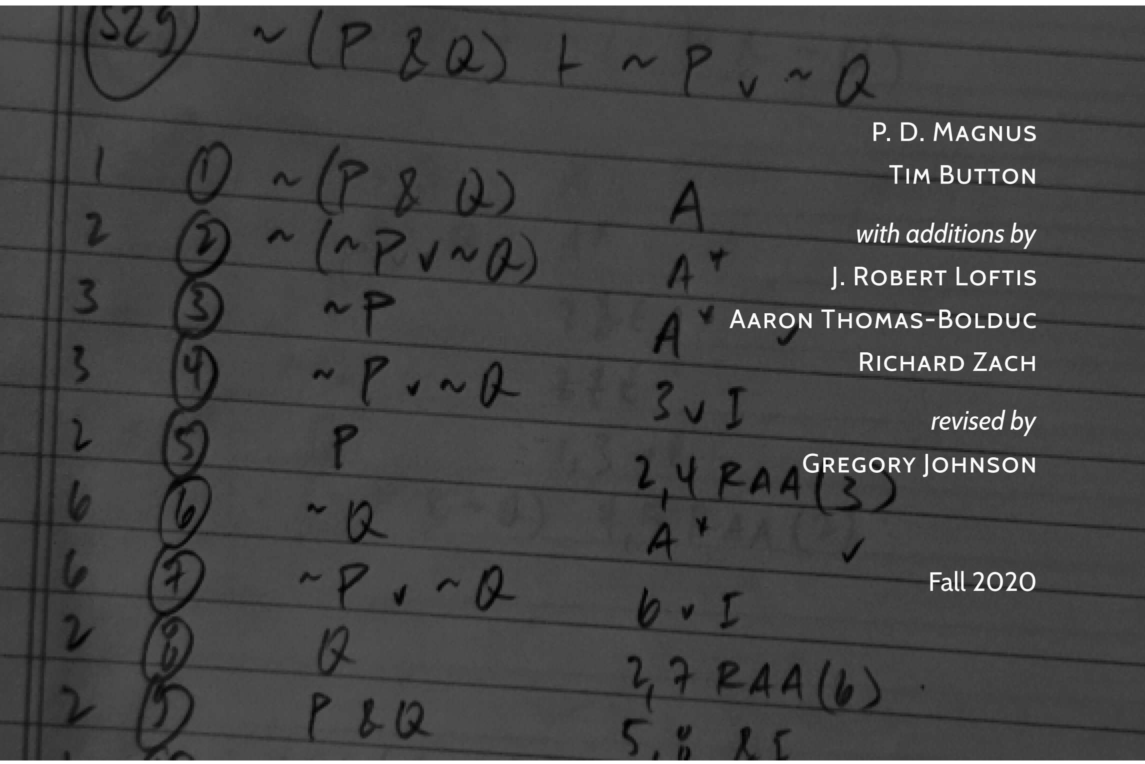
forall x

THE MISSISSIPPI STATE EDITION

solutions to selected exercises



P. D. Magnus
Tim Button

with additions by
J. Robert Loftis
Aaron Thomas-Bolduc
Richard Zach

revised by
Gregory Johnson

Fall 2020

This booklet is based on the solutions booklet `forall`*x*: *Cambridge*, by

Tim Button
University of Cambridge

used under a CC BY 4.0 license, which is based in turn on `forall`*x*, by

P.D. Magnus
University at Albany, State University of New York

used under a CC BY 4.0 license,
which was remixed & expanded by

Aaron Thomas-Bolduc & Richard Zach
University of Calgary

It includes additional material from `forall`*x* by P.D. Magnus, used under a CC BY 4.0 license, and from `forall`*x*: *Lorain County Remix*, by Cathal Woods and J. Robert Loftis, used with permission.

Contents

1

Arguments

Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

2

Valid arguments

A. Which of the following arguments is valid? Which is invalid?

1. Socrates is a man.
2. All men are carrots.

∴ Socrates is a carrot. Valid

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.

∴ Abe Lincoln was born in Illinois. Valid

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.

∴ Abe Lincoln will not die. Invalid

Abe Lincoln might die for some other reason: someone else might pull the trigger; he might die of old age.

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.

∴ Abe Lincoln was from France. Valid

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

∴ The world will not end today. Valid

1. Joe is now 19 years old.
2. Joe is now 87 years old.

∴ Bob is now 20 years old. Valid

An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. It is impossible for all the premises to be true; so it is certainly impossible that the premises are all true and the conclusion is false.

B. Could there be:

1. A valid argument that has one false premise and one true premise? Yes.
   Example: the first argument, above.
2. A valid argument that has only false premises? Yes.
   Example: Socrates is a frog, all frogs are excellent pianists, therefore Socrates is an excellent pianist.
3. A valid argument with only false premises and a false conclusion? Yes.
   The same example will suffice.
4. An invalid argument that can be made valid by the addition of a new premise? Yes.
   Plenty of examples, but let me offer a more general observation. We can *always* make an invalid argument valid, by adding a contradiction into the premises. For an argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. If the premises are contradictory, then it is impossible for them all to be true (and the conclusion false).
5. A valid argument that can be made invalid by the addition of a new premise? No.
   An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. Adding another premise will only make it harder for the premises all to be true together.

In each case: if so, give an example; if not, explain why not.

3

Other logical notions

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon. Contingent
2. Someone once crossed the Rubicon. Contingent
3. No one has ever crossed the Rubicon. Contingent
4. If Caesar crossed the Rubicon, then someone has. Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon. Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar. Contingent

B. For each of the following: Is it a necessary truth, a necessary falsehood, or contingent?

1. Elephants dissolve in water.
2. Wood is a light, durable substance useful for building things.
3. If wood were a good building material, it would be useful for building things.
4. I live in a three story building that is two stories tall.
5. If gerbils were mammals they would nurse their young.

C. Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.
   If you put an elephant in water, it will disintegrate.
2. All mammals dissolve in water.
   If you put an elephant in water, it will disintegrate.
3. George Bush was the 43rd president.
   Barack Obama is the 44th president.

4. Barack Obama is the 44th president.
   Barack Obama was president immediately after the 43rd president.
5. Elephants dissolve in water.
   All mammals dissolve in water.

D. Which of the following pairs of sentences are necessarily equivalent?

1. Thelonious Monk played piano.
   John Coltrane played tenor sax.
2. Thelonious Monk played gigs with John Coltrane.
   John Coltrane played gigs with Thelonious Monk.
3. All professional piano players have big hands.
   Piano player Bud Powell had big hands.
4. Bud Powell suffered from severe mental illness.
   All piano players suffer from severe mental illness.
5. John Coltrane was deeply religious.
   John Coltrane viewed music as an expression of spirituality.

E. Consider the following sentences:

G1 There are at least four giraffes at the wild animal park.

G2 There are exactly seven gorillas at the wild animal park.

G3 There are not more than two Martians at the wild animal park.

G4 Every giraffe at the wild animal park is a Martian.

Now consider each of the following collections of sentences. Which are jointly possible? Which are jointly impossible?

1. Sentences G2, G3, and G4 — Jointly possible
2. Sentences G1, G3, and G4 — Jointly impossible
3. Sentences G1, G2, and G4 — Jointly possible
4. Sentences G1, G2, and G3 — Jointly possible

F. Consider the following sentences.

M1 All people are mortal.

M2 Socrates is a person.

M3 Socrates will never die.

M4 Socrates is mortal.

Which combinations of sentences are jointly possible? Mark each "possible" or "impossible."

1. Sentences M1, M2, and M3
2. Sentences M2, M3, and M4
3. Sentences M2 and M3
4. Sentences M1 and M4
5. Sentences M1, M2, M3, and M4

G. Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument that has one false premise and one true premise
   Yes: 'All whales are mammals (*true*). All mammals are plants (*false*). So all whales are plants.'
2. A valid argument that has a false conclusion
   Yes. (See example from previous exercise.)
3. A valid argument, the conclusion of which is a necessary falsehood
   Yes: '$1 + 1 = 3$. So $1 + 2 = 4$.'
4. An invalid argument, the conclusion of which is a necessary truth
   No. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.
5. A necessary truth that is contingent
   No. If a sentence is a necessary truth, it cannot possibly be false, but a contingent sentence can be false.
6. Two necessarily equivalent sentences, both of which are necessary truths
   Yes: '4 is even', '4 is divisible by 2'.
7. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
   No. A necessary truth cannot possibly be false, while a contingent sentence can be false. So in any situation in which the contingent sentence is false, it will have a different truth value from the necessary truth. Thus they will not necessarily have the same truth value, and so will not be equivalent.
8. Two necessarily equivalent sentences that together are jointly impossible
   Yes: '$1 + 1 = 4$' and '$1 + 1 = 3$'.
9. A jointly possible collection of sentences that contains a necessary falsehood
   No. If a sentence is necessarily false, there is no way to make it true, let alone it along with all the other sentences.

10. A jointly impossible set of sentences that contains a necessary truth
    Yes: '$1 + 1 = 4$' and '$1 + 1 = 2$'.

H. Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid argument, whose premises are all necessary truths, and whose conclusion is contingent
2. A valid argument with true premises and a false conclusion
3. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
4. A jointly possible collection of sentences, all of which are contingent
5. A false necessary truth
6. A valid argument with false premises
7. A necessarily equivalent pair of sentences that are not jointly possible
8. A necessary truth that is also a necessary falsehood
9. A jointly possible collection of sentences that are all necessary falsehoods

5

Connectives

A. Using the symbolization key given, symbolize each English sentence in TFL.

M: Those creatures are men in suits.
C: Those creatures are chimpanzees.
G: Those creatures are gorillas.

1. Those creatures are not men in suits.
   $\neg M$
2. Those creatures are men in suits, or they are not.
   $(M \lor \neg M)$
3. Those creatures are either gorillas or chimpanzees.
   $(G \lor C)$
4. Those creatures are neither gorillas nor chimpanzees.
   $\neg(C \lor G)$
5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.
   $(C \rightarrow \neg(G \lor M))$
6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.
   $(M \lor (C \lor G))$

B. Using the symbolization key given, symbolize each English sentence in TFL.

A: Mister Ace was murdered.
B: The butler did it.
C: The cook did it.

D: The Duchess is lying.
E: Mister Edge was murdered.
F: The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.
   $(A \lor E)$
2. If Mister Ace was murdered, then the cook did it.
   $(A \rightarrow C)$
3. If Mister Edge was murdered, then the cook did not do it.
   $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.
   $(B \lor D)$
5. The cook did it only if the Duchess is lying.
   $(C \rightarrow D)$
6. If the murder weapon was a frying pan, then the culprit must have been the cook.
   $(F \rightarrow C)$
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
   $(\neg F \rightarrow (C \lor B))$
8. Mister Ace was murdered if and only if Mister Edge was not murdered.
   $(A \leftrightarrow \neg E)$
9. The Duchess is lying, unless it was Mister Edge who was murdered.
   $(D \lor E)$
10. If Mister Ace was murdered, he was done in with a frying pan.
    $(A \rightarrow F)$
11. Since the cook did it, the butler did not.
    $(C \,\&\, \neg B)$
12. Of course the Duchess is lying!
    D

C. Using the symbolization key given, symbolize each English sentence in TFL.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.

$(E_1 \,\&\, E_2)$

2. If Ava is a firefighter, then she is satisfied with her career.
   $(F_1 \rightarrow S_1)$
3. Ava is a firefighter, unless she is an electrician.
   $(F_1 \vee E_1)$
4. Harrison is an unsatisfied electrician.
   $(E_2 \,\&\, \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
   $\neg(E_1 \vee E_2)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
   $((E_1 \,\&\, E_2) \,\&\, \neg(S_1 \vee S_2))$
7. Harrison is satisfied only if he is a firefighter.
   $(S_2 \rightarrow F_2)$
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
   $((\neg E_1 \rightarrow \neg E_2) \,\&\, (E_1 \rightarrow E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
   $(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
    $((E_2 \,\&\, F_2) \rightarrow S_2)$
11. It cannot be that Harrison is both an electrician and a firefighter.
    $\neg(E_2 \,\&\, F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
    $((F_2 \,\&\, F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D. Using the symbolization key given, translate each English-language sentence into TFL.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.
   $J_1 \,\&\, J_2$
2. Neither Miles Davis nor John Coltrane played tuba.

$\neg(M_2 \vee J_3)$ or $\neg M_2 \,\&\, \neg J_3$

3. John Coltrane did not play both tenor sax and tuba.
   $\neg(J_1 \,\&\, J_3)$ or $\neg J_1 \vee \neg J_3$
4. John Coltrane did not play tenor sax unless he also played soprano sax.
   $\neg J_1 \vee J_2$
5. John Coltrane did not play tuba, but Miles Davis did.
   $\neg J_3 \,\&\, M_2$
6. Miles Davis played trumpet only if he also played tuba.
   $M_1 \rightarrow M_2$
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
   $M_1 \rightarrow (J_1 \vee (J_2 \vee J_3))$
8. If John Coltrane played tuba then Miles Davis played neither trumpet nor tuba.
   $J_3 \rightarrow \neg(M_1 \vee M_2)$ or $J_3 \rightarrow (\neg M_1 \,\&\, \neg M_2)$
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.
   $(J_3 \,\&\, M_2) \leftrightarrow (\neg J_1 \wedge \neg M_1)$ or $(J_3 \,\&\, M_2) \leftrightarrow \neg(J_1 \vee M_1)$

E. Give a symbolization key and symbolize the following English sentences in TFL.

A: Alice is a spy.
B: Bob is a spy.
C: The code has been broken.
G: The German embassy will be in an uproar.

1. Alice and Bob are both spies.
   $(A \,\&\, B)$
2. If either Alice or Bob is a spy, then the code has been broken.
   $((A \vee B) \rightarrow C)$
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
   $(\neg(A \vee B) \rightarrow \neg C)$
4. The German embassy will be in an uproar, unless someone has broken the code.
   $(G \vee C)$
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
   $((C \vee \neg C) \,\&\, G)$
6. Either Alice or Bob is a spy, but not both.

$((A \vee B) \,\&\, \neg(A \,\&\, B))$

F. Give a symbolization key and symbolize the following English sentences in TFL.

F: There is food to be found in the pridelands.
R: Rafiki will talk about squashed bananas.
A: Simba is alive.
K: Scar will remain as king.

1. If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.
$(F \rightarrow R)$
2. Rafiki will talk about squashed bananas unless Simba is alive.
$(R \vee A)$
3. Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.
$((R \vee \neg R) \,\&\, F)$
4. Scar will remain as king if and only if there is food to be found in the pridelands.
$(K \leftrightarrow F)$
5. If Simba is alive, then Scar will not remain as king.
$(A \rightarrow \neg K)$

G. For each argument, write a symbolization key and symbolize all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

   P: Dorothy plays the Piano in the morning.
   C: Roger wakes up cranky.
   D: Dorothy is distracted.

   $(P \rightarrow C)$, $(P \vee D)$, $(\neg C \rightarrow D)$
2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.

   T_1: It rains on Tuesday
   T_2: It snows on Tuesday
   S: Neville is sad on Tuesday

C: Neville is cold on Tuesday

$(T_1 \vee T_2)$, $(T_1 \rightarrow S)$, $(T_2 \rightarrow C)$, $(S \vee C)$

3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

   Z: Zoog remembered to do his chores
   C: Things are clean
   N: Things are neat

   $(Z \rightarrow (C \,\&\, \neg N))$, $(\neg Z \rightarrow (N \,\&\, \neg C))$, $((N \vee C) \,\&\, \neg(N \,\&\, C))$.

H. For each argument, write a symbolization key and translate the argument as well as possible into TFL. The part of the passage in italics is there to provide context for the argument, and doesn't need to be symbolized.

1. It is going to rain soon. I know because my leg is hurting, and my leg hurts if it's going to rain.
2. *Spider-man tries to figure out the bad guy's plan.* If Doctor Octopus gets the uranium, he will blackmail the city. I am certain of this because if Doctor Octopus gets the uranium, he can make a dirty bomb, and if he can make a dirty bomb, he will blackmail the city.
3. *A westerner tries to predict the policies of the Chinese government.* If the Chinese government cannot solve the water shortages in Beijing, they will have to move the capital. They don't want to move the capital. Therefore they must solve the water shortage. But the only way to solve the water shortage is to divert almost all the water from the Yangzi river northward. Therefore the Chinese government will go with the project to divert water from the south to the north.

I. We symbolized an *exclusive or* using '$\vee$', '&', and '$\neg$'. How could you symbolize an *exclusive or* using only two connectives? Is there any way to symbolize an *exclusive or* using only one connective?

For two connectives, we could offer any of the following:

$$\neg(\mathsf{A} \leftrightarrow \mathsf{B})$$
$$(\neg\mathsf{A} \leftrightarrow \mathsf{B})$$
$$(\neg(\neg\mathsf{A} \,\&\, \neg\mathsf{B}) \,\&\, \neg(\mathsf{A} \,\&\, \mathsf{B}))$$

But if we wanted to symbolize it using only one connective, we would have to introduce a new primitive connective.

6

Sentences of TFL

A. For each of the following: (a) Is it a sentence of TFL, strictly speaking? (b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

1. (A) — (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ — (a) no (b) yes
3. $\neg\neg\neg\neg F$ — (a) yes (b) yes
4. $\neg \,\&\, S$ — (a) no (b) no
5. $(G \,\&\, \neg G)$ — (a) yes (b) yes
6. $(A \rightarrow (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$ — (a) no (b) yes
7. $[(Z \leftrightarrow S) \rightarrow W] \,\&\, [J \vee X]$ — (a) no (b) yes
8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \,\&\, D)$ — (a) no (b) no

B. Are there any sentences of TFL that contain no atomic sentences? Explain your answer.

No. Atomic sentences contain atomic sentences (trivially). And every more complicated sentence is built up out of less complicated sentences, that were in turn built out of less complicated sentences, . . . , that were ultimately built out of atomic sentences.

C. What is the scope of each connective in the sentence

$$\big[(H \rightarrow I) \vee (I \rightarrow H)\big] \,\&\, (J \vee K)$$

The scope of the left-most instance of '$\rightarrow$' is '$(H \rightarrow I)$'.
The scope of the right-most instance of '$\rightarrow$' is '$(I \rightarrow H)$'.
The scope of the left-most instance of '$\vee$ is '$\big[(H \rightarrow I) \vee (I \rightarrow H)\big]$'
The scope of the right-most instance of '$\vee$' is '$(J \vee K)$'
The scope of the conjunction is the entire sentence; so conjunction is the main logical connective of the sentence.

9

Complete truth tables

A. Complete truth tables for each of the following:

1. $A \rightarrow A$

A	A	$\rightarrow$	A
T	T	**T**	T
F	F	**T**	F

2. $C \rightarrow \neg C$

C	C	$\rightarrow$	$\neg$	C
T	T	**F**	F	T
F	F	**T**	T	F

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	$\neg$	$(A$	$\leftrightarrow$	$\neg$	$B)$
T	T	T	T	T	**T**	T	T	F	F	T
T	F	T	F	F	**T**	F	T	T	T	F
F	T	F	F	T	**T**	F	F	T	F	T
F	F	F	T	F	**T**	T	F	F	T	F

4. $(A \rightarrow B) \vee (B \rightarrow A)$

A	B	$(A$	$\rightarrow$	$B)$	$\vee$	$(B$	$\rightarrow$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	T	T	**T**	T	F	F
F	F	F	T	F	**T**	F	T	F

5. $(A \mathbin{\&} B) \rightarrow (B \vee A)$

A	B	$(A$	$\&$	$B)$	$\rightarrow$	$(B$	$\vee$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	F	T	**T**	T	T	F
F	F	F	F	F	**T**	F	F	F

6. $\neg(A \vee B) \leftrightarrow (\neg A \mathbin{\&} \neg B)$

A	B	$\neg$	$(A$	$\vee$	$B)$	$\leftrightarrow$	$(\neg$	A	$\&$	$\neg$	$B)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

7. $\left[(A \mathbin{\&} B) \mathbin{\&} \neg(A \mathbin{\&} B)\right] \mathbin{\&} C$

A	B	C	$[(A$	$\&$	$B)$	$\&$	$\neg$	$(A$	$\&$	$B)]$	$\&$	C
T	T	T	T	T	T	F	F	T	T	T	**F**	T
T	T	F	T	T	T	F	F	T	T	T	**F**	F
T	F	T	T	F	F	F	T	T	F	F	**F**	T
T	F	F	T	F	F	F	T	T	F	F	**F**	F
F	T	T	F	F	T	F	T	F	F	T	**F**	T
F	T	F	F	F	T	F	T	F	F	T	**F**	F
F	F	T	F	F	F	F	T	F	F	F	**F**	T
F	F	F	F	F	F	F	T	F	F	F	**F**	F

8. $[(A \mathbin{\&} B) \mathbin{\&} C] \rightarrow B$

A	B	C	$[(A$	$\&$	$B)$	$\&$	$C]$	$\rightarrow$	B
T	T	T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	F	F	**T**	T
T	F	T	T	F	F	F	T	**T**	F
T	F	F	T	F	F	F	F	**T**	F
F	T	T	F	F	T	F	T	**T**	T
F	T	F	F	F	T	F	F	**T**	T
F	F	T	F	F	F	F	T	**T**	F
F	F	F	F	F	F	F	F	**T**	F

9. $\neg[(C \vee A) \vee B]$

A	B	C	$\neg$	$[(C$	$\vee$	$A)$	$\vee$	$B]$
T	T	T	**F**	T	T	T	T	T
T	T	F	**F**	F	T	T	T	T
T	F	T	**F**	T	T	T	T	F
T	F	F	**F**	F	T	T	T	F
F	T	T	**F**	T	T	F	T	T
F	T	F	**F**	F	F	F	T	T
F	F	T	**F**	T	T	F	T	F
F	F	F	**T**	F	F	F	F	F

B. Check all the claims made in introducing the new notational conventions in §9.4, i.e. show that:

1. '$((A \,\&\, B) \,\&\, C)$' and '$(A \,\&\, (B \,\&\, C))$' have the same truth table

A	B	C	$(A$	$\&$	$B)$	$\&$	C	A	$\&$	$(B$	$\&$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**F**	F	T	**F**	F	F	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table

A	B	C	$(A$	$\vee$	$B)$	$\vee$	C	A	$\vee$	$(B$	$\vee$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**T**	F	T	**T**	T	T	F
T	F	T	T	T	F	**T**	T	T	**T**	F	T	T
T	F	F	T	T	F	**T**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**T**	F	F	**T**	T	T	F
F	F	T	F	F	F	**T**	T	F	**T**	F	T	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

3. '$((A \vee B) \,\&\, C)$' and '$(A \vee (B \,\&\, C))$' do not have the same truth table

A	B	C	$(A \vee B)$			$\&$	C	A	$\vee$	$(B$	$\&$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**T**	T	F	F
T	F	T	T	T	F	**T**	T	T	**T**	F	F	T
T	F	F	T	T	F	**F**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

4. '$((A \rightarrow B) \rightarrow C)$' and '$(A \rightarrow (B \rightarrow C))$' do not have the same truth table

A	B	C	$(A \rightarrow B)$			$\rightarrow$	C	A	$\rightarrow$	$(B$	$\rightarrow$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**T**	T	T	**T**	F	T	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	T	T
F	F	F	F	T	F	**F**	F	F	**T**	F	T	F

Also, check whether:

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table
   Indeed they do:

A	B	C	$(A \leftrightarrow B)$			$\leftrightarrow$	C	A	$\leftrightarrow$	$(B$	$\leftrightarrow$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**T**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	F	T
F	F	F	F	T	F	**F**	F	F	**F**	F	T	F

C. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1. $\neg(S \leftrightarrow (P \rightarrow S))$

$\neg$	(S	$\leftrightarrow$	(P	$\rightarrow$	S))
F	T	T	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F
T	F	F	F	T	F

2. $\neg[(X \,\&\, Y) \vee (X \vee Y)]$

$\neg$	[(X	&	Y)	$\vee$	(X	$\vee$	Y)]
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	F
F	F	F	T	T	F	T	T
T	F	F	F	F	F	F	F

3. $(A \rightarrow B) \leftrightarrow (\neg B \leftrightarrow \neg A)$

(A	$\rightarrow$	B)	$\leftrightarrow$	($\neg$	B	$\leftrightarrow$	$\neg$	A)
T	T	T	T	F	T	T	F	T
T	F	F	T	T	F	F	F	T
F	T	T	F	F	T	F	T	F
F	T	F	T	T	F	T	T	F

4. $[C \leftrightarrow (D \vee E)] \,\&\, \neg C$

[C	$\leftrightarrow$	(D	$\vee$	E)]	&	$\neg$	C
T	T	T	T	T	F	F	T
T	T	T	T	F	F	F	T
T	T	F	T	T	F	F	T
T	F	F	F	F	F	F	T
F	F	T	T	T	F	T	F
F	F	T	T	F	F	T	F
F	F	F	T	T	F	T	F
F	T	F	F	F	T	T	F

5. $\neg(G \,\&\, (B \,\&\, H)) \leftrightarrow (G \vee (B \vee H))$

¬	(G	&	(B	&	H))	↔	(G	∨	(B	∨	H))
F	T	T	T	T	T	F	T	T	T	T	T
T	T	F	T	F	F	T	T	T	T	T	F
T	T	F	F	F	T	T	T	T	F	T	T
T	T	F	F	F	F	T	T	T	F	F	F
T	F	F	T	T	T	T	F	T	T	T	T
T	F	F	T	F	F	T	F	T	T	T	F
T	F	F	F	F	T	T	F	T	F	T	T
T	F	F	F	F	F	F	F	F	F	F	F

D. Write complete truth tables for the following sentences and mark the column that represents the possible truth values for the whole sentence.

1. $(D \& \neg D) \to G$

(D	&	¬	D)	→	G
T	F	F	T	T	T
T	F	F	T	T	F
F	F	T	F	T	T
F	F	T	F	T	F

2. $(\neg P \vee \neg M) \leftrightarrow M$

(¬	P	∨	¬	M)	↔	M
F	T	F	F	T	T	T
F	T	T	T	F	F	F
T	F	T	F	T	T	T
T	F	T	T	F	T	F

3. $\neg\neg(\neg A \& \neg B)$

¬	¬	(¬	A	&	¬	B)
F	T	F	T	F	F	T
F	T	F	T	F	T	F
F	T	T	F	F	F	T
T	F	T	F	T	T	F

4. $[(D \& R) \rightarrow I] \rightarrow \neg(D \vee R)$

[(D	&	R)	→	I]	→	¬	(D	∨	R)
T	T	T	T	T	F	F	T	T	T
T	T	T	F	F	T	F	T	T	T
T	F	F	T	T	F	F	T	T	F
T	F	F	T	F	F	F	T	T	F
F	F	T	T	T	F	F	F	T	T
F	F	T	T	F	F	F	F	T	T
F	F	F	T	T	T	T	F	F	F
F	F	F	T	F	T	T	F	F	F

5. $\neg[(D \leftrightarrow O) \leftrightarrow A] \rightarrow (\neg D \& O)$

¬	[(D	↔	O)	↔	A]	→	(¬	D	&	O)
F	T	T	T	T	T	T	F	T	F	T
T	T	T	T	F	F	F	F	T	F	T
T	T	F	F	F	T	F	F	T	F	F
F	T	F	F	T	F	T	F	T	F	F
T	F	F	T	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T	F	T	T
F	F	T	F	T	T	T	T	F	F	F
T	F	T	F	F	F	T	T	F	F	F

If you want additional practice, you can construct truth tables for any of the sentences and arguments in the exercises for the previous chapter.

11

Entailment & validity

A. Revisit your answers to §10**A**. Determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

1. $A \to A$ — Tautology
2. $C \to \neg C$ — Neither
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ — Tautology
4. $(A \to B) \vee (B \to A)$ — Tautology
5. $(A \& B) \to (B \vee A)$ — Tautology
6. $\neg(A \vee B) \leftrightarrow (\neg A \& \neg B)$ — Tautology
7. $\big[(A \& B) \& \neg(A \& B)\big] \& C$ — Contradiction
8. $[(A \& B) \& C] \to B$ — Tautology
9. $\neg\big[(C \vee A) \vee B\big]$ — Neither

B. Use truth tables to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \to A$, $\neg A \to \neg A$, $A \& A$, $A \vee A$ — Jointly consistent (see line 1)

A	$A \to A$	$\neg\ A \to \neg\ A$	$A \,\&\, A$	$A \vee A$
T	T **T** T	F T **T** F T	T **T** T	T **T** T
F	F **T** F	T F **T** T F	F **F** F	F **F** F

2. $A \vee B$, $A \to C$, $B \to C$ — Jointly consistent (see line 1)

A	B	C	$A \vee B$	$A \to C$	$B \to C$
T	T	T	T **T** T	T **T** T	T **T** T
T	T	F	T **T** T	T **F** F	T **F** F
T	F	T	T **T** T	T **T** T	F **T** T
T	F	F	T **T** F	T **F** F	F **T** F
F	T	T	F **T** F	F **T** T	T **T** T
F	T	F	F **T** T	F **T** F	T **F** F
F	F	T	F **F** F	F **T** T	F **T** T
F	F	F	F **F** F	F **T** F	F **T** F

3. $B \,\&\, (C \vee A)$, $A \to B$, $\neg(B \vee C)$ Jointly inconsistent

A	B	C	$B \,\&\, (C \vee A)$	$A \to B$	$\neg\,(B \vee C)$
T	T	T	T **T** T T T	T **T** T	**F** T T T
T	T	F	T **T** F T T	T **T** T	**F** T T F
T	F	T	F **F** T T T	T **F** F	**F** F T T
T	F	F	F **F** F T T	T **F** F	**T** F F F
F	T	T	T **T** T T F	F **T** T	**F** T T T
F	T	F	T **F** F F F	F **T** T	**F** T T F
F	F	T	F **F** T T F	F **T** F	**F** F T T
F	F	F	F **F** F F F	F **T** F	**T** F F F

4. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$ Jointly consistent (see line 8)

A	B	C	$A \leftrightarrow (B \vee C)$	$C \to \neg\, A$	$A \to \neg\, B$
T	T	T	T **T** T T T	T **F** F T	T **F** F T
T	T	F	T **T** T T F	F **T** F T	T **F** F T
T	F	T	T **T** F T T	T **F** F T	T **T** T F
T	F	F	T **F** F F F	F **T** F T	T **T** T F
F	T	T	F **F** T T T	T **T** T F	F **T** F T
F	T	F	F **F** T T F	F **T** T F	F **T** F T
F	F	T	F **F** F T T	T **T** T F	F **T** T F
F	F	F	F **T** F F F	F **T** T F	F **T** T F

C. Use truth tables to determine whether each argument is valid or invalid.

1. $A \to A \therefore A$ Invalid (see line 2)

A	$A \to A$	A
T	T **T** T	T
F	F **T** F	F

2. $A \to (A \,\&\, \neg A) \therefore \neg A$ Valid

A	$A \to (A \,\&\, \neg A)$	$\neg A$
T	T **F** T F F T	**F** T
F	F **T** F F T F	**T** F

3. $A \vee (B \to A) \therefore \neg A \to \neg B$ Valid

A	B	$A \vee (B \to A)$	$\neg A \to \neg B$
T	T	T **T** T T T	F T **T** F T
T	F	T **T** F T T	F T **T** T F
F	T	F **F** T F F	T F **F** F T
F	F	F **T** F T F	T F **T** T F

4. $A \vee B, B \vee C, \neg A \therefore B \,\&\, C$ Invalid (see line 6)

A	B	C	$A \vee B$	$B \vee C$	$\neg A$	$B \,\&\, C$
T	T	T	T **T** T	T **T** T	**F** T	T **T** T
T	T	F	T **T** T	T **T** F	**F** T	T **F** F
T	F	T	T **T** F	F **T** T	**F** T	F **F** T
T	F	F	T **T** F	F **F** F	**F** T	F **F** F
T	T	T	F **T** T	T **T** T	**T** F	T **T** T
T	T	F	F **T** T	T **T** F	**T** F	T **F** F
T	F	T	F **F** F	F **T** T	**T** F	F **F** T
T	F	F	F **F** F	F **F** F	**T** F	F **F** F

5. $(B \,\&\, A) \to C, (C \,\&\, A) \to B \therefore (C \,\&\, B) \to A$ Invalid (see line 5)

A	B	C	$(B \,\&\, A) \to C$	$(C \,\&\, A) \to B$	$(C \,\&\, B) \to A$
T	T	T	T T T **T** T	T T T **T** T	T T T **T** T
T	T	F	T T T **F** F	F F T **T** T	F F T **T** T
T	F	T	F F T **T** T	T T T **F** F	T F F **T** T
T	F	F	F F T **T** F	F F T **T** F	F F F **T** T
F	T	T	T F F **T** T	T F F **T** T	T T T **F** F
F	T	F	T F F **T** F	F F F **T** T	F F T **T** F
F	F	T	F F F **T** T	T F F **T** F	T F F **T** F
F	F	F	F F F **T** F	F F F **T** F	F F F **T** F

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence, using a complete truth table.

1. $\neg B \,\&\, B$ Contradiction
2. $\neg D \vee D$ Tautology
3. $(A \,\&\, B) \vee (B \,\&\, A)$ Contingent

4. $\neg[A \rightarrow (B \rightarrow A)]$ Contradiction
5. $A \leftrightarrow [A \rightarrow (B \,\&\, \neg B)]$ Contradiction
6. $[(A \,\&\, B) \leftrightarrow B] \rightarrow (A \rightarrow B)$ Contingent

E. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write "equivalent." Otherwise write, "Not equivalent."

1. A and $\neg A$
2. $A \,\&\, \neg A$ and $\neg B \leftrightarrow B$
3. $[(A \vee B) \vee C]$ and $[A \vee (B \vee C)]$
4. $A \vee (B \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C)$
5. $[A \,\&\, (A \vee B)] \rightarrow B$ and $A \rightarrow B$

F. Determine whether each the following sentences are logically equivalent using complete truth tables. If the two sentences really are equivalent, write "equivalent." Otherwise write, "not equivalent."

1. $A \rightarrow A$ and $A \leftrightarrow A$
2. $\neg(A \rightarrow B)$ and $\neg A \rightarrow \neg B$
3. $A \vee B$ and $\neg A \rightarrow B$
4. $(A \rightarrow B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$
5. $A \leftrightarrow (B \leftrightarrow C)$ and $A \,\&\, (B \,\&\, C)$

G. Determine whether each collection of sentences is jointly consistent or jointly inconsistent using a complete truth table.

1. $A \,\&\, \neg B$, $\neg(A \rightarrow B)$, $B \rightarrow A$

A	&	¬	B	¬	(A	→	B)	B	→	A	Consistent
T	F	F	T	F	T	T	T	T	T	T	
T	**T**	T	F	**T**	T	F	F	F	**T**	T	
F	F	F	T	F	F	T	T	T	F	F	
F	F	T	F	F	F	T	F	F	T	F	

2. $A \vee B$, $A \rightarrow \neg A$, $B \rightarrow \neg B$

A	∨	B	A	→	¬	A	B	→	¬	B	Inconsistent
T	T	T	T	F	F	T	T	F	F	T	
T	T	F	T	F	F	T	F	T	T	F	
F	T	T	F	T	T	F	T	F	F	T	
F	F	F	F	T	T	F	F	T	T	F	

3. $\neg(\neg A \vee B)$, $A \rightarrow \neg C$, $A \rightarrow (B \rightarrow C)$ Consistent

¬	(¬	A	∨	B)	A	→	¬	C	A	→	(B	→	C)
F	F	T	T	T	T	F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	F	T	F	T	F	F
T	F	T	F	F	T	F	F	T	T	T	F	T	T
T	F	T	F	F	T	**T**	T	F	T	**T**	F	T	F
F	T	F	T	T	F	T	F	T	F	F	T	T	T
F	T	F	T	T	F	T	T	F	F	T	T	F	F
F	T	F	T	F	F	T	F	T	F	T	F	T	T
F	T	F	T	F	F	T	T	F	F	T	F	T	F

4. $A \rightarrow B$, $A \& \neg B$ Inconsistent
5. $A \rightarrow (B \rightarrow C)$, $(A \rightarrow B) \rightarrow C$, $A \rightarrow C$ Consistent

H. Determine whether each collection of sentences is jointly consistent or jointly inconsistent, using a complete truth table.

1. $\neg B$, $A \rightarrow B$, A Inconsistent
2. $\neg(A \vee B)$, $A \leftrightarrow B$, $B \rightarrow A$ Consistent
3. $A \vee B$, $\neg B$, $\neg B \rightarrow \neg A$ Inconsistent
4. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \rightarrow B$ Consistent
5. $(A \vee B) \vee C$, $\neg A \vee \neg B$, $\neg C \vee \neg B$ Consistent

I. Determine whether each argument is valid or invalid, using a complete truth table.

1. $A \rightarrow B$, $B \therefore A$ Invalid
2. $A \leftrightarrow B$, $B \leftrightarrow C \therefore A \leftrightarrow C$ Valid
3. $A \rightarrow B$, $A \rightarrow C \therefore B \rightarrow C$ Invalid.
4. $A \rightarrow B$, $B \rightarrow A \therefore A \leftrightarrow B$ Valid

J. Determine whether each argument is valid or invalid, using a complete truth table.

1. $A \vee \left[A \rightarrow (A \leftrightarrow A)\right] \therefore A$ Invalid
2. $A \vee B$, $B \vee C$, $\neg B \therefore A \& C$ Valid
3. $A \rightarrow B$, $\neg A \therefore \neg B$ Invalid
4. A, $B \therefore \neg(A \rightarrow \neg B)$ Valid
5. $\neg(A \& B)$, $A \vee B$, $A \leftrightarrow B \therefore C$ Valid

K. Answer each of the questions below and justify your answer.

1. Suppose that A and B are logically equivalent. What can you say about $\mathsf{A} \leftrightarrow \mathsf{B}$?
   A and B have the same truth value on every line of a complete truth table, so $\mathsf{A} \leftrightarrow \mathsf{B}$ is true on every line. It is a tautology.
2. Suppose that $(\mathsf{A} \,\&\, \mathsf{B}) \to \mathsf{C}$ is neither a tautology nor a contradiction. What can you say about whether $\mathsf{A}, \mathsf{B} \mathrel{.\mathord{\cdot}.} \mathsf{C}$ is valid?
   Since the sentence $(\mathsf{A} \,\&\, \mathsf{B}) \to \mathsf{C}$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, A and B are true and C is false. So the argument is invalid.
3. Suppose that A, B and C are jointly inconsistent. What can you say about $(\mathsf{A} \,\&\, \mathsf{B} \,\&\, \mathsf{C})$?
   Since the sentences are jointly inconsistent, there is no valuation on which they are all true. So their conjunction is false on every valuation. It is a contradiction
4. Suppose that A is a contradiction. What can you say about whether $\mathsf{A}, \mathsf{B} \vDash \mathsf{C}$?
   Since A is false on every line of a complete truth table, there is no line on which A and B are true and C is false. So the entailment holds.
5. Suppose that C is a tautology. What can you say about whether $\mathsf{A}, \mathsf{B} \vDash \mathsf{C}$?
   Since C is true on every line of a complete truth table, there is no line on which A and B are true and C is false. So the entailment holds.
6. Suppose that A and B are logically equivalent. What can you say about $(\mathsf{A} \vee \mathsf{B})$?
   Not much. Since A and B are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is logically equivalent to them.
7. Suppose that A and B are *not* logically equivalent. What can you say about $(\mathsf{A} \vee \mathsf{B})$?
   A and B have different truth values on at least one line of a complete truth table, and $(\mathsf{A} \vee \mathsf{B})$ will be true on that line. On other lines, it might be true or false. So $(\mathsf{A} \vee \mathsf{B})$ is either a tautology or it is contingent; it is *not* a contradiction.

L. Consider the following principle:

- Suppose A and B are logically equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

Is this principle correct? Explain your answer.

The principle is correct. Since A and B are logically equivalent, they have the same truth table. So every valuation that makes A true also makes B true, and every valuation that makes A false also makes B false. So if no valuation makes all the premises true and the conclusion false, when A was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace A with B.

12

Truth table shortcuts

A. Using shortcuts, determine whether each sentence is a tautology, a contradiction, or neither.

1. $\neg B \,\&\, B$ Contradiction

B	$\neg$	B	$\&$	B
T	F		**F**	
F			**F**	

2. $\neg D \vee D$ Tautology

D	$\neg$	D	$\vee$	D
T			**T**	
F	T		**T**	

3. $(A \,\&\, B) \vee (B \,\&\, A)$ Neither

A	B	$(A$	$\&$	$B)$	$\vee$	$(B$	$\&$	$A)$
T	T		T		**T**			
T	F		F		**F**		F	
F	T		F		**F**		F	
F	F		F		**F**		F	

4. $\neg[A \rightarrow (B \rightarrow A)]$ Contradiction

A	B	$\neg[$	A	$\rightarrow$	$(B$	$\rightarrow$	$A)]$
T	T	**F**		T		T	
T	F	**F**		T		T	
F	T	**F**		T			
F	F	**F**		T			

5. $A \leftrightarrow [A \rightarrow (B \,\&\, \neg B)]$ Contradiction

A	B	A	$\leftrightarrow$	$[A$	$\rightarrow$	$(B$	$\&$	$\neg$	$B)]$
T	T		**F**		F		F	F	
T	F		**F**		F		F		
F	T		**F**		T				
F	F		**F**		T				

6. $\neg(A \,\&\, B) \leftrightarrow A$ Neither

A	B	$\neg$	$(A$	$\&$	$B)$	$\leftrightarrow$	A
T	T	F		T		**F**	
T	F	T		F		**T**	
F	T	T		F		**F**	
F	F	T		F		**F**	

7. $A \rightarrow (B \vee C)$ Neither

A	B	C	A	$\rightarrow$	$(B$	$\vee$	$C)$
T	T	T		**T**		T	
T	T	F		**T**		T	
T	F	T		**T**		T	
T	F	F		**F**		F	
F	T	T		**T**			
F	T	F		**T**			
F	F	T		**T**			
F	F	F		**T**			

8. $(A \,\&\, \neg A) \rightarrow (B \vee C)$ Tautology

A	B	C	$(A$	$\&$	$\neg$	$A)$	$\rightarrow$	$(B$	$\vee$	$C)$
T	T	T		**F**	F		T			
T	T	F		**F**	F		T			
T	F	T		**F**	F		T			
T	F	F		**F**	F		T			
F	T	T		**F**			T			
F	T	F		**F**			T			
F	F	T		**F**			T			
F	F	F		**F**			T			

9. $(B \,\&\, D) \leftrightarrow [A \leftrightarrow (A \vee C)]$ Neither

A	B	C	D	$(B$	$\&$	$D)$	$\leftrightarrow$	$[A$	$\leftrightarrow$	$(A$	$\vee$	$C)]$
T	T	T	T		T		**T**		T		T	
T	T	T	F		F		**F**		T		T	
T	T	F	T		T		**T**		T		T	
T	T	F	F		F		**F**		T		T	
T	F	T	T		F		**F**		T		T	
T	F	T	F		F		**F**		T		T	
T	F	F	T		F		**F**		T		T	
T	F	F	F		F		**F**		T		T	
F	T	T	T		T		**F**		F		T	
F	T	T	F		F		**T**		F		T	
F	T	F	T		T		**T**		T		F	
F	T	F	F		F		**F**		T		F	
F	F	T	T		F		**T**		F		T	
F	F	T	F		F		**T**		F		T	
F	F	F	T		F		**F**		T		F	
F	F	F	F		F		**F**		T		F	

13

Partial truth tables

A. Use complete or partial truth tables (as appropriate) to determine whether these pairs of sentences are logically equivalent:

1. A, $\neg A$ — Not logically equivalent

A	A	$\neg A$
T	T	F

2. A, $A \vee A$ — Logically equivalent

A	A	$A \vee A$
T	T	T
T	T	T

3. $A \rightarrow A$, $A \leftrightarrow A$ — Logically equivalent

A	$A \rightarrow A$	$A \leftrightarrow A$
T	T	T
F	T	T

4. $A \vee \neg B$, $A \rightarrow B$ — Not logically equivalent

A	B	$A \vee \neg B$	$A \rightarrow B$
T	F	T	F

5. $A \,\&\, \neg A$, $\neg B \leftrightarrow B$ — Logically equivalent

A	B	A	$\&$	$\neg$	A	$\neg$	B	$\leftrightarrow$	B
T	T		**F**	F		F		**F**	
T	F		**F**	F		T		**F**	
F	T		**F**			F		**F**	
F	F		**F**			T		**F**	

6. $\neg(A \& B)$, $\neg A \vee \neg B$ Logically equivalent

A	B	$\neg$	$(A \& B)$	$\neg A$	$\vee$	$\neg B$
T	T	**F**	T	F	**F**	F
T	F	**T**	F	F	**T**	T
F	T	**T**	F	T	**T**	F
F	F	**T**	F	T	**T**	T

7. $\neg(A \rightarrow B)$, $\neg A \rightarrow \neg B$ Not logically equivalent

A	B	$\neg$	$(A \rightarrow B)$	$\neg A$	$\rightarrow$	$\neg B$
T	T	**F**	T	F	**T**	F

8. $(A \rightarrow B)$, $(\neg B \rightarrow \neg A)$ Logically equivalent

A	B	$(A \rightarrow B)$	$(\neg B$	$\rightarrow$	$\neg A)$
T	T	T	F	**T**	
T	F	F	T	**F**	F
F	T	T	F	**T**	
F	F	T	T	**T**	T

B. Use complete or partial truth tables (as appropriate) to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \& B$, $C \rightarrow \neg B$, C Jointly inconsistent

A	B	C	$A \& B$	$C \rightarrow$	$\neg B$	C
T	T	T	T	**F**	F	T
T	T	F	T	**T**		F
T	F	T	F	**T**	T	T
T	F	F	F	**T**		F
F	T	T	F	**F**	F	T
F	T	F	F	**T**		F
F	F	T	F	**T**	T	T
F	F	F	F	**T**		F

2. $A \rightarrow B$, $B \rightarrow C$, A, $\neg C$ Jointly inconsistent

A	B	C	$A \rightarrow B$	$B \rightarrow C$	A	$\neg C$
T	T	T	T	T	T	F
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

3. $A \vee B$, $B \vee C$, $C \rightarrow \neg A$ Jointly consistent

A	B	C	$A \vee B$	$B \vee C$	$C \rightarrow \neg A$
F	T	T	T	T	**T** T

4. A, B, C, $\neg D$, $\neg E$, F Jointly consistent

A	B	C	D	E	F	A	B	C	$\neg D$	$\neg E$	F
T	T	T	F	F	T	T	T	T	T	T	T

C. Use complete or partial truth tables (as appropriate) to determine whether each argument is valid or invalid:

1. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \therefore A$ Invalid

A	$A \vee \big[A \rightarrow (A \leftrightarrow A)\big]$	A
F	**T** T	F

2. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$ Invalid

A	B	$A \leftrightarrow \neg\,(B \leftrightarrow A)$	A
F	F	**T** F T	F

3. $A \rightarrow B, B \therefore A$ Invalid

A	B	$A \rightarrow B$	B	A
F	T	T	T	F

4. $A \vee B, B \vee C, \neg B \therefore A \,\&\, C$ Valid

A	B	C	$A \vee B$	$B \vee C$	$\neg B$	$A \& C$
T	T	T				T
T	T	F			F	F
T	F	T				T
T	F	F	T	F	T	F
F	T	T			F	F
F	T	F			F	F
F	F	T	F		T	F
F	F	F	F		T	F

5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$ Valid

A	B	C	$A \leftrightarrow B$	$B \leftrightarrow C$	$A \leftrightarrow C$
T	T	T			T
T	T	F	T	F	F
T	F	T			T
T	F	F	F		F
F	T	T	F		F
F	T	F			T
F	F	T	T	F	F
F	F	F			T

D. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $A \rightarrow \neg A$ Contingent

A	A	$\rightarrow$	$\neg$	A
T	*T*	**F**	*F*	*T*
F	*F*	**T**	*T*	*F*

2. $A \rightarrow (A \& (A \vee B))$ Tautology

A	B	A	$\rightarrow$	$(A$	$\&$	$(A$	$\vee$	$B))$
T	*T*	*T*	**T**	*T*	*T*	*T*	*T*	*T*
T	*F*	*T*	**T**	*T*	*T*	*T*	*T*	*F*
F	*T*	*F*	**T**	*F*	*F*	*F*	*T*	*T*
F	*F*	*F*	**T**	*F*	*F*	*F*	*F*	*F*

3. $(A \to B) \leftrightarrow (B \to A)$ Contingent

A	B	$(A$	$\to$	$B)$	$\leftrightarrow$	$(B$	$\to$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**F**	F	T	T
F	T	F	T	T	**F**	T	F	F
F	F	F	T	F	**T**	F	T	F

4. $A \to \neg(A \,\&\, (A \vee B))$ Contingent

A	B	A	$\to$	$\neg$	$(A$	$\&$	$(A$	$\vee$	$B))$
T	T	T	**F**	F	T	T	T	T	T
T	F	T	**F**	F	T	T	T	T	F
F	T	F	**T**	T	F	F	F	T	T
F	F	F	**T**	T	F	F	F	F	F

5. $\neg B \to [(\neg A \,\&\, A) \vee B]$ Contingent

A	B	$\neg$	B	$\to$	$((\neg$	A	$\&$	$A)$	$\vee$	$B)$
T	T	F	T	**T**	F	T	F	T	T	T
T	F	T	F	**F**	F	T	F	T	F	F
F	T	F	T	**T**	T	F	F	F	T	T
F	F	T	F	**F**	T	F	F	F	F	F

6. $\neg(A \vee B) \leftrightarrow (\neg A \,\&\, \neg B)$ Tautology

A	B	$\neg$	$(A$	$\vee$	$B)$	$\leftrightarrow$	$(\neg$	A	$\&$	$\neg$	$B)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

7. $[(A \,\&\, B) \,\&\, C] \to B$ Tautology

A	B	C	$((A$	$\&$	$B)$	$\&$	$C)$	$\to$	B
T	T	T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	F	F	**T**	T
T	F	T	T	F	F	F	T	**T**	F
T	F	F	T	F	F	F	F	**T**	F
F	T	T	F	F	T	F	T	**T**	T
F	T	F	F	F	T	F	F	**T**	T
F	F	T	F	F	F	F	T	**T**	F
F	F	F	F	F	F	F	F	**T**	F

8. $\neg[(C \vee A) \vee B]$ Contingent

A	B	C	$\neg$	$((C$	$\vee$	$A)$	$\vee$	$B)$
T	T	T	**F**	T	T	T	T	T
T	T	F	**F**	F	T	T	T	T
T	F	T	**F**	T	T	T	T	F
T	F	F	**F**	F	T	T	T	F
F	T	T	**F**	T	T	F	T	T
F	T	F	**F**	F	F	F	T	T
F	F	T	**F**	T	T	F	T	F
F	F	F	**T**	F	F	F	F	F

9. $[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)] \,\&\, C$ Contradiction

A	B	C	$((A$	$\&$	$B)$	$\&$	$\neg$	$(A$	$\&$	$B))$	$\&$	C
T	T	T	T	T	T	F	F	T	T	T	**F**	T
T	T	F	T	T	T	F	F	T	T	T	**F**	F
T	F	T	T	F	F	F	T	T	F	F	**F**	T
T	F	F	T	F	F	F	T	T	F	F	**F**	F
F	T	T	F	F	T	F	T	F	F	T	**F**	T
F	T	F	F	F	T	F	T	F	F	T	**F**	F
F	F	T	F	F	F	F	T	F	F	F	**F**	T
F	F	F	F	F	F	F	T	F	F	F	**F**	F

10. $(A \,\&\, B)] \rightarrow [(A \,\&\, C) \vee (B \,\&\, D)]$ Contingent

A	B	C	D	$((A$	$\&$	$B))$	$\rightarrow$	$((A$	$\&$	$C)$	$\vee$	$(B$	$\&$	$D))$
T	T	T	T	T	T	T	**T**	T	T	T	T	T	T	T
T	T	F	F	T	T	T	**F**	T	F	F	F	T	F	F

E. Determine whether each sentence is a tautology, a contradiction, or a contingent sentence. Justify your answer with a complete or partial truth table where appropriate.

1. $\neg(A \vee A)$ Contradiction
2. $(A \rightarrow B) \vee (B \rightarrow A)$ Tautology
3. $[(A \rightarrow B) \rightarrow A] \rightarrow A$ Tautology
4. $\neg[(A \rightarrow B) \vee (B \rightarrow A)]$ Contradiction
5. $(A \,\&\, B) \vee (A \vee B)$ Contingent

6. $\neg(A \,\&\, B) \leftrightarrow A$ Contingent
7. $A \rightarrow (B \vee C)$ Contingent
8. $(A \,\&\, \neg A) \rightarrow (B \vee C)$ Tautology
9. $(B \,\&\, D) \leftrightarrow [A \leftrightarrow (A \vee C)]$ Contingent
10. $\neg[(A \rightarrow B) \vee (C \rightarrow D)]$ Contingent

F. Determine whether each the following pairs of sentences are logically equivalent using complete truth tables. If the two sentences really are logically equivalent, write "equivalent." Otherwise write, "not equivalent."

1. A and $A \vee A$
2. A and $A \,\&\, A$
3. $A \vee \neg B$ and $A \rightarrow B$
4. $(A \rightarrow B)$ and $(\neg B \rightarrow \neg A)$
5. $\neg(A \,\&\, B)$ and $\neg A \vee \neg B$
6. $((U \rightarrow (X \vee X)) \vee U)$ and $\neg(X \,\&\, (X \,\&\, U))$
7. $((C \,\&\, (N \leftrightarrow C)) \leftrightarrow C)$ and $(\neg\neg\neg N \rightarrow C)$
8. $[(A \vee B) \,\&\, C]$ and $[A \vee (B \,\&\, C)]$
9. $((L \,\&\, C) \,\&\, I)$ and $L \vee C$

G. Determine whether each collection of sentences is jointly consistent or jointly inconsistent. Justify your answer with a complete or partial truth table where appropriate.

1. $A \rightarrow A$, $\neg A \rightarrow \neg A$, $A \,\&\, A$, $A \vee A$ Consistent
2. $A \rightarrow \neg A$, $\neg A \rightarrow A$ Inconsistent
3. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$ Consistent
4. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$, $\neg C$ Inconsistent
5. $B \,\&\, (C \vee A)$, $A \rightarrow B$, $\neg(B \vee C)$ Inconsistent
6. $(A \leftrightarrow B) \rightarrow B$, $B \rightarrow \neg(A \leftrightarrow B)$, $A \vee B$ Consistent
7. $A \leftrightarrow (B \vee C)$, $C \rightarrow \neg A$, $A \rightarrow \neg B$ Consistent
8. $A \leftrightarrow B$, $\neg B \vee \neg A$, $A \rightarrow B$ Consistent
9. $A \leftrightarrow B$, $A \rightarrow C$, $B \rightarrow D$, $\neg(C \vee D)$ Consistent
10. $\neg(A \,\&\, \neg B)$, $B \rightarrow \neg A$, $\neg B$ Consistent

H. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1. $A \rightarrow (A \,\&\, \neg A) \therefore \neg A$ Valid
2. $A \vee B$, $A \rightarrow B$, $B \rightarrow A \therefore A \leftrightarrow B$ Valid

3. $A \vee (B \rightarrow A) \therefore \neg A \rightarrow \neg B$ Valid

4. $A \vee B, A \rightarrow B, B \rightarrow A \therefore A \& B$ Valid

5. $(B \& A) \rightarrow C, (C \& A) \rightarrow B \therefore (C \& B) \rightarrow A$ Invalid

6. $\neg(\neg A \vee \neg B), A \rightarrow \neg C \therefore A \rightarrow (B \rightarrow C)$ Invalid

7. $A \& (B \rightarrow C), \neg C \& (\neg B \rightarrow \neg A) \therefore C \& \neg C$ Valid

8. $A \& B, \neg A \rightarrow \neg C, B \rightarrow \neg D \therefore A \vee B$ Invalid

9. $A \rightarrow B \therefore (A \& B) \vee (\neg A \& \neg B)$ Invalid

10. $\neg A \rightarrow B, \neg B \rightarrow C, \neg C \rightarrow A \therefore \neg A \rightarrow (\neg B \vee \neg C)$ Invalid

I. Determine whether each argument is valid or invalid. Justify your answer with a complete or partial truth table where appropriate.

1. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$ Invalid

2. $A \vee B, B \vee C, \neg A \therefore B \& C$ Invalid

3. $A \rightarrow C, E \rightarrow (D \vee B), B \rightarrow \neg D \therefore (A \vee C) \vee (B \rightarrow (E \& D))$ Invalid

4. $A \vee B, C \rightarrow A, C \rightarrow B \therefore A \rightarrow (B \rightarrow C)$ Invalid

5. $A \rightarrow B, \neg B \vee A \therefore A \leftrightarrow B$ Valid

15

Basic rules for TFL

A. The following two 'proofs' are *incorrect*. Explain the mistakes they make.

1	$(\neg L \wedge A) \vee L$	
2	$\quad \neg L \,\&\, A$	
3	$\quad \neg L$	:&E 3
4	$\quad A$	:&E 1
5	$\quad L$	
6	$\quad \bot$	¬E 5, 3
7	$\quad A$	X 6
8	A	:∨E 1, 2–4, 5–7

&E on line 4 can't be applied to line 1, since it is not of the form $\mathsf{A} \,\&\, \mathsf{B}$. '$A$' could be obtained by &E, but from line 2.

⊥I on line 5 illicitly refers to a line from a closed subproof (line 3).

1	$A \,\&\, (B \,\&\, C)$	
2	$(B \vee C) \to D$	
3	B	:&E 1
4	$B \vee C$	:∨I 3
5	D	:→E 4, 2

&E on line 3 should yield '$B \,\&\, C$'. 'B' could then be obtained by &E again.

The citation for line 5 is the wrong way round: it should be '→E 2, 4'.

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into *bona fide* proofs. Additionally, write down the argument that corresponds to each proof.

1	$P \& S$	
2	$S \to R$	
3	P	:&E 1
4	S	:&E 1
5	R	:$\to$E 2, 4
6	$R \vee E$	:$\vee$I 5

Corresponding argument:
$P \& S, S \to R \therefore R \vee E$

1	$A \to D$	
2	$\quad A \& B$	
3	$\quad A$	:&E 2
4	$\quad D$	:$\to$E 1, 3
5	$\quad D \vee E$	:$\vee$I 4
6	$(A \& B) \to (D \vee E)$	:$\to$I 2–5

Corresponding argument:
$A \to D \therefore (A \& B) \to (D \vee E)$

1	$\neg L \to (J \vee L)$	
2	$\neg L$	
3	$J \vee L$	:$\to$E 1, 2
4	$\quad J$	
5	$\quad J \& J$	:&I 4, 4
6	$\quad J$	:&E 5
7	$\quad L$	
8	$\quad \bot$	$\neg$E 7, 2
9	$\quad J$	X 8
10	J	:$\vee$E 3, 4–6, 7–9

Corresponding argument:
$\neg L \to (J \vee L), \neg L \therefore J$

C. Give a proof for each of the following arguments:

1. $J \to \neg J \therefore \neg J$

1	$J \to \neg J$	
2	$\quad J$	
3	$\quad \neg J$	:$\to$E 1, 2
4	$\quad \bot$	$\neg$E 2, 3
5	$\neg J$	:$\neg$I 2–4

2. $Q \to (Q \& \neg Q) \therefore \neg Q$

1	$Q \to (Q \& \neg Q)$	
2	$\quad Q$	
3	$\quad Q \& \neg Q$	:$\to$E 1, 2
4	$\quad \neg Q$	:&E 3
5	$\quad \bot$	$\neg$E 2, 4
6	$\neg Q$	:$\neg$I 2–5

3. $A \to (B \to C) \therefore (A \,\&\, B) \to C$

1	$A \to (B \to C)$	
2	$\quad A \,\&\, B$	
3	$\quad A$	:&E 2
4	$\quad B \to C$	:→E 1, 3
5	$\quad B$	:&E 2
6	$\quad C$	:→E 4, 5
7	$(A \,\&\, B) \to C$	:→I 2–6

4. $K \,\&\, L \therefore K \leftrightarrow L$

1	$K \,\&\, L$	
2	$\quad K$	
3	$\quad L$	:&E 1
4	$\quad L$	
5	$\quad K$	:&E 1
6	$K \leftrightarrow L$	:↔I 2–3, 4–5

5. $(C \,\&\, D) \vee E \therefore E \vee D$

1	$(C \,\&\, D) \vee E$	
2	$\quad C \,\&\, D$	
3	$\quad D$	:&E 2
4	$\quad E \vee D$	:∨I 3
5	$\quad E$	
6	$\quad E \vee D$	:∨I 5
7	$E \vee D$	:∨E 1, 2–4, 5–6

6. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

1	$A \leftrightarrow B$	
2	$B \leftrightarrow C$	
3	$\quad A$	
4	$\quad B$	$:\leftrightarrow$E 1, 3
5	$\quad C$	$:\leftrightarrow$E 2, 4
6	$\quad C$	
7	$\quad B$	$:\leftrightarrow$E 2, 6
8	$\quad A$	$:\leftrightarrow$E 1, 7
9	$A \leftrightarrow C$	$:\leftrightarrow$I 3–5, 6–8

7. $\neg F \rightarrow G, F \rightarrow H \therefore G \vee H$

1	$\neg F \rightarrow G$	
2	$F \rightarrow H$	
3	$\quad \neg(G \vee H)$	
4	$\quad\quad F$	
5	$\quad\quad H$	$:\rightarrow$E 2, 4
6	$\quad\quad G \vee H$	$:\vee$I 5
7	$\quad\quad \bot$	$\neg$E 3, 6
8	$\quad \neg F$	$:\neg$I 4–7
9	$\quad G$	$:\rightarrow$E 1, 8
10	$\quad G \vee H$	$:\vee$I 9
11	$\quad \bot$	$\neg$E 3, 10
12	$G \vee H$	IP 3–11

8. $(Z \,\&\, K) \vee (K \,\&\, M), K \rightarrow D \therefore D$

1	$(Z \& K) \vee (K \& M)$	
2	$K \rightarrow D$	
3	$\quad Z \& K$	
4	$\quad K$	:&E 3
5	$\quad K \& M$	
6	$\quad K$	:&E 5
7	K	:∨E 1, 3–4, 5–6
8	D	:→E 2, 7

9. $P \& (Q \vee R), P \rightarrow \neg R \therefore Q \vee E$

1	$P \& (Q \vee R)$	
2	$P \rightarrow \neg R$	
3	P	:&E 1
4	$\neg R$	:→E 2, 3
5	$Q \vee R$	:&E 1
6	$\quad Q$	
7	$\quad Q \vee E$	:∨I 6
8	$\quad R$	
9	$\quad \bot$	¬E 8, 4
10	$\quad Q \vee E$	X 9
11	$Q \vee E$	:∨E 5, 6–7, 8–10

10. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$

1	$\mid S \leftrightarrow T$	
2	$\mid \quad \mid S$	
3	$\mid \quad \mid T$	:$\leftrightarrow$E 1, 2
4	$\mid \quad \mid T \vee S$	:$\vee$I 3
5	$\mid \quad \mid T \vee S$	
6	$\mid \quad \mid \quad \mid T$	
7	$\mid \quad \mid \quad \mid S$	:$\leftrightarrow$E 1, 6
8	$\mid \quad \mid \quad \mid S$	
9	$\mid \quad \mid \quad \mid S \,\&\, S$	:&I 8, 8
10	$\mid \quad \mid \quad \mid S$	:&E 9
11	$\mid \quad \mid S$	:$\vee$E 5, 6–7, 8–10
12	$\mid S \leftrightarrow (T \vee S)$	:$\leftrightarrow$I 2–4, 5–11

11. $\neg(P \rightarrow Q) \therefore \neg Q$

1	$\mid \neg(P \rightarrow Q)$	
2	$\mid \quad \mid Q$	
3	$\mid \quad \mid \quad \mid P$	
4	$\mid \quad \mid \quad \mid Q \,\&\, Q$	:&I 2, 2
5	$\mid \quad \mid \quad \mid Q$	:&E 4
6	$\mid \quad \mid P \rightarrow Q$	:$\rightarrow$I 3–5
7	$\mid \quad \mid \bot$	$\neg$E 6, 1
8	$\mid \neg Q$	:$\neg$I 2–7

12. $\neg(P \rightarrow Q) \therefore P$

$$
\begin{array}{rll}
1 & \vert\ \underline{\neg(P \to Q)} & \\
2 & \vert\ \ \vert\ \underline{\neg P} & \\
3 & \vert\ \ \vert\ \ \vert\ \underline{P} & \\
4 & \vert\ \ \vert\ \ \vert\ \bot & \neg\mathrm{E}\ 3, 2 \\
5 & \vert\ \ \vert\ \ \vert\ Q & \mathrm{X}\ 4 \\
6 & \vert\ \ \vert\ P \to Q & :\!\to\!\mathrm{I}\ 3\text{–}5 \\
7 & \vert\ \ \vert\ \bot & \neg\mathrm{E}\ 6, 1 \\
8 & \vert\ P & \mathrm{IP}\ 2\text{–}7
\end{array}
$$

16

Proof-theoretic concepts

A. Show that each of the following sentences is a theorem:

1. $O \to O$

$$\begin{array}{rlll}
1 & \vert & \quad \vert\ O & \\
2 & \vert & \quad \vert\ O & \text{R 1} \\
3 & \vert & O \to O & :\!\to\!\text{I 1–2}
\end{array}$$

2. $N \vee \neg N$

$$\begin{array}{rlll}
1 & \vert & \quad \vert\ \neg(N \vee \neg N) & \\
2 & \vert & \quad \vert \quad \vert\ N & \\
3 & \vert & \quad \vert \quad \vert\ N \vee \neg N & :\!\vee\text{I 2} \\
4 & \vert & \quad \vert \quad \vert\ \bot & \neg\text{E 1, 3} \\
5 & \vert & \quad \vert\ \neg N & :\!\neg\text{I 2–4} \\
6 & \vert & \quad \vert\ N \vee \neg N & :\!\vee\text{I 5} \\
7 & \vert & \quad \vert\ \bot & \neg\text{E 1, 6} \\
8 & \vert & N \vee \neg N & \text{IP 1–7}
\end{array}$$

3. $J \leftrightarrow [J \vee (L \,\&\, \neg L)]$

1	J	
2	$J \vee (L \,\&\, \neg L)$	:$\vee$I 1
3	$J \vee (L \,\&\, \neg L)$	
4	$L \,\&\, \neg L$	
5	L	:&E 4
6	$\neg L$	:&E 4
7	$\bot$	$\neg$E 5, 6
8	$\neg(L \,\&\, \neg L)$	:$\neg$I 4–7
9	J	DS 3, 8
10	$J \leftrightarrow [J \vee (L \,\&\, \neg L)]$	:$\leftrightarrow$I 1–2, 3–9

4. $((A \rightarrow B) \rightarrow A) \rightarrow A$

1	$(A \rightarrow B) \rightarrow A$	
2	$\neg A$	
3	$\neg(A \rightarrow B)$	MT 1, 2
4	A	
5	$\bot$	$\neg$E 4, 2
6	B	X 5
7	$A \rightarrow B$	:$\rightarrow$I 4–6
8	$\bot$	$\neg$E 7, 3
9	A	IP 1–8
10	$((A \rightarrow B) \rightarrow A) \rightarrow A$	:$\rightarrow$I 1–9

B. Provide proofs to show each of the following:

1. $C \rightarrow (E \,\&\, G), \neg C \rightarrow G \vdash G$

1	$C \to (E \,\&\, G)$	
2	$\neg C \to G$	
3	$\quad \neg G$	
4	$\quad\quad C$	
5	$\quad\quad E \,\&\, G$	:$\to$E 1, 4
6	$\quad\quad G$	:&E 5
7	$\quad\quad \bot$	$\neg$E 3, 6
8	$\quad \neg C$	:$\neg$I 4–7
9	$\quad G$	:$\to$E 2, 8
10	$\quad \bot$	$\neg$E 3, 9
11	G	IP 3–10

2. $M \,\&\, (\neg N \to \neg M) \vdash (N \,\&\, M) \lor \neg M$

1	$M \,\&\, (\neg N \to \neg M)$	
2	M	:&E 1
3	$\neg N \to \neg M$	:&E 1
4	$\quad \neg N$	
5	$\quad \neg M$	:$\to$E 3, 4
6	$\quad \bot$	$\neg$E 2, 5
7	N	IP 4–6
8	$N \,\&\, M$	:&I 7, 2
9	$(N \,\&\, M) \lor \neg M$	:$\lor$I 8

3. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \to M) \vdash Y \to Z$

1	$(Z \,\&\, K) \leftrightarrow (Y \,\&\, M)$	
2	$D \,\&\, (D \rightarrow M)$	
3	D	:&E 2
4	$D \rightarrow M$	:&E 2
5	M	:→E 4, 3
6	$\quad Y$	
7	$\quad Y \,\&\, M$	:&I 6, 5
8	$\quad Z \,\&\, K$	:↔E 1, 7
9	$\quad Z$	:&E 8
10	$Y \rightarrow Z$	:→I 6–9

4. $(W \vee X) \vee (Y \vee Z), X \rightarrow Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \rightarrow Y$	
3	$\neg Z$	
4	$\quad W \vee X$	
5	$\quad\quad W$	
6	$\quad\quad W \vee Y$	:∨I 5
7	$\quad\quad X$	
8	$\quad\quad Y$	:→E 2, 7
9	$\quad\quad W \vee Y$	:∨I 8
10	$\quad W \vee Y$	:∨E 4, 5–6, 7–9
11	$\quad Y \vee Z$	
12	$\quad Y$	DS 11, 3
13	$\quad W \vee Y$	:∨I 12
14	$W \vee Y$	:∨E 1, 4–10, 11–13

C. Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E$, $E \leftrightarrow R$

1	$R \leftrightarrow E$	
2	$\quad E$	
3	$\quad R$	:$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	:$\leftrightarrow$E 1, 4
6	$E \leftrightarrow R$	:$\leftrightarrow$I 2–3, 4–5

1	$E \leftrightarrow R$	
2	$\quad E$	
3	$\quad R$	:$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	:$\leftrightarrow$E 1, 4
6	$R \leftrightarrow E$	:$\leftrightarrow$I 4–5, 2–3

2. G, $\neg\neg\neg\neg G$

1	G	
2	$\quad \neg\neg\neg G$	
3	$\quad \neg G$	DNE 2
4	$\quad \bot$	$\neg$E 1, 3
5	$\neg\neg\neg\neg G$	:$\neg$I 2–4

1	$\neg\neg\neg\neg G$	
2	$\neg\neg G$	DNE 1
3	G	DNE 2

3. $T \rightarrow S$, $\neg S \rightarrow \neg T$

1	$T \rightarrow S$	
2	$\quad \neg S$	
3	$\quad \neg T$	MT 1, 2
4	$\neg S \rightarrow \neg T$	:$\rightarrow$I 2–3

1	$\neg S \rightarrow \neg T$	
2	$\quad T$	
3	$\quad\quad \neg S$	
4	$\quad\quad \neg T$	:$\rightarrow$E 1, 3
5	$\quad\quad \bot$	$\neg$E 2, 4
6	$\quad \neg\neg S$	:$\neg$I 3–5
7	$\quad S$	DNE 6
8	$T \rightarrow S$	:$\rightarrow$I 2–7

4. $U \rightarrow I$, $\neg(U \& \neg I)$

1	$U \rightarrow I$	
2	$\quad U \& \neg I$	
3	$\quad U$	:&E 2
4	$\quad \neg I$	:&E 2
5	$\quad I$	:$\rightarrow$E 1, 3
6	$\quad \bot$	$\neg$E 5, 4
7	$\neg(U \& \neg I)$	:$\neg$I 2–6

1	$\neg(U \mathbin{\&} \neg I)$	
2	U	
3	$\neg I$	
4	$U \mathbin{\&} \neg I$	:&I 2, 3
5	$\bot$	¬E 4, 1
6	$\neg\neg I$	:¬I 3–5
7	I	DNE 6
8	$U \to I$	:→I 2–7

5. $\neg(C \to D), C \,\&\, \neg D$

1	$C \,\&\, \neg D$	
2	C	:&E 1
3	$\neg D$	:&E 1
4	$\quad C \to D$	
5	$\quad D$	:→E 4, 2
6	$\quad \bot$	¬E 5, 3
7	$\neg(C \to D)$	:¬I 4–6

1	$\neg(C \to D)$	
2	$\quad D$	
3	$\quad\quad C$	
4	$\quad\quad D$	R 2
5	$\quad C \to D$	:→I 3–4
6	$\quad \bot$	¬E 5, 1
7	$\neg D$	:¬I 2–6
8	$\quad \neg C$	
9	$\quad\quad C$	
10	$\quad\quad \bot$	¬E 9, 8
11	$\quad\quad D$	X 10
12	$\quad C \to D$	:→I 9–11
13	$\quad \bot$	¬E 12, 1
14	$\neg\neg C$	:¬I 8–13
15	C	DNE 14
16	$C \,\&\, \neg D$	:&I 15, 7

6. $\neg G \leftrightarrow H, \neg(G \leftrightarrow H)$

1	$\neg G \leftrightarrow H$	
2	$\quad G \leftrightarrow H$	
3	$\quad\quad G$	
4	$\quad\quad H$	:↔E 2, 3
5	$\quad\quad \neg G$	:↔E 1, 4
6	$\quad\quad \bot$	¬E 3, 5
7	$\quad\quad \neg G$	
8	$\quad\quad H$	:↔E 1, 7
9	$\quad\quad G$	:↔E 2, 8
10	$\quad\quad \bot$	¬E 9, 7
11	$\quad \bot$	LEM 3–6, 7–10
12	$\neg(G \leftrightarrow H)$	:¬I 2–11

1	$\vert\ \neg(G \leftrightarrow H)$	
2	$\vert\ \vert\ \neg G$	
3	$\vert\ \vert\ \vert\ \neg H$	
4	$\vert\ \vert\ \vert\ \vert\ G$	
5	$\vert\ \vert\ \vert\ \vert\ \bot$	$\neg$E 4, 2
6	$\vert\ \vert\ \vert\ \vert\ H$	X 5
7	$\vert\ \vert\ \vert\ \vert\ H$	
8	$\vert\ \vert\ \vert\ \vert\ \bot$	$\neg$E 7, 3
9	$\vert\ \vert\ \vert\ \vert\ G$	X 8
10	$\vert\ \vert\ \vert\ G \leftrightarrow H$	:$\leftrightarrow$I 4–6, 7–9
11	$\vert\ \vert\ \vert\ \bot$	$\neg$E 10, 1
12	$\vert\ \vert\ H$	IP 3–11
13	$\vert\ \vert\ H$	
14	$\vert\ \vert\ \vert\ G$	
15	$\vert\ \vert\ \vert\ \vert\ G$	
16	$\vert\ \vert\ \vert\ \vert\ H$	R 13
17	$\vert\ \vert\ \vert\ \vert\ H$	
18	$\vert\ \vert\ \vert\ \vert\ G$	R 14
19	$\vert\ \vert\ \vert\ G \leftrightarrow H$	:$\leftrightarrow$I 15–16, 17–18
20	$\vert\ \vert\ \vert\ \bot$	$\neg$E 19, 1
21	$\vert\ \vert\ \neg G$	:$\neg$I 14–20
22	$\vert\ \neg G \leftrightarrow H$	:$\leftrightarrow$I 2–12, 13–21

D. If you know that $A \vdash B$, what can you say about $(A \,\&\, C) \vdash B$? What about $(A \vee C) \vdash B$? Explain your answers.

If $A \vdash B$, then $(A \,\&\, C) \vdash B$. After all, if $A \vdash B$, then there is some proof with assumption A that ends with B, and no undischarged assumptions other than A. Now, if we start a proof with assumption $(A \,\&\, C)$, we can obtain A by &E. We can now copy and paste the original proof of B from A, adding 1 to every line number and line number citation. The result will be a proof of B from assumption A.

However, we cannot prove much from $(A \vee C)$. After all, it might be impossible to prove B from C.

E. In this chapter, we claimed that it is just as hard to show that two sentences are not provably equivalent, as it is to show that a sentence is not a theorem. Why did we claim this? (*Hint*: think of a sentence that would be a theorem iff A and B were provably equivalent.)

Consider the sentence $A \leftrightarrow B$. Suppose we can show that this is a theorem. So we can prove it, with no assumptions, in m lines, say. Then if we assume A and copy and paste the proof of $A \leftrightarrow B$ (changing the line numbering), we will have a deduction of this shape:

$$
\begin{array}{l|l l}
1 & \underline{A} & \\
m+1 & A \leftrightarrow B & \\
m+2 & B & :\leftrightarrow\mathrm{E}\ m+1,\ 1
\end{array}
$$

This will show that $A \vdash B$. In exactly the same way, we can show that $B \vdash A$. So if we can show that $A \leftrightarrow B$ is a theorem, we can show that A and B are provably equivalent.

Conversely, suppose we can show that A and B are provably equivalent. Then we can prove B from the assumption of A in m lines, say, and prove A from the assumption of B in n lines, say. Copying and pasting these proofs together (changing the line numbering where appropriate), we obtain:

$$
\begin{array}{l|l l}
1 & \quad\vert\ \underline{A} & \\
m & \quad\vert\ B & \\
m+1 & \quad\vert\ \underline{B} & \\
m+n & \quad\vert\ A & \\
m+n+1 & A \leftrightarrow B & :\leftrightarrow\mathrm{I}\ 1\text{–}m,\ m+1\text{–}m+n
\end{array}
$$

Thus showing that $A \leftrightarrow B$ is a theorem.

There was nothing special about A and B in this. So what this shows is that the problem of showing that two sentences are provably equivalent is, essentially, the same problem as showing that a certain kind of sentence (a biconditional) is a theorem.

19

Additional rules for TFL

A. The following proofs are missing their citations (rule and line numbers). Add them wherever they are required:

1	$W \to \neg B$	
2	$A \,\&\, W$	
3	$B \vee (J \,\&\, K)$	
4	W	:&E 2
5	$\neg B$	:→E 1, 4
6	$J \,\&\, K$	DS 3, 5
7	K	:&E 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\quad \neg L$	
4	$\quad \neg O$	DS 2, 3
5	$\quad L$	:↔E 1, 4
6	$\quad \bot$	¬E 5, 3
7	$\neg\neg L$	:¬I 3–6
8	L	DNE 7

1	$Z \to (C \,\&\, \neg N)$	
2	$\neg Z \to (N \,\&\, \neg C)$	
3	$\quad \neg(N \vee C)$	
4	$\quad \neg N \,\&\, \neg C$	DeM 3
5	$\quad \neg N$	:&E 4
6	$\quad \neg C$	:&E 4
7	$\quad\quad Z$	
8	$\quad\quad C \,\&\, \neg N$	:→E 1, 7
9	$\quad\quad C$	:&E 8
10	$\quad\quad \bot$	¬E 9, 6
11	$\quad \neg Z$	:¬I 7–10
12	$\quad N \,\&\, \neg C$	:→E 2, 11
13	$\quad N$	:&E 12
14	$\quad \bot$	¬E 13, 5
15	$\neg\neg(N \vee C)$	:¬I 3–14
16	$N \vee C$	DNE 15

B. Give a proof for each of these arguments:

1. $E \vee F, F \vee G, \neg F \therefore E \,\&\, G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	E	DS 1, 3
5	G	DS 2, 3
6	$E \,\&\, G$	:&I 4, 5

2. $M \vee (N \rightarrow M) \therefore \neg M \rightarrow \neg N$

1	$M \vee (N \rightarrow M)$	
2	$\quad \neg M$	
3	$\quad N \rightarrow M$	DS 1, 2
4	$\quad \neg N$	MT 3, 2
5	$\neg M \rightarrow \neg N$	:→I 2–4

3. $(M \vee N) \,\&\, (O \vee P), N \rightarrow P, \neg P \therefore M \,\&\, O$

1	$(M \vee N) \,\&\, (O \vee P)$	
2	$N \rightarrow P$	
3	$\neg P$	
4	$\neg N$	MT 2, 3
5	$M \vee N$	:&E 1
6	M	DS 5, 4
7	$O \vee P$	:&E 1
8	O	DS 7, 3
9	$M \,\&\, O$	:&I 6, 8

4. $(X \,\&\, Y) \vee (X \,\&\, Z), \neg(X \,\&\, D), D \vee M \therefore M$

1	$(X \,\&\, Y) \vee (X \,\&\, Z)$	
2	$\neg(X \,\&\, D)$	
3	$D \vee M$	
4	$\quad X \,\&\, Y$	
5	$\quad X$	:&E 4
6	$\quad X \,\&\, Z$	
7	$\quad X$	:&E 6
8	X	:∨E 1, 4–5, 6–7
9	$\quad D$	
10	$\quad X \,\&\, D$	:&I 8, 9
11	$\quad \bot$	¬E 10, 2
12	$\neg D$	:¬I 9–11
13	M	DS 3, 12

20

Soundness and Completeness

Practice exercises

A. Use either a derivation or a truth table for each of the following.

1. Show that $A \rightarrow [((B \,\&\, C) \vee D) \rightarrow A]$ is a tautology.
2. Show that $A \rightarrow (A \rightarrow B)$ is not a tautology
3. Show that the sentence $A \rightarrow \neg A$ is not a contradiction.
4. Show that the sentence $A \leftrightarrow \neg A$ is a contradiction.
5. Show that the sentence $\neg(W \rightarrow (J \vee J))$ is contingent
6. Show that the sentence $\neg(X \vee (Y \vee Z)) \vee (X \vee (Y \vee Z))$ is not contingent
7. Show that the sentence $B \rightarrow \neg S$ is equivalent to the sentence $\neg\neg B \rightarrow \neg S$
8. Show that the sentence $\neg(X \vee O)$ is not equivalent to the sentence $X \,\&\, O$
9. Show that the sentences $\neg(A \vee B)$, C, $C \rightarrow A$ are jointly inconsistent.
10. Show that the sentences $\neg(A \vee B)$, $\neg B$, $B \rightarrow A$ are jointly consistent
11. Show that $\neg(A \vee (B \vee C)) \therefore \neg C$ is valid.
12. Show that $\neg(A \,\&\, (B \vee C)) \therefore \neg C$ is invalid.

B. Use either a derivation or a truth table for each of the following.

1. Show that $A \rightarrow (B \rightarrow A)$ is a tautology
2. Show that $\neg(((N \leftrightarrow Q) \vee Q) \vee N)$ is not a tautology

3. Show that $Z \vee (\neg Z \leftrightarrow Z)$ is contingent
4. show that $(L \leftrightarrow ((N \rightarrow N) \rightarrow L)) \vee H$ is not contingent
5. Show that $(A \leftrightarrow A) \,\&\, (B \,\&\, \neg B)$ is a contradiction
6. Show that $(B \leftrightarrow (C \vee B))$ is not a contradiction.
7. Show that $((\neg X \leftrightarrow X) \vee X)$ is equivalent to X
8. Show that $F \,\&\, (K \,\&\, R)$ is not equivalent to $(F \leftrightarrow (K \leftrightarrow R))$
9. Show that the sentences $\neg(W \rightarrow W)$, $(W \leftrightarrow W) \,\&\, W$, $E \vee (W \rightarrow \neg(E \,\&\, W))$ are inconsistent.
10. Show that the sentences $\neg R \vee C$, $(C \,\&\, R) \rightarrow \neg R$, $(\neg(R \vee R) \rightarrow R)$ are consistent.
11. Show that $\neg\neg(C \leftrightarrow \neg C), ((G \vee C) \vee G) \therefore ((G \rightarrow C) \,\&\, G)$ is valid.
12. Show that $\neg\neg L, (C \rightarrow \neg L) \rightarrow C) \therefore \neg C$ is invalid.