forall x

THE MISSISSIPPI STATE EDITION

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Preface

I will begin by quoting E. J. Lemmon.

> It is not easy, and perhaps not even useful, to explain briefly what logic is. Like most subjects, it comprises many different kinds of problem and has no exact boundaries; at one end, it shades off into mathematics, at another, into philosophy. The best way to find out what logic is is to do some. (1965, p. 1)

He then continues, "None the less, a few very general remarks about the subject may help to set the stage for the rest of this book." Following his lead, here are some general remarks. First, formal logic is the study of a formal language. Unlike natural languages (such as English, Spanish, and Mandarin), in a formal language, every part of the language is precisely defined. Using a formal language limits what we can do. Natural languages are extremely flexible and adaptable; the formal language that we will use is not. The trade-off, however, is that our formal language is very precise, and it makes clear some of the fundamental aspects of human reasoning.

Second, this textbook covers *truth-functional logic* (TFL). (For reference, TFL also goes by other names: *propositional logic* or *sentential logic*.) In truth functional logic, individual statements (e.g., "the cat is on the mat" or "Westerby is talking with Ricardo") are treated as units that can be combined into more complex statements with 'or', 'and', 'not', 'if ... then ...', and 'if and only if'. The study of truth functional logic, then, is the study of the properties of these more complex statements and the logical relationships between them.

Incidentally, the title *forallx* (i.e., "for all x") is a reference to *first-order logic*, a branch of logic that is often studied following TFL. This textbook does not cover first-order logic, but, in any event, this is a symbolic expression in first-order logic: $\forall x(Kx \rightarrow Gx)$, and it is read, "for all x, if

x is K, then x is G." Hence, the name of the textbook. (If, for instance, K stands for "is a king," and G stands for "is greedy," then $\forall x(Kx \rightarrow Gx)$ means "for all x, if x is a king, then x is greedy," or "everyone who is a king is greedy.")

For instructors

This textbook covers truth functional propositional logic only. The rules introduced in chapter 14 are similar to those used in Allen and Hand's *Logic Primer*. (See section B.3 in the appendix for the exact list.) Proofs are constructed using Fitch notation, not the Lemmon-style system used by Allen and Hand.

This book is based on a text originally written by P. D. Magnus and revised and expanded by Tim Button, J. Robert Loftis, Aaron Thomas-Bolduc, and Richard Zach. I have made additional revisions, taken out chapters that are not needed for the 1000-level logic course at Mississippi State University, and added instructions for using the logic software Carnap (http://carnap.io/), which can be used in conjunction with Parts 3 and 4. The resulting text is licensed under a Creative Commons Attribution-ShareAlike 4.0 license.

Part 1

Key notions of logic

1 Arguments

In everyday discourse, the word ‘argument’ generally refers to something along the lines of a belligerent shouting match. Logic, however, is not concerned with arguments in that sense of the word. An argument, as we will understand it, is something more like this:

1. It is raining heavily.
2. If you do not take an umbrella, you will get soaked.
3. Therefore, you should take an umbrella.

Here, we have a series of sentences. The first two sentences are the *premises* of the argument, and the final sentence is the *conclusion* of the argument. If you believe the premises, then the argument, perhaps, provides you with a reason to believe the conclusion. We will say that an argument is any collection of premises, together with a conclusion.

Chapters 1 - 3 cover some basic logical notions that apply to arguments in a natural language like English. It is important to begin with a clear understanding of what arguments are and of what it means for an argument to be valid. Later we will translate arguments from English into a formal language. We want formal validity, as defined in the formal language, to have at least some of the important features of natural-language validity.

In the example just given, we used individual sentences to express both of the argument’s premises, and we used a third sentence to express the argument’s conclusion. Many arguments are expressed in this way, but a single sentence can contain a complete argument, as is shown here:

> Joan was wearing sunglasses, and so it must have been sunny.

This argument has one premise and a conclusion, which are separated by the ‘and’.

Many arguments start with premises, and end with a conclusion, but not all do. The argument with which this section began might equally have been presented with the conclusion at the beginning, like so:

> You should take an umbrella. After all, it is raining heavily. And if you do not take an umbrella, you will get soaked.

Equally, it might have been presented with the conclusion in the middle:

> It is raining heavily. Accordingly, you should take an umbrella, given that if you do not take an umbrella, you will get soaked.

When approaching an argument, we want to know whether or not the conclusion follows from the premises. So the first thing to do is to identify the premise or premises and the conclusion. As a guide, these words are often used to indicate an argument's conclusion:

so, therefore, hence, thus, accordingly, consequently

By contrast, these expressions often indicate that we are dealing with a premise, rather than a conclusion:

since, because, given that

1.1 Sentences

To be perfectly general, we can define an ARGUMENT as a series of sentences. The sentences at the beginning of the series are premises. The final sentence in the series is the conclusion. If the premises are true and the argument is a good one, then you have a reason to accept the conclusion.

To be a premise or conclusion of an argument, it must possible for the sentence to be true or false. So, for our purposes, by SENTENCE, we mean a declarative sentence.

Questions 'Are you sleepy yet?' is an interrogative sentence. Although you might be sleepy or you might be alert, the question itself is neither true nor false. For this reason, questions will not count as sentences in logic. Suppose you answer the question: 'I am not sleepy.' This is either true or false, and so it is a sentence in the logical sense. Generally, *questions* will not count as sentences, but *answers* will.

Imperatives Commands, for instance, 'Wake up!', 'Sit up straight', and so on, are imperative sentences. Although it might be good for you to sit up straight or it might not, the command is neither true nor false. Note, however, that commands are not always phrased as imperatives. 'You will respect my authority' *is* either true or false— either you will or you will not— and so it counts as a sentence in the logical sense.

Exclamations 'Ouch!' is sometimes called an exclamatory sentence, but it is neither true nor false. We will treat 'Ouch, I hurt my toe!' as meaning the same thing as 'I hurt my toe.' The 'ouch' does not add anything that could be true or false.

1.2 Truth values

As we said, arguments consist of premises and a conclusion, and certain kinds of English sentence cannot be used to express a premise or conclusion of an argument. Those are *questions*, *imperatives* (or *commands*), and *exclamations*. The common feature of these three kinds of sentence is that they are not *assertoric*: they cannot be true or false. It does not even make sense to ask whether a question is true. It only makes sense to ask whether it is true that someone asked the question or whether the answer to a question is true.

The general point is that, the premises and conclusion of an argument must be capable of having a TRUTH VALUE. The two truth values that concern us are 'true' and 'false'.

Practice exercises

At the end of some chapters, there are exercises that review and explore the material covered in the chapter. There is no substitute for actually working through some problems, because learning logic is more about developing a way of thinking than it is about memorizing facts.

So here's the first exercise. Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

2 Valid arguments

In chapter 1, we gave a very permissive account of what an argument is. To see just how permissive it is, consider the following:

1. There is a bassoon-playing dragon in the *Cathedra Romana*.
2. Therefore, Salvador Dali was a poker player.

We have been given a premise and a conclusion. So we have an argument. Admittedly, it is a *terrible* argument, but it is still an argument.

2.1 Two ways that arguments can go wrong

It is worth pausing to ask what makes the argument so weak. In fact, there are two sources of weakness. First, the argument's premise is obviously false. The Pope's throne is only ever occupied by a hat-wearing man. Second, the conclusion does not follow from the premise of the argument. Even if there were a bassoon-playing dragon in the Pope's throne, we would not be able to draw any conclusion about Dali's predilection for poker.

What about the first argument discussed in chapter 1? The premises of this argument might well be false. It might be sunny outside; or it might be that you can avoid getting soaked without taking an umbrella. But even if both premises were true, it does not necessarily show you that you should take an umbrella. Perhaps you enjoy walking in the rain, and you would like to get soaked. So, even if both premises were true, the conclusion might nonetheless be false.

Consider a third argument:

1. You are reading this book.
2. This is a logic book.
3. Therefore, you are a logic student.

This is not a terrible argument. The premises are true, and most people who read this book are logic students. Yet, it is possible for someone besides a logic student to read it. If your roommate picked up the book and began looking through it, he or she would not immediately become a logic student. So the premises of this argument, even though they are true, do not guarantee the truth of the conclusion.

The general point is that, for any argument, there are two ways that it might go wrong:

- One or more of the premises might be false.
- The conclusion might not follow with certainty from the premises.

It is often important to determine whether or not the premises of an argument are true. However, that is normally a task best left to experts in the field: as it might be historians, scientists, or whomever. In our role as *logicians*, we are more concerned with arguments *in general*. Hence, we are (usually) more concerned with the second way in which arguments can go wrong. That is, we are interested in whether or not a conclusion *follows from* some premises.

2.2 Validity

As logicians, we want to be able to determine when the conclusion of an argument follows from the premises. One way to put this is as follows. We want to know whether, if all the premises were true, the conclusion would also have to be true. This motivates the definition of valid.

Valid

An argument is VALID if and only if it is impossible for all of the premises to be true and the conclusion false.

Consider this example:

1. Oranges are either fruits or musical instruments.
2. Oranges are not fruits.
3. Therefore, oranges are musical instruments.

The conclusion of this argument is ridiculous. Nevertheless, it follows from the premises. *If* both premises are true, *then* the conclusion just has to be true. So the argument is valid.

That example illustrates that valid arguments do not need to have true premises or true conclusions. Conversely, having true premises and a true conclusion is not enough to make an argument valid. Consider this example:

1. London is in England.
2. Beijing is in China.
3. Therefore, Paris is in France.

The premises and conclusion of this argument are, as a matter of fact, all true, but the argument is invalid. If Paris were to declare independence from the rest of France, then the conclusion would be false, even though both of the premises would remain true. Thus, it is *possible* for the premises of this argument to be true and the conclusion false. So the argument is invalid.

The important thing to remember is that validity is not about the actual truth or falsity of the sentences in the argument. It is about whether it is *possible* for all the premises to be true and the conclusion false. Going a step further, however, we will say that an argument is SOUND if and only if it is both valid and all of its premises are true.

2.3 Inductive arguments

Many good arguments are invalid. Consider this one:

1. In January 2016, it rained in London.
2. In January 2017, it rained in London.
3. In January 2018, it rained in London.
4. In January 2019, it rained in London.
5. Therefore, it rains every January in London.

This argument generalizes from observations about several cases to a conclusion about all cases. Such arguments are called INDUCTIVE arguments. The argument could be made stronger by adding additional premises before drawing the conclusion: In January 2015, it rained in

London; In January 2014, it rained in London; and so on. But, however many premises of this form we add, the argument will remain invalid. Even if it has rained in London in every January thus far, it remains *possible* that London will stay dry next January.

The point of all this is that inductive arguments—even good inductive arguments—are not (deductively) valid. They are not *watertight*. Unlikely though it might be, it is *possible* for their conclusion to be false, even when all of their premises are true. In this book, we will set aside the question of what makes for a good inductive argument. Our interest is simply in sorting the valid arguments from the invalid ones.

Practice exercises

A. Which of the following arguments are valid? Which are invalid?

1. Socrates is a man.
2. All men are carrots.
3. Therefore, Socrates is a carrot.

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.
3. Therefore, Abe Lincoln was born in Illinois.

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.
3. Therefore, Abe Lincoln will not die.

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.
3. Therefore, Abe Lincoln was from France.

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.
3. Therefore, The world will not end today.

1. Joe is now 19 years old.
2. Joe is now 87 years old.

3. Therefore, Bob is now 20 years old.

B. For each statement, determine if it is possible or not. If it is possible, given an example as illustration. If it is not possible, then explain why it isn't.

1. A valid argument that has one false premise and one true premise
2. A valid argument that has only false premises
3. A valid argument with only false premises and a false conclusion
4. An invalid argument that can be made valid by the addition of a new premise
5. A valid argument that can be made invalid by the addition of a new premise

3 Other logical notions

The concept of a valid argument is central to logic. In this section, we will introduce some other important concepts that apply just to sentences, not to full arguments.

3.1 Joint possibility

Consider these two sentences:

B1. Jane's only brother is shorter than her.
B2. Jane's only brother is taller than her.

Logic alone cannot tell us which, if either, of these sentences is true. Yet we can say that *if* B1 is true, *then* B2 must be false. Similarly, if B2 is true, then B1 must be false. It is impossible that both sentences are true at the same time. In other words, these sentences are inconsistent. On the other hand, G1 and G2 can both be true at the same time.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

One of these sentences may be false and the other true, but it is *possible* that they are both true at the same time. These observations motivate the following definitions.

> Sentences are JOINTLY POSSIBLE if and only if it is possible for them all to be true together.
> Sentences are JOINTLY IMPOSSIBLE if and only if it is *not* possible for them all to be true together.

So, G1 and G2 are *jointly possible* while B1 and B2 are *jointly impossible*.

We can investigate the joint possibility of any number of sentences. For example, let's add two more sentences to G1 and G2:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two extra-terrestrials at the wild animal park.
G4. Every giraffe at the wild animal park is an extra-terrestrial.

Together, G1 and G4 entail that there are at least four extra-terrestrials giraffes at the park. This conflicts with G3, which implies that there are no more than two extra-terrestrial giraffes there. So the sentences G1–G4 are jointly impossible. They cannot all be true together. (Note that the sentences G1, G3 and G4 are jointly impossible. G1, G2, and G3, meanwhile, are jointly possible.)

3.2 Necessary truths, necessary falsehoods, and contingency

In assessing arguments for validity, we care about what would be true *if* the premises were true, but some sentences just *must* be true. Consider these sentences:

a. It is raining.
b. Either it is raining here, or it is not.
c. It is both raining here and not raining here.

In order to know if sentence *a* is true, you would need to look outside or check the weather channel. It might be true; it might be false. A sentence which is capable of being true and capable of being false (in different circumstances, of course) is called CONTINGENT.

Sentence *b* is different. You do not need to look outside to know that it is true. Regardless of what the weather is like, it is either raining or it is not. That is a NECESSARY TRUTH.

Equally, you do not need to check the weather to determine whether or not sentence *c* is true. It must be false, simply as a matter of logic. It might be raining here and not raining across town; it might be raining now but stop raining even as you finish this sentence; but it is impossible for it to be both raining and not raining in the same place and at the same time. So, whatever the world is like, it is not both raining here and not raining here. It is a NECESSARY FALSEHOOD.

Finally, one thing to note is that a sentence might always be true and still be contingent. For instance, if there never were a time when the universe contained fewer than seven objects, then the sentence 'At least seven objects exist' would always be true. Yet the sentence is contingent. The universe *could have been* much, much smaller than it is, and then the sentence would be false.

Necessary equivalence

We can also ask about the logical relations *between* two sentences. For example:

> John went to the store after he washed the dishes.
> John washed the dishes before he went to the store.

These two sentences are both contingent, since John might not have gone to the store or washed dishes at all. Yet they must have the same truth-value. That is, they must either both be true or both be false. When two sentences necessarily have the same truth value, we say that they are NECESSARILY EQUIVALENT.

Summary of logical notions

> An argument is (deductively) VALID if it is impossible for the premises to be true and the conclusion false. It is INVALID otherwise.

> A collection of sentences is JOINTLY POSSIBLE if it is possible for all these sentences to be true together; it is JOINTLY IMPOSSIBLE otherwise.

> A NECESSARY TRUTH is a sentence that must be true; it could not possibly be false.

> A NECESSARY FALSEHOOD is a sentence that must be false; it could not possibly be true.

> A CONTINGENT SENTENCE is neither a necessary truth nor a necessary falsehood. It may be true or it may not.

Two sentences are NECESSARILY EQUIVALENT if they must have the same truth value. (I.e., they must both be true or they both must be false.)

3.3 Practice exercises

A. Determine if each sentence is a necessary truth, a necessary falsehood, or contingent.

1. Caesar crossed the Rubicon.
2. Someone once crossed the Rubicon.
3. No one has ever crossed the Rubicon.
4. If Caesar crossed the Rubicon, then someone has.
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
6. If anyone has ever crossed the Rubicon, it was Caesar.
7. Elephants dissolve in water.
8. Wood is a light, durable substance useful for building things.
9. If wood is a good building material, it is useful for building things.
10. I live in a three story building that is two stories tall.
11. If gerbils are mammals, they nurse their young.

B. Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.
   If you put an elephant in water, it will dissolve.
2. All mammals dissolve in water.
   If you put an elephant in water, it will dissolve.
3. George Bush was the 43rd president.
   Barack Obama was the 44th president.
4. Barack Obama was the 44th president.
   Barack Obama was president immediately after the 43rd president.
5. Elephants dissolve in water.
   All mammals dissolve in water.
6. Thelonious Monk played piano.
   John Coltrane played tenor sax.

7. Thelonious Monk played with John Coltrane.
   John Coltrane played with Thelonious Monk.
8. All professional pianists begin playing as young children.
   The professional pianist Bud Powell began playing as a young child.
9. Bud Powell suffered from severe mental illness.
   All professional pianists suffer from severe mental illness.
10. John Coltrane was deeply religious.
    John Coltrane viewed music as an expression of spirituality.

C.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two Martians at the wild animal park.
G4. Every giraffe at the wild animal park is a Martian.

Determine if each set of sentences is jointly possible or jointly impossible.

1. Sentences G2, G3, and G4
2. Sentences G1, G3, and G4
3. Sentences G1, G2, and G4
4. Sentences G1, G2, and G3

D.

M1. All people are mortal.
M2. Socrates is a person.
M3. Socrates will never die.
M4. Socrates is mortal.

Determine if each set of sentences is jointly possible or jointly impossible.

1. Sentences M1, M2, and M3
2. Sentences M2, M3, and M4
3. Sentences M2 and M3
4. Sentences M1 and M4

5. Sentences M1, M2, M3, and M4

E. For each statement, determine whether or not it is possible. If it is possible, give an example that illustrates the statement. If it is not possible, explain why not.

1. A valid argument that has one false premise and one true premise
2. A valid argument that has a false conclusion
3. A valid argument, the conclusion of which is a necessary falsehood
4. An invalid argument, the conclusion of which is a necessary truth
5. A necessary truth that is contingent
6. Two necessarily equivalent sentences, both of which are necessary truths
7. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
8. Two necessarily equivalent sentences that together are jointly impossible
9. A jointly possible collection of sentences that contains a necessary falsehood
10. A jointly impossible set of sentences that contains a necessary truth
11. A valid argument, whose premises are all necessary truths, and whose conclusion is contingent
12. A valid argument with true premises and a false conclusion
13. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
14. A jointly possible collection of sentences, all of which are contingent
15. A false necessary truth
16. A valid argument with false premises
17. A necessarily equivalent pair of sentences that are not jointly possible
18. A necessary truth that is also a necessary falsehood
19. A jointly possible collection of sentences that are all necessary falsehoods

3.4 Answers

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon.
   Contingent
2. Someone once crossed the Rubicon.
   Contingent
3. No one has ever crossed the Rubicon.
   Contingent
4. If Caesar crossed the Rubicon, then someone has.
   Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
   Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar.
   Contingent
7. Elephants dissolve in water.
   Contingent
8. Wood is a light, durable substance useful for building things.
   Contingent
9. If wood is a good building material, it is useful for building things.
   Necessarily true
10. I live in a three story building that is two stories tall.
    Necessarily false
11. If gerbils are mammals, they nurse their young.
    This sentence is necessarily true. (*Mammalia* is defined as the class of animals wherein the females have mammaries and nurse their young. Hence, 'If gerbils are mammals, they nurse their young' is necessarily true.)

B.

1. Elephants dissolve in water.
   If you put an elephant in water, it will dissolve.

These sentences are necessarily equivalent.

2. All mammals dissolve in water.
   If you put an elephant in water, it will dissolve.
   These sentences are *not* necessarily equivalent.
3. George Bush was the 43rd president.
   Barack Obama was the 44th president.
   These sentences are *not* necessarily equivalent.
4. Barack Obama was the 44th president.
   Barack Obama was president immediately after the 43rd president.
   These sentences are necessarily equivalent.
5. Elephants dissolve in water.
   All mammals dissolve in water.
   These sentences are *not* necessarily equivalent.
6. Thelonious Monk played piano.
   John Coltrane played tenor sax.
   These sentences are *not* necessarily equivalent.
7. Thelonious Monk played with John Coltrane.
   John Coltrane played with Thelonious Monk.
   These sentences are necessarily equivalent.
8. All professional pianists begin playing as young children.
   The professional pianist Bud Powell began playing as a young child.
   These sentences are *not* necessarily equivalent.
9. Bud Powell suffered from severe mental illness.
   All professional pianists suffer from severe mental illness.
   These sentences are *not* necessarily equivalent.
10. John Coltrane was deeply religious.
    John Coltrane viewed music as an expression of spirituality.
    These sentences are *not* necessarily equivalent.

C.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two Martians at the wild animal park.

G4. Every giraffe at the wild animal park is a Martian.

1. Sentences G2, G3, and G4 — Jointly possible
2. Sentences G1, G3, and G4 — Jointly impossible
3. Sentences G1, G2, and G4 — Jointly possible
4. Sentences G1, G2, and G3 — Jointly possible

D.

M1. All people are mortal.
M2. Socrates is a person.
M3. Socrates will never die.
M4. Socrates is mortal.

1. Sentences M1, M2, and M3 — Jointly impossible
2. Sentences M2, M3, and M4 — Jointly impossible
3. Sentences M2 and M3 — Jointly possible
   Person, at least in the philosophical sense, is different than *human being* (although the two concepts generally overlap). *Person* means, basically, *moral agent*, and so, for instance, God, if he exists, is a person. Consequently, just the sentence 'Socrates is a person' doesn't tell us whether or not Socrates will die.
4. Sentences M1 and M4 — Jointly possible
5. Sentences M1, M2, M3, and M4 — Jointly impossible

E.

1. A valid argument that has one false premise and one true premise
   Yes, this is possible.
   'All whales are mammals (*true*). All mammals are plants (*false*). Therefore, all whales are plants.'
2. A valid argument that has a false conclusion
   Yes, this is possible. See example from previous exercise.
3. A valid argument, the conclusion of which is a necessary falsehood
   Yes, this is possible. '$1+1=3$. Therefore, $1+2=4$.'
4. An invalid argument, the conclusion of which is a necessary truth

No, this is not possible. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.

5. A necessary truth that is contingent
   No, this is not possible. If a sentence is a necessary truth, it cannot possibly be false, but a contingent sentence can be false.
6. Two necessarily equivalent sentences, both of which are necessary truths
   Yes, this is possible. '4 is even', '4 is divisible by 2'.
7. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
   No, this is not possible. A necessary truth cannot possibly be false, while a contingent sentence can be false. So in any situation in which the contingent sentence is false, it will have a different truth value from the necessary truth. Thus, they will not necessarily have the same truth value, and so they will not be equivalent.
8. Two necessarily equivalent sentences that together are jointly impossible
   Yes, this is possible. '$1 + 1 = 4$' and '$1 + 1 = 3$'.
9. A jointly possible collection of sentences that contains a necessary falsehood
   No, this is not possible. If a sentence is necessarily false, there is no way to make it true, let alone make it true along with all the other sentences.
10. A jointly impossible set of sentences that contains a necessary truth
    Yes, this is possible. '$1 + 1 = 4$' and '$1 + 1 = 2$'.
11. A valid argument, whose premises are all necessary truths, and whose conclusion is contingent
12. A valid argument with true premises and a false conclusion
13. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
14. A jointly possible collection of sentences, all of which are contingent
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16. A valid argument with false premises
17. A necessarily equivalent pair of sentences that are not jointly possible
18. A necessary truth that is also a necessary falsehood
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Part 2

Truth-functional logic

4 Symbolization, part 1

4.1 Validity in virtue of form

Consider this argument:

1. It is raining outside.
2. If it is raining outside, then Jenny is miserable.
3. Therefore, Jenny is miserable.

and this one:

1. Jenny is a student.
2. If Jenny is a student, then John is a spy.
3. Therefore, John is a spy.

Both arguments are valid, and there is a straightforward sense in which we can say that they share a common structure. We might express the structure this way:

1. A
2. If A, then C
3. Therefore, C

This looks like an excellent argument *structure*. Indeed, any argument with this structure or form will be valid. Now, consider this argument:

1. Jenny is either happy or sad.
2. Jenny is not happy.
3. Therefore, Jenny is sad.

Again, this argument is valid, and this is its structure:

1. A or B
2. not A
3. Therefore, B

Here is another example:

1. It's not the case that Jim both studied often and acted in lots of plays.
2. Jim acted in lots of plays.
3. Therefore, Jim did not study often.

This valid argument has a structure which we might represent this way:

1. not (A and B)
2. A
3. Therefore, not B

These examples illustrate an important idea, which we might describe as *validity in virtue of form*. These arguments are valid, but in each case, that has nothing to do with the specific meaning of 'Jenny is sad', 'John is a spy', or 'Jim acted in lots of plays'. Instead, these arguments are valid in virtue of the meanings of just these words: 'and', 'or', 'not,' and 'if..., then...'.

4.2 Validity for special reasons

There are plenty of arguments that are valid, but not for reasons relating to their structure. This an example:

1. Juanita is a vixen
2. Therefore, Juanita is a fox

It is impossible for the premise to be true and the conclusion false. So the argument is valid. However, the validity is not related to the form of the argument. Here is an invalid argument with the same form:

1. Juanita is a vixen
2. Therefore, Juanita is a cathedral

This suggests that the validity of the previous argument *is* keyed to the meaning of the words 'vixen' and 'fox'. But, whether or not that is right, it is not simply the structure of the argument that makes it valid. Equally, consider the argument:

1. The sculpture is green all over.
2. Therefore, the sculpture is not red all over.

Again, it seems impossible for the premise to be true and the conclusion false, for nothing can be both green all over and red all over. So the argument is valid, but here is an invalid argument with the same form:

1. The sculpture is green all over.
2. Therefore, the sculpture is not shiny all over.

The argument is invalid, since it is possible to be green all over and shiny all over. Plausibly, the first argument about the sculpture is valid because of the way that colors (or color-words) interact, but, whether or not that is right, it is not simply the structure of the argument that makes it valid.

The important point here is that we will be interested only in arguments that are valid or invalid because of their structure.

4.3 Atomic sentences and symbolization

We isolated the form of the arguments, in §4.1 by replacing sentences and subsentences of sentences with individual letters. 'It is raining outside' is a subsentence of 'If it is raining outside, then Jenny is miserable', and we replaced that subsentence with 'A'.

This kind of representation—letters standing for sentences or subsentences—is central to the formal language that we develop in this book. We start with some *atomic sentences*. Notice that if we extract 'it is raining outside' and 'Jenny is miserable' from 'If it is raining outside, then Jenny is miserable', both 'it is raining outside' and 'Jenny is miserable' are, themselves, complete sentences. That is, they contain a subject, verb, and direct object. If we extract any part of 'it is raining outside', however, we will not have a complete sentence. Thus, in terms of sentences 'it is raining outside' is an atom, or, as we will call it, an *atomic sentence*. It's the smallest collection of words that still constitute a sentence.

Similarly, 'Jenny is miserable', 'Jenny is a student', 'John is a spy', and 'Jenny is happy' are atomic sentences. On the other hand, 'If it is raining outside, then Jenny is miserable' and 'Jenny is either happy or sad' are not atomic sentences. They are both sentences that are constructed out of two atomic sentences.

Atomic sentences are the basic building blocks used to form more complex sentences. We will use uppercase Roman letters for atomic sentences of TFL. There are only twenty-six letters of the alphabet, but there is no limit to the number of atomic sentences that we might want to consider. By adding subscripts to letters, we obtain new atomic sentences. Here, for instance, are five different atomic sentences of TFL:

$$A, R, R_1, R_2, A_{234}$$

We will use atomic sentences to represent, or *symbolize*, certain English sentences. To do this, we provide a SYMBOLIZATION KEY, such as the following:

A: It is raining outside
C: Jenny is miserable

In doing this, we are not fixing this symbolization *once and for all*. We are just saying that, for the time being, we will think of the atomic sentence 'A' as symbolizing the English sentence 'It is raining outside', and the atomic sentence of TFL, 'C', as symbolizing the English sentence 'Jenny is miserable'. Later, when we are dealing with different sentences or different arguments, we can provide a new symbolization key; as it might be:

A: Jenny is a student.
C: John is a spy.

5 Logical operators

At this point, we should clarify the task at hand. Truth-functional propositional logic is a branch of logic that focuses on the relationships between atomic sentences. One part of truth-functional propositional logic (or *TFL* for short) is a formal language. This formal language consists of sentence letters, which stand for atomic sentences of English (although we won't always be concerned about the specific English sentences that they might represent), and the *logical operators* 'and', 'or', 'not', 'if ..., then ...' and 'if and only if'. A logical operator is a word or phrase that modifies a sentence or connects two sentences to form a more complex sentence. We call these operators *truth-functional* because the truth of the complex sentences depends entirely on the truth of the atomic sentences of which they are composed. (They are also sometimes referred to as *connectives* because, except in the case of 'not', these operators connect two simpler sentences.)

In addition to symbolizing English sentences with sentence letters, we also want to symbolize the truth-functional operators. The symbols that we will use are shown in table 5.1. The operators listed there are not the only connectives in English. Others are, for example, 'unless', 'neither ... nor ...', 'necessarily', and 'because'. As we will see, the first two can be expressed with the connectives that are in table 5.1. The last two, however, cannot. Although they are logical operators, 'necessarily' and 'because' are not truth functional.

Once we have introduced these logical operators (in this chapter and in chapter 8) and explained what can and cannot be a sentence in TFL (which we will do in chapter 6) our formal language will be complete. Although the formal language is central, truth-functional propositional logic does not consist only of a formal language. There is also a *deductive system*, which we will explore in part 4.

SYMBOL	THE SENTENCE'S NAME	ITS MEANING
$\neg$	negation	'It is not the case that…'
&	conjunction	'Both… and …'
$\vee$	disjunction	'Either… or …'
$\rightarrow$	conditional	'If … then …'
$\leftrightarrow$	biconditional	'… if and only if …'

Table 5.1

5.1 Negation

Consider how we might symbolize these sentences:

1. Mary is in Barcelona.
2. It is not the case that Mary is in Barcelona.
3. Mary is not in Barcelona.

To begin, we need an atomic sentence. This will be our symbolization key:

B: Mary is in Barcelona.

B is sentence 1. Sentence 2 is partially symbolized as 'It is not the case that B'. In order to complete the symbolization, we need a symbol for 'it is not the case that'. Or, in other words, a symbol that, when added to B will express 'the negation of B'. We will use '$\neg$' and symbolize sentence 2 as '$\neg B$'.

Sentence 3 also contains the word 'not', and it is obviously equivalent to sentence 2. As such, we can also symbolize it as '$\neg B$'.

Negation

A sentence can be symbolized as $\neg$A if it can be paraphrased in English as 'It is not the case that … '

Here are a few more examples:

4. The cog can be replaced.
5. The cog is irreplaceable.
6. The cog is not irreplaceable.

For these, we will use this representation key:

R: The cog is replaceable

Sentence 4 is symbolized by 'R'. Sentence 5 can be reworded as *it is not the case that the cog is replaceable*. So even though sentence 5 does not contain the word 'not', we will symbolize it '$\neg R$'.

Sentence 6 can be paraphrased as 'It is not the case that the cog is irreplaceable.' That sentence can then be paraphrased as 'It is not the case that it is not the case that the widget is replaceable'. So we symbolize this English sentence as '$\neg\neg R$'.

But some care is needed when handling negations. Consider:

7. Jane is happy.
8. Jane is unhappy.

If we 'H' stand for 'Jane is happy', then we can symbolize sentence 7 as 'H'. It would be a mistake, however, to symbolize sentence 8 with '$\neg H$'. '$\neg H$' means 'Jane is not happy', but 'Jane is not happy' does not have the same meaning as 'Jane is unhappy'. After all, Jane might be neither happy nor unhappy; her affect might just be neutral. In order to symbolize sentence 8, then, we would need a different sentence letter.

5.2 Conjunction

Let's start with these sentences:

9. Adam is athletic.
10. Barbara is athletic.
11. Adam is athletic, and Barbara is also athletic.

We will need separate sentence letters to symbolize sentences 9 and 10, and so we will use these:

A: Adam is athletic.
B: Barbara is athletic.

Sentence 9 is symbolized as 'A', and sentence 10 as 'B'. Sentence 11 expresses 'A and B'. To symbolize the 'and'. We will use '&' (which is called the *ampersand*). Thus, sentence 11 becomes '$(A \& B)$'. When two sentences are connected with an '&', the resulting sentence is called a CONJUNCTION. The two sentences that are combined with the '&' are the CONJUNCTS of the conjunction. So, 'A' and 'B' are the conjuncts of the conjunction '$(A \& B)$'.

Notice that we don't need to symbolize the word 'also' in sentence 11. Words like 'both' and 'also' function to draw our attention to the fact that two things are being conjoined. Maybe they affect the emphasis of a sentence, but we will not (and cannot) symbolize such terms in TFL.

Let's look at some trickier conjunctions.

12. Barbara is athletic and smart.
13. Barbara and Adam are both athletic.
14. Although Lisa is smart, she is not athletic.
15. Adam is athletic, but Barbara is more athletic than him.

In each of these cases, we must, first, state each atomic sentence precisely, then it will be obvious what sentence letters we need and how to use the '&'.

The first, 'Barbara is athletic and smart' is actually expressing two atomic sentences: 'Barbara is athletic' and 'Barbara is smart'. Sentence 13 also contains two atomic sentences: 'Barbara is athletic' and 'Adam is athletic'. Notice that sentence 14 does not contain an 'and' at all. *Although* may have a slightly different meaning in English than *and*, but broadly speaking, they have the same meaning and perform the same role in sentences. As far as TFL is concerned, they are both conjunctions. Here, the conjunction is combining these two atomic sentences: 'Lisa is smart' and 'Lisa is athletic'. When symbolizing this sentence, though, we will also have to include the '$\neg$' to symbolize the 'not' in the second one.

We will get to sentence 15 in a moment, but right now, this will be our expanded symbolization key:

A: Adam is athletic.
B: Barbara is athletic.
C: Barbara is smart.
D: Lisa is smart.
E: Lisa is athletic.

With this key, we symbolize sentences 12 - 14 as follows.

12. $(B \& C)$
13. $(B \& A)$
14. $(D \& \neg E)$

Notice that we have lost all sorts of nuance by expressing sentence 14 as a sentence in TFL. There is a distinct difference in tone between the English version of sentence 14 and $(D \& \neg E)$, which is read as 'Both Lisa is smart and it is not the case that Lisa is athletic'. TFL does not (and cannot) preserve those sorts of nuances.

Sentence 15 raises a different issue. You might think, at this point, that there is some trick to representing this sentence with two of the letters given in the symbolization key above. The first half of sentence 15 is symbolized as 'A', but there is no way to use 'B' for 'Barbara is athletic' and then symbolize 'more than him' separately in TFL. (We cannot write $B > A$ in TFL.) Instead, we need a new sentence letter. Let the TFL sentence 'F' symbolize the English sentence 'Barbara is more athletic than Adam'. Now we can symbolize sentence 15 by '$(A \& F)$'.

> **Conjunction**
>
> A sentence can be symbolized as $(\mathsf{A} \& \mathsf{B})$ if it can be paraphrased any of these ways in English:
>
> 'Both…, and…',
> '…, and…',
> '…, but …',
> '…, although …',
> '…, as well as …'

Parentheses

You might be wondering why we put parentheses around the conjunctions. It is to help us make the meaning of the TFL expression precise. Consider these two sentences in English:

16. It's not the case that you will get both soup and salad.
17. You will not get soup but you will get salad.

For these, we will use this symbolization key:

S_1: You will get soup.
S_2: You will get salad.

Sentence 16 can be paraphrased as 'This is not the case: you will get soup and you will get salad'. We can symbolize the *you will get soup and you will get salad* part as '$(S_1 \,\&\, S_2)$'. To symbolize the full sentence, we simply add the negation symbol *outside* the parentheses: '$\neg(S_1 \,\&\, S_2)$'.

Sentence 17, meanwhile, also includes a 'not', but that 'not' only applies to S_1. You *will not* get soup, and you *will* get salad. The first part, 'you will not get soup' is symbolized as '$\neg S_1$', and the full sentence becomes '$(\neg S_1 \,\&\, S_2)$'.

Sentences 16 and 17 are different, and how we symbolize them differs accordingly. If we didn't use parentheses, then they would both be $\neg S_1 \,\&\, S_2$, which obviously isn't what we want. With the parentheses, we can show that, in 16, the entire conjunction is negated, while in 17 just one conjunct is negated. Brackets help us to keep track of the *scope* of the negation.

5.3 Disjunction

We will start with these sentences:

18. Either Mary will play a video game, or she will watch a movie.
19. Either Mary or Omar will play a video game.

And for these sentences, we will use this symbolization key:

F: Mary will play a video game.
O: Omar will play a video game.
M: Mary will watch a movie.

To represent the 'or' in sentences 18 and 19, we will use the symbol '$\vee$' (which we call the *wedge*, not v). Sentence 18, then, is written as '$(F \vee M)$'. When two sentences are connected with an '$\vee$', the resulting sentence is called a DISJUNCTION. 'F' and 'M' are the DISJUNCTS of the disjunction '$(F \vee M)$'.

Sentence 19 is only slightly more complicated. We can paraphrase it as 'Either Mary will play a video game, or Omar will play a video game', and symbolize it as '$(F \vee O)$'.

Disjunction

A sentence can be symbolized as $(A \lor B)$ if it can be paraphrased in English as 'Either…, or….' Each of the disjuncts must be a sentence.

The inclusive or

Sometimes in English, the word 'or' is used in a way that excludes the possibility that both disjuncts are true. This is called an EXCLUSIVE OR. An *exclusive or* is clearly intended when it says, on a restaurant menu, 'Entrees come with either soup or salad'. This means that, with your entree, you may have soup or you may have salad, but you cannot have both.

At other times, the word 'or' allows for the possibility that both disjuncts might be true. If Mary doesn't spend too much time with video games or movies, then she might say, "I will get an A in Logic or I will get an A in Twentieth Century U.S. History". She probably means that she will get an A in at least one *or both* of those courses. (After all, if she did end up getting an A in both, then we wouldn't insist that she was wrong when she said, "I will get an A in Logic or I will get an A in Twentieth Century U.S. History".)

When the intended meaning is that that one or the other or both of the disjuncts are true, then the INCLUSIVE OR is being used. The TFL symbol '$\lor$' always symbolizes an *inclusive or*.

Negation and disjunction

Last, let's look at these examples:

20. Either you will not have soup, or you will not have salad.
21. You will have neither soup nor salad.
22. You can have either soup or salad, but not both.

Using S_1 and S_2 again, sentence 20 is symbolized by '$(\neg S_1 \lor \neg S_2)$'.

Sentences 21 and 22 are a little trickier. Sentence 21 can be paraphrased as 'This is not the case: you have the soup or you will have the salad'. (If it helps, this is equivalent to 'You will not have the soup and you will not have the salad'.) But sticking with the disjunction, as our paraphrased sentence shows, we are negating the entire disjunction. Hence, we symbolize sentence 21 as '$\neg(S_1 \lor S_2)$'.

Because we are translating the sentence into TFL, the 'or' in sentence 22 has to be interpreted as the inclusive-or. The full sentence, however, expresses the meaning of the exclusive-or: one or the other, but not both. So how do we express that in TFL? We can break the sentence into two parts. The first part, 'you can have soup or you can have salad', is symbolized as '$(S_1 \lor S_2)$'. The second part says that you cannot have both. We can paraphrase this as: 'This not the case: you can have soup and you can have salad'. This, we symbolize as '$\neg(S_1 \,\&\, S_2)$'. Now we just need to put the two parts together. As we saw above, 'but' can usually be symbolized with '&'. Therefore, sentence 22 is '$((S_1 \lor S_2) \,\&\, \neg(S_1 \,\&\, S_2))$'.

This last example demonstrates that, although the TFL symbol '$\lor$' always stands for *inclusive or*, we can still express the *exclusive or* in TFL. We just have to use '$\neg$', '&', and '$\lor$'.

5.4 Conditional

We will start with this sentence:

23. If Jean is in Paris, then Jean is in France.

And we will use this symbolization key:

P: Jean is in Paris.
F: Jean is in France

Sentence 23 has this form: 'if P, then F', and any sentence with this form is called a CONDITIONAL. We will use '$\rightarrow$' to symbolize 'if . . . , then . . . '. Thus, sentence 23 becomes '$(P \rightarrow F)$'.

In a conditional, what goes before the '$\rightarrow$' is called the ANTECEDENT, and what comes after the '$\rightarrow$' is called the CONSEQUENT. So, in sentence 23, 'Jean is in Paris' is the antecedent, and 'Jean is in France' is the consequent.

If Jean is in Paris, then she is in France	If A, then B.
Jean is in France if she is in Paris.	B if A.
Whenever Jean is in Paris, she is in France.	Whenever A, B.
Jean is in France provided that she is in Paris.	B provided that A.
Provided that Jean is in Paris, she is in France.	Provided that A, B.
Jean is in Paris only if she is in France.	A only if B.

Table 5.2: The most common way of expressing a conditional in English is as 'If Jean is in Paris, then she is in France.' This table lists some alternative but equivalent ways of expressing the same sentence.

Conditional

A sentence can be symbolized as $\mathsf{A} \rightarrow \mathsf{B}$ if it can be paraphrased in English as 'If A, then B'.

Many English expressions can be represented using the conditional, and the most common alternatives to 'if A, then B' are listed in table 5.2. If you think about it, you'll see that all six of the sentences in the table have the same meaning, and so they can all be symbolized as '$P \rightarrow F$' (or generally, as '$\mathsf{A} \rightarrow \mathsf{B}$').

5.5 Biconditional

All of the logical operators that we have discussed so far are ones with which you were already familiar because you are an English speaker. The biconditional, which is mostly commonly expressed as ... *if and only if* ..., is one that you might not have really noticed before—even if you used it on occasion. We'll start with the basic case.

24. The Bearcats won if and only if they scored more points than the Razorbacks.

And this will be our symbolization key:

B: The Bearcats won.
R: The Bearcats scored more points than the Razorbacks.

The symbol '$\leftrightarrow$' will stand for 'if and only if', and so we can symbolize sentence 24 with the TFL sentence '$B \leftrightarrow R$'.

Now, let's probe a little further into the meaning of 'if and only if' with a different example.

25. If Mary has a sunburn, then she went to the beach.
26. If she went to the beach, then Mary has a sunburn.
27. If Mary has a sunburn, then she went to the beach, and if she went to the beach, then Mary has a sunburn.
28. Mary has a sunburn if and only if she went to the beach.

We will use this symbolization key:

S: Mary has a sunburn.
B: Mary wen to the beach.

From the previous section, you know how to symbolize sentences 25 and 26. (But notice that sentences 25 and 26 have different meanings.)

25. $(S \to B)$
26. $(B \to S)$

Sentence 27, then, is a conjunction created by combining 25 and 26:

27. $(S \to B) \,\&\, (B \to S)$

Maybe it is apparent to you right away, or maybe you need to ponder it (and we will return to this in chapter 8), but sentence 27 has the same meaning as sentence 28. Thus, $(S \to B) \,\&\, (B \to S)$ is equivalent to $(S \leftrightarrow B)$. We call sentences that have the form $\mathsf{A} \leftrightarrow \mathsf{B}$ BICONDITIONALS, because they are equivalent to the conditional in both directions.

The expression 'if and only if' occurs a lot in philosophy, mathematics, and logic, and sometimes you will see it abbreviated 'iff'. (Although even when 'iff' is written, we still say 'if and only if.') So 'if' with only *one* 'f' is the English conditional. But 'iff' with *two* 'f's is the English biconditional.

Biconditional

A sentence can be symbolized as $\mathsf{A} \leftrightarrow \mathsf{B}$ if it can be paraphrased in English as 'A iff B'—that is, as 'A if and only if B'.

A word of caution. Ordinary speakers of English often use 'if …, then…' when they really mean to use something more like '…if and only if …'. Perhaps your parents told you when you were a child: 'if you don't eat your vegetables, you won't get any dessert'. Suppose that you ate your vegetables, but that your parents refused to give you any dessert, on the grounds that they were only committed to the *conditional* (roughly 'if you get dessert, then you will have eaten your vegetables'), rather than the biconditional (roughly, 'you get dessert if and only if you eat your vegetables'). Despite the valuable lesson in truth functional propositional logic, you would have been upset. So, be aware of this when interpreting what people say, and in your own writing, make sure you use *if and only if* if and only if you mean to use it.

5.6 Unless

We have now introduced all of the logical operators of TFL. We can use them together to symbolize many kinds of sentences. An especially difficult case is when we use the English-language connective 'unless'. Take this sentence:

29. Unless you wear a jacket, you will catch a cold. (Or equivalently, 'You will catch a cold unless you wear a jacket'.)

To symbolize 29, we will use this symbolization key:

J: You will wear a jacket.
D: You will catch a cold.

Sentence 29 mean that if you do not wear a jacket, then you will catch a cold. With this in mind, we might symbolize it as '$\neg J \to D$'. Alternatively, it means that if you do not catch a cold, then you must have worn a jacket. With this in mind, we can symbolize it as '$\neg D \to J$'. And, finally, it also means that either you will wear a jacket or you will catch a cold. Hence, we can symbolize it as '$J \lor D$'.

All three ways of symbolizing sentence 29 are correct. Indeed, in chapter 10 we will see that all three symbolizations are equivalent in TFL. Following the somewhat standard practice, however, we will define *unless* as a disjunction.

Unless

If a sentence can be paraphrased as 'Unless A, B,' then it can be symbolized as '$\mathsf{A} \vee \mathsf{B}$'.

There is a complication with treating 'unless' as a disjunction, however. As we said earlier, 'or' has an inclusive and an exclusive meaning, but in TFL, 'or' is always inclusive. Ordinary speakers of English, however, often use 'unless' to mean something more like the exclusive-or. Suppose someone says: 'I will go running unless it rains'. They probably mean 'either I will go running or it will rain, but not both'. So, it can be argued that the conditional—e.g., 'if it does not rain, then I will go running' ($\neg R_a \rightarrow R_u$)—captures the meaning of 'unless' better than does the disjunction.

5.7 The turnstile

The final symbol that we need in not actually a symbol of TFL, but it is a useful symbol to have when displaying arguments in TFL. The symbol '$\vdash$' is called the *turnstile*. The purpose of the turnstile is to separate the sentences that are the premises of an argument from the sentence that is the conclusion, and it can be read as *therefore*. Here is an example,

$$(P \rightarrow C), (P \vee D) \vdash (\neg C \rightarrow D)$$

In this argument, the premises are '$(P \rightarrow C)$' and '$(P \vee D)$', and the conclusion is '$(\neg C \rightarrow D)$'.

Like the metavariables '$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$', '$\vdash$' is a symbol of our metalanguage, augmented English. The difference between the object language and the metalanguage is explained in §7.2.

5.8 Expressions in TFL

Some expressions in TFL will use only one logical operator, but many will contain multiple logical operators. These, for instance, each contain two or three:

$\neg C \vee D$	"Not C or D."
$(B \vee D) \rightarrow (C \,\&\, F)$	"If B or D, then C and F."
$P \leftrightarrow (R \,\&\, S)$	"P if and only if both R and S."
$\neg(Q \,\&\, R)$	"It is not the case that both Q and R."
	or "Not both Q and R."

Your first task is to recognize each logic operator and recall it's meaning. The next step is to translate expressions from English to TFL or vice versa when the expressions in TFL contain multiple logical operators. This takes practice, and you can consult the exercises in the next section to help you develop this skill.

5.9 Practice exercises

A. Using the symbolization key given, translate each English sentence into TFL.

A: Those creatures are aliens.
C: Those creatures are centaurs.
V: Those creatures are vampires.

Always use capital letters for the atomic sentences, and, in this case, be especially careful to distinguish between V and $\vee$.

1. Those creatures are not aliens.
2. Those creatures are aliens, or they are not.
3. Those creatures are either vampires or centaurs.
4. Those creatures are neither vampires nor centaurs.
5. If those creatures are centaurs, then it is not the case that they are vampires or aliens.
6. Either those creatures are aliens, or they are both centaurs and vampires.

B. Using the symbolization key given, translate each English sentence into TFL.

A: Mr. Adams was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mr. Edwards was murdered.
F: The murder weapon was a frying pan.

1. Either Mr. Adams or Mr. Edwards was murdered.
2. If Mr. Adams was murdered, then the cook did it.
3. If Mr. Edwards was murdered, then the cook did not do it.
4. Either the butler did it, or the Duchess is lying.
5. The cook did it only if the Duchess is lying.
6. If the murder weapon was not a frying pan, then the cook did not do it.

7. If the murder weapon was not a frying pan, then either the cook or the butler did it.
8. Mr. Adams was murdered if and only if Mr. Edwards was not murdered.
9. It is not the case that either the Duchess is lying or Mr. Edwards was not murdered.
10. If Mr. Adams was murdered, he was killed with a frying pan.
11. The cook did it, and the butler did not.
12. Of course the Duchess is lying!

C. Using the symbolization key given, translate each English sentence into TFL.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
2. If Ava is a firefighter, then she is satisfied with her career.
3. Ava is a firefighter, unless she is an electrician.
4. Harrison is an unsatisfied electrician.
5. Neither Ava nor Harrison is an electrician.
6. Both Ava and Harrison are electricians, but Ava is satisfied with her career and Harrison is not satisfied with his career.
7. Harrison is satisfied with his career only if he is a firefighter.
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
10. If Harrison is both an electrician and a firefighter, then he is satisfied with his career.
11. It is not the case that Harrison is both an electrician and a firefighter.
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.

D. Using the symbolization key given, translate each English-language sentence into TFL.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.
2. Neither Miles Davis nor John Coltrane played tuba.
3. John Coltrane did not play both tenor sax and tuba.
4. John Coltrane did not play tenor sax unless he also played soprano sax.
5. John Coltrane did not play tuba, but Miles Davis did.
6. Miles Davis played trumpet only if he also played tuba.
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
8. It is not the case that if John Coltrane played tuba then Miles Davis played trumpet or tuba.
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.

E. Give a symbolization key, and then translate the following English sentences into TFL.

1. It is not the case that Alice and Bob are both spies.
2. If either Alice or Bob is a spy, then the code has been broken.
3. If neither Alice nor Bob is a spy, then the code has not been unbroken.
4. The letter is in the German embassy, unless someone has broken the code.
5. Either the code has been broken or it has not, but the letter is in German embassy regardless.
6. Either Alice or Bob is a spy, but not both.

F. For each argument, first, make a symbolization key, and then translate all of the sentences of the argument into TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cross. Dorothy plays piano in the morning unless she is distracted. So, if Roger does not wake up cross, then Dorothy must be distracted.
2. It will either rain or snow on Tuesday. If it rains, Neville will be gloomy. If it snows, Neville will be cold. Therefore, Neville will either be gloomy or cold on Tuesday.
3. If Zoey remembered to do her chores, then the house is clean but not neat. If she forgot, then the house is neat but not clean. Therefore, the house is either neat or clean; but not both.

5.10 Answers

A.

A: Those creatures are aliens.
C: Those creatures are centaurs.
V: Those creatures are vampires.

1. Those creatures are not aliens.
   $\neg A$
2. Those creatures are aliens, or they are not.
   $(A \vee \neg A)$
3. Those creatures are either vampires or centaurs.
   $(V \vee C)$
4. Those creatures are neither vampires nor centaurs.
   $\neg(C \vee V)$
5. If those creatures are centaurs, then it is not the case that they are vampires or aliens.
   $(C \to \neg(V \vee A))$
6. Either those creatures are aliens, or they are both centaurs and vampires.
   $(A \vee (C \,\&\, V))$

B.

A: Mr. Adams was murdered.
B: The butler did it.

C: The cook did it.
D: The Duchess is lying.
E: Mr. Edwards was murdered.
F: The murder weapon was a frying pan.

1. Either Mr. Adams or Mr. Edwards was murdered.
   $(A \vee E)$
2. If Mr. Adams was murdered, then the cook did it.
   $(A \rightarrow C)$
3. If Mr. Edwards was murdered, then the cook did not do it.
   $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.
   $(B \vee D)$
5. The cook did it only if the Duchess is lying. (See table 5.2.)
   $(C \rightarrow D)$
6. If the murder weapon was not a frying pan, then the cook did not do it.
   $(\neg F \rightarrow \neg C)$
7. If the murder weapon was not a frying pan, then either the cook or the butler did it.
   $(\neg F \rightarrow (C \vee B))$
8. Mr. Adams was murdered if and only if Mr. Edwards was not murdered.
   $(A \leftrightarrow \neg E)$
9. It is not the case that either the Duchess is lying or Mr. Edwards was not murdered.
   $\neg(D \vee \neg E)$
10. If Mr. Adams was murdered, he was killed with a frying pan.
    $(A \rightarrow F)$
11. The cook did it, and the butler did not.
    $(C \,\&\, \neg B)$
12. Of course the Duchess is lying!
    D

C.

E_1: Ava is an electrician.

E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
   $(E_1 \& E_2)$
2. If Ava is a firefighter, then she is satisfied with her career.
   $(F_1 \rightarrow S_1)$
3. Ava is a firefighter, unless she is an electrician.
   $(F_1 \vee E_1)$
4. Harrison is an unsatisfied electrician.
   $(E_2 \& \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
   $\neg(E_1 \vee E_2)$
6. Both Ava and Harrison are electricians, but Ava is satisfied with her career and Harrison is not satisfied with his career.
   $((E_1 \& E_2) \& (S_1 \& \neg S_2))$
7. Harrison is satisfied with his career only if he is a firefighter.
   $(S_2 \rightarrow F_2)$
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
   $((\neg E_1 \rightarrow \neg E_2) \& (E_1 \rightarrow E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
   $(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he is satisfied with his career.
    $((E_2 \& F_2) \rightarrow S_2)$
11. It is not the case that Harrison is both an electrician and a firefighter.
    $\neg(E_2 \& F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
    $((F_2 \& F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.
   $J_1 \& J_2$
2. Neither Miles Davis nor John Coltrane played tuba.
   $\neg(M_2 \vee J_3)$ or $\neg M_2 \& \neg J_3$
3. John Coltrane did not play both tenor sax and tuba.
   $\neg(J_1 \& J_3)$ or $\neg J_1 \vee \neg J_3$
4. If John Coltrane did not play tenor sax, then he played soprano sax.
   $\neg J_1 \rightarrow J_2$
5. John Coltrane did not play tuba, but Miles Davis did.
   $\neg J_3 \& M_2$
6. Miles Davis played trumpet only if he also played tuba.
   $M_1 \rightarrow M_2$
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
   $M_1 \rightarrow (J_1 \vee (J_2 \vee J_3))$ or $M_1 \rightarrow ((J_1 \vee J_2) \vee J_3)$
8. It is not the case that if John Coltrane played tuba, then Miles Davis played trumpet or tuba.
   $\neg(J_3 \rightarrow (M_1 \vee M_2))$
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.
   $(J_3 \& M_2) \leftrightarrow (\neg J_1 \& \neg M_1)$ or $(J_3 \& M_2) \leftrightarrow \neg(J_1 \vee M_1)$

E.

A: Alice is a spy.

B: Bob is a spy.
C: The code has been broken.
L: The letter is in German embassy.

1. It is not the case that Alice and Bob are both spies.
   $\neg(A \& B)$
2. If either Alice or Bob is a spy, then the code has been broken.
   $((A \vee B) \rightarrow C)$
3. If neither Alice nor Bob is a spy, then the code has not been unbroken.
   $\neg(A \vee B) \rightarrow \neg C$
4. The letter is in the German embassy, unless someone has broken the code.
   $(L \vee C)$
5. Either the code has been broken or it has not, but the letter is in German embassy regardless.
   $((C \vee \neg C) \& L)$
6. Either Alice or Bob is a spy, but not both.
   $((A \vee B) \& \neg(A \& B))$

F. For each argument, write a symbolization key and symbolize all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cross. Dorothy plays piano in the morning unless she is distracted. So, if Roger does not wake up cross, then Dorothy must be distracted.

   P: Dorothy plays the piano in the morning.
   C: Roger wakes up cross.
   D: Dorothy is distracted.

   $(P \rightarrow C), (P \vee D) \vdash (\neg C \rightarrow D)$

2. It will either rain or snow on Tuesday. If it rains, Neville will be gloomy. If it snows, Neville will be cold. Therefore, Neville will either be gloomy or cold on Tuesday.

   T_1: It rains on Tuesday

T_2: It snows on Tuesday
G: Neville is gloomy on Tuesday
C: Neville is cold on Tuesday

$(T_1 \vee T_2), (T_1 \rightarrow G), (T_2 \rightarrow C) \vdash (G \vee C)$

3. If Zoey remembered to do her chores, then the house is clean but not neat. If she forgot, then the house is neat but not clean. Therefore, the house is either neat or clean; but not both.

   Z: Zoey remembered to do her chores
   C: The house is clean.
   N: The house is neat.

   $(Z \rightarrow (C \,\&\, \neg N)), (\neg Z \rightarrow (N \,\&\, \neg C)) \vdash ((N \vee C) \,\&\, \neg(N \,\&\, C))$.

6 Sentences of TFL

"Bring with thee airs from heaven or blasts from hell" is a sentence of English. '$(A \vee B)$' is a sentence of TFL. Oddly, although we can identify sentences of English when we encounter them, there is not a formal definition of *sentence of English* that will tell us, for any possible combination of words and punctuation, whether or not it is a sentence of English. It is possible, however, to provide such a definition for sentences of TFL, and we will examine that definition in this chapter.

6.1 Expressions

You have been introduced to the symbols of TFL in the previous two chapters. They are also summarized in table 6.1. We define an EXPRESSION OF TFL as any string of symbols of TFL. Take any of the symbols of TFL and write them down, in any order, and you have an expression of TFL.

6.2 Sentences

Many expressions of TFL will be total gibberish. We want to know when an expression of TFL amounts to a *sentence.* To that end, we have these seven rules. They are one part of the grammar of TFL.

atomic sentences	$A, B, C, \ldots, Z$
with subscripts if needed	$A_1, A_2, A_3, A_4, \ldots, J_{10}, J_{11}, \ldots$
logical operators	$\neg, \&, \vee, \rightarrow, \leftrightarrow$
brackets	(,)

Table 6.1: The three types of symbols of TFL

Sentences of TFL

1. Every atomic sentence is a sentence.
2. If A is a sentence, then $\neg\mathsf{A}$ is a sentence.
3. If A and B are sentences, then $(\mathsf{A} \,\&\, \mathsf{B})$ is a sentence.
4. If A and B are sentences, then $(\mathsf{A} \vee \mathsf{B})$ is a sentence.
5. If A and B are sentences, then $(\mathsf{A} \rightarrow \mathsf{B})$ is a sentence.
6. If A and B are sentences, then $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a sentence.
7. Nothing else is a sentence.

Notice that A and A are different fonts. A is an atomic sentence in TFL. A is not, actually, part of TFL. Rather, it stands for any sentence in TFL. That sentence could be A or it could be $(B \rightarrow D)$ or anything else. This use of *metavariables* is explained more fully in §7.3.

To use 1 – 7, you have to drill down to each part of the sentence. For instance, (4) tells us that if A and B are sentences, then $(\mathsf{A} \vee \mathsf{B})$ is a sentence. Thus, to know if $(\mathsf{A} \vee \mathsf{B})$ is a sentence, we have to determine if A and B themselves are sentences of TFL. Let's look at an example:

$$(P \leftrightarrow T) \vee Q$$

According to (4), this is a sentence if '$(P \leftrightarrow T)$' is a sentence and if 'Q' is a sentence. Are they? According to (1), every atomic sentence—that is, every individual capital letter in this font: $A, B, C, \ldots$—is a sentence of TFL, and so 'Q' is a sentence. And according to (6), '$(P \leftrightarrow T)$' is as long as 'P' and 'T' are both sentences—which they are since both are atomic sentences. Thus, '$(P \leftrightarrow T) \vee Q$' is a sentence of TFL.

Ultimately, you want to be able to just look at an expression and tell whether or not it is a correctly formed sentence of TFL, and with time you will be able to do so. Here are some examples of sentences of TFL:

1. $((P \,\&\, R) \rightarrow (S \rightarrow T))$
2. $(P \,\&\, (R \rightarrow (S \rightarrow T)))$
3. $((P \leftrightarrow \neg S) \vee \neg(T \leftrightarrow R))$

4. $((R \leftrightarrow T) \& \neg(P \lor (Q \lor \neg T)))$
5. $\neg(P \rightarrow \neg(R \lor (S \leftrightarrow T)))$
6. $((S \lor P) \& \neg(R \lor \neg\neg R))$

These, on the other hand, are ***not*** sentences of TFL because each violates one or more of 1 – 7:

1. $P\neg$
2. $(R \& \lor S)$
3. $(R \& Q \rightarrow)$
4. $(P\neg Q)$
5. $(P, Q \leftrightarrow T)$
6. $(P \& Q \& R)$

You will learn to recognize sentences of TFL more quickly if you write neatly and space the atomic sentences, logical operators, and parentheses as is shown in this textbook. Spaces are not actually part of TFL, and so technically, you don't need to use them. But just as you would never add or drop spaces when writing sentences in English, you should always do the same when using TFL.

6.3 The main logical operator

Consider this sentence:

> Dr. Wilson is in his office, and if she is teaching today, then Dr. Cook is in her office.

This sentence contains two logical operators, 'and' and 'if . . . , then. . . ', and one of them is the MAIN LOGICAL OPERATOR of the sentence. The main logical operator determines, at the most general level, what kind of sentence it is—a conjunction, a disjunction, a conditional, a biconditional, or a negation. The sentence above is a conjunction. Thus, the 'and' is the main logic operator, and these are the two conjuncts:

1. Dr. Wilson is in his office.
2. If she is teaching today, then Dr. Cook is in her office.

The second conjunct is a conditional, but the *if…, then…* only applies to *she is teaching today* and *Dr. Cook is in her office*, not to the whole sentence.

Now let's look at this sentence:

> If today is not Saturday, then Amy is at work and Kate is at school.

This is a conditional. The antecedent is *today is not Saturday*, and the consequent is *Amy is at work and Kate is at school.* So, although there are three logical operators in this sentence, the main one is the *if ..., then ...* (and so if we translated this sentence into TFL, the main logical operator would be the '$\rightarrow$').

Even though the sentence is a conditional, the *not* and the *and* each have a role. But their roles are limited to only a part of the sentence. Take the *not.* It applies to (i.e., negates) *today is Saturday.* The *and*'s job, meanwhile, is to create a conjunction by joining *Amy is at work* and *Kate is at school.*

Now, let's turn to expressions in TFL. Although identifying the main logical operator in a long expression in TFL can seem confusing at first, because we are using parentheses, you'll find that it's not too difficult. Let's start with this example: $((P \,\&\, Q) \vee R)$. This is a disjunction. One disjunct is $(P \,\&\, Q)$ and the other is R. Hence, the main logical operator is the '$\vee$'.

Let's change the expression to $\neg((P \,\&\, Q) \vee R)$. This is a negation, and so the main logical operator is the '$\neg$'. Notice that the '$\neg$' is outside of the brackets that enclose the entire '$(P \,\&\, Q) \vee R$'. That means that the '$\neg$' applies to the entire sentence. Hence, it is the main logical operator. Here are some other examples:

1. $((P \,\&\, R) \rightarrow (\neg Q \,\&\, S))$ The main logical operator is the '$\rightarrow$'.
2. $(((T \rightarrow P) \,\&\, R) \vee (S \leftrightarrow Q))$ The main logical operator is the '$\vee$'.
3. $\neg\neg\neg D$ The main logical operator is the first '$\neg$'.
4. $(P \,\&\, \neg(\neg Q \vee R))$ The main logical operator is the '&'.
5. $((\neg E \vee F) \leftrightarrow \neg G)$ The main logical operator is the '$\leftrightarrow$'.

Ultimately, you want to be able to identify the main logical operator by just looking at a sentence and seeing what kind of sentence it is. If it is a conjunction, then part of the sentence will be one conjunct and the rest will be the other conjunct (and nothing will be left over). If it's a disjunction, then part of the sentence will be one disjunct and the rest will the other disjunct, again with nothing left over. If it's a conditional, then part of the sentence will be the antecedent and the rest will be the consequent. And if it's a negation, then the whole sentence (minus the 'not' itself) is being negated.

Alternatively, when the sentence includes the outermost brackets, you can find the main logical operator by using this method:

(1) If the first symbol in the sentence is '$\neg$', then that is the main logical operator.

(2) Otherwise, start counting the brackets by following one of these two procedures. (The open-bracket is '(' and the closed bracket is ')'.)

  (2a) Start from the left, and begin counting. For each open-bracket add 1, and for each closing-bracket, subtract 1. When your count is at exactly 1, the next operator you come to (*apart* from a '$\neg$') is the main logical operator.

  (2b) If starting at the left-side of the sentence doesn't seem to work, follow the same procedure, but begin at the far right (and work left).

As we will discuss in the next section, in some cases, it is acceptable to omit the outermost brackets in a sentence of TFL. For instance, although it is not strictly allowable according to the rules given in §6.2, because it will not introduce any confusion or ambiguity, we can write '$(P \& R) \rightarrow Q$' instead of '$((P \& R) \rightarrow Q)$'. When the outermost brackets are dropped, we add (3) and (4) to our method.

(3) When the outermost brackes are omitted, (2a) and (2b) can still be used, but stop when your count gets to zero instead of 1.

(4) For sentences that contain two or more atomic sentences, if '$\neg$' is the main logical operator, then the outermost brackets have to be used. (When '$\neg$' is the main logical operator—as it is in this example: $\neg((P \& Q) \vee R)$—the '$\neg$' will be outside the outermost brackets.)

(5) For sentences that contain two or more atomic sentences, when the outermost brackets are omitted, (1) no longer applies and '$\neg$' won't be the main logical operator.

Scope

Finally, let's define the SCOPE of a logical operator. Basically, the scope of a logical operator is the part of the sentence to which the operator applies (or as Lemmon says, "what a particular occurrence of a connective controls"). We give the precise definition in terms of the main logical operator for the whole sentence and the main logical operators for any sub-sentences contained therein.

Scope

The SCOPE of a logical operator is the sentence or sub-sentence for which that logical operator is the main logical operator. Alternatively, Lemmon defines the *scope of a logical operator* as 'the shortest sentence in which the logical operator appears'.

The scope of the main logical operator is always the entire sentence. The scope of every other logical operator is a sub-sentence. Consider this sentence:

$$(\neg(R \,\&\, T) \leftrightarrow (P \rightarrow \neg Q))$$

The main logical operator is the '$\leftrightarrow$'. Therefore, the scope of '$\leftrightarrow$' is the entire sentence. The scope of the '$\neg$' is '$\neg(R \,\&\, T)$', which means that '$\neg$' is the main logical operator for that sub-sentence. Similarly, the '&' is the main logical operator for just the '$(R \,\&\, T)$', and so the scope of the '&' is '$(R \,\&\, T)$'. The '$\rightarrow$' is the main logical operator for '$(P \rightarrow \neg Q)$'. And the '$\neg$' is the main logical operator for '$\neg Q$'. Hence, the scope of each are those respective sub-sentences.

6.4 Bracketing conventions

Brackets (i.e., parentheses) are required for any sentence of TFL containing two or more atomic sentences. For instance, even in a simple sentence such as '$(Q \& R)$', they are required. One reason for this is because the rules given in §6.2 require it. Those rules don't make a distinction between sentences containing only two atomic sentences and sentences containing more than two. They just tell us to use brackets in any sentence containing a '&', '$\vee$', '$\rightarrow$', or '$\leftrightarrow$'. Another reason for using brackets is that we might make '$(Q \& R)$' a sub-sentence in a more complex sentence. For example, we might want to negate '$(Q \& R)$', which would give us '$\neg(Q \& R)$'. If we just had '$Q \& R$' without the brackets and put a '$\neg$' in front of it, we would have '$\neg Q \& R$'. But '$\neg Q \& R$' is different than '$\neg(Q \& R)$'.

That said, there are some convenient conventions that we can use as long as we are careful. First, as long as the entire sentence is not being negated, we can omit the *outermost* brackets of a sentence. Thus, we allow ourselves to write '$Q \& R$' instead of '$(Q \& R)$' when '$Q \& R$' is the whole sentence. We must remember, however, to put brackets around it when we want to embed the sentence into a more complex sentence.

Second, it can be a bit difficult to stare at long sentences with many nested pairs of brackets. To make things a bit easier on the eyes, we will allow ourselves to use square brackets, '[' and ']', in addition to rounded ones. So, there is no logical difference, for example, between '$(P \vee Q)$' and '$[P \vee Q]$'.

Combining these two conventions, we can rewrite

$$(((H \rightarrow I) \vee (I \rightarrow H)) \& (J \vee K))$$

this way:

$$\big[(H \rightarrow I) \vee (I \rightarrow H)\big] \& (J \vee K)$$

The scope of each connective is now much easier to identify.

6.5 Practice exercises

A. For each of the following, (a) is it a sentence of TFL, strictly speaking, and (b) is it a sentence of TFL, allowing for our relaxed bracketing conventions? If, by either of those standards, it is a sentence of TFL, then (c) what is the main logical operator?

1. (A)

2. $J_{374} \vee \neg J_{374}$
3. $\neg\neg\neg\neg F$
4. $\neg \,\&\, S$
5. $(G \,\&\, \neg G)$
6. $(A \to (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$
7. $[(Z \leftrightarrow S) \to W] \,\&\, [J \vee X]$
8. $(F \leftrightarrow \neg D \to J) \vee (C \,\&\, D)$

B. What is the scope of each connective in this sentence?

$$\big[(H \to I) \vee (I \to H)\big] \,\&\, (J \vee K)$$

6.6 Answers

A. For each of the following, (a) is it a sentence of TFL, strictly speaking, and (b) is it a sentence of TFL, allowing for our relaxed bracketing conventions? If, by either of those standards, it is a sentence of TFL, then (c) what is the main logical operator?

1. (A) — (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ — (a) no (b) yes (c) the '$\vee$'
3. $\neg\neg\neg\neg F$ — (a) yes (b) yes (c) the first '$\neg$'
4. $\neg \,\&\, S$ — (a) no (b) no
5. $(G \,\&\, \neg G)$ — (a) yes (b) yes (c) the '&'
6. $(A \to (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$ — (a) no (b) yes (c) the '$\vee$'
7. $[(Z \leftrightarrow S) \to W] \,\&\, [J \vee X]$ — (a) no (b) yes (c) the '&'
8. $(F \leftrightarrow \neg D \to J) \vee (C \,\&\, D)$ — (a) no (b) no

B. $\big[(H \to I) \vee (I \to H)\big] \,\&\, (J \vee K)$

The scope of the left-most '$\to$' is '$(H \to I)$'.
The scope of the right-most '$\to$' is '$(I \to H)$'.
The scope of the left-most '$\vee$ is '$\big[(H \to I) \vee (I \to H)\big]$'.
The scope of the right-most '$\vee$' is '$(J \vee K)$'.
The scope of the '&' is the entire sentence, and so the '&' is the main logical operator and the sentence is a conjunction.

7 Use and mention

7.1 Quotation conventions

Consider these two sentences:

1. Justin Trudeau is the Prime Minister.
2. 'Justin Trudeau' is composed of two uppercase letters and eleven lowercase letters

When we want to talk about the Prime Minister of Canada, which we are doing in sentence 1, we *use* his name. When we want to talk about the Prime Minister's name, as we are in sentence 2 we *mention* that name.

There is a general point here. When we want to talk about things in the world, we just *use* words. When we want to talk about words, we have to *mention* those words. To indicate that we are mentioning them rather than using them, we put them in single quotation marks, or use italics.

Let's compare sentences 1 and 2 to sentences 3 and 4:

3. 'Justin Trudeau' is the Prime Minister.
4. Justin Trudeau is composed of two uppercase letters and eleven lowercase letters.

Sentence 1 is correct. Justin Trudeau, the man, is the Prime Minister of Canada. According to sentence 3, the phrase 'Justin Trudeau' is the Prime Minister, which is false. The same problem is illustrated by sentences 2 and 4. Sentence 2 is fine. According to 4, however, Justin Trudeau (the man) is made of letters, which is false.

7.2 Object language and metalanguage

Since we are describing a formal language, we are often *mentioning* expressions from TFL. When we talk about a language, the language that we are talking about is called the OBJECT LANGUAGE. The language that we use to talk about the object language is called the METALANGUAGE.

Imagine for a moment that we are talking about German. In that case, German is our object language, and English—the language we are using to talk about German—is the metalanguage.

5. Schnee ist weiß is a German sentence.
6. 'Schnee ist weiß' is a German sentence.

Sentence 6 is correct. There were are saying that the clause at the beginning of the sentence is a German sentence. You can probably tell that sentence 5 is attempting to express the same idea, but, as it is, it is just a sentence stating 'Snow is white is a German sentence'—in two different languages, no less.

Of course, we aren't concerned with German here. For the most part, the object language in this chapter has been the formal language of TFL. The metalanguage is English. And just as we saw with sentence 6, when we are referring to a sentence in the object language, we need to indicate that we are mentioning it, not using it.

7. 'D' is an atomic sentence of TFL.
8. '$\neg(\neg Q \vee R)$' is a sentence of TFL ***if*** '$(\neg Q \vee R)$' is a sentence of TFL.

The general point is that, whenever we want to talk in English about some specific expression of TFL, we need to indicate that we are *mentioning* the expression, rather than *using* it. We can either deploy quotation marks, or we can adopt some similar convention, such as placing it centrally in the page.

7.3 Metavariables

Sometimes we discuss specific expressions of TFL like 'D' and '$\neg(\neg Q \vee R)$'. Other times, however, we want to say something about an arbitrary expression of TFL, not a specific one. To do that, we use these uppercase letters:

$$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$$

You probably noticed that we used these letters in our definition of a sentence of TFL in §6.2. For instance, this is one rule in that definition:

3. If A and B are sentences of TFL, then $(\mathsf{A} \,\&\, \mathsf{B})$ is a sentence of TFL.

We used 'A' and 'B' because those symbols must stand for any possible sentence of TFL, not just 'A' and 'B'. For instance, 'A' might stand for '$(P \vee Q)$' or '$((R \to T) \,\&\, \neg Q)$' or anything else.

'$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$' do not belong to TFL. Rather, they are part of the metalanguage—that is, English—that we use to talk about expressions of TFL.

> A METAVARIABLE is a variable in the metalanguage (i.e., English) that represents any sentence in our formal language of TFL. The symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$ are used for the metavariables.

7.4 Quotation conventions for arguments

One of our main purposes for using TFL is to study arguments, and that will be our concern in Parts 3 and 4. In English, the premises of an argument are often expressed by individual sentences, and the conclusion by a further sentence. Since we can symbolize English sentences, we can symbolize English arguments using TFL. Thus we might ask whether the argument whose premises are the TFL sentences 'A' and '$A \to C$', and whose conclusion is the TFL sentence 'C', is valid. However, it is quite a mouthful to write that every time. So instead we will introduce another bit of abbreviation. This:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{C}$$

abbreviates:

the argument with premises $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ and conclusion C

To avoid unnecessary clutter, we will not regard this as requiring quotation marks around it. (Note, then, that '$\vdash$' is a symbol of our augmented *metalanguage* and not a new symbol of TFL.)

Part 3

Truth tables

8 Characteristic truth tables

8.1 A quick introduction to truth tables

Consider this sentence:

> Either the key is on the table, or Jane is on the train and the key is not on the table.

Now, ask yourself, when is this sentence true and when is it false?

(a) If *the key is on the table* is true, then the sentence will be true regardless of whether *Jane is on the train* is true or false.
(b) If *the key is on the table* is false, then the sentence will be true as long as *Jane is on the train* is true.
(c) But if both *the key is on the table* is false and *Jane is on the train* is false, then the sentence will be false.

We worked that out by thinking about the different possible scenarios, and the logic of 'or' and 'and'. An alternative, and a somewhat easier method to use for more complex sentences, is to create a truth table. Truth tables tell us when a sentence is true or false, and, as we will see in chapters 10 and 11, they allow us to perform other analyses as well.

This is a truth table—in fact, it is the truth table for the above sentence:

J	K	$(K$	$\vee$	$(J$	$\&$	$\neg$	$K)$
T	T	T	**T**	T	F	F	T
T	F	F	**T**	T	T	T	F
F	T	T	**T**	F	F	F	T
F	F	F	**F**	F	F	T	F

To begin thinking about truth tables, notice the following features of this table.

J	K	(K	$\vee$	(J	&	$\neg$	K)
T	T	T	T	T	F	F	T
T	F	F	T	T	T	T	F
F	T	T	T	F	F	F	T
F	F	F	F	F	F	T	F

Figure 8.1: In a truth table, the columns under the atomic sentences on the right are copied from the columns on the far left.

1. The sentence for which we are creating the truth table, in this case, '$K \vee (J \& \neg K)$', is at the top of the truth table, to the right of the vertical line.
2. The atomic sentences that are in '$K \vee (J \& \neg K)$' are at the top of the truth table to the left of the vertical line, and they are in alphabetical order there.
3. The 'T's and 'F's below the horizontal line stand for 'true' and 'false'. (See the definition of TRUTH VALUES at the beginning of the next section.)
4. Below the 'J' and 'K' on the left are the different possible combinations of true and false. Below the 'J', each 'T' is a scenario where J is true, and each 'F' is a scenario where J is false. (And likewise for K.) So, on the first line (right below the horizontal line), J is true, and K is true. On the second line, J is true, and K is false. On the third line, J is false, and K is true. And on the fourth line, J is false, and K is false.
5. The columns of Ts and Fs that are to the left of the vertical line are repeated on the right side of the table under each 'J' and 'K', respectively. (See figure 8.1.)

Those are the basic features of every truth table. The task when creating a truth table is knowing how, and in what order, to fill in the columns under the logical operators. In this chapter, we will examine simple sentences containing only one logic operator. These truth tables are the "characteristic truth tables" for each logical operator. In the next chapter, we will explore how to create truth tables for more complex sentences.

8.2 The characteristic truth tables

You were introduced five logical operators in chapter 5, and now we need to explain when sentences using each one are true and when they are false.

Truth values

Truth values are the logical values that a sentence can have: *true* and *false*.

Conjunction For any sentences A and B, the conjunction (A & B) is true if and only if both A and B are true. If one or both of A and B are false, then the sentence (A & B) is false. We can summarize this in the *characteristic truth table for conjunction*:

A	B	A	&	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	F	F

Looking at line 1, we see that, when A is true and B is true, there is a 'T' under the '&', which indicates that (A & B) is true. On line 2, meanwhile, A is true and B is false, which means that (A & B) is false.

Lines 3 and 4 represent the final two combinations of 'true' and 'false' for A and B. On line 3, A is false and B is true. When that is the case, (A&B) is false. And then on line 4, A and B are both false. In that scenario, (A & B) is, again, false.

Note that conjunction is *symmetrical*. The truth value for (A & B) is always the same as the truth value for (B & A).

Negation For any sentence A, if A is true, then ¬A is false. And likewise, if A is false, then ¬A is true. This is represented in the *characteristic truth table for negation*:

A	¬	A
T	F	T
F	T	F

Disjunction Recall that '$\vee$' always represents the inclusive-or. So, for any sentences A and B, the disjunction $(\mathsf{A} \vee \mathsf{B})$ is true when A is true or B is true or both are true. The only instance when $(\mathsf{A} \vee \mathsf{B})$ is false is when both A and B are false. This is represented in the *characteristic truth table for disjunction*:

A	B	A	∨	B
T	T	T	T	T
T	F	T	T	F
F	T	F	T	T
F	F	F	F	F

This is a good time to explain another point. We are, in this chapter, simply stipulating when each of these types of sentences are true and false. This amounts to a definition for each logical operator in TFL. (Thus, the meaning of '$\vee$' is what is given in the above truth table.) We have reasons for defining them these ways, and there is a consensus that these are the best definitions. But, in the end, these are the correct truth tables for each logical operator because these are the ways that we have chosen to set them.

Conceivably, we could say that $(\mathsf{A} \vee \mathsf{B})$ is false when A and B are both false *and* when A and B both are true. That would agree with the way that we, at least some of the time, use *or* in English. But that's not what we've chosen to do, and so the way that $(\mathsf{A} \vee \mathsf{B})$ is defined in the truth table above is going to apply from this point forward (and similarly for all of the other connectives).

Conditional The conditional is interesting and, for some, philosophically contentious. One way to think about the conditional is as rule: if the antecedent happens, then the consequent has to happen. So, for instance, take this conditional:

> If it is Wednesday, then I am on campus by 10:00 am.

This sentence is obviously true when (1) it is Wednesday and I am on campus by 10:00 am. Conversely, this sentence is false when (2) it is Wednesday, but I am not on campus by 10:00 am. (If that happens, the rule has been broken.) Those two scenarios are represented by lines 1 and 2 in the characteristic truth table for the conditional, which is as follows.

A	B	A	→	B
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

For the other two scenarios, we have to concentrate a bit.

(3) Our conditional is also true when it is not Wednesday (let's say it's Tuesday), but I'm on campus by 10:00 am. In this case, the rule *if it is Wednesday, then I am on campus by 10:00 am* hasn't been broken; it just doesn't apply. So, when the antecedent is false and the consequent is true, the conditional is true. That's represented by line 3 of the characteristic truth table for the conditional.

(4) Similarly, when it is not Wednesday and I am not on campus by 10:00 am, the rule hasn't been broken. It is still in force. It just hasn't been invoked at all. So even though the antecedent didn't happen and the consequent didn't happen, the conditional is still true. (In other words, let's say, it's Saturday and, at 10:00 am, I am at home in bed. It's false that 'it is Wednesday' and it's false that 'I am on campus by 10:00 am', but it's still true that *if it is Wednesday, then I am on campus by 10:00 am.*) This scenario is represented on line 4 of the characteristic truth table.

Those four scenarios are pretty straightforward. The conditional is philosophically contentious, however, because every conditional is not as simple as 'if it is Wednesday, then I am on campus by 10:00 am'. Take a conditional where the antecedent is always false: 'if the queen of England is on the moon, then Mississippi State University is in Starkville.' This isn't much of a rule, but it is a conditional. And since the antecedent is false and the consequent is true, the sentence is true. Even stranger, consider this conditional: 'if the queen of England is on the moon, then pigs can fly.' Now, the antecedent is always false and the consequent is always false (at least in our world), but, as is shown on line 4 of the characteristic truth table, the sentence is true.

Sometimes the truth values for the antecedent, the consequent, and the whole conditional make sense (as in our first example) and sometimes they seem odd. That has generated philosophical debate, but it actually does not present a problem for us. The conditional is precisely defined by its characteristic truth table. We, then, simply use that definition, and we don't have to make any decisions about whether a particular conditional is odd or should really be true or false.

Finally, notice that, unlike the conjunction and the disjunction, the conditional is *asymmetrical.* You cannot switch the antecedent and consequent without changing the meaning of the sentence. This is because $\mathsf{A} \to \mathsf{B}$ ('if it is Wednesday, then I am on campus by 10:00 am') has a different truth table than $\mathsf{B} \to \mathsf{A}$ ('if I am on campus by 10:00 am, then it is Wednesday').

Biconditional As we said in §5.5, the biconditional is equivalent to the conjunction of a conditional running in each direction—that is, to $(\mathsf{A} \to \mathsf{B})\ \&\ (\mathsf{A} \to \mathsf{B})$. Consequently, on every line where both $\mathsf{A} \to \mathsf{B}$ is true and $\mathsf{B} \to \mathsf{A}$ is true, $\mathsf{A} \leftrightarrow \mathsf{B}$ is true. On every line where either $\mathsf{A} \to \mathsf{B}$ is false or $\mathsf{B} \to \mathsf{A}$ is false, $\mathsf{A} \leftrightarrow \mathsf{B}$ is false. That yields the following characteristic truth table for the biconditional.

A	B	A	↔	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	T	F

9 Complete truth tables

In chapter 8, we examined the characteristic truth tables for the logical operators of TFL. The characteristic truth tables show us when a sentence with only one of the logical operators is true and when it is false. That, in effect, is a definition for each logical operator. Now that we have those definitions, we can investigate when other, more complex sentences are true and false—for instance, ones like '$(H \& I) \rightarrow H$' and '$(M \& (N \vee P))$', which we will go through in this chapter. Once we understand how to create truth tables, we can investigate other properties of sentences of TFL, which we will do in chapters 10 and 11.

Before we begin, we will define VALUATION.

> **Valuation**
>
> A *valuation* is any assignment of truth values to particular atomic sentences of TFL. Each row of a truth table represents a possible valuation. The entire truth table represents all possible valuations.

Thus, the truth table provides us with a way of finding the truth values of complex sentences on each possible valuation—that is, for every combination of 'true' and 'false' for every atomic sentence.

9.1 An example

Consider the sentence '$(H \& I) \rightarrow H$', which contains three atomic sentences, although only two different ones. We set up the truth table for this sentence by putting H and I on the left side of the vertical line and '$(H \& I) \rightarrow H$' on the right. (Although H appears twice in '$(H \& I) \rightarrow H$', we only need one H on the left.) Below the H and I on the left side, we put every combination of 'T' and 'F'.

Since we have two atomic sentences on the left, there are four combinations of true and false. For consistency, the Ts and Fs should always be listed this way: (a) in the column next to the vertical line, they alternate T, F, T, F; (b) in the next column (to the left), they alternate in pairs: T, T, F, F; and (c) if there are more than two atomic sentences, then more columns and more rows are needed, but the pattern remains the same. (See table 9.1.)

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T					
T	F					
F	T					
F	F					

Once the left side of the truth table is completed, we begin filling in the right side. First, we copy the truth values for the atomic sentences. For the H, that gives us this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T				T
T	F	T				T
F	T	F				F
F	F	F				F

Adding the truth values for I, we have this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T		T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

Now , there are two columns that remain. The one under the '&' and the one under the '→'. '$(H \& I) \rightarrow H$', is a conditional. Therefore, the '→' is the main logical operator, and the column under the '→' will be filled in last. So, right now, we focus on '$(H \& I)$'. This is a conjunction, and to determine the truth values for just this sub-sentence, we turn to the characteristic truth table for conjunction. As is shown on p. 69, when 'H' and 'I' are both true, we put a 'T' below the '&'.

H	I	(H	&	I)	→	H
T	T	T	**T**	T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

On the second line, 'H' is true and 'I' is false. That means that '$(H \& I)$' is false, and so we put 'F' on the second line below the '&'.

H	I	(H	&	I)	→	H
T	T	T	**T**	T		T
T	F	T	**F**	F		T
F	T	F		T		F
F	F	F		F		F

Following the characteristic truth table for conjunction, we fill in the truth values for the third and fourth lines, and that completes the column under the '&'.

H	I	(H	&	I)	→	H
T	T	T	**T**	T		T
T	F	T	**F**	F		T
F	T	F	**F**	T		F
F	F	F	**F**	F		F

Now, we complete the truth table by filling in the column under the '→'. In this conditional, '$(H \& I)$' is the antecedent and the 'H' after the '→' is the consequent. Therefore, we need to look at the truth values below the '&' and the 'H', and we need to refer to the characteristic truth table for the conditional (p. 71). On the first line, '$(H \& I)$' is true and 'H' is true, and so we put a 'T' beneath the '→'.

H	I	(H	&	I)	→	H
T	T	T	T	T	**T**	T
T	F	T	F	F		T
F	T	F	F	T		F
F	F	F	F	F		F

On the second row, '$(H \& I)$' is false and 'H' is true. (That's the scenario on the third line of the characteristic truth table for the conditional (p. 71), not the second.) A conditional is true when the antecedent is false and the consequent is true, and so we put a 'T' in the second row beneath the '→'.

H	I	(H	&	I)	→	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T		F
F	F	F	F	F		F

On the third and fourth rows, '$(H \& I)$' is false and 'H' is false, and so again, we put 'T' below the '→' on each line. (On both of these lines, the antecedent is false and the consequent is false, and so these correspond to line four in the characteristic truth table for the conditional.)

H	I	(H	&	I)	→	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Since the '→' is the main logical operator, we've now determine the truth values for this sentence. The column of 'T's beneath the '→' tells us that the sentence '$(H \& I) \rightarrow H$' is true regardless of the truth values of 'H' and 'I'. Those atomic sentences can be true or false in any combination, and the full sentence, '$(H \& I) \rightarrow H$', remains true. Since we have considered all four possible assignments of truth and falsity to 'H' and 'I', we can say that '$(H \& I) \rightarrow H$' is true on every *valuation*.

In the truth table for any sentence, the most important column is the one beneath the *main logical operator* for the sentence, since this tells us the truth value of the entire sentence. We have emphasized it in the last truth table above by putting this column in bold. When you work through truth tables yourself, you should similarly emphasize it underlining or circling it.

9.2 Building complete truth tables

A COMPLETE TRUTH TABLE has a line for every possible combination of *true* and *false* for the atomic sentences that compose the full sentence. Each line represents a *valuation*, and a complete truth table has a line for all the different valuations.

The size of the complete truth table depends on the number of different atomic sentences in the table. A sentence that contains only one atomic sentence requires only two rows, even if the same letter is repeated many times, as in the sentence '$[(C \leftrightarrow C) \rightarrow C] \,\&\, \neg(C \rightarrow C)$'. The complete truth table requires only two lines because there are only two possibilities: 'C' can be true or it can be false. The truth table for this sentence looks like this:

C	[(	C	$\leftrightarrow$	C	)	$\rightarrow$	C	]	&	$\neg$	(	C	$\rightarrow$	C	)
T		T	T	T		T	T		**F**	F		T	T	T	
F		F	T	F		F	F		**F**	F		F	T	F	

Looking at the column underneath the main logical operator, we see that the sentence is false on both rows of the table; i.e., the sentence is false regardless of whether 'C' is true or false. In other words, it is false on every valuation.

A sentence that contains two atomic sentences requires four lines for a complete truth table. A sentence that contains three atomic sentences requires eight lines, as shown in the truth table for '$M \,\&\, (N \vee P)$'. Notice that the 'T's and 'F's in the columns below 'N' and 'P' (on the left side) follow the same pattern as the example in the previous section. The column under the 'M', meanwhile, has four 'T's and then four 'F's.

M	N	P	M	&	$(N$	$\vee$	$P)$
T	T	T	T	**T**	T	T	T
T	T	F	T	**T**	T	T	F
T	F	T	T	**T**	F	T	T
T	F	F	T	**F**	F	F	F
F	T	T	F	**F**	T	T	T
F	T	F	F	**F**	T	T	F
F	F	T	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F

THE LEFT SIDE OF THE TRUTH TABLE

COLUMN	PATTERN
first (next to the vertical line)	T, F, T, F, ...
second	T, T, F, F, ...
third	T, T, T, T, F, F, F, F, ...
fourth	8 Ts, 8 Fs, ...
fifth	16 Ts, 16 Fs, ...

Table 9.1: Every truth table for the same sentence should be the same. To ensure that they are, the columns on the left side of the truth table should be filled in using the patterns given in this table. The first column is the one closest to the vertical line.

By inspecting this truth table, we can see that '$M \& (N \vee P)$' can be true or false, depending on the truth values of 'M', 'N', and 'P'.

A complete truth table for a sentence that contains four different atomic sentences requires 16 lines. If the sentence has five different letters, the truth table will have 32 lines. If it has six letters, it will have 64 lines, and so on. The rule here is this: for n different atomic sentences, the truth table for the sentence must have 2^n lines.

9.3 Some more examples

1. To create a truth table for '$(P \leftrightarrow Q) \rightarrow (P \vee Q)$', first, we fill in the columns below each P and Q. Next, we fill in the columns under the '$\leftrightarrow$' and the '$\vee$' (in either order).

P	Q	$(P$	$\leftrightarrow$	$Q)$	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T	T	T		T	T	T
T	F	T	F	F		T	T	F
F	T	F	F	T		F	T	T
F	F	F	T	F		F	F	F

Once we have those columns complete, we finish the truth table by filling in the column under the '$\rightarrow$', which we do by looking at the column under the '$\leftrightarrow$' and the column under the '$\vee$'.

P	Q	$(P$	$\leftrightarrow$	$Q)$	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	T	T	F
F	T	F	F	T	**T**	F	T	T
F	F	F	T	F	**F**	F	F	F

2. To make a truth table for '$P \,\&\, \neg Q$', after we have filled in the columns below the P and Q, we fill in the column under the $\neg$. To do that, we look at the column under the Q.

P	Q	$(P$	$\&$	$\neg$	$Q)$
T	T	T		**F**	T
T	F	T		**T**	F
F	T	F		**F**	T
F	F	F		**T**	F

Then, to complete the truth table, we fill in the column under the '&'—which we do by looking at the column under the P and the column under the '$\neg$'.

P	Q	$(P$	$\&$	$\neg$	$Q)$
T	T	T	**F**	F	T
T	F	T	**T**	T	F
F	T	F	**F**	F	T
F	F	F	**F**	T	F

3. For '$\neg(P \rightarrow Q)$', after we have filled in the columns under the 'P' and the 'Q', we fill in the column under the '$\rightarrow$'.

P	Q	$\neg$	$(P$	$\rightarrow$	$Q)$
T	T		T	T	T
T	F		T	F	F
F	T		F	T	T
F	F		F	T	F

Then, to complete the table, we fill in the column under the '$\neg$'. To fill in that column, we look at the column under the '$\rightarrow$'.

P	Q	$\neg$	$(P$	$\rightarrow$	$Q)$
T	T	**F**	T	T	T
T	F	**T**	T	F	F
F	T	**F**	F	T	T
F	F	**F**	F	T	F

4. In '$(P \& \neg Q) \vee Q$', the '$\vee$' is the main logical operator, and so we will fill in the column under it last. First (after we have filled in the columns under the 'P' and the 'Q'), we fill in the column under the '$\neg$'.

P	Q	$(P$	$\&$	$\neg$	$Q)$	$\vee$	Q
T	T	T		F	T		T
T	F	T		T	F		F
F	T	F		F	T		T
F	F	F		T	F		F

Next, while looking at the column under the 'P' and under the '$\neg$', we fill in the column under the '&'.

P	Q	$(P$	$\&$	$\neg$	$Q)$	$\vee$	Q
T	T	T	**F**	F	T		T
T	F	T	**T**	T	F		F
F	T	F	**F**	F	T		T
F	F	F	**F**	T	F		F

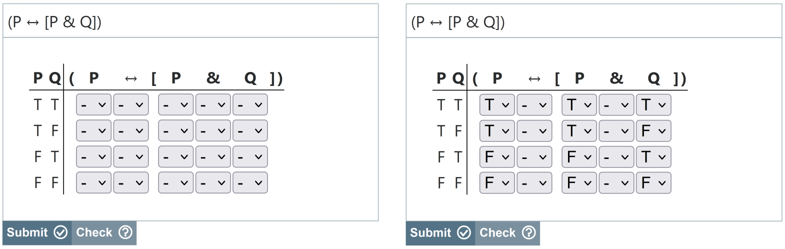
Then last, we fill in the column under the '$\vee$' while looking at the column under the '&' and under the 'Q'.

P	Q	$(P$	$\&$	$\neg$	$Q)$	$\vee$	Q
T	T	T	F	F	T	**T**	T
T	F	T	T	T	F	**T**	F
F	T	F	F	F	T	**T**	T
F	F	F	F	T	F	**F**	F

9.4 Truth tables in Carnap

You should practice making truth tables on paper, but you also need to make them using the software package Carnap (https://carnap.io/).

Using Carnap is pretty straightforward, and it's made easier because the left side of the truth table is completed for you. (See figure 9.1a.) On the right side, below each atomic sentence and connective, you have the option of selecting a 'T' or an 'F'. (See figure 9.1b.)



(a) *(b)*

Figure 9.1

Most often (although not always), the problems in Carnap will be set up so that you will only be able to submit your answers when they are correct. At those times, once the truth table is complete, you will select 'Check'. Carnap will tell you "Success!" or "Something's not quite right." It is easy to make a mistake when filling in a truth table, and so if something is not quite right, then you have to inspect every truth value until you find the mistake. Then select 'Check' again. Once Carnap confirms that the truth table is correct, select 'Submit'. **Don't forget to submit after you complete every truth table correctly.**

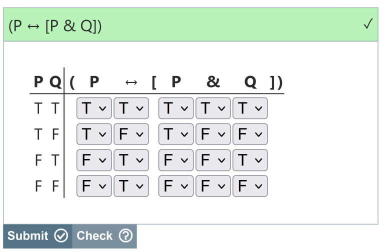


Figure 9.2: A completed and verified truth table in Carnap.

9.5 Practice exercises

A. Make a complete truth table for each sentence.

1. $A \to A$
2. $C \to \neg C$
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
4. $(A \to B) \lor (B \to A)$
5. $(A \,\&\, B) \to (B \lor A)$
6. $\neg(A \lor B) \leftrightarrow (\neg A \,\&\, \neg B)$
7. $\big[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)\big] \,\&\, C$
8. $[(A \,\&\, B) \,\&\, C] \to B$
9. $\neg\big[(C \lor A) \lor B\big]$

B. Check whether each of these statements is true.

1. '$((A \,\&\, B) \,\&\, C)$' and '$(A \,\&\, (B \,\&\, C))$' have the same truth table
2. '$((A \lor B) \lor C)$' and '$(A \lor (B \lor C))$' have the same truth table
3. '$((A \lor B) \,\&\, C)$' and '$(A \lor (B \,\&\, C))$' do not have the same truth table
4. '$((A \to B) \to C)$' and '$(A \to (B \to C))$' do not have the same truth table
5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table

C. Make truth tables for the following sentences, and mark the column that represents the possible truth values for the whole sentence.

1. $\neg(N \leftrightarrow (P \to N))$
2. $\neg[(X \,\&\, Y) \lor (X \lor Y)]$
3. $(A \to B) \leftrightarrow (\neg B \leftrightarrow \neg A)$
4. $[C \leftrightarrow (D \lor E)] \,\&\, \neg C$
5. $\neg(C \,\&\, (B \,\&\, H)) \leftrightarrow (C \lor (B \lor H))$
6. $(D \,\&\, \neg D) \to G$
7. $(\neg P \lor \neg R) \leftrightarrow R$
8. $\neg\neg(\neg A \,\&\, \neg B)$
9. $[(D \,\&\, H) \to J] \to \neg(D \lor H)$
10. $\neg[(D \leftrightarrow F) \leftrightarrow G] \to (\neg D \,\&\, F)$

9.6 Answers

A.

1. $A \to A$

A	A	$\to$	A
T	T	**T**	T
F	F	**T**	F

2. $C \to \neg C$

C	C	$\to$	$\neg$	C
T	T	**F**	F	T
F	F	**T**	T	F

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	$\neg$	$(A$	$\leftrightarrow$	$\neg$	$B)$
T	T	T	T	T	**T**	T	T	F	F	T
T	F	T	F	F	**T**	F	T	T	T	F
F	T	F	F	T	**T**	F	F	T	F	T
F	F	F	T	F	**T**	T	F	F	T	F

4. $(A \to B) \lor (B \to A)$

A	B	$(A$	$\to$	$B)$	$\lor$	$(B$	$\to$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	T	T	**T**	T	F	F
F	F	F	T	F	**T**	F	T	F

5. $(A \& B) \to (B \lor A)$

A	B	$(A$	$\&$	$B)$	$\to$	$(B$	$\lor$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	F	T	**T**	T	T	F
F	F	F	F	F	**T**	F	F	F

6. $\neg(A \lor B) \leftrightarrow (\neg A \& \neg B)$

A	B	$\neg$	$(A$	$\vee$	$B)$	$\leftrightarrow$	$(\neg$	A	$\&$	$\neg$	$B)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

7. $[(A \& B) \& \neg(A \& B)] \& C$

A	B	C	$[(A$	$\&$	$B)$	$\&$	$\neg$	$(A$	$\&$	$B)]$	$\&$	C
T	T	T	T	T	T	F	F	T	T	T	**F**	T
T	T	F	T	T	T	F	F	T	T	T	**F**	F
T	F	T	T	F	F	F	T	T	F	F	**F**	T
T	F	F	T	F	F	F	T	T	F	F	**F**	F
F	T	T	F	F	T	F	T	F	F	T	**F**	T
F	T	F	F	F	T	F	T	F	F	T	**F**	F
F	F	T	F	F	F	F	T	F	F	F	**F**	T
F	F	F	F	F	F	F	T	F	F	F	**F**	F

8. $[(A \& B) \& C] \to B$

A	B	C	$[(A$	$\&$	$B)$	$\&$	$C]$	$\to$	B
T	T	T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	F	F	**T**	T
T	F	T	T	F	F	F	T	**T**	F
T	F	F	T	F	F	F	F	**T**	F
F	T	T	F	F	T	F	T	**T**	T
F	T	F	F	F	T	F	F	**T**	T
F	F	T	F	F	F	F	T	**T**	F
F	F	F	F	F	F	F	F	**T**	F

9. $\neg[(C \vee A) \vee B]$

A	B	C	$\neg$[(	C	$\vee$	A)	$\vee$	B]
T	T	T	**F**	T	T	T	T	T
T	T	F	**F**	F	T	T	T	T
T	F	T	**F**	T	T	T	T	F
T	F	F	**F**	F	T	T	T	F
F	T	T	**F**	T	T	F	T	T
F	T	F	**F**	F	F	F	T	T
F	F	T	**F**	T	T	F	T	F
F	F	F	**T**	F	F	F	F	F

B.

1. '$((A \& B) \& C)$' and '$(A \& (B \& C))$' have the same truth table

A	B	C	(A	&	B)	&	C	A	&	(B	&	C)
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**F**	F	T	**F**	F	F	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table

A	B	C	(A	$\vee$	B)	$\vee$	C	A	$\vee$	(B	$\vee$	C)
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**T**	F	T	**T**	T	T	F
T	F	T	T	T	F	**T**	T	T	**T**	F	T	T
T	F	F	T	T	F	**T**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**T**	F	F	**T**	T	T	F
F	F	T	F	F	F	**T**	T	F	**T**	F	T	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

3. '$((A \vee B) \& C)$' and '$(A \vee (B \& C))$' do not have the same truth table

A	B	C	$(A$	$\lor$	$B)$	$\&$	C	A	$\lor$	$(B$	$\&$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**T**	T	F	F
T	F	T	T	T	F	**T**	T	T	**T**	F	F	T
T	F	F	T	T	F	**F**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

4. '$((A \to B) \to C)$' and '$(A \to (B \to C))$' do not have the same truth table

A	B	C	$(A$	$\to$	$B)$	$\to$	C	A	$\to$	$(B$	$\to$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**T**	T	T	**T**	F	T	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	T	T
F	F	F	F	T	F	**F**	F	F	**T**	F	T	F

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table.

A	B	C	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	C	A	$\leftrightarrow$	$(B$	$\leftrightarrow$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**T**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	F	T
F	F	F	F	T	F	**F**	F	F	**F**	F	T	F

C. Complete truth tables for each sentence with the column that represents the possible truth values for the whole sentence in bold.

1. $\neg(N \leftrightarrow (P \rightarrow N))$

N	P	¬	$(N$	↔	$(P$	→	$N))$
T	T	**F**	T	T	T	T	T
T	F	**F**	T	T	F	T	T
F	T	**F**	F	T	T	F	F
F	F	**T**	F	F	F	T	F

2. $\neg[(X \& Y) \vee (X \vee Y)]$

X	Y	¬	[(X	&	Y)	∨	(X	∨	Y)]
T	T	**F**	T	T	T	T	T	T	T
T	F	**F**	T	F	F	T	T	T	F
F	T	**F**	F	F	T	T	F	T	T
F	F	**T**	F	F	F	F	F	F	F

3. $(A \rightarrow B) \leftrightarrow (\neg B \leftrightarrow \neg A)$

A	B	(A	→	B)	↔	(¬	B	↔	¬	A)
T	T	T	T	T	**T**	F	T	T	F	T
T	F	T	F	F	**T**	T	F	F	F	T
F	T	F	T	T	**F**	F	T	F	T	F
F	F	F	T	F	**T**	T	F	T	T	F

4. $[C \leftrightarrow (D \vee E)] \& \neg C$

C	D	E	[C	↔	(D	∨	E)]	&	¬	C
T	T	T	T	T	T	T	T	**F**	F	T
T	T	F	T	T	T	T	F	**F**	F	T
T	F	T	T	T	F	T	T	**F**	F	T
T	F	F	T	F	F	F	F	**F**	F	T
F	T	T	F	F	T	T	T	**F**	T	F
F	T	F	F	F	T	T	F	**F**	T	F
F	F	T	F	F	F	T	T	**F**	T	F
F	F	F	F	T	F	F	F	**T**	T	F

5. $\neg(C \& (B \& H)) \leftrightarrow (C \vee (B \vee H))$

C	B	H	¬	(C	&	(B	&	H))	↔	(C	∨	(B	∨	H))
T	T	T	F	T	T	T	T	T	**F**	T	T	T	T	T
T	T	F	T	T	F	T	F	F	**T**	T	T	T	T	F
T	F	T	T	T	F	F	F	T	**T**	T	T	F	T	T
T	F	F	T	T	F	F	F	F	**T**	T	T	F	F	F
F	T	T	T	F	F	T	T	T	**T**	F	T	T	T	T
F	T	F	T	F	F	T	F	F	**T**	F	T	T	T	F
F	F	T	T	F	F	F	F	T	**T**	F	T	F	T	T
F	F	F	T	F	F	F	F	F	**F**	F	F	F	F	F

6. $(D \& \neg D) \rightarrow G$

D	G	(D	&	¬	D)	→	G
T	T	T	F	F	T	**T**	T
T	F	T	F	F	T	**T**	F
F	T	F	F	T	F	**T**	T
F	F	F	F	T	F	**T**	F

7. $(\neg P \vee \neg R) \leftrightarrow R$

P	R	(¬	P	∨	¬	R)	↔	R
T	T	F	T	F	F	T	**F**	T
T	F	F	T	T	T	F	**F**	F
F	T	T	F	T	F	T	**T**	T
F	F	T	F	T	T	F	**F**	F

8. $\neg\neg(\neg A \& \neg B)$

A	B	¬	¬	(¬	A	&	¬	B)
T	T	**F**	T	F	T	F	F	T
T	F	**F**	T	F	T	F	T	F
F	T	**F**	T	T	F	F	F	T
F	F	**T**	F	T	F	T	T	F

9. $[(D \& H) \rightarrow J] \rightarrow \neg(D \vee H)$

D	H	J	[(D	&	H)	→	J]	→	¬	(D	∨	H)
T	T	T	T	T	T	T	T	**F**	F	T	T	T
T	T	F	T	T	T	F	F	**T**	F	T	T	T
T	F	T	T	F	F	T	T	**F**	F	T	T	F
T	F	F	T	F	F	T	F	**F**	F	T	T	F
F	T	T	F	F	T	T	T	**F**	F	F	T	T
F	T	F	F	F	T	T	F	**F**	F	F	T	T
F	F	T	F	F	F	T	T	**T**	T	F	F	F
F	F	F	F	F	F	T	F	**T**	T	F	F	F

10. $\neg[(D \leftrightarrow F) \leftrightarrow G] \rightarrow (\neg D \& F)$

D	F	G	¬	[(D	↔	F)	↔	G]	→	(¬	D	&	F)
T	T	T	F	T	T	T	T	T	**T**	F	T	F	T
T	T	F	T	T	T	T	F	F	**F**	F	T	F	T
T	F	T	T	T	F	F	F	T	**F**	F	T	F	F
T	F	F	F	T	F	F	T	F	**T**	F	T	F	F
F	T	T	T	F	F	T	F	T	**T**	T	F	T	T
F	T	F	F	F	F	T	T	F	**T**	T	F	T	T
F	F	T	F	F	T	F	T	T	**T**	T	F	F	F
F	F	F	T	F	T	F	F	F	**F**	T	F	F	F

10 Six concepts

As we did in the previous chapter, we begin with the definition of VALUATION.

> **Valuation**
>
> A *valuation* is any assignment of truth values to particular atomic sentences of TFL. Each row of a truth table represents a possible valuation. The entire truth table represents all possible valuations.

Let's say that we are going to create a truth table for '$P \vee \neg Q$'. (See figure 10.1.) On the first line of the truth table, you may recall, we make P = 'true' and Q = 'true'. That is one valuation. On the second line, we make P = 'true' and Q = 'false'. That is another valuation. The assignments of 'true' and 'false' to 'P' and 'Q' on lines 3 and 4, then, are the remaining possible valuations, when we have a sentence containing only two atomic sentences.

In the previous chapter, we used truth tables to determine—for each possible valuation—the truth value of any TFL sentence. In this chapter, we will extend this type of analysis. We will examine six properties that apply (or may apply) to either single TFL sentences (*tautology*, *contradiction*, and *contingent*) or sets of TFL sentences (*equivalent*, *jointly*

	P	Q	P	$\vee$	$\neg$	Q
	T	T	T	**T**	F	T
This is one valuation. →	T	F	T	**T**	T	F
	F	T	F	**F**	F	T
	F	F	F	**T**	T	F

Figure 10.1: For the TFL sentence '$P \vee \neg Q$', P = 'true' and Q = 'true' is one valuation. On this valuation, '$P \vee \neg Q$' is true.

consistent, and *jointly inconsistent*). For each, we use a truth table to determine which property applies.

10.1 Tautologies and contradictions

In §3.2, we said that a *necessary truth* is a sentence that must be true, a *necessary falsehood* is a sentence that must be false, and a sentence that is neither a necessary truth or a necessary falsehood is *contingent*. The first two, *necessary truth* and *necessary falsehood*, have surrogates in TFL. We will start with the surrogate for necessary truth.

Tautology

A sentence of TFL is a TAUTOLOGY if and only if it is true on every valuation.

We can determine whether a sentence is a tautology using a truth table. If the sentence is true on every line of a complete truth table (that is, if there is a 'T' on every line under the main connective), then it is true on every valuation. And if it is true on every valuation, it is a tautology. The example from §9.1, '$(H \& I) \rightarrow H$', for instance, is a tautology.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Tautology is only a surrogate, however, for *necessary truth*. There are some necessary truths that we cannot adequately symbolize in TFL. An example is '$2 + 2 = 4$'. This *must* be true, but if we try to symbolize it in TFL, the best we can offer is an atomic sentence, perhaps,

F: $2 + 2 = 4$

But an atomic sentence by itself cannot be a tautology. (To see this, try making a truth table for just 'F'.) Still, if we can adequately symbolize some English sentence as a TFL sentence, and that TFL sentence is a tautology, then the English sentence expresses a necessary truth.

We have a similar surrogate for *necessary falsehood*.

Contradiction

A sentence of TFL is a CONTRADICTION if and only if it is false on every valuation.

Again, we can determine whether a sentence is a contradiction with a truth table. If the sentence is false on every line of a complete truth table, then it is false on every valuation, and so it is a contradiction. The standard example of a contradiction is '$P \& \neg P$'. Since we have only one letter in this sentence, it is only a two line truth table, but on each line, the sentence is false.

P	P	$\&$	$\neg$	P
T	T	**F**	F	T
F	F	**F**	T	F

Similarly, although its truth table has four lines, '$(P \vee Q) \leftrightarrow (\neg P \& \neg Q)$' is a contradiction.

P	Q	$(P$	$\vee$	$Q)$	$\leftrightarrow$	$(\neg$	P	$\&$	$\neg$	$Q)$
T	T	T	T	T	**F**	F	T	F	F	T
T	F	T	T	F	**F**	F	T	F	T	F
F	T	F	T	T	**F**	T	F	F	F	T
F	F	F	F	F	**F**	T	F	T	T	F

In §3.2, we defined CONTINGENT as "a sentence that is capable of being true and capable of being false (in different circumstances, of course)." A truth table, then, provides us with those different circumstances.

Contingent

A sentence that is true on at least one valuation and false on at least one valuation is contingent.

Or, we can also say: any sentence that is neither a tautology nor a contradiction is contingent.

'$\neg(P \vee Q)$', for instance, is contingent.

P	Q	$\neg$	$(P$	$\vee$	$Q)$
T	T	**F**	T	T	T
T	F	**F**	T	T	F
F	T	**F**	F	T	T
F	F	**T**	F	F	F

10.2 Equivalence

There are several possible logical relationships that can exist between two or more sentences of TFL. We we examine three relationships, and we will focus on pairs of sentences. The first logical relationship is EQUIVALENCE.

Equivalent

A and B are EQUIVALENT if and only if, for every valuation, their truth values agree (that is, if and only if there is no valuation for which they have opposite truth values).

Equivalently, if $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology, then A and B are EQUIVALENT.

Recall from 7.3, that A stands for any possible sentence of TFL (as do B, C, D, etc.). Hence, 'A' can stand for '$P \vee Q$' or '$(P \leftrightarrow \neg R) \,\&\, T$' or anything else.

Consider the sentences '$\neg(P \vee Q)$' and '$\neg P \& \neg Q$'. Are they equivalent? To find out, we construct a truth table containing both sentences.

P	Q	$\neg$	$(P$	$\vee$	$Q)$	$\neg$	P	$\&$	$\neg$	Q
T	T	**F**	T	T	T	F	T	**F**	F	T
T	F	**F**	T	T	F	F	T	**F**	T	F
F	T	**F**	F	T	T	T	F	**F**	F	T
F	F	**T**	F	F	F	T	F	**T**	T	F

Looking at the columns for the main logical operators ('$\neg$' for the first sentence, '&' for the second), we see that on the first three rows, both sentences are false. On the final row, both are true. Since they match on every row—that is, on every valuation for 'P' and 'Q'—the two sentences are equivalent.

10.3 Consistency

In §3.1, we said that sentences are *jointly possible* if and only if it is possible for all of them to be true at once. The surrogate for this concept in TFL is JOINTLY CONSISTENT.

Jointly consistent

A and B are JOINTLY CONSISTENT if and only if there is some valuation that makes them both true *and* they are not equivalent.

Equivalently, if

(1) there is at least one valuation that makes $(\mathsf{A} \mathbin{\&} \mathsf{B})$ true, and
(2) $(\mathsf{A} \leftrightarrow \mathsf{B})$ is *not* a tautology,

then A and B are JOINTLY CONSISTENT.

The requirement that the two sentences not be equivalent is not always included, but we will distinguish between sentences that are jointly consistent from those that are equivalent.

This was one of the examples in §3.1:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

These are jointly possible because it is possible for them both to be true at the same time. It takes nothing away from their joint possibility that they can also be false at the same time or one can be false while the other is true. Applying that same observation to *jointly consistent*, all we need is one line where both sentences are true. (More than one line is fine also, although the truth values for the two sentences shouldn't match on every line. If they do, then the sentences are equivalent.) '$(P \lor Q)$' and '$(P \mathbin{\&} \neg Q)$' have one line where they are both true, and so they are jointly consistent.

P	Q	P	$\vee$	Q	P	&	$\neg$	Q
T	T	T	**T**	T	T	**F**	F	T
T	F	T	**T**	F	T	**T**	T	F
F	T	F	**T**	T	F	**F**	F	T
F	F	F	**F**	F	F	**F**	T	F

And finally, in §3.1, we also said that sentences are *jointly impossible* if and only if it is *not* possible for all of them to be true at once. The surrogate for this concept in TFL is JOINTLY INCONSISTENT.

Jointly inconsistent

A and B are JOINTLY INCONSISTENT if and only if there is no valuation that makes them both true.

There are three ways that two sentences can be jointly inconsistent.

(1) One each line, the truth value for one sentence is 'T' and the truth value for the other sentence is 'F'. For instance, the truth values for '$P \vee Q$' and '$\neg P \& \neg Q$' never match. On each line, one is true and the other is false. Hence, two sentences are jointly inconsistent in this way when $\neg(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology.

P	Q	P	$\vee$	Q	$\neg$	P	&	$\neg$	Q
T	T	T	**T**	T	F	T	**F**	F	T
T	F	T	**T**	F	F	T	**F**	T	F
F	T	F	**T**	T	T	F	**F**	F	T
F	F	F	**F**	F	T	F	**T**	T	F

(2) When the truth value for one sentence is 'T', then the truth value for the other sentence is 'F', but both sentences can be false at the same time. For example, '$\neg(\neg P \vee Q)$' and '$(\neg P \& \neg Q)$' are never both true on the same line, but they are false on the same line. For two sentences that are jointly inconsistent in this way, this criterion is satisfied: $\neg(\mathsf{A} \& \mathsf{B})$ is a tautology.

P	Q	$\neg$	$(\neg$	P	$\vee$	$Q)$	$\neg$	P	$\&$	$\neg$	Q
T	T	**F**	F	T	T	T	F	T	**F**	F	T
T	F	**T**	F	T	F	F	F	T	**F**	T	F
F	T	**F**	T	F	T	T	T	F	**F**	F	T
F	F	**F**	T	F	T	F	T	F	**T**	T	F

(3) Both sentences are false on every line. For example, the truth values for '$\neg P \& P$' and '$\neg Q \& Q$' are always the same. On each line, both sentences are false. So, for sentences that are jointly inconsistent in this way, both of these criteria must be satisfied: $\neg$(A & B) is a tautology and (A $\leftrightarrow$ B) is a tautology. (And the latter, recall, means that these sentences are equivalent, and so here *jointly inconsistent* and *equivalent* overlap.)

P	Q	$\neg$	P	$\&$	P	$\neg$	Q	$\&$	Q
T	T	F	T	**F**	T	F	T	**F**	T
T	F	F	T	**F**	T	T	F	**F**	F
F	T	T	F	**F**	F	F	T	**F**	T
F	F	T	F	**F**	F	T	F	**F**	F

10.4 Practice exercises

A. Revisit your answers to the exercises in part A of chapter 9, and determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

B. Create a truth table for each sentence, and then determine whether the sentence is a **tautology**, a **contradiction**, or is **contingent**.

1. $\neg B \& B$
2. $\neg D \vee D$
3. $(A \& B) \vee (B \& A)$
4. $\neg[A \rightarrow (B \rightarrow A)]$
5. $A \leftrightarrow [A \rightarrow (B \& \neg B)]$
6. $[(A \& B) \leftrightarrow B] \rightarrow (A \rightarrow B)$

C. For each set of sentences, create a truth table and then determine whether the sentences are **jointly consistent** or **jointly inconsistent**.

1. $A \to A$, $\neg A \to \neg A$, $A \,\&\, A$, $A \vee A$
2. $A \vee B$, $A \to C$, $B \to C$
3. $B \,\&\, (C \vee A)$, $A \to B$, $\neg(B \vee C)$
4. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$
5. $A \,\&\, \neg B$, $\neg(A \to B)$, $B \to A$
6. $A \vee B$, $A \to \neg A$, $B \to \neg B$
7. $\neg(\neg A \vee B)$, $A \to \neg C$, $A \to (B \to C)$
8. $A \to B$, $A \,\&\, \neg B$
9. $A \to (B \to C)$, $(A \to B) \to C$, $A \to C$
10. $\neg B$, $A \to B$, A
11. $\neg(A \vee B)$, $A \leftrightarrow B$, $B \to A$
12. $A \vee B$, $\neg B$, $\neg B \to \neg A$
13. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \to B$
14. $(A \vee B) \vee C$, $\neg A \vee \neg B$, $\neg C \vee \neg B$

D. For each pair of sentences, create a truth table and then determine whether the pair of sentences are **equivalent** or are not.

1. A and $\neg A$
2. $A \,\&\, \neg A$ and $\neg B \leftrightarrow B$
3. $[(A \vee B) \vee C]$ and $[A \vee (B \vee C)]$
4. $A \vee (B \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C)$
5. $[A \,\&\, (A \vee B)] \to B$ and $A \to B$
6. $A \to A$ and $A \leftrightarrow A$
7. $\neg(A \to B)$ and $\neg A \to \neg B$
8. $A \vee B$ and $\neg A \to B$
9. $(A \to B) \to C$ and $A \to (B \to C)$
10. $A \leftrightarrow (B \leftrightarrow C)$ and $A \,\&\, (B \,\&\, C)$

E.

1. Suppose that A and B are equivalent. What can you say about $\mathsf{A} \leftrightarrow \mathsf{B}$?

2. Suppose that A and B are jointly inconsistent. What can you say about $(\mathsf{A} \,\&\, \mathsf{B})$?
3. Suppose that A and B are equivalent. What can you say about $(\mathsf{A} \lor \mathsf{B})$?
4. Suppose that A and B are *not* equivalent. What can you say about $(\mathsf{A} \lor \mathsf{B})$?
5. Consider this principle:

   > Suppose A and B are equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

   Is this principle correct? Explain your answer.

10.5 Answers

A. From chapter 9

1. $A \rightarrow A$	tautology
2. $C \rightarrow \neg C$	neither
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$	tautology
4. $(A \rightarrow B) \lor (B \rightarrow A)$	tautology
5. $(A \,\&\, B) \rightarrow (B \lor A)$	tautology
6. $\neg(A \lor B) \leftrightarrow (\neg A \,\&\, \neg B)$	tautology
7. $\big[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)\big] \,\&\, C$	contradiction
8. $[(A \,\&\, B) \,\&\, C] \rightarrow B$	tautology
9. $\neg\big[(C \lor A) \lor B\big]$	neither

B. Use a truth table to determine whether each sentence is a tautology, a contradiction, or a contingent sentence.

1. $\neg B \,\&\, B$	Contradiction
2. $\neg D \lor D$	Tautology
3. $(A \,\&\, B) \lor (B \,\&\, A)$	Contingent
4. $\neg[A \rightarrow (B \rightarrow A)]$	Contradiction
5. $A \leftrightarrow [A \rightarrow (B \,\&\, \neg B)]$	Contradiction
6. $[(A \,\&\, B) \leftrightarrow B] \rightarrow (A \rightarrow B)$	Contingent

C. Use a truth table to determine whether each set of sentences are jointly consistent or jointly inconsistent.

1. $A \to A$, $\neg A \to \neg A$, $A \,\&\, A$, $A \vee A$
   These sentences are jointly consistent. (See line 1.)

A	A	$\to$	A	$\neg$	A	$\to$	$\neg$	A	A	$\&$	A	A	$\vee$	A
T	T	**T**	T	F	T	**T**	F	T	T	**T**	T	T	**T**	T
F	F	**T**	F	T	F	**T**	T	F	F	**F**	F	F	**F**	F

2. $A \vee B$, $A \to C$, $B \to C$
   These sentences are jointly consistent. (See line 1.)

A	B	C	A	$\vee$	B	A	$\to$	C	B	$\to$	C
T	T	T	T	**T**	T	T	**T**	T	T	**T**	T
T	T	F	T	**T**	T	T	**F**	F	T	**F**	F
T	F	T	T	**T**	T	T	**T**	T	F	**T**	T
T	F	F	T	**T**	F	T	**F**	F	F	**T**	F
F	T	T	F	**T**	F	F	**T**	T	T	**T**	T
F	T	F	F	**T**	T	F	**T**	F	T	**F**	F
F	F	T	F	**F**	F	F	**T**	T	F	**T**	T
F	F	F	F	**F**	F	F	**T**	F	F	**T**	F

3. $B \,\&\, (C \vee A)$, $A \to B$, $\neg(B \vee C)$
   These sentences are jointly inconsistent.

A	B	C	B	$\&$	$(C$	$\vee$	$A)$	A	$\to$	B	$\neg$	$(B$	$\vee$	$C)$
T	T	T	T	**T**	T	T	T	T	**T**	T	**F**	T	T	T
T	T	F	T	**T**	F	T	T	T	**T**	T	**F**	T	T	F
T	F	T	F	**F**	T	T	T	T	**F**	F	**F**	F	T	T
T	F	F	F	**F**	F	T	T	T	**F**	F	**T**	F	F	F
F	T	T	T	**T**	T	T	F	F	**T**	T	**F**	T	T	T
F	T	F	T	**F**	F	F	F	F	**T**	T	**F**	T	T	F
F	F	T	F	**F**	T	T	F	F	**T**	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F	F	**T**	F	**T**	F	F	F

4. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$ These sentences are jointly consistent. (See line 8.)

A	B	C	A	$\leftrightarrow$	$(B$	$\vee$	$C)$	C	$\rightarrow$	$\neg$	A	A	$\rightarrow$	$\neg$	B
T	T	T	T	**T**	T	T	T	T	**F**	F	T	T	**F**	F	T
T	T	F	T	**T**	T	T	F	F	**T**	F	T	T	**F**	F	T
T	F	T	T	**T**	F	T	T	T	**F**	F	T	T	**T**	T	F
T	F	F	T	**F**	F	F	F	F	**T**	F	T	T	**T**	T	F
F	T	T	F	**F**	T	T	T	T	**T**	T	F	F	**T**	F	T
F	T	F	F	**F**	T	T	F	F	**T**	T	F	F	**T**	F	T
F	F	T	F	**F**	F	T	T	T	**T**	T	F	F	**T**	T	F
F	F	F	F	**T**	F	F	F	F	**T**	T	F	F	**T**	T	F

5. $A \& \neg B$, $\neg(A \rightarrow B)$, $B \rightarrow A$
   These sentences are jointly consistent. (See line 2.)

A	B	A	&	¬	B	¬	(A	→	B)	B	→	A
T	T	T	F	F	T	F	T	T	T	T	T	T
T	F	T	**T**	T	F	**T**	T	F	F	F	**T**	T
F	T	F	F	F	T	F	F	T	T	T	F	F
F	F	F	F	T	F	F	F	T	F	F	T	F

6. $A \vee B$, $A \rightarrow \neg A$, $B \rightarrow \neg B$
   These sentences are jointly inconsistent.

A	B	A	∨	B	A	→	¬	A	B	→	¬	B
T	T	T	T	T	T	F	F	T	T	F	F	T
T	F	T	T	F	T	F	F	T	F	T	T	F
F	T	F	T	T	F	T	T	F	T	F	F	T
F	F	F	F	F	F	T	T	F	F	T	T	F

7. $\neg(\neg A \vee B)$, $A \rightarrow \neg C$, $A \rightarrow (B \rightarrow C)$
   These sentences are jointly consistent.

A	B	C	¬	(¬	A	∨	B)	A	→	¬	C	A	→	(B	→	C)
T	T	T	F	F	T	T	T	T	F	F	T	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T	T	F	T	F	T	F	F
T	F	T	T	F	T	F	F	T	F	F	T	T	T	F	T	T
T	F	F	**T**	F	T	F	F	T	**T**	T	F	T	**T**	F	T	F
F	T	T	F	T	F	T	T	F	T	F	T	F	F	T	T	T
F	T	F	F	T	F	T	T	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	F	T	F	T	F	T	F	T	T
F	F	F	F	T	F	T	F	F	T	T	F	F	T	F	T	F

8. $A \rightarrow B$, $A \,\&\, \neg B$ — Inconsistent
9. $A \rightarrow (B \rightarrow C)$, $(A \rightarrow B) \rightarrow C$, $A \rightarrow C$ — Consistent
10. $\neg B$, $A \rightarrow B$, A — Inconsistent
11. $\neg(A \vee B)$, $A \leftrightarrow B$, $B \rightarrow A$ — Consistent
12. $A \vee B$, $\neg B$, $\neg B \rightarrow \neg A$ — Inconsistent
13. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \rightarrow B$ — Consistent
14. $(A \vee B) \vee C$, $\neg A \vee \neg B$, $\neg C \vee \neg B$ — Consistent

D. Use a truth table to determine whether each set of sentences are equivalent or not.

1. A and $\neg A$
2. $A \,\&\, \neg A$ and $\neg B \leftrightarrow B$
3. $[(A \vee B) \vee C]$ and $[A \vee (B \vee C)]$
4. $A \vee (B \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C)$
5. $[A \,\&\, (A \vee B)] \rightarrow B$ and $A \rightarrow B$
6. $A \rightarrow A$ and $A \leftrightarrow A$
7. $\neg(A \rightarrow B)$ and $\neg A \rightarrow \neg B$
8. $A \vee B$ and $\neg A \rightarrow B$
9. $(A \rightarrow B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$
10. $A \leftrightarrow (B \leftrightarrow C)$ and $A \,\&\, (B \,\&\, C)$

E.

1. Suppose that A and B are equivalent. What can you say about $\mathsf{A} \leftrightarrow \mathsf{B}$?

   A and B have the same truth value on every line of a complete truth table, so $\mathsf{A} \leftrightarrow \mathsf{B}$ is true on every line. It is a tautology.

2. Suppose that A and B are jointly inconsistent. What can you say about $(\mathsf{A} \,\&\, \mathsf{B})$?

   Since the sentences are jointly inconsistent, there is no valuation on which they are both true. So their conjunction is false on every valuation. It is a contradiction

3. Suppose that A and B are equivalent. What can you say about $(\mathsf{A} \vee \mathsf{B})$?

Not much. Since A and B are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is equivalent to them.

4. Suppose that A and B are *not* equivalent. What can you say about $(A \lor B)$?

   A and B have different truth values on at least one line of a complete truth table, and $(A \lor B)$ will be true on that line. On other lines, it might be true or false. So $(A \lor B)$ is either a tautology or it is contingent; it is *not* a contradiction.

5. Consider this principle:

   Suppose A and B are logically equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

   Is this principle correct? Explain your answer.

   The principle is correct. Since A and B are logically equivalent, they have the same truth table. So every valuation that makes A true also makes B true, and every valuation that makes A false also makes B false. So if no valuation makes all the premises true and the conclusion false, when A was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace A with B.

11 Truth tables and validity

11.1 Validity

Having examined the logical relations between two sentences in §10.2 and §10.3, we can now go a step further and consider the relationship between the premises and the conclusion of an argument. Recall the definition of VALID.

valid

An argument is VALID when (and only when) it is the case that if the premises are true, then the conclusion has to be true.

When using a truth table to determine if an argument is valid, we list the premise or premises first, then, the turnstile symbol ('$\vdash$'), and, finally, the conclusion. We will use '$\neg L \rightarrow (M \vee L), \neg L \vdash M$' as our example.

The symbol '$\vdash$' is used to separate the premises from the conclusion in arguments in TFL. It can be read as *therefore*.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T		**T**
T	F	T	F	**T**	T	T	F	**T**	F		**T**
F	T	F	T	**T**	F	T	T	**F**	T		**F**
F	F	T	F	**F**	F	F	F	**T**	F		**F**

Once the truth table is completed for '$\neg L \rightarrow (M \vee L), \neg L \vdash M$', we investigate whether this argument satisfies (or violates) the definition of *valid*. Ask yourself, "When both premises are true, is the conclusion true?" And "Is there any line (that is, any valuation) where both premises are true and the conclusion is false?" If the answer to the first question is always "yes," then the argument is valid. If the answer to the second question is ever "no," then the argument is invalid.

As you can see, there is only one row where both '$\neg L \rightarrow (M \vee L)$' and '$\neg L$' are true, and so that is the row that mainly concerns us. On that row, the conclusion is also true. Hence, '$\neg L \rightarrow (M \vee L), \neg L \vdash M$' is valid.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T		**T**
T	F	T	F	**T**	T	T	F	**T**	F	✓	**T**
F	T	F	T	**T**	F	T	T	**F**	T		**F**
F	F	T	F	**F**	F	F	F	**T**	F		**F**

When using truth tables to determine if an argument is valid, we will put '✓' and '✗' in the column under the turnstile. As just shown, when all of the premises are true and the conclusion is true, we put a '✓' on that line beneath the turnstile. If, on a line, all of the premise are true and the conclusion is false, then we put a '✗' beneath the turnstile.

Also (and **importantly!**), when there is a line where one or more of the premises are false, we put a '✓' beneath the turnstile (whether the conclusion is true or false). An argument is valid when it is the case that *if the premises are true*, then the conclusion has to be true. If there are valuations (i.e., lines) where both premises are not true doesn't matter. Such lines don't violate our definition of *valid*, and so they get a '✓'.

Completing our truth table we have this:

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T	✓	**T**
T	F	T	F	**T**	T	T	F	**T**	F	✓	**T**
F	T	F	T	**T**	F	T	T	**F**	T	✓	**F**
F	F	T	F	**F**	F	F	F	**T**	F	✓	**F**

Now, let's make one small (but significant) change to the argument: $\neg L \rightarrow (M \vee L), \neg L \vdash \neg M$. The premises are the same, but now the conclusion is $\neg M$ instead of M.

The truth values for the premises are the same, and the truth values for the conclusion have, on each line, flipped from T to F or vice versa. Now, when we evaluate each line, what do we find? As before, on lines 1, 3, and 4, one of the premises is false, and so they get a '✓'. On line 2, the premises are true and the conclusion is false. That line gets a '✗'! Because there is a line where the premises are true and the conclusion is false, '$\neg L \rightarrow (M \vee L), \neg L \vdash \neg M$' is not valid.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	$\neg M$
T	T	F	T	**T**	T	T	T	**F**	T	✓	**F**
T	F	T	F	**T**	T	T	F	**T**	F	✗	**F**
F	T	F	T	**T**	F	T	T	**F**	T	✓	**T**
F	F	T	F	**F**	F	F	F	**T**	F	✓	**T**

good and bad lines

Let's call lines that violate the definition of VALID *bad lines* and the lines that do not *good lines*.

(1) Any line where all of the premises are true and the conclusion is false **is a bad line**. Put an '✗' on that line.

(2) Any line where all of the premises are true and the conclusion is true **is a good line**. Put a '✓' on that line.

(3) Any line where the conclusion is true cannot be a bad line. (So, whatever the case may be with the premises, **it's a good line**.) Put a '✓' on that line.

(4) Any line where at least one premise is false cannot be a bad line. So, **it's a good line**. Put a '✓' on that line.

11.2 Some examples

Here are some examples using truth tables to determine whether an argument is valid. As a reminder, the definition of valid is given in §11.1, and we can also use 1 – 4 on p. 107 (which are consequences of the definition). We will begin with arguments that have only one premise and then do some with multiple premises.

1. First we will determine if '$P \& Q \vdash Q$' is valid. The premise, '$P \& Q$', is only true on line 1. Since it is false on lines 2 – 4, we know that those are good lines. (See guideline 4.) On line 1, '$P \& Q$' is true and the conclusion, 'Q', is true, and so that is also a good line. (See guideline 2.) Since every line is a good line, this argument is valid.

P	Q	$P \& Q$	$\vdash$	Q
T	T	T **T** T	✓	**T**
T	F	T **F** F	✓	**F**
F	T	F **F** T	✓	**T**
F	F	F **F** F	✓	**F**

2. In '$\neg(P \vee Q) \vdash \neg P \& Q$', the premise is false on lines 1 – 3, and so we know that those are good lines. On line 4, the premise is true and the conclusion is false, which means that line 4 is a bad line. (See guideline 1.) Since it has at least one bad line, this argument is not valid.

P	Q	$\neg\ (P \vee Q)$	$\vdash$	$\neg\ P \ \&\ Q$
T	T	**F** T T T	✓	F T **F** T
T	F	**F** T T F	✓	F T **F** F
F	T	**F** F T T	✓	T F **T** T
F	F	**T** F F F	✗	T F **F** F

3. Now an argument with two premises: '$P \to Q, \neg Q \vdash \neg P$'. Since both premise are not true on lines 1, 2, and 3, those are all good lines. On line 4, both premises are true and the conclusion is true, and so that is a good line. Since every line is a good line, this argument is valid.

P	Q	$P \to Q$	$\neg\ Q$	$\vdash$	$\neg\ P$
T	T	T **T** T	**F** T	✓	**F** T
T	F	T **F** F	**T** F	✓	**F** T
F	T	F **T** T	**F** T	✓	**T** F
F	F	F **T** F	**T** F	✓	**T** F

4. Next, consider '$P \to Q, P \to \neg Q \vdash P$'. Since the second premise is false on line 1 and the first premise is false on line 2, those are good lines. On line 3, both of the premises are true and the conclusion is false. That's a bad line. And then the same is also the case on line 4, and so that is a bad line also. Since two of the lines in this truth table are bad lines, the argument is invalid.

P	Q	P	$\rightarrow$	Q,	P	$\rightarrow$	$\neg$	Q	$\vdash$	P
T	T	T	**T**	T	T	**F**	F	T	✓	**T**
T	F	T	**F**	F	T	**T**	T	F	✓	**T**
F	T	F	**T**	T	F	**T**	F	T	✗	**F**
F	F	F	**T**	F	F	**T**	T	F	✗	**F**

5. In the last argument, we have three premises. One of the premises is false on each of lines 1, 2, 4, 5, 7, and 8, and so those are all good lines. On line 3, all of the premises are true and the conclusion is true, and so that is a good line. On line 6, all of the premises are true but the conclusion is false, and so that is a bad line. Since one of the lines is a bad line, this argument is invalid.

P	Q	R	P	$\vee$	Q,	P	$\rightarrow$	R,	Q	$\rightarrow$	$\neg$	R	$\vdash$	R
T	T	T	T	**T**	T	T	**T**	T	T	**F**	F	T	✓	**T**
T	T	F	T	**T**	T	T	**F**	F	T	**T**	T	F	✓	**F**
T	F	T	T	**T**	F	T	**T**	T	F	**T**	F	T	✓	**T**
T	F	F	T	**T**	F	T	**F**	F	F	**T**	T	F	✓	**F**
F	T	T	F	**T**	T	F	**T**	T	T	**F**	F	T	✓	**T**
F	T	F	F	**T**	T	F	**T**	F	T	**T**	T	F	✗	**F**
F	F	T	F	**F**	F	F	**T**	T	F	**T**	F	T	✓	**T**
F	F	F	F	**F**	F	F	**T**	F	F	**T**	T	F	✓	**F**

11.3 ‘$\vdash$’ versus ‘$\rightarrow$’

When using truth tables to determine whether an argument is valid, it may help you to notice a similarity between ‘$\vdash$’ and ‘$\rightarrow$’. As you know, a conditional is true under every circumstance except when the antecedent is true and the consequent if false. (So, when we have a ‘T’ under the antecedent and an ‘F’ under the consequent, we put an ‘F’ under the ‘$\rightarrow$’.) Meanwhile, in an argument, when all of the premises are true and the conclusion is false, the argument is invalid. (So, for a specific line, when we have a ‘T’ under every premise and an ‘F’ under the conclusion, we put a ‘✗’ under the ‘$\vdash$’.)

The reasoning here is similar. In both cases, we are violating the principle—of either the conditional or of a valid argument—when we have a false sentence that follows from a sentence or a set of sentences that are all true. Thus, if $\mathsf{A} \rightarrow \mathsf{C}$ is false, then $\mathsf{A} \vdash \mathsf{C}$ is invalid (and if $\mathsf{A} \vdash \mathsf{C}$ is invalid, then $\mathsf{A} \rightarrow \mathsf{C}$ is false). Conversely, whenever $\mathsf{A} \rightarrow \mathsf{C}$ is true, then $\mathsf{A} \vdash \mathsf{C}$ is valid (and vice versa).

11.4 The limits of this type of analysis

We have seen in chapters 10 and 11 that truth tables are a useful tool for analyzing sentences—whether those are individual sentences, pairs of sentences, or arguments. There are limitations to this type of analysis, however, and it worth understanding some of those limitations.

First, consider this argument:

1. Daisy has four legs.
   Therefore, Daisy has more than two legs.

To symbolize this argument in TFL, we would have to use two different atomic sentences—perhaps 'F' for the premise and 'T' for the conclusion. The English version of this argument is clearly valid, but '$F \vdash T$' is just as clearly invalid.

F	T	T	$\vdash$	F
T	T	T	✓	T
T	F	F	✓	T
F	T	T	✗	F
F	F	F	✓	F

Hence, we should keep in mind that while some English sentences can be effectively translated into TFL, not all can be.

Next, consider this sentence:

2. John is neither bald nor not-bald.

This is symbolized in TFL as '$\neg(B \vee \neg B)$', and, as you can see from the truth table, it is a contradiction.

B	$\neg$	$(B$	$\vee$	$\neg$	$B)$
T	**F**	T	T	F	T
F	**F**	F	T	T	F

But sentence 2 does not seem like a contradiction. After all, someone could very well add “John is on the borderline of baldness,” which would (it seems) mean that sentence 2 is true. But since TFL cannot represent a place between ‘B’ and ‘$\neg B$’, it cannot treat *John is neither bald nor not-bald* as true.

Third, let’s think about this statement:

3. It’s not the case that, if God exists, he answers evil prayers.

Symbolizing this in TFL, we have ‘$\neg(G \rightarrow E)$’. As we can see from the truth table, ‘$\neg(G \rightarrow E)$’ $\vdash$‘G’ is valid.

E	G	$\neg$	$(G$	$\rightarrow$	$E)$	$\vdash$	G
T	T	**F**	T	T	T	✓	**T**
T	F	**F**	F	T	T	✓	**F**
F	T	**T**	T	F	F	✓	**T**
F	F	**F**	F	T	F	✓	**F**

So sentence 3 seems to entail that God exists. But that’s not what we expect. An atheist could believe that ‘It’s not the case that, if God exists, he answers evil prayers’ without accepting that God does, in fact, exist.

It might be that sentence 3, despite appearances, does not express what we mean. We can try rephrasing it this way:

4. If God exists, he does not answer evil prayers.

This we symbolize as ‘$G \rightarrow \neg E$’. Now, as shown in the truth table on the left, ‘G’ does not follow from this premise. (That is, the argument ‘$G \rightarrow \neg E \vdash G$’ is invalid.) But, at the same time, from the premise ‘$\neg G$’ (i.e., ‘God does not exist’), it follows that ‘if God exists, he answers evil prayers’.

E	G	$(G$	$\rightarrow$	$\neg$	$E)$	$\vdash$	G
T	T	T	**F**	F	T	✓	**T**
T	F	F	**T**	F	T	✗	**F**
F	T	T	**T**	T	F	✓	**T**
F	F	F	**T**	T	F	✗	**F**

E	G	$\neg$	G	$\vdash$	$(G$	$\rightarrow$	$E)$
T	T	**F**	T	✓	T	**T**	T
T	F	**T**	F	✓	F	**T**	T
F	T	**F**	T	✓	T	**F**	F
F	F	**T**	F	✓	F	**T**	F

(We can also put these final two points as follows. When ‘G’ is false, ‘$G \rightarrow \neg E$’ is true, and so we don’t have to be committed to the existence of

God to accept that 'If God exists, he does not answer evil prayers'. But if 'G' is false, then '$G \rightarrow E$'—i.e., 'If God exists, he answers evil prayers'—is true.)

In different ways, these three examples illustrate some of the limitations of using a language like TFL that can only handle truth-functional connectives. These limitations give rise to some interesting questions in philosophical logic, however. The case of John's baldness (or non-baldness) raises the general question of what logic we should use when dealing with *vague* discourse. The case of God answering evil prayers illustrates some of the *paradoxes of material implication*.

Part of the purpose of studying truth-functional propositional logic is to equip ourselves with the tools to explore these questions of philosophical logic. But we have to walk before we can run; and so we have to become proficient using TFL before we can adequately discuss its limits and consider alternatives.

11.5 Practice exercises

A. Create a truth table for each argument and then determine if the argument is valid or invalid.

1. $A \rightarrow A \vdash A$
2. $A \rightarrow (A \& \neg A) \vdash \neg A$
3. $A \vee (B \rightarrow A) \vdash \neg A \rightarrow \neg B$
4. $A \vee B, B \vee C, \neg A \vdash B \& C$
5. $(B \& A) \rightarrow C, (C \& A) \rightarrow B \vdash (C \& B) \rightarrow A$
6. $A \rightarrow B, B \vdash A$
7. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$
8. $A \rightarrow B, A \rightarrow C \vdash B \rightarrow C$
9. $A \rightarrow B, B \rightarrow A \vdash A \leftrightarrow B$
10. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \vdash A$
11. $A \vee B, B \vee C, \neg B \vdash A \& C$
12. $A \rightarrow B, \neg A \vdash \neg B$
13. $A, B \vdash \neg(A \rightarrow \neg B)$
14. $\neg(A \& B), A \vee B, A \leftrightarrow B \vdash C$

B.

1. Suppose that $(A \& B) \rightarrow C$ is neither a tautology nor a contradiction. Is it possible to determine if $A, B \vdash C$ is valid or not? Explain.
2. Suppose that A is a contradiction. Is $A, B \vdash C$ valid or invalid? Explain.
3. Suppose that C is a tautology. Is $A, B \vdash C$ valid or invalid? Explain.

11.6 Answers

A. Use truth tables to determine whether each argument is valid or invalid.

1. $A \rightarrow A \vdash A$

   This argument is invalid.

A	A	$\rightarrow$	A	$\vdash$	A
T	T	**T**	T	✓	T
F	F	**T**	F	✗	F

2. $A \rightarrow (A \& \neg A) \vdash \neg A$

   This argument is valid.

A	A	$\rightarrow$	$(A$	$\&$	$\neg$	$A)$	$\vdash$	$\neg$	A
T	T	**F**	T	F	F	T	✓	**F**	T
F	F	**T**	F	F	T	F	✓	**T**	F

3. $A \vee (B \rightarrow A) \vdash \neg A \rightarrow \neg B$

   This argument is valid.

A	B	A	$\vee$	$(B$	$\rightarrow$	$A)$	$\vdash$	$\neg$	A	$\rightarrow$	$\neg$	B
T	T	T	**T**	T	T	T	✓	F	T	**T**	F	T
T	F	T	**T**	F	T	T	✓	F	T	**T**	T	F
F	T	F	**F**	T	F	F	✓	T	F	**F**	F	T
F	F	F	**T**	F	T	F	✓	T	F	**T**	T	F

4. $A \vee B, B \vee C, \neg A \vdash B \& C$

   This argument is invalid.

A	B	C	A	$\vee$	$B,$	B	$\vee$	$C,$	$\neg$	A	$\vdash$	B	$\&$	C
T	T	T	T	**T**	T	T	**T**	T	**F**	T	✓	T	**T**	T
T	T	F	T	**T**	T	T	**T**	F	**F**	T	✓	T	**F**	F
T	F	T	T	**T**	F	F	**T**	T	**F**	T	✓	F	**F**	T
T	F	F	T	**T**	F	F	**F**	F	**F**	T	✓	F	**F**	F
T	T	T	F	**T**	T	T	**T**	T	**T**	F	✓	T	**T**	T
T	T	F	F	**T**	T	T	**T**	F	**T**	F	✗	T	**F**	F
T	F	T	F	**F**	F	F	**T**	T	**T**	F	✓	F	**F**	T
T	F	F	F	**F**	F	F	**F**	F	**T**	F	✓	F	**F**	F

5. $(B \& A) \to C, (C \& A) \to B \vdash (C \& B) \to A$
   This argument is invalid.

$(B$	$\&$	$A)$	$\to$	$C,$	$(C$	$\&$	$A)$	$\to$	B	$\vdash$	$(C$	$\&$	$B)$	$\to$	A
T	T	T	**T**	T	T	T	T	**T**	T	✓	T	T	T	**T**	T
T	T	T	**F**	F	F	F	T	**T**	T	✓	F	F	T	**T**	T
F	F	T	**T**	T	T	T	T	**F**	F	✓	T	F	F	**T**	T
F	F	T	**T**	F	F	F	T	**T**	F	✓	F	F	F	**T**	T
T	F	F	**T**	T	T	F	F	**T**	T	✗	T	T	T	**F**	F
T	F	F	**T**	F	F	F	F	**T**	T	✓	F	F	T	**T**	F
F	F	F	**T**	T	T	F	F	**T**	F	✓	T	F	F	**T**	F
F	F	F	**T**	F	F	F	F	**T**	F	✓	F	F	F	**T**	F

6. $A \to B, B \vdash A$ Invalid
7. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$ Valid
8. $A \to B, A \to C \vdash B \to C$ Invalid
9. $A \to B, B \to A \vdash A \leftrightarrow B$ Valid
10. $A \vee \left[A \to (A \leftrightarrow A)\right] \vdash A$ Invalid
11. $A \vee B, B \vee C, \neg B \vdash A \& C$ Valid
12. $A \to B, \neg A \vdash \neg B$ Invalid
13. $A, B \vdash \neg(A \to \neg B)$ Valid
14. $\neg(A \& B), A \vee B, A \leftrightarrow B \vdash C$ Valid

B.

1. Suppose that $(\mathsf{A} \& \mathsf{B}) \to \mathsf{C}$ is neither a tautology nor a contradiction. Is it possible to determine if $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ is valid or not?

Since the sentence $(A \& B) \to C$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, A and B are true and C is false. Hence, the argument, '$A, B \vdash C$', is invalid.

2. Suppose that A is a contradiction. Is $A, B \vdash C$ valid or invalid?

   Since A is false on every line of a truth table, there is no line on which A and B are true and C is false. Hence, the argument is valid. (Although that would be kind of an odd argument since we know that one of the premises is a contradiction.)

3. Suppose that C is a tautology. Is $A, B \vdash C$ valid or invalid?

   Since C is true on every line of a complete truth table, there is no line on which A and B are true and C is false. Hence, the argument is valid.

12 Truth table shortcuts

With practice, you will become adept at quickly filling out truth tables. There are, however, some shortcuts that will (1) save you some time and (2) reinforce the meaning the concepts that can be tested using truth tables.

12.1 Testing for validity

As we said in §11.1, when we use truth tables to test for validity, we are checking for *bad* lines: lines where the premises are all true and the conclusion is false. Consequently,

- Any line where the conclusion is true is not a bad line.
- Any line where some premise is false is not a bad line.

Since *all* we are doing is looking for bad lines, if we find a line where the conclusion is true, we do not need to evaluate anything else on that line. That line definitely isn't bad. Likewise, if we find a line where some premise is false, we do not need to evaluate anything else on that line.

With this in mind, consider how we might investigate wither this argument is valid:

$$\neg L \to (J \vee L), \neg L \vdash J$$

The first step is evaluating the conclusion. If we find that the conclusion is true on some line, then that is not a bad line, and so we can simply ignore the rest of the line.

J	L	$\neg$	L	$\to$	$(J$	$\vee$	$L),$	$\neg$	L	$\vdash$	J
T	T									✓	T
T	F									✓	T
F	T			?				?		?	F
F	F			?				?		?	F

The blank spaces under $\neg L \to (J \vee L)$ and $\neg L$ indicate that we are not going to bother doing any more investigation since the line is not bad. The question-marks indicate that we need to keep investigating. On those lines, it is possible that the premises are true and the conclusion is false.

The easiest premise to evaluate is the second ($\neg L$), and so we do that next.

J	L	$\neg$	L	$\to$	$(J$	$\vee$	$L)$,	$\neg$	L	$\vdash$	J
T	T									✓	T
T	F									✓	T
F	T							F		✓	F
F	F			?				T		?	F

Now we see that we no longer need to consider the third line. It will not be a bad line, because at least one of the premises is false on that line, namely, $\neg L$. Finally, we complete the fourth line:

J	L	$\neg$	L	$\to$	$(J$	$\vee$	$L)$,	$\neg$	L	$\vdash$	J
T	T									✓	T
T	F									✓	T
F	T							F		✓	F
F	F	T		**F**		F		T		✓	F

Since the fourth line tells us that the first premise is false, the truth table has no bad lines. Hence, the argument is valid: any valuation for which all the premises are true is a valuation for which the conclusion is true.

Let us check whether the following argument is valid using the same method.

$$A \vee B, \neg(A \,\&\, C), \neg(B \,\&\, \neg D) \vdash (\neg C \vee D)$$

Again, we first determine the truth value of the conclusion. Since this is a disjunction, it is true whenever either disjunct is true. We can speed things along by noting that the conclusion will be true whenever 'D' is true. Then we only have to determine the truth value for '$\neg C$' on the lines where D is false.

Once we have the truth values for the conclusion, we can, as we did in the last example, ignore every line apart from the lines where the conclusion is false.

A	B	C	D	$A \vee B,$	$\neg(A \& C),$	$\neg(B \& \neg D)$	$\vdash$	$(\neg$	C	$\vee$	$D)$
T	T	T	T				✓			**T**	
T	T	T	F	?	?	?	?	F		**F**	
T	T	F	T				✓			**T**	
T	T	F	F				✓	T		**T**	
T	F	T	T				✓			**T**	
T	F	T	F	?	?	?	?	F		**F**	
T	F	F	T				✓			**T**	
T	F	F	F				✓	T		**T**	
F	T	T	T				✓			**T**	
F	T	T	F	?	?	?	?	F		**F**	
F	T	F	T				✓			**T**	
F	T	F	F				✓	T		**T**	
F	F	T	T				✓			**T**	
F	F	T	F	?	?	?	?	F		**F**	
F	F	F	T				✓			**T**	
F	F	F	F				✓	T		**T**	

We must now evaluate the premises. The first premise is the simplest, and so we start there. Of the four lines where the conclusion is false, there are three where $A \vee B$ is true. So the truth values for the next premise have to be determined for those three lines. (The second premise is simpler to evaluate than the third, so it's next.)

On those three lines, there is only one where the first two premise are true. With a little bit more work, we find that the third premise is false on that line. There is no line where the premises are true and the conclusion is false! The argument is valid.

A	B	C	D	A	$\vee$	B,	$\neg$	$(A$	$\&$	$C)$,	$\neg$	$(B$	$\&$	$\neg$	$D)$	$\vdash$	$(\neg$	C	$\vee$	$D)$
T	T	T	T													✓			**T**	
T	T	T	F		**T**		**F**		T							✓	F		**F**	
T	T	F	T													✓			**T**	
T	T	F	F													✓	T		**T**	
T	F	T	T													✓			**T**	
T	F	T	F		**T**		**F**		T							✓	F		**F**	
T	F	F	T													✓			**T**	
T	F	F	F													✓	T		**T**	
F	T	T	T													✓			**T**	
F	T	T	F		**T**		**T**		F		**F**		T	T		✓	F		**F**	
F	T	F	T													✓			**T**	
F	T	F	F													✓	T		**T**	
F	F	T	T													✓			**T**	
F	F	T	F		**F**											✓	F		**F**	
F	F	F	T													✓			**T**	
F	F	F	F													✓	T		**T**	

If we had used no shortcuts, we would have had to write 256 'T's or 'F's on this table. Using shortcuts, we only had to write 37. We have saved ourselves a *lot* of work.

12.2 Partial truth tables

In the previous section, we saw how an incomplete truth table—although one that still had all of the lines—could be enough to determine if an argument is valid or invalid. That's one method where we use less than the full truth table. Another is where we create a one line truth table. This is called a PARTIAL TRUTH TABLE.

We can also use partial truth tables to determine if a sentence is not a tautology or is not a contradiction, and to determine if a set of sentences are not consistent or are consistent.

Tautology To show that a sentence is a tautology, we need to show that it is true on every valuation. That is to say, we need to know that it is true on every line of the truth table. To do that, we need a complete truth table.

To show that a sentence is *not* a tautology, however, we only need one line: a line on which the sentence is false. Therefore, in order to show that some sentence is not a tautology, it is enough to provide a single valuation—a single line of the truth table—that makes the sentence false.

Suppose that we want to show that the sentence '$(U \& T) \rightarrow (S \& W)$' is *not* a tautology. We set up a PARTIAL TRUTH TABLE. We have only left space for one line, rather than 16, since we are only looking for one line on which the sentence is false. Let us suppose that the sentence is false

S	T	U	W	$(U$	$\&$	$T)$	$\rightarrow$	$(S$	$\&$	$W)$
							F			

The main logical operator of the sentence is a conditional. In order for the conditional to be false, the antecedent must be true and the consequent must be false. So we put those in the table.

S	T	U	W	$(U$	$\&$	$T)$	$\rightarrow$	$(S$	$\&$	$W)$
					T		**F**		F	

For the '$(U \& T)$' to be true, both 'U' and 'T' must be true. Knowing that, we can set the truth values for these atomic sentences on the left side of the truth table.

S	T	U	W	$(U$	$\&$	$T)$	$\rightarrow$	$(S$	$\&$	$W)$
	T	T		T	T	T	**F**		F	

Now we just need to see if we can make '$(S \& W)$' false, which requires at least one of 'S' and 'W' to be false. Since the truth values for 'S' and 'W' have not been set yet, we can make both 'S' and 'W' false if we want. With that, we finish the table in this way:

S	T	U	W	$(U$	$\&$	$T)$	$\rightarrow$	$(S$	$\&$	$W)$
F	T	T	F	T	T	T	**F**	F	F	F

We now have a partial truth table that shows that '$(U \& T) \rightarrow (S \& W)$' is not a tautology. Put otherwise, we have shown that there is a valuation that makes '$(U \& T) \rightarrow (S \& W)$' false, namely, the valuation where 'S' is false, 'T' is true, 'U' is true and 'W' is false (which would be line 10 in a full truth table).

To be clear, we use this method in an *attempt* to show that a sentence is not a tautology. If a sentence is a tautology, then we won't be able to find an assignment of 'true' and 'false' for every sentence letter that makes the full sentence false.

Contradiction Showing that a sentence is a contradiction requires a complete truth table: we need to show that the sentence is false on every line of the truth table.

On the other hand, to show that a sentence is *not* a contradiction, all we need to do is find a valuation that makes the sentence true, and so a single line of a truth table will suffice. We can illustrate this with the same example.

S	T	U	W	$(U$	$\&$	$T)$	$\to$	$(S$	$\&$	$W)$
							T			

One way for this sentence to be true is for the antecedent to be false. Since the antecedent is a conjunction, we can just make one of the conjuncts false. Let's make 'U' false. Then, we can assign whatever truth value we like to the other atomic sentences. With S = *false*, T = *true*, U = *false*, and W = *false*, we have shown that '$(U \& T) \to (S \& W)$' is not contradiction.

S	T	U	W	$(U$	$\&$	$T)$	$\to$	$(S$	$\&$	$W)$
F	T	F	F	F	F	T	**T**	F	F	F

Equivalent To show that two sentences are logically equivalent, we must show that the sentences have the same truth value on every valuation. So this requires a complete truth table.

To show that two sentences are *not* logically equivalent, we only need to show that there is a valuation on which they have different truth values. So this requires only a one-line partial truth table. We make the table so that one sentence is true and the other false.

Consistent To show that some sentences are jointly consistent, we must show that there is a valuation that makes all of the sentences true. This requires only a partial truth table with a single line.

TO CHECK	THAT IT IS	THAT IT IS NOT
tautology	complete	one-line partial
contradiction	complete	one-line partial
equivalent	complete	one-line partial
consistent	one-line partial	complete
valid	complete	one-line partial

Table 12.1: The kind of truth table required to check each of these logical notions.

To show that some sentences are jointly inconsistent, we must show that there is no valuation which makes all of the sentence true. So this requires a complete truth table: You must show that on every row of the table at least one of the sentences is false.

Valid To show that an argument is valid, we must show that there is no valuation that makes all of the premises true and the conclusion false. This requires a truth table with all of the requisite lines, although we can take the shortcuts that were described in the first section of this chapter.

To show that argument is invalid, we must show that there is a valuation that makes all of the premises true and the conclusion false. So this requires only a one-line partial truth table where all of the premises are true and the conclusion is false.

12.3 Practice exercises

A. If it is possible, use a partial truth table to show that the pair of sentences are **not equivalent**. If it can't be shown that they are not equivalent, then create a full truth table showing that they are equivalent.

1. A, $\neg A$
2. A, $A \vee A$
3. $A \rightarrow A$, $A \leftrightarrow A$
4. $A \vee \neg B$, $A \rightarrow B$
5. $A \& \neg A$, $\neg B \leftrightarrow B$
6. $\neg(A \& B)$, $\neg A \vee \neg B$
7. $\neg(A \rightarrow B)$, $\neg A \rightarrow \neg B$
8. $(A \rightarrow B)$, $(\neg B \rightarrow \neg A)$

9. $((U \rightarrow (X \vee X)) \vee U)$ and $\neg(X \,\&\, (X \,\&\, U))$
10. $((C \,\&\, (N \leftrightarrow C)) \leftrightarrow C)$ and $(\neg\neg\neg N \rightarrow C)$
11. $[(A \vee B) \,\&\, C]$ and $[A \vee (B \,\&\, C)]$
12. $((L \,\&\, C) \,\&\, I)$ and $L \vee C$

B. If it is possible, use a partial truth table to show that the set of sentences are **consistent**. If it can't be shown that they are consistent, then create a full truth table showing that they are not consistent.

1. $A \,\&\, B$, $C \rightarrow \neg B$, C
2. $A \rightarrow B$, $B \rightarrow C$, A, $\neg C$
3. $A \vee B$, $B \vee C$, $C \rightarrow \neg A$
4. A, B, C, $\neg D$, $\neg E$, F
5. $A \,\&\, (B \vee C)$, $\neg(A \,\&\, C)$, $\neg(B \,\&\, C)$
6. $A \rightarrow B$, $B \rightarrow C$, $\neg(A \rightarrow C)$
7. $A \rightarrow A$, $\neg A \rightarrow \neg A$, $A \,\&\, A$, $A \vee A$
8. $A \rightarrow \neg A$, $\neg A \rightarrow A$
9. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$
10. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$, $\neg C$
11. $B \,\&\, (C \vee A)$, $A \rightarrow B$, $\neg(B \vee C)$
12. $(A \leftrightarrow B) \rightarrow B$, $B \rightarrow \neg(A \leftrightarrow B)$, $A \vee B$
13. $A \leftrightarrow (B \vee C)$, $C \rightarrow \neg A$, $A \rightarrow \neg B$
14. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \rightarrow B$
15. $A \leftrightarrow B$, $A \rightarrow C$, $B \rightarrow D$, $\neg(C \vee D)$
16. $\neg(A \,\&\, \neg B)$, $B \rightarrow \neg A$, $\neg B$

C. If it is possible, use a partial truth table to show that the argument is **invalid**. If it can't be shown that the argument is invalid, then, using the shortcuts explained in §12.1, create a full truth table showing that it is valid.

1. $A \vee \left[A \rightarrow (A \leftrightarrow A)\right] \vdash A$
2. $A \leftrightarrow \neg(B \leftrightarrow A) \vdash A$
3. $A \rightarrow B, B \vdash A$
4. $A \vee B, B \vee C, \neg B \vdash A \,\&\, C$
5. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$
6. $A \rightarrow (A \,\&\, \neg A) \vdash \neg A$
7. $A \vee B$, $A \rightarrow B$, $B \rightarrow A \vdash A \leftrightarrow B$

8. $A \vee (B \to A) \vdash \neg A \to \neg B$
9. $A \vee B,\ A \to B,\ B \to A \vdash A \& B$
10. $(B \& A) \to C,\ (C \& A) \to B \vdash (C \& B) \to A$
11. $\neg(\neg A \vee \neg B),\ A \to \neg C \vdash A \to (B \to C)$
12. $A \& (B \to C),\ \neg C \& (\neg B \to \neg A) \vdash C \& \neg C$
13. $A \& B,\ \neg A \to \neg C,\ B \to \neg D \vdash A \vee B$
14. $A \to B \vdash (A \& B) \vee (\neg A \& \neg B)$
15. $\neg A \to B, \neg B \to C, \neg C \to A \vdash \neg A \to (\neg B \vee \neg C)$
16. $A \leftrightarrow \neg(B \leftrightarrow A) \vdash A$
17. $A \vee B,\ B \vee C,\ \neg A \vdash B \& C$
18. $A \to C,\ E \to (D \vee B),\ B \to \neg D \vdash (A \vee C) \vee (B \to (E \& D))$
19. $A \vee B,\ C \to A,\ C \to B \vdash A \to (B \to C)$
20. $A \to B,\ \neg B \vee A \vdash A \leftrightarrow B$

D. If it is possible, use a partial truth table to show that the sentence is **not a tautology** or **not a contradiction**. If it can't be shown that it is not a tautology or not a contradiction, then give a full truth table showing that the sentence is a tautology, contradiction, or contingent. (Note that if the sentence is not a tautology *and* not a contradiction, then it is contingent.)

1. $A \to \neg A$
2. $A \to (A \& (A \vee B))$
3. $(A \to B) \leftrightarrow (B \to A)$
4. $A \to \neg(A \& (A \vee B))$
5. $\neg B \to [(\neg A \& A) \vee B]$
6. $\neg(A \vee B) \leftrightarrow (\neg A \& \neg B)$
7. $[(A \& B) \& C] \to B$
8. $\neg\big[(C \vee A) \vee B\big]$
9. $\big[(A \& B) \& \neg(A \& B)\big] \& C$
10. $(A \& B)] \to [(A \& C) \vee (B \& D)]$
11. $\neg(A \vee A)$
12. $(A \to B) \vee (B \to A)$
13. $[(A \to B) \to A] \to A$
14. $\neg[(A \to B) \vee (B \to A)]$

15. $(A \& B) \vee (A \vee B)$
16. $\neg(A \& B) \leftrightarrow A$
17. $A \rightarrow (B \vee C)$
18. $(A \& \neg A) \rightarrow (B \vee C)$
19. $(B \& D) \leftrightarrow [A \leftrightarrow (A \vee C)]$
20. $\neg[(A \rightarrow B) \vee (C \rightarrow D)]$

Part 4

Natural deduction for TFL

13 Natural deduction

13.1 Natural deduction versus truth tables

An argument is valid when (and only when) it is impossible for all of the premises to be true and the conclusion to be false. And we have seen that truth tables can be used to determine whether an argument is valid. In the next chapter, you will learn another method for verifying that an argument is valid. Before we turn to this new method, however, let's review the strengths and weakness of truth tables.

1. The truth table method for determining if an argument is valid focuses directly on the definition of *valid*. Each line of a complete truth table corresponds to a valuation. So, given an argument in TFL, truth tables reveal whether or not the conclusion is true when all of the premises true.

2. Truth tables also allow us to easily and rigorously set the meaning for each logical operator. As we discussed in §5.3, in English, 'or' can take the inclusive-or meaning (one or the other, or both) or the exclusive-or meaning (one or the other, but not both), and, at different times, both meanings are used in English. We can discuss which English meaning is closest to the meaning of '$\vee$' in TFL (it's the inclusive-or), but, in the end, we just set the meaning of the symbol '$\vee$' with this truth table:

A	B	$\mathsf{A} \vee \mathsf{B}$
T	T	T
T	F	T
F	T	T
F	F	F

Hence, this is the definition for this logical operator: '$\vee$' is the symbol that connects A and B in the ways shown in the truth table.

And, then, the same goes for the other logical operators.

3. To create a truth table, the number of lines needed is 2^n, where n is the number of different letters in the argument. So, an argument with four different sentence letters will require a 16 line truth table, one with five letters will require 32 lines, one with six different letters will require 64 lines, and so on. Hence, while a truth table can be used to determine if any argument is valid or invalid, one of the weakness of this method is that it is difficult to use when the argument contains more than four different sentence letters.

4. But what is typically seen as the biggest weakness of using truth tables to determine if an argument is valid is that it doesn't reveal to us *why* the argument is valid. It doesn't, in other words, lay out the reasoning that demonstrates why (and how) the conclusion follows from the premises.

As an alternative to truth tables, we can use a *natural deduction system.* Such a system allows us to verify that an argument is valid and to see why it is valid. We do this by making explicit the reasoning process that takes us from the premises to the conclusion. We begin with twelve basic rules. (For instance, this is one of the rules: if we know that '*P or Q*' is true; and we also know that '*not P*' is true, then we can assert that '*Q*' is true.) The rules can be combined, and with just a small number of them, we hope to be able to show how we get from the premises to the conclusion for all of the valid arguments that can be represented in TFL. There are different deduction systems that can be used with TFL. A *natural* deduction system is one that, for the most part, reflects the ways that we naturally reason—at least insofar as the reasoning involves 'and', 'or', 'not', 'if . . . , then . . . ', and 'if and only if'.

13.2 Truth functional propositional logic

We have reached a point where it is useful to summarize what TFL is. As you know, the symbols of TFL are the sentence letters that represent atomic sentences, the logical operators $\neg$, &, $\vee$, $\rightarrow$, and $\leftrightarrow$, and brackets. These, then, can be combined into sentences using the rules given in chapter 6. And, then, in chapter 8, truth tables were used to set the meaning of the logical operators.

Truth tables also give us a method for verifying that an argument satisfies the definition of *valid*. *Valid* is a concept and is not, strictly speaking a part of TFL. Rather it is a property of some arguments that can, to an extent, be studied and explicated using TFL. Similarly, as you have seen, *tautology*, *contradiction*, *contingent*, *equivalent*, *jointly consistent*, and *jointly inconsistent* are concepts that can be explained using TFL.

The final part of TFL is a system of natural deduction, which sets the rules for how sentences containing the logical operators can be combined or taken apart. The natural deduction system introduced in the next chapter consists of twelve rules of derivation. (That is, twelve rules for deriving new statements from those that we already have.)

13.3 Fitch

The modern development of natural deduction dates from simultaneous but unrelated papers by Gerhard Gentzen and Stanisław Jaśkowski that were published in 1934. The natural deduction system that we will use, however, is based largely on work by Frederic Fitch that was first published in 1952. Consequently, the format that is used in the next chapter for writing proofs is called *Fitch notation*.

14 The rules of derivation

14.1 Proofs

A PROOF is a list of sentences. The sentence or sentences at the beginning of the list are assumptions. These are the premises of the argument. (We call them *assumptions* because, at least within the proof, they do not require any justification. We just state them.) Every other sentence in the list follows from earlier sentences by a specific rule. The final sentence is the conclusion of the argument.

As an illustration, consider this argument:

$$\neg(A \vee B) \vdash \neg A \mathbin{\&} \neg B$$

We start the proof by numbering the line and writing the premise:

1	$\neg(A \vee B)$	:PR

Every line in a proof is numbered so that we can refer to it later if we need to do so. We have also indicated that this is a premise by putting 'PR' at the end of the line. And we have drawn a line underneath the premise. Everything written above the line is an *initial assumption* (i.e., a premise). Everything written below the line will either be a sentence that can be derived from that assumption, or it will be a new assumption that we introduce. The colon that is right before 'PR' is, technically, optional, but it has to be used in Carnap to separate the TFL sentence from the 'PR' (or the rule) that is written at the end of each line.

The conclusion of this argument is '$\neg A \mathbin{\&} \neg B$'; and so we want our proof to end—on some line, we'll call it n—with that sentence:

1	$\neg(A \vee B)$	:PR
2	$\ldots$	
	$\ldots$	
	$\ldots$	
n	$\neg A \,\&\, \neg B$	

It doesn't matter how many lines it takes to arrive at the conclusion, although, generally, we would prefer a shorter proof over a longer one.

Suppose we have this argument:

$$A \vee B, \neg(A \,\&\, C), \neg(B \,\&\, \neg D) \vdash \neg C \vee D$$

This argument has three premises, and so we start by listing them, numbering each line, and drawing a line under the final premise:

1	$A \vee B$	:PR
2	$\neg(A \,\&\, C)$	:PR
3	$\neg(B \,\&\, \neg D)$	:PR

This, meanwhile, will be the final line of the proof:

n	$\neg C \vee D$

Setting up the premises and the conclusion is, however, the easy part. The real task—and the interesting part—is determining each of the steps that get us from the premise or premises to the conclusion.

To do that, we will use a NATURAL DEDUCTION system. In this system, there are two rules for each logical operator: an INTRODUCTION rule, which allows us to derive a new sentence that has the logical operator as the main connective, and an ELIMINATION rule, which allows us to extract a sub-sentence from a sentence that has that logical operator as the main connective. (Table 14.1 contains a list of the rules.) These rules can then be combined to demonstrate each step that must be taken to get from the premises to the conclusion. All of the rules introduced in this chapter are also summarized on pp. 196 - 197.

THE RULES OF DERIVATION	
conjunction introduction rule	conjuction elimination rule
disjunction introduction rule	disjunction elimination rule
conditional introduction rule	conditional elimination rule
biconditional introduction rule	biconditional elimination rule
negation introduction rule	negation elimination rule
reiteration rule	
double negation rule	

Table 14.1

14.2 Conjunction introduction and elimination

Let's say that we know that Sarah is swimming. We also, as it happens, know that Amy is reading. We are, therefore, justified in stating, "Sarah is swimming and Amy is reading." This reasoning process, which we all do naturally, is part of our natural deduction system. It is called the CONJUNCTION INTRODUCTION RULE.

conjunction introduction rule

m	A	
n	B	
	A & B	:&I m, n

The 'm' and 'n' will never appear in an actual proof. In a proof, the lines are numbered 1, 2, 3, etc. The 'm' and 'n' are used here to indicate that A and B can be on any lines in the proof.

The 'A' and 'B' can occur in either order, and the conjunction can be 'A & B' or 'B & A'.

For the example about Sarah and Amy, we can use this symbolization key:

S: Sarah is swimming.
R: Amy is reading.

Let's say that 'S' and 'R' are our premises (although they don't have to be to use the conjunction introduction rule), and so they are on lines 1 and 2. Then on any subsequent line—but, in this case, it will be line 3—we can get '$S \& R$' by using the conjunction introduction rule (&I).

1	S	:PR
2	R	:PR
3	$S \& R$	:&I 1, 2

Every line of our proof must either be an assumption (and remember that a premise is an assumption), or it must be justified by some rule. Therefore, on line 3, we put '&I 1, 2' to indicate that '$S \& R$' was obtained by applying the conjunction introduction rule to lines 1 and 2.

The conjunction introduction rule introduces a sentence with '&' as the main connective. We also have a rule that eliminates that connective. Suppose someone tells you that *Jeff is eating and Mary is sleeping*. Assuming that whoever told you this is reliable, you are entitled to infer simply that *Jeff is eating*. You are also entitled to infer that *Mary is sleeping*. These are applications of the CONJUNCTION ELIMINATION RULE (which is actually two similar rules).

conjunction elimination rule

m	A & B	
	A	:&E m

and equally:

m	A & B	
	B	:&E m

When you have a conjunction on one line of a proof, you can use the conjunction elimination rule to obtain either of the conjuncts on a new line. You can only, however, apply this rule when the '&' is the main logical operator. So, for instance, you cannot use the conjunction elimination rule to obtain 'D' from '$C \vee (D \& E)$'. **Each of the rules of derivation can only be applied to the main logical operator of a sentence.**

With just these two rules, we can undertake the proof for this argument, which requires more steps than did the previous example:

$$(A \vee B) \,\&\, (G \,\&\, H) \vdash (A \vee B) \,\&\, H$$

The main logical operator in both the premise and conclusion of this argument is '&', and so we will use both of our conjunction rules in the proof. We begin by writing down the premise, and we put a line below it. Everything after this line must follow from our premise by the application of our rules.

1	$(A \vee B) \,\&\, (G \,\&\, H)$	:PR

From the premise, we can eliminate the main connective (and only the main connective) using the conjunction elimination rule. Using this rule twice gives us this:

1	$(A \vee B) \,\&\, (G \,\&\, H)$	:PR
2	$(A \vee B)$	:&E 1
3	$(G \,\&\, H)$	:&E 1

Now that '$G \,\&\, H$' is on its own line, we can use the conjunction elimination rule again to get H on a line by itself.

1	$(A \vee B) \,\&\, (G \,\&\, H)$	:PR
2	$(A \vee B)$	:&E 1
3	$(G \,\&\, H)$	:&E 1
4	H	:&E 3

In our final step, we use the conjunction introduction rule to get the conclusion, '$(A \vee B) \,\&\, H$'.

1	$(A \vee B) \,\&\, (G \,\&\, H)$	:PR
2	$(A \vee B)$	:&E 1
3	$(G \,\&\, H)$	:&E 1
4	H	:&E 3
5	$(A \vee B) \,\&\, H$	:&I 2, 4

And we're done. Notice that there is nothing in this representation of the proof to indicate that the last line is the conclusion. It's only because we began with '$(A \vee B) \& (G \& H) \vdash (A \vee B) \& H$' that we know that we have arrived at the conclusion that we wanted.

14.3 Disjunction intro and elim

Disjunction elimination For the disjunction rules, let's start with this example:

> *Sarah is swimming or Jeff is eating a burrito.*

When is this true? Recall from §5.3 that we are using the inclusive-or. So, *Sarah is swimming or Jeff is eating a burrito* is true when either

(a) *Sarah is swimming* is true, or
(b) *Jeff is eating a burrito* is true, or
(c) both are true.

That's a lot of options, but if all we know is that Sarah is swimming or Jeff is eating a burrito, then we don't know precisely what either one of them are doing. But let's say that someone (whom we trust completely) tells us that, actually, Sarah is *not* swimming. That piece of information about Sarah, then, let's us safely infer that Jeff is eating a burrito. This reasoning process is captured by the DISJUNCTION ELIMINATION RULE.

disjunction elimination rule

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\neg\mathsf{B}$	
	A	$:\vee\mathrm{E}\ m, n$

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\neg\mathsf{A}$	
	B	$:\vee\mathrm{E}\ m, n$

disjunction introduction rule

m	A	
	$\mathsf{A} \vee \mathsf{B}$	$:\vee\mathrm{I}\ m$

m	A	
	$\mathsf{B} \vee \mathsf{A}$	$:\vee\mathrm{I}\ m$

Disjunction introduction *Sarah is swimming or Jeff is eating a burrito* is true when *Sarah is swimming* is false (she isn't swimming) and *Jeff is eating a burrito* is true. In other words, the disjunction will be true as long as one of the disjuncts is true. This feature of disjunctions lets us make an inference that we don't use often in our everyday lives. It is a very simple inference, however. Take any sentence. We'll use *you are studying logic.* That's true. Since *you are studying logic* is true, each one of these sentences is also true:

> You are studying logic, or you are studying German.
> You are studying logic, or your mother is in Fiji.
> You are studying logic, or a dragon is on the moon.

The idea is that, if we know that a sentence is true, we can create a longer sentence by adding 'or *any sentence whatsoever*' and the disjunction will also be true. This feature of the disjunction underlies the DISJUNCTION INTRODUCTION RULE (which, again, is two similar rules).

Introduction and elimination

When you use the rules that are given in this chapter, you are applying the *patterns* given in the definition of each rule. Every pattern is different, and so you have make sure that you understand each one. But you don't, actually, have to think beyond the patterns—although you can, and some people find it helpful to think about how the rules of derivation conform to what you learned about each type of TFL sentence in the chapter on characteristic truth tables (chapter 8).

It can also be useful to understand why we call these *introduction* and *elimination* rules. (Or, at least, it's useful to not misunderstand why we use these terms.) The introduction rules are given that name because, in each case, we introduce a logical operator. For instance, if a proof begins this way:

1	P	:PR
2	Q	:PR

then, clearly, there is not a '&' in the proof yet. The conjunction introduction rule, however, lets us introduce one:

3	$P \& Q$	:&I 1, 2

The elimination rules, meanwhile, all, in a way, eliminate a logical operator. That's not a perfect explanation, however, because the TFL sentence containing the logical operator still exists (and can be used). But, nonetheless, when we use an elimination rule we take a subsentence that is on one side of a logical operator and put it on a line by itself—so the logical operator (and the other side of the sentence) is gone from what's on the new line. For instance, if we have this:

1	$R \& T$	:PR

then we can put the 'R' on a new line by itself:

2	R	:&E 1

The '$R \& T$' is still on line 1 and can be used again in the proof, and so it definitely hasn't been eliminated. Hence, we might think of the elimination rules as *extraction* rules rather than elimination rules, but *elimination* is the commonly used term, and so we will stick with it.

Double negation

The DOUBLE NEGATION RULE is a rule of convenience that sometimes compliments the disjunction-elimination rule. (There are also times when it will be used in the proofs that are discussed in 17.1, but, for the material in this chapter, it will only be used with the disjunction-elimination rule.) First, notice that the disjunction-elimination rule is very specific. To use it, we need, on one line of our proof, a sentence with the form '$\mathsf{A} \vee \mathsf{B}$', and on another line of our proof, we need one side of the disjunction (either A or B) with a '$\neg$' in front of it—that is, either $\neg\mathsf{A}$ or $\neg\mathsf{B}$.

This presents a problem if we have these two sentences somewhere in a proof:

m	$\neg P \vee Q$
n	P

You might think that, given those two lines, we can put 'Q' on a new line like this:

$$
\begin{array}{r|ll}
m & \neg P \lor Q & \\
n & P & \\
 & Q & :\lor\text{E } m, n
\end{array}
$$

In a sense, this is the right idea for the disjunction elimination rule—one side of the disjunction '$\neg P \lor Q$' has to be true and the 'P' on line n means that '$\neg P$' is false. Hence, we should be allowed to put 'Q' on a new line. The disjunction-elimination rule, however, does not permit this. To see why, let's distinguish between NEGATION and DENIAL.

negation and denial

The NEGATION of a sentence is the sentence with a 'not' added to it.
The DENIAL of a sentence is the sentence with either a 'not' added or a 'not' removed.

For example,

1. the negation of 'today is Tuesday' is 'today is not Tuesday'.

In TFL,

2. the negation of A is $\neg\mathsf{A}$. The negation of $\neg\mathsf{A}$ is $\neg\neg\mathsf{A}$.

Meanwhile,

3. the denial of 'it is not raining' is either (a) 'it is raining' or (b) 'it is not not raining'.

In TFL,

4. the denial of $\neg\mathsf{A}$ is either A or $\neg\neg\mathsf{A}$.

To use the disjunction elimination rule, we must have the ***negation*** of one side of the disjunction on another line. In the example above, we have the denial of $\neg P$, not its negation on line n. The DOUBLE NEGATION

RULE helps us correct this so that we can use the disjunction elimination rule more often.

Before we see how the double negation rule can help us with our derivation, let's introduce the rule. The first version of the double negation rule allows us to add two *not*s (i.e., *not not*) to a sentence in TFL—which, of course, will not change the sentence's truth value. The second version of the double negation rule allows us to remove two *not*s, although needing to do this is less common.

double negation rule

m	A	
n	$\neg\neg\mathsf{A}$	:DN m

m	$\neg\neg\mathsf{A}$	
n	A	:DN m

Let's say that this is the argument for which we need to provide a proof: '$\neg P \vee Q, P \vdash Q$'. After the premises, we use the double negation rule to get '$\neg\neg P$' from line 2.

1	$\neg P \vee Q$	:PR
2	P	:PR
3	$\neg\neg P$	:DN 2

The 'P' on line 2 and the '$\neg\neg P$' on line 3 have exactly the same meaning. The only difference between 'P' and '$\neg\neg P$' is their form. But now that we have '$\neg\neg P$', we have the negation of what is on the left side of the disjunction (i.e., '$\neg P$'). That allows us to use the disjunction elimination rule, and we can get the conclusion.

1	$\neg P \vee Q$	:PR
2	P	:PR
3	$\neg\neg P$	:DN 2
4	Q	:$\vee$E 1, 3

Remember that, in this chapter, you will only use the double negation rule right before you use the disjunction elimination rule, and you will only use it some of the time with the disjunction elimination rule.

when not to use (left) and when to use (right) the double negation rule

m	A $\vee$ B	
n	$\neg$B	
	A	:$\vee$E m, n

m	A $\vee$ $\neg$B	
n	B	
p	$\neg\neg$B	:DN n
	A	:$\vee$E m, p

14.4 Conditional elimination

For the conditional, we will cover the elimination rule now and the introduction rule in §14.7. Consider the following argument:

1. If the envelope is on the table, then Aleksander is in the safe house.
2. The envelope is on the table.
3. Therefore, Aleksander is in the safe house.

In this argument—which is valid—we have a conditional and then, on a separate line, the antecedent of that conditional ('the envelope is on the table'). That allows us to safely infer the antecedent ('Aleksander is in the safe house'). In short, if we have a conditional and we know that the antecedent of the conditional is true, then we know that the consequent has to be true. (See also the discussion of the conditional on p. 71.) Deriving the consequent of the conditional in this way is an application of the CONDITIONAL ELIMINATION RULE. This rule is also sometimes called *modus ponens*. When we use the rule, the conditional and the antecedent of the conditional can be separated from one another, and they can appear in any order.

conditional elimination rule

m	$A \to B$	
n	A	
	B	$:\to$E m, n

biconditional elimination rule

m	$A \leftrightarrow B$	
n	A	
	B	$:\leftrightarrow$E m, n

m	$A \leftrightarrow B$	
n	B	
	A	$:\leftrightarrow$E m, n

14.5 Biconditional intro and elim

The BICONDITIONAL ELIMINATION RULE is similar to the conditional elimination rule but a bit more flexible. If you have a biconditional and the left side of the biconditional on another line, you can put the right side on a new line. Similarly, if you have the right side, you can put the left side on a new line. Notice the difference between the conditional elimination and the biconditional elimination rule. There are two ways to use the biconditional elimination rule. There is only one way to use the conditional elimination rule.

In chapter 8, we said that the biconditional is "the conjunction of a conditional running in each direction." This is the foundation for the BICONDITIONAL INTRODUCTION RULE. If we have both conditionals, $A \to B$ and $B \to A$, then we can put $A \leftrightarrow B$ on a new line.

biconditional introduction rule

$$
\begin{array}{r|ll}
m & \mathsf{A} \rightarrow \mathsf{B} & \\
n & \mathsf{B} \rightarrow \mathsf{A} & \\
 & \mathsf{A} \leftrightarrow \mathsf{B} & :\leftrightarrow\mathrm{I}\ m, n
\end{array}
$$

14.6 Some examples

We will now review some proofs that use the rules that are covered in §§14.2 – 14.5.

1. A proof of '$P \,\&\, Q, \neg R \vdash Q \,\&\, \neg R$' requires the conjunction introduction rule and the conjunction elimination rule.

$$
\begin{array}{r|ll}
1 & P \,\&\, Q & :\mathrm{PR} \\
2 & \neg R & :\mathrm{PR} \\
\hline
3 & Q & :\&\mathrm{E}\ 1 \\
4 & Q \,\&\, \neg R & :\&\mathrm{I}\ 2, 3
\end{array}
$$

2. For a proof of '$P \vee Q, R \& \neg Q \vdash P \vee S$', we need the conjunction elimination rule, disjunction introduction rule, and the disjunction elimination rule.

$$
\begin{array}{r|ll}
1 & P \vee Q & :\mathrm{PR} \\
2 & R \,\&\, \neg Q & :\mathrm{PR} \\
\hline
3 & \neg Q & :\&\mathrm{E}\ 2 \\
4 & P & :\vee\mathrm{E}\ 1, 3 \\
5 & P \vee S & :\vee\mathrm{I}\ 4
\end{array}
$$

3. For '$C \,\&\, (D \vee \neg F), F \,\&\, G \vdash C \,\&\, (D \vee H)$', we use all five of the rules introduced in §§14.2 and 14.3.

1	$C \,\&\, (D \vee \neg F)$	:PR
2	$F \,\&\, G$	:PR
3	C	:&E 1
4	$D \vee \neg F$	:&E 1
5	F	:&E 2
6	$\neg\neg F$	:DN 5
7	D	:∨E 4, 6
8	$D \vee H$	:∨I 7
9	$C \,\&\, (D \vee H)$	:&I 3, 8

4. For a proof of ‘$P \to Q, R \,\&\, P \vdash Q \,\&\, R$’, we use the conjunction introduction rule, the conjunction elimination rule, and the conditional elimination rule.

1	$P \to Q$	:PR
2	$R \,\&\, P$	:PR
3	P	:&E 2
4	R	:&E 2
5	Q	:→E 1, 3
6	$Q \,\&\, R$	:&I 4, 5

5. For a proof of ‘$R \leftrightarrow T, P \vee T, \neg P \vdash R$’, we use the disjunction elimination rule and the biconditional elimination rule.

1	$R \leftrightarrow T$	:PR
2	$P \vee T$	:PR
3	$\neg P$	:PR
4	T	:∨E 2, 3
5	R	:↔E 1, 4

6. And last, a proof for

$$(S \to T) \lor \neg R, (T \to S) \lor Q, R \,\&\, \neg Q \vdash T \leftrightarrow S$$

requires the conjunction elimination rule, the double negation rule, the disjunction elimination rule, and the biconditional introduction rule.

1	$(S \to T) \lor \neg R$	:PR
2	$(T \to S) \lor Q$	:PR
3	$R \,\&\, \neg Q$	:PR
4	R	:&E 3
5	$\neg Q$	:&E 3
6	$\neg\neg R$	:DN 4
7	$S \to T$	:∨E 1, 6
8	$T \to S$	:∨E 2, 5
9	$T \leftrightarrow S$	:↔I 7, 8

14.7 Conditional introduction

The CONDITIONAL INTRODUCTION RULE is a little bit more complicated than the conditional elimination rule, but, with some thought (and some practice), it is easily grasped. We'll start with this symbolization key for the sentence letters G and L:

G: Kate's German class meets today.
L: Kate's logic class meets today.

And this is our argument:

$$G \lor L \vdash \neg G \to L$$

Maybe you can see that this argument is valid. But if you can't right now, that's fine. We will go through the proof for this argument, and in the process explain the conditional introduction rule. We start by listing the premise.

1 | $G \lor L$:PR

Next, we need to make a new assumption: 'Kate's German class is *not* meeting today'. We might say that we're making this assumption "for the sake of argument" or to see where it leads. To indicate that this is an assumption that we have supplied, we put '$\neg G$' on line 2 this way:

1 | $G \lor L$:PR
2 | | $\neg G$:AS

You will notice right away that the '$\neg G$' is indented. Whenever we make an assumption ourselves, we must indent it and the lines that follow. This creates a SUBPROOF that is set off from the rest of the proof. The assumption is cited with 'AS', and we put a line under the assumption just as we do with the final premise. With this extra assumption in place, we next use the disjunction elimination rule to get L on line 3.

1 | $G \lor L$:PR
2 | | $\neg G$:AS
3 | | L :$\lor$E 1, 2

The idea for the first three lines of this proof are, first, we know that *Kate's German class meets today or her logic class meets today.* (Or, at least, we are assuming that '$G \lor L$' is true because that is the premise that we were given). Next, on line 2, we are, in effect, asking, "What if her German class is not meeting today?" That is, what will follow if we make this assumption? Well, one thing that will follow is that Kate's logic class must be meeting.

So, on line 2, we have asked, What if *Kate's German class is not meeting today*? On line 3, we have one answer: *Kate's logic class is meeting today.* Therefore, on line 4, we can use the conditional introduction rule to put these two together as *if Kate's German class is not meeting today, then Kate's logic class is meeting today.*

$$
\begin{array}{r|l l}
1 & G \lor L & :\text{PR} \\
\hline
2 & \quad \begin{array}{|l} \neg G \\ \hline \end{array} & :\text{AS} \\
3 & \quad \begin{array}{|l} L \end{array} & :\lor\text{E } 1, 2 \\
4 & \neg G \to L & :\to\text{I } 2\text{–}3
\end{array}
$$

For this final step, we have gone back to the original vertical line of the proof.

When we use the conditional introduction rule, the assumption that we make will always be the antecedent of the conditional. The last line of the subproof, meanwhile, will always be the consequent of the conditional.

conditional introduction rule

Begin by making an assumption, A. From that assumption, derive B. Once that is done, you know that *if* A, then B, and you can put the conditional on the line after the subproof.

$$
\begin{array}{r|l l}
i & \quad \begin{array}{|l} \mathsf{A} \\ \hline \end{array} & :\text{AS} \\
j & \quad \begin{array}{|l} \mathsf{B} \end{array} & \\
 & \mathsf{A} \to \mathsf{B} & :\to\text{I } i\text{–}j
\end{array}
$$

There can be as many or as few lines as needed between lines i and j.

Subproofs Lines i through j are called a SUBPROOF. These are the rules for subproofs:

1. Once a subproof has been closed, none of the lines in the subproof can be used again. (The conditional $\mathsf{A} \to \mathsf{B}$ can be used later in the proof because it is outside of the subproof.)
2. A subproof is closed by the application of the conditional introduction rule—or, as you will see shortly, the negation introduction or the negation elimination rules.
3. When we close a subproof, the assumption made at the beginning of the subproof has been *discharged*.
4. A proof is not complete until every assumption that we have made (and so not counting the premises) is discharged.

14.8 Some more examples

Let's go through a few more examples, each of which uses the conditional introduction rule.

1. Suppose we want a proof of this argument:

$$P \to Q, Q \to R \vdash P \to R$$

We start by listing both of our premises. Next, since we want $(P \to R)$, we assume the antecedent of that conditional.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS

Now, even though it is an assumption that we've introduced, since 'P' is on a line by itself (and the subproof has not yet been closed), we can use it for our next step. With 'P' and the '$P \to Q$' on line 1, we can use $\to$E to get 'Q'.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:$\to$E 1, 3

With the Q on line 4 and $Q \to R$ on line 2, we can use $\to$E and get R. So, by assuming 'P', we were able to get 'R'. Last, we apply the $\to$I rule, which discharges our assumption and completes the proof.

1	$P \rightarrow Q$	:PR
2	$Q \rightarrow R$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:$\rightarrow$E 1, 3
5	$\quad R$	:$\rightarrow$E 2, 4
6	$P \rightarrow R$	:$\rightarrow$I 3–5

2. Next, let's construct a proof for this argument: '$F \rightarrow (G \& H) \vdash F \rightarrow G$'. We proceed this way:

1	$F \rightarrow (G \& H)$	:PR
2	$\quad F$	:AS
3	$\quad G \& H$	:$\rightarrow$E 1, 2
4	$\quad G$	:&E 3
5	$F \rightarrow G$	:$\rightarrow$I 2–4

3. As you know, the biconditional elimination rule is similar to the conditional elimination rule. (But they are not the same. See p. 144 to compare them.) We should also, however, be able to start with a biconditional, say '$B \leftrightarrow C$' and derive either of the conditionals: '$B \rightarrow C$' or '$C \rightarrow B$'. This is easily done with the conditional introduction rule.

1	$B \leftrightarrow C$	:PR
2	$\quad B$	:AS
3	$\quad C$	:$\leftrightarrow$E 1, 2
4	$B \rightarrow C$	:$\rightarrow$I 2–3

   And, with a similar proof, we can also derive '$C \rightarrow B$'.

4. In the proof for '$\neg P \vee (R \& Q) \vdash P \rightarrow Q$', we will use the conditional introduction rule as well as double negation rule and disjunction-elimination rule.

1	$\neg P \vee (R \,\&\, Q)$	:PR
2	$\quad P$	:AS
3	$\quad \neg\neg P$	:DN 2
4	$\quad R \,\&\, Q$	:$\vee$E 1, 3
5	$\quad Q$	:&E 4
6	$P \rightarrow Q$	:$\rightarrow$I 2–5

14.9 Negation introduction and elimination

Here is a simple mathematical argument in English:

1. Assume that there is some greatest natural number. Call it G.
2. That number plus one is also a natural number.
3. $G+1$ is greater than G.
4. Thus, G is the greatest natural number (accordingt to 1), and there is a natural number greater than G (according to 3).
5. The previous line is a contradiction.
6. Therefore, the assumption that we made on line 1 is false. There is no greatest natural number.

This type of argument is traditionally called a *reductio*. Its full Latin name is *reductio ad absurdum*, which means 'reduction to absurdity' (although *absurdity* in the sense that we generally use the word today isn't part of this). In a reductio, we assume something for the sake of argument—for example, that there is a greatest natural number. Then we show that the assumption leads to two contradictory sentences—for example, 'G is the greatest natural number' and 'G is not the greatest natural number.' In this way, we have shown that the original assumption must be false, which means that the denial of the assumption is true.

Our two negation rules (which are basically the same rule) formalize this reasoning process.

negation introduction rule

m	| | A	:AS
n	| | B	
p	| | ¬B	
	| ¬A	:¬I m–p

negation elimination rule

m	| | ¬A	:AS
n	| | B	
p	| | ¬B	
	| A	:¬E m–p

Notice that, just as we do when using the conditional introduction rule, we begin by making an assumption. The subproof that follows is indented, and the assumption that we made must be discharged by applying either the negation introduction rule or the negation elimination rule.

When using either of the negation rules, the last two lines of the subproof must be an explicit contradiction: B on one line and its negation, ¬B, on the next line (or vice versa). Those two lines cannot be separated. When you cite the rule, however, the lines that you give are the lines for the whole subproof (starting with the assumption), not just the two lines containing the contradiction.

Reiteration

To get a contradiction on the last two lines of a subproof, you will usually have to move a sentence that is on an earlier line to the last or second-to-last line of the subproof. This is done with the REITERATION RULE. Just as the double negation rule is a rule of convenience that sometimes compliments the disjunction-elimination rule, the reiteration rule is a rule of convenience that compliments the negation elimination and negation introduction rules.

reiteration rule

m	| A	
n	| A	:R m

To demonstrate both the negation elimination rule and the reiteration rule, we will go through the proof for this argument: '$\neg P \rightarrow \neg Q, Q \vdash P$'. Looking at the argument, you'll notice that our conclusion is 'P', but we cannot get 'P' by using &E, ∨E, →E, or ↔E. That tells us that we will need to use one of our negation rules.

After the premises, we make the assumption that we need for negation elimination. Since, ultimately, we want 'P', we will assume '$\neg P$' so that, once we discharge that assumption (and close the subproof), we will have the 'P' that we are after.

$$
\begin{array}{r|l|l}
1 & \neg P \rightarrow \neg Q & :\mathrm{PR} \\
2 & Q & :\mathrm{PR} \\
\hline
3 & \quad \vert\ \neg P & :\mathrm{AS}
\end{array}
$$

We then use the conditional elimination rule to get $\neg Q$ on line 4.

$$
\begin{array}{r|l|l}
1 & \neg P \rightarrow \neg Q & :\mathrm{PR} \\
2 & Q & :\mathrm{PR} \\
\hline
3 & \quad \vert\ \neg P & :\mathrm{AS} \\
4 & \quad \vert\ \neg Q & :\rightarrow\mathrm{E}\ 1, 3
\end{array}
$$

The Q on line 2 and $\neg Q$ on line 4 are a contradiction, but to use the negation elimination rule we need to have Q on line 5. To get it there, we use the reiteration rule.

$$
\begin{array}{r|l|l}
1 & \neg P \rightarrow \neg Q & :\mathrm{PR} \\
2 & Q & :\mathrm{PR} \\
\hline
3 & \quad \vert\ \neg P & :\mathrm{AS} \\
4 & \quad \vert\ \neg Q & :\rightarrow\mathrm{E}\ 1, 3 \\
5 & \quad \vert\ Q & :\mathrm{R}\ 2
\end{array}
$$

Now that $\neg Q$ and Q are on consecutive lines, we can use ¬E to discharge the assumption that we made, and that gives us the conclusion we are after: P.

1	$\neg P \rightarrow \neg Q$	:PR
2	Q	:PR
3	$\quad \neg P$	:AS
4	$\quad \neg Q$	:→E 1, 3
5	$\quad Q$	:R 2
6	P	:¬E 3–5

We just used the negation elimination rule. The negation introduction rule is, essentially, the same. Whether you use the negation introduction rule or negation elimination rule is just a function of whether you want '¬A' or 'A' on the line after the subproof.

14.10 Even more examples

The negation introduction rule or the negation elimination rule is used in each of these proofs.

1. $P \rightarrow Q, \neg Q \vdash \neg P$

1	$P \rightarrow Q$	:PR
2	$\neg Q$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:→E 1, 3
5	$\quad \neg Q$	:R 2
6	$\neg P$	:¬I 3–5

2. $P \rightarrow \neg Q \vdash \neg(P \,\&\, Q)$

$$\begin{array}{lll}
1 & \mid P \to \neg Q & :\mathrm{PR} \\
2 & \mid \quad \mid P \,\&\, Q & :\mathrm{AS} \\
3 & \mid \quad \mid P & :\&\mathrm{E}\ 2 \\
4 & \mid \quad \mid \neg Q & :{\to}\mathrm{E}\ 1, 3 \\
5 & \mid \quad \mid Q & :\&\mathrm{E}\ 2 \\
6 & \mid \neg(P \,\&\, Q) & :\neg\mathrm{I}\ 2\text{–}5
\end{array}$$

3. $B \,\&\, C, \neg(B \,\&\, D) \vdash \neg D$

$$\begin{array}{lll}
1 & \mid B \,\&\, C & :\mathrm{PR} \\
2 & \mid \neg(B \,\&\, D) & :\mathrm{PR} \\
3 & \mid \quad \mid D & :\mathrm{AS} \\
4 & \mid \quad \mid B & :\&\mathrm{E}\ 1 \\
5 & \mid \quad \mid B \,\&\, D & :\&\mathrm{I}\ 3, 4 \\
6 & \mid \quad \mid \neg(B \,\&\, D) & :\mathrm{R}\ 2 \\
7 & \mid \neg D & :\neg\mathrm{I}\ 3\text{–}6
\end{array}$$

4. A proof for '$\neg P \vdash P \to Q$' requires two subproofs. First, we assume 'P' so that we can use the conditional introduction rule at the end of the proof. Then, we assume '$\neg Q$' so that we can use the negation elimination rule and get 'Q' on the last line of the first subproof.

$$\begin{array}{lll}
1 & \mid \neg P & :\mathrm{PR} \\
2 & \mid \quad \mid P & :\mathrm{AS} \\
3 & \mid \quad \mid \quad \mid \neg Q & :\mathrm{AS} \\
4 & \mid \quad \mid \quad \mid P & :\mathrm{R}\ 2 \\
5 & \mid \quad \mid \quad \mid \neg P & :\mathrm{R}\ 1 \\
6 & \mid \quad \mid Q & :\neg\mathrm{E}\ 3\text{–}5 \\
7 & \mid P \to Q & :{\to}\mathrm{I}\ 2\text{–}6
\end{array}$$

14.11 Invalid arguments

In this chapter, we have taken it for granted that each argument that we have encountered has been valid. The purpose of providing a proof is (1) to confirm that it is valid and (2) to show why it is valid—that is, to lay out each step that takes us from the premises to the conclusion. If an argument is invalid, however, we are stuck. It is impossible to provide a correct proof of an invalid argument using the rules given in this chapter. At the same time, not being able to provide a proof for an argument doesn't mean that the argument is invalid. Perhaps the proof is just too complicated for us to figure out.

In chapter 13, we discussed some reasons to prefer natural deduction to truth tables for checking that an argument is valid. To show that an argument is invalid, however, creating a truth table is not merely a superior method, it is our only option.

14.12 Practice exercises

A. Give a proof for each argument.

1. $\neg P \rightarrow (Q \vee P), \neg P \vdash Q$
2. $D \,\&\, H, H \leftrightarrow J \vdash J \vee N$
3. $G \,\&\, (H \,\&\, J), (H \vee J) \rightarrow K \vdash K$
4. $P \,\&\, (Q \vee R), P \rightarrow \neg R \vdash Q \vee S$
5. $(P \vee \neg Q) \leftrightarrow R, R \,\&\, Q \vdash P \,\&\, R$
6. $(R \,\&\, T) \rightarrow Q, R \vee \neg P, P \,\&\, T \vdash Q$
7. $S \rightarrow T, Q \,\&\, \neg R, \neg R \leftrightarrow (T \rightarrow S) \vdash S \leftrightarrow T$
8. $(L \vee M) \rightarrow N, P \leftrightarrow N, L \vdash L \,\&\, P$

B. Give a proof for each argument.

1. $P \rightarrow (Q \rightarrow R) \vdash (P \,\&\, Q) \rightarrow R$
2. $Q \rightarrow R \vdash (Q \,\&\, S) \rightarrow (R \vee T)$
3. $M \,\&\, (\neg N \rightarrow \neg M) \vdash (N \,\&\, M) \vee \neg M$
4. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \rightarrow M) \vdash Y \rightarrow Z$

5. $C \to (E \& G), \neg C \to G \vdash G$
6. $\neg(P \to Q) \vdash \neg Q$
7. $S \leftrightarrow T \vdash S \leftrightarrow (T \vee S)$
8. $D \vee F, D \to G, F \to H \vdash G \vee H$
9. $(W \vee X) \vee (Y \vee Z), X \to Y, \neg Z \vdash W \vee Y$

C. If you know that a proof can be given for $\mathsf{A} \vdash \mathsf{B}$ (that is, you know that the argument is valid), then is it possible to know if $(\mathsf{A} \& \mathsf{C}) \vdash \mathsf{B}$ is valid? Is it possible to know if $(\mathsf{A} \to \mathsf{B}) \to \mathsf{B} \vdash \mathsf{B}$ is valid? Explain your answers.

14.13 Answers

A.

1. $\neg P \to (Q \vee P), \neg P \vdash Q$

1	$\neg P \to (Q \vee P)$	:PR
2	$\neg P$	:PR
3	$Q \vee P$	:$\to$E 1, 2
4	Q	:$\vee$E 2, 3

2. $D \& H, H \leftrightarrow J \vdash J \vee N$

1	$D \& H$	:PR
2	$H \leftrightarrow J$	:PR
3	H	:&E 1
4	J	:$\leftrightarrow$E 2, 3
5	$J \vee N$	:$\vee$I 4

3. $G \,\&\, (H \,\&\, J), (H \vee J) \rightarrow K \vdash K$

1	$G \,\&\, (H \,\&\, J)$	:PR
2	$(H \vee J) \rightarrow K$	:PR
3	$H \,\&\, J$	:&E 1
4	H	:&E 3
5	$H \vee J$	:∨I 4
6	K	:→I 2, 5

4. $P \,\&\, (Q \vee R), P \rightarrow \neg R \vdash Q \vee S$

1	$P \,\&\, (Q \vee R)$	:PR
2	$P \rightarrow \neg R$	:PR
3	P	:&E 1
4	$\neg R$	:→I 2, 3
5	$Q \vee R$	:&E 1
6	Q	:∨E 4, 5
7	$Q \vee S$	:∨I 6

5. $(P \vee \neg Q) \leftrightarrow R, R \,\&\, Q \vdash P \,\&\, R$

1	$(P \vee \neg Q) \leftrightarrow R$	:PR
2	$R \,\&\, Q$	:PR
3	R	:&E 2
4	$P \vee \neg Q$	:↔E 1, 3
5	Q	:&E 2
6	$\neg\neg Q$	:DN 5
7	P	:∨E 4, 6
8	$P \,\&\, R$	:&I 3, 7

6. $(R \& T) \to Q, R \lor \neg P, P \& T \vdash Q$

1	$(R \& T) \to Q$	:PR
2	$R \lor \neg P$	:PR
3	$P \& T$	:PR
4	P	:&E 3
5	$\neg\neg P$	:DN 4
6	R	:$\lor$E 2, 5
7	T	:&E 3
8	$R \& T$	:&I 6, 7
9	Q	:$\to$E 1, 8

7. $S \to T, Q \& \neg R, \neg R \leftrightarrow (T \to S) \vdash S \leftrightarrow T$

1	$S \to T$	:PR
2	$Q \& \neg R$	:PR
3	$\neg R \leftrightarrow (T \to S)$	:PR
4	$\neg R$	:&E 2
5	$T \to S$	:$\leftrightarrow$E 3, 4
6	$S \leftrightarrow T$	:$\leftrightarrow$I 1, 5

8. $(L \lor M) \to N, P \leftrightarrow N, L \vdash L \& P$

1	$(L \lor M) \to N$	:PR
2	$P \leftrightarrow N$	:PR
3	L	:PR
4	$L \lor M$	:$\lor$I 3
5	N	:$\to$E 1, 4
6	P	:$\leftrightarrow$E 2, 5
7	$L \& P$	:&I 3, 6

B.

1. $P \to (Q \to R) \vdash (P \,\&\, Q) \to R$

1	$P \to (Q \to R)$	:PR
2	$\quad P \,\&\, Q$	:AS
3	$\quad P$	:&E 2
4	$\quad Q \to R$	:→E 1, 3
5	$\quad Q$	:&E 2
6	$\quad R$	:→E 4, 5
7	$(P \,\&\, Q) \to R$	:→I 2–6

2. $Q \to R \vdash (Q \,\&\, S) \to (R \vee T)$

1	$Q \to R$	:PR
2	$\quad Q \,\&\, S$	:AS
3	$\quad Q$	:&E 2
4	$\quad R$	:→E 1, 3
5	$\quad R \vee T$	:∨I 4
6	$(Q \,\&\, S) \to (R \vee T)$	:→I 2–5

3. $M \,\&\, (\neg N \to \neg M) \vdash (N \,\&\, M) \vee \neg M$

1	$M \,\&\, (\neg N \to \neg M)$	:PR
2	M	:&E 1
3	$\neg N \to \neg M$	:&E 1
4	$\quad \neg N$	:AS
5	$\quad \neg M$	:→E 3, 4
6	$\quad M$	:R 2
7	N	:¬E 4–6
8	$N \,\&\, M$	:&I 2, 7
9	$(N \,\&\, M) \vee \neg M$	:∨I 8

4. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \to M) \vdash Y \to Z$

1	$(Z \,\&\, K) \leftrightarrow (Y \,\&\, M)$	:PR
2	$D \,\&\, (D \to M)$	:PR
3	D	:&E 2
4	$D \to M$	:&E 2
5	M	:→E 3, 4
6	$\quad Y$	:AS
7	$\quad Y \,\&\, M$	:&I 5, 6
8	$\quad Z \,\&\, K$	:↔E 1, 7
9	$\quad Z$	:&E 8
10	$Y \to Z$	:→I 6–9

5. $C \to (E \& G), \neg C \to G \vdash G$

$$
\begin{array}{r|llll}
1 & C \to (E \& G) & & & :\text{PR} \\
2 & \neg C \to G & & & :\text{PR} \\
\hline
3 & & \neg G & & :\text{AS} \\
4 & & & C & :\text{AS} \\
5 & & & E \& G & :\to\text{E } 1, 4 \\
6 & & & G & :\&\text{E } 5 \\
7 & & & \neg G & :\text{R } 3 \\
8 & & \neg C & & :\neg\text{I } 4\text{–}7 \\
9 & & G & & :\to\text{E } 2, 8 \\
10 & & \neg G & & :\text{R } 3 \\
11 & G & & & :\neg\text{E } 3\text{–}10
\end{array}
$$

6. $\neg(P \to Q) \vdash \neg Q$

$$
\begin{array}{r|llll}
1 & \neg(P \to Q) & & & :\text{PR} \\
\hline
2 & & Q & & :\text{AS} \\
3 & & & P & :\text{AS} \\
4 & & & Q & :\text{R } 2 \\
5 & & P \to Q & & :\to\text{I } 3\text{–}4 \\
6 & & \neg(P \to Q) & & :\text{R } 1 \\
7 & \neg Q & & & :\neg\text{I } 2\text{–}6
\end{array}
$$

7. $S \leftrightarrow T \vdash S \leftrightarrow (T \vee S)$

1	$S \leftrightarrow T$	:PR
2	$\quad \mid S$	:AS
3	$\quad \mid T$	:$\leftrightarrow$E 1, 2
4	$\quad \mid T \vee S$	:$\vee$I 3
5	$S \rightarrow (T \vee S)$	:$\rightarrow$I 2–4
6	$\quad \mid T \vee S$	:AS
7	$\quad \mid \quad \mid \neg S$	:AS
8	$\quad \mid \quad \mid T$	:$\vee$E 6, 7
9	$\quad \mid \quad \mid S$	:$\leftrightarrow$E 1, 8
10	$\quad \mid \quad \mid \neg S$	:R 7
11	$\quad \mid S$	:$\neg$E 7–10
12	$(T \vee S) \rightarrow S$	:$\rightarrow$I 6–11
13	$S \leftrightarrow (T \vee S)$	:$\leftrightarrow$I 5, 12

8. $D \lor F, D \to G, F \to H \vdash G \lor H$

1	$D \lor F$	:PR
2	$D \to G$	:PR
3	$F \to H$	:PR
4	$\quad \neg(G \lor H)$	:AS
5	$\quad\quad \neg D$	:AS
6	$\quad\quad F$	:$\lor$E 1, 5
7	$\quad\quad H$	:$\to$E 3, 6
8	$\quad\quad G \lor H$	:$\lor$I 7
9	$\quad\quad \neg(G \lor H)$	:R 4
10	$\quad D$	:$\neg$E 5–9
11	$\quad G$	:$\to$E 2, 10
12	$\quad G \lor H$	:$\lor$I 11
13	$\quad \neg(G \lor H)$	:R 4
14	$G \lor H$	:$\neg$E 4–13

9. $(W \vee X) \vee (Y \vee Z), X \rightarrow Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	:PR
2	$X \rightarrow Y$	:PR
3	$\neg Z$	:PR
4	$\mid\ \neg(W \vee Y)$	:AS
5	$\mid\ \mid\ \neg(W \vee X)$	:AS
6	$\mid\ \mid\ Y \vee Z$	:$\vee$E 1, 5
7	$\mid\ \mid\ Y$	:$\vee$E 3, 6
8	$\mid\ \mid\ W \vee Y$	:$\vee$I 7
9	$\mid\ \mid\ \neg(W \vee Y)$	:R 4
10	$\mid\ W \vee X$	:$\neg$E 5–9
11	$\mid\ \mid\ X$	:AS
12	$\mid\ \mid\ Y$	:$\rightarrow$E 2, 11
13	$\mid\ \mid\ W \vee Y$	:$\vee$I 12
14	$\mid\ \mid\ \neg(W \vee Y)$	:R 4
15	$\mid\ \neg X$	:$\neg$I 11–14
16	$\mid\ W$	:$\vee$E 10, 15
17	$\mid\ W \vee Y$	:$\vee$I 16
18	$\mid\ \neg(W \vee Y)$	:R 4
19	$W \vee Y$	:$\neg$E 4–18

C. If $\mathsf{A} \vdash \mathsf{B}$ is valid, then $(\mathsf{A} \,\&\, \mathsf{C}) \vdash \mathsf{B}$ is valid. We know this because if $\mathsf{A} \vdash \mathsf{B}$, then there is some proof with assumption A that ends with B, and no undischarged assumptions other than A. Now, if we start a proof with assumption $(\mathsf{A} \,\&\, \mathsf{C})$, we can obtain A by &E. We can now copy and paste the original proof of B from A, adding 1 to every line number and line number citation. The result will be a proof of B from assumption A.

We also know that $(\mathsf{A} \to \mathsf{B}) \to \mathsf{A} \vdash \mathsf{B}$ is valid. Since, there is some proof with assumption A that ends with B, we can assume A, derive B, and then use the conditional-introduction rule to get $(\mathsf{A} \to \mathsf{B})$. We use the condtional-elimination rule on $(\mathsf{A} \to \mathsf{B})$ and the premise, and get B, the conclusion.

15 Proofs in Carnap

Creating proofs in Carnap is not difficult. To type the connectives, use the symbols on the right in table 15.1.

Carnap will number the lines automatically. After the TFL sentence on each line, there has to be a colon (':') before the 'PR', 'AS', or the rule. Carnap is flexible with the spacing on a line, but as a guideline, put a tab space between the sentence and ':PR', ':AS', or the rule (:→E, :∨I, etc.). Also indent subproofs with a tab space. (Carnap will let you use more or fewer spaces, but a subproof has to be indented some amount.)

To create a proof, you are given an interface like the one shown in figure 15.1. As you can see, the argument is given at the top. In this case, the premises are $P \rightarrow \neg Q$ and $R \,\&\, P$, and the conclusion is $\neg Q$. (The premises are separated by commas. The premises and the conclusion are separated by the turnstile ('⊢').)

Begin by listing the premises, and don't forget to put ':PR' after each one. If there is a problem with a line—either the sentence isn't formed correctly, the rule you've cited isn't being used correctly, or there's some other mistake—Carnap will put ? or ⚠ at the end of the line. When the line is ok, you will get a '+'. We finish this proof using the &E and →E rules (figure 15.3). When the proof is correct, the box containing the argument will turn green, and the proof can be submitted.

Our next example, $(P \vee Q) \vdash (\neg P \rightarrow Q)$, requires a subproof. We begin

TFL OPERATOR	IN CARNAP
¬	~
&	&
∨	v (lowercase v)
→	-> (dash, greater than sign)
↔	<->

Table 15.1

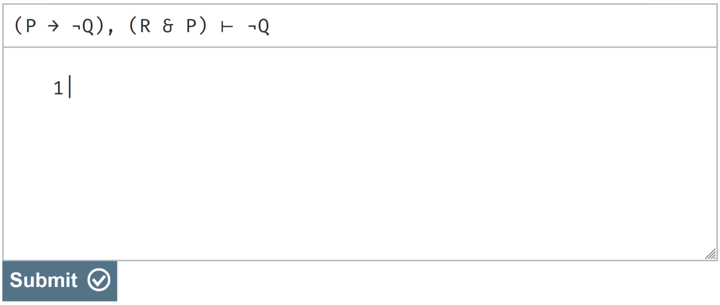


Figure 15.1

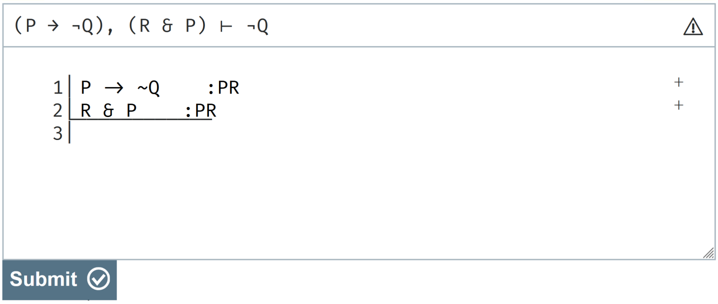


Figure 15.2

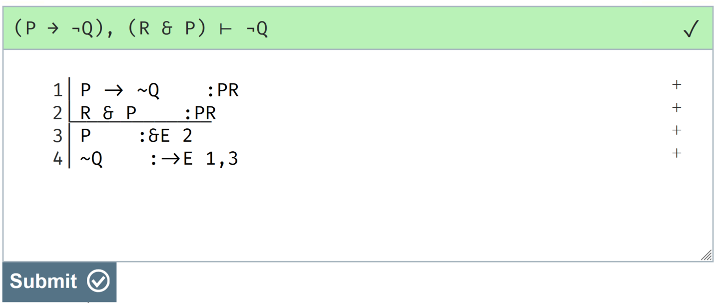


Figure 15.3

as before. To create the subproof, put a tab space before $\neg P$ and put ':AS' at the end of the line (figure 15.4). Since the next line is also part of the subproof, we again need a tab before the Q. We end the subproof (and discharge the assumption) with the →I rule. $\neg P \to Q$ is not indented (so no tabs or spaces before the $\neg P$). That's the conclusion, and so if everything is correct, Carnap will give you the green bar and you can submit the proof (figure 15.5).

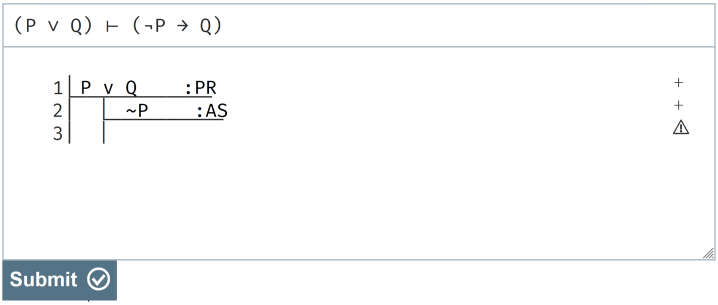


Figure 15.4

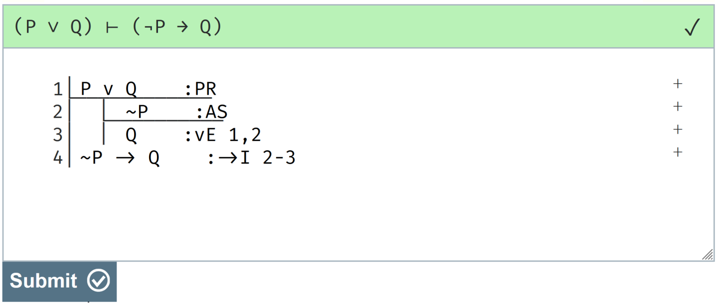


Figure 15.5

Although creating proofs in Carnap is not difficult, you do have to be careful. Creating a program that can verify proofs that use only the rules of derivation given in chapter 14 is relatively simple because there are only a small number of rules and, to produce proofs of valid arguments, we follow those rules very strictly. But, as a consequence, Carnap is not designed to understand what you are trying to do if you deviate from the rules, even if it is a minor deviation or an innocent mistake. So, some reminders:

1. As long as '¬' is not the main logical operator, you can drop the outermost parentheses. All other parentheses have to be used.
2. Capitalize 'PR', 'AS','E', 'I' (in the rules), and all atomic sentences.

3. Don't forget the ':' right before PR, AS, or the rule that you are citing.
4. There is no space between the &, ∨, →, ↔, or ¬ and the 'E' or 'I'.
5. There is a space (and no punctuation) after the 'E' or 'I'.
6. There is a comma between the two lines that have to be cited for &I, ∨E, →E, and ↔E (e.g., '→E 2,4').
7. There is a dash between the two lines that have to be cited for →I, ¬I, and ¬E (e.g., '¬E 4-6').

16 Some strategies

There is no simple recipe for constructing proofs, and there is no substitute for practice. Here, however, are some questions to ask yourself and some strategies to keep in mind.

1. Do you know all of the rules? **If you don't have them memorized yet, then they should be written on a sheet of paper that you have next to you while you're working.**
2. Are there steps that you can take without making an assumption? If yes, is it worth taking those steps?
3. If you're not sure how to proceed, but you can do conjunction elimination, conditional elimination, disjunction elimination, or biconditional elimination, then do them just to see what happens.

The theme for 4 – 7 is "think ahead." Some amount of trial and error is often necessary, but, especially when you are constructing a proof that will contain a subproof, it's important to think about how each step that you take will affect the later parts of your proof.

4. If an assumption is needed, is it for →I, ¬I, or ¬E? **Don't make an assumption if you don't know which of these rules you plan to use when you close the subproof.**
5. If an assumption is needed, what should it be? (If you want to get $P \to Q$, then you're going to use →I and your assumption should be P.)
6. If you make an assumption, then you should know what you want on either the last line or the last two lines of your subproof.
   a. If you're using →I, then you will need the consequent of the conditional on the last line of the subproof.

b. If you're using ¬I or ¬E, then you need a contradiction on the last two lines of your subproof, although that can be any contradiction. It doesn't have to be related to the assumption.

7. Sometimes it is useful to work backwards from the conclusion. The conclusion, of course, will be the last line of your proof, and you can, if you wish, put it at the bottom of the proof anytime. For example, let's say that you need to provide a proof for this argument: $P \rightarrow (\neg Q \rightarrow R) \vdash (P \,\&\, \neg Q) \rightarrow R$. You can begin this way:

$$
\begin{array}{l|ll}
1 & P \rightarrow (\neg Q \rightarrow R) & :\text{PR} \\
\hline
 & & \\
 & & \\
 & (P \,\&\, \neg Q) \rightarrow R &
\end{array}
$$

Knowing that you need to arrive at a conditional, you also know these three things: (1) you need to use the conditional-introduction rule, (2) what your assumption should be, and (3) what will be on the last line of your subproof.

$$
\begin{array}{l|l|ll}
1 & \multicolumn{2}{l}{P \rightarrow (\neg Q \rightarrow R)} & :\text{PR} \\
\hline
2 & & P \,\&\, \neg Q & :\text{AS} \\
\cline{3-3}
 & & & \\
 & & R & \\
 & \multicolumn{2}{l}{(P \,\&\, \neg Q) \rightarrow R} & :\rightarrow\text{I}
\end{array}
$$

Sketching out a proof in this way is easy to do when you are writing on paper. If you are doing it in Carnap, be careful of the spacing that you put on each blank line.

8. The negation introduction and negation elimination rules are a last resort. Use them when you can't use any of the other rules.

When you do use them, always have in mind that, when you complete the subproof, you will have the opposite of the assumption. Hence, a good guideline is to make the assumption the opposite of the conclusion.

(If you have to make two assumptions—and both assumptions will be discharged with one of these rules—this guideline only applies to the first assumption. Determining the best choice for a second assumption sometimes takes a little trial and error.)

9. **Persist**. Try different things. If one approach fails, then try something else.

17 Proof-theoretic concepts

17.1 Theorems

You are familiar with arguments that have this form:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{C}$$

We may also, however, have a sentence for which it is possible to give a proof with no premises: $\vdash \mathsf{C}$. In this case, we say that C is a THEOREM.

Theorem

C is a THEOREM if and only if $\vdash \mathsf{C}$

One such sentence is '$\neg(P \,\&\, \neg P)$'. To show that this sentence is a theorem, we give a proof that has no premises and no undischarged assumptions. To get started, we do, however, have to make an assumption. We will assume '$P \,\&\, \neg P$'. Once we show that this assumption leads to contradiction, we can discharge it and we will have '$\neg(P \,\&\, \neg P)$'. This is the proof:

1			$P \,\&\, \neg P$	
2			P	:&E 1
3			$\neg P$	:&E 1
4		$\neg(P \,\&\, \neg P)$	:¬I 1–3

This theorem, '$\vdash \neg(P \,\&\, \neg P)$' is an instance of what is sometimes called *the law of non-contradiction*.

To show that a sentence is a theorem, we just have to find a suitable proof. On the other hand, it is not possible to show that a sentence is *not* a theorem this same way. To show that a sentence is not a theorem with our natural deduction system, we would have to demonstrate, not just that certain proof strategies fail, but that *no* proof is possible. Even if we fail in trying to give a proof for a sentence in a thousand different ways, perhaps the proof is just too long and complex for us to figure out.

17.2 Equivalent, consistent, and inconsistent

In §10.2, we defined *equivalent* in terms of truth tables, namely, if two sentences have the same truth value on every line of a truth table, then they are equivalent. We can also show that two sentences are equivalent using our natural deduction system. To indicate that we have shown that the two sentences are equivalent with a derivation (or actually with two derivations), we will call this equivalence PROVABLY EQUIVALENT.

Provably equivalent

Two sentences A and B are PROVABLY EQUIVALENT iff each can be derived from the other. I.e., $\mathsf{A} \vdash \mathsf{B}$ and $\mathsf{B} \vdash \mathsf{A}$.
(Equivalently, A and B are PROVABLY EQUIVALENT if $\vdash \mathsf{A} \leftrightarrow \mathsf{B}$.)

As in the case of showing that a sentence is a theorem, it is relatively easy to show that two sentences are provably equivalent: it just requires a pair of proofs. Showing that sentences are *not* provably equivalent is not possible for t he same reason that it isn't possible to show that a sentence is not a theorem. Even if we fail to produce two proofs showing that two sentences are provably equivalent, that doesn't mean that the proofs don't exist. It just means that we've failed to figure out what they are.

We also, in §10.3, defined *jointly inconsistent* using truth tables: sentences are jointly inconsistent if there is no line on a truth table where they are all true. Again, we can show that two or more sentences are jointly inconsistent with our natural deduction system.

Provably inconsistent

The sentences $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are PROVABLY INCONSISTENT iff, from them, a contradiction can be derived. I.e. $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash (\mathsf{B} \,\&\, \neg\mathsf{B})$.

TO CHECK	THAT IT IS	THAT IT IS NOT
theorem	one proof	*not possible with proofs*
equivalent	two proofs	*not possible with proofs*
inconsistent	one proof	*not possible with proofs*
consistent	*not possible with proofs*	one proof

Table 17.1: This table summarizes what is required to check each of these logical notions.

To show that a set of sentences are provably inconsistent, we use the sentences as premises and then derive a contradiction. (Any contradiction will do.) For instance, this proof demonstrates that $P \& Q$ and $\neg P \vee \neg Q$ are provably inconsistent.

1	$P \& Q$	:PR
2	$\neg P \vee \neg Q$	:PR
3	P	:&E 1
4	$\neg\neg P$	:DN 3
5	$\neg Q$	:$\vee$E 2, 4
6	Q	:&E 1

Showing that some set of sentences are *not* provably inconsistent is, as you might guess at this point, not possible. Doing so would require showing, not just that we have failed to derive a contradiction from a set a sentences, but that no such derivation is possible.

Table 17.1 summarizes what we have covered in this chapter. As we will discuss in the next chapter, when the presence (or the absence) of a logical property cannot be demonstrated using our natural deduction system, we have to resort to using a truth table.

17.3 Practice exercises

A. Give a proof for each of these theorems.

1. $\vdash O \to O$
2. $\vdash S \to (S \vee R)$
3. $\vdash N \vee \neg N$
4. $\vdash \neg((R \vee T) \& (\neg R \& \neg T))$
5. $\vdash (R \leftrightarrow M) \to (M \to R)$
6. $\vdash J \leftrightarrow [J \vee (L \& \neg L)]$
7. $\vdash (P \to Q) \vee (Q \to P)$

B. Show that each of the following pairs of sentences are provably equivalent. (To indicate that the inference from the premise to the conclusion goes from the first sentence to the second and vice versa, we use the symbols $\dashv\vdash$.)

1. $T \to S \dashv\vdash \neg S \to \neg T$
2. $R \to Q \dashv\vdash \neg(R \& \neg Q)$

17.4 Answers

A.

1. $\vdash O \to O$

$$\begin{array}{r|l|ll}
1 & & \underline{O} & :\text{AS} \\
2 & & O & :\text{R } 1 \\
3 & O \to O & & :{\to}\text{I } 1\text{–}2
\end{array}$$

2. $\vdash S \to (S \vee R)$

$$\begin{array}{r|l|ll}
1 & & \underline{S} & :\text{AS} \\
2 & & S & :\text{R } 1 \\
3 & & S \vee R & :{\vee}\text{I } 2 \\
4 & S \to (S \vee R) & & :{\to}\text{I } 1\text{–}3
\end{array}$$

3. $\vdash N \lor \neg N$

1	$\mid\ \mid\ \neg(N \lor \neg N)$	:AS
2	$\mid\ \mid\ \mid\ N$	:AS
3	$\mid\ \mid\ \mid\ N \lor \neg N$	:$\lor$I 2
4	$\mid\ \mid\ \mid\ \neg(N \lor \neg N)$	:R 1
5	$\mid\ \mid\ \neg N$	:$\neg$I 2–4
6	$\mid\ \mid\ N \lor \neg N$	:$\lor$I 5
7	$\mid\ \mid\ \neg(N \lor \neg N)$	:R 1
8	$\mid\ N \lor \neg N$	:$\neg$E 1–7

4. $\vdash \neg((R \lor T) \,\&\, (\neg R \,\&\, \neg T))$

1	$\mid\ \mid\ (R \lor T) \,\&\, (\neg R \,\&\, \neg T)$	:AS
2	$\mid\ \mid\ R \lor T$	:&E 1
3	$\mid\ \mid\ \neg R \,\&\, \neg T$	:&E 1
4	$\mid\ \mid\ \neg R$	:&E 3
5	$\mid\ \mid\ \neg T$	:&E 3
6	$\mid\ \mid\ T$	:$\lor$E 2, 4
7	$\mid\ \neg((R \lor T) \,\&\, (\neg R \,\&\, \neg T))$	:$\neg$I 1–6

5. $\vdash (R \leftrightarrow M) \rightarrow (M \rightarrow R)$

1	$\mid\ \mid\ R \leftrightarrow M$	:AS
2	$\mid\ \mid\ \mid\ M$	:AS
3	$\mid\ \mid\ \mid\ R$	:$\leftrightarrow$E 1, 2
4	$\mid\ \mid\ M \rightarrow R$	:$\rightarrow$I 2–3
5	$\mid\ (R \leftrightarrow M) \rightarrow (M \rightarrow R)$	:$\rightarrow$I 1–4

6. $\vdash J \leftrightarrow (J \lor (L \& \neg L))$

1	$\quad\quad J$	:AS
2	$\quad\quad J \lor (L \& \neg L)$	:∨I 1
3	$J \to (J \lor (L \& \neg L))$	:→I 1–2
4	$\quad\quad J \lor (L \& \neg L)$	:AS
5	$\quad\quad\quad\quad L \& \neg L$	:AS
6	$\quad\quad\quad\quad L$	:&E 5
7	$\quad\quad\quad\quad \neg L$	:&E 5
8	$\quad\quad \neg(L \& \neg L)$	:¬E 5–7
9	$\quad\quad J$	:∨E 4, 8
10	$(J \lor (L \& \neg L)) \to J$	:→I 4–9
11	$J \leftrightarrow (J \lor (L \& \neg L))$	:↔I 3, 10

7. $\vdash (P \to Q) \lor (Q \to P)$

1	$\neg((P \to Q) \lor (Q \to P))$	:AS
2	P	:AS
3	$\neg Q$	:AS
4	Q	:AS
5	P	:R 2
6	$Q \to P$	:→I 4–5
7	$(P \to Q) \lor (Q \to P)$	:∨I 6
8	$\neg((P \to Q) \lor (Q \to P))$	:R 1
9	Q	:¬E 3–8
10	$P \to Q$	:→I 2–9
11	$(P \to Q) \lor (Q \to P)$	:∨I 10
12	$\neg((P \to Q) \lor (Q \to P))$	:R 1
13	$(P \to Q) \lor (Q \to P)$	:¬E 1–12

B.

1. $T \to S \dashv\vdash \neg S \to \neg T$

1	$T \to S$	:PR
2	$\neg S$	:AS
3	T	:AS
4	S	:→E 1, 3
5	$\neg S$	:R 2
6	$\neg T$	:¬I 2–5
7	$\neg S \to \neg T$	:→I 2–6

1	$\neg S \to \neg T$	:PR
2	$\mid\; T$	:AS
3	$\mid\;\mid\; \neg S$	:AS
4	$\mid\;\mid\; \neg T$	:→E 1, 3
5	$\mid\;\mid\; T$	:R 2
6	$\mid\; S$	:¬E 3–5
7	$T \to S$	:→I 2–6

2. $R \to Q \dashv\vdash \neg(R \& \neg Q)$

1	$R \to Q$	:PR
2	$\mid\; R \& \neg Q$	:AS
3	$\mid\; R$	:&E 2
4	$\mid\; \neg Q$	:&E 2
5	$\mid\; Q$	:→E 1, 3
6	$\neg(R \& \neg Q)$	:¬I 2–5

1	$\neg(R \& \neg Q)$	:PR
2	$\mid\; R$	:AS
3	$\mid\;\mid\; \neg Q$	:AS
4	$\mid\;\mid\; R \& \neg Q$	:&I 2, 3
5	$\mid\;\mid\; \neg(R \& \neg Q)$	:R 1
6	$\mid\; Q$	:¬E 3–5
7	$R \to Q$	:→I 2–6

18 Soundness and completeness

We have two ways of checking or verifying that an argument is valid: (1) using truth tables and (2) using the natural deduction system to provide a proof. Consequently, we also have two ways of characterizing the concept of *validity*. (See table 18.1.) You might think that we can take it for granted that, with respect to determining if an argument is valid, both methods will always give us the same result, but that is not exactly the case. (We, right now, can take it for granted, but that's only because the requisite work to show that the two methods will always agree has already been done.) If you think about it for a moment, you'll notice that the two methods don't have anything in common, and so, it is not intuitively obvious that they will always produce the same result. But they do.

How do we know that the truth table method and the natural deduction method will always agree? Demonstrating that they will goes beyond the scope of this book. But we will review the two properties that a logic system (like TFL) must have for the two methods to always be in agreement. To begin, let us define two new terms.

> *p-valid*: being valid because a proof can be given using the rules in our natural deduction system. (*p-valid* is short for *proof-valid*. This is also sometimes called *syntactically valid*).
> *tt-valid*: being valid because there is no line in a truth table where the premises are true and the conclusion is false. (This is also sometimes called *semantically valid*).

First, it must be the case that every argument that is p-valid is tt-valid. This property is called SOUNDNESS.

	TRUTH TABLE (SEMANTIC) DEFINITION	PROOF-THEORETIC (SYNTACTIC) DEFINITION
Tautology	A sentence whose truth table has a T on every line under the main connective	A sentence that can be derived without any premises. I.e., a theorem.
Contradiction	A sentence whose truth table has an F on every line under the main connective	A sentence whose negation can be derived without any premises
Contingent sentence	A sentence whose truth table has both T and F (in any combination) under the main connective	A sentence that is not a theorem or contradiction
Equivalent sentences	The columns under the main connective for both sentences are identical.	The sentences can be derived from each other
Inconsistent sentences	Sentences that do not have a single line in their truth tables where, in the column under the main connective, they all have a T.	Sentences from which one can derive a contradiction
Consistent sentences	Sentences that have at least one line in their truth tables where, in the column under the main connective, they all have a T.	Sentences that are not inconsistent
Valid argument	An argument whose truth table has no lines where there is a T under each main connective for the premises and an F under the main connective for the conclusion.	An argument where one can derive the conclusion from the premises

Table 18.1: The two ways of defining each of these logical concepts in TFL.

Soundness

SOUNDNESS is a property of a logic system iff, for any argument, if the argument is p-valid, then the argument tt-valid.

Equivalently, SOUNDNESS is a property of a logic system iff, for any sentence, if a sentence is a theorem, then it is a tautology.

Soundness is a property of TFL because every argument for which we can give a proof (and hence show that it is valid that way) will also be valid by the truth table method.

> *Soundness*, the property of logical systems that we are discussing here, is different than the *sound*, the property of individual arguments, that is defined on p. 9.

Soundness is the property that goes in this direction: p-valid ⇒ tt-valid. The other direction, tt-valid ⇒ p-valid, is called COMPLETENESS.

> Like '→', '⇒' can be read as 'if ..., then ...'. Since 'p-valid ⇒ tt-valid' is not an expression in TFL, we shouldn't use the '→' symbol in it. Instead, we are using the *metalogical arrow* to express the relationship between *p-valid* and *tt-valid*.

Completeness

COMPLETENESS is a property of a logic system iff, for any argument, if the argument is tt-valid, the the argument is p-valid.

Equivalently, COMPLETENESS is a property of a logic system iff, for any sentence, if the sentence is a tautology, then it is a theorem.

Proving that a logic system is complete is generally harder than proving soundness. Proving soundness for a logic system amounts to showing that all of the rules of the deduction system work the way they are supposed to work. Showing that a logic system is complete means showing that all of the rules that are needed have been included, and none have been left out. Again, showing this is beyond the scope of this book. The important point is that, happily, TFL is both sound and complete. This is not the case for all formal languages (or all logical systems). Because it is true of TFL, we can choose to give proofs or give truth tables—whichever is easier for the task at hand.

Some people are naturally drawn to truth tables because they can be produced mechanically, and that seems easier. But, as we mentioned in chapter 13, when arguments contain more than three letters, their truth table become quite large. Also, providing a proof informs us of the steps that must be taken to get from the premises to the conclusion. It illustrates *why* an argument is valid in a way that a truth table cannot. Comparing proofs also gives us insight into how arguments are similar or different, and that, in turn, informs us about the similarities and differences between various reasoning strategies. Truth tables, meanwhile, tell us nothing but whether an argument is valid or invalid.

It also bears mentioning that TFL is the standard first step into formal logic, but more complex systems of logic cannot employ truth tables and so derivations must be used. It is wise, therefore, to master derivations in TFL before moving onto to other branches of logic.

At the same time, there are some logical properties, the presence (or really the absence) of which, can only only be established with truth tables. In each of these cases, we might surmise from our failure to find a proof that the property is present, but our failure might just be a consequence of not trying hard enough. This is true for showing that (1) an argument is invalid, (2) a sentence is *not* a theorem, (3) a sentence is *not* a contradiction, (4) a sentence is contingent (which is to say that it's *not* a theorem and *not* a contradiction), (4) two sentences are *not* equivalent, and (5) two or more sentences are consistent (which is to say that they are *not* inconsistent). If we wish to show that any of those properties apply, then we have to resort to truth tables.

18.1 Practice exercises

A. For each of the following, if the argument is valid, give a proof. If it is not valid, make a truth table showing that it is not.

1. $\neg(P \lor Q) \vdash P \,\&\, Q$

TO VERIFY	THAT IT IS	THAT IT IS NOT
Tautology	proof or a truth table	truth table
Contradiction	proof or a truth table	truth table
Contingent	truth table	proof or a truth table
Equivalent	proof or a truth table	truth table
Consistent	truth table	proof or a truth table
Valid	proof or a truth table	truth table

Table 18.2: This table summarizes what is required to check each of these logical properties.

2. $\neg(A \vee B) \vdash C \rightarrow \neg A$
3. $\neg(A \vee B) \vdash B \rightarrow A$
4. $\neg(A \vee (B \vee C)) \vdash \neg C$
5. $\neg(A \& (B \vee C)) \vdash \neg C$
6. $((\neg X \leftrightarrow X) \vee X) \vdash X$
7. $F \& (K \& R) \vdash F \leftrightarrow (K \leftrightarrow R)$
8. $\neg L, K \rightarrow \neg L \vdash \neg K$
9. $L, K \rightarrow \neg L \vdash \neg K$
10. $\vdash A \rightarrow [((B \& C) \vee D) \rightarrow A]$
11. $\vdash A \rightarrow (A \rightarrow B)$

18.2 Answers

A.

1. $\neg(P \vee Q) \vdash P \& Q$ is invalid.

P	Q	¬	(P	∨	Q)	⊢	(P	&	Q)
T	T	**F**	T	T	T	✓	T	**T**	T
T	F	**F**	T	T	F	✓	T	**F**	F
F	T	**F**	F	T	T	✓	F	**F**	T
F	F	**T**	F	F	F	✗	F	**F**	F

2. $\neg(A \vee B) \vdash C \rightarrow \neg A$ is valid.
3. $\neg(A \vee B) \vdash B \rightarrow A$ is valid.
4. $\neg(A \vee (B \vee C)) \vdash \neg C$ is valid.

5. $\neg(A \,\&\, (B \vee C)) \vdash \neg C$ is invalid.

A	B	C	¬	(A	&	(B	∨	C))	⊢	¬	C
T	T	T	**F**	T	T	T	T	T	✓	**F**	T
T	T	F	**F**	T	T	T	T	F	✓	**T**	F
T	F	T	**F**	T	T	F	T	T	✓	**F**	T
T	F	F	**T**	T	F	F	F	F	✓	**T**	F
F	T	T	**T**	F	F	T	T	T	✗	**F**	T
F	T	F	**T**	F	F	T	T	F	✓	**T**	F
F	F	T	**T**	F	F	F	T	T	✗	**F**	T
F	F	F	**T**	F	F	F	F	F	✓	**T**	F

6. $(\neg X \leftrightarrow X) \vee X \vdash X$ is valid.
7. $F \,\&\, (K \,\&\, R) \vdash F \leftrightarrow (K \leftrightarrow R)$ is valid.
8. $\neg L, K \rightarrow \neg L \vdash \neg K$ is invalid.

K	L	¬	L,	(K	→	¬	L)	⊢	¬	K
T	T	**F**	T	T	**F**	F	T	✓	**F**	T
T	F	**T**	F	T	**T**	T	F	✗	**F**	T
F	T	**F**	T	F	**T**	F	T	✓	**T**	F
F	F	**T**	F	F	**T**	T	F	✓	**T**	F

9. $L, K \rightarrow \neg L \vdash \neg K$ is valid.
10. $\vdash A \rightarrow [((B \,\&\, C) \vee D) \rightarrow A]$ is theorem.
11. $\vdash A \rightarrow (A \rightarrow B)$ is not a theorem—and so not a tautology.

A	B	⊢	A	→	(A	→	B)
T	T	✓	T	**T**	T	T	T
T	F	✗	T	**F**	T	F	F
F	T	✓	F	**T**	F	T	T
F	F	✓	F	**T**	F	T	F

Appendices

A Symbolic notation

A.1 Alternative nomenclature

Truth-functional logic. TFL goes by other names. Sometimes it is called *sentential logic*, because this branch of logic deals fundamentally with sentences. Sometimes it is called *propositional logic* because it might also be thought to deal fundamentally with propositions. We have used with *truth-functional logic* to emphasize that it deals only with assignments of truth and falsity to sentences and that its connectives are all truth-functional.

Formulas. In §6, we defined *sentences* of TFL. These are also sometimes called 'formulas' (or 'well-formed formulas') since in TFL there is no distinction between a formula and a sentence.

Valuations. Some texts call valuations *truth-assignments* or *truth-value assignments*.

A.2 Alternative symbols

In the history of formal logic, different symbols have been used at different times and by different authors. Often, authors were forced to use notation that their printers could typeset. This appendix presents some common symbols, so that you can recognize them if you encounter them in an article or in another book.

Negation. Two commonly used symbols are the *hoe*, '$\neg$', and the *swung dash* or *tilda*, '$\sim$.' In some more advanced formal systems it is necessary to distinguish between two kinds of negation; the distinction is sometimes represented by using both '$\neg$' and '$\sim$'. Older texts

SYMBOLS OF FORMAL LOGIC	
negation	$\neg$, $\sim$
conjunction	$\wedge$, &, $\bullet$
disjunction	$\vee$
conditional	$\rightarrow$, $\supset$
biconditional	$\leftrightarrow$, $\equiv$

Table A.1

sometimes indicate negation by a line over the formula being negated, e.g., $\overline{A \& B}$.

Disjunction. The symbol '$\vee$' is typically used to symbolize inclusive disjunction. One etymology is from the Latin word 'vel', meaning 'or'.

Conjunction. Conjunction is often symbolized with the *ampersand*, '&'. The ampersand is a decorative form of the Latin word 'et', which means 'and'. (Its etymology still lingers in certain fonts, particularly in italic fonts; thus an italic ampersand might appear as '*&*'.) This symbol is commonly used in natural English writing (e.g. 'Smith & Sons'), and so even though it is a natural choice, many logicians use a different symbol to avoid confusion between the object and metalanguage—as a symbol in a formal system, the ampersand is not the English word '&'. The most common choice now is '$\wedge$', which is a counterpart to the symbol used for disjunction. Sometimes a single dot, '•', is used. In some older texts, there is no symbol for conjunction at all; 'A and B' is simply written 'AB'.

Conditional. There are two common symbols for the conditional (which can also be called the *material conditional*): the *arrow*, '$\rightarrow$', and the *hook*, '$\supset$'.

Biconditional. The *double-headed arrow*, '$\leftrightarrow$', is used in systems that use the arrow to represent the biconditional. Systems that use the hook for the conditional typically use the *triple bar*, '$\equiv$', for the biconditional.

B Quick reference

B.1 Characteristic Truth Tables

A	$\neg A$
T	F
F	T

A	B	$A \& B$	$A \vee B$	$A \to B$	$A \leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

A	$\neg A$
1	0
0	1

A	B	$A \& B$	$A \vee B$	$A \to B$	$A \leftrightarrow B$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

B.2 Symbolization

SENTENTIAL CONNECTIVES

It is not the case that P.	$\neg P$
Either P, or Q.	$(P \vee Q)$
Neither P, nor Q.	$\neg(P \vee Q)$ or $(\neg P \& \neg Q)$
Both P, and Q.	$(P \& Q)$
If P, then Q.	$(P \to Q)$
P only if Q.	$(P \to Q)$
P if and only if Q.	$(P \leftrightarrow Q)$
P unless Q.	$(P \vee Q)$

B.3 Rules of derivation for TFL

When you have what is in blue, then, on a new line, you can put what is in red. m, n, p, and q stand for lines numbers. m and n don't have to be consecutive line numbers. The p and q in the negation-introduction and negation-elimination rules are consecutive line numbers.

CONJUNCTION INTRO

m	A	
n	B	
	$\mathsf{A} \,\&\, \mathsf{B}$	:&I m, n

m	A	
n	B	
	$\mathsf{B} \,\&\, \mathsf{A}$	:&I m, n

CONJUNCTION ELIM

m	$\mathsf{A} \,\&\, \mathsf{B}$	
	A	:&E m

m	$\mathsf{A} \,\&\, \mathsf{B}$	
	B	:&E m

DISJUNCTION INTRO

m	A	
	$\mathsf{A} \vee \mathsf{B}$	:∨I m

m	A	
	$\mathsf{B} \vee \mathsf{A}$	:∨I m

DISJUNCTION ELIM

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\neg\mathsf{B}$	
	A	:∨E m, n

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\neg\mathsf{A}$	
	B	:∨E m, n

DOUBLE NEGATION

m	A	
	$\neg\neg\mathsf{A}$	:DN m

CONDITIONAL ELIM

$$\begin{array}{r|ll} m & A \to B & \\ n & A & \\ & B & :\to\text{E } m, n \end{array}$$

CONDITIONAL INTRO

$$\begin{array}{r|l} \begin{array}{r} m \\ n \end{array} & \begin{array}{|l} A \\ \hline B \end{array} \quad \begin{array}{l} :\text{AS} \\ \end{array} \\ & A \to B \qquad :\to\text{I } m\text{–}n \end{array}$$

BICONDITIONAL INTRO

$$\begin{array}{r|ll} m & A \to B & \\ n & B \to A & \\ & A \leftrightarrow B & :\leftrightarrow\text{I } m, n \end{array}$$

NEGATION INTRO

$$\begin{array}{r|l} \begin{array}{r} m \\ p \\ q \end{array} & \begin{array}{|l} A \\ \hline B \\ \neg B \end{array} \quad \begin{array}{l} :\text{AS} \\ \\ \end{array} \\ & \neg A \qquad :\neg\text{I } m\text{–}q \end{array}$$

BICONDITIONAL ELIM

$$\begin{array}{r|ll} m & A \leftrightarrow B & \\ n & B & \\ & A & :\leftrightarrow\text{E } m, n \end{array}$$

$$\begin{array}{r|ll} m & A \leftrightarrow B & \\ n & A & \\ & B & :\leftrightarrow\text{E } m, n \end{array}$$

NEGATION ELIM

$$\begin{array}{r|l} \begin{array}{r} m \\ p \\ q \end{array} & \begin{array}{|l} \neg A \\ \hline B \\ \neg B \end{array} \quad \begin{array}{l} :\text{AS} \\ \\ \end{array} \\ & A \qquad :\neg\text{E } m\text{–}q \end{array}$$

REITERATION

$$\begin{array}{r|ll} m & A & \\ & A & :\text{R } m \end{array}$$

Glossary

antecedent The sentence on the left side of a conditional.

argument a connected series of sentences, divided into premises and conclusion.

atomic sentence A sentence used to represent a basic sentence; a single letter in TFL, or a predicate symbol followed by names in FOL.

biconditional The symbol $\leftrightarrow$, used to represent words and phrases that function like the English phrase "if and only if"; or a sentence formed using this connective..

complete truth table A table that gives all the possible truth values for a sentence of TFL or sentences in TFL, with a line for every possible valuation of all atomic sentences.

completeness A property held by logical systems if and only if tt-valid implies p-valid.

conclusion the last sentence in an argument.

conclusion indicator a word or phrase such as "therefore" used to indicate that what follows is the conclusion of an argument.

conditional The symbol $\rightarrow$, used to represent words and phrases that function like the English phrase "if ..., then ..."; a sentence formed by using this symbol.

conjunct A sentence joined to another by a conjunction.

conjunction The symbol &, used to represent words and phrases that function like the English word "and"; or a sentence formed using that symbol.

connective A word or phrase used to modify a sentence, or a word or phrase used to combine two sentences into a more complex sentence.

consequent The sentence on the right side of a conditional.

contingent sentence A sentence that is neither a necessary truth nor a necessary falsehood; a sentence that in some situations is true and in others false.

contradiction (of TFL) A sentence that has only Fs in the column under the main logical operator of its complete truth table; a sentence that is false on every valuation.

disjunct A sentence joined to another by a disjunction.

disjunction The connective $\vee$, used to represent words and phrases that function like the English word "or" in its inclusive sense; or a sentence formed by using this connective.

equivalence (in TFL) A property held by pairs of sentences if and only if the complete truth table for those sentences has identical columns under the two main logical operators, i.e., if the sentences have the same truth value on every valuation.

invalid A property of arguments that holds when it is possible for the premises to be true without the conclusion being true; the opposite of valid.

joint consistency (in TFL) A property held by sentences if and only if the complete truth table for those sentences contains one line on which all the sentences are true, i.e., if some valuation makes all the sentences true.

joint possibility A property possessed by some sentences when they can all be true at the same time.

logical validity (in TFL) A property held by arguments if and only if the complete truth table for the argument contains no rows where the premises are all true and the conclusion false, i.e., if no valuation makes all premises true and the conclusion false.

main connective The last connective that you add when you assemble a sentence using the recursive definition..

metalanguage The language logicians use to talk about the object language. In this textbook, the metalanguage is English, supplemented by certain symbols like metavariables and technical terms like "valid.".

metavariables A variable in the metalanguage that can represent any sentence in the object language..

necessary equivalence A property held by a pair of sentences that must always have the same truth value.

necessary falsehood A sentence that must be false.

necessary truth A sentence that must be true.

negation The symbol ¬, used to represent words and phrases that function like the English word "not".

object language A language that is constructed and studied by logicians. In this textbook, the object languages are TFL and FOL..

premise a sentence in an argument other than the conclusion.

premise indicator a word or phrase such as "because" used to indicate that what follows is the premise of an argument.

proof A sequence of sentences. The first sentences of the sequence are assumptions; these are the premises of the argument. Every sentence later in the sequence follows from earlier sentences by one of the rules of TFL. The final sentence of the sequence is the conclusion of the argument.

provable equivalence A property held by pairs of statements if and only if there is a derivation which takes you from each one to the other one.

provable inconsistency Sentences are provably inconsistent iff a contradiction can be derived from them.

scope A property of connectives. The sentence or subsentence for which that connective is the main logical operator.

sentence of TFL A string of symbols in TFL that can be built up according to the recursive rules given on p. 54.

sound A property of arguments that holds if the argument is valid and has all true premises.

soundness A property held by logical systems if and only if p-valid implies tt-valid.

symbolization key A list that shows which English sentences are represented by which atomic sentences in TFL.

tautology A sentence that has only Ts in the column under the main logical operator of its complete truth table; a sentence that is true on every valuation.

theorem A sentence that can be proved without any premises.

truth value One of the two logical values sentences can have: True and False.

valid A property of arguments where it is impossible for the premises to be true and the conclusion false.

valuation An assignment of truth values to particular atomic sentence of TFLs.