12 Truth table shortcuts

With practice, you will become adept at quickly filling out truth tables. There are, however, some shortcuts that will (1) save you some time and (2) reinforce the meaning the concepts that can be tested using truth tables.

12.1 Testing for validity

As we said in §11.1, when we use truth tables to test for validity, we are checking for *bad* lines: lines where the premises are all true and the conclusion is false. Consequently,

- Any line where the conclusion is true is not a bad line.
- Any line where some premise is false is not a bad line.

Since *all* we are doing is looking for bad lines, if we find a line where the conclusion is true, we do not need to evaluate anything else on that line. That line definitely isn't bad. Likewise, if we find a line where some premise is false, we do not need to evaluate anything else on that line.

With this in mind, consider how we might investigate wither this argument is valid:

$$\neg L \rightarrow (J \vee L), \neg L \vdash J$$

The first step is evaluating the conclusion. If we find that the conclusion is true on some line, then that is not a bad line, and so we can simply ignore the rest of the line.

J	L	$\neg L \rightarrow (J \vee L),$	$\neg L$	$\vdash$	J
T	T			✓	T
T	F			✓	T
F	T	?	?	?	F
F	F	?	?	?	F

The blank spaces under $\neg L \rightarrow (J \vee L)$ and $\neg L$ indicate that we are not going to bother doing any more investigation since the line is not bad. The question-marks indicate that we need to keep investigating. On those lines, it is possible that the premises are true and the conclusion is false.

The easiest premise to evaluate is the second ($\neg L$), and so we do that next.

J	L	$\neg$	L	$\rightarrow$	$(J$	$\vee$	$L),$	$\neg$	L	$\vdash$	J
T	T									✓	T
T	F									✓	T
F	T							F		✓	F
F	F			?				T		?	F

Now we see that we no longer need to consider the third line. It will not be a bad line, because at least one of the premises is false on that line, namely, $\neg L$. Finally, we complete the fourth line:

J	L	$\neg$	L	$\rightarrow$	$(J$	$\vee$	$L),$	$\neg$	L	$\vdash$	J
T	T									✓	T
T	F									✓	T
F	T							F		✓	F
F	F	T		**F**		F		T		✓	F

Since the fourth line tells us that the first premise is false, the truth table has no bad lines. Hence, the argument is valid: any valuation for which all the premises are true is a valuation for which the conclusion is true.

Let us check whether the following argument is valid using the same method.

$$A \vee B, \neg(A \,\&\, C), \neg(B \,\&\, \neg D) \vdash (\neg C \vee D)$$

Again, we first determine the truth value of the conclusion. Since this is a disjunction, it is true whenever either disjunct is true. We can speed things along by noting that the conclusion will be true whenever 'D' is true. Then we only have to determine the truth value for '$\neg C$' on the lines where D is false.

Once we have the truth values for the conclusion, we can, as we did in the last example, ignore every line apart from the lines where the conclusion is false.

A	B	C	D	$A \vee B$,	$\neg(A \& C)$,	$\neg(B \& \neg D)$	$\vdash$	$(\neg$	C	$\vee$	$D)$
T	T	T	T				✓			**T**	
T	T	T	F	?	?	?	?	F		**F**	
T	T	F	T				✓			**T**	
T	T	F	F				✓	T		**T**	
T	F	T	T				✓			**T**	
T	F	T	F	?	?	?	?	F		**F**	
T	F	F	T				✓			**T**	
T	F	F	F				✓	T		**T**	
F	T	T	T				✓			**T**	
F	T	T	F	?	?	?	?	F		**F**	
F	T	F	T				✓			**T**	
F	T	F	F				✓	T		**T**	
F	F	T	T				✓			**T**	
F	F	T	F	?	?	?	?	F		**F**	
F	F	F	T				✓			**T**	
F	F	F	F				✓	T		**T**	

We must now evaluate the premises. The first premise is the simplest, and so we start there. Of the four lines where the conclusion is false, there are three where $A \vee B$ is true. So the truth values for the next premise have to be determined for those three lines. (The second premise is simpler to evaluate than the third, so it's next.)

On those three lines, there is only one where the first two premise are true. With a little bit more work, we find that the third premise is false on that line. There is no line where the premises are true and the conclusion is false! The argument is valid.

A	B	C	D	A	$\lor$	$B,$	$\neg$	$(A$	$\&$	$C),$	$\neg$	$(B$	$\&$	$\neg$	$D)$	$\vdash$	$(\neg$	C	$\lor$	$D)$
T	T	T	T													✓			**T**	
T	T	T	F		**T**		**F**		T							✓	F		**F**	
T	T	F	T													✓			**T**	
T	T	F	F													✓	T		**T**	
T	F	T	T													✓			**T**	
T	F	T	F		**T**		**F**		T							✓	F		**F**	
T	F	F	T													✓			**T**	
T	F	F	F													✓	T		**T**	
F	T	T	T													✓			**T**	
F	T	T	F		**T**		**T**		F		**F**		T	T		✓	F		**F**	
F	T	F	T													✓			**T**	
F	T	F	F													✓	T		**T**	
F	F	T	T													✓			**T**	
F	F	T	F		**F**											✓	F		**F**	
F	F	F	T													✓			**T**	
F	F	F	F													✓	T		**T**	

If we had used no shortcuts, we would have had to write 256 'T's or 'F's on this table. Using shortcuts, we only had to write 37. We have saved ourselves a *lot* of work.

12.2 Partial truth tables

In the previous section, we saw how an incomplete truth table—although one that still had all of the lines—could be enough to determine if an argument is valid or invalid. That's one method where we use less than the full truth table. Another is where we create a one line truth table. This is called a PARTIAL TRUTH TABLE.

We can also use partial truth tables to determine if a sentence is not a tautology or is not a contradiction, and to determine if a set of sentences are not consistent or are consistent.

Tautology To show that a sentence is a tautology, we need to show that it is true on every valuation. That is to say, we need to know that it is true on every line of the truth table. To do that, we need a complete truth table.

To show that a sentence is *not* a tautology, however, we only need one line: a line on which the sentence is false. Therefore, in order to show that some sentence is not a tautology, it is enough to provide a single valuation – a single line of the truth table – that makes the sentence false.

Suppose that we want to show that the sentence '$(U \& T) \rightarrow (S \& W)$' is *not* a tautology. We set up a PARTIAL TRUTH TABLE. We have only left space for one line, rather than 16, since we are only looking for one line on which the sentence is false. Let us suppose that the sentence is false

S	T	U	W	$($	U	$\&$	T	$)$	$\rightarrow$	$($	S	$\&$	W	$)$
									F					

The main logical operator of the sentence is a conditional. In order for the conditional to be false, the antecedent must be true and the consequent must be false. So we put those in the table.

S	T	U	W	$($	U	$\&$	T	$)$	$\rightarrow$	$($	S	$\&$	W	$)$
						T			**F**			F		

For the '$(U \& T)$' to be true, both 'U' and 'T' must be true. Knowing that, we can set the truth values for these atomic sentences on the left side of the truth table.

S	T	U	W	$($	U	$\&$	T	$)$	$\rightarrow$	$($	S	$\&$	W	$)$
	T	T			T	T	T		**F**			F		

Now we just need to see if we can make '$(S \& W)$' false, which requires at least one of 'S' and 'W' to be false. Since the truth values for 'S' and 'W' have not been set yet, we can make both 'S' and 'W' false if we want. With that, we finish the table in this way:

S	T	U	W	$($	U	$\&$	T	$)$	$\rightarrow$	$($	S	$\&$	W	$)$
F	T	T	F		T	T	T		**F**		F	F	F	

We now have a partial truth table that shows that '$(U \,\&\, T) \rightarrow (S \,\&\, W)$' is not a tautology. Put otherwise, we have shown that there is a valuation that makes '$(U \,\&\, T) \rightarrow (S \,\&\, W)$' false, namely, the valuation where 'S' is false, 'T' is true, 'U' is true and 'W' is false (which would be line 10 in a full truth table).

To be clear, we use this method in an *attempt* to show that a sentence is not a tautology. If a sentence is a tautology, then we won't be able to find an assignment of 'true' and 'false' for every sentence letter that makes the full sentence false.

Contradiction Showing that a sentence is a contradiction requires a complete truth table: we need to show that the sentence is false on every line of the truth table.

On the other hand, to show that a sentence is *not* a contradiction, all we need to do is find a valuation that makes the sentence true, and so a single line of a truth table will suffice. We can illustrate this with the same example.

S	T	U	W	(	U	&	T	)	$\rightarrow$	(	S	&	W	)
									T					

One way for this sentence to be true is for the antecedent to be false. Since the antecedent is a conjunction, we can just make one of the conjuncts false. Let's make 'U' false. Then, we can assign whatever truth value we like to the other atomic sentences. With $S = \textit{false}$, $T = \textit{true}$, $U = \textit{false}$, and $W = \textit{false}$, we have shown that '$(U \,\&\, T) \rightarrow (S \,\&\, W)$' is not contradiction.

S	T	U	W	(	U	&	T	)	$\rightarrow$	(	S	&	W	)
F	T	F	F		F	F	T		**T**		F	F	F	

Equivalent To show that two sentences are logically equivalent, we must show that the sentences have the same truth value on every valuation. So this requires a complete truth table.

To show that two sentences are *not* logically equivalent, we only need to show that there is a valuation on which they have different truth values.

TO CHECK	THAT IT IS	THAT IT IS NOT
tautology	complete	one-line partial
contradiction	complete	one-line partial
equivalent	complete	one-line partial
consistent	one-line partial	complete
valid	complete	one-line partial

Table 12.1: The kind of truth table required to check each of these logical notions.

So this requires only a one-line partial truth table. We make the table so that one sentence is true and the other false.

Consistent To show that some sentences are jointly consistent, we must show that there is a valuation that makes all of the sentences true. This requires only a partial truth table with a single line.

To show that some sentences are jointly inconsistent, we must show that there is no valuation which makes all of the sentence true. So this requires a complete truth table: You must show that on every row of the table at least one of the sentences is false.

Valid To show that an argument is valid, we must show that there is no valuation that makes all of the premises true and the conclusion false. This requires a truth table with all of the requisite lines, although we can take the shortcuts that were described in the first section of this chapter.

To show that argument is invalid, we must show that there is a valuation that makes all of the premises true and the conclusion false. So this requires only a one-line partial truth table where all of the premises are true and the conclusion is false.

12.3 Practice exercises

A. If it is possible, use a partial truth table to show that the pair of sentences are **not equivalent**. If it can't be shown that they are not equivalent, then create a full truth table showing that they are equivalent.

1. $A, \neg A$
2. $A, A \vee A$
3. $A \rightarrow A, A \leftrightarrow A$
4. $A \vee \neg B, A \rightarrow B$
5. $A \,\&\, \neg A, \neg B \leftrightarrow B$
6. $\neg(A \,\&\, B), \neg A \vee \neg B$
7. $\neg(A \rightarrow B), \neg A \rightarrow \neg B$
8. $(A \rightarrow B), (\neg B \rightarrow \neg A)$
9. $((U \rightarrow (X \vee X)) \vee U)$ and $\neg(X \,\&\, (X \,\&\, U))$
10. $((C \,\&\, (N \leftrightarrow C)) \leftrightarrow C)$ and $(\neg\neg\neg N \rightarrow C)$
11. $[(A \vee B) \,\&\, C]$ and $[A \vee (B \,\&\, C)]$
12. $((L \,\&\, C) \,\&\, I)$ and $L \vee C$

B. If it is possible, use a partial truth table to show that the set of sentences are **consistent**. If it can't be shown that they are consistent, then create a full truth table showing that they are not consistent.

1. $A \,\&\, B, C \rightarrow \neg B, C$
2. $A \rightarrow B, B \rightarrow C, A, \neg C$
3. $A \vee B, B \vee C, C \rightarrow \neg A$
4. $A, B, C, \neg D, \neg E, F$
5. $A \,\&\, (B \vee C), \neg(A \,\&\, C), \neg(B \,\&\, C)$
6. $A \rightarrow B, B \rightarrow C, \neg(A \rightarrow C)$
7. $A \rightarrow A, \neg A \rightarrow \neg A, A \,\&\, A, A \vee A$
8. $A \rightarrow \neg A, \neg A \rightarrow A$
9. $A \vee B, A \rightarrow C, B \rightarrow C$
10. $A \vee B, A \rightarrow C, B \rightarrow C, \neg C$
11. $B \,\&\, (C \vee A), A \rightarrow B, \neg(B \vee C)$
12. $(A \leftrightarrow B) \rightarrow B, B \rightarrow \neg(A \leftrightarrow B), A \vee B$
13. $A \leftrightarrow (B \vee C), C \rightarrow \neg A, A \rightarrow \neg B$
14. $A \leftrightarrow B, \neg B \vee \neg A, A \rightarrow B$
15. $A \leftrightarrow B, A \rightarrow C, B \rightarrow D, \neg(C \vee D)$
16. $\neg(A \,\&\, \neg B), B \rightarrow \neg A, \neg B$

C. If it is possible, use a partial truth table to show that the argument is

invalid. If it can't be shown that the argument is invalid, then, using the shortcuts explained in §12.1, create a full truth table showing that it is valid.

1. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \vdash A$
2. $A \leftrightarrow \neg(B \leftrightarrow A) \vdash A$
3. $A \rightarrow B, B \vdash A$
4. $A \vee B, B \vee C, \neg B \vdash A \,\&\, C$
5. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$
6. $A \rightarrow (A \,\&\, \neg A) \vdash \neg A$
7. $A \vee B, A \rightarrow B, B \rightarrow A \vdash A \leftrightarrow B$
8. $A \vee (B \rightarrow A) \vdash \neg A \rightarrow \neg B$
9. $A \vee B, A \rightarrow B, B \rightarrow A \vdash A \,\&\, B$
10. $(B \,\&\, A) \rightarrow C, (C \,\&\, A) \rightarrow B \vdash (C \,\&\, B) \rightarrow A$
11. $\neg(\neg A \vee \neg B), A \rightarrow \neg C \vdash A \rightarrow (B \rightarrow C)$
12. $A \,\&\, (B \rightarrow C), \neg C \,\&\, (\neg B \rightarrow \neg A) \vdash C \,\&\, \neg C$
13. $A \,\&\, B, \neg A \rightarrow \neg C, B \rightarrow \neg D \vdash A \vee B$
14. $A \rightarrow B \vdash (A \,\&\, B) \vee (\neg A \,\&\, \neg B)$
15. $\neg A \rightarrow B, \neg B \rightarrow C, \neg C \rightarrow A \vdash \neg A \rightarrow (\neg B \vee \neg C)$
16. $A \leftrightarrow \neg(B \leftrightarrow A) \vdash A$
17. $A \vee B, B \vee C, \neg A \vdash B \,\&\, C$
18. $A \rightarrow C, E \rightarrow (D \vee B), B \rightarrow \neg D \vdash (A \vee C) \vee (B \rightarrow (E \,\&\, D))$
19. $A \vee B, C \rightarrow A, C \rightarrow B \vdash A \rightarrow (B \rightarrow C)$
20. $A \rightarrow B, \neg B \vee A \vdash A \leftrightarrow B$

D. If it is possible, use a partial truth table to show that the sentence is **not a tautology** or **not a contradiction**. If it can't be shown that it is not a tautology or not a contradiction, then give a full truth table showing that the sentence is a tautology, contradiction, or contingent. (Note that if the sentence is not a tautology *and* not a contradiction, then it is contingent.)

1. $A \rightarrow \neg A$
2. $A \rightarrow (A \,\&\, (A \vee B))$
3. $(A \rightarrow B) \leftrightarrow (B \rightarrow A)$
4. $A \rightarrow \neg(A \,\&\, (A \vee B))$

5. $\neg B \rightarrow [(\neg A \,\&\, A) \vee B]$
6. $\neg(A \vee B) \leftrightarrow (\neg A \,\&\, \neg B)$
7. $[(A \,\&\, B) \,\&\, C] \rightarrow B$
8. $\neg\Big[(C \vee A) \vee B\Big]$
9. $\Big[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)\Big] \,\&\, C$
10. $(A \,\&\, B)] \rightarrow [(A \,\&\, C) \vee (B \,\&\, D)]$
11. $\neg(A \vee A)$
12. $(A \rightarrow B) \vee (B \rightarrow A)$
13. $[(A \rightarrow B) \rightarrow A] \rightarrow A$
14. $\neg[(A \rightarrow B) \vee (B \rightarrow A)]$
15. $(A \,\&\, B) \vee (A \vee B)$
16. $\neg(A \,\&\, B) \leftrightarrow A$
17. $A \rightarrow (B \vee C)$
18. $(A \,\&\, \neg A) \rightarrow (B \vee C)$
19. $(B \,\&\, D) \leftrightarrow [A \leftrightarrow (A \vee C)]$
20. $\neg[(A \rightarrow B) \vee (C \rightarrow D)]$