can see that this argument is valid. (That is, you can see that if the premise is true, then the conclusion has to be true.) But if you can't right now, that's ok. Let's move to the natural deduction format and fill in the steps that take us from that premise to the conclusion. We start by listing the premise.

1	G	PR

The next thing that we need to do is to make an *additional* assumption: 'today is Tuesday'. (We might say that we're making this assumption "for the sake of argument" or to see where it leads). To indicate that we are no longer dealing *merely* with our initial assumption (the premise 'G'), but with some additional assumption, we continue our proof as follows:

1	G		PR
2		T	AS

We are *not* claiming, on line 2, to have proved 'L' from line 1, so we do not need to write in any justification for the additional assumption on line 2. The 'AS' just indicates that 'T' is an assumption. We also, however, draw a line under it (to indicate that it is an assumption) and indent it with a further vertical line (to indicate that it is additional).

With this extra assumption in place, we can use & I:

$$
\begin{array}{r|l l}
1 & G & \text{PR} \\
\hline
2 & \quad | \; T & \text{AS} \\
3 & \quad | \; G \,\&\, T & \&\text{I } 1, 2
\end{array}
$$

So we have now shown that, on the additional assumption, 'G', we can obtain 'G & T'. We can therefore conclude that, 'if T' is the case , then so is G & T'. Or, more briefly: '$T \rightarrow (G \,\&\, T)$':

$$
\begin{array}{r|l l}
1 & G & \text{PR} \\
\hline
2 & \quad | \; T & \text{AS} \\
3 & \quad | \; G \,\&\, T & \&\text{I } 1, 2 \\
4 & T \rightarrow (G \,\&\, T) & \rightarrow\text{I } 2\text{–}3
\end{array}
$$

For this final step, we have dropped back to the original vertical line. When we introduce the conditional, we *discharged* the assumption that we made ('T'), which will always be the antecedent of the conditional.

Although this is a little more involved than the conjunction introduction or disjunction introduction rules, what we are doing here is straightforward. First, we make an assumption, A. From that assumption, we prove B. Once we've done that, we know that *if* A, then B, and we have our conditional.

$$
\begin{array}{r|l l}
i & \quad | \; \mathsf{A} & \\
j & \quad | \; \mathsf{B} & \\
 & \mathsf{A} \rightarrow \mathsf{B} & \rightarrow\text{I } i\text{–}j
\end{array}
$$

There can be as many or as few lines as needed between lines i and j.

Lines i through j are called a SUBPROOF, and once a subproof has been discharged, none of the lines in the subproof can be used again. The conditional $\mathsf{A} \to \mathsf{B}$ can be used later in the proof (if it's needed) because it is outside of the subproof. But nothing from lines i through j can be used. Moreover, a proof is not complete until every assumption that has been introduced is discharged. That is to say, every subproof must be closed by the application of the →I rule or the ¬I or ¬E rules. Thus, we stipulate:

> To cite individual lines when applying a rule, those lines must (1) come before the application of the rule, but (2) not occur within a closed subproof.
> A proof is not complete until every additional assumption (not counting the premises) is discharged.

Let's go through a second example. Suppose we want a proof of this argument:

$$P \to Q, Q \to R \therefore P \to R$$

We start by listing both of our premises. Next, since we want $(P \to R)$, we assume the antecedent of that conditional.

$$
\begin{array}{rlll}
1 & \mid & P \to Q & \text{PR} \\
2 & \mid & \underline{Q \to R} & \text{PR} \\
3 & \mid & \quad \mid \underline{P} & \text{AS}
\end{array}
$$

Now, even though it is an assumption that we've introduced, since 'P' is on a line by itself (and the subproof has not yet been closed), we can use it for our next step. With 'P', we can use →E on the first premise. This gives us 'Q'.

$$
\begin{array}{rlll}
1 & \mid & P \to Q & \text{PR} \\
2 & \mid & \underline{Q \to R} & \text{PR} \\
3 & \mid & \quad \mid \underline{P} & \text{AS} \\
4 & \mid & \quad \mid Q & \to\text{E } 1, 3
\end{array}
$$

With the Q on line 4, we can use →E on the second premise, and that gives us R So, by assuming 'P', we were able to prove 'R'. Last, we apply the →I rule, which discharges 'P' and completes the proof:

$$
\begin{array}{lll}
1 & P \to Q & \text{PR} \\
2 & Q \to R & \text{PR} \\
3 & \quad P & \text{AS} \\
4 & \quad Q & \to\text{E } 1, 3 \\
5 & \quad R & \to\text{E } 2, 4 \\
6 & P \to R & \to\text{I } 3\text{–}5
\end{array}
$$

One more example. Consider how you would prove: $F \to (G \,\&\, H) \therefore F \to G$. Perhaps it is tempting to write down the premise and then apply the & E rule to the conjunction $(G \,\&\, H)$. This is not allowed, however. The rules of proof can only be applied to the main connective of a sentence. (That's the '→' in this sentence, not the ' & '.) To use & E, we need to get $(G \,\&\, H)$ on a line by itself, and so we proceed this way:

$$
\begin{array}{lll}
1 & F \to (G \,\&\, H) & \text{PR} \\
2 & \quad F & \text{AS} \\
3 & \quad G \,\&\, H & \to\text{E } 1, 2 \\
4 & \quad G & \&\text{E } 3 \\
5 & F \to G & \to\text{I } 2\text{–}4
\end{array}
$$

15.6 Biconditional

We said in chapter 8 that the biconditional is "the conjunction of a conditional running in each direction. The BICONDITIONAL INTRODUCTION RULE then is, basically this: from A →B and B → A, infer A ↔ B. That's the idea, but we don't actually ever show the conditionals, we just need one subproof that begins with the assumption A and ends with B and a second subproof that does the opposite. (Both subproofs are required.)

The subproofs can come in any order, and the second subproof does not need to come immediately after the first. Notice how both subproofs are cited after the '↔I'.

$$
\begin{array}{rlll}
m & \vert & \quad \vert\ \underline{\mathsf{A}} & \text{AS} \\
n & \vert & \quad \vert\ \mathsf{B} & \\
p & \vert & \quad \vert\ \underline{\mathsf{B}} & \text{AS} \\
q & \vert & \quad \vert\ \mathsf{A} & \\
 & \vert & \mathsf{A} \leftrightarrow \mathsf{B} & \leftrightarrow\text{I } m\text{–}n,\ p\text{–}q
\end{array}
$$

The BICONDITIONAL ELIMINATION RULE (↔E) is a bit more flexible than the conditional rule. If you have the left-hand subsentence of the biconditional, you can derive the right-hand subsentence. If you have the right-hand subsentence, you can derive the left-hand subsentence. This is the rule:

$$
\begin{array}{rlll}
m & \vert & \mathsf{A} \leftrightarrow \mathsf{B} & \\
n & \vert & \mathsf{A} & \\
 & \vert & \mathsf{B} & \leftrightarrow\text{E } m,\ n
\end{array}
$$

$$
\begin{array}{rlll}
m & \vert & \mathsf{A} \leftrightarrow \mathsf{B} & \\
n & \vert & \mathsf{B} & \\
 & \vert & \mathsf{A} & \leftrightarrow\text{E } m,\ n
\end{array}
$$

15.7 Negation

Here is a simple mathematical argument in English:

> Assume there is some greatest natural number. Call it A.
> That number plus one is also a natural number.
> $A + 1$ is greater than A.
> Thus, A is the greatest natural number, and there is a natural number greater than A.

The previous line is a contradiction.
∴ There is no greatest natural number.

This argument form is traditionally called a *reductio*. Its full Latin name is *reductio ad absurdum*, which means 'reduction to absurdity.' In a reductio, we assume something for the sake of argument—for example, that there is a greatest natural number. Then we show that the assumption leads to two contradictory sentences—for example, that A is the greatest natural number and that it is not. In this way, we show that the original assumption must be false. Our negation rules (which are basically the same rule) formalize this reasoning process.

This is the NEGATION INTRODUCTION (¬I) rule:

$$\begin{array}{l|l|ll}
m & & \underline{\mathsf{A}} & \text{AS} \\
n & & \mathsf{B} & \\
p & & \neg\mathsf{B} & \\
 & \neg\mathsf{A} & & \neg\text{I } m\text{–}p
\end{array}$$

The ¬E rule is similar. We begin by assuming ¬A, show that it leads to a contradiction, and can then infer A.

$$\begin{array}{l|l|ll}
m & & \underline{\neg\mathsf{A}} & \text{AS} \\
n & & \mathsf{B} & \\
p & & \neg\mathsf{B} & \\
 & \mathsf{A} & & \neg\text{E } m\text{–}p
\end{array}$$

When using these rules, the last two lines of the subproof must be an explicit contradiction: A on one line and its negation, ¬A, on the next line (or vice versa). Those two lines cannot be separated.

To see how the rule works, suppose we want to prove the law of non-contradiction: $\neg(G \,\&\, \neg G)$. We can prove this without any premises by immediately starting a subproof. We want to apply ¬I to the subproof, so we assume $(G \,\&\, \neg G)$. We then get an explicit contradiction by &E. The proof looks like this:

1	$\quad G \,\&\, \neg G$	AS
2	$\quad G$	& E 1
3	$\quad \neg G$	& E 1
4	$\neg(G \,\&\, \neg G)$	$\neg$I 1–3

15.8 Reiteration & double negation

In addition to the rules for each logical operator, we also have a REITERATION RULE and a DOUBLE NEGATION RULE. If you already have shown something in the course of a proof, the reiteration rule allows you to repeat it on a new line, which is sometimes useful.

m	A	
n	A	R m

The double negation rule allows you to add two *not*s to an atomic sentence (which, of course, will not change the sentence's truth value). This is sometimes useful when using the disjunction or conditional elimination rules. It also allows you to remove two 'nots', although needing to do that is less common.

m	A	
n	$\neg\neg$A	DN m

m	$\neg\neg$A	
n	A	DN m

Chapter 16

Proof-theoretic concepts

In this chapter we will introduce some new vocabulary. The following expression:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{C}$$

means that there is some proof which starts with assumptions among $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ and ends with C (and contains no undischarged assumptions other than those we started with).

The symbol '$\vdash$' is called the *single turnstile*. This is not the double turnstile symbol ('$\vDash$') that we introduced in chapter 10 to symbolize entailment. The single turnstile, '$\vdash$', concerns the existence of proofs; the double turnstile, '$\vDash$', concerns the existence of valuations (or interpretations, when used for FOL). *They are very different notions.*

Armed with our '$\vdash$' symbol, we can introduce some more terminology. To say that there is a proof of A with no undischarged assumptions, we write: $\vdash \mathsf{A}$. In this case, we say that A is a THEOREM.

> A is a THEOREM iff $\vdash \mathsf{A}$

To illustrate this, suppose we want to show that '$\neg(A \,\&\, \neg A)$' is a theorem. So we need a proof of '$\neg(A \,\&\, \neg A)$' which has *no* undischarged assumptions. However, since we want to prove a sentence whose main logical operator is a negation, we will want to start with a *subproof* within which we assume '$A \,\&\, \neg A$', and show that this assumption leads to contradiction. All told, then, the proof looks like this:

1	$\quad A \,\&\, \neg A$	
2	$\quad A$	$\&$E 1
3	$\quad \neg A$	$\&$E 1
4	$\quad \bot$	$\neg$E 3, 2
5	$\neg(A \,\&\, \neg A)$	$\neg$I 1–4

We have therefore proved '$\neg(A \,\&\, \neg A)$' on no (undischarged) assumptions. This particular theorem is an instance of what is sometimes called *the Law of Non-Contradiction*.

To show that something is a theorem, you just have to find a suitable proof. It is typically much harder to show that something is *not* a theorem. To do this, you would have to demonstrate, not just that certain proof strategies fail, but that *no* proof is possible. Even if you fail in trying to prove a sentence in a thousand different ways, perhaps the proof is just too long and complex for you to make out. Perhaps you just didn't try hard enough.

Here is another new bit of terminology:

> Two sentences A and B are PROVABLY EQUIVALENT iff each can be proved from the other; i.e., both $\mathsf{A} \vdash \mathsf{B}$ and $\mathsf{B} \vdash \mathsf{A}$.

As in the case of showing that a sentence is a theorem, it is relatively easy to show that two sentences are provably equivalent: it just requires a pair of proofs. Showing that sentences are *not* provably equivalent would be much harder: it is just as hard as showing that a sentence is not a theorem.

Here is a third, related, bit of terminology:

> The sentences $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are PROVABLY INCONSISTENT iff a contradiction can be proved from them, i.e. $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \bot$. If they are not INCONSISTENT, we call them PROVABLY CONSISTENT.

It is easy to show that some sentences are provably inconsistent: you just need to prove a contradiction from assuming all the sentences. Showing that some sentences are not provably inconsistent is much harder. It would require more than just providing a proof or two; it would require showing that no proof of a certain kind is *possible*.

This table summarises whether one or two proofs suffice, or whether we must reason about all possible proofs.

	Yes	**No**
theorem?	one proof	all possible proofs
inconsistent?	one proof	all possible proofs
equivalent?	two proofs	all possible proofs
consistent?	all possible proofs	one proof

Practice exercises

A. Show that each of the following sentences is a theorem:

1. $O \rightarrow O$
2. $N \lor \neg N$
3. $J \leftrightarrow [J \lor (L \,\&\, \neg L)]$
4. $((A \rightarrow B) \rightarrow A) \rightarrow A$

B. Provide proofs to show each of the following:

1. $C \rightarrow (E \,\&\, G), \neg C \rightarrow G \vdash G$
2. $M \,\&\, (\neg N \rightarrow \neg M) \vdash (N \,\&\, M) \lor \neg M$
3. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \rightarrow M) \vdash Y \rightarrow Z$
4. $(W \lor X) \lor (Y \lor Z), X \rightarrow Y, \neg Z \vdash W \lor Y$

C. Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E$, $E \leftrightarrow R$
2. G, $\neg\neg\neg\neg G$
3. $T \rightarrow S$, $\neg S \rightarrow \neg T$
4. $U \rightarrow I$, $\neg(U \,\&\, \neg I)$
5. $\neg(C \rightarrow D), C \,\&\, \neg D$
6. $\neg G \leftrightarrow H$, $\neg(G \leftrightarrow H)$

D. If you know that $\mathsf{A} \vdash \mathsf{B}$, what can you say about $(\mathsf{A} \,\&\, \mathsf{C}) \vdash \mathsf{B}$? What about $(\mathsf{A} \lor \mathsf{C}) \vdash \mathsf{B}$? Explain your answers.

E. In this chapter, we claimed that it is just as hard to show that two sentences are not provably equivalent, as it is to show that a sentence is not a theorem. Why did we claim this? (*Hint*: think of a sentence that would be a theorem iff A and B were provably equivalent.)

Chapter 17

Proofs in Carnap

Creating proofs in Carnap is not difficult. To type the connectives, use the following.

symbol	**in Carnap**
$\neg$	~
&	&
$\vee$	v (lowercase v)
$\rightarrow$	-> (dash, greater than sign)
$\leftrightarrow$	<->

Carnap will number the lines automatically. After the sentence on each line, there has to be a colon (':') before the 'PR', 'AS', or the rule. Carnap is flexible with the spacing on a line, but as a guideline, put two spaces between the sentence and ':PR', ':AS', or the rule (:$\rightarrow$E, :$\vee$I, etc.). Also indent subproofs with two spaces. (Carnap will let you use more or fewer spaces, but a subproof has to be indented some amount.)

To illustrate creating proofs in Carnap, we will go through two examples. To produce a proof, you are given an interface like the one shown in figure 17.1. The argument for which you are creating the proof is given at the top. In this case, the premises are $P \rightarrow Q$ and R & P, and the conclusion is Q. (The premises and conclusion are separated by the turnstile ('$\vdash$'). See chapter 16 for further explanation.)

Begin by listing the premises, and don't forget to put ':PR' after each one. If there is a problem with a line—either the sentence isn't formed correctly, the rule

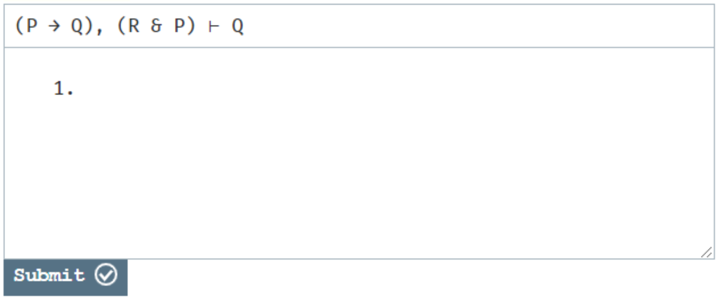


Figure 17.1

you've cited isn't being used correctly, or there's some other mistake—Carnap will put ? or ⚠ at the end of the line. When the line is ok, you will get a +.

Once you complete each line, Carnap will give you the typographically correct version on the right (figure 17.2). We complete the proof using the & E and →E rules (figure 17.3). When the proof is correct, the box containing the argument will turn green, and the proof can be submitted.

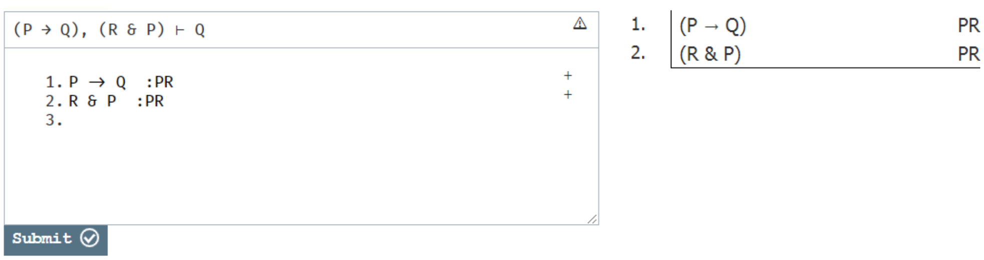


Figure 17.2

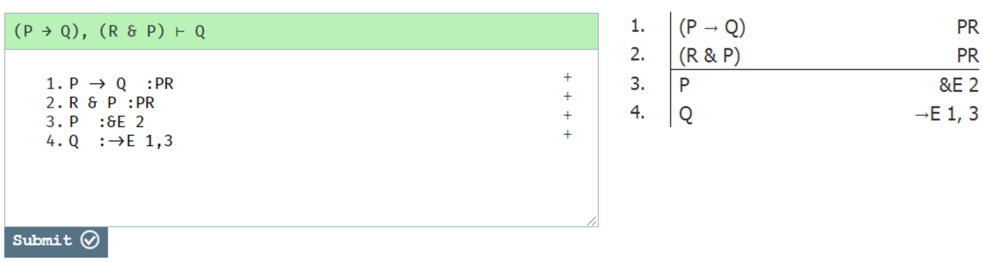


Figure 17.3

Our next example, $(A \lor B) \vdash (\neg A \to B)$, requires a subproof. We begin as before. To create the subproof, put two spaces before $\neg A$ and put ':AS' at the end of the line (figure 17.4). Since the next line is also part of the subproof, we again need two spaces before the B. We end the subproof (and discharge the assumption) with the $\rightarrow$I rule. $\neg A \rightarrow B$ is not indented (so no spaces before the $\neg A$). That's the conclusion, and so if everything is correct, Carnap will give you the green bar and you can submit the proof (figure 17.5).

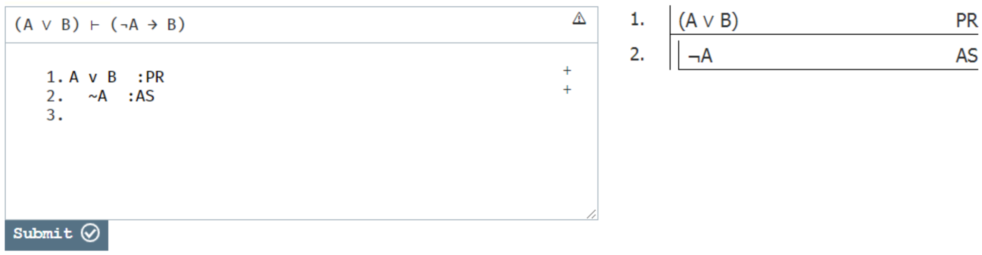


Figure 17.4

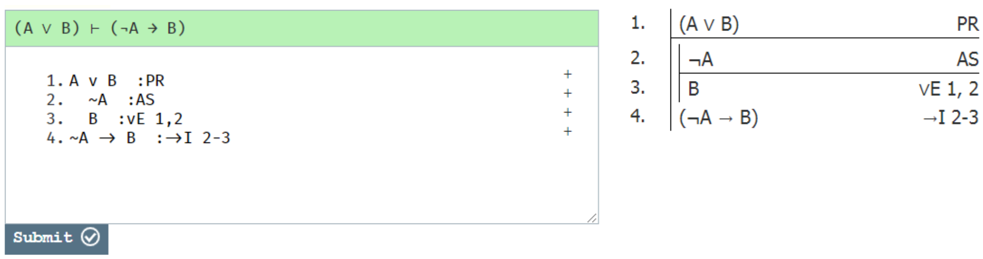


Figure 17.5

As I said at the beginning of this chapter, creating proofs in Carnap is not difficult. You do have to be careful, however. Programming a language like TFL is relatively simple because there are only a small number of rules and, to produce proofs of valid arguments, we follow those rules very strictly. But as a consequence, Carnap is not designed to understand what you are trying to do if you deviate from the rules, even if it is a minor deviation or an innocent mistake. So, some reminders:

1. Capitalize 'PR', 'AS', the 'E' and 'I' in the rules, and all atomic sentences.
2. Don't forget the ':' right before PR, AS, or the rule that you are citing.
3. There is no space between the & , $\vee$, $\rightarrow$, $\leftrightarrow$, or $\neg$ and the 'E' or 'I'.
4. There is a space (and no punctuation) after the 'E' or 'I'.

5. There is a comma between the two lines that have to be cited for &I, ∨E, →E, and ↔E (e.g., '→E 2,4').
6. There is a dash between the two lines that have to be cited for →I, ¬I, and ¬E (e.g., '¬E 4-6').
7. The ↔I rule requires dashes and a comma. (Check the format on p. 103.)

Chapter 18

Proof strategies

There is no simple recipe for proofs, and there is no substitute for practice. Here, though, are some rules of thumb and strategies to keep in mind.

Work backwards from what you want. The ultimate goal is to obtain the conclusion. Look at the conclusion and ask what the introduction rule is for its main logical operator. This gives you an idea of what should happen *just before* the last line of the proof. Then you can treat this line as if it were your goal. Ask what you could do to get to this new goal.

For example: If your conclusion is a conditional $\mathsf{A} \rightarrow \mathsf{B}$, plan to use the $\rightarrow$I rule. This requires starting a subproof in which you assume A. The subproof ought to end with B. So, what can you do to get B?

Work forwards from what you have. When you are starting a proof, look at the premises; later, look at the sentences that you have obtained so far. Think about the elimination rules for the main operators of these sentences. These will tell you what your options are.

For a short proof, you might be able to eliminate the premises and introduce the conclusion. A long proof is formally just a number of short proofs linked together, so you can fill the gap by alternately working back from the conclusion and forward from the premises.

Try proceeding indirectly. If you cannot find a way to show A directly, try starting by assuming $\neg\mathsf{A}$. If a contradiction follows, then you will be able to obtain $\neg\neg\mathsf{A}$ by $\neg$I, and then A by DNE.

Persist. Try different things. If one approach fails, then try something else.

Chapter 19

Additional rules for TFL

19.1 Derived rules

The rules of our natural deduction system are systematic. There is an introduction and an elimination rule for each logical operator. But why these basic rules rather than others? Some natural deduction systems have this rule:

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\mathsf{A} \rightarrow \mathsf{C}$	
o	$\mathsf{B} \rightarrow \mathsf{C}$	
	C	DIL m, n, o

Let's call this rule DILEMMA (DIL) It might seem as if there will be some proofs that we cannot do without this rule. But that is not the case. Any proof that you can do using the Dilemma rule can be done with basic rules of our natural deduction system. It will just take more steps. Consider this proof:

1	$A \lor B$	PR
2	$A \to C$	PR
3	$B \to C$	PR
4	$\quad \neg C$	AS
5	$\quad\quad A$	AS
6	$\quad\quad C$	$\to$E 2, 5
7	$\quad\quad \neg C$	R 4
8	$\quad \neg A$	$\neg$I 5–7
9	$\quad\quad B$	AS
10	$\quad\quad C$	$\to$E 3, 9
11	$\quad\quad \neg C$	R 4
12	$\quad B$	$\lor$E 1, 8
13	$\quad \neg B$	$\neg$I 9–11
14	C	$\neg$E 4–13

This proof demonstrates that the dilemma rule is not really necessary. Adding it to the list of basic rules would not allow us to derive anything that we could not derive without it. Nevertheless, the it would be convenient. It would allow us to do in one line what requires eleven lines with the basic rules (and subproofs within subproofs!). So we will add it to the proof system as a derived rule.

A DERIVED RULE is a rule of proof that does not make any new proofs possible. Anything that can be proven with a derived rule can be proven without it. You can think of a short proof using a derived rule as shorthand for a longer proof that uses only the basic rules. Anytime you use the dilemma rule, you could always take ten extra lines and prove the same thing without it.

For the sake of convenience, we will add several other derived rules, all of which can be used in Carnap. One is MODUS TOLLENS (MT).

$$\begin{array}{r|ll} m & \mathsf{A} \to \mathsf{B} & \\ n & \neg\mathsf{B} & \\ & \neg\mathsf{A} & \text{MT } m, n \end{array}$$

We leave the proof of this rule as an exercise. Note that if we had already proven the MT rule, then the proof of the DIL rule could have been done in only five lines. We will also add the HYPOTHETICAL SYLLOGISM (HS) as a derived rule. We have already given a proof of it on p. 102.

$$\begin{array}{r|ll} m & \mathsf{A} \to \mathsf{B} & \\ n & \mathsf{B} \to \mathsf{C} & \\ & \mathsf{A} \to \mathsf{C} & \text{HS } m, n \end{array}$$

19.2 Rules of replacement

We will now introduce some derived rules that may be applied to part of a sentence. These are called RULES OF REPLACEMENT, because they can be used to replace part of a sentence with a logically equivalent expression. One simple rule of replacement is COMMUTIVITY (abbreviated Comm), which says that we can swap the order of conjuncts in a conjunction or the order of disjuncts in a disjunction. We define the rule this way:

$$\begin{array}{rl} (\mathsf{A} \,\&\, \mathsf{B}) \Longleftrightarrow (\mathsf{B} \,\&\, \mathsf{A}) & \\ (\mathsf{A} \vee \mathsf{B}) \Longleftrightarrow (\mathsf{B} \vee \mathsf{A}) & \\ (\mathsf{A} \leftrightarrow \mathsf{B}) \Longleftrightarrow (\mathsf{B} \leftrightarrow \mathsf{A}) & \text{Comm} \end{array}$$

The bold arrow means that you can take a subformula on one side of the arrow and replace it with the subformula on the other side. **The arrow is double-headed because rules of replacement work in both directions**. (In this case, and in the other cases below, the name 'Comm' applies to each of the three versions, even though they use different logical operators.)

Consider this argument: $(M \vee P) \to (P \,\&\, M)$ $\therefore$ $(P \vee M) \to (M \,\&\, P)$.

Although it is obviously valid, a proof of it using only the basic rules would be quite long. With the Comm rule, however, we can easily provide a proof:

1	$(M \lor P) \to (P \,\&\, M)$	PR
2	$(P \lor M) \to (P \,\&\, M)$	Comm 1
3	$(P \lor M) \to (M \,\&\, P)$	Comm 2

Two more replacement rules are called DE MORGAN'S LAWS, named for the 19th-century British logician August De Morgan. (Although De Morgan did discover these laws, he was not the first to do so.) The rules capture useful relations between negation, conjunction, and disjunction, and we demonstrated that the sentences in the first one are logically equivalent in §10.2. Here are the rules, which we abbreviate 'DeM':

$$\neg(\mathsf{A} \lor \mathsf{B}) \iff (\neg\mathsf{A} \,\&\, \neg\mathsf{B})$$
$$\neg(\mathsf{A} \,\&\, \mathsf{B}) \iff (\neg\mathsf{A} \lor \neg\mathsf{B}) \quad \text{DeM}$$

In §5.6, we discussed different ways of translating 'you will catch a cold unless you wear a jacket'. We found that multiple translations were plausible: '$\neg J \to D$', '$\neg D \to J$', and '$J \lor D$', and I said that these were all equivalent. The MATERIAL CONDITIONAL rule (MC) is used for this equivalence. It takes two forms:

$$(\mathsf{A} \to \mathsf{B}) \iff (\neg\mathsf{A} \lor \mathsf{B})$$
$$(\mathsf{A} \lor \mathsf{B}) \iff (\neg\mathsf{A} \to \mathsf{B}) \quad \text{MC}$$

Now consider this argument: $\neg(P \to Q) \mathrel{.\,\dot{}\,.} P \,\&\, \neg Q$. We could prove this argument using only the basic rules, but that would take 17 lines and five subproofs. With DN, DeM, and MC, the proof is much simpler:

1	$\neg(P \to Q)$	PR
2	$\neg(\neg P \lor Q)$	MC 1
3	$\neg\neg P \,\&\, \neg Q$	DeM 2
4	$P \,\&\, \neg Q$	DN 3

A final replacement rule captures the relation between conditionals and biconditionals. We will call this rule BICONDITIONAL EXCHANGE and abbreviate it $\leftrightarrow$ex.

$$[(\mathsf{A} \rightarrow \mathsf{B}) \,\&\, (\mathsf{B} \rightarrow \mathsf{A})] \Longleftrightarrow (\mathsf{A} \leftrightarrow \mathsf{B}) \qquad \leftrightarrow\text{ex}$$

Chapter 20

Soundness and completeness

In §16, we saw that we could use derivations to test for the same concepts we used truth tables to test for. Not only could we use derivations to prove that an argument is valid, we could also use them to test if a sentence is a tautology or a pair of sentences are equivalent. We also started using the single turnstile the same way we used the double turnstile. If we could prove that A was a tautology with a truth table, we wrote $\vDash \mathsf{A}$, and if we could prove it using a derivation, we wrote $\vdash \mathsf{A}$.

You may have wondered at that point if the two kinds of turnstiles always worked the same way. If you can show that A is a tautology using truth tables, can you also always show that it is true using a derivation? Is the reverse true? Are these things also true for tautologies and pairs of equivalent sentences? As it turns out, the answer to all these questions and many more like them is yes. We can show this by defining all these concepts separately and then proving them equivalent. That is, we imagine that we actually have two notions of validity, valid$_{\vDash}$ and valid$_{\vdash}$ and then show that the two concepts always work the same way.

To begin with, we need to define all of our logical concepts separately for truth tables and derivations. A lot of this work has already been done. We handled all of the truth table definitions in §10. We have also already given syntactic definitions for tautologies (theorems) and pairs of logically equivalent sentences. The other definitions follow naturally. For most logical properties we can devise a test using derivations, and those that we cannot test for directly can be defined in terms of the concepts that we can define.

For instance, we defined a theorem as a sentence that can be derived without any premises (p. 106). Since the negation of a contradiction is a tautology, we can define a SYNTACTIC CONTRADICTION IN TFL as a sentence whose negation can be derived without any premises. The syntactic definition of a contingent sentence is a little different. We don't have any practical, finite method for proving that a sentence is contingent using derivations, the way we did using truth tables. So we have to content ourselves with defining "contingent sentence" negatively. A sentence is SYNTACTICALLY CONTINGENT IN TFL if it is not a theorem or a contradiction.

A collection of sentences are PROVABLY INCONSISTENT IN TFL if and only if one can derive a contradiction from them. Consistency, on the other hand, is like contingency, in that we do not have a practical finite method to test for it directly. So again, we have to define a term negatively. A collection of sentences is PROVABLY CONSISTENT IN TFL if and only if they are not provably inconsistent.

Finally, an argument is PROVABLY VALID IN TFL if and only if there is a derivation of its conclusion from its premises. All of these definitions are given in Table 20.1.

All of our concepts have now been defined both semantically and syntactically. How can we prove that these definitions always work the same way? A full proof here goes well beyond the scope of this book. However, we can sketch what it would be like. We will focus on showing the two notions of validity to be equivalent. From that the other concepts will follow quickly. The proof will have to go in two directions. First we will have to show that things which are syntactically valid will also be semantically valid. In other words, everything that we can prove using derivations could also be proven using truth tables. Put symbolically, we want to show that valid$_\vdash$ implies valid$_\vDash$. Afterwards, we will need to show things in the other directions, valid$_\vDash$ implies valid$_\vdash$

This argument from $\vdash$ to $\vDash$ is the problem of SOUNDNESS. A proof system is SOUND if there are no derivations of arguments that can be shown invalid by truth tables. Demonstrating that the proof system is sound would require showing that *any* possible proof is the proof of a valid argument. It would not be enough simply to succeed when trying to prove many valid arguments and to fail when trying to prove invalid ones.

The proof that we will sketch depends on the fact that we initially defined a sentence of TFL using a recursive definition (see p. 37). We could have also used recursive definitions to define a proper proof in TFL and a proper truth table. (Although we didn't.) If we had these definitions, we could then use a

Concept	Truth table (semantic) definition	Proof-theoretic (syntactic) definition
Tautology	A sentence whose truth table only has Ts under the main connective	A sentence that can be derived without any premises.
Contradiction	A sentence whose truth table only has Fs under the main connective	A sentence whose negation can be derived without any premises
Contingent sentence	A sentence whose truth table contains both Ts and Fs under the main connective	A sentence that is not a theorem or contradiction
Equivalent sentences	The columns under the main connectives are identical.	The sentences can be derived from each other
Inconsistent sentences	Sentences which do not have a single line in their truth table where they are all true.	Sentences from which one can derive a contradiction
Consistent sentences	Sentences which have at least one line in their truth table where they are all true.	Sentences which are not inconsistent
Valid argument	An argument whose truth table has no lines where there are all Ts under main connectives for the premises and an F under the main connective for the conclusion.	An argument where one can derive the conclusion from the premises

Table 20.1: Two ways to define logical concepts.

recursive proof to show the soundness of TFL. A recursive proof works the same way as a recursive definition. With the recursive definition, we identified a group of base elements that were stipulated to be examples of the thing we were trying to define. In the case of a TFL sentence, the base class was the set of sentence letters A, B, C, ... We just announced that these were sentences. The second step of a recursive definition is to say that anything that is built up from your base class using certain rules also counts as an example of the thing you are defining. In the case of a definition of a sentence, the rules corresponded to the five sentential connectives (see p. 37). Once you have established a recursive definition, you can use that definition to show that all the members of the class you have defined have a certain property. You simply prove that the property is true of the members of the base class, and then you prove that the rules for extending the base class don't change the property. This is what it means to give a recursive proof.

Even though we don't have a recursive definition of a proof in TFL, we can sketch how a recursive proof of the soundness of TFL would go. Imagine a base class of one-line proofs, one for each of our eleven rules of inference. The members of this class would look like this $\mathsf{A}, \mathsf{B} \vdash \mathsf{A} \,\&\, \mathsf{B}$; $\mathsf{A} \,\&\, \mathsf{B} \vdash \mathsf{A}$; $\mathsf{A} \lor \mathsf{B}, \neg\mathsf{A} \vdash \mathsf{B}$ ... etc. Since some rules have a couple different forms, we would have to have add some members to this base class, for instance $\mathsf{A} \,\&\, \mathsf{B} \vdash \mathsf{B}$ Notice that these are all statements in the metalanguage. The proof that TFL is sound is not a part of TFL, because TFL does not have the power to talk about itself.

You can use truth tables to prove to yourself that each of these one-line proofs in this base class is valid$_{\vDash}$. For instance the proof $\mathsf{A}, \mathsf{B} \vdash \mathsf{A} \,\&\, \mathsf{B}$ corresponds to a truth table that shows $\mathsf{A}, \mathsf{B} \vDash \mathsf{A} \,\&\, \mathsf{B}$ This establishes the first part of our recursive proof.

The next step is to show that adding lines to any proof will never change a valid$_{\vDash}$ proof into an invalid$_{\vDash}$ one. We would need to do this for each of our eleven basic rules of inference. So, for instance, for &I we need to show that for any proof $\mathsf{A}_1, \ldots, \mathsf{A}_n \vdash \mathsf{B}$ adding a line where we use &I to infer $\mathsf{C} \,\&\, \mathsf{D}$, where $\mathsf{C} \,\&\, \mathsf{D}$ can be legitimately inferred from $\mathsf{A}_1, \ldots, \mathsf{A}_n, \mathsf{B}$, would not change a valid proof into an invalid proof. But wait, if we can legitimately derive $\mathsf{C} \,\&\, \mathsf{D}$ from these premises, then C and D must be already available in the proof. They are either already among $\mathsf{A}_1, \ldots, \mathsf{A}_n, \mathsf{B}$, or can be legitimately derived from them. As such, any truth table line in which the premises are true must be a truth table line in which C and D are true. According to the characteristic truth table for

&, this means that C & D is also true on that line. Therefore, C & D validly follows from the premises. This means that using the &E rule to extend a valid proof produces another valid proof.

In order to show that the proof system is sound, we would need to show this for the other inference rules. Since the derived rules are consequences of the basic rules, it would suffice to provide similar arguments for the 11 other basic rules. This tedious exercise falls beyond the scope of this book.

So we have shown that $\mathsf{A} \vdash \mathsf{B}$ implies $\mathsf{A} \vDash \mathsf{B}$. What about the other direction, that is why think that *every* argument that can be shown valid using truth tables can also be proven using a derivation.

This is the problem of completeness. A proof system has the property of COMPLETENESS if and only if there is a derivation of every semantically valid argument. Proving that a system is complete is generally harder than proving that it is sound. Proving that a system is sound amounts to showing that all of the rules of your proof system work the way they are supposed to. Showing that a system is complete means showing that you have included *all* the rules you need, that you haven't left any out. Showing this is beyond the scope of this book. The important point is that, happily, the proof system for TFL is both sound and complete. This is not the case for all proof systems or all formal languages. Because it is true of TFL, we can choose to give proofs or give truth tables—whichever is easier for the task at hand.

Now that we know that the truth table method is interchangeable with the method of derivations, you can chose which method you want to use for any given problem. Students often prefer to use truth tables, because they can be produced purely mechanically, and that seems 'easier'. However, we have already seen that truth tables become impossibly large after just a few sentence letters. On the other hand, there are a couple situations where using proofs simply isn't possible. We syntactically defined a contingent sentence as a sentence that couldn't be proven to be a tautology or a contradiction. There is no practical way to prove this kind of negative statement. We will never know if there isn't some proof out there that a statement is a contradiction and we just haven't found it yet. We have nothing to do in this situation but resort to truth tables. Similarly, we can use derivations to prove two sentences equivalent, but what if we want to prove that they are *not* equivalent? We have no way of proving that we will never find the relevant proof. So we have to fall back on truth tables again.

Table 20.2 summarizes when it is best to give proofs and when it is best to

Logical property	**To prove it present**	**To prove it absent**
Being a tautology	Derive the sentence	Find the false line in the truth table for the sentence
Being a contradiction	Derive the negation of the sentence	Find the true line in the truth table for the sentence
Contingency	Find a false line and a true line in the truth table for the sentence	Prove the sentence or its negation
Equivalence	Derive each sentence from the other	Find a line in the truth tables for the sentence where they have different values
Consistency	Find a line in truth table for the sentence where they all are true	Derive a contradiction from the sentences
Validity	Derive the conclusion from the premises	Find no line in the truth table where the premises are true and the conclusion false.

Table 20.2: When to provide a truth table and when to provide a proof.

give truth tables.

Practice exercises

A. Use either a derivation or a truth table for each of the following.

1. Show that $A \rightarrow [((B \,\&\, C) \vee D) \rightarrow A]$ is a tautology.
2. Show that $A \rightarrow (A \rightarrow B)$ is not a tautology
3. Show that the sentence $A \rightarrow \neg A$ is not a contradiction.
4. Show that the sentence $A \leftrightarrow \neg A$ is a contradiction.

5. Show that the sentence $\neg(W \rightarrow (J \vee J))$ is contingent
6. Show that the sentence $\neg(X \vee (Y \vee Z)) \vee (X \vee (Y \vee Z))$ is not contingent
7. Show that the sentence $B \rightarrow \neg S$ is equivalent to the sentence $\neg\neg B \rightarrow \neg S$
8. Show that the sentence $\neg(X \vee O)$ is not equivalent to the sentence $X \ \& \ O$
9. Show that the sentences $\neg(A \vee B)$, C, $C \rightarrow A$ are jointly inconsistent.
10. Show that the sentences $\neg(A \vee B)$, $\neg B$, $B \rightarrow A$ are jointly consistent
11. Show that $\neg(A \vee (B \vee C)) \therefore \neg C$ is valid.
12. Show that $\neg(A \ \& \ (B \vee C)) \therefore \neg C$ is invalid.

B. Use either a derivation or a truth table for each of the following.

1. Show that $A \rightarrow (B \rightarrow A)$ is a tautology
2. Show that $\neg(((N \leftrightarrow Q) \vee Q) \vee N)$ is not a tautology
3. Show that $Z \vee (\neg Z \leftrightarrow Z)$ is contingent
4. show that $(L \leftrightarrow ((N \rightarrow N) \rightarrow L)) \vee H$ is not contingent
5. Show that $(A \leftrightarrow A) \ \& \ (B \ \& \ \neg B)$ is a contradiction
6. Show that $(B \leftrightarrow (C \vee B))$ is not a contradiction.
7. Show that $((\neg X \leftrightarrow X) \vee X)$ is equivalent to X
8. Show that $F \ \& \ (K \ \& \ R)$ is not equivalent to $(F \leftrightarrow (K \leftrightarrow R))$
9. Show that the sentences $\neg(W \rightarrow W)$, $(W \leftrightarrow W) \ \& \ W$, $E \vee (W \rightarrow \neg(E \ \& \ W))$ are inconsistent.
10. Show that the sentences $\neg R \vee C$, $(C \ \& \ R) \rightarrow \neg R$, $(\neg(R \vee R) \rightarrow R)$ are consistent.
11. Show that $\neg\neg(C \leftrightarrow \neg C), ((G \vee C) \vee G) \therefore ((G \rightarrow C) \ \& \ G)$ is valid.
12. Show that $\neg\neg L, (C \rightarrow \neg L) \rightarrow C) \therefore \neg C$ is invalid.

Appendices

Appendix A

Symbolic notation

1.1 Alternative nomenclature

Truth-functional logic. TFL goes by other names. Sometimes it is called *sentential logic*, because it deals fundamentally with sentences. Sometimes it is called *propositional logic*, on the idea that it deals fundamentally with propositions. We have stuck with *truth-functional logic*, to emphasize the fact that it deals only with assignments of truth and falsity to sentences, and that its connectives are all truth-functional.

First-order logic. FOL goes by other names. Sometimes it is called *predicate logic*, because it allows us to apply predicates to objects. Sometimes it is called *quantified logic*, because it makes use of quantifiers.

Formulas. Some texts call formulas *well-formed formulas.* Since 'well-formed formula' is such a long and cumbersome phrase, they then abbreviate this as *wff.* This is both barbarous and unnecessary (such texts do not countenance 'ill-formed formulas'). We have stuck with 'formula'.

In §6, we defined *sentences* of TFL. These are also sometimes called 'formulas' (or 'well-formed formulas') since in TFL, unlike FOL, there is no distinction between a formula and a sentence.

Valuations. Some texts call valuations *truth-assignments*, or *truth-value assignments.*

Expressive adequacy. Some texts describe TFL as *truth-functionally complete*, rather than expressively adequate.

***n*-place predicates.** We have chosen to call predicates 'one-place', 'two-place', 'three-place', etc. Other texts respectively call them 'monadic', 'dyadic', 'triadic', etc. Still other texts call them 'unary', 'binary', 'ternary', etc.

Names. In FOL, we have used 'a', 'b', 'c', for names. Some texts call these 'constants'. Other texts do not mark any difference between names and variables in the syntax. Those texts focus simply on whether the symbol occurs *bound* or *unbound*.

Domains. Some texts describe a domain as a 'domain of discourse', or a 'universe of discourse'.

1.2 Alternative symbols

In the history of formal logic, different symbols have been used at different times and by different authors. Often, authors were forced to use notation that their printers could typeset. This appendix presents some common symbols, so that you can recognize them if you encounter them in an article or in another book.

Negation. Two commonly used symbols are the *hoe*, '$\neg$', and the *swung dash* or *tilda*, '$\sim$.' In some more advanced formal systems it is necessary to distinguish between two kinds of negation; the distinction is sometimes represented by using both '$\neg$' and '$\sim$'. Older texts sometimes indicate negation by a line over the formula being negated, e.g., $\overline{A \mathbin{\&} B}$. Some texts use '$x \neq y$' to abbreviate '$\neg x = y$'.

Disjunction. The symbol '$\vee$' is typically used to symbolize inclusive disjunction. One etymology is from the Latin word 'vel', meaning 'or'.

Conjunction. Conjunction is often symbolized with the *ampersand*, '&'. The ampersand is a decorative form of the Latin word 'et', which means 'and'. (Its etymology still lingers in certain fonts, particularly in italic fonts; thus an italic ampersand might appear as '*&*'.) This symbol is commonly used in natural English writing (e.g. 'Smith & Sons'), and so even though it is a natural choice, many logicians use a different symbol to avoid confusion between the object and metalanguage: as a symbol in a formal system, the ampersand is not the English word '&'. The most common choice now is '$\wedge$', which is a counterpart to the symbol used for disjunction. Sometimes a single dot, '$\bullet$', is used. In some older texts, there is no symbol for conjunction at all; 'A and B' is simply written 'AB'.

Material Conditional. There are two common symbols for the material conditional: the *arrow*, '$\rightarrow$', and the *hook*, '$\supset$'.

Material Biconditional. The *double-headed arrow*, '$\leftrightarrow$', is used in systems that use the arrow to represent the material conditional. Systems that use the hook for the conditional typically use the *triple bar*, '$\equiv$', for the biconditional.

Quantifiers. The universal quantifier is typically symbolized as a rotated 'A', and the existential quantifier as a rotated, 'E'. In some texts, there is no separate symbol for the universal quantifier. Instead, the variable is just written in parentheses in front of the formula that it binds. For example, they might write '$(x)Px$' where we would write '$\forall x\, Px$'.

These alternative typographies are summarised below:

negation	$\neg$, $\sim$
conjunction	$\wedge$, &, $\bullet$
disjunction	$\vee$
conditional	$\rightarrow$, $\supset$
biconditional	$\leftrightarrow$, $\equiv$
universal quantifier	$\forall x$, (x)

Appendix B

Quick reference

2.1 Characteristic Truth Tables

A	$\neg\mathsf{A}$
T	F
F	T

A	B	$\mathsf{A} \,\&\, \mathsf{B}$	$\mathsf{A} \vee \mathsf{B}$	$\mathsf{A} \rightarrow \mathsf{B}$	$\mathsf{A} \leftrightarrow \mathsf{B}$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

A	$\neg\mathsf{A}$
1	0
0	1

A	B	$\mathsf{A} \,\&\, \mathsf{B}$	$\mathsf{A} \vee \mathsf{B}$	$\mathsf{A} \rightarrow \mathsf{B}$	$\mathsf{A} \leftrightarrow \mathsf{B}$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

2.2 Symbolization

Sentential Connectives

It is not the case that P.	$\neg P$
Either P, or Q.	$(P \vee Q)$
Neither P, nor Q.	$\neg(P \vee Q)$ or $(\neg P \,\&\, \neg Q)$
Both P, and Q.	$(P \,\&\, Q)$
If P, then Q.	$(P \rightarrow Q)$
P only if Q.	$(P \rightarrow Q)$
P if and only if Q.	$(P \leftrightarrow Q)$
Unless P, Q. P unless Q.	$(P \vee Q)$

Predicates

All Fs are Gs.	$\forall x(Fx \rightarrow Gx)$
Some Fs are Gs.	$\exists x(Fx \,\&\, Gx)$
Not all Fs are Gs.	$\neg\forall x(Fx \rightarrow Gx)$ or $\exists x(Fx \,\&\, \neg Gx)$
No Fs are Gs.	$\forall x(Fx \rightarrow \neg Gx)$ or $\neg\exists x(Fx \,\&\, Gx)$

Identity

Only j is G.	$\forall x(Gx \leftrightarrow x = j)$
Everything besides j is G.	$\forall x(x \neq j \rightarrow Gx)$
The F is G.	$\exists x(Fx \,\&\, \forall y(Fy \rightarrow x = y) \,\&\, Gx)$
It is not the case that the F is G. (wide)	$\neg\exists x(Fx \,\&\, \forall y(Fy \rightarrow x = y) \,\&\, Gx)$
The F is non-G. (narrow)	$\exists x(Fx \,\&\, \forall y(Fy \rightarrow x = y) \,\&\, \neg Gx)$

Using identity to symbolize quantities

There are at least _____ Fs.

one: $\exists xFx$
two: $\exists x_1 \exists x_2(Fx_1 \ \& \ Fx_2 \ \& \ x_1 \neq x_2)$
$three$: $\exists x_1 \exists x_2 \exists x_3(Fx_1 \ \& \ Fx_2 \ \& \ Fx_3 \ \& \ x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_2 \neq x_3)$
$four$: $\exists x_1 \exists x_2 \exists x_3 \exists x_4(Fx_1 \ \& \ Fx_2 \ \& \ Fx_3 \ \& \ Fx_4 \ \& \ x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_1 \neq x_4 \ \& \ x_2 \neq x_3 \ \& \ x_2 \neq x_4 \ \& \ x_3 \neq x_4)$
n: $\exists x_1 \cdots \exists x_n(Fx_1 \ \& \ \cdots \ \& \ Fx_n \ \& \ x_1 \neq x_2 \ \& \ \cdots \ \& \ x_{n-1} \neq x_n)$

There are at most _____ Fs.

One way to say 'at most n things are F' is to put a negation sign in front of one of the symbolizations above and say $\neg$'at least $n+1$ things are F.' Equivalently:

one: $\forall x_1 \forall x_2 \big[(Fx_1 \ \& \ Fx_2) \rightarrow x_1 = x_2\big]$
two: $\forall x_1 \forall x_2 \forall x_3 \big[(Fx_1 \ \& \ Fx_2 \ \& \ Fx_3) \rightarrow (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3)\big]$
$three$: $\forall x_1 \forall x_2 \forall x_3 \forall x_4 \big[(Fx_1 \ \& \ Fx_2 \ \& \ Fx_3 \ \& \ Fx_4) \rightarrow (x_1 = x_2 \vee x_1 = x_3 \vee x_1 = x_4 \vee x_2 = x_3 \vee x_2 = x_4 \vee x_3 = x_4)\big]$
n: $\forall x_1 \cdots \forall x_{n+1} \big[(Fx_1 \ \& \ \cdots \ \& \ Fx_{n+1}) \rightarrow (x_1 = x_2 \vee \cdots \vee x_n = x_{n+1})\big]$

There are exactly _____ Fs.

One way to say 'exactly n things are F' is to conjoin two of the symbolizations above and say 'at least n things are F' & 'at most n things are F.' The following equivalent formulae are shorter:

$zero$: $\forall x \neg Fx$
one: $\exists x \big[Fx \ \& \ \neg \exists y(Fy \ \& \ x \neq y)\big]$
two: $\exists x_1 \exists x_2 \big[Fx_1 \ \& \ Fx_2 \ \& \ x_1 \neq x_2 \ \& \ \neg \exists y \big(Fy \ \& \ y \neq x_1 \ \& \ y \neq x_2\big)\big]$
$three$: $\exists x_1 \exists x_2 \exists x_3 \big[Fx_1 \ \& \ Fx_2 \ \& \ Fx_3 \ \& \ x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_2 \neq x_3 \ \& \ \neg \exists y(Fy \ \& \ y \neq x_1 \ \& \ y \neq x_2 \ \& \ y \neq x_3)\big]$
n: $\exists x_1 \cdots \exists x_n \big[Fx_1 \ \& \ \cdots \ \& \ Fx_n \ \& \ x_1 \neq x_2 \ \& \ \cdots \ \& \ x_{n-1} \neq x_n \ \& \ \neg \exists y(Fy \ \& \ y \neq x_1 \ \& \ \cdots \ \& \ y \neq x_n)\big]$

2.3 Basic rules for TFL

Reiteration

$$\begin{array}{r|ll} m & \mathsf{A} & \\ & \mathsf{A} & \text{R } m \end{array}$$

Conjunction Introduction

$$\begin{array}{r|ll} m & \mathsf{A} & \\ n & \mathsf{B} & \\ & \mathsf{A} \mathbin{\&} \mathsf{B} & \& \text{I } m, n \end{array}$$

Conjunction Elimination

$$\begin{array}{r|ll} m & \mathsf{A} \mathbin{\&} \mathsf{B} & \\ & \mathsf{A} & \& \text{E } m \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} \mathbin{\&} \mathsf{B} & \\ & \mathsf{B} & \& \text{E } m \end{array}$$

Disjunction Introduction

$$\begin{array}{r|ll} m & \mathsf{A} & \\ & \mathsf{A} \vee \mathsf{B} & \vee\text{I } m \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} & \\ & \mathsf{B} \vee \mathsf{A} & \vee\text{I } m \end{array}$$

Disjunction Elimination

$$\begin{array}{r|ll} m & \mathsf{A} \vee \mathsf{B} & \\ n & \neg\mathsf{B} & \\ & \mathsf{A} & \vee\text{E } m, n \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} \vee \mathsf{B} & \\ n & \neg\mathsf{A} & \\ & \mathsf{B} & \vee\text{E } m, n \end{array}$$

Conditional Introduction

$$\begin{array}{r|ll} m & \quad \begin{array}{|l} \mathsf{A} \\ \hline \end{array} & \text{AS} \\ n & \quad \begin{array}{|l} \mathsf{B} \end{array} & \\ & \mathsf{A} \rightarrow \mathsf{B} & \rightarrow\text{I } m\text{–}n \end{array}$$

Conditional Elimination

$$\begin{array}{r|ll} m & \mathsf{A} \rightarrow \mathsf{B} & \\ n & \mathsf{A} & \\ & \mathsf{B} & \rightarrow\text{E } m, n \end{array}$$

Biconditional Introduction

$$\begin{array}{r|ll} m & \quad \begin{array}{|l} \mathsf{A} \\ \hline \end{array} & \text{AS} \\ n & \quad \begin{array}{|l} \mathsf{B} \end{array} & \\ p & \quad \begin{array}{|l} \mathsf{B} \\ \hline \end{array} & \text{AS} \\ q & \quad \begin{array}{|l} \mathsf{A} \end{array} & \\ & \mathsf{A} \leftrightarrow \mathsf{B} & \leftrightarrow\text{I } m\text{–}n,\ p\text{–}q \end{array}$$

Biconditional Elimination

$$\begin{array}{r|ll}
m & A \leftrightarrow B & \\
n & B & \\
 & A & \leftrightarrow\text{E } m, n
\end{array}$$

$$\begin{array}{r|ll}
m & A \leftrightarrow B & \\
n & A & \\
 & B & \leftrightarrow\text{E } m, n
\end{array}$$

Negation Introduction

$$\begin{array}{r|l|ll}
m & & \underline{A} & \text{AS} \\
n-1 & & B & \\
n & & \neg B & \\
\multicolumn{1}{r|}{} & \multicolumn{2}{l}{\neg A} & \neg\text{I } m\text{–}n
\end{array}$$

Negation Elimination

$$\begin{array}{r|l|ll}
m & & \underline{\neg A} & \text{AS} \\
n-1 & & B & \\
n & & \neg B & \\
\multicolumn{1}{r|}{} & \multicolumn{2}{l}{A} & \neg\text{E } m\text{–}n
\end{array}$$

2.4 Derived rules for TFL

Dilemma

$$\begin{array}{r|ll}
m & A \vee B & \\
n & A \rightarrow C & \\
p & B \rightarrow C & \\
 & C & \text{DIL } m, n, p
\end{array}$$

Modus Tollens

$$\begin{array}{r|ll}
m & A \rightarrow B & \\
n & \neg B & \\
 & \neg A & \text{MT } m, n
\end{array}$$

Hypothetical Syllogism

$$\begin{array}{r|ll}
m & A \rightarrow B & \\
n & B \rightarrow C & \\
 & A \rightarrow C & \text{HS } m, n
\end{array}$$

2.5 Quantifier rules

Existential Introduction

$$
\begin{array}{l|ll}
m & \mathsf{Ac} & \\
 & \exists\mathsf{xAx} & \exists\mathrm{I}\ m
\end{array}
$$

Note that x may replace some or all occurrences of c in Ac.

Existential Elimination

$$
\begin{array}{l|lll}
m & \exists\mathsf{xAx} & & \\
n & & \multicolumn{1}{|l}{\mathsf{Ac}^*} & \\
\cline{3-3}
p & & \multicolumn{1}{|l}{\mathsf{B}} & \\
 & \mathsf{B} & & \exists\mathrm{E}\ m,\ n\text{–}p
\end{array}
$$

* c must not appear in ∃xAx, in B, or in any undischarged assumption.

Universal Introduction

$$
\begin{array}{l|ll}
m & \mathsf{Ac}^* & \\
 & \forall\mathsf{xAx} & \forall\mathrm{I}\ m
\end{array}
$$

* c must not occur in any undischarged assumptions.

Universal Elimination

$$
\begin{array}{l|ll}
m & \forall\mathsf{xAx} & \\
 & \mathsf{Ac} & \forall\mathrm{E}\ m
\end{array}
$$

2.6 Identity rules

Identity introduction

$$\begin{array}{l|ll} & c = c & \text{=I} \end{array}$$

Identity elimination

$$\begin{array}{l|ll} m & a = b & \\ n & A(\ldots a \ldots a \ldots) & \\ & A(\ldots b \ldots a \ldots) & \text{=E } m, n \end{array}$$

$$\begin{array}{l|ll} m & a = b & \\ n & A(\ldots b \ldots b \ldots) & \\ & A(\ldots a \ldots b \ldots) & \text{=E } m, n \end{array}$$

2.7 Replacement rules

Commutivity (Comm)

$$(A \,\&\, B) \Longleftrightarrow (B \,\&\, A)$$
$$(A \vee B) \Longleftrightarrow (B \vee A)$$
$$(A \leftrightarrow B) \Longleftrightarrow (B \leftrightarrow A)$$

DeMorgan (DeM)

$$\neg(A \vee B) \Longleftrightarrow (\neg A \,\&\, \neg B)$$
$$\neg(A \,\&\, B) \Longleftrightarrow (\neg A \vee \neg B)$$

Double Negation (DN)

$$\neg\neg A \Longleftrightarrow A$$

Material Conditional (MC)

$$(A \rightarrow B) \Longleftrightarrow (\neg A \vee B)$$
$$(A \vee B) \Longleftrightarrow (\neg A \rightarrow B)$$

Biconditional Exchange (↔ex)

$$[(A \rightarrow B) \,\&\, (B \rightarrow A)] \Longleftrightarrow (A \leftrightarrow B)$$

Quantifier Negation (QN)

$$\neg\forall x A \Longleftrightarrow \exists x \neg A$$
$$\neg\exists x A \Longleftrightarrow \forall x \neg A$$

Glossary

antecedent The sentence on the left side of a conditional.

argument a connected series of sentences, divided into premises and conclusion.

atomic sentence A sentence used to represent a basic sentence; a single letter in TFL, or a predicate symbol followed by names in FOL.

biconditional The symbol $\leftrightarrow$, used to represent words and phrases that function like the English phrase "if and only if"; or a sentence formed using this connective..

complete truth table A table that gives all the possible truth values for a sentence of TFL or sentences in TFL, with a line for every possible valuation of all atomic sentences.

completeness A property held by logical systems if and only if $\vDash$ implies $\vdash$.

conclusion the last sentence in an argument.

conclusion indicator a word or phrase such as "therefore" used to indicate that what follows is the conclusion of an argument.

conditional The symbol $\rightarrow$, used to represent words and phrases that function like the English phrase "if … then …"; a sentence formed by using this symbol.

conjunct A sentence joined to another by a conjunction.

conjunction The symbol $\&$, used to represent words and phrases that function like the English word "and"; or a sentence formed using that symbol.

connective A logical operator in TFL used to combine atomic sentences into larger sentences.

consequent The sentence on the right side of a conditional.

contingent sentence A sentence that is neither a necessary truth nor a necessary falsehood; a sentence that in some situations is true and in others false.

contradiction (of TFL) A sentence that has only Fs in the column under the main logical operator of its complete truth table; a sentence that is false on every valuation.

disjunct A sentence joined to another by a disjunction.

disjunction The connective $\vee$, used to represent words and phrases that function like the English word "or" in its inclusive sense; or a sentence formed by using this connective.

equivalence (in TFL) A property held by pairs of sentences if and only if the complete truth table for those sentences has identical columns under the two main logical operators, i.e., if the sentences have the same truth value on every valuation.

invalid A property of arguments that holds when it is possible for the premises to be true without the conclusion being true; the opposite of valid.

joint consistency (in TFL) A property held by sentences if and only if the complete truth table for those sentences contains one line on which all the sentences are true, i.e., if some valuation makes all the sentences true.

joint possibility A property possessed by some sentences when they can all be true at the same time.

logical validity (in TFL) A property held by arguments if and only if the complete truth table for the argument contains no rows where the premises are all true and the conclusion false, i.e., if no valuation makes all premises true and the conclusion false.

main connective The last connective that you add when you assemble a sentence using the recursive definition..

metalanguage The language logicians use to talk about the object language. In this textbook, the metalanguage is English, supplemented by certain symbols like metavariables and technical terms like "valid.".

metavariables A variable in the metalanguage that can represent any sentence in the object language..

necessary equivalence A property held by a pair of sentences that must always have the same truth value.

necessary falsehood A sentence that must be false.

necessary truth A sentence that must be true.

negation The symbol $\neg$, used to represent words and phrases that function like the English word "not".

object language A language that is constructed and studied by logicians. In this textbook, the object languages are TFL and FOL..

premise a sentence in an argument other than the conclusion.

premise indicator a word or phrase such as "because" used to indicate that what follows is the premise of an argument.

proof A sequence of sentences. The first sentences of the sequence are assumptions; these are the premises of the argument. Every sentence later in the sequence follows from earlier sentences by one of the rules of TFL. The final sentence of the sequence is the conclusion of the argument.

provable equivalence A property held by pairs of statements if and only if there is a derivation which takes you from each one to the other one.

provable inconsistency Sentences are provably inconsistent iff a contradiction can be derived from them.

scope The sentences that are joined by a connective. These are the sentences the connective was applied to when the sentence was assembled using a recursive definition..

sentence of TFL A string of symbols in TFL that can be built up according to the recursive rules given on p. 37..

sound A property of arguments that holds if the argument is valid and has all true premises.

soundness A property held by logical systems if and only if $\vdash$ implies $\vDash$.

symbolization key A list that shows which English sentences are represented by which atomic sentences in TFL.

tautology A sentence that has only Ts in the column under the main logical operator of its complete truth table; a sentence that is true on every valuation.

theorem A sentence that can be proved without any premises.

truth value One of the two logical values sentences can have: True and False.

valid A property of arguments where it is impossible for the premises to be true and the conclusion false.

valuation An assignment of truth values to particular atomic sentence of TFLs.

In the Introduction to his volume *Symbolic Logic*, Charles Lutwidge Dodson advised: "When you come to any passage you don't understand, *read it again*: if you *still* don't understand it, *read it again*: if you fail, even after *three* readings, very likely your brain is getting a little tired. In that case, put the book away, and take to other occupations, and next day, when you come to it fresh, you will very likely find that it is *quite* easy."

The same might be said for this volume, although readers are forgiven if they take a break for snacks after *two* readings.