forall x

THE MISSISSIPPI STATE EDITION

FALL 2025

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Preface

The logician E. J. Lemmon begins his textbook *Beginning Logic* this way:

> It is not easy, and perhaps not even useful, to explain briefly what logic is. Like most subjects, it comprises many different kinds of problem and has no exact boundaries; at one end, it shades off into mathematics, at another, into philosophy. The best way to find out what logic is is to do some. (1965, p. 1)

He then continues, "None the less, a few very general remarks about the subject may help to set the stage for the rest of this book." Following his lead, here are some general remarks. First, formal logic is the study of formal languages. Unlike natural languages (such as English, Spanish, and Mandarin), in a formal language, every part of the language is precisely defined. Using a formal language limits what we can do. Natural languages are extremely flexible and adaptable; by and large, formal languages are not. The trade-off, however, is that formal languages are very precise, and the ones covered in this textbook make clear some of the fundamental aspects of human reasoning.

Second, this textbook is designed for a single semester introduction to formal (i.e., deductive) logic. It primarily covers *truth-functional logic*. (For reference, truth-functional logic also goes by other names: *propositional logic* and *sentential logic*.) In truth-functional logic, individual statements (e.g., "the missing painting is in storage" or "Westerby is talking with Ricardo") are treated as units that can be combined into more complex statements with *or*, *and*, *not*, *if* ...*then* ..., and *if and only if*. The study of truth-functional logic, then, is the study of the properties of these more complex statements and the logical relationships between them.

The final part of the book introduces *first-order logic*, which, in addition to the content of truth-functional logic, uses *names*, *variables*, *predicates*, the *identity relation*, and what are called *quantifiers*. These additional elements of the language make it much more expressive than truth-functional language, which broadens the logical analysis that can be undertaken.

Incidentally, the title *forallx* (i.e., "for all x") is a reference to *first-order logic*. This is a symbolic expression in first-order logic: $\forall x(Kx \rightarrow Gx)$, and it is read, "for all x, if x is K, then x is G." Hence, the name of the textbook. (If, for instance, K stands for "is a king," and G stands for "is greedy," then $\forall x(Kx \rightarrow Gx)$ means "for all x, if x is a king, then x is greedy," or "everyone who is a king is greedy.")

For instructors

This textbook is for a single semester course in formal logic. It is assumed that the students will have no prior exposure to formal logic (or to informal logic for that matter). The textbook covers truth-functional (i.e., propositional) logic and introduces first-order logic. The coverage of the former is designed to guide students as they develop the ability to evaluate arguments using truth tables and proofs. The coverage of the later is only intended to expose students, in the last two or three weeks of the semester, to the syntax of FOL, translation to and from FOL, and simple proofs using the rules of derivation for FOL.

The rules of derivation for TFL are listed in section in the appendix, and proofs are constructed using Fitch notation.

This book is based on a text originally written by P. D. Magnus and revised and expanded by Tim Button, J. Robert Loftis, Aaron Thomas-Bolduc, and Richard Zach. I have made additional revisions, taken out chapters that are not needed for the 1000-level logic course at Mississippi State University, and added instructions for using the logic software Carnap (http://carnap.io/), which can be used in conjunction with Parts - .

G. S. J.

Part 1

Key concepts of logic

1 Arguments

1.1 We begin here.

In everyday discourse, the word *argument* typically refers to a verbal disagreement between two people. In logic and philosophy, however, it has a different and special meaning (although plenty of people do argue, in the everyday sense of the word, in logic and philosophy). We will use *argument* to refer to a set of sentences like these:

1. It is raining heavily.
2. Jeff doesn't want to get soaked.
3. If Jeff doesn't take an umbrella, he will get soaked.

---

4. Therefore, Jeff will take an umbrella.

In this set, the first three sentences support—or justify—the fourth sentence. The sentences providing support are the *premises*. The sentence that is supported by (or justified by) the premises is the *conclusion*. Together, *premises* and a *conclusion* comprise an *argument*.

> Argument
>
> An ARGUMENT is a set of sentences. One or more of the sentences provide support for another sentence in the set. The sentences providing support are PREMISES. The sentence being supported is the CONCLUSION.

That's the definition of an argument, but a broader analysis must include the idea that arguments can be good or bad—or somewhere in between. A good argument is one in which the premises do, in fact, support the conclusion. For such an argument, if the premises are true, then we have good reason to believe that the conclusion is true. On the other hand, a bad (or a weak) argument is still an argument. It is just one in which the premises provide little support for the conclusion.

In the definition of an argument, we said that each premise and the conclusion is a sentence. And, as we saw, the premises and the conclusion in the example are all individual sentences. All arguments can be expressed this way and many are, but a single sentence, like this one, can also contain a complete argument:

> Joan was wearing sunglasses when she came inside, and so it must be sunny.

This argument has one premise and a conclusion. The premise and the conclusion could both be individual sentences, but here they are just independent clauses separated by the 'and'. (The premise is before the 'and', and the conclusion is after it.)

Many arguments also start with the premises and end with a conclusion. But not all arguments are expressed in this order. For instance, here we have the same argument about Joan, but the conclusion is at the beginning:

> It must be sunny because Joan was wearing sunglasses when she came inside.

When approaching an argument, we want to know whether or not the conclusion is supported by the premises. Therefore, first, we must identify the premise or premises (the sentences providing support) and the conclusion (and the sentence being supported). As a guide, these words are often used to indicate that a sentence or clause is the conclusion of an argument:

> so, therefore, hence, thus, accordingly, consequently

And these often indicate that what follows are premises:

> since, because, given that

So that we can undertake a more detailed and precise analysis of some kinds of arguments, in chapter , we will begin introducing a formal language: truth functional logic. But before we get there, in this chapter and chapter , we will cover some basic logical notions that apply to arguments in a natural language like English. Then, in chapter , we will examine logical notions that apply to just sentences (not full arguments), and still in a natural language like English.

1.2 Sentences

Only sentences that can be true or false can be the premises or the conclusion of an argument, and for the purpose of studying truth-funcitonal logic, we will define SENTENCE as a statement that can be true or false.

The following types of sentences cannot be true or false, and so they are not, in this sense, sentences and cannot be part of an argument.

Questions 'Are you sleepy yet?' is, obviously, a question. Although you might be sleepy or you might be alert, the question itself is neither true nor false. For this reason, questions will not count as sentences in logic.

Imperatives Imperative sentences are, essentially, commands (although they can be nicer than what we usually think of as a command). For instance, 'Wake up!', 'Sit up straight', and 'Please, tell me how to set the table' are all imperatives. Although it might be a good idea for you to sit up, and you may or may not do it, the command is neither true nor false. Note, however, that commands are not always phrased as imperatives. As Cartman might say,"You will respect my authority." This is a command, but it is also true or false – either you will or you will not respect Cartman's authority – and so it counts as a sentence in logic.

Exclamations Some exclamatory sentences can be true or false (and so they are also declarative sentences) and some cannot be. 'It's Friday!' is an exclamation, and it is true or false. It can be part of an argument. On the other hand, a sentence such as 'Ouch!' is neither true nor false, and so it cannot be part of an argument.

1.3 Truth values

We define SENTENCE in this restrictive way because all of the premises and the conclusion in an argument must be capable of having a TRUTH

VALUE. Although more advanced "non-classical" logic systems introduce more options, the two truth values that concern us are just 'true' and 'false'.

It may be the case that we would have great difficulty or be unable to establish whether a particular premise or conclusion is true or false, but nonethess, we can coherently think about each one being true or false (which we can't do for, say, a question).

> truth values
>
> TRUTH VALUES are the logical values that a sentence can have, *true* and *false*.

Practice exercises

Identify the conclusion of each of these arguments:

1. All men are mortal, and Socrates is a man. Therefore, Socrates is mortal.
2. Carlos must be on the train because, once the code was given to Petra and the asset was safe, he had to leave Budapest.
3. If the book isn't on the desk, then Stephen is in the archives. And if Stephen is in the dinning hall, then Patricia is stealing the document. So, if Patricia isn't stealing the document, then the book is still on the desk.
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence, Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

The answers are on p. 14

2 Validity and other standards

2.1 Validity

Consider this argument:

1. You are reading this book.
2. This is a logic book.
3. Therefore, you are a logic student.

> When we list the premises and the conclusion of an argument this way, the final line is always the conclusion. All of the lines before the final one are the premises.

If the premises of this argument are true—which, as it turns out, they are—it is very likely that the conclusion is true. But it is possible that someone besides a logic student is reading this book. If, say, the roommate of the book's owner picked it up and began looking through it, he or she would not immediately become a logic student. So, for this argument, we can say that, if the premises are true, then it is *likely*, but not certain, that the conclusion is also true.

Now, take this one:

1. Paris is in France, or it is in Germany.
2. Paris is not in Germany.
3. Therefore, Paris is in France.

In this case, if the premises are true—which, again, they are—then the conclusion has to be true. There is no way for the premises to be true and the conclusion to be false.

Here is another example,

1. Paris is in Sweden, or it is in Spain.
2. Paris is not in Sweden.
3. Therefore, Paris is in Spain.

Although this argument might strike you as a bit odd, we can say almost the exact same thing about this one as we did for the previous one:

> In this argument, if the premises are true, then the conclusion has to be true. There is no way for the premises to be true and the conclusion to be false.

We have to drop the bit about the premises being true because the first one is false. But nonetheless, *if the premises are true*, then the conclusion has to be true.

This brings us to an important definition as well as an important point about doing logic. First the definition.

Valid

These are two equivalent definitions of VALID (or DEDUCTIVELY VALID):

1. An argument is VALID when, and only when, it is the case that, if the premises are true, then the conclusion has to be true.
2. An argument VALID when, and only when, it is impossible for all of the premises to be true and the conclusion to be false.

Every argument that does not satisfy the definition of *valid* is INVALID (or DEDUCTIVELY INVALID).

Typically, the study of logic focuses on determining when the conclusion of an argument follows from the premises with certainty. From the perspective of logic, whether the premises actually are true is less important. Of course, determining whether or not they are true can be important for many reasons, but this task is normally left to historians, scientists, or the Hardy boys.

Here is another argument to consider:

1. Paris is a large city in France, or Paris is a large city on Jupiter.
2. Paris is not a large city in France.
3. Therefore, Paris is a large city on Jupiter.

This argument is valid. *If* both premises are true (they're not, but if they were), then the conclusion has to be true. Now, let's think about this argument:

1. London is in England.
2. Beijing is in China.
3. Therefore, Paris is in France.

The premises and conclusion of this argument are all true, but the argument is invalid. If Paris were, somehow, to become independence from the rest of France, then the conclusion would be false, even though both of the premises would remain true. Thus, it is *possible* for the premises of this argument to be true and the conclusion false. Hence, the argument is *invalid*. Here is another example of an invalid argument:

1. If you are in Paris, then you are in France.
2. You are in France.
3. Therefore, you are in Paris.

The important point to remember is that validity is not about the actual truth or falsity of the sentences in the argument. It is about whether it is *possible* or *impossible* for all of the premises to be true and the conclusion to be false. (Or, to say the same thing in a different way, whether the conclusion has to be true *if* all of the premises are true.)

We can, however, classify the arguments that are valid and have all true premises. We call these SOUND. The second argument on p. 7 is sound.

Sound

An argument is SOUND when, and only when, it is valid and has all true premises.

2.2 Inductively strong arguments

Many good arguments are invalid. Consider this one:

1. In January 2019, it rained in London.
2. In January 2020, it rained in London.
3. In January 2021, it rained in London.
4. In January 2022, it rained in London.
5. In January 2023, it rained in London.
6. In January 2024, it rained in London.
7. In January 2025, it rained in London.
8. Therefore, next January, it will rain in London.

This argument generalizes from observations about several recent past cases to a conclusion about the next one. It could be made even stronger by adding additional premises, for instance: 'In January 2018, it rained in London,' 'In January 2017, it rained in London,' and so on. But, however many premises like this we add, the argument will remain invalid. Even if it has rained in London every January for the past 10,000 years, it remains *possible* that it won't rain in London next January. Hence, this argument is invalid. But, at the same time, you might think, "but it's still a good argument!" It is, and we have a way of classifying such arguments.

Inductively strong

An argument is INDUCTIVELY STRONG when (and only when) [1] it is not valid and [2] it is the case that if the premises are true, then their being true makes it likely that the conclusion is true.

When an argument is invalid, the probability that the conclusion will be true (if the premises are true) can range from very high to zero. To simplify matters, we can say that the options are *inductively strong*, *medium*, or *weak*, although since this is a continuum, we could be much more fine grained than this. (But we won't. See table .)

Whereas a valid argument that has all true premises is *sound*, an inductively strong argument that has all true premises is *reliable*.

Reliable

An argument is RELIABLE when (and only when) it is inductively strong and has all true premises.

The premises being true,	These are all invalid.
make it very probable that the conclusion will be true.	inductively strong
...	
make it somewhat probable that the conclusion will be true.	inductively medium
...	
do not make it very likely that the conclusion will be true.	inductively weak

Table 2.1: Every argument is valid or invalid. Invalid arguments can have any degree of inductive strength, depending on how likely the conclusion is to be true given the premises.

In this textbook, we are going to set aside the analysis of inductively strong arguments and focus just on valid versus invalid arguments.

Practice exercises

A. Determine if each of the following arguments is valid or invalid.

(1)
1. Socrates is a man.
2. All men are carrots.
3. Therefore, Socrates is a carrot.

(2)
1. Either today is Labor Day, or the building is full.
2. The building isn't full.
3. Therefore, today is Labor Day.

(3)
1. If the green van is missing, then Claire is at the beach.
2. The green van is missing.
3. Therefore, Claire is at the beach.

(4) If Jones decided that she is going to get divorced, then she called a lawyer. Jones just called a lawyer. Hence, she has decided that she's going to get divorced.

(5)
1. Jeff is playing basketball, or Mary is watching television.
2. Mary is watching television.

3. Therefore, Jeff is playing basketball.

(6) 1. 160 12th graders at Central High School were asked if they planned to go to college next year.
2. 75 percent said that they were planning to go to college the following year.
3. Therefore, about 75 percent of all the 12th graders at Central High School are probably going to college next year.

(7) 1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is in New Jersey, then Mary stole the painting.

(8) 1. As vacation destinations, Florence and Lisbon have many similarities: nice weather, historical attractions, and great restaurants.
2. Sarah enjoyed visiting Florence.
3. Therefore, Sarah will probably enjoy visiting Lisbon.

(9) 1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is not in New Jersey, then Mary did not steal the painting.

(10) 1. Amy is on campus.
2. Therefore, Amy is on campus, or she is on the moon.

(11) 1. Jack is taking a nap.
2. Therefore, Jack is taking a nap, and Kate is reading.

(12) 1. If Roger is in the bank, then Steven is waiting in the apartment.
2. Roger is not in the bank.
3. Therefore, Steven is not waiting in the apartment.

(13) 1. If Joan is at work, then Kate is sleeping.
2. Therefore, if Kate is not sleeping, then Joan is not at work.

(14) 1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is taking a nap, then Mary is in the library.

(15) 1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.

3. Therefore, if Mary is in the library, then Claire is taking a nap.

(16)
1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is not taking a nap, then Mary is not in the library.

(17)
1. George is an architect, or Susan is a lawyer.
2. George is not an architect.
3. Therefore, Susan is a lawyer.

(18)
1. Amy is walking in the park, or Sarah is playing basketball.
2. Amy is walking in the park.
3. Therefore, Sarah is not playing basketball.

(19)
1. George is mowing the lawn.
2. Therefore, George is mowing the lawn, and Fred is looking for his coat.

(20)
1. Almost all sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

(21)
1. All sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

B. For each statement, determine if it is possible or not. If it is possible, given an example as illustration. If it is not possible, then explain why it isn't.

1. A valid argument that has one false premise and one true premise
2. A valid argument that has a false conclusion
3. A valid argument that has only false premises
4. A valid argument with only false premises and a false conclusion

5. An invalid argument that can be made valid by the addition of a new premise
6. A valid argument that can be made invalid by the addition of a new premise

2.3 Answers

These are the answers for the chapter practice problems :

1. Therefore, Socrates is mortal.
2. Carlos must be on the train.
3. So, if Patricia isn't stealing the document, then the book is still on the desk.
4. Hence, Colonel Mustard did it in the kitchen with the lead-piping.

A.

(1)
1. Socrates is a man.
2. All men are carrots.
3. Therefore, Socrates is a carrot.

This argument is valid.

(2)
1. Either today is Labor Day, or the building is full.
2. The building isn't full.
3. Therefore, today is Labor Day.

This argument is valid.

(3)
1. If the green van is missing, then Claire is at the beach.
2. The green van is missing.
3. Therefore, Claire is at the beach.

This argument is valid.

(4) If Jones decided that she is going to get divorced, then she called a lawyer. Jones just called a lawyer. Hence, she has decided that she's going to get divorced.

This argument is invalid.

(5)
1. Jeff is playing basketball, or Mary is watching television.
2. Mary is watching television.
3. Therefore, Jeff is playing basketball.

This argument is invalid.

(6)
1. 240 12th graders at Central High School were asked if they planned to go to college next year.
2. 75 percent said that they were planning to go to college the following year.
3. Therefore, about 75 percent of all the 12th graders at Central High School are probably going to college next year.

This argument is invalid.

(7)
1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is in New Jersey, then Mary stole the painting.

This argument is invalid.

(8)
1. As vacation destinations, Florence and Lisbon have many similarities: nice weather, historical attractions, and great restaurants.
2. Sarah enjoyed visiting Florence.
3. Therefore, Sarah will probably enjoy visiting Lisbon.

This argument is invalid.

(9)
1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is not in New Jersey, then Mary did not steal the painting.

This argument is valid.

(10)
1. Amy is on campus.
2. Therefore, Amy is on campus, or she is on the moon.

This argument is valid.

(11)
1. Jack is taking a nap.
2. Therefore, Jack is taking a nap, and Kate is reading.

This argument is invalid.

(12) 1. If Roger is in the bank, then Steven is waiting in the apartment.
2. Roger is not in the bank.
3. Therefore, Steven is not waiting in the apartment.

This argument is invalid.

(13) 1. If Joan is at work, then Kate is sleeping.
2. Therefore, if Kate is not sleeping, then Joan is not at work.

This argument is valid.

(14) 1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is taking a nap, then Mary is in the library.

This argument is invalid.

(15) 1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Mary is in the library, then Claire is taking a nap.

This argument is valid.

(16) 1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is not taking a nap, then Mary is not in the library.

This argument is valid.

(17) 1. George is an architect, or Susan is a lawyer.
2. George is not an architect.
3. Therefore, Susan is a lawyer.

This argument is valid.

(18) 1. Amy is walking in the park, or Sarah is playing basketball.
2. Amy is walking in the park.
3. Therefore, Sarah is not playing basketball.

For the way that we will define 'or' in the logic system that is developed in this textbook, this argument is invalid.

(19)
1. George is mowing the lawn.
2. Therefore, George is mowing the lawn, and Fred is looking for his coat.

This argument is invalid.

(20)
1. Almost all sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

This argument is invalid.

(21)
1. All sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

This argument is valid.

B.

(1) A valid argument that has one false premise and one true premise
Yes, this is possible.

1. All whales are mammals. (*true*)
2. All mammals are plants. (*false*)
3. Therefore, all whales are plants.

(2) A valid argument that has a false conclusion
Yes, this is possible. See the example from the previous exercise.

(3) A valid argument that has only false premises
Yes, this is possible.

1. Starkville, Mississippi is larger than Atlanta. (*false*)
2. Atlanta is larger than New York City. (*false*)
3. Therefore, Starkville, Mississippi is larger than New York City.

(4) A valid argument with only false premises and a false conclusion
Yes, this is possible. See the example from the previous exercise.

(5) An invalid argument that can be made valid by the addition of a new premise
Yes, this is possible. This argument is invalid:

1. At State University, all members of the men's basketball team can dunk.
2. Jeff is not a member of the men's basketball team.
3. Therefore, Jeff cannot dunk.

But when an additional premise is added, it becomes valid:

1. At State University, only members of the men's basketball team can dunk.
2. At State University, all members of the men's basketball team can dunk.
3. Jeff is not a member of the men's basketball team.
4. Therefore, Jeff cannot dunk.

(6) A valid argument that can be made invalid by the addition of a new premise
No, this is not possible. For an argument to be valid, it must be the case that if all of the premises are true, then the conclusion has to be true. For instance, here is a valid argument:

1. If you are in Paris, then you are in France.
2. You are in Paris.
3. Therefore, you are in France.

Adding a premise such as "You are cold and wet" isn't going to change anything. If you add a premise that creates a contradiction—for instance "You are not in Paris."— then you've just made it impossible for all of the premises to be true at the same time, but that doesn't violate the definition of *valid*.

3 Other concepts of logic

Valid is a central concept in logic. In this section, we will introduce some other important concepts that apply just to sentences, not to full arguments.

3.1 Concepts for single sentences

Consider these three sentences about you:

a. I am in Mississippi.
b. Either I am in Mississippi, or I am not in Mississippi.
c. I am in Mississippi, and I am not in Mississippi.

In order to know if sentence (*a*) is true, you would need to check where you are. Depending on where that is, the sentence might be true, or it might be false. A sentence that is capable of being true and capable of being false (in different circumstances, of course) is CONTINGENT.

Sentence (*b*) is different. You do not need to check your location to know that it is true. Wherever you are, this sentence will be true. Thus, this sentence is a NECESSARY TRUTH.

Similarly, you do not need to check anything to determine whether or not sentence (*c*) is true. It must be false, simply as a matter of logic. You might be jumping back and forth over the Mississippi state line, or you might even be standing with one foot in Mississippi and one foot in Tennessee (in which case you are still in Mississippi). It is impossible, however, for you, at any one moment, to be both in and not in Mississippi. Therefore, this sentence is a NECESSARY FALSEHOOD.

sentences: necessary and contingent

A NECESSARY TRUTH is a sentence that must be true; it could not possibly be false.

A NECESSARY FALSEHOOD is a sentence that must be false; it could not possibly be true.

A CONTINGENT SENTENCE is neither a necessary truth nor a necessary falsehood. It may be true or it may not.

An easily misunderstood aspect of this analysis is that a sentence might always be true and still be contingent. For instance, this sentence is true:

d. Mary Todd married Abraham Lincoln in 1842.

And there is no way, now, that it will ever be false. But it could have been false. Todd and Lincoln could have gotten married in a different year, or Todd could have married someone else or no one at all. Thus, it is contingent. A full analysis of this (and other) contingent truths would be too lengthy to undertake here, but hopefully you can see that things could have worked out in such a way that (*d*) would be false.

This is in contrast to sentences like these:

e. Today, in Starkville, Mississippi, it is Thursday, or it is not Thursday.
f. Five plus seven equals twelve.
g. Every oncologist is a doctor.

These sentences cannot be false, and there is no way to imagine a possible series of events that would make them false. Hence, they are not contingent. They are necessary truths.

3.2 Joint possibility

Consider these two sentences:

B1. Jane's only brother is shorter than her.
B2. Jane's only brother is taller than her.

Without knowing Jane and her brother, we have no way of knowing which, if either, of these sentences is true. Yet we can say that *if* B1 is true, *then* B2 must be false. Similarly, if B2 is true, then B1 must be false. It is impossible that both sentences are true at the same time. On the other hand, G1 and G2 can both be true at the same time.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

Both of these sentences might be false. One of them might be false while the other is true. But it is *possible* that they are both true.

jointly possible and impossible

A set of sentences are JOINTLY POSSIBLE when, and only when, it is possible for them all to be true at the same time.

A set of sentences are JOINTLY IMPOSSIBLE when, and only when, it is *not* possible for them all to be true at the same time.

So, G1 and G2 are *jointly possible* while B1 and B2 are *jointly impossible*.

We can investigate the joint possibility of any number of sentences. For example, let's add two more sentences to G1 and G2:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two extra-terrestrials at the wild animal park.
G4. Every giraffe at the wild animal park is an extra-terrestrial.

Together, G1 and G4 entail that there are at least four extra-terrestrials giraffes at the park. This conflicts with G3, which states that there are no more than two extra-terrestrials there. So, G1–G4 are jointly impossible. They cannot all be true together. (Notice also that just G1, G3 and G4 are jointly impossible, while G1, G2, and G3 are jointly possible.)

3.3 Necessary equivalence

Sentences G1 and G2 (which we said were jointly possible) can both be true at the same time. They can also both be false, or one false and the other true. A stronger relationship holds between these two sentences:

> John went to the store after he washed the dishes.
> John washed the dishes before he went to the store.

These two sentences must have the same truth value. That is, they must either both be true or both be false. It is impossible for one to be true and one to be false (at the same time). When two sentences *must* have the same truth value, they are NECESSARILY EQUIVALENT.

> necessarily equivalent
>
> Two sentences are NECESSARILY EQUIVALENT if they must have the same truth value. That is, they must both be true or they both must be false.

3.4 Practice exercises

A. Determine if each sentence is a necessary truth, a necessary falsehood, or contingent.

1. Caesar crossed the Rubicon.
2. No one has ever crossed the Rubicon.
3. If Caesar crossed the Rubicon, then someone has.
4. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
5. If anyone has ever crossed the Rubicon, it was Caesar.
6. Elephants dissolve in water.
7. Wood is a light, durable substance useful for building things.
8. If wood is a good building material, it is useful for building things.
9. I live in a three story building that is two stories tall.
10. If gerbils are mammals, they nurse their young.

B. Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.
   If you put an elephant in water, it will dissolve.
2. All mammals dissolve in water.
   If you put an elephant in water, it will dissolve.
3. George Bush was the 43rd president.
   Barack Obama was the 44th president.
4. Barack Obama was the 44th president.
   Barack Obama was president immediately after the 43rd president.
5. Thelonious Monk played with John Coltrane.
   John Coltrane played with Thelonious Monk.

C.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two Martians at the wild animal park.
G4. Every giraffe at the wild animal park is a Martian.

Determine if each set of sentences is jointly possible or jointly impossible.

1. Sentences G2, G3, and G4
2. Sentences G1, G3, and G4
3. Sentences G1, G2, and G4
4. Sentences G1, G2, and G3

D.

M1. All people are mortal.
M2. Socrates is a person.
M3. Socrates will never die.
M4. Socrates is mortal.

Determine if each set of sentences is jointly possible or jointly impossible.

1. Sentences M1, M2, and M3
2. Sentences M2, M3, and M4
3. Sentences M2 and M3
4. Sentences M1 and M4
5. Sentences M1, M2, M3, and M4

3.5 Answers

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon.
   Contingent
2. No one has ever crossed the Rubicon.
   Contingent
3. If Caesar crossed the Rubicon, then someone has.
   Necessarily true
4. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
   Necessarily false
5. If anyone has ever crossed the Rubicon, it was Caesar.
   Contingent
6. Elephants dissolve in water.
   Contingent
7. Wood is a light, durable substance useful for building things.
   Contingent
8. If wood is a good building material, it is useful for building things.
   Necessarily true
9. I live in a three story building that is two stories tall.
   Necessarily false
10. If gerbils are mammals, they nurse their young.
    This sentence is necessarily true. (*Mammalia* is defined as the class of animals wherein the females have mammaries and nurse their young. Hence, 'If gerbils are mammals, they nurse their young' is necessarily true.)

B.

1. Elephants dissolve in water.
   If you put an elephant in water, it will dissolve.
   These sentences are necessarily equivalent.

2. All mammals dissolve in water.
   If you put an elephant in water, it will dissolve.
   These sentences are *not* necessarily equivalent.
3. George Bush was the 43rd president.
   Barack Obama was the 44th president.
   These sentences are *not* necessarily equivalent.
4. Barack Obama was the 44th president.
   Barack Obama was president immediately after the 43rd president.
   These sentences are necessarily equivalent.
5. Thelonious Monk played with John Coltrane.
   John Coltrane played with Thelonious Monk.
   These sentences are necessarily equivalent.

C.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two Martians at the wild animal park.
G4. Every giraffe at the wild animal park is a Martian.

1.	Sentences G2, G3, and G4	Jointly possible
2.	Sentences G1, G3, and G4	Jointly impossible
3.	Sentences G1, G2, and G4	Jointly possible
4.	Sentences G1, G2, and G3	Jointly possible

D.

M1. All people are mortal.
M2. Socrates is a person.
M3. Socrates will never die.
M4. Socrates is mortal.

1.	Sentences M1, M2, and M3	Jointly impossible
2.	Sentences M2, M3, and M4	Jointly impossible
3.	Sentences M2 and M3	Jointly possible

Person, at least in the philosophical sense, is different than *human being* (although the two concepts generally overlap). *Person* means, basically, *moral agent*, and so, for instance, God, if he exists, is a person. Consequently, just the sentence 'Socrates is a person' doesn't tell us whether or not Socrates will die.

4. Sentences M1 and M4 — Jointly possible
5. Sentences M1, M2, M3, and M4 — Jointly impossible

Part 2

Truth-functional logic

4 Form and symbolization

4.1 Validity in virtue of form

Let's examine two arguments. This is the first one:

1. If Claire is in the safe house, then the agency still has the document.
2. Claire is in the safe house.
3. Therefore, the agency still has the document.

This is the second:

1. If Mary escaped to Tangier, then Simon is a double agent.
2. Mary escaped to Tangier.
3. Therefore, Simon is a double agent.

Both arguments are valid, and, as perhaps you can see, they share a common form. We can represent the form by itself this way:

1. If A, then B
2. A
3. Therefore, B

Any argument with this form will be valid. It doesn't matter what English sentences are put in the places of A and B.

Here is another valid argument:

1. Seoul is larger than London.
2. London is larger than Chicago.
3. Therefore, Seoul is larger than Chicago.

This argument also has a particular form that makes it valid, and we can represent its form like this:

1. C is larger than D.
2. D is larger than F.
3. Therefore, C is larger than F.

For the first argument form that we examined, A and B could be any sentences. Here, C, D, and F are names (not full sentences), and we can put any names (for anything) in the places of C, D, and F, and the argument will remain valid.

In contrast, this argument is valid, but there is no particular form that makes it so.

1. Kate is an oncologist.
2. Therefore, Kate is a doctor.

Unlike the previous three examples, this argument is not valid because of the form of the argument. Rather, it's valid because of the particular meanings of *oncologist* and *doctor*.

These examples illustrate the idea of VALIDITY IN VIRTUE OF FORM. The arguments about Claire, Mary, Simon, Seoul, London, and Chicago are valid, but—unlike the argument about Kate—their being valid has nothing to do with the specific meaning of 'Claire is in the safe house', 'Simon is a double agent', 'Seoul', 'London', or 'Chicago.' (Whether the arguments are sound depends on these meanings, but not whether the arguments are valid.) Instead, these arguments are valid in virtue of the meanings of just these words: *if, then* and *is larger than*.

valid in virtue of form

Let us define STRUCTURE WORDS as *if-then*, *and*, *or*, *not*, *if and only if*, and comparative adjectives followed by *than* (e.g., *larger than*, *faster than*, *older than*).
An argument is VALID IN VIRTUE ITS FORM when the argument contains properly organized structure words. When it does, any sentences or names can be substituted into the argument and it will remain valid.

This is not a perfect definition. *Form* in formal logic is much broader than just the use of these "structure words." This is a good definition with which to start, however.

And, although valid for reasons other than the argument's form is an interesting topic, our focus will be on arguments that are valid because of their form—and in fact, only valid in virtue of some forms. In part 5, we will broaden the analysis a bit, but in parts – of this textbook, we will be interested in arguments where the form is set by the use of *if-then*, *and*, *or*, *not*, and *if and only if*.

> Going forward, we will set aside arguments that are valid because they employ comparative adjectives. To make a final point about them, though, arguments that use comparative adjectives are valid because these terms denote a TRANSATIVE RELATION. Such a relation exists when the relation between two elements in a series applies to any elements in the series as long as the elements are taken in order.

Here are some examples of arguments that are valid in virtue of *or*, *not*, and *and*. This one:

1. Claire is either a student, or she is a spy.
2. Claire is not a student.
3. Therefore, Claire is a spy.

has this form:

1. G or H
2. not G
3. Therefore, H

And this argument:

1. It's not the case that Jeff both studies often and acts in lots of plays.
2. Jeff acts in lots of plays.
3. Therefore, Jeff does not study often.

has a form that we can represent like this:

1. not (K and L)
2. L
3. Therefore, not K

4.2 Atomic sentences and symbolization

Consider this sentence again:

(a) If Claire is in the safe house, then the agency has the document.

'Claire is in the safe house' and 'The agency has the document' are *subsentences* of sentence (a). To specify the structure of the first argument in this chapter, we replaced the sentences and subsentences in it with individual letters. 'Claire is in the safe house' was replaced with 'A', and 'The agency has the document' was replaced with 'B'. This kind of representation—letters standing for sentences or subsentences—is one important part of the formal language developed in this textbook.

Atomic sentences

Again, we define SENTENCE as a statement that can be true or false. When no smaller sentence can be extracted from a sentence, it is an ATOMIC SENTENCE.

In English, an atomic sentence is a declarative sentence that does not contain any subsentences.

In our logic system, an atomic sentence is a single capital letter in this font: $A, B, C, D, E\ldots$

Atomic sentences are the basic building blocks used to form more complex sentences. We will use uppercase Roman letters for atomic sentences in our logic system. If, by chance, we ever need more than twenty-six different atomic sentences, we can obtain additional ones by adding subscripts to letters. Here, for instance, are five different atomic sentences:

$$M, P, P_1, P_2, M_{17}$$

We will use atomic sentences to represent, or *symbolize*, certain English sentences. To do this, we provide a SYMBOLIZATION KEY, such as the following.

A: Claire is in the safe house.
B: The agency has the document.

When we do this, we are not fixing this symbolization once and for all. We are just, for the time being, letting 'A' symbolize the English sentence 'Claire is in the safe house', and 'B' symbolize the English sentence 'The agency has the document'. Later, when we are dealing with different sentences or different arguments, we can provide a new symbolization key; for instance,

A: Jake stole the code.
B: Jake is on the train.

5 Logical operators

TRUTH-FUNCTIONAL LOGIC is a branch of logic that focuses on the relationships between atomic sentences. One part of truth-functional logic (or 'TFL' for short) is a formal language. This formal language consists of atomic sentences of TFL (see §) and the LOGICAL OR TRUTH-FUNCTIONAL OPERATORS 'and', 'or', 'not', 'if …, then …' and 'if and only if'.

A logical operator is a word or phrase that modifies a sentence or connects two sentences to form a more complex sentence. We call these operators *truth-functional* because the truth of the complex sentences depends entirely on the truth of the atomic sentences of which they are composed. (These operators are also sometimes referred to as *connectives* since, except for 'not', they all connect two simpler sentences.)

In addition to symbolizing English sentences with sentence letters, we also want to symbolize the logical operators. The symbols that we will use are shown in table . The operators listed there are not the only ones that we have in English. Others are, for example, 'unless', 'neither … nor …', 'necessarily', and 'because'. As we will see, the first two can be expressed with the connectives that are in table . The last two, however, cannot. Although they are logical operators, 'necessarily' and 'because' are not truth functional.

SYMBOL	THE SENTENCE'S NAME	ITS MEANING
$\neg$	negation	'It is not the case that ….'
&	conjunction	'Both …, and ….'
$\vee$	disjunction	'Either …, or ….'
$\rightarrow$	conditional	'If …, then ….'
$\leftrightarrow$	biconditional	'… if and only if ….'

Table 5.1: The logical operators of truth functional logic. See table on p. 153 for instructions about how to type these symbols.

Once we have introduced these logical operators (in this chapter and in chapter) and have explained what can and cannot be a sentence in TFL (which we will do in chapter), our formal language will be complete. Although the formal language is central, truth-functional logic does not consist only of a formal language. There is also a *deductive system*, which we will explore in part .

5.1 Negation

Consider how we might symbolize these sentences:

1. Mary is in Barcelona.
2. It is not the case that Mary is in Barcelona.
3. Mary is not in Barcelona.

To begin, we need an atomic sentence. This will be our symbolization key:

M: Mary is in Barcelona.

M is sentence , and so we don't need to do anything else there. The second sentence is partially symbolized as 'It is not the case that M'. In order to complete the symbolization, we need a symbol for 'it is not the case that'. Or, put differently, we need a symbol that, when added to M, will express 'the negation of M'. We will use '$\neg$' and symbolize sentence as $\neg M$.

Sentence is worded a little bit differently, but it is equivalent to sentence . As such, we can also symbolize it as $\neg M$.

> Negation
>
> A sentence can be symbolized as $\neg$**A** if it can be paraphrased in English as 'It is not the case that …'.

Here are a few more examples:

4. The cog can be replaced.
5. The cog is irreplaceable.
6. The cog is not irreplaceable.

For these, we will use this symbolization key:

R: The cog is replaceable

Sentence is symbolized just by R. Sentence can be reworded as *it is not the case that the cog is replaceable*. So even though sentence does not contain the word 'not', we will symbolize it $\neg R$. Sentence , you will notice, is the denial of sentence . So, we symbolize as $\neg\neg R$.

Finally, consider these English sentences:

7. Jane is happy.
8. Jane is unhappy.

If we use H stand for 'Jane is happy', then we can symbolize sentence as H. It would be a mistake, however, to symbolize sentence with $\neg H$. $\neg H$ means 'Jane is not happy', but 'Jane is not happy' does not have the same meaning as 'Jane is unhappy'. After all, Jane might be neither happy nor unhappy; her mood might just be neutral. In order to symbolize sentence , we would need a different sentence letter.

5.2 Conjunction

Let's start with these sentences:

9. Sam is on the train, and Kate is on the train.
10. Sam is not on the train, but Kate is on train.
11. It is not the case that both Sam and Kate are on the train.

Even though they are both about being on the train, we need these two sentence letters to translate sentences , , and :

S: Sam is on the train.
K: Kate is on the train.

Sentence can be partially symbolized as (S and K). For the 'and', we will use '&', which is called the *ampersand*. Thus, sentence becomes $(S \,\&\, K)$.

When two sentences are connected with an '&', the resulting sentence is called a CONJUNCTION. The two sentences that are combined with the '&' are the CONJUNCTS. So, S and K are the conjuncts of $(S \,\&\, K)$.

For sentence , we symbolize 'Sam is not on the train' as $\neg S$. 'But' may have a slightly different meaning in English than 'and', but, grammatically, 'but' and 'and' do the same job: they join two sentences as a conjunction. Thus, sentence is symbolized as $(\neg S \& K)$.

Sentence is a little more complicated. We know from the previous section that 'it is not the case that' is symbolized with the '$\neg$'. Therefore, we can partially symbolize this sentence like this:

$\neg$ both Sam and Kate are on the train

'Both Sam and Kate are on the train' is equivalent to sentence , and so we symbolize it as $(S \& K)$. Putting this together, we get $\neg(S \& K)$.

Notice the difference between sentences and in English and how we translated them into truth-functional logic. In sentence , the negation only applies to 'Sam is on the train'. Thus, the '$\neg$' is right next to the S. In sentence , the negation applies to the whole 'both Sam and Kate are on the train'. Consequently, we need to have the '$\neg$' outside the parentheses to indicate that the whole $(S \& K)$ is being negated.

> There is a method for distributing the '$\neg$' inside the parentheses, but it's not as simple as just moving the '$\neg$' next to both sentence letters. In other words, $\neg(S \& K) \neq (\neg S \& \neg K)$.

Finally, we don't symbolize the 'both' in , other than to use the '&' between the S and the K. The 'both' indicates where the conjunction begins, which is important for distinguishing sentences and , and it is also sometimes used in English just for emphasis. It doesn't, however, have any separate meaning in truth-functional logic.

Parentheses

Although we will relax this requirement later, in truth-functional logic, conjunctions, as well as the types of sentences that we will discuss in sections - , should be enclosed in parentheses. (A negation [for instance, $\neg P$] should not, though.) The purpose of the parentheses is to let us be perfectly explicit about how each logical operator is related to each sentence letter in a sentence.

Conjunction

A sentence can be symbolized as $(\mathsf{A} \mathbin{\&} \mathsf{B})$ if it can be paraphrased any of these ways in English:

> 'Both …, and….'
> '…, and ….'
> '…, but ….'
> '…, although ….'
> '…, as well as ….'

5.3 Disjunction

We will start with these sentences:

12. Tom is at the train station, or Kate driving to Santa Fe.
13. Kate is driving to Santa Fe, and either Tom is at the train station or Sarah is at the train station.

And we will use this symbolization key:

K: Kate is driving to Santa Fe.
S: Sarah is at the train station.
T: Tom is at the train station.

To represent the 'or' in sentences and , we will use the symbol '$\lor$' (which we call the *wedge*, not *v*). Sentence , then, is written as $(T \lor K)$. When two sentences are connected with an '$\lor$', the resulting sentence is called a DISJUNCTION. T and K are the DISJUNCTS of $(T \lor K)$.

In sentence , we have an *and* and an *or*. But this way of symbolizing the sentence is wrong:

$$(K \mathbin{\&} T \lor S)$$

We have to determine, and reveal in the TFL sentence, if the T goes with the K like this:

$$((K \mathbin{\&} T) \lor S)$$

Or with the S like this:

$(K \,\&\, (T \vee S))$

In the English sentence, the 'either' indicates where the disjunction begins. Hence, $(T \vee S)$ is the disjunction, and $K \,\&$ ... precedes it, giving us this: $(K \,\&\, (T \vee S))$.

Disjunction

A sentence can be symbolized as $(\mathbf{A} \vee \mathbf{B})$ if it can be paraphrased in English as 'Either…, or….'

The inclusive or

Sometimes in English, the word 'or' is used in a way that excludes the possibility that both disjuncts are true. This is called the EXCLUSIVE OR. An *exclusive or* is clearly intended when it says on a restaurant menu "Entrees come with either soup or salad." This means that, with your entree, you may have soup or you may have salad, but you cannot have both.

At other times, the word 'or' allows for the possibility that both disjuncts might be true. For instance, Amy might say, "I am going to get an A in Logic or an A in German III." She probably means that she will get an A in one or both of these courses. (After all, if she ends up getting an A in both, we wouldn't insist that she was wrong when she said, "I am going to get an A in Logic or an A in German III," and we wouldn't expect her to try to get a B in one if she knew that she was going to get an A in the other.)

When, by "A or B," we mean that A is true or B is true or *both* are true, then we are using the INCLUSIVE OR. The TFL symbol '$\vee$' always symbolizes an *inclusive or*.

Negation and disjunction

Think about these sentences:

14. Either Kate is not at the train station, or Sarah is not at the train station.
15. Neither Kate nor Sarah is at the train station.

16. Either Kate is at the train station or Sarah is at the train station, but both are not.

Sentence is symbolized as $(\neg K \vee \neg S)$. Sentences and are a little trickier.

According to sentence , neither one is at the train station. Therefore, we can paraphrase it like this:

> It is not the case that either Amy is at the train station or Sarah is at the train station.

As our paraphrased sentence shows, we are negating the entire disjunction. Hence, we symbolize sentence as $\neg(K \vee S)$.

Sentence expresses the meaning of the *exclusive or*: one or the other, but not both. The '$\vee$', however, represents the *inclusive or*: one or the other, or both. Therefore, to represent in TFL, we need to break the sentence into two parts.

The first part, 'Kate is at the train station or Sarah is at the train station', is symbolized as $(K \vee S)$. The second part, which states that both won't be there, is paraphrased this way: 'It is not the case that both Kate is at the train station and Sarah is at the train station'. We symbolize this as $\neg(K \,\&\, S)$. We put the two parts together with an 'and', and sentence becomes $((K \vee S) \,\&\, \neg(K \,\&\, S))$.

These last two examples demonstrate that we can sometimes symbolize English sentences that, at first, appear not to be using the logical operators of TFL. (That is, the ones listed in table .) We can do this as long as we can figure out a way to paraphrase the English sentence so that it is using some combination of '*and*', '*or*' (i.e., the inclusive-or), '*not*', '*if* …, *then* …', and '*if and only if*'.

5.4 Conditional

We will start with this sentence:

17. If Jean is in Paris, then Jean is in France.

And we will use this symbolization key:

> P: Jean is in Paris.

F: Jean is in France

Sentence has this form: if P, then F, and we call this type of sentence a CONDITIONAL. We will use '→' to symbolize *if* …, *then* …. Thus, sentence becomes $(P \to F)$.

In a conditional, what goes before the '→' is called the ANTECEDENT, and what comes after the '→' is called the CONSEQUENT. So, in sentence , 'Jean is in Paris' is the antecedent, and 'Jean is in France' is the consequent.

Conditional

A sentence can be symbolized as $(\mathsf{A} \to \mathsf{B})$ if it can be paraphrased in English as 'If A, then B'.

Many English expressions can be represented using the conditional, and the most common alternatives to 'if **A**, then **B**' are listed in table . If you think about it, you'll see that all six of the sentences in the table have the same meaning, and so they can all be symbolized as $(\mathsf{A} \to \mathsf{B})$. (Or, in this case, as $(P \to F)$.)

5.5 Biconditional

All of the logical operators that we have discussed so far are ones with which you were already familiar because you are an English speaker. The biconditional, which is mostly commonly expressed as '…*if and only if* …', is one that you might not have really noticed before – even if you have used it on occasion. We'll start with the basic case.

If Jean is in Paris, then she is in France	If **A**, then **B**.
Jean is in France if she is in Paris.	**B** if **A**.
Whenever Jean is in Paris, she is in France.	Whenever **A**, **B**.
Jean is in France provided that she is in Paris.	**B** provided that **A**.
Provided that Jean is in Paris, she is in France.	Provided that **A**, **B**.
Jean is in Paris only if she is in France.	**A** only if **B**.

Table 5.2: The most common way of expressing a conditional in English is as 'If Jean is in Paris, then she is in France.' This table lists some alternative but equivalent ways of expressing the same sentence.

18. The Bulldogs won if and only if they scored more points than the Razorbacks.

And this will be our symbolization key:

P: The Bulldogs scored more points than the Razorbacks.
W: The Bulldogs won.

The symbol '$\leftrightarrow$' will stand for 'if and only if', and so we can symbolize sentence as $(W \leftrightarrow P)$.

That's how we translate an English sentence containing *if and only if* into TFL. But let's think about what *if and only if* means. If sentence is true, then which of these conditionals must also be true?

19. If the Bulldogs won, then they scored more points than the Razorbacks.
20. If the Bulldogs scored more points than the Razorbacks, then they won.

The answer is both must be true.

From the previous section, you know that we symbolize sentences and like this:

. $(W \rightarrow P)$
. $(P \rightarrow W)$

So, sentence is equivalent to $((W \rightarrow P) \,\&\, (P \rightarrow W))$, which is why we call $(W \leftrightarrow P)$ a BICONDITIONAL: it is equivalent to the two conditionals that have their antecendent and consequent switched.

The expression 'if and only if' occurs frequently in philosophy, mathematics, and logic, and sometimes you will see it abbreviated 'iff'. (Although even when 'iff' is written, we still say 'if and only if.')

Biconditional

A sentence can be symbolized as $(\mathsf{A} \leftrightarrow \mathsf{B})$ if it can be paraphrased in English as 'A if and only if B'.

5.6 Unless

We have now introduced all of the logical operators of TFL. We can use them together to symbolize many kinds of sentences. An especially difficult case is when we use the English-language connective 'unless'. Take this sentence:

21. You will catch a cold unless you wear a jacket.

To symbolize , we will use this symbolization key:

J: You will wear a jacket.
D: You will catch a cold.

One meaning of sentence is that if you do not wear a jacket, then you will catch a cold. This we symbolize as $(\neg J \rightarrow D)$. Alternatively, the sentence can mean that if you do not catch a cold, then you must have worn a jacket. This is symbolized as $(\neg D \rightarrow J)$. And, finally, it can also mean that either you will wear a jacket or you will catch a cold. This, we symbolize as $(J \vee D)$.

All three ways of symbolizing sentence are correct. In fact, in chapter , we will see that these three TFL sentences are equivalent. Following the somewhat standard practice, however, we will define *unless* as a disjunction.

> Unless
>
> If a sentence can be paraphrased as 'A unless B,' then it can be symbolized as '$(\mathsf{A} \vee \mathsf{B})$'.

There is a complication with treating 'unless' as a disjunction, however. As we said earlier, 'or' has an inclusive and an exclusive meaning, but in TFL, 'or' is always inclusive. Speakers of English, however, often use 'unless' to mean something more like the exclusive-or. Suppose someone says: 'I will go running unless it snows'. They probably mean 'either I will go running or it will snow, but not both'. So, it can be argued that the conditional – i.e., 'if it does not snow, then I will go running' or $(\neg S \rightarrow R)$ – captures the meaning of *unless* better than does the disjunction.

5.7 The turnstile

The final symbol that we need is, technically, not a symbol of TFL, but it is useful to have when stating an argument. The symbol '$\vdash$' is called the *turnstile*. The purpose of the turnstile is to separate the sentences that are the premises of an argument from the sentence that is the conclusion, and it can be read as *therefore*. Here is an example,

$$(P \to Q), (P \vee S) \vdash (\neg Q \to S)$$

In this argument, the premises are $(P \to Q)$ and $(P \vee S)$, and the conclusion is $(\neg Q \to S)$.

5.8 Practice exercises

A. Using the symbolization key given, translate each English sentence into TFL.

A: Mr. Adams was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mr. Edwards was murdered.
F: The murder weapon was a frying pan.

1. Either Mr. Adams or Mr. Edwards was murdered.
2. If Mr. Adams was murdered, then the cook did it.
3. If Mr. Edwards was murdered, then the cook did not do it.
4. Either the butler did it, or the Duchess is lying.
5. The cook did it only if the Duchess is lying. (Check table .)
6. If the murder weapon was not a frying pan, then the cook did not do it.
7. If the murder weapon was not a frying pan, then either the cook or the butler did it.
8. Mr. Adams was murdered if and only if Mr. Edwards was not murdered.
9. It is not the case that either the Duchess is lying or Mr. Edwards was not murdered.

10. The cook did it, and the butler did not.

B. Using this symbolization key, translate each English sentence into TFL.

C: The code has been broken.
E: The letter is in German embassy.
G: Gorka is a spy.
L: Lena is a spy.

1. It is not the case that both Gorka and Lena are spies.
2. If either Gorka or Lena is a spy, then the code has been broken.
3. If the letter is in the embassy and the code hasn't been broken, then Lena is not a spy.
4. If the code has been broken and the letter isn't in the embassy, then Gorka is a spy.
5. Lena is a spy if and only if either Gorka is a spy or the code has been broken.
6. Either Lena is a spy and the code has been broken, or the letter is not in the German embassy and Gorka is a spy.
7. If neither Gorka nor Lena is a spy, then the code has not been broken.
8. The letter is in the German embassy, unless someone has broken the code.
9. Either the code has been broken or it has not, but the letter is in German embassy regardless.
10. Either Gorka or Lena is a spy, but not both.

5.9 Answers

A.

1. Either Mr. Adams or Mr. Edwards was murdered.
   $(A \vee E)$
2. If Mr. Adams was murdered, then the cook did it.
   $(A \rightarrow C)$
3. If Mr. Edwards was murdered, then the cook did not do it.
   $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.

$(B \lor D)$

5. The cook did it only if the Duchess is lying. (See table .)
   $(C \to D)$
6. If the murder weapon was not a frying pan, then the cook did not do it.
   $(\neg F \to \neg C)$
7. If the murder weapon was not a frying pan, then either the cook or the butler did it.
   $(\neg F \to (C \lor B))$
8. Mr. Adams was murdered if and only if Mr. Edwards was not murdered.
   $(A \leftrightarrow \neg E)$
9. It is not the case that either the Duchess is lying or Mr. Edwards was not murdered.
   $\neg(D \lor \neg E)$
10. The cook did it, and the butler did not.
    $(C \,\&\, \neg B)$

B.

1. It is not the case that both Gorka and Lena are spies.
   $\neg(G \,\&\, L)$
2. If either Gorka or Lena is a spy, then the code has been broken.
   $((G \lor L) \to C)$
3. If the letter is in the embassy and the code hasn't been broken, then Lena is not a spy.
   $((E \,\&\, \neg C) \to \neg L)$
4. If the code has been broken and the letter isn't in the embassy, then Gorka is a spy.
   $((C \,\&\, \neg E) \to G)$
5. Lena is a spy if and only if either Gorka is a spy or the code has been broken.
   $(L \leftrightarrow (G \lor C))$
6. Either Lena is a spy and the code has been broken, or the letter is not in the German embassy and Gorka is a spy.
   $((L \,\&\, C) \lor (\neg E \,\&\, G))$

7. If neither Gorka nor Lena is a spy, then the code has not been broken.
   $\neg(G \vee L) \rightarrow \neg C$
8. The letter is in the German embassy, unless someone has broken the code.
   $(E \vee C)$
9. Either the code has been broken or it has not, but the letter is in German embassy regardless.
   $((C \vee \neg C) \,\&\, E)$
10. Either Gorka or Lena is a spy, but not both.
    $((G \vee L) \,\&\, \neg(G \,\&\, L))$

6 Sentences of TFL

'Bring with thee airs from heaven or blasts from hell' is a sentence of English. $(P \lor Q)$ is a sentence of TFL. Oddly, although we can identify sentences of English when we encounter them, there is not a formal definition of *sentence of English* that will tell us, for any possible combination of words and punctuation, whether or not it is a sentence of English. It is possible, however, to provide such a definition for sentences of TFL, and we will examine that definition in this chapter.

6.1 Expressions

You have been introduced to the symbols of TFL in the previous two chapters. They are also summarized in table . We define an EXPRESSION OF TFL as any string of symbols of TFL. Take any of the symbols of TFL and write them down, in any order, and you have an expression of TFL.

6.2 Sentences

Many expressions of TFL will be total gibberish. We want to know when an expression of TFL amounts to a *sentence*. To that end, we have the following seven rules, which are one part of the grammar of TFL.

atomic sentences	$A, B, C, \ldots, Z$
with subscripts if needed	$A_1, A_2, A_3, A_4, \ldots, J_{10}, J_{11}, \ldots$
logical operators	$\neg, \&, \lor, \rightarrow, \leftrightarrow$
parentheses	(,)

Table 6.1: The three types of symbols of TFL

Sentences of TFL

1. Every atomic sentence is a sentence.
2. If A is a sentence, then $\neg\mathsf{A}$ is a sentence.
3. If A and B are sentences, then $(\mathsf{A} \,\&\, \mathsf{B})$ is a sentence.
4. If A and B are sentences, then $(\mathsf{A} \lor \mathsf{B})$ is a sentence.
5. If A and B are sentences, then $(\mathsf{A} \rightarrow \mathsf{B})$ is a sentence.
6. If A and B are sentences, then $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a sentence.
7. Nothing else is a sentence.

Notice that A and A are different fonts. A is an atomic sentence in TFL. A (which is not, actually, part of TFL) is a *metavariable*. It stands for any sentence of TFL. The sentence that it stands for could be anything: A or $(B \rightarrow D)$ or $((P \leftrightarrow Q) \lor T)$ or anything else. This use of metavariables is explained more fully in §.

From the previous chapter, you have the basic idea about how the logical operators are used. The simplest cases are when '&', '$\lor$', '$\rightarrow$', or '$\leftrightarrow$' only connect two atomic sentences or when '$\neg$' is before a single atomic sentence, as we find here:

$$(P \leftrightarrow Q)$$
$$\neg R$$

But our logic system must be more complex than this to be useful. To see how we can create more complex sentences in TFL, we will start with these two sentences:

$$(P \,\&\, S)$$
$$(R \rightarrow T)$$

Now, let's see how we would express "$(P \,\&\, S)$ *or* $(R \rightarrow T)$." Although $(P \,\&\, S)$ and $(R \rightarrow T)$ are both composed of two atomic sentences and a logical operator, we can treat each as a unit and combine them with a '$\lor$' like this:

$$((P \,\&\, S) \lor (R \rightarrow T))$$

Similarly, we can express '*not* $(P \,\&\, S)$' by treating the $(P \,\&\, S)$ as a unit and adding a '$\neg$':

$$\neg(P \,\&\, S)$$

We can even, if we need to do so, express '*not* $\neg(P \,\&\, S)$' by implementing the same procedure again. Now, $\neg(P \,\&\, S)$ is the unit and we add a '$\neg$' to it:

$$\neg\neg(P \,\&\, S)$$

These procedures can be used to form an infinite number of sentences of TFL. We just have to be sure that we are following rules 1 – 7, given on p. 50. And when following these rules, we must remember that A and B are units that can stand for either single atomic sentences or longer sentences. In fact, they can stand for sentences of any length.

Ultimately, you want to be able to just look at an expression and tell whether or not it is a correctly formed sentence of TFL, and with time you will be able to do so. Here are some examples of sentences of TFL:

1. $((P \,\&\, R) \to (S \to T))$
2. $(P \,\&\, (R \to (S \to T)))$
3. $((P \leftrightarrow \neg S) \vee \neg(T \leftrightarrow R))$
4. $((R \leftrightarrow T) \,\&\, \neg(P \vee (Q \vee \neg T)))$
5. $\neg(P \to \neg(R \vee (S \leftrightarrow T)))$
6. $((S \vee P) \,\&\, \neg(R \vee \neg\neg R))$

These, on the other hand, are **not** sentences of TFL because each violates one or more of 1 – 7:

1. $(P\neg \,\&\, R)$
2. $((R \,\&\, \vee S) \to Q)$
3. $(R \,\&\, Q \to)$
4. $((P\neg Q) \,\&\, (R \leftrightarrow T))$
5. $(P, Q \leftrightarrow T)$
6. $(P \,\&\, Q \,\&\, R)$

You will learn to recognize sentences of TFL more quickly if you write neatly and space the atomic sentences, logical operators, and parentheses as is shown in this textbook. Spaces are not actually part of TFL, and so technically, you don't need to use them. But just as you would never add or drop spaces when writing sentences in English, you should not add or drop spaces when using TFL.

6.3 The main logical operator

Setting aside the '$\neg$' for a moment, each of the other logical operators, as you know, combine two sentences (which may, themselves, be composed of multiple sentences). The two sentences that are joined together by a logical operator comprise that operator's SCOPE.

Scope

For '&', '$\lor$', '$\to$', and '$\leftrightarrow$', the SCOPE of the logical operator is the two sentences that the logical operator is conjoining.

For '$\neg$', the SCOPE is the sentence being negated by this operator.

So, the scope of the '&' in $(P \,\&\, Q)$ is the P and the Q And the scope of the '&' in $((P \,\&\, Q) \to T)$ is still just the P and the Q. The scope of the '$\to$' in $((P \,\&\, Q) \to T)$ is $(P \,\&\, Q)$ and T.

Here is a more complex example:

$$(((T \to P) \,\&\, R) \lor (S \leftrightarrow Q))$$

The scope of the '$\to$' is T and P.
The scope of the '&' is $(T \to P)$ and R.
The scope of the '$\leftrightarrow$' is S and Q.
The scope of the '$\lor$' is $((T \to P) \,\&\, R)$ and $(S \leftrightarrow Q)$.

When the scope of a logical operator is the whole sentence (besides that operator itself), then that logical operator is the MAIN LOGICAL OPERATOR. So, in the previous example, the '$\lor$' is the main logical operator. This means that the sentence is a disjunction. (The subsentences are different kinds of sentences, but the whole sentence is a disjunction.)

Main Logical Operator (MLO)

The MAIN LOGICAL OPERATOR is the logical operator whose scope is the entire sentence.

Being able to identify the main logical operator is very important for what you will be learning in parts 3 and 4 of this textbook, and so when you see a sentence of TFL, you always want to determine the scope of each logical operator and the identify the main logical operator.

Here are some more examples:

$((P \,\&\, R) \rightarrow (\neg Q \,\&\, S))$	The MLO is the '$\rightarrow$'.
$(((S \vee N) \leftrightarrow Q) \,\&\, (T \rightarrow R))$	The MLO is the '&'.
$(((\neg D \vee N) \rightarrow R) \leftrightarrow T)$	The MLO is the '$\leftrightarrow$'.
$(P \,\&\, (T \leftrightarrow (Q \vee R)))$	The MLO is the '&'.
$((\neg E \vee F) \leftrightarrow G)$	The MLO is the '$\leftrightarrow$'.

Unlike the other operators, the '$\neg$', doesn't connect two sentences; it just negates a sentence (which may be composed of multiple subsentences). Hence, the scope of the '$\neg$' is the sentence that is being negated. For each of these examples, the scope of the '$\neg$' is the whole sentence (besides the '$\neg$' itself):

$$\neg Q$$
$$\neg(P \vee T)$$
$$\neg(R \vee (S \rightarrow N))$$
$$\neg((P \,\&\, R) \vee (S \rightarrow T))$$

And since, in each of these examples, the scope of the '$\neg$' is the whole sentence, the '$\neg$' is the main logical operator in each case.

Once the main logical operator has been identified, we know what kind of sentence we have and what its components are. (This will be super important when we get to chapter .) If it is a conjunction, then part of the sentence will be one conjunct and the rest will be the other conjunct (and nothing will be left over). If it's a disjunction, then part of the sentence will be one disjunct and the rest will the other disjunct, again with nothing left over. If it's a conditional, then part of the sentence will be the antecedent and the rest will be the consequent. And if it's a negation, then the whole sentence (minus the 'not' itself) is being negated.

6.4 Using parentheses

Parentheses are required for any sentence of TFL containing two or more atomic sentences. One reason for this is because the rules given on p. 50 require it. Another reason for using parentheses is that we might make a sentence, say, $(Q \& R)$, a sub-sentence in a more complex sentence. For example, we might want to negate $(Q \& R)$, which would give us $\neg(Q \& R)$. If we just had $Q \& R$ without the parentheses and put a '$\neg$' in front of it, we would have $\neg Q \& R$, which has a different meaning than $\neg(Q \& R)$.

That said, there are some convenient conventions that we can use as long as we are careful. First, if the entire sentence is not about to become a sub-sentence, we can omit the sentence's *outermost* parentheses. Thus, we allow ourselves to write $Q \& R$ instead of $(Q \& R)$ when $Q \& R$ is the whole sentence. We must remember, however, to put parentheses around it when we want to embed the sentence into a more complex one.

Second, it can be a bit difficult to stare at long sentences with many nested pairs of parentheses. To make things a bit easier on the eyes, we will allow ourselves to use square brackets, '[' and ']', in addition to rounded ones. So, there is no logical difference, for example, between $(P \vee Q)$ and $[P \vee Q]$.

Combining these two conventions, we can rewrite this sentence:

$$(((H \rightarrow I) \vee (I \rightarrow H)) \& (J \vee K))$$

like this:

$$[(H \rightarrow I) \vee (I \rightarrow H)] \& (J \vee K)$$

The scope of each logical operator is now much easier to identify.

6.5 Metavariables

Sometimes we refer to specific sentences of TFL like P and $\neg(\neg Q \vee R)$. Other times, however, we want to say something about an arbitrary sentence of TFL, not a specific one. To do this, we use these uppercase letters:

$$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$$

You probably noticed that we used these letters in our definition of a sentence of TFL in section . For instance, this is one rule in that definition:

3. If A and B are sentences of TFL, then $(\mathsf{A} \,\&\, \mathsf{B})$ is a sentence of TFL.

We use A and B (and other capital letters in this font) when we want the letter to stand for any possible sentence of TFL. Hence, A can stand for A (or B) or for $(P \vee Q)$ or $((R \to T) \,\&\, \neg Q)$ or anything else that counts as a sentence of TFL.

$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$, etc. are not actually part of TFL. Rather, they are a part of English, and we use them to talk about expressions of TFL.

Metavariable

A METAVARIABLE is a variable in the metalanguage (i.e., English) that represents any sentence in TFL. The symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$ are used for the metavariables.

When we use one language to talk about another one, the language about which we are talking is called the OBJECT LANGUAGE, and the language that we use to talk about the object language is called the METALANGUAGE

6.6 Practice exercises

A. For each of the following, (a) is it, strictly speaking, a sentence of TFL, and (b) is it a sentence of TFL when we allow ourselves to drop the outermost parentheses and use square brackets? If, by either of these standards, it is a sentence of TFL, then (c) what is the main logical operator?

1. (A)
2. $J_{374} \vee \neg J_{374}$
3. $\neg\neg\neg\neg F$
4. $\neg \,\&\, S$
5. $(G \,\&\, \neg G)$
6. $(A \to (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$
7. $[(Z \leftrightarrow S) \to W] \,\&\, [J \vee X]$

8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \,\&\, D)$

B. What is the scope of each connective in this sentence?

$$[(H \rightarrow I) \vee (G \leftrightarrow M)] \,\&\, (J \vee K)$$

6.7 Answers

A.

1. (A) (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ (a) no (b) yes (c) the '$\vee$'
3. $\neg\neg\neg\neg F$ (a) yes (b) yes (c) the first '$\neg$'
4. $\neg \,\&\, S$ (a) no (b) no
5. $(G \,\&\, \neg G)$ (a) yes (b) yes (c) the '&'
6. $(A \rightarrow (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$ (a) no (b) yes (c) the '$\vee$'
7. $[(Z \leftrightarrow S) \rightarrow W] \,\&\, [J \vee X]$ (a) no (b) yes (c) the '&'
8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \,\&\, D)$ (a) no (b) no

B. $[(H \rightarrow I) \vee (G \leftrightarrow M)] \,\&\, (J \vee K)$

The scope of the '$\rightarrow$' is H and I.
The scope of the '$\leftrightarrow$' is G and M.
The scope of the first '$\vee$' is $(H \rightarrow I)$ and $(G \leftrightarrow M)$.
The scope of the second '$\vee$' is J and K.
The scope of the '&' is $[(H \rightarrow I) \vee (G \leftrightarrow M)]$ and $(J \vee K)$. Therefore, the '&' is the main logical operator, and the sentence is a conjunction.

Part 3

Truth tables

7 Characteristic truth tables

7.1 A quick introduction to truth tables

Consider this sentence:

> Either the key is on the table, or Jane is on the train and the key is not on the table.

Now, ask yourself, when is this sentence true and when is it false?

(a) If *the key is on the table* is true, then the sentence will be true regardless of whether *Jane is on the train* is true or false.
(b) If *the key is on the table* is false, then the sentence will be true as long as *Jane is on the train* is true.
(c) But if both *the key is on the table* is false and *Jane is on the train* is false, then the sentence will be false.

We worked that out by thinking about the different possible scenarios, and the logic of 'or' and 'and'. An alternative, and a somewhat easier method, is to create a truth table. Truth tables tell us when a sentence is true or false, and, as we will see in chapters and , they allow us to perform other analyses as well.

This is the truth table for the above sentence:

J	K	K	$\vee$	$(J$	$\&$	$\neg$	$K)$
T	T	T	**T**	T	F	F	T
T	F	F	**T**	T	T	T	F
F	T	T	**T**	F	F	F	T
F	F	F	**F**	F	F	T	F

To begin thinking about truth tables, notice the following features of this one.

1. The sentence for which we are creating the truth table, in this case, $(K \vee (J \,\&\, \neg K))$, is at the top of the truth table, to the right of the vertical line.
2. The atomic sentences that are in $(K \vee (J \,\&\, \neg K))$ are at the top of the truth table to the left of the vertical line, and they are placed in alphabetical order.
3. The Ts and Fs below the horizontal line stand for 'true' and 'false'. (See the definition of TRUTH VALUES at the beginning of the next section.)
4. Below the J and K on the left are the different possible combinations of true and false. Below the J, each T is a scenario where J is true, and each F is a scenario where J is false. (And likewise for K.) So, on the first line (right below the horizontal line), J is true, and K is true. On the second line, J is true, and K is false. On the third line, J is false, and K is true. And on the fourth line, J is false, and K is false.
5. The columns of Ts and Fs that are to the left of the vertical line are repeated on the right side of the table under each J and K, respectively.

J	K	K	$\vee$	$(J$	$\&$	$\neg$	$K)$
T	T	T	T	T	F	F	T
T	F	F	T	T	T	T	F
F	T	T	T	F	F	F	T
F	F	F	F	F	F	T	F

These are the basic features of every truth table. The task when creating a truth table is knowing how, and in what order, to fill in the columns under the logical operators. In this chapter, we will examine simple sentences containing only one logic operator. These are the "characteristic truth tables" for each logical operator. In the next chapter, we will explore how to create truth tables for more complex sentences.

7.2 The characteristic truth tables

You were introduced to five logical operators in chapter , and now we need to explain when sentences using each one are true and when they are false.

> **Truth values**
>
> *Truth values* are the logical values that a sentence can have: *true* and *false*.

Conjunction For any sentences **A** and **B**, the conjunction (**A** & **B**) is true if and only if both **A** and **B** are true. If one or both of **A** and **B** are false, then the sentence (**A** & **B**) is false. We can summarize this in the *characteristic truth table for conjunction*:

A	B	A	&	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	F	F

Looking at line 1, we see that, when **A** is true and **B** is true, there is a 'T' under the '&', which indicates that (**A** & **B**) is true. On line 2, meanwhile, **A** is true and **B** is false, which means that (**A** & **B**) is false.

Lines 3 and 4 represent the final two combinations of 'true' and 'false' for **A** and **B**. On line 3, **A** is false and **B** is true. When that is the case, (**A** & **B**) is false. And then on line 4, **A** and **B** are both false. In that scenario, (**A** & **B**) is, again, false.

Note that conjunction is *symmetrical*. The truth value for (**A** & **B**) is always the same as the truth value for (**B** & **A**).

Negation For any sentence **A**, if **A** is true, then ¬**A** is false. And likewise, if **A** is false, then ¬**A** is true. This is represented in the *characteristic truth table for negation*:

A	¬	A
T	F	T
F	T	F

Disjunction Recall that '$\vee$' always represents the inclusive-or. So, for any sentences A and B, the disjunction $(\mathsf{A} \vee \mathsf{B})$ is true when A is true or B is true or both are true. The only instance when $(\mathsf{A} \vee \mathsf{B})$ is false is when both A and B are false. This is represented in the *characteristic truth table for disjunction*:

A	B	A	∨	B
T	T	T	T	T
T	F	T	T	F
F	T	F	T	T
F	F	F	F	F

This is a good time to explain another point. We are, in this chapter, simply stipulating when each of these types of sentences are true and false. This amounts to a definition for each logical operator in TFL. (Thus, the meaning of '$\vee$' is what is given in the above truth table.) We have reasons for defining them these ways, and there is a consensus that these are the best definitions. But, in the end, these are the correct truth tables for each logical operator because these are the ways that we have chosen to set them.

Conceivably, we could say that $(\mathsf{A} \vee \mathsf{B})$ is false when A and B are both false *and* when A and B both are true. That would agree with the way that we, at least some of the time, use *or* in English. But that's not what we've chosen to do, and so the way that $(\mathsf{A} \vee \mathsf{B})$ is defined in the truth table above is going to apply from this point forward (and similarly for all of the other connectives).

Conditional The conditional is interesting and, for some, philosophically contentious. One way to think about the conditional is as rule: if the antecedent happens, then the consequent has to happen. So, for instance, take this conditional:

> If it is Wednesday, then I am on campus by 10:00 am.

This sentence is obviously true when (1) it is Wednesday and I am on campus by 10:00 am. Conversely, this sentence is false when (2) it is Wednesday, but I am not on campus by 10:00 am. (If that happens, the rule has been broken.) Those two scenarios are represented by lines 1 and 2 in the characteristic truth table for the conditional, which is as follows.

A	B	A	→	B
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

For the other two scenarios, we have to concentrate a bit.

(3) Our conditional is also true when it is not Wednesday (let's say it's Tuesday), but I'm on campus by 10:00 am. In this case, the rule *if it is Wednesday, then I am on campus by 10:00 am* hasn't been broken; it just doesn't apply. So, when the antecedent is false and the consequent is true, the conditional is true. That's represented by line 3 of the characteristic truth table for the conditional.
(4) Similarly, when it is not Wednesday and I am not on campus by 10:00 am, the rule hasn't been broken. It is still in force. It just hasn't been invoked at all. So even though the antecedent didn't happen and the consequent didn't happen, the conditional is still true. (In other words, let's say, it's Saturday and, at 10:00 am, I am at home in bed. It's false that 'it is Wednesday' and it's false that 'I am on campus by 10:00 am', but it's still true that *if it is Wednesday, then I am on campus by 10:00 am*.) This scenario is represented on line 4 of the characteristic truth table.

Those four scenarios are pretty straightforward. The conditional is philosophically contentious, however, because every conditional is not as simple as 'if it is Wednesday, then I am on campus by 10:00 am'. Take a conditional where the antecedent is always false: 'if the King of England is on the moon, then Mississippi State University is in Starkville.' This isn't much of a rule, but it is a conditional. And since, in our current world, the antecedent is false and the consequent is true, the sentence is true. Even stranger, consider this conditional: 'if the King of England is on the moon, then pigs can fly.' Now, the antecedent is always false and the consequent is always false (at least in our world), but, as is shown on line 4 of the characteristic truth table, the sentence is true.

Sometimes the truth values for the antecedent, the consequent, and the whole conditional make sense (as in our first example) and sometimes they seem odd. That has generated philosophical debate, but it actually does not present a problem for us. The conditional is precisely defined by its characteristic truth table. We, then, simply use that definition, and we don't have to make any decisions about whether a particular conditional is odd or should really be considered true or false.

Finally, notice that, unlike the conjunction and the disjunction, the conditional is *asymmetrical*. You cannot switch the antecedent and consequent without changing the meaning of the sentence. This is because $\mathsf{A} \to \mathsf{B}$ (e.g., 'if it is Wednesday, then I am on campus by 10:00 am') has a different truth table than $\mathsf{B} \to \mathsf{A}$ (e.g., 'if I am on campus by 10:00 am, then it is Wednesday').

Biconditional As we said in §, the biconditional is equivalent to the conjunction of a conditional running in each direction—that is, to $(\mathsf{A} \to \mathsf{B})$ & $(\mathsf{A} \to \mathsf{B})$. Consequently, on every line where both $\mathsf{A} \to \mathsf{B}$ is true and $\mathsf{B} \to \mathsf{A}$ is true, $\mathsf{A} \leftrightarrow \mathsf{B}$ is true. On every line where either $\mathsf{A} \to \mathsf{B}$ is false or $\mathsf{B} \to \mathsf{A}$ is false, $\mathsf{A} \leftrightarrow \mathsf{B}$ is false. That yields the following characteristic truth table for the biconditional.

A	B	A	$\leftrightarrow$	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	T	F

7.3 The counterfactual conditional

As we have seen, in English, *or* can be used as the *inclusive or* or as the *exclusive or*, but in TFL we have only the *inclusive or* (pp. 40, 62). The conditional is in a somewhat similar situation. In the section on the conditional in this chapter, I used the example 'If it is Wednesday, then I am on campus by 10:00 am' to illustrate that the characteristic truth table for the conditional fits with the way that we treat conditional statements in English. This is only partially true. 'If it is Wednesday, then I am on campus by 10:00 am' is an *indicative conditional*, which is the type of conditional that is often, although not always, used in English.

indicative conditional

An INDICATIVE CONDITIONAL has indicative sentences as the antecedent and the consequent. Truth-functional logic uses the indicative conditional.
An *indicative sentence* is one that expresses a fact—or expresses what seems to be the case, e.g., 'today is Wednesday' or 'Jones is at the train station'.

By way of introducing the other type of conditional (which is not a part of TFL), consider these two sentences:

1. If Theodore Roosevelt had won the 1912 presidential election, he would have become the first person to serve a third term as president of the United States.
2. If Theodore Roosevelt had won the 1912 presidential election, he would have transformed into an adult male moose.

One would assume (correctly) that the first sentence is true and the second one is false. But if we evaluate them using the characteristic truth table for the conditional (p. 63), we find that both are true. Both have false antecedents—Theodore Roosevelt did not win the 1912 presidential election—and both have false consequents. (He didn't become the first person to serve a third term, and he also didn't transform into a moose.)

So, what is going on here? These are *counterfactual conditionals* (also called *subjunctive conditionals*), not indicative conditionals. They ask us to treat a statement that is false (because it didn't happen) as if it had happened (i.e., to treat something as counter to the facts). Then, once we've made this move—in this case, we are imagining that Teddy Roosevelt did win the 1912 election—we can evaluate whether the consequent would have happened or not. For sentence 1, it would have, and so this sentence is true. (But notice that the reason why this sentence is true is different than the reason that we get from consulting the characteristic truth table for the conditional.) Sentence 2, on the other hand, is false because if Roosevelt had won, he would not have transformed into a moose.

8 Complete truth tables

In chapter , we examined the characteristic truth tables for the logical operators of TFL. The characteristic truth tables show us when a sentence with only one of the logical operators is true and when it is false. That, in effect, is a definition for each logical operator. Now that we have those definitions, we can investigate when other, more complex sentences are true and false—for instance, ones like this one: '$(H \;\&\; I) \to H$' and '$(M \;\&\; (N \lor P))$', which we will go through in this chapter. Once we understand how to create truth tables, we can investigate other properties of sentences of TFL, which we will do in chapters and .

Before we begin, we will define TRUTH-VALUE ASSIGNMENT.

> Truth-value assignment
>
> A *truth-value assignment* is any assignment of truth values to atomic sentences of TFL. **The left side of each row of a truth table represents a possible truth-value assignment.** The entire left side of the truth table represents all possible truth-value assignments. (See also the truth table on p. 83.)

Thus, the truth table provides us with a way of finding the truth values of complex sentences on each possible truth-value assignment—that is, for every combination of 'true' and 'false' for every atomic sentence.

8.1 An example

Consider the sentence '$(H \;\&\; I) \to H$', which contains three atomic sentences, although only two different ones. We set up the truth table for this sentence by putting H and I on the left side of the vertical line and '$(H \;\&\; I) \to H$' on the right. (Although H appears twice in '$(H \;\&\; I) \to H$', we only need one H on the left.) Below the H and I on the left side, we put every combination of 'T' and 'F'.

Since we have two atomic sentences on the left, there are four combinations of true and false. For consistency, the Ts and Fs should always be listed this way: (a) in the column next to the vertical line, they alternate T, F, T, F; (b) in the next column (to the left), they alternate in pairs: T, T, F, F; and (c) if there are more than two atomic sentences, then more columns and more rows are needed, but the pattern remains the same. (See table on p. 73.)

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T					
T	F					
F	T					
F	F					

Once the left side of the truth table is completed, we begin filling in the right side. First, we copy the truth values for the atomic sentences. For the H, that gives us this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T				T
T	F	T				T
F	T	F				F
F	F	F				F

Adding the truth values for I, we have this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T		T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

8.2 Truth tables and scope

Now , there are two columns that remain. The one under the '&' and the one under the '→'. You may have thought that a logical operator's scope was of only passing importance when we discussed it in §. It is, however, extremely important for understanding how to complete truth tables for sentences that contain more than one logical operator.

To determine the order in which we complete a truth table, we follow this rule.

rule 1 for sentences containing mulitiple logical operators

We can only fill in the column under a logical operator when the columns for everything in that logical operator's scope are complete.

In '$(H \& I) \rightarrow H$', the scope of the '&' is the 'H' and the 'I' inside the parentheses. Since the columns under the 'H' and the 'I' are complete, we can fill in the column under the '&'. To do this, we turn to the characteristic truth table for the conjunction. As is shown on p. 61, when 'H' and 'I' are both true, we put a 'T' below the '&'.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	**T**	T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

On the second line, 'H' is true and 'I' is false. That means that '$(H \& I)$' is false, and so we put 'F' on the second line below the '&'.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	T	T		T
T	F	T	**F**	F		T
F	T	F		T		F
F	F	F		F		F

Following the characteristic truth table for conjunction, we fill in the truth values for the third and fourth lines, and that completes the column under the '&'.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	**T**	T		T
T	F	T	**F**	F		T
F	T	F	**F**	T		F
F	F	F	**F**	F		F

The scope of the '→' is the '$(H \& I)$' and the 'H'. Since the columns under both are complete, we can finish the truth table by filling in the column under the '→'. But first, we need a second rule.

rule 2 for sentences containing mulitiple logical operators

For any sentence or sub-sentence (other than an atomic sentence), the column under its main logical operator represents the truth value for the sentence. (For an atomic sentence, the column under the letter represents its truth value.)

Since the '&' is the main logical operator in '$(H \& I)$', the column below the '&' is one of the columns that we must consult to fill in the column under the '→'. The other is the column below the 'H' after the '→'. Using the truth values in those two columns, we then refer to the characteristic truth table for the conditional on p. 63.

On the first line, '$(H \& I)$' is true and 'H' is true, and so we put a 'T' beneath the '→'.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	T	T	**T**	T
T	F	T	F	F		T
F	T	F	F	T		F
F	F	F	F	F		F

On the second row, '$(H \& I)$' is false and 'H' is true. (That's the truth-value assignment given on the third line of the characteristic truth table for the conditional, not the second.) A conditional is true when the antecedent is false and the consequent is true, and so we put a 'T' in the second row beneath the '→'.

H	I	$(H$	$\&$	$I)$	$\to$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T		F
F	F	F	F	F		F

On the third and fourth rows, '$(H \,\&\, I)$' is false and 'H' is false, and so again, we put 'T' below the '$\to$' on each line. (On both of these lines, the antecedent is false and the consequent is false, and so these correspond to line four in the characteristic truth table for the conditional.)

H	I	$(H$	$\&$	$I)$	$\to$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Since the '$\to$' is the main logical operator for '$(H \,\&\, I) \to H$', we've now determine the truth values for this sentence. The column of 'T's beneath the '$\to$' tells us that '$(H \,\&\, I) \to H$' is true regardless of the truth values of 'H' and 'I'. Those atomic sentences can be true or false in any combination, and the full sentence, '$(H \,\&\, I) \to H$', remains true. Since we have considered all four possible assignments of truth and falsity to 'H' and 'I', we can say that '$(H \,\&\, I) \to H$' is true on every *truth-value assignment*.

In the truth table for any sentence, the most important column is the one beneath the main logical operator for the sentence because this column tells us the truth value of the entire sentence. We have emphasized it in the last truth table above by putting this column in bold. When you work through truth tables yourself, you should similarly emphasize it with underlining or circling.

8.3 Building complete truth tables

A COMPLETE TRUTH TABLE has a line for every possible combination of *true* and *false* for the atomic sentences that compose the full sentence.

Each line represents a *truth-value assignment*, and a complete truth table has a line for all of the different truth-value assignments.

The size of the complete truth table depends on the number of *different* atomic sentences in the table. The truth table for a TFL sentence that contains only one atomic sentence, perhaps repeated multiple times, needs only two rows. Here is an example of such a sentence:

$$((P \,\&\, \neg P) \rightarrow P)$$

The truth table for this — or any sentence containing only one atomic sentence — is only two lines because there are only two possibilities: 'P' can be true or it can be false.

P	((P	&	¬	P)	→	P)
T	T	F	F	T	**T**	T
F	F	F	T	F	**T**	F

As we have seen, for a sentence that contains two different atomic sentences, we need four lines for a complete truth table.

The complete truth table for a sentence that contains three different atomic sentences, meanwhile, needs eight lines, as shown right below. Notice that the 'T's and 'F's in the columns below 'N' and 'P' (on the left side) follow the same pattern as the example in the previous section. The column under the 'M', meanwhile, has four 'T's and then four 'F's.

M	N	P	M	&	(N	∨	P)
T	T	T	T	**T**	T	T	T
T	T	F	T	**T**	T	T	F
T	F	T	T	**T**	F	T	T
T	F	F	T	**F**	F	F	F
F	T	T	F	**F**	T	T	T
F	T	F	F	**F**	T	T	F
F	F	T	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F

A complete truth table for a sentence that contains four different atomic sentences has to have 16 lines. If the sentence has five different letters, the truth table must have 32 lines. If it has six letters, it will have 64 lines, and so on following this rule: a truth table for a TFL sentence with n different atomic sentences must have 2^n rows.

THE LEFT SIDE OF THE TRUTH TABLE

COLUMN	PATTERN
first (next to the vertical line)	T, F, T, F, ...
second	T, T, F, F, ...
third	T, T, T, T, F, F, F, F, ...
fourth	8 Ts, 8 Fs, ...
fifth	16 Ts, 16 Fs, ...

Table 8.1: Every truth table for the same sentence should be the same. To ensure that they are, the columns on the left side of the truth table should be filled in using the patterns given in this table. The first column is the one closest to the vertical line.

8.4 Some more examples

1. To create a truth table for '$(P \leftrightarrow Q) \to (P \lor Q)$', first, we fill in the columns below each P and Q. Next, since the truth values for everything in the scope of the '$\leftrightarrow$' and the '$\lor$' are complete, we can fill in the columns below those two logical operators (in either order).

P	Q	(P	$\leftrightarrow$	Q)	$\to$	(P	$\lor$	Q)
T	T	T	T	T		T	T	T
T	F	T	F	F		T	T	F
F	T	F	F	T		F	T	T
F	F	F	T	F		F	F	F

Once we have those columns complete, we finish the truth table by filling in the column under the '$\to$', which we do by looking at the column under the '$\leftrightarrow$' and the column under the '$\lor$'.

P	Q	(P	$\leftrightarrow$	Q)	$\to$	(P	$\lor$	Q)
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	T	T	F
F	T	F	F	T	**T**	F	T	T
F	F	F	T	F	**F**	F	F	F

2. To make a truth table for '$P \& \neg Q$', after we have filled in the columns below the P and Q, we fill in the column under the $\neg$. To do that, we look at the column under the Q.

P	Q	(P	&	$\neg$	Q)
T	T	T		**F**	T
T	F	T		**T**	F
F	T	F		**F**	T
F	F	F		**T**	F

Then, to complete the truth table, we fill in the column under the '&'—which we do by looking at the column under the P and the column under the '$\neg$'.

P	Q	(P	&	$\neg$	Q)
T	T	T	**F**	F	T
T	F	T	**T**	T	F
F	T	F	**F**	F	T
F	F	F	**F**	T	F

3. For '$\neg(P \to Q)$', after we have filled in the columns under the 'P' and the 'Q', we fill in the column under the '$\to$'.

P	Q	$\neg$	(P	$\to$	Q)
T	T		T	T	T
T	F		T	F	F
F	T		F	T	T
F	F		F	T	F

Then, to complete the table, we fill in the column under the '$\neg$'. To fill in this column, we look at the column under the '$\to$'.

P	Q	$\neg$	(P	$\to$	Q)
T	T	**F**	T	T	T
T	F	**T**	T	F	F
F	T	**F**	F	T	T
F	F	**F**	F	T	F

4. $(P \& \neg Q) \vee Q$

The scope of the '&': 'P' and '$\neg Q$'. Order: 2nd.
The scope of the $\neg$: 'Q'. Order: 1st.
The scope of the $\vee$: '$(P \& \neg Q)$' and 'Q'. Order: 3rd.

P	Q	(P	&	$\neg$	Q)	$\vee$	Q
T	T	T		F	T		T
T	F	T		T	F		F
F	T	F		F	T		T
F	F	F		T	F		F

Next, while looking at the column under the 'P' and under the '$\neg$', we fill in the column under the '&'.

P	Q	(P	&	$\neg$	Q)	$\vee$	Q
T	T	T	**F**	F	T		T
T	F	T	**T**	T	F		F
F	T	F	**F**	F	T		T
F	F	F	**F**	T	F		F

Then last, we fill in the column under the '$\vee$' while looking at the column under the '&' and under the 'Q'.

P	Q	(P	&	$\neg$	Q)	$\vee$	Q
T	T	T	F	F	T	**T**	T
T	F	T	T	T	F	**T**	F
F	T	F	F	F	T	**T**	T
F	F	F	F	T	F	**F**	F

8.5 Truth tables in Carnap

You should practice making truth tables on paper, but you also need to make them on the online site Carnap. Using Carnap is pretty straightforward, and it's made easier because the left side of the truth table is completed for you. (See figure .) On the right side, below each atomic sentence and connective, you have the option of selecting a 'T' or an 'F'. (See figure .)

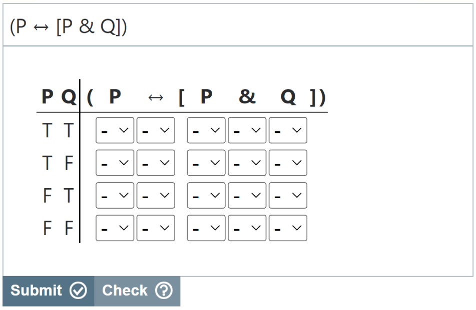

Figure 8.1

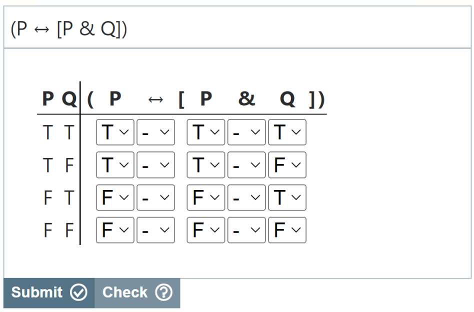

Figure 8.2

Most often (although not always), the problems in Carnap will be set up so that you will only be able to submit your answers when they are correct. At those times, once the truth table is complete, you will select 'Check'. Carnap will tell you "Success!" or "Something's not quite right." It is easy to make a mistake when filling in a truth table, and so if something is not quite right, then you have to inspect every truth value until you find the mistake. Then select 'Check' again. Once Carnap confirms that the truth table is correct, select 'Submit'. **Don't forget to submit after you complete every truth table correctly.**

8.6 Practice exercises

A. Make a complete truth table for each sentence.

1. $S \rightarrow \neg S$
2. $\neg(P \leftrightarrow \neg Q)$

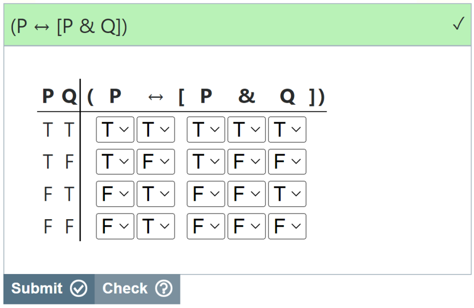

Figure 8.3: A completed and verified truth table in Carnap.

3. $(P \,\&\, Q) \rightarrow (Q \vee P)$
4. $\neg(P \vee Q) \leftrightarrow (\neg P \,\&\, \neg Q)$
5. $[(P \,\&\, Q) \,\&\, S] \rightarrow Q$
6. $\neg[(S \vee P) \vee Q]$

B. Check whether each of these statements is true.

1. '$((P \,\&\, Q) \,\&\, S)$' and '$(P \,\&\, (Q \,\&\, S))$' have the same truth table
2. '$((P \vee Q) \vee S)$' and '$(P \vee (Q \vee S))$' have the same truth table
3. '$((P \vee Q) \,\&\, S)$' and '$(P \vee (Q \,\&\, S))$' do not have the same truth table
4. '$((P \rightarrow Q) \rightarrow S)$' and '$(P \rightarrow (Q \rightarrow S))$' do not have the same truth table
5. '$((P \leftrightarrow Q) \leftrightarrow S)$' and '$(P \leftrightarrow (Q \leftrightarrow S))$' have the same truth table

C. Make truth tables for the following sentences, and mark the column that represents the possible truth values for the whole sentence.

1. $(D \,\&\, \neg D) \rightarrow G$
2. $\neg(N \leftrightarrow (P \rightarrow N))$
3. $\neg\neg(\neg P \,\&\, \neg Q)$
4. $[R \leftrightarrow (S \vee W)] \,\&\, \neg R$
5. $\neg(P \,\&\, (Q \,\&\, S)) \leftrightarrow (P \vee (Q \vee S))$
6. $\neg[(D \leftrightarrow F) \leftrightarrow G] \rightarrow (\neg D \,\&\, F)$

8.7 Answers

A.

1. $S \rightarrow \neg S$

S	S	$\rightarrow$	$\neg$	S
T	T	**F**	F	T
F	F	**T**	T	F

2. $\neg(P \leftrightarrow \neg Q)$

P	Q	$\neg$	$(P$	$\leftrightarrow$	$\neg$	$Q)$
T	T	**T**	T	F	F	T
T	F	**F**	T	T	T	F
F	T	**F**	F	T	F	T
F	F	**T**	F	F	T	F

3. $(P \,\&\, Q) \rightarrow (Q \vee P)$

P	Q	$(P$	$\&$	$Q)$	$\rightarrow$	$(Q$	$\vee$	$P)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	F	T	**T**	T	T	F
F	F	F	F	F	**T**	F	F	F

4. $\neg(P \vee Q) \leftrightarrow (\neg P \,\&\, \neg Q)$

P	Q	$\neg$	$(P$	$\vee$	$Q)$	$\leftrightarrow$	$(\neg$	P	$\&$	$\neg$	$Q)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

5. $[(P \,\&\, Q) \,\&\, S] \rightarrow Q$

P	Q	S	$[(P$	$\&$	$Q)$	$\&$	$S]$	$\rightarrow$	Q
T	T	T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	F	F	**T**	T
T	F	T	T	F	F	F	T	**T**	F
T	F	F	T	F	F	F	F	**T**	F
F	T	T	F	F	T	F	T	**T**	T
F	T	F	F	F	T	F	F	**T**	T
F	F	T	F	F	F	F	T	**T**	F
F	F	F	F	F	F	F	F	**T**	F

6. $\neg[(S \vee P) \vee Q]$

P	Q	S	$\neg[($	S	$\vee$	$P)$	$\vee$	$Q]$
T	T	T	**F**	T	T	T	T	T
T	T	F	**F**	F	T	T	T	T
T	F	T	**F**	T	T	T	T	F
T	F	F	**F**	F	T	T	T	F
F	T	T	**F**	T	T	F	T	T
F	T	F	**F**	F	F	F	T	T
F	F	T	**F**	T	T	F	T	F
F	F	F	**T**	F	F	F	F	F

B.

1. ‘$((P \,\&\, Q) \,\&\, S)$’ and ‘$(P \,\&\, (Q \,\&\, S))$’ have the same truth table

P	Q	S	$(P$	$\&$	$Q)$	$\&$	S	P	$\&$	$(Q$	$\&$	$S)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**F**	F	T	**F**	F	F	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

2. ‘$((P \vee Q) \vee S)$’ and ‘$(P \vee (Q \vee S))$’ have the same truth table

P	Q	S	$(P$	$\vee$	$Q)$	$\vee$	S	P	$\vee$	$(Q$	$\vee$	$S)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**T**	F	T	**T**	T	T	F
T	F	T	T	T	F	**T**	T	T	**T**	F	T	T
T	F	F	T	T	F	**T**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**T**	F	F	**T**	T	T	F
F	F	T	F	F	F	**T**	T	F	**T**	F	T	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

3. ‘$((P \vee Q)\ \&\ S)$’ and ‘$(P \vee (Q\ \&\ S))$’ do not have the same truth table

P	Q	S	$(P$	$\vee$	$Q)$	$\&$	S	P	$\vee$	$(Q$	$\&$	$S)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**T**	T	F	F
T	F	T	T	T	F	**T**	T	T	**T**	F	F	T
T	F	F	T	T	F	**F**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

4. ‘$((P \to Q) \to S)$’ and ‘$(P \to (Q \to S))$’ do not have the same truth table

P	Q	S	$(P$	$\to$	$Q)$	$\to$	S	P	$\to$	$(Q$	$\to$	$S)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**T**	T	T	**T**	F	T	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	T	T
F	F	F	F	T	F	**F**	F	F	**T**	F	T	F

5. ‘$((P \leftrightarrow Q) \leftrightarrow S)$’ and ‘$(P \leftrightarrow (Q \leftrightarrow S))$’ have the same truth table.

P	Q	S	$(P$	$\leftrightarrow$	$Q)$	$\leftrightarrow$	S	P	$\leftrightarrow$	$(Q$	$\leftrightarrow$	$S)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**T**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	F	T
F	F	F	F	T	F	**F**	F	F	**F**	F	T	F

C.

1. $(D \,\&\, \neg D) \to G$

D	G	(D	&	¬	D)	→	G
T	T	T	F	F	T	**T**	T
T	F	T	F	F	T	**T**	F
F	T	F	F	T	F	**T**	T
F	F	F	F	T	F	**T**	F

2. $\neg(N \leftrightarrow (P \to N))$

N	P	¬	(N	↔	(P	→	N))
T	T	**F**	T	T	T	T	T
T	F	**F**	T	T	F	T	T
F	T	**F**	F	T	T	F	F
F	F	**T**	F	F	F	T	F

3. $\neg\neg(\neg P \,\&\, \neg Q)$

P	Q	¬	¬	(¬	P	&	¬	Q)
T	T	**F**	T	F	T	F	F	T
T	F	**F**	T	F	T	F	T	F
F	T	**F**	T	T	F	F	F	T
F	F	**T**	F	T	F	T	T	F

4. $[R \leftrightarrow (S \vee W)] \,\&\, \neg R$

R	S	W	[R	↔	(S	∨	W)]	&	¬	R
T	T	T	T	T	T	T	T	**F**	F	T
T	T	F	T	T	T	T	F	**F**	F	T
T	F	T	T	T	F	T	T	**F**	F	T
T	F	F	T	F	F	F	F	**F**	F	T
F	T	T	F	F	T	T	T	**F**	T	F
F	T	F	F	F	T	T	F	**F**	T	F
F	F	T	F	F	F	T	T	**F**	T	F
F	F	F	F	T	F	F	F	**T**	T	F

Figure 8.4: Mistakes happen. Try to minimize them by not rushing and always consulting the characteristic truth tables in chapter . *Peanuts* © *1969*

5. $\neg(P \,\&\, (Q \,\&\, S)) \leftrightarrow (P \vee (Q \vee S))$

P	*Q*	*S*	¬	(P	&	(Q	&	S))	↔	(P	∨	(Q	∨	S))
T	T	T	F	T	T	T	T	T	**F**	T	T	T	T	T
T	T	F	T	T	F	T	F	F	**T**	T	T	T	T	F
T	F	T	T	T	F	F	F	T	**T**	T	T	F	T	T
T	F	F	T	T	F	F	F	F	**T**	T	T	F	F	F
F	T	T	T	F	F	T	T	T	**T**	F	T	T	T	T
F	T	F	T	F	F	T	F	F	**T**	F	T	T	T	F
F	F	T	T	F	F	F	F	T	**T**	F	T	F	T	T
F	F	F	T	F	F	F	F	F	**F**	F	F	F	F	F

6. $\neg[(D \leftrightarrow F) \leftrightarrow G] \rightarrow (\neg D \,\&\, F)$

D	*F*	*G*	¬	[(D	↔	F)	↔	G]	→	(¬	D	&	F)
T	T	T	F	T	T	T	T	T	**T**	F	T	F	T
T	T	F	T	T	T	T	F	F	**F**	F	T	F	T
T	F	T	T	T	F	F	F	T	**F**	F	T	F	F
T	F	F	F	T	F	F	T	F	**T**	F	T	F	F
F	T	T	T	F	F	T	F	T	**T**	T	F	T	T
F	T	F	F	F	F	T	T	F	**T**	T	F	T	T
F	F	T	F	F	T	F	T	T	**T**	T	F	F	F
F	F	F	T	F	T	F	F	F	**F**	T	F	F	F

9 Six concepts

As we did in the previous chapter, we begin with the definition of TRUTH-VALUE ASSIGNMENT.

truth-value assignment

A *truth-value assignment* is any assignment of truth values to atomic sentences of TFL. The left side of each row of a truth table represents a possible truth-value assignment. The entire left side of the truth table represents all possible truth-value assignments.

Let's say that we are going to create a truth table for $(P \lor \neg Q)$. On the first line of the truth table, you may recall, we make P = 'true' and Q = 'true'. This is one truth-value assignment (and it is dark purple on the first line of the truth table below).

On the second line, we make P = 'true' and Q = 'false'. This is another truth-value assignment (lighter purple). The assignments of 'true' and 'false' to P and Q on lines 3 and 4, then, are the remaining possible truth-value assignments (the darker and lighter pink respectively), when we have a sentence containing only two atomic sentences.

	P	Q	P	$\lor$	$\neg$	Q
	T	T	T	**T**	F	T
one truth-value assignment →	T	F	T	**T**	T	F
	F	T	F	**F**	F	T
	F	F	F	**T**	T	F

In the previous chapter, we used truth tables to determine — for each possible truth-value assignment — the truth value of any TFL sentence. In this chapter, we will extend this type of analysis. We will examine six properties that apply (or may apply) to either single TFL sentences (*tautology*, *contradiction*, and *contingent*) or sets of TFL sentences (*equivalent*, *jointly consistent*, and *jointly inconsistent*). For each, we use a truth table to determine which property the sentence or set of sentences has.

9.1 Individual sentences: Tautology and contradiction

In §, we said that a *necessary truth* is a sentence that must be true, a *necessary falsehood* is a sentence that must be false, and a sentence that is neither a necessary truth or a necessary falsehood is *contingent*. The first two, *necessary truth* and *necessary falsehood*, have surrogates in TFL. We will start with the one for necessary truth.

tautology

A sentence of TFL is a TAUTOLOGY if and only if it is true on every truth-value assignment.

We can determine whether a sentence is a tautology using a truth table. If the sentence is true on every line of a complete truth table (that is, if there is a 'T' on every line under the main logical operator), then it is true on every truth-value assignment. And if it is true on every truth-value assignment, it is a tautology. The example from §, $((H \,\&\, I) \rightarrow H)$, for instance, is a tautology.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Tautology is only a surrogate, however, for *necessary truth*. There are some necessary truths that we cannot adequately symbolize in TFL. An example is 'Five plus two equals seven'. This sentence *must* be true, but if we try to symbolize it in TFL, we will just have an atomic sentence such as this:

S: Five plus two equals seven.

But an atomic sentence by itself cannot be a tautology. (To see this, try making a truth table for just S.) Still, if we can adequately symbolize some English sentence as a TFL sentence, and that TFL sentence is a tautology, then the English sentence expresses a necessary truth.

We have a similar surrogate for *necessary falsehood*.

> contradiction
>
> A sentence of TFL is a CONTRADICTION if and only if it is false on every truth-value assignment.

Again, we can determine whether a sentence is a contradiction with a truth table. If the sentence is false on every line of a complete truth table, then it is false on every truth-value assignment, and so it is a contradiction. The standard example of a contradiction is $(P \;\&\; \neg P)$. Since we have only one letter in this sentence, it is only a two line truth table, but on each line, the sentence is false.

P	P	$\&$	$\neg$	P
T	T	**F**	F	T
F	F	**F**	T	F

Similarly, although its truth table has four lines, $((P \vee Q) \leftrightarrow (\neg P \;\&\; \neg Q))$ is a contradiction.

P	Q	$(P$	$\vee$	$Q)$	$\leftrightarrow$	$(\neg$	P	$\&$	$\neg$	$Q)$
T	T	T	T	T	**F**	F	T	F	F	T
T	F	T	T	F	**F**	F	T	F	T	F
F	T	F	T	T	**F**	T	F	F	F	T
F	F	F	F	F	**F**	T	F	T	T	F

In §, we defined CONTINGENT as “a sentence that is capable of being true and capable of being false (in different circumstances, of course).” A truth table, then, provides us with those different circumstances.

contingent

A sentence that is true on at least one truth-value assignment and false on at least one truth-value assignment is CONTINGENT.

Or, we can also say: any sentence that is neither a tautology nor a contradiction is CONTINGENT.

$\neg(P \vee Q)$, for instance, is contingent.

P	Q	$\neg$	$(P$	$\vee$	$Q)$
T	T	**F**	T	T	T
T	F	**F**	T	T	F
F	T	**F**	F	T	T
F	F	**T**	F	F	F

9.2 Equivalence

There are several possible logical relationships that can exist between two or more sentences of TFL. We we examine three of them, and we will focus on pairs of sentences (although these relationships can be applied to larger sets of sentences). The first logical relationship is EQUIVALENCE.

equivalent

A and B are EQUIVALENT if and only if, for every truth-value assignment, their truth values agree (that is, if and only if there is no truth-value assignment for which they have opposite truth values).

Alternatively, we can say that if $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology, then A and B are EQUIVALENT.

Recall from , that A stands for any possible sentence of TFL (as do B, C, D, etc.). Hence, A can stand for $(P \vee Q)$ or $((P \leftrightarrow \neg R) \,\&\, T)$ or anything else.

Consider the sentences $\neg(P \vee Q)$ and $(\neg P \,\&\, \neg Q)$. Are they equivalent? To find out, we construct a truth table containing both sentences.

P	Q	$\neg$	$(P$	$\vee$	$Q)$	$\neg$	P	$\&$	$\neg$	Q
T	T	**F**	T	T	T	F	T	**F**	F	T
T	F	**F**	T	T	F	F	T	**F**	T	F
F	T	**F**	F	T	T	T	F	**F**	F	T
F	F	**T**	F	F	F	T	F	**T**	T	F

Looking at the columns for the main logical operators ('¬' for the first sentence, '&' for the second), we see that on the first three rows, both sentences are false. On the final row, both are true. Since they match on every row—that is, on every truth-value assignment for P and Q—the two sentences are equivalent.

9.3 Consistency

In §, we said that sentences are *jointly possible* if and only if it is possible for all of them to be true at once. The surrogate for this concept in TFL is JOINTLY CONSISTENT.

jointly consistent

A and B are JOINTLY CONSISTENT if and only if there is some truth-value assignment that makes them both true *and* they are not equivalent.

Equivalently, if

(1) there is at least one truth-value assignment that makes $(\mathsf{A} \& \mathsf{B})$ true, and
(2) $(\mathsf{A} \leftrightarrow \mathsf{B})$ is *not* a tautology,

then A and B are JOINTLY CONSISTENT.

The requirement that the two sentences not be equivalent is not always included, but we will distinguish between sentences that are jointly consistent from those that are equivalent.

This was one of the examples in §:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

These are jointly possible because it is possible for them both to be true at the same time. It takes nothing away from their joint possibility that they can also be false at the same time or one can be false while the other is true. Applying that same observation to *jointly consistent*, all we need is one line where both sentences are true. (More than one line is fine also, although the truth values for the two sentences shouldn't match on every line. If they do, then the sentences are equivalent.) $(P \vee Q)$ and $(P \& \neg Q)$ have one line where they are both true, and so they are jointly consistent.

P	Q	P	$\vee$	Q	P	$\&$	$\neg$	Q
T	T	T	**T**	T	T	**F**	F	T
T	F	T	**T**	F	T	**T**	T	F
F	T	F	**T**	T	F	**F**	F	T
F	F	F	**F**	F	F	**F**	T	F

And finally, in §, we also said that sentences are *jointly impossible* if and only if it is *not* possible for all of them to be true at once. The surrogate for this concept in TFL is JOINTLY INCONSISTENT.

jointly inconsistent

A and **B** are JOINTLY INCONSISTENT if and only if there is no truth-value assignment that makes them both true.

There are three ways that two sentences can be jointly inconsistent.

(1) One each line, the truth value for one sentence is 'T' and the truth value for the other sentence is 'F'. For instance, the truth values for $(P \vee Q)$ and $(\neg P \& \neg Q)$ never match. On each line, one is true and the other is false. Hence, two sentences are jointly inconsistent in this way when $\neg(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology.

P	Q	P	$\vee$	Q	$\neg$	P	$\&$	$\neg$	Q
T	T	T	**T**	T	F	T	**F**	F	T
T	F	T	**T**	F	F	T	**F**	T	F
F	T	F	**T**	T	T	F	**F**	F	T
F	F	F	**F**	F	T	F	**T**	T	F

(2) When the truth value for one sentence is 'T', then the truth value for the other sentence is 'F', but both sentences can be false at the same time. For example, $\neg(\neg P \vee Q)$ and $(\neg P \,\&\, \neg Q)$ are never both true on the same line, but they are false on the same line. For two sentences that are jointly inconsistent in this way, this criterion is satisfied: $\neg(\mathsf{A} \,\&\, \mathsf{B})$ is a tautology.

P	Q	$\neg$	$(\neg$	P	$\vee$	$Q)$	$\neg$	P	$\&$	$\neg$	Q
T	T	**F**	F	T	T	T	F	T	**F**	F	T
T	F	**T**	F	T	F	F	F	T	**F**	T	F
F	T	**F**	T	F	T	T	T	F	**F**	F	T
F	F	**F**	T	F	T	F	T	F	**T**	T	F

(3) Both sentences are false on every line. For example, the truth values for $(\neg P \,\&\, P)$ and $(\neg Q \,\&\, Q)$ are always the same. On each line, both sentences are false. So, for sentences that are jointly inconsistent in this way, both of these criteria must be satisfied: $\neg(\mathsf{A} \,\&\, \mathsf{B})$ is a tautology and $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology. (And the latter, recall, means that these sentences are equivalent, and so here *jointly inconsistent* and *equivalent* overlap.)

P	Q	$\neg$	P	$\&$	P	$\neg$	Q	$\&$	Q
T	T	F	T	**F**	T	F	T	**F**	T
T	F	F	T	**F**	T	T	F	**F**	F
F	T	T	F	**F**	F	F	T	**F**	T
F	F	T	F	**F**	F	T	F	**F**	F

9.4 Practice exercises

A. Create a truth table for each sentence, and then determine whether the sentence is a **tautology**, a **contradiction**, or is **contingent**.

1. $\neg Q \,\&\, Q$
2. $\neg D \vee D$
3. $(P \,\&\, Q) \vee (Q \,\&\, P)$
4. $\neg[P \rightarrow (Q \rightarrow P)]$
5. $P \leftrightarrow [P \rightarrow (Q \,\&\, \neg Q)]$

6. $(P \rightarrow Q) \lor (Q \rightarrow P)$
7. $[(P \,\&\, Q) \leftrightarrow Q] \rightarrow (P \rightarrow Q)$

B. For each set of sentences, create a truth table and then determine whether the sentences are **jointly consistent** or **jointly inconsistent**.

1. $P \,\&\, \neg Q$, $\neg(P \rightarrow Q)$, $Q \rightarrow P$
2. $P \lor Q$, $P \rightarrow \neg P$, $Q \rightarrow \neg Q$
3. $P \rightarrow Q$, $P \,\&\, \neg Q$
4. $P \rightarrow (Q \rightarrow S)$, $(P \rightarrow Q) \rightarrow S$, $P \rightarrow S$
5. $\neg(P \lor Q)$, $P \leftrightarrow Q$, $Q \rightarrow P$
6. $P \lor Q$, $\neg Q$, $\neg Q \rightarrow \neg P$

C. For each pair of sentences, create a truth table and then determine whether the sentences are **equivalent** or are not.

1. $P \,\&\, \neg P$ and $\neg Q \leftrightarrow Q$
2. $[(P \lor Q) \lor S]$ and $[P \lor (Q \lor S)]$
3. $P \lor (Q \,\&\, S)$ and $(P \lor Q) \,\&\, (P \lor S)$
4. $\neg(P \rightarrow Q)$ and $\neg P \rightarrow \neg Q$
5. $P \lor Q$ and $\neg P \rightarrow Q$
6. $(P \rightarrow Q) \rightarrow S$ and $P \rightarrow (Q \rightarrow S)$

D.

1. Suppose that A and B are equivalent. What can you say about $\mathsf{A} \leftrightarrow \mathsf{B}$?
2. Suppose that A and B are jointly inconsistent. What can you say about $(\mathsf{A} \,\&\, \mathsf{B})$?
3. Suppose that A and B are equivalent. What can you say about $(\mathsf{A} \lor \mathsf{B})$?
4. Suppose that A and B are *not* equivalent. What can you say about $(\mathsf{A} \lor \mathsf{B})$?
5. Consider this principle:

   > Suppose A and B are equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

   Is this principle correct? Explain your answer.

9.5 Answers

A. Use a truth table to determine whether each sentence is a tautology, a contradiction, or contingent.

1. $\neg Q \,\&\, Q$ is a contradiction.

Q	$\neg$	Q	$\&$	Q
T	F	T	**F**	T
F	T	F	**F**	F

2. $\neg D \lor D$ is a tautology.

D	$\neg$	D	$\lor$	D
T	F	T	**T**	T
F	T	F	**T**	F

3. $(P \& Q) \lor (Q \& P)$ is contingent.

P	Q	$(P$	$\&$	$Q)$	$\lor$	$(Q$	$\&$	$P)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**F**	F	F	T
F	T	F	F	T	**F**	T	F	F
F	F	F	F	F	**F**	F	F	F

4. $\neg[P \rightarrow (Q \rightarrow P)]$ is a contradiction.

P	Q	$\neg$	$[P$	$\rightarrow$	$(Q$	$\rightarrow$	$P)]$
T	T	**F**	T	T	T	T	T
T	F	**F**	T	T	F	T	T
F	T	**F**	F	T	T	F	F
F	F	**F**	F	T	F	T	F

5. $P \leftrightarrow [P \rightarrow (Q \& \neg Q)]$ is a contradiction.

P	Q	P	$\leftrightarrow$	$[P$	$\rightarrow$	$(Q$	$\&$	$\neg$	$Q)]$
T	T	T	**F**	T	F	T	F	F	T
T	F	T	**F**	T	F	F	F	T	F
F	T	F	**F**	F	T	T	F	F	T
F	F	F	**F**	F	T	F	F	T	F

6. $(P \rightarrow Q) \lor (Q \rightarrow P)$ is a tautology.

P	Q	$(P$	$\rightarrow$	$Q)$	$\lor$	$(Q$	$\rightarrow$	$P)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	T	T	**T**	T	F	F
F	F	F	T	F	**T**	F	T	F

7. $[(P \& Q) \leftrightarrow Q] \rightarrow (P \rightarrow Q)$ is contingent.

P	Q	[(P	&	Q)	$\leftrightarrow$	Q]	$\rightarrow$	(P	$\rightarrow$	Q)
T	T	T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	T	F	**F**	T	F	F
F	T	F	F	T	F	T	**T**	F	T	T
F	F	F	F	F	T	F	**T**	F	T	F

B.
Use a truth table to determine whether the sentences in each set are **jointly consistent** or **jointly inconsistent**.

1. $P \& \neg Q, \neg(P \rightarrow Q), Q \rightarrow P$
   These sentences are jointly consistent. (See line 2.)

P	Q	P	&	¬	Q	¬	(P	→	Q)	Q	→	P
T	T	T	F	F	T	F	T	T	T	T	T	T
T	F	T	**T**	T	F	**T**	T	F	F	F	**T**	T
F	T	F	F	F	T	F	F	T	T	T	F	F
F	F	F	F	T	F	F	F	T	F	F	T	F

2. $P \vee Q, P \rightarrow \neg P, Q \rightarrow \neg Q$
   These sentences are jointly inconsistent.

P	Q	P	∨	Q	P	→	¬	P	Q	→	¬	Q
T	T	T	T	T	T	F	F	T	T	F	F	T
T	F	T	T	F	T	F	F	T	F	T	T	F
F	T	F	T	T	F	T	T	F	T	F	F	T
F	F	F	F	F	F	T	T	F	F	T	T	F

3. $P \rightarrow Q, P \& \neg Q$
   These sentences are jointly inconsistent.

P	Q	P	→	Q	P	&	¬	Q
T	T	T	T	T	T	F	F	T
T	F	T	F	F	T	T	T	F
F	T	F	T	T	F	F	F	T
F	F	F	T	F	F	F	T	F

4. $P \rightarrow (Q \rightarrow S), (P \rightarrow Q) \rightarrow S, P \rightarrow S$
   These sentences are jointly consistent.

P	Q	S	P	$\to$	$(Q$	$\to$	$S)$	$(P$	$\to$	$Q)$	$\to$	S	P	$\to$	S
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	F	F	T	F	F
T	F	T	T	T	F	T	T	T	F	F	T	T	T	T	T
T	F	F	T	T	F	T	F	T	F	F	T	F	T	F	F
F	T	T	F	T	T	T	T	F	T	T	T	T	F	T	T
F	T	F	F	T	T	F	F	F	T	T	F	F	F	T	F
F	F	T	F	T	F	T	T	F	T	F	T	T	F	T	T
F	F	F	F	T	F	T	F	F	T	F	F	F	F	T	F

5. $\neg(P \vee Q), P \leftrightarrow Q, Q \to P$
   These sentences are jointly consistent.

P	Q	$\neg$	$(P$	$\vee$	$Q)$	P	$\leftrightarrow$	Q	Q	$\to$	P
T	T	F	T	T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	F	F	F	T	T
F	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	F	F	F	T	F	F	T	F

6. $P \vee Q, \neg Q, \neg Q \to \neg P$
   These sentences are jointly inconsistent.

P	Q	P	$\vee$	Q	$\neg$	Q	$\neg$	Q	$\to$	$\neg$	P
T	T	T	T	T	F	T	F	T	T	F	T
T	F	T	T	F	T	F	T	F	F	F	T
F	T	F	T	T	F	T	F	T	T	T	F
F	F	F	F	F	T	F	T	F	T	T	F

C. Use a truth table to determine whether the sentences in each set are **equivalent** or not.

1. $P \mathbin{\&} \neg P$ and $\neg Q \leftrightarrow Q$ are equivalent.

P	Q	P	$\&$	$\neg$	P	$\neg$	Q	$\leftrightarrow$	Q
T	T	T	**F**	F	T	F	T	**F**	T
T	F	T	**F**	F	T	T	F	**F**	F
F	T	F	**F**	T	F	F	T	**F**	T
F	F	F	**F**	T	F	T	F	**F**	F

2. $[(P \vee Q) \vee S]$ and $[P \vee (Q \vee S)]$ are equivalent.

P	Q	S	$[(P$	$\vee$	$Q)$	$\vee$	$S]$	$[P$	$\vee$	$(Q$	$\vee$	$S)]$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**T**	F	T	**T**	T	T	F
T	F	T	T	T	F	**T**	T	T	**T**	F	T	T
T	F	F	T	T	F	**T**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**T**	F	F	**T**	T	T	F
F	F	T	F	F	F	**T**	T	F	**T**	F	T	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

3. $P \vee (Q \,\&\, S)$ and $(P \vee Q) \,\&\, (P \vee S)$ are equivalent.

P	Q	S	P	$\vee$	$(Q$	$\&$	$S)$	$(P$	$\vee$	$Q)$	$\&$	$(P$	$\vee$	$S)$
T	T	T	T	**T**	T	T	T	T	T	T	**T**	T	T	T
T	T	F	T	**T**	T	F	F	T	T	T	**T**	T	T	F
T	F	T	T	**T**	F	F	T	T	T	F	**T**	T	T	T
T	F	F	T	**T**	F	F	F	T	T	F	**T**	T	T	F
F	T	T	F	**T**	T	T	T	F	T	T	**T**	F	T	T
F	T	F	F	**F**	T	F	F	F	T	T	**F**	F	F	F
F	F	T	F	**F**	F	F	T	F	F	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F	F	F	F	**F**	F	F	F

4. $\neg(P \rightarrow Q)$ and $\neg P \rightarrow \neg Q$ are not equivalent.

P	Q	$\neg$	$(P$	$\rightarrow$	$Q)$	$\neg$	P	$\rightarrow$	$\neg$	Q
T	T	F	T	T	T	F	T	T	F	T
T	F	T	T	F	F	F	T	T	T	F
F	T	F	F	T	T	T	F	F	F	T
F	F	F	F	T	F	T	F	T	T	F

5. $P \vee Q$ and $\neg P \rightarrow Q$ are equivalent.

P	Q	P	$\vee$	Q	$\neg$	P	$\rightarrow$	Q
T	T	T	T	T	F	T	T	T
T	F	T	T	F	F	T	T	F
F	T	F	T	T	T	F	T	T
F	F	F	F	F	T	F	F	F

6. $(P \to Q) \to S$ and $P \to (Q \to S)$ are not equivalent.

P	Q	S	$(P$	$\to$	$Q)$	$\to$	S	P	$\to$	$(Q$	$\to$	$S)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T	F	T	F	F
T	F	T	T	F	F	T	T	T	T	F	T	T
T	F	F	T	F	F	T	F	T	T	F	T	F
F	T	T	F	T	T	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F	F	T	T	F	F
F	F	T	F	T	F	T	T	F	T	F	T	T
F	F	F	F	T	F	F	F	F	T	F	T	F

D.

1. Suppose that **A** and **B** are equivalent. What can you say about **A** $\leftrightarrow$ **B**?

    A and **B** have the same truth value on every line of a complete truth table, so **A** $\leftrightarrow$ **B** is true on every line. It is a tautology.

2. Suppose that **A** and **B** are jointly inconsistent. What can you say about (**A** & **B**)?

    Since the sentences are jointly inconsistent, there is no truth-value assignment on which they are both true. So their conjunction is false on every truth-value assignment. It is a contradiction

3. Suppose that **A** and **B** are equivalent. What can you say about (**A** $\vee$ **B**)?

    Not much. Since **A** and **B** are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is equivalent to them.

4. Suppose that **A** and **B** are *not* equivalent. What can you say about (**A** $\vee$ **B**)?

    A and **B** have different truth values on at least one line of a complete truth table, and (**A** $\vee$ **B**) will be true on that line. On other lines, it might be true or false. So (**A** $\vee$ **B**) is either a tautology or it is contingent; it is *not* a contradiction.

5. Consider this principle:

    Suppose **A** and **B** are logically equivalent. Suppose an argument contains **A** (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced **A** with **B**.

    Is this principle correct? Explain your answer.

    The principle is correct. Since **A** and **B** are logically equivalent, they have the same truth table. So every truth-value assignment that makes A true also makes B true, and every truth-value assignment that makes A false also makes B false. So if no truth-value assignment makes all the premises true and the conclusion false, when A was among the premises or the conclusion, then no truth-value assignment makes all the premises true and the conclusion false, when we replace A with B.

10 Truth tables and validity

10.1 Validity

Having examined the logical relations between two sentences in § and §, we can now go a step further and consider the relationship between the premises and the conclusion of an argument. Recall the definition of VALID.

> valid
>
> An argument is VALID when (and only when) it is the case that if the premises are true, then the conclusion has to be true.

When using a truth table to determine if an argument is valid, we list the premise or premises first, then, the turnstile symbol ('$\vdash$'), and, finally, the conclusion. We will use '$\neg L \rightarrow (M \vee L), \neg L \vdash M$' as our example.

> The symbol '$\vdash$' is used to separate the premises from the conclusion in arguments in TFL. It can be read as *therefore*.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T		**T**
T	F	T	F	**T**	T	T	F	**T**	F		**T**
F	T	F	T	**T**	F	T	T	**F**	T		**F**
F	F	T	F	**F**	F	F	F	**T**	F		**F**

Once the truth table is completed for '$\neg L \rightarrow (M \vee L), \neg L \vdash M$', we investigate whether this argument satisfies (or violates) the definition of *valid*. Ask yourself, "When both premises are true, is the conclusion true?" And "Is there any line (that is, any truth-value assignment) where both premises are true and the conclusion is false?" If the answer to the first question is always "yes," then the argument is valid. If the answer to the second question is ever "no," then the argument is invalid.

As you can see, there is only one row where both '$\neg L \rightarrow (M \vee L)$' and '$\neg L$' are true, and so that is the row that mainly concerns us. On that row, the conclusion is also true. Hence, '$\neg L \rightarrow (M \vee L), \neg L \vdash M$' is valid.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T		**T**
T	F	T	F	**T**	T	T	F	**T**	F	✓	**T**
F	T	F	T	**T**	F	T	T	**F**	T		**F**
F	F	T	F	**F**	F	F	F	**T**	F		**F**

When using truth tables to determine if an argument is valid, we will put '✓' and '✗' in the column under the turnstile. As just shown, when all of the premises are true and the conclusion is true, we put a '✓' on that line beneath the turnstile. If, on a line, all of the premise are true and the conclusion is false, then we put a '✗' beneath the turnstile.

Also (and **importantly!**), when there is a line where one or more of the premises are false, we put a '✓' beneath the turnstile—whether the conclusion is true or false. An argument is valid when it is the case that *if the premises are true*, then the conclusion has to be true. It doesn't matter if there are truth-value assignments (i.e., lines) where both premises are not true. Such lines don't violate our definition of *valid*, and so they get a '✓'.

Completing our truth table we have this:

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T	✓	**T**
T	F	T	F	**T**	T	T	F	**T**	F	✓	**T**
F	T	F	T	**T**	F	T	T	**F**	T	✓	**F**
F	F	T	F	**F**	F	F	F	**T**	F	✓	**F**

Now, let's make one small (but significant) change to the argument: $\neg L \rightarrow (M \vee L), \neg L \vdash \neg M$. The premises are the same, but now the conclusion is $\neg M$ instead of M.

The truth values for the premises are the same, and the truth values for the conclusion have, on each line, flipped from T to F or vice versa. Now, when we evaluate each line, what do we find? As before, on lines 1, 3, and 4, one of the premises is false, and so they get a '✓'. On line

2, the premises are true and the conclusion is false. That line gets a '✕'! Because there is a line where the premises are true and the conclusion is false, '$\neg L \rightarrow (M \vee L), \neg L \vdash \neg M$' is not valid.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	$\neg M$
T	T	F	T	**T**	T	T	T	**F**	T	✓	**F**
T	F	T	F	**T**	T	T	F	**T**	F	✕	**F**
F	T	F	T	**T**	F	T	T	**F**	T	✓	**T**
F	F	T	F	**F**	F	F	F	**T**	F	✓	**T**

good and bad lines

Let's call lines that violate the definition of VALID *bad lines* and the lines that do not *good lines*.

(1) Any line where all of the premises are true and the conclusion is false **is a bad line**. Put an '✕' on that line.

(2) Any line where all of the premises are true and the conclusion is true **is a good line**. Put a '✓' on that line.

(3) Any line where the conclusion is true cannot be a bad line. (So, whatever the case may be with the premises, **it's a good line**.) Put a '✓' on that line.

(4) Any line where at least one premise is false cannot be a bad line. So, **it's a good line**. Put a '✓' on that line.

10.2 Some examples

Here are some examples using truth tables to determine whether an argument is valid. As a reminder, the definition of valid is given in §, and we can also use 1 – 4 on p. 99 (which are consequences of the definition). We will begin with arguments that have only one premise and then do some with multiple premises.

1. First we will determine if '$P \& Q \vdash Q$' is valid. The premise, '$P \& Q$', is only true on line 1. Since it is false on lines 2 – 4, we know that

those are good lines. (See guideline 4.) On line 1, 'P & Q' is true and the conclusion, 'Q', is true, and so that is also a good line. (See guideline 2.) Since every line is a good line, this argument is valid.

P	Q	P	&	Q	$\vdash$	Q
T	T	T	**T**	T	✓	**T**
T	F	T	**F**	F	✓	**F**
F	T	F	**F**	T	✓	**T**
F	F	F	**F**	F	✓	**F**

2. In '$\neg(P \vee Q) \vdash \neg P \,\&\, Q$', the premise is false on lines 1 – 3, and so we know that those are good lines. On line 4, the premise is true and the conclusion is false, which means that line 4 is a bad line. (See guideline 1.) Since it has at least one bad line, this argument is not valid.

P	Q	$\neg$	$(P$	$\vee$	$Q)$	$\vdash$	$\neg$	P	&	Q
T	T	**F**	T	T	T	✓	F	T	**F**	T
T	F	**F**	T	T	F	✓	F	T	**F**	F
F	T	**F**	F	T	T	✓	T	F	**T**	T
F	F	**T**	F	F	F	×	T	F	**F**	F

3. Now an argument with two premises: '$P \rightarrow Q, \neg Q \vdash \neg P$'. Since both premise are not true on lines 1, 2, and 3, those are all good lines. On line 4, both premises are true and the conclusion is true, and so that is a good line. Since every line is a good line, this argument is valid.

P	Q	P	$\rightarrow$	Q	$\neg$	Q	$\vdash$	$\neg$	P
T	T	T	**T**	T	**F**	T	✓	**F**	T
T	F	T	**F**	F	**T**	F	✓	**F**	T
F	T	F	**T**	T	**F**	T	✓	**T**	F
F	F	F	**T**	F	**T**	F	✓	**T**	F

4. Next, consider '$P \rightarrow Q, P \rightarrow \neg Q \vdash P$'. Since the second premise is false on line 1 and the first premise is false on line 2, those are good lines. On line 3, both of the premises are true and the conclusion is false. That's a bad line. And then the same is also the case on line 4, and so that is a bad line also. Since two of the lines in this truth table are bad lines, the argument is invalid.

P	Q	P	$\rightarrow$	$Q,$	P	$\rightarrow$	$\neg$	Q	$\vdash$	P
T	T	T	**T**	T	T	**F**	F	T	✓	**T**
T	F	T	**F**	F	T	**T**	T	F	✓	**T**
F	T	F	**T**	T	F	**T**	F	T	✗	**F**
F	F	F	**T**	F	F	**T**	T	F	✗	**F**

5. In the last argument, we have three premises. One of the premises is false on each of lines 1, 2, 4, 5, 7, and 8, and so those are all good lines. On line 3, all of the premises are true and the conclusion is true, and so that is a good line. On line 6, all of the premises are true but the conclusion is false, and so that is a bad line. Since one of the lines is a bad line, this argument is invalid.

P	Q	R	P	$\vee$	$Q,$	P	$\rightarrow$	$R,$	Q	$\rightarrow$	$\neg$	R	$\vdash$	R
T	T	T	T	**T**	T	T	**T**	T	T	**F**	F	T	✓	**T**
T	T	F	T	**T**	T	T	**F**	F	T	**T**	T	F	✓	**F**
T	F	T	T	**T**	F	T	**T**	T	F	**T**	F	T	✓	**T**
T	F	F	T	**T**	F	T	**F**	F	F	**T**	T	F	✓	**F**
F	T	T	F	**T**	T	F	**T**	T	T	**F**	F	T	✓	**T**
F	T	F	F	**T**	T	F	**T**	F	T	**T**	T	F	✗	**F**
F	F	T	F	**F**	F	F	**T**	T	F	**T**	F	T	✓	**T**
F	F	F	F	**F**	F	F	**T**	F	F	**T**	T	F	✓	**F**

10.3 ‘$\vdash$’ versus ‘$\rightarrow$’

When using truth tables to determine whether an argument is valid, it may help you to notice a similarity between ‘$\vdash$’ and ‘$\rightarrow$’. As you know, a conditional is true under every circumstance except when the antecedent is true and the consequent if false. (So, when we have a ‘T’ under the antecedent and an ‘F’ under the consequent, we put an ‘F’ under the ‘$\rightarrow$’.) Meanwhile, in an argument, when all of the premises are true and the conclusion is false, the argument is invalid. (So, for a specific line, when we have a ‘T’ under every premise and an ‘F’ under the conclusion, we put a ‘✗’ under the ‘$\vdash$’.)

The reasoning here is similar. In both cases, we are violating the principle – of either the conditional or of a valid argument – when we have a false sentence that follows from a sentence or a set of sentences that are all true. Thus, if $\mathsf{A} \to \mathsf{C}$ is false, then $\mathsf{A} \vdash \mathsf{C}$ is invalid (and if $\mathsf{A} \vdash \mathsf{C}$ is invalid, then $\mathsf{A} \to \mathsf{C}$ is false). Conversely, whenever $\mathsf{A} \to \mathsf{C}$ is true, then $\mathsf{A} \vdash \mathsf{C}$ is valid (and vice versa).

10.4 Practice exercises

A. Create a truth table for each argument and then determine if the argument is valid or invalid.

1. $P \vee (Q \to P) \vdash \neg P \to \neg Q$
2. $P \to Q, Q \vdash P$
3. $P \to Q, Q \to P \vdash P \leftrightarrow Q$
4. $P \to Q, \neg P \vdash \neg Q$
5. $P \vee Q, Q \vee S, \neg P \vdash Q \,\&\, S$
6. $(Q \,\&\, P) \to S, (S \,\&\, P) \to Q \vdash (S \,\&\, Q) \to P$
7. $P \leftrightarrow Q, Q \leftrightarrow S \vdash P \leftrightarrow S$
8. $P \to Q, P \to S \vdash Q \to S$
9. $P \vee Q, Q \vee S, \neg Q \vdash P \,\&\, S$
10. $\neg(P \,\&\, Q), P \vee Q, P \leftrightarrow Q \vdash S$

B.

1. Suppose that $(\mathsf{A} \,\&\, \mathsf{B}) \to \mathsf{C}$ is neither a tautology nor a contradiction. Is it possible to determine if $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ is valid or not? Explain.
2. Suppose that A is a contradiction. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid? Explain.
3. Suppose that C is a tautology. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid? Explain.

10.5 Answers

A.

1. $P \vee (Q \to P) \vdash \neg P \to \neg Q$
   This argument is valid.

P	Q	P	$\vee$	$(Q$	$\to$	$P)$	$\vdash$	$\neg$	P	$\to$	$\neg$	Q
T	T	T	**T**	T	T	T	✓	F	T	**T**	F	T
T	F	T	**T**	F	T	T	✓	F	T	**T**	T	F
F	T	F	**F**	T	F	F	✓	T	F	**F**	F	T
F	F	F	**T**	F	T	F	✓	T	F	**T**	T	F

2. $P \to Q, Q \vdash P$
   This argument is invalid.

P	Q	P	$\to$	$Q,$	Q	$\vdash$	P
T	T	T	T	T	T	✓	T
T	F	T	F	F	F	✓	T
F	T	F	T	T	T	×	F
F	F	F	T	F	F	✓	F

3. $P \to Q, Q \to P \vdash P \leftrightarrow Q$
   This argument is valid.

P	Q	P	$\to$	$Q,$	Q	$\to$	P	$\vdash$	P	$\leftrightarrow$	Q
T	T	T	T	T	T	T	T	✓	T	T	T
T	F	T	F	F	F	T	T	✓	T	F	F
F	T	F	T	T	T	F	F	✓	F	F	T
F	F	F	T	F	F	T	F	✓	F	T	F

4. $P \to Q, \neg P \vdash \neg Q$
   This argument is invalid.

P	Q	P	$\to$	$Q,$	$\neg$	P	$\vdash$	$\neg$	Q
T	T	T	T	T	F	T	✓	F	T
T	F	T	F	F	F	T	✓	T	F
F	T	F	T	T	T	F	×	F	T
F	F	F	T	F	T	F	✓	T	F

5. $P \vee Q, Q \vee S, \neg P \vdash Q \,\&\, S$
   This argument is invalid.

P	Q	S	P	$\vee$	$Q,$	Q	$\vee$	$S,$	$\neg$	P	$\vdash$	Q	$\&$	S
T	T	T	T	**T**	T	T	**T**	T	**F**	T	✓	T	**T**	T
T	T	F	T	**T**	T	T	**T**	F	**F**	T	✓	T	**F**	F
T	F	T	T	**T**	F	F	**T**	T	**F**	T	✓	F	**F**	T
T	F	F	T	**T**	F	F	**F**	F	**F**	T	✓	F	**F**	F
T	T	T	F	**T**	T	T	**T**	T	**T**	F	✓	T	**T**	T
T	T	F	F	**T**	T	T	**T**	F	**T**	F	×	T	**F**	F
T	F	T	F	**F**	F	F	**T**	T	**T**	F	✓	F	**F**	T
T	F	F	F	**F**	F	F	**F**	F	**T**	F	✓	F	**F**	F

6. $(Q \,\&\, P) \rightarrow S, (S \,\&\, P) \rightarrow Q \vdash (S \,\&\, Q) \rightarrow P$
   This argument is invalid.

P	Q	S	$(Q$	$\&$	$P)$	$\rightarrow$	$S,$	$(S$	$\&$	$P)$	$\rightarrow$	Q	$\vdash$	$(S$	$\&$	$Q)$	$\rightarrow$	P
T	T	T	T	T	T	**T**	T	T	T	T	**T**	T	✓	T	T	T	**T**	T
T	T	F	T	T	T	**F**	F	F	F	T	**T**	T	✓	F	F	T	**T**	T
T	F	T	F	F	T	**T**	T	T	T	T	**F**	F	✓	T	F	F	**T**	T
T	F	F	F	F	T	**T**	F	F	F	T	**T**	F	✓	F	F	F	**T**	T
F	T	T	T	F	F	**T**	T	T	F	F	**T**	T	×	T	T	T	**F**	F
F	T	F	T	F	F	**T**	F	F	F	F	**T**	T	✓	F	F	T	**T**	F
F	F	T	F	F	F	**T**	T	T	F	F	**T**	F	✓	T	F	F	**T**	F
F	F	F	F	F	F	**T**	F	F	F	F	**T**	F	✓	F	F	F	**T**	F

7. $P \leftrightarrow Q, Q \leftrightarrow S \vdash P \leftrightarrow S$
   This argument is valid.

P	Q	S	P	$\leftrightarrow$	$Q,$	Q	$\leftrightarrow$	S	$\vdash$	P	$\leftrightarrow$	S
T	T	T	T	T	T	T	T	T	✓	T	T	T
T	T	F	T	T	T	T	F	F	✓	T	F	F
T	F	T	T	F	F	F	F	T	✓	T	T	T
T	F	F	T	F	F	F	T	F	✓	T	F	F
F	T	T	F	F	T	T	T	T	✓	F	F	T
F	T	F	F	F	T	T	F	F	✓	F	T	F
F	F	T	F	T	F	F	F	T	✓	F	F	T
F	F	F	F	T	F	F	T	F	✓	F	T	F

8. $P \rightarrow Q, P \rightarrow S \vdash Q \rightarrow S$
   This argument is invalid.

P	Q	S	P	$\rightarrow$	$Q,$	P	$\rightarrow$	S	$\vdash$	Q	$\rightarrow$	S
T	T	T	T	T	T	T	T	T	✓	T	T	T
T	T	F	T	T	T	T	F	F	✓	T	F	F
T	F	T	T	F	F	T	T	T	✓	F	T	T
T	F	F	T	F	F	T	F	F	✓	F	T	F
F	T	T	F	T	T	F	T	T	✓	T	T	T
F	T	F	F	T	T	F	T	F	×	T	F	F
F	F	T	F	T	F	F	T	T	✓	F	T	T
F	F	F	F	T	F	F	T	F	✓	F	T	F

9. $P \vee Q, Q \vee S, \neg Q \vdash P \& S$
   This argument is valid.

P	Q	S	P	$\vee$	$Q,$	Q	$\vee$	$S,$	$\neg$	Q	$\vdash$	P	$\&$	S
T	T	T	T	T	T	T	T	T	F	T	✓	T	T	T
T	T	F	T	T	T	T	T	F	F	T	✓	T	F	F
T	F	T	T	T	F	F	T	T	T	F	✓	T	T	T
T	F	F	T	T	F	F	F	F	T	F	✓	T	F	F
F	T	T	F	T	T	T	T	T	F	T	✓	F	F	T
F	T	F	F	T	T	T	T	F	F	T	✓	F	F	F
F	F	T	F	F	F	F	T	T	T	F	✓	F	F	T
F	F	F	F	F	F	F	F	F	T	F	✓	F	F	F

10. $\neg(P \& Q), P \vee Q, P \leftrightarrow Q \vdash S$
    This argument is valid.

P	Q	S	$\neg$	$(P$	$\&$	$Q),$	P	$\vee$	$Q,$	P	$\leftrightarrow$	Q	$\vdash$	S
T	T	T	F	T	T	T	T	T	T	T	T	T	✓	T
T	T	F	F	T	T	T	T	T	T	T	T	T	✓	F
T	F	T	T	T	F	F	T	T	F	T	F	F	✓	T
T	F	F	T	T	F	F	T	T	F	T	F	F	✓	F
F	T	T	T	F	F	T	F	T	T	F	F	T	✓	T
F	T	F	T	F	F	T	F	T	T	F	F	T	✓	F
F	F	T	T	F	F	F	F	F	F	F	T	F	✓	T
F	F	F	T	F	F	F	F	F	F	F	T	F	✓	F

B.

1. Suppose that $(\mathsf{A} \& \mathsf{B}) \to \mathsf{C}$ is neither a tautology nor a contradiction. Is it possible to determine if $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ is valid or not?

   Since the sentence $(\mathsf{A} \& \mathsf{B}) \to \mathsf{C}$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, A and B are true and C is false. Hence, the argument, '$\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$', is invalid.

2. Suppose that A is a contradiction. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid?

   Since A is false on every line of a truth table, there is no line on which A and B are true and C is false. Hence, the argument is valid. (This would be an odd argument, however, since we know that one of the premises is a contradiction.)

3. Suppose that C is a tautology. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid?

   Since C is true on every line of a complete truth table, there is no line on which A and B are true and C is false. Hence, the argument is valid.

Part 4

Natural deduction for TFL

11 Natural deduction

11.1 Natural deduction versus truth tables

An argument is valid when (and only when) it is impossible for all of the premises to be true and the conclusion to be false. And we have seen that truth tables can be used to determine whether an argument is valid. In the next chapter, you will learn another method for verifying that an argument is valid. Before we turn to this new method, however, let's review the strengths and weakness of truth tables.

1. The truth table method for determining if an argument is valid focuses directly on the definition of *valid*. Each line of a complete truth table corresponds to a truth-value assignment. Thus, given an argument in TFL, truth tables reveal whether or not the conclusion is true when all of the premises true.

2. Truth tables also allow us to easily and rigorously set the meaning for each logical operator. As we discussed in §, in English, 'or' can take the inclusive-or meaning (one or the other, or both) or the exclusive-or meaning (one or the other, but not both), and, at different times, both meanings are used in English. We can discuss which English meaning is closest to the meaning of '$\vee$' in TFL (it's the inclusive-or), but, in the end, we just set the meaning of the symbol '$\vee$' with this truth table:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Hence, the definition for the '$\vee$' is simple this: the operator that connects **A** and **B** to yield the truth values shown in this truth table. And, then, the same goes for the other logical operators.

3. To create a truth table, the number of lines needed is 2^n, where n is the number of different letters in the argument. So, an argument with four different sentence letters will require a 16 line truth table, one with five letters will require 32 lines, one with six different letters will require 64 lines, and so on. Hence, while a truth table can be used to determine if any argument is valid or invalid, one of the weakness of this method is that it is difficult to use when the argument contains more than four different sentence letters.
4. But what is typically seen as the biggest weakness of using truth tables to determine if an argument is valid is that it doesn't reveal to us *why* the argument is valid. It doesn't, in other words, lay out the reasoning that demonstrates why (and how) the conclusion follows from the premises.

As an alternative to truth tables, we have a *natural deduction system*. Such a system allows us to verify that an argument is valid and to see why it is valid. We do this by making explicit the reasoning process that takes us from the premises to the conclusion. We begin with twelve basic rules—which we call *rules of derivation*. (For instance, this is one of the rules: if we know that '$P \vee Q$' is true; and we also know that '¬P' is true, then we can assert that 'Q' is true.) The rules can be combined, and with just these twelve, we hope to be able to show how we get from the premises to the conclusion for all of the valid arguments that can be represented in TFL.

There are different natural deduction systems that can be used with TFL. But all, for the most part, reflect the ways that we naturally reason—at least insofar as the reasoning involves 'and', 'or', 'not', 'if ..., then ...', and 'if and only if'.

11.2 Truth functional propositional logic

We have reached a point where it is useful to summarize what TFL is. As you know, the symbols of TFL are the sentence letters that represent atomic sentences, the logical operators ¬, &, ∨, →, and ↔, and parentheses. These, then, can be combined into sentences using the rules given in chapter . And, then, in chapter , truth tables were used to set the meaning of the logical operators.

Truth tables also give us a method for determining if an argument is *valid*. *Valid* is a concept and is not, strictly speaking a part of TFL. Rather it is a property of some arguments that can, to an extent, be studied and explicated using TFL. Similarly, as you have seen, *tautology*, *contradiction*, *contingent*, *equivalent*, *jointly consistent*, and *jointly inconsistent* are concepts that can be explained using TFL.

The final part of TFL is the system of natural deduction, which sets the rules for how sentences containing the logical operators can be combined or taken apart.

11.3 Fitch

The modern development of natural deduction dates from simultaneous but unrelated papers by Gerhard Gentzen and Stanisław Jaskowski that were published in 1934. The natural deduction system that we will use, however, is based largely on work by Frederic Fitch that was first published in 1952. Consequently, the format that is used in the next chapter for writing proofs is called *Fitch notation*.

12 The rules of derivation

12.1 Proofs

As was explained in the previous chapter, creating a PROOF is one way of demonstrating that an argument is valid. (And, as you know, using a truth table is the other way.) A proof is a list of sentences. The sentence or sentences at the beginning of the list are the premises of the argument. Every other sentence in the list follows from earlier sentences by a specific rule (with one exception, which we will get to in §). The final sentence is the conclusion of the argument.

As an illustration, consider this argument:

$$\neg(P \vee Q) \vdash \neg P \,\&\, \neg Q$$

We start the proof by numbering the line and writing the premise:

$$\begin{array}{rl|l} 1 & \neg(P \vee Q) & :\text{PR} \\ \hline \end{array}$$

Every line in a proof is numbered so that we can refer to it later if we need to do so. We have also indicated that this is a premise by putting 'PR' at the end of the line. And we have drawn a line underneath the premise. Everything written below the line will either be a sentence that can be derived from that premise, or it will be a new assumption that we introduce. The colon that is right before 'PR' is, technically, optional, but it has to be used in Carnap to separate the TFL sentence from the 'PR' (or the rule) that is written at the end of each line.

The conclusion of this argument is '$\neg P \,\&\, \neg Q$'; and so we want our proof to end—on some line, we'll call it n—with that sentence:

1	$\neg(P \vee Q)$	:PR
2	...	
	...	
	...	
n	$\neg P \,\&\, \neg Q$	

It doesn't matter how many lines it takes to arrive at the conclusion, although, generally, we prefer a shorter proof over a longer one.

Now, suppose we have this argument:

$$P \vee Q, \neg(P \,\&\, S), \neg(Q \,\&\, \neg T) \vdash \neg S \vee T$$

This argument has three premises, and so we start by listing them, numbering each line, and drawing a line under the final premise:

1	$P \vee Q$	:PR
2	$\neg(P \,\&\, S)$	:PR
3	$\neg(Q \,\&\, \neg T)$	:PR

This, meanwhile, will be the final line of the proof:

n	$\neg S \vee T$

Setting up the premises and the conclusion is, however, the easy part. The real task—and the interesting part—is determining each of the steps that get us from the premise or premises to the conclusion.

To do that, we will use a NATURAL DEDUCTION system. In this system, there are two rules for each logical operator: an INTRODUCTION rule, which allows us to derive a new sentence that has the logical operator as the main connective, and an ELIMINATION rule, which allows us to extract a sub-sentence from a sentence that has that logical operator as the main connective. (Table contains a list of the rules.) These rules can then be combined to demonstrate each step that must be taken to get from the premises to the conclusion. All of the rules introduced in this chapter are also summarized on pp. 208 - 209.

THE RULES OF DERIVATION	
conjunction introduction rule	conjuction elimination rule
disjunction introduction rule	disjunction elimination rule
conditional introduction rule	conditional elimination rule
biconditional introduction rule	biconditional elimination rule
negation introduction rule	negation elimination rule
reiteration rule	
double negation rule	

Table 12.1

12.2 Conjunction introduction and elimination

Let's say that we know that Sarah is swimming. We also, as it happens, know that Amy is reading. We are, therefore, justified in stating, "Sarah is swimming and Amy is reading." This reasoning process, which we all do naturally, is part of our natural deduction system. It is called the CONJUNCTION INTRODUCTION RULE.

conjunction introduction rule

m	**A**	
n	**B**	
	A & **B**	:&I *m*, *n*

If we have **A** on a line and **B** on a line, then we can put **A** & **B** on a new line. The '**A**' and '**B**' can occur in either order, and the conjunction can be '**A** & **B**' or '**B** & **A**'.

The '*m*' and '*n*' will never appear in an actual proof. In a proof, the lines are numbered 1, 2, 3, etc. The '*m*' and '*n*' are used in the statement of the rule to indicate that **A** and **B** can be on any lines in the proof. If you look ahead, you will see that some of rules given in § - consist of three lines and some have two.

(a) For each of the rules that consist of three lines, you can add what is on the last line (of the rule) to your proof when, and only when, you have what is given on the first two lines (of the rule).

(b) For each of the rules that consist of two lines, you can add what is on the last line (of the rule) to your proof when, and only when, you have what is given on the first line (of the rule).

Returning to the example about Sarah and Amy, we can use this symbolization key:

S: Sarah is swimming.
R: Amy is reading.

Let's say that 'S' and 'R' are our premises (although they don't have to be to use the conjunction introduction rule), and so they are on lines 1 and 2. Then on any subsequent line – but, in this case, it will be line 3 – we can get 'S & R' by using the conjunction introduction rule.

1	S	:PR
2	R	:PR
3	$S \mathbin{\&} R$	:&I 1, 2

To show that this application of the conjunction introduction rule is our justification for the new 'S & R' on line 3, we put '&I 1, 2' on the far right. This indicate that 'S & R' was obtained by applying the conjunction introduction rule to the 'S' on line 1 and the 'R' on line 2.

The conjunction introduction rule introduces a sentence with '&' as the main connective. We also have a rule that let's us extract what is on one side of a conjunction and put it on a new line. Suppose someone tells you that *Jeff is eating and Mary is sleeping*. Assuming that whoever told you this is reliable, you are entitled to infer simply that *Jeff is eating*. You are also entitled to infer that *Mary is sleeping*. These are applications of the CONJUNCTION ELIMINATION RULE (which is actually two similar rules).

conjunction elimination rule

$$\begin{array}{r|ll} m & \mathsf{A} \,\&\, \mathsf{B} & \\ & \mathsf{A} & :\&\text{E } m \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} \,\&\, \mathsf{B} & \\ & \mathsf{B} & :\&\text{E } m \end{array}$$

If we have $\mathsf{A} \,\&\, \mathsf{B}$ on a line, then, on a new line, we can put either A by itself or B by itself.

When you have a conjunction on one line of a proof, you can use the conjunction elimination rule to obtain either of the conjuncts on a new line. You can only, however, apply this rule when the '&' is the main logical operator. So, for instance, you cannot use the conjunction elimination rule to obtain 'T' from '$S \vee (T \,\&\, W)$' (because '$\vee$' is the main logical operator). The same holds for all of the other rules. **Each of the rules of derivation can only be applied to the main logical operator of a sentence.**

We will now construct a couple of proofs using the two rules just introduced. First, one for this argument: '$P \,\&\, Q, \neg R \vdash Q \,\&\, \neg R$'.

$$\begin{array}{r|ll} 1 & P \,\&\, Q & :\text{PR} \\ 2 & \neg R & :\text{PR} \\ \hline \end{array}$$

After we have listed the premises, we use the conjunction elimination rule to get 'Q' on a line by itself.

$$\begin{array}{r|ll} 1 & P \,\&\, Q & :\text{PR} \\ 2 & \neg R & :\text{PR} \\ \hline 3 & Q & :\&\text{E } 1 \end{array}$$

And then, to finish the proof, we use the conjunction introduction rule to get '$Q \,\&\, \neg R$'.

$$\begin{array}{r|ll} 1 & P \,\&\, Q & :\text{PR} \\ 2 & \neg R & :\text{PR} \\ \hline 3 & Q & :\&\text{E } 1 \\ 4 & Q \,\&\, \neg R & :\&\text{I } 2, 3 \end{array}$$

Notice that there is nothing in this representation of the proof to indicate that the last line is the conclusion. It's only because we began with '$P \,\&\, Q, \neg R \vdash Q \,\&\, \neg R$' that we know that, on line 4, we have arrived at the conclusion that we want.

Next, we will take up the proof for this argument:

$$(N \vee P) \,\&\, (Q \,\&\, S) \vdash (N \vee P) \,\&\, S$$

After listing the premise, we can use the conjunction elimination rule twice to get '$N \vee P$' and '$Q \,\&\, S$' on lines by themselves.

$$\begin{array}{r|ll} 1 & (N \vee P) \,\&\, (Q \,\&\, S) & :\text{PR} \\ \hline 2 & (N \vee P) & :\&\text{E } 1 \\ 3 & (Q \,\&\, S) & :\&\text{E } 1 \end{array}$$

Now that '$Q \,\&\, S$' is on its own line, we can use the conjunction elimination rule again to get S on a line by itself.

$$\begin{array}{r|ll} 1 & (N \vee P) \,\&\, (Q \,\&\, S) & :\text{PR} \\ \hline 2 & (N \vee P) & :\&\text{E } 1 \\ 3 & (Q \,\&\, S) & :\&\text{E } 1 \\ 4 & S & :\&\text{E } 3 \end{array}$$

In our final step, we use the conjunction introduction rule to get the conclusion, '$(N \vee P) \,\&\, S$'.

1	$(N \vee P) \,\&\, (Q \,\&\, S)$	:PR
2	$(N \vee P)$	:&E 1
3	$(Q \,\&\, S)$	:&E 1
4	S	:&E 3
5	$(N \vee P) \,\&\, S$	:&I 2, 4

12.3 Disjunction intro and elim

Disjunction elimination For the disjunction rules, let's start with this example:

> *Sarah is swimming or Jeff is eating a burrito.*

When is this true? Recall from § that we are using the inclusive-or. So, *Sarah is swimming or Jeff is eating a burrito* is true when either

(a) only *Sarah is swimming* is true, or
(b) only *Jeff is eating a burrito* is true, or
(c) both are true.

That's a lot of options, but if all we know is that Sarah is swimming *or* Jeff is eating a burrito, then we don't know precisely what either one of them is doing. But let's say that someone (whom we trust completely) tells us that, actually, Sarah is *not* swimming. This piece of information about Sarah, then, let's us safely infer that Jeff is eating a burrito. This reasoning process is an example of the DISJUNCTION ELIMINATION RULE.

disjunction elimination rule

m	$\mathsf{A} \lor \mathsf{B}$	
n	$\neg\mathsf{B}$	
	A	$:\lor\mathrm{E}\ m, n$

m	$\mathsf{A} \lor \mathsf{B}$	
n	$\neg\mathsf{A}$	
	B	$:\lor\mathrm{E}\ m, n$

If we have '$\mathsf{A} \lor \mathsf{B}$' on a line and, on another line, we have what is either before or after the '$\lor$' with a '$\neg$' before it (i.e., '$\neg\mathsf{B}$' or '$\neg\mathsf{A}$'), then, on a new line, we can put what is on the other side of the '$\lor$'.

Disjunction introduction *Sarah is swimming or Jeff is eating a burrito* is true when *Sarah is swimming* is false (she isn't swimming) and *Jeff is eating a burrito* is true. In other words, the disjunction will be true as long as one of the disjuncts is true. This feature of disjunctions lets us make an inference that we don't use often in our everyday lives. It is a very simple inference, however. Take any sentence. We'll use *you are studying logic*. That's true. Since *you are studying logic* is true, each one of these sentences is also true:

> You are studying logic, or you are studying German.
> You are studying logic, or your mother is in Fiji.
> You are studying logic, or a dragon is on the moon.

The idea is that, if we know that a sentence is true, we can create a longer sentence by adding 'or *any sentence whatsoever*' and the disjunction will also be true. This feature of the disjunction underlies the DISJUNCTION INTRODUCTION RULE (which, again, is two similar rules).

disjunction introduction rule

m	A	
	$\mathsf{A} \lor \mathsf{B}$	$:\lor\text{I } m$

m	A	
	$\mathsf{B} \lor \mathsf{A}$	$:\lor\text{I } m$

If we have 'A' on a line, then on a new line we can repeat the 'A' and add '$\lor$' and anything else.

Now, we will construct a proof for '$P \lor Q, R \,\&\, \neg Q \vdash P \lor S$'. In this proof, we will use the conjunction elimination rule, disjunction introduction rule, and the disjunction elimination rule.

After listing the premises, we use the conjunction elimination rule to get '$\neg Q$' on a line by itself.

1	$P \lor Q$	:PR
2	$R \,\&\, \neg Q$	:PR
3	$\neg Q$	:&E 2

Once we have the '$\neg Q$' on a line, we can (with the '$P \lor Q$' on line 1) use the disjunction elimination rule to get 'P' on line 4.

1	$P \lor Q$	:PR
2	$R \,\&\, \neg Q$	:PR
3	$\neg Q$	:&E 2
4	P	:$\lor$E 1, 3

And then last, we use the disjunction introduction rule to get the conclusion.

1	$P \vee Q$	:PR
2	$R \mathbin{\&} \neg Q$	:PR
3	$\neg Q$	:&E 2
4	P	:∨E 1, 3
5	$P \vee S$	:∨I 4

Introduction and elimination

When you use the rules in this chapter, you are applying the *patterns* given in the definition of each rule. Every pattern is different, and so you have to make sure that you understand each one. But you don't, actually, have to think beyond the patterns—although you can, and some people find it helpful to think about how the rules of derivation conform to what you learned about each logical operator in chapter .

It can also be useful to understand why we call these *introduction* and *elimination* rules. (Or, at least, it's useful not to misunderstand why we use these terms.) The introduction rules are given this name because, in each case, we introduce a logical operator. For instance, if a proof begins this way:

1	P	:PR
2	Q	:PR

then, clearly, there is not an '&' in the proof yet. The conjunction introduction rule, however, lets us introduce one:

3	$P \mathbin{\&} Q$	:&I 1, 2

And similarly (although each in its own way), the disjunction introduction, biconditional introduction, and conditional introduction rules let us create sentences with the '∨', '↔,' and '→' as the main logical operator.

The elimination rules, meanwhile, in a way, eliminate a logical operator. It might, however, be more useful to think of these as *extraction* rules because we use them to extract a part of a sentence and put it on a new line. For instance, if we have this:

$$\begin{array}{r|ll} 1 & R \mathbin{\&} T & \text{:PR} \\ \hline \end{array}$$

then we can use the conjunction elimination rule to take the R and put it on a line by itself:

$$\begin{array}{r|ll} 2 & R & \text{:\&E 1} \end{array}$$

We might think of this as having eliminated the '&', but the (R & T) is still on line 1 and can be used again in the proof. Hence, *extraction rules* is a little more accurate than *elimination rules*, but *elimination* is the commonly used term, and so we will stick with it.

Double negation

The DOUBLE NEGATION RULE is a rule of convenience that sometimes compliments the disjunction-elimination rule. (There are also times when it will be used in the proofs that are discussed in , but, for the material in this chapter, it will only be used with the disjunction-elimination rule.) First, notice that the disjunction-elimination rule is very specific. To use it, we need, on one line of our proof, a sentence with the form '$\mathsf{A} \vee \mathsf{B}$', and on another line of our proof, we need one side of the disjunction (either A or B) with a '¬' in front of it; that is, either $\neg\mathsf{A}$ or $\neg\mathsf{B}$.

This presents a problem if we have these two sentences somewhere in a proof:

$$\begin{array}{r|l} m & \neg P \vee Q \\ n & P \end{array}$$

You might think that, given those two lines, we can put 'Q' on a new line like this:

$$\begin{array}{r|ll} m & \neg P \vee Q & \\ n & P & \\ & Q & :\vee\text{E } m, n \end{array}$$

In a sense, this is the right idea for the disjunction elimination rule. One side of the disjunction '$\neg P \vee Q$' has to be true, and the 'P' on line n means that '$\neg P$' is false. Hence, we should be allowed to put 'Q' on a new line. The disjunction elimination rule, however, does not permit this. To see why, let's distinguish between NEGATION and DENIAL.

> negation and denial
>
> The NEGATION of a sentence is the sentence with a 'not' added to it.
> The DENIAL of a sentence is the sentence with either a 'not' added or a 'not' removed.

For example,

1. the negation of 'today is Tuesday' is 'today is not Tuesday'.

In TFL,

2. the negation of P is $\neg P$. The negation of $\neg P$ is $\neg\neg P$.

Meanwhile,

3. the denial of 'it is not raining' is either (a) 'it is raining' or (b) 'it is not not raining'.

In TFL,

4. the denial of $\neg P$ is either P or $\neg\neg P$.

To use the disjunction elimination rule, we must have the **negation** of one side of the disjunction on another line. In the example above, we have the denial of $\neg P$, not its negation, on line n. The DOUBLE NEGATION RULE helps us correct this so that we can use the disjunction elimination rule more often.

Before we see how the double negation rule can help us with our derivation, let's introduce the rule. The first version of the double negation rule allows us to add two *nots* (i.e., *not not*) to a sentence in TFL—which, of course, will not change the sentence's truth value. The second version of the double negation rule allows us to remove two *not*s, although needing to do this is less common.

double negation rule

m	A	
n	$\neg\neg\mathsf{A}$	:DN m

m	$\neg\neg\mathsf{A}$	
n	A	:DN m

Let's say that this is the argument for which we need to provide a proof: '$\neg P \vee Q, P \vdash Q$'. After the premises, we use the double negation rule to get '$\neg\neg P$' from line 2.

1	$\neg P \vee Q$	:PR
2	P	:PR
3	$\neg\neg P$	:DN 2

The 'P' on line 2 and the '$\neg\neg P$' on line 3 have exactly the same meaning. The only difference between 'P' and '$\neg\neg P$' is their form. But now that we have '$\neg\neg P$', we have the negation of what is on the left side of the disjunction (which is '$\neg P$'). That allows us to use the disjunction elimination rule, and we can get the conclusion.

1	$\neg P \vee Q$	:PR
2	P	:PR
3	$\neg\neg P$	:DN 2
4	Q	:$\vee$E 1, 3

Remember that, in this chapter, you will only use the double negation rule right before you use the disjunction elimination rule, and you will only use it some of the time with the disjunction elimination rule.

when not to use (left) and when to use (right) the double negation rule

m	$A \lor B$	
n	$\neg B$	
	A	$:\lor\text{E } m, n$

m	$A \lor \neg B$	
n	B	
p	$\neg\neg B$	$:\text{DN } n$
	A	$:\lor\text{E } m, p$

12.4 Conditional elimination

For the conditional, we will cover the elimination rule now and the introduction rule in §. Consider the following argument:

1. If the envelope is on the table, then Aleksander is in the safe house.
2. The envelope is on the table.
3. Therefore, Aleksander is in the safe house.

In this argument—which is valid—we have a conditional and then, on a separate line, the antecedent of that conditional ('the envelope is on the table'). This allows us to safely infer the consequent ('Aleksander is in the safe house'). In short, if we have a conditional and we know that the antecedent of the conditional is true, then we know that the consequent has to be true. (See also the discussion of the conditional on p. 63.) Deriving the consequent of the conditional in this way is an application of the CONDITIONAL ELIMINATION RULE. This rule is also sometimes called *modus ponens*. When we use the rule, the conditional and the antecedent of the conditional can be separated from one another, and they can appear in any order.

conditional elimination rule

$$\begin{array}{r|l}
m & \mathsf{A} \to \mathsf{B} \\
n & \mathsf{A} \\
 & \mathsf{B} \qquad\qquad :\to\text{E } m, n
\end{array}$$

If we have '$\mathsf{A} \to \mathsf{B}$' on a line and, on a different line, we have just 'A' (i.e., the antecedent of the conditional), then, on a new line, we can put 'B' (i.e., the consequent of the conditional).

biconditional elimination rule

If we have '$\mathsf{A} \leftrightarrow \mathsf{B}$' on a line and, on a different line, we have 'A', then, on a new line, we can put 'B'.

$$\begin{array}{r|l}
m & \mathsf{A} \leftrightarrow \mathsf{B} \\
n & \mathsf{A} \\
 & \mathsf{B} \qquad\qquad :\leftrightarrow\text{E } m, n
\end{array}$$

Or, if we have '$\mathsf{A} \leftrightarrow \mathsf{B}$' on a line and, on a different line, we have 'B', then, on a new line, we can put 'A'.

$$\begin{array}{r|l}
m & \mathsf{A} \leftrightarrow \mathsf{B} \\
n & \mathsf{B} \\
 & \mathsf{A} \qquad\qquad :\leftrightarrow\text{E } m, n
\end{array}$$

12.5 Biconditional intro and elim

The BICONDITIONAL ELIMINATION RULE is similar to the conditional elimination rule but a bit more flexible. If you have a biconditional on one line and the left side of the biconditional on another line, you can put the right side on a new line. Or, if you have the right side, you can put the left side on a new line. Notice the difference between the conditional elimination rule and the biconditional elimination rule. There are two ways to use the biconditional elimination rule. There is only one way to use the conditional elimination rule.

In chapter , we said that the biconditional is "the conjunction of a conditional running in each direction." This is the basis for the BICONDITIONAL INTRODUCTION RULE. If we have both conditionals, $\mathsf{A} \to \mathsf{B}$ and $\mathsf{B} \to \mathsf{A}$ on separate lines in our proof, then we can put $\mathsf{A} \leftrightarrow \mathsf{B}$ on a new line.

biconditional introduction rule

m	$\mathsf{A} \to \mathsf{B}$	
n	$\mathsf{B} \to \mathsf{A}$	
	$\mathsf{A} \leftrightarrow \mathsf{B}$	$:\leftrightarrow\text{I } m, n$

m	$\mathsf{A} \to \mathsf{B}$	
n	$\mathsf{B} \to \mathsf{A}$	
	$\mathsf{B} \leftrightarrow \mathsf{A}$	$:\leftrightarrow\text{I } m, n$

If we have '$\mathsf{A} \to \mathsf{B}$' and '$\mathsf{B} \to \mathsf{A}$' on two lines of our proof, then, on a new line, we can put either '$\mathsf{A} \leftrightarrow \mathsf{B}$' or '$\mathsf{B} \leftrightarrow \mathsf{A}$'.

12.6 Some examples

We will now look at some proofs that use the rules that are covered in sections – . There are also practice exercises using these rules on p. 143.

1. For a proof of '$P \to Q, R \,\&\, P \vdash Q \,\&\, R$', we use the conjunction introduction rule, the conjunction elimination rule, and the conditional elimination rule.

1	$P \to Q$	:PR
2	$R \mathbin{\&} P$	:PR
3	P	:&E 2
4	R	:&E 2
5	Q	:→E 1, 3
6	$Q \mathbin{\&} R$	:&I 4, 5

2. For a proof of '$R \leftrightarrow T, P \vee T, \neg P \vdash R$', we use the disjunction elimination rule and the biconditional elimination rule.

1	$R \leftrightarrow T$	:PR
2	$P \vee T$	:PR
3	$\neg P$	:PR
4	T	:∨E 2, 3
5	R	:↔E 1, 4

3. For '$C \mathbin{\&} (D \vee \neg F), F \mathbin{\&} G \vdash C \mathbin{\&} (D \vee H)$', we use all five of the rules introduced in §§ and .

1	$C \mathbin{\&} (D \vee \neg F)$	:PR
2	$F \mathbin{\&} G$	:PR
3	C	:&E 1
4	$D \vee \neg F$	:&E 1
5	F	:&E 2
6	$\neg\neg F$	:DN 5
7	D	:∨E 4, 6
8	$D \vee H$	:∨I 7
9	$C \mathbin{\&} (D \vee H)$	:&I 3, 8

4. For a proof of '$(R \& T) \rightarrow Q, T \& S, R \vdash Q$', we use the conjunction elimination, conjunction introduction, and conditional elimination rules.

1	$(R \,\&\, T) \rightarrow Q$	:PR
2	$T \,\&\, S$	:PR
3	R	:PR
4	T	:&E 2
5	$R \,\&\, T$	:&I 3, 4
6	Q	:→E 1, 5

5. For a proof of '$P \leftrightarrow (R \vee S), T \rightarrow R, Q \& T \vdash P$', we use the conjunction elimination, conditional elimination, disjunction introduction, and biconditional introduction rules.

1	$P \leftrightarrow (R \vee S)$	:PR
2	$T \rightarrow R$	:PR
3	$Q \,\&\, T$	:PR
4	T	:&E 3
5	R	:→E 2, 4
6	$R \vee S$	:∨I 5
7	P	:→E 1, 5

6. And last, a proof for this argument:

$$(S \rightarrow T) \vee \neg R, (T \rightarrow S) \vee Q, R \,\&\, \neg Q \vdash T \leftrightarrow S$$

requires the conjunction elimination rule, the double negation rule, the disjunction elimination rule, and the biconditional introduction rule.

1	$(S \to T) \vee \neg R$	:PR
2	$(T \to S) \vee Q$	:PR
3	$R \,\&\, \neg Q$	:PR
4	R	:&E 3
5	$\neg Q$	:&E 3
6	$\neg\neg R$	:DN 4
7	$S \to T$	:∨E 1, 6
8	$T \to S$	:∨E 2, 5
9	$T \leftrightarrow S$	:↔I 7, 8

12.7 Conditional introduction

The CONDITIONAL INTRODUCTION RULE is a little bit more complicated than the conditional elimination rule, but, with some thought (and some practice), it is easily grasped. We'll start with this symbolization key for the sentence letters G and L:

G: Kate's German class meets today.
L: Kate's logic class meets today.

And this is our argument:

$$G \vee L \vdash \neg G \to L$$

We will go through the proof for this argument, and in the process explain the conditional introduction rule. We start by listing the premise.

1	$G \vee L$	:PR

Next, we need to make a new assumption: 'Kate's German class is *not* meeting today'. We might say that we're making this assumption "for the sake of argument" or to see where it leads. To indicate that this is an assumption that we have supplied, we put '$\neg G$' on line 2 this way:

1	$G \lor L$	:PR
2	$\quad \neg G$	:AS

You will notice right away that the '$\neg G$' is indented. Whenever we make an assumption ourselves, we must indent it and the lines that follow. This creates a SUBPROOF that is set off from the rest of the proof. The assumption is cited with 'AS', and we put a line under the assumption just as we do with the final premise. With this assumption in place, we next use the disjunction elimination rule to get L on line 3.

1	$G \lor L$	:PR
2	$\quad \neg G$	:AS
3	$\quad L$	:$\lor$E 1, 2

The idea for the first three lines of this proof are, first, we know that *Kate's German class meets today or her logic class meets today*. (Or, at least, we are assuming that '$G \lor L$' is true because that is the premise that we were given). Next, on line 2, we are, in effect, asking, "What if her German class is not meeting today?" That is, what will follow if we make this assumption? Well, one thing that will follow is that Kate's logic class must be meeting today.

So, on line 2, we have asked, What if *Kate's German class is not meeting today*? On line 3, we have one answer: *Kate's logic class is meeting today*. Therefore, on line 4, we can use the conditional introduction rule to put these two together as *if Kate's German class is not meeting today, then Kate's logic class is meeting today*.

1	$G \lor L$	:PR
2	$\quad \neg G$	:AS
3	$\quad L$	:$\lor$E 1, 2
4	$\neg G \to L$	:$\to$I 2–3

For this final step, we have gone back to the original vertical line of the proof.

When we use the conditional introduction rule, the assumption that we make will always be the antecedent of the conditional. The last line of the subproof, meanwhile, will always be the consequent of the conditional.

conditional introduction rule

i	$\quad \mid \mathsf{A}$	:AS
j	$\quad \mid \mathsf{B}$	
	$\mathsf{A} \rightarrow \mathsf{B}$	:$\rightarrow$I i–j

We begin by making an assumption: 'A'. We then derive 'B'. Once that is done, we know that *if* A, *then* B, and we can put the conditional on the line after the subproof.
There can be as many or as few lines as needed between lines i and j.
The lines cited are the range for the subproof, beginning with the line where the assumption is.

To simplify matters at the beginning of section , I only use the term *premises* to refer to the premises of an argument. A premise, however, is just a type of assumption. It is an assumption because we are taking it as given, and so it requires no justification—just like the assumption that we make at the beginning of a subproof.

Subproofs Lines i through j are called a SUBPROOF. These are the rules for subproofs:

1. Once a subproof has been closed, none of the lines in the subproof can be used again. (The conditional $\mathsf{A} \rightarrow \mathsf{B}$ can be used later in the proof because it is outside of the subproof.)
2. A subproof is closed by the application of the conditional introduction rule—or, as you will see shortly, the negation introduction or the negation elimination rules.
3. When we close a subproof, the assumption made at the beginning of the subproof has been *discharged*.
4. A proof is not complete until every assumption that we have made (and so not counting the premises) is discharged.

12.8 Some more examples

Each of these examples uses the conditional introduction rule.

1. We will start with a proof for this argument:

$$P \to Q, Q \to R \vdash P \to R$$

We start by listing both of our premises. Next, since we want '$(P \to R)$', we assume the antecedent of that conditional.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS

Now, even though it is an assumption that we've introduced, since 'P' is on a line by itself (and the subproof has not yet been closed), we can use it for our next step. With 'P' and the '$P \to Q$' on line 1, we can use the conditional elimination rule to get 'Q'.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:$\to$E 1, 3

With the 'Q' on line 4 and '$Q \to R$' on line 2, we can use the conditional elimination rule again; this time to get 'R'. So, by assuming 'P', we were able to get 'R'. Last, we apply the conditional introduction rule, which discharges our assumption and completes the proof.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:$\to$E 1, 3
5	$\quad R$	:$\to$E 2, 4
6	$P \to R$	:$\to$I 3–5

2. Next, let's construct a proof for this argument: '$F \to (G \,\&\, H) \vdash F \to G$'. We proceed this way:

1	$F \to (G \,\&\, H)$	:PR
2	$\quad F$	:AS
3	$\quad G \,\&\, H$	:$\to$E 1, 2
4	$\quad G$	:&E 3
5	$F \to G$	:$\to$I 2–4

3. As you know, the biconditional elimination rule is similar to the conditional elimination rule. (But they are not the same. See p. 127 to compare them.) We should also, however, be able to start with a biconditional, say '$M \leftrightarrow P$' and derive either of the conditionals: '$M \to P$' or '$P \to M$'. This is easily done with the conditional introduction rule.

1	$M \leftrightarrow P$	:PR
2	$\quad M$	:AS
3	$\quad P$	:$\leftrightarrow$E 1, 2
4	$M \to P$	:$\to$I 2–3

And, with a similar proof, we can also derive '$P \to M$'.

4. In the proof for '$\neg P \vee (R \& Q) \vdash P \rightarrow Q$', we will use the conditional introduction rule as well as double negation rule and disjunction-elimination rule.

1	$\neg P \vee (R \& Q)$	:PR
2	$\quad P$	:AS
3	$\quad \neg\neg P$	:DN 2
4	$\quad R \& Q$	:$\vee$E 1, 3
5	$\quad Q$	:&E 4
6	$P \rightarrow Q$	:$\rightarrow$I 2–5

12.9 Negation introduction and elimination

Here is a simple mathematical argument in English:

1. Assume that there is some greatest natural number. Call it G.
2. That number plus one is also a natural number.
3. $G + 1$ is greater than G.
4. Thus, G is the greatest natural number (accordingt to 1), and there is a natural number greater than G (according to 3).
5. The previous line is a contradiction.
6. Therefore, the assumption that we made on line 1 is false. There is no greatest natural number.

This type of argument is traditionally called a *reductio*. Its full Latin name is *reductio ad absurdum*, which means 'reduction to absurdity' (although *absurdity* in the sense that we generally use the word today isn't part of this). In a reductio, we assume something for the sake of argument—for example, that there is a greatest natural number. Then we show that the assumption leads to two contradictory sentences—for example, 'G is the greatest natural number' and 'G is not the greatest natural number.' In this way, we have shown that the original assumption must be false, which means that the denial of the assumption is true.

Our two negation rules (which are basically the same rule) formalize this reasoning process.

negation introduction rule

$$
\begin{array}{l|l|ll}
m & & \quad \mathsf{A} & \quad :\text{AS} \\
\hline
n & & \quad \mathsf{B} & \\
p & & \quad \neg\mathsf{B} & \\
 & \neg\mathsf{A} & & \quad :\neg\text{I } m\text{–}p
\end{array}
$$

Assume **A**. Derive a contradiction. (That is, get **B** and ¬**B** on the last two lines of the subproof). Exit the subproof, and put ¬**A** on the first line after the subproof.

There can be as many or as few lines as needed between lines m and n, but n and p have to be consecutive lines. **B** and ¬**B** can be any contradiction that it is possible to derive; they can be in either order; and **A** can be one half of the contradiction (although it doesn't have to be).

negation elimination rule

$$
\begin{array}{l|l|ll}
m & & \quad \neg\mathsf{A} & \quad :\text{AS} \\
\hline
n & & \quad \mathsf{B} & \\
p & & \quad \neg\mathsf{B} & \\
 & \mathsf{A} & & \quad :\neg\text{E } m\text{–}p
\end{array}
$$

Assume ¬**A**. Derive a contradiction. (That is, get **B** and ¬**B** on the last two lines of the subproof). Exit the subproof, and put **A** on the first line after the subproof.

Notice that, just as we do when using the conditional introduction rule, we begin by making an assumption. The subproof that follows is indented, and the assumption that we made must be discharged by applying either the negation introduction rule or the negation elimination rule.

When using either of the negation rules, the last two lines of the subproof must be an explicit contradiction: **B** on one line and its negation, ¬**B**, on the next line (or vice versa). Those two lines cannot be separated. When you cite the rule, however, the lines that you give are the lines for the whole subproof (starting with the assumption), not just the two lines containing the contradiction.

Reiteration

To get a contradiction on the last two lines of a subproof, we will usually have to move a sentence that is on an earlier line to the last or second-to-last line of the subproof. This is done with the REITERATION RULE. Just as the double negation rule is a rule of convenience that sometimes compliments the disjunction elimination rule, the reiteration rule is a rule of convenience that compliments the negation elimination and negation introduction rules.

reiteration rule

m	A	
n	A	:R m

To demonstrate both the negation elimination rule and the reiteration rule, we will go through the proof for this argument: '$\neg P \to \neg Q, Q \vdash P$'. Looking at the argument, you'll notice that our conclusion is 'P', but we cannot get 'P' by using &E, ∨E, →E, or ↔E. This means that we will need to use one of our negation rules.

After the premises, we make the assumption that we need for negation elimination. Since, ultimately, we want 'P', we will assume '$\neg P$' so that, once we discharge that assumption (and close the subproof), we will have the 'P' that we are after.

1	$\neg P \to \neg Q$	:PR
2	Q	:PR
3	$\quad \neg P$	:AS

We then use the conditional elimination rule to get $\neg Q$ on line 4.

$$
\begin{array}{r|l l}
1 & \neg P \to \neg Q & :\text{PR} \\
2 & \underline{Q} & :\text{PR} \\
3 & \quad \underline{\neg P} & :\text{AS} \\
4 & \quad \neg Q & :{\to}\text{E } 1, 3
\end{array}
$$

The Q on line 2 and $\neg Q$ on line 4 are a contradiction, but to use the negation elimination rule we need to have 'Q' on line 5. To get it there, we use the reiteration rule.

$$
\begin{array}{r|l l}
1 & \neg P \to \neg Q & :\text{PR} \\
2 & \underline{Q} & :\text{PR} \\
3 & \quad \underline{\neg P} & :\text{AS} \\
4 & \quad \neg Q & :{\to}\text{E } 1, 3 \\
5 & \quad Q & :\text{R } 2
\end{array}
$$

Now that '$\neg Q$' and 'Q' are on consecutive lines, we can use the negation elimination rule to discharge the assumption that we made, and that gives us the conclusion we are after: 'P'.

$$
\begin{array}{r|l l}
1 & \neg P \to \neg Q & :\text{PR} \\
2 & \underline{Q} & :\text{PR} \\
3 & \quad \underline{\neg P} & :\text{AS} \\
4 & \quad \neg Q & :{\to}\text{E } 1, 3 \\
5 & \quad Q & :\text{R } 2 \\
6 & P & :\neg\text{E } 3\text{–}5
\end{array}
$$

We just used the negation elimination rule. The negation introduction rule is, essentially, the same. Whether you use the negation introduction rule or negation elimination rule is just a function of whether you want '$\neg\mathbf{A}$' or '$\mathbf{A}$' on the line after the subproof.

12.10 Even more examples

The negation introduction rule or the negation elimination rule is used in each of these proofs.

1. $P \to Q, \neg Q \vdash \neg P$

1	$P \to Q$		:PR
2	$\neg Q$		:PR
3		P	:AS
4		Q	:$\to$E 1, 3
5		$\neg Q$	:R 2
6	$\neg P$		:$\neg$I 3–5

2. $P \to \neg Q \vdash \neg(P \,\&\, Q)$

1	$P \to \neg Q$		:PR
2		$P \,\&\, Q$	:AS
3		P	:&E 2
4		$\neg Q$	:$\to$E 1, 3
5		Q	:&E 2
6	$\neg(P \,\&\, Q)$		:$\neg$I 2–5

3. $Q \& R, \neg(Q \& S) \vdash \neg S$

1	$Q \& R$	:PR
2	$\neg(Q \& S)$	:PR
3	$\quad S$	:AS
4	$\quad Q$	:&E 1
5	$\quad Q \& S$	:&I 3, 4
6	$\quad \neg(Q \& S)$	:R 2
7	$\neg S$	:¬I 3–6

4. The proof for '$\neg P \vdash P \rightarrow Q$' requires two subproofs. First, we assume 'P' so that we can use the conditional introduction rule at the end of the proof. Then, we assume '$\neg Q$' so that we can use the negation elimination rule and get 'Q' on the last line of the first subproof.

1	$\neg P$	:PR
2	$\quad P$	:AS
3	$\quad\quad \neg Q$	:AS
4	$\quad\quad P$	:R 2
5	$\quad\quad \neg P$	:R 1
6	$\quad Q$	:¬E 3–5
7	$P \rightarrow Q$	:→I 2–6

5. The proof for '$Q \vee S, Q \rightarrow T, S \rightarrow T \vdash T$' also requires two subproofs.

1	$Q \vee S$	:PR
2	$Q \rightarrow T$	:PR
3	$S \rightarrow T$	:PR
4	$\quad \neg T$	:AS
5	$\quad\quad \neg Q$	:AS
6	$\quad\quad S$	:$\vee$E 1, 5
7	$\quad\quad T$	:$\rightarrow$E 3, 6
8	$\quad\quad \neg T$	:R 4
9	$\quad Q$	:$\neg$E 5–8
10	$\quad T$	:$\rightarrow$E 2, 9
11	$\quad \neg T$	:R 4
12	T	:$\neg$E 4–11

12.11 Invalid arguments

In this chapter, we have taken it for granted that each argument that we have encountered has been valid. The purpose of providing a proof is (1) to confirm that it is valid and (2) to show why it is valid—that is, to lay out each step that takes us from the premises to the conclusion. If an argument is invalid, however, we are stuck. It is impossible to provide a correct proof of an invalid argument using the rules given in this chapter. At the same time, not being able to provide a proof for an argument doesn't mean that the argument is invalid. Perhaps the proof is just too complicated for us to figure out.

In chapter , we discussed some reasons to prefer natural deduction to truth tables for checking that an argument is valid. To show that an argument is invalid, however, creating a truth table is not merely a superior method, it is our only option.

12.12 Practice exercises

A. Give a proof for each argument using the rules from §§ – .

1. $\neg P \rightarrow (Q \lor P), \neg P \vdash Q$
   Note: After you list the premises on lines 1 and 2, notice that $\neg P$ is the antecedent of the sentence on line 1 and $\neg P$ is by itself on line 2. This means using the conditional elimination rule is an option.

2. $P \rightarrow (Q \lor \neg P), P \vdash Q$
   Note: This problem is similar to the one right above, but it's not exactly the same. If you're not sure how to do this one, look at the subsection on the double negation rule (pp. 123 - 126).

3. $D \,\&\, H, H \leftrightarrow J \vdash J \lor N$

4. $R \,\&\, S, (S \lor Q) \rightarrow T \vdash T$
   Note: Since $(S \lor Q)$ is the antecedent of the conditional on line 2, to use the conditional elimination rule, you need to have $(S \lor Q)$ on a line by itself. Since $(S \lor Q)$ is a disjunction, you will need to use the disjunction introduction rule to get it. (Check how that rule works if you don't remember: p. 121.)

5. $G \,\&\, (H \,\&\, J), (H \lor M) \rightarrow K \vdash K$

6. $P \,\&\, (Q \lor R), P \rightarrow \neg R \vdash Q$

7. $(P \lor \neg Q) \leftrightarrow R, R \,\&\, Q \vdash P \,\&\, R$

8. $(R \,\&\, T) \rightarrow Q, R \lor \neg P, P \,\&\, T \vdash Q$
   Note: After you list the premises for this proof, you'll see that the conclusion is the consequent of the conditional on line 1. To use the conditional elimination rule, you need to use the two other sentences (on lines 2 and 3) to get the antecedent on a line by itself. That will take several steps. Think about whether you need to use the double negation rule for one of those steps.

9. $S \rightarrow T, Q \,\&\, \neg R, \neg R \leftrightarrow (T \rightarrow S) \vdash S \leftrightarrow T$

10. $(L \lor M) \rightarrow N, P \leftrightarrow N, L \vdash L \,\&\, P$

11. $(P \,\&\, R) \leftrightarrow (S \lor T), P \rightarrow Q, T \vdash Q$

12. $S \rightarrow T, T \rightarrow S, T \leftrightarrow (T \leftrightarrow S) \vdash S$

B. Give a proof for each argument.

1. $P \rightarrow (Q \rightarrow R) \vdash (P \,\&\, Q) \rightarrow R$

2. $Q \rightarrow R \vdash (Q \,\&\, S) \rightarrow (R \lor T)$

3. $M \,\&\, (\neg N \rightarrow \neg M) \vdash (N \,\&\, M) \vee \neg M$
4. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \rightarrow M) \vdash Y \rightarrow Z$
5. $C \rightarrow (E \,\&\, G), \neg C \rightarrow G \vdash G$
6. $\neg(P \rightarrow Q) \vdash \neg Q$
7. $S \leftrightarrow T \vdash S \leftrightarrow (T \vee S)$
8. $D \vee F, D \rightarrow G, F \rightarrow H \vdash G \vee H$

12.13 Answers

A.

1. $\neg P \rightarrow (Q \vee P), \neg P \vdash Q$

$$
\begin{array}{r|ll}
1 & \neg P \rightarrow (Q \vee P) & :\text{PR} \\
2 & \neg P & :\text{PR} \\
\hline
3 & Q \vee P & :\rightarrow\text{E } 1, 2 \\
4 & Q & :\vee\text{E } 2, 3
\end{array}
$$

2. $P \rightarrow (Q \vee \neg P), P \vdash Q$

$$
\begin{array}{r|ll}
1 & P \rightarrow (Q \vee \neg P) & :\text{PR} \\
2 & P & :\text{PR} \\
\hline
3 & Q \vee \neg P & :\rightarrow\text{E } 1, 2 \\
4 & \neg\neg P & :\text{DN } 2 \\
5 & Q & :\vee\text{E } 3, 4
\end{array}
$$

3. $D \,\&\, H, H \leftrightarrow J \vdash J \vee N$

$$
\begin{array}{r|ll}
1 & D \,\&\, H & :\text{PR} \\
2 & H \leftrightarrow J & :\text{PR} \\
\hline
3 & H & :\&\text{E } 1 \\
4 & J & :\leftrightarrow\text{E } 2, 3 \\
5 & J \vee N & :\vee\text{I } 4
\end{array}
$$

4. $R \,\&\, S, (S \vee Q) \to T \vdash T$

1	$R \,\&\, S$	:PR
2	$(S \vee Q) \to T$	:PR
3	S	:&E 1
4	$S \vee Q$	:∨I 3
5	T	:→E 2, 4

5. $G \,\&\, (H \,\&\, J), (H \vee M) \to K \vdash K$

Note: In this proof, just like in the previous one, you need to use the disjunction introduction rule to get an antecedent—in this case, '$(H \vee M)$'—on a line by itself (so that you can then use the conditional elimination rule). This is a good trick to remember.

1	$G \,\&\, (H \,\&\, J)$	:PR
2	$(H \vee M) \to K$	:PR
3	$H \,\&\, J$	:&E 1
4	H	:&E 3
5	$H \vee M$	:∨I 4
6	K	:→I 2, 5

6. $P \,\&\, (Q \vee R), P \to \neg R \vdash Q$

1	$P \,\&\, (Q \vee R)$	:PR
2	$P \to \neg R$	:PR
3	P	:&E 1
4	$\neg R$	:→I 2, 3
5	$Q \vee R$	:&E 1
6	Q	:∨E 4, 5

7. $(P \vee \neg Q) \leftrightarrow R, R \,\&\, Q \vdash P \,\&\, R$

1	$(P \vee \neg Q) \leftrightarrow R$	:PR
2	$R \,\&\, Q$	:PR
3	R	:&E 2
4	$P \vee \neg Q$	:$\leftrightarrow$E 1, 3
5	Q	:&E 2
6	$\neg\neg Q$	:DN 5
7	P	:$\vee$E 4, 6
8	$P \,\&\, R$	:&I 3, 7

8. $(R \,\&\, T) \rightarrow Q, R \vee \neg P, P \,\&\, T \vdash Q$

1	$(R \,\&\, T) \rightarrow Q$	:PR
2	$R \vee \neg P$	:PR
3	$P \,\&\, T$	:PR
4	P	:&E 3
5	$\neg\neg P$	:DN 4
6	R	:$\vee$E 2, 5
7	T	:&E 3
8	$R \,\&\, T$	:&I 6, 7
9	Q	:$\rightarrow$E 1, 8

9. $S \rightarrow T, Q \,\&\, \neg R, \neg R \leftrightarrow (T \rightarrow S) \vdash S \leftrightarrow T$

1	$S \rightarrow T$	:PR
2	$Q \,\&\, \neg R$	:PR
3	$\neg R \leftrightarrow (T \rightarrow S)$	:PR
4	$\neg R$	:&E 2
5	$T \rightarrow S$	:$\leftrightarrow$E 3, 4
6	$S \leftrightarrow T$	:$\leftrightarrow$I 1, 5

10. $(L \vee M) \rightarrow N, P \leftrightarrow N, L \vdash L \,\&\, P$

1	$(L \vee M) \rightarrow N$	:PR
2	$P \leftrightarrow N$	:PR
3	L	:PR
4	$L \vee M$	:$\vee$I 3
5	N	:$\rightarrow$E 1, 4
6	P	:$\leftrightarrow$E 2, 5
7	$L \,\&\, P$	:&I 3, 6

11. $(P \,\&\, R) \leftrightarrow (S \vee T), P \rightarrow Q, T \vdash Q$

1	$(P \,\&\, R) \leftrightarrow (S \vee T)$	:PR
2	$P \rightarrow Q$	:PR
3	T	:PR
4	$S \vee T$	:$\vee$I 3
5	$P \,\&\, R$	:$\leftrightarrow$E 1, 4
6	P	:&E 5
7	Q	:$\rightarrow$E 2, 6

12. $S \rightarrow T, T \rightarrow S, T \leftrightarrow (T \leftrightarrow S) \vdash S$

1	$S \rightarrow T$	:PR
2	$T \rightarrow S$	:PR
3	$T \leftrightarrow (T \leftrightarrow S)$	:PR
4	$T \leftrightarrow S$	:$\leftrightarrow$I 1, 2
5	T	:$\leftrightarrow$E 3, 4
6	S	:$\rightarrow$E 2, 5

B.

1. $P \to (Q \to R) \vdash (P \,\&\, Q) \to R$

1	$P \to (Q \to R)$	:PR
2	$\quad \mid P \,\&\, Q$	:AS
3	$\quad \mid P$	:&E 2
4	$\quad \mid Q \to R$	:→E 1, 3
5	$\quad \mid Q$	:&E 2
6	$\quad \mid R$	:→E 4, 5
7	$(P \,\&\, Q) \to R$	:→I 2–6

2. $Q \to R \vdash (Q \,\&\, S) \to (R \lor T)$

1	$Q \to R$	:PR
2	$\quad \mid Q \,\&\, S$	:AS
3	$\quad \mid Q$	:&E 2
4	$\quad \mid R$	:→E 1, 3
5	$\quad \mid R \lor T$	:∨I 4
6	$(Q \,\&\, S) \to (R \lor T)$	:→I 2–5

3. $M \,\&\, (\neg N \to \neg M) \vdash (N \,\&\, M) \lor \neg M$

1	$M \,\&\, (\neg N \to \neg M)$	:PR
2	M	:&E 1
3	$\neg N \to \neg M$	:&E 1
4	$\quad \mid \neg N$	:AS
5	$\quad \mid \neg M$	:→E 3, 4
6	$\quad \mid M$	:R 2
7	N	:¬E 4–6
8	$N \,\&\, M$	:&I 2, 7
9	$(N \,\&\, M) \lor \neg M$	:∨I 8

4. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \rightarrow M) \vdash Y \rightarrow Z$

1	$(Z \,\&\, K) \leftrightarrow (Y \,\&\, M)$	:PR
2	$D \,\&\, (D \rightarrow M)$	:PR
3	D	:&E 2
4	$D \rightarrow M$	:&E 2
5	M	:→E 3, 4
6	$\quad Y$	:AS
7	$\quad Y \,\&\, M$	:&I 5, 6
8	$\quad Z \,\&\, K$	:↔E 1, 7
9	$\quad Z$	:&E 8
10	$Y \rightarrow Z$	:→I 6–9

5. $C \rightarrow (E \,\&\, G), \neg C \rightarrow G \vdash G$

1	$C \rightarrow (E \,\&\, G)$	:PR
2	$\neg C \rightarrow G$	:PR
3	$\quad \neg G$	:AS
4	$\quad\quad C$	:AS
5	$\quad\quad E \,\&\, G$	:→E 1, 4
6	$\quad\quad G$	:&E 5
7	$\quad\quad \neg G$	:R 3
8	$\quad \neg C$	:¬I 4–7
9	$\quad G$	:→E 2, 8
10	$\quad \neg G$	:R 3
11	G	:¬E 3–10

6. $\neg(P \to Q) \vdash \neg Q$

1	$\neg(P \to Q)$	:PR
2	$\mid\ Q$	:AS
3	$\mid\ \mid\ P$	:AS
4	$\mid\ \mid\ Q$	:R 2
5	$\mid\ P \to Q$	:$\to$I 3–4
6	$\mid\ \neg(P \to Q)$	:R 1
7	$\neg Q$	:$\neg$I 2–6

7. $S \leftrightarrow T \vdash S \leftrightarrow (T \lor S)$

1	$S \leftrightarrow T$	:PR
2	$\mid\ S$	:AS
3	$\mid\ T$	:$\leftrightarrow$E 1, 2
4	$\mid\ T \lor S$	:$\lor$I 3
5	$S \to (T \lor S)$	:$\to$I 2–4
6	$\mid\ T \lor S$	:AS
7	$\mid\ \mid\ \neg S$	:AS
8	$\mid\ \mid\ T$	:$\lor$E 6, 7
9	$\mid\ \mid\ S$	:$\leftrightarrow$E 1, 8
10	$\mid\ \mid\ \neg S$	:R 7
11	$\mid\ S$	:$\neg$E 7–10
12	$(T \lor S) \to S$	:$\to$I 6–11
13	$S \leftrightarrow (T \lor S)$	:$\leftrightarrow$I 5, 12

8. $D \vee F, D \rightarrow G, F \rightarrow H \vdash G \vee H$

1	$D \vee F$	:PR
2	$D \rightarrow G$	:PR
3	$F \rightarrow H$	:PR
4	$\quad \neg(G \vee H)$	:AS
5	$\quad\quad \neg D$	:AS
6	$\quad\quad F$	:$\vee$E 1, 5
7	$\quad\quad H$	:$\rightarrow$E 3, 6
8	$\quad\quad G \vee H$	:$\vee$I 7
9	$\quad\quad \neg(G \vee H)$	:R 4
10	$\quad D$	:$\neg$E 5–9
11	$\quad G$	:$\rightarrow$E 2, 10
12	$\quad G \vee H$	:$\vee$I 11
13	$\quad \neg(G \vee H)$	:R 4
14	$G \vee H$	:$\neg$E 4–13

13 Proofs in Carnap

Creating proofs in Carnap is not difficult. To type the connectives, use the symbols on the right in table .

Carnap will number the lines automatically. After the TFL sentence on each line, there has to be a colon (':') before the 'PR', 'AS', or the rule. Carnap is flexible with the spacing on a line, but as a guideline, put a tab space between the sentence and ':PR', ':AS', or the rule (:$\rightarrow$E, :$\vee$I, etc.). Also indent subproofs with a tab space. (Carnap will let you use more or fewer spaces, but a subproof has to be indented some amount.)

To create a proof, you are given an interface like the one shown in figure . As you can see, the argument is given at the top. In this case, the premises are $P \rightarrow \neg Q$ and $R \,\&\, P$, and the conclusion is $\neg Q$. (The premises are separated by commas. The premises and the conclusion are separated by the turnstile ('$\vdash$').)

Begin by listing the premises, and don't forget to put ':PR' after each one. If there is a problem with a line—the sentence isn't formed correctly, the rule you've cited isn't being used correctly, or there's some other mistake—Carnap will put ? or ⚠ at the end of the line. When the line is ok, you will get a '+'. We finish this proof using the &E and $\rightarrow$E rules (figure). When the proof is correct, the box containing the argument will turn green, and the proof can be submitted.

Our next example, $(P \vee Q) \vdash (\neg P \rightarrow Q)$, requires a subproof. We

TFL OPERATOR	IN CARNAP
$\neg$	~
&	&
$\vee$	v (lowercase v)
$\rightarrow$	-> (dash, greater than sign)
$\leftrightarrow$	<->

Table 13.1

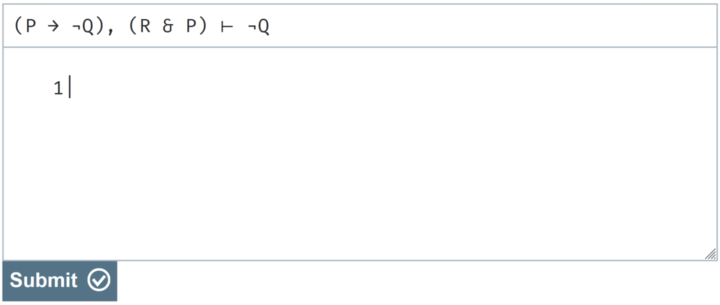

Figure 13.1

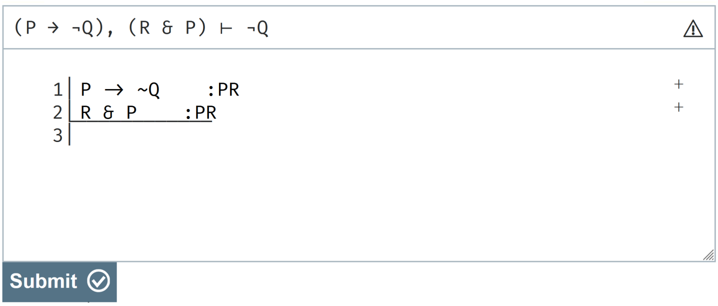

Figure 13.2

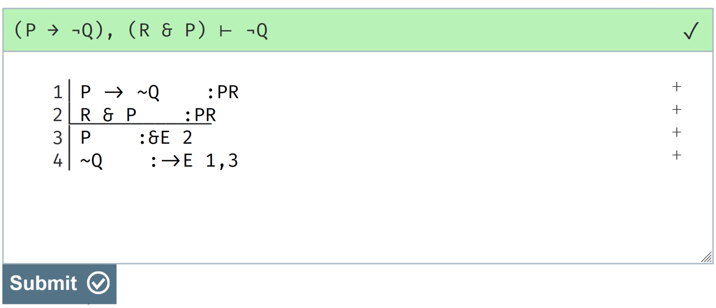

Figure 13.3

begin as before. To create the subproof, put a tab space before ¬*P* and put ':AS' at the end of the line (figure). Since the next line is also part of the subproof, we again need a tab before the *Q*. We end the subproof (and discharge the assumption) with the →I rule. ¬*P* → *Q* is not indented (so no tabs or spaces before the ¬*P*). That's the conclusion, and so if everything is correct, Carnap will give you the green bar and you can submit the proof (figure).

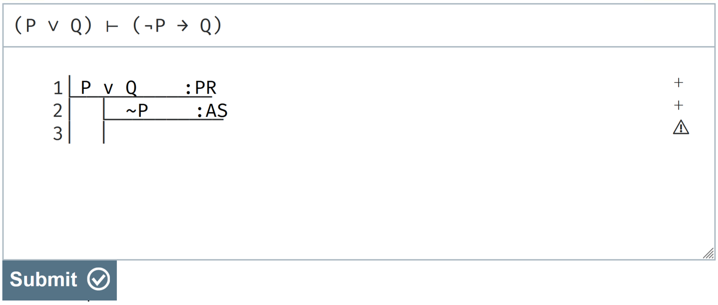

Figure 13.4

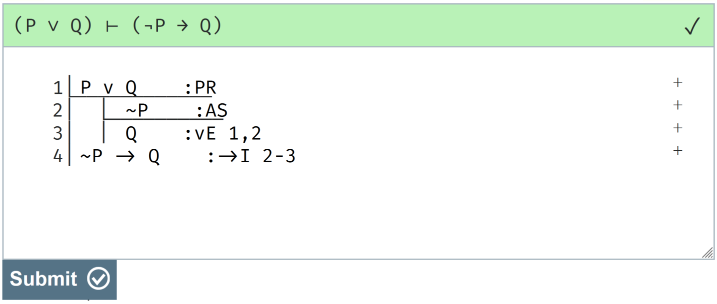

Figure 13.5

Although creating proofs in Carnap is not difficult, you do have to be careful. Creating a program that can verify proofs that use only the rules of derivation given in chapter is relatively simple because there are only a small number of rules and, to produce proofs of valid arguments, we follow those rules very strictly. But, as a consequence, Carnap is not designed to understand what you are trying to do if you deviate from the rules, even if it is a minor deviation or an innocent mistake. So, some reminders:

1. As long as '¬' is not the main logical operator, you can drop the outermost parentheses. All other parentheses have to be used.
2. Capitalize 'PR', 'AS','E', 'I' (in the rules), and all atomic sentences.
3. Don't forget the ':' right before PR, AS, or the rule that you are citing.

4. There is no space between the &, ∨, →, ↔, or ¬ and the 'E' or 'I'.
5. There is a space (and no punctuation) after the 'E' or 'I'.
6. There is a comma between the two lines that have to be cited for &I, ∨E, →E, and ↔E (e.g., '→E 2,4').
7. There is a dash between the two lines that have to be cited for →I, ¬I, and ¬E (e.g., '¬E 4-6').

14 Some strategies

There is no simple recipe for constructing proofs, and there is no substitute for practice. Here, however, are some questions to ask yourself and some strategies to keep in mind.

1. Do you know all of the rules? **If you don't have them memorized yet, then they should be written on a sheet of paper that you have next to you while you're working.**
2. Are there steps that you can take without making an assumption? If yes, is it worth taking those steps?
3. If you're not sure how to proceed, but you can do conjunction elimination, conditional elimination, disjunction elimination, or biconditional elimination, then do them just to see what happens.

The theme for 4 – 7 is "think ahead." Some amount of trial and error is often necessary, but, especially when you are constructing a proof that will contain a subproof, it's important to think about how each step that you take will affect the later parts of your proof.

4. If an assumption is needed, is it for $\to$I, $\neg$I, or $\neg$E? **Don't make an assumption if you don't know which of these rules you plan to use when you close the subproof.**
5. If an assumption is needed, what should it be? (If you want to get $P \to Q$, then you're going to use $\to$I and your assumption should be P.)
6. If you make an assumption, then you should know what you want on either the last line or the last two lines of your subproof.
    a. If you're using $\to$I, then you will need the consequent of the conditional on the last line of the subproof.

b. If you're using ¬I or ¬E, then you need a contradiction on the last two lines of your subproof, although that can be any contradiction. It doesn't have to be related to the assumption.

7. Sometimes it is useful to work backwards from the conclusion. The conclusion, of course, will be the last line of your proof, and you can, if you wish, put it at the bottom of the proof anytime. For example, let's say that you need to provide a proof for this argument: $P \to (\neg Q \to R) \vdash (P \,\&\, \neg Q) \to R$. You can begin this way:

1	$P \to (\neg Q \to R)$	:PR
	$(P \,\&\, \neg Q) \to R$	

Knowing that you need to arrive at a conditional, you also know these three things: (1) you need to use the conditional-introduction rule, (2) what your assumption should be, and (3) what will be on the last line of your subproof.

1	$P \to (\neg Q \to R)$	:PR
2	$\quad P \,\&\, \neg Q$	:AS
	$\quad R$	
	$(P \,\&\, \neg Q) \to R$	:→I

Sketching out a proof in this way is easy to do when you are writing on paper. If you are doing it in Carnap, be careful about the spacing that you put on each blank line.

8. The negation introduction and negation elimination rules are a last resort. Use them when you can't use any of the other rules. When you do use them, always have in mind that, when you complete the subproof, you will have the opposite of the assumption. Hence, a good guideline is to make the assumption the opposite of the conclusion.

   (If you have to make two assumptions – and both assumptions will be discharged with one of these rules – this guideline only applies to the first assumption. Determining the best choice for a second assumption sometimes takes a little trial and error.)

9. **Persist**. Try different things. If one approach fails, then try something else.

15 Proof-theoretic concepts

15.1 Theorems

You are familiar with arguments that have this form:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{C}$$

We may also, however, have a sentence for which it is possible to give a proof with no premises: $\vdash \mathsf{C}$. In this case, we say that C is a THEOREM.

> Theorem
>
> C is a THEOREM if and only if $\vdash \mathsf{C}$

One such sentence is $\neg(P \,\&\, \neg P)$. To show that this sentence is a theorem, we give a proof that has no premises and no undischarged assumptions. To get started, we do, however, have to make an assumption. We will assume $(P \,\&\, \neg P)$. Once we show that this assumption leads to contradiction, we can discharge it and we will have $\neg(P \,\&\, \neg P)$. This is the proof:

1	$\quad P \,\&\, \neg P$	:AS
2	$\quad P$	:&E 1
3	$\quad \neg P$	:&E 1
4	$\neg(P \,\&\, \neg P)$	:¬I 1–3

This theorem, $\vdash \neg(P \,\&\, \neg P)$, is an instance of what is called *the law of non-contradiction*.

To show that a sentence is a theorem, we just have to find a suitable proof. On the other hand, it is not possible to show that a sentence is *not* a theorem this same way. To show that a sentence is not a theorem with our natural deduction system, we would have to demonstrate, not just that certain proof strategies fail, but that *no* proof is possible. Even if we fail in trying to give a proof for a sentence in a thousand different ways, perhaps the proof is just too long and complex for us to figure out.

15.2 Equivalent, consistent, and inconsistent

In section , we defined *equivalent* in terms of truth tables, namely, if two sentences have the same truth value on every line of a truth table, then they are equivalent. We can also show that two sentences are equivalent using our natural deduction system. To indicate that we have shown that the two sentences are equivalent with a derivation (or actually with two derivations), we will call this equivalence PROVABLY EQUIVALENT.

Provably equivalent

Two sentences A and B are PROVABLY EQUIVALENT iff each can be derived from the other. I.e., $\mathsf{A} \vdash \mathsf{B}$ and $\mathsf{B} \vdash \mathsf{A}$.

Equivalently, A and B are PROVABLY EQUIVALENT if $\vdash (\mathsf{A} \leftrightarrow \mathsf{B})$.

As in the case of showing that a sentence is a theorem, it is relatively easy to show that two sentences are provably equivalent: it just requires a pair of proofs. Showing that sentences are *not* provably equivalent is not possible for t he same reason that it isn't possible to show that a sentence is not a theorem. Even if we fail to produce two proofs showing that two sentences are provably equivalent, that doesn't mean that the proofs don't exist. It just means that we've failed to figure out what they are.

We also, in section , defined *jointly inconsistent* using truth tables: sentences are jointly inconsistent if there is no line on a truth table where they are all true. Again, we can show that two or more sentences are jointly inconsistent with our natural deduction system.

Provably inconsistent

The sentences $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are PROVABLY INCONSISTENT iff, from them, a contradiction can be derived. I.e. $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash (\mathsf{B} \,\&\, \neg\mathsf{B})$.

To show that a set of sentences are provably inconsistent, we use the sentences as premises and then derive a contradiction. (Any contradiction will do.) For instance, this proof demonstrates that $(P \& Q)$ and $(\neg P \vee \neg Q)$ are provably inconsistent.

1	$P \,\&\, Q$	:PR
2	$\neg P \vee \neg Q$	:PR
3	P	:&E 1
4	$\neg\neg P$	:DN 3
5	$\neg Q$	:∨E 2, 4
6	Q	:&E 1
7	$Q \,\&\, \neg Q$	:&I 5, 6

Showing that some set of sentences are *not* provably inconsistent is, as you might guess at this point, not possible. Doing so would require showing, not just that we have failed to derive a contradiction from a set a sentences, but that no such derivation is possible.

Table summarizes what we have covered in this chapter. As we will discuss in the next chapter, when the presence (or the absence) of a logical property cannot be demonstrated using our natural deduction system, we have to resort to using a truth table.

TO CHECK	THAT IT IS	THAT IT IS NOT
theorem	one proof	not possible with proofs
equivalent	two proofs	not possible with proofs
inconsistent	one proof	not possible with proofs
consistent	not possible with proofs	one proof

Table 15.1: This table summarizes what is required to check each of these logical notions.

15.3 Practice exercises

A. Give a proof for each of these theorems.

1. $\vdash O \to O$
2. $\vdash S \to (S \lor R)$
3. $\vdash N \lor \neg N$
4. $\vdash \neg((R \lor T) \,\&\, (\neg R \,\&\, \neg T))$
5. $\vdash (R \leftrightarrow M) \to (M \to R)$

B. Show that each of the following pairs of sentences are provably equivalent. (To indicate that the inference from the premise to the conclusion goes from the first sentence to the second and vice versa, we use the symbols $\dashv\vdash$.)

1. $T \to S \dashv\vdash \neg S \to \neg T$
2. $R \to Q \dashv\vdash \neg(R \,\&\, \neg Q)$

15.4 Answers

A.

1. $\vdash O \to O$

1	$\quad\vert\ \underline{O}$	:AS
2	$\quad\vert\ O$	:R 1
3	$O \to O$	:→I 1–2

2. $\vdash S \to (S \vee R)$

1	$\quad S$	:AS
2	$\quad S$	:R 1
3	$\quad S \vee R$	:$\vee$I 2
4	$S \to (S \vee R)$	:$\to$I 1–3

3. $\vdash N \vee \neg N$

1	$\quad \neg(N \vee \neg N)$	:AS
2	$\quad\quad N$	:AS
3	$\quad\quad N \vee \neg N$	:$\vee$I 2
4	$\quad\quad \neg(N \vee \neg N)$	:R 1
5	$\quad \neg N$	:$\neg$I 2–4
6	$\quad N \vee \neg N$	:$\vee$I 5
7	$\quad \neg(N \vee \neg N)$	:R 1
8	$N \vee \neg N$	:$\neg$E 1–7

4. $\vdash \neg((R \vee T) \& (\neg R \& \neg T))$

1	$\quad (R \vee T) \& (\neg R \& \neg T)$	:AS
2	$\quad R \vee T$	:&E 1
3	$\quad \neg R \& \neg T$	:&E 1
4	$\quad \neg R$	:&E 3
5	$\quad \neg T$	:&E 3
6	$\quad T$	:$\vee$E 2, 4
7	$\neg((R \vee T) \& (\neg R \& \neg T))$	:$\neg$I 1–6

5. $\vdash (R \leftrightarrow M) \rightarrow (M \rightarrow R)$

1	$\quad\vert\quad\vert\ \underline{R \leftrightarrow M}$	:AS
2	$\quad\vert\quad\vert\quad\vert\ \underline{M}$	:AS
3	$\quad\vert\quad\vert\quad\vert\ R$	:$\leftrightarrow$E 1, 2
4	$\quad\vert\quad\vert\ M \rightarrow R$	:$\rightarrow$I 2–3
5	$\quad\vert\ (R \leftrightarrow M) \rightarrow (M \rightarrow R)$	:$\rightarrow$I 1–4

B.

1. $T \rightarrow S \dashv\vdash \neg S \rightarrow \neg T$

1	$\quad\vert\ \underline{T \rightarrow S}$	:PR
2	$\quad\vert\quad\vert\ \underline{\neg S}$	:AS
3	$\quad\vert\quad\vert\quad\vert\ \underline{T}$	:AS
4	$\quad\vert\quad\vert\quad\vert\ S$	:$\rightarrow$E 1, 3
5	$\quad\vert\quad\vert\quad\vert\ \neg S$	:R 2
6	$\quad\vert\quad\vert\ \neg T$	:$\neg$I 2–5
7	$\quad\vert\ \neg S \rightarrow \neg T$	:$\rightarrow$I 2–6

1	$\quad\vert\ \underline{\neg S \rightarrow \neg T}$	:PR
2	$\quad\vert\quad\vert\ \underline{T}$	:AS
3	$\quad\vert\quad\vert\quad\vert\ \underline{\neg S}$	:AS
4	$\quad\vert\quad\vert\quad\vert\ \neg T$	:$\rightarrow$E 1, 3
5	$\quad\vert\quad\vert\quad\vert\ T$	:R 2
6	$\quad\vert\quad\vert\ S$	:$\neg$E 3–5
7	$\quad\vert\ T \rightarrow S$	:$\rightarrow$I 2–6

2. $R \rightarrow Q \dashv\vdash \neg(R \,\&\, \neg Q)$

1	$R \rightarrow Q$	:PR
2	$\quad R \,\&\, \neg Q$	:AS
3	$\quad R$	:&E 2
4	$\quad \neg Q$	:&E 2
5	$\quad Q$	:→E 1, 3
6	$\neg(R \,\&\, \neg Q)$	:¬I 2–5

1	$\neg(R \,\&\, \neg Q)$	:PR
2	$\quad R$	:AS
3	$\quad\quad \neg Q$	:AS
4	$\quad\quad R \,\&\, \neg Q$	:&I 2, 3
5	$\quad\quad \neg(R \,\&\, \neg Q)$	:R 1
6	$\quad Q$	:¬E 3–5
7	$R \rightarrow Q$	:→I 2–6

16 Soundness and completeness

We have two ways of checking or verifying that an argument is valid: (1) using truth tables and (2) using the natural deduction system to provide a proof. Consequently, we also have two ways of characterizing the concept of *validity*. (See table .) You might think that we can take it for granted that, with respect to determining if an argument is valid, both methods will always give us the same result, but that is not exactly the case. (We, right now, can take it for granted, but that's only because the requisite work to show that the two methods will always agree has already been done.) If you think about it for a moment, you'll notice that the two methods don't have anything in common, and so, it is not intuitively obvious that they will always produce the same result. But they do.

How do we know that the truth table method and the natural deduction method will always agree? Demonstrating that they will goes beyond the scope of this book. But we will review the two properties that a logic system (like TFL) must have for the two methods to always be in agreement. To begin, let us define two new terms.

> *p-valid*: being valid because a proof can be given using the rules in our natural deduction system. (*p-valid* is short for *proof-valid*. This is also sometimes called *syntactically valid*).
> *tt-valid*: being valid because there is no line in a truth table where the premises are true and the conclusion is false. (This is also sometimes called *semantically valid*).

First, it must be the case that every argument that is p-valid is tt-valid. This property is called SOUNDNESS.

	TRUTH TABLE (SEMANTIC) DEFINITION	PROOF-THEORETIC (SYNTACTIC) DEFINITION
Tautology	A sentence whose truth table has a T on every line under the main connective	A sentence that can be derived without any premises. I.e., a theorem.
Contradiction	A sentence whose truth table has an F on every line under the main connective	A sentence whose negation can be derived without any premises
Contingent sentence	A sentence whose truth table has both T and F (in any combination) under the main connective	A sentence that is not a theorem or contradiction
Equivalent sentences	The columns under the main connective for both sentences are identical.	The sentences can be derived from each other
Inconsistent sentences	Sentences that do not have a single line in their truth tables where, in the column under the main connective, they all have a T.	Sentences from which one can derive a contradiction
Consistent sentences	Sentences that have at least one line in their truth tables where, in the column under the main connective, they all have a T.	Sentences that are not inconsistent
Valid argument	An argument whose truth table has no lines where there is a T under each main connective for the premises and an F under the main connective for the conclusion.	An argument where one can derive the conclusion from the premises

Table 16.1: The two ways of defining each of these logical concepts in TFL.

soundness

SOUNDNESS is a property of a logic system iff, for any argument, if the argument is p-valid, then the argument tt-valid.

Equivalently, SOUNDNESS is a property of a logic system iff, for any sentence, if the sentence is a theorem, then it is a tautology.

Soundness is a property of TFL because every argument for which we can give a proof (and hence show that it is valid that way) will also be valid by the truth table method.

> *Soundness*, the property of logical systems that we are discussing here, is different than *sound*, the property of individual arguments, that is defined on p. 9.

Soundness is the property that goes in this direction: p-valid ⇒ tt-valid. The other direction, tt-valid ⇒ p-valid, is called COMPLETENESS.

> Like '→', '⇒' can be read as 'if . . . , then . . .'.
>
> Since 'p-valid ⇒ tt-valid' is not an expression in TFL, we shouldn't use the '→' symbol in it. Instead, we are using the *metalogical arrow* to express the relationship between *p-valid* and *tt-valid*.

completeness

COMPLETENESS is a property of a logic system iff, for any argument, if the argument is tt-valid, the the argument is p-valid.

Equivalently, COMPLETENESS is a property of a logic system iff, for any sentence, if the sentence is a tautology, then it is a theorem.

Proving that a logic system is complete is generally harder than proving soundness. Proving soundness for a logic system amounts to showing that all of the rules of the deduction system work the way they are supposed to work. Showing that a logic system is complete means showing that all of the rules that are needed have been included, and none have been left out. Again, showing this is beyond the scope of this book. The important point is that, happily, TFL is both sound and complete. This is not the case for all formal languages (or all logical systems). Because it is true of TFL, we can choose to give proofs or give truth tables – whichever is easier for the task at hand.

Some people are naturally drawn to truth tables because they can be produced mechanically, and that seems easier. But, as we mentioned in chapter , when arguments contain more than three letters, their truth table become quite large. Also, providing a proof informs us of the steps that must be taken to get from the premises to the conclusion. It illustrates *why* an argument is valid in a way that a truth table cannot. Comparing proofs also gives us insight into how arguments are similar or different, and that, in turn, informs us about the similarities and differences between various reasoning strategies. Truth tables, meanwhile, tell us nothing but whether an argument is valid or invalid.

It also bears mentioning that TFL is the standard first step into formal logic, but more complex systems of logic cannot employ truth tables and so derivations must be used. It is wise, therefore, to master derivations in TFL before moving onto to other branches of logic.

At the same time, there are some logical properties, the presence (or really the absence) of which, can only only be established with truth tables. In each of these cases, we might surmise from our failure to find a proof that the property is present, but our failure might just be a consequence of not trying hard enough. This is true for showing that (1) an argument is invalid, (2) a sentence is *not* a theorem, (3) a sentence is *not* a contradiction, (4) a sentence is contingent (which is to say that it's *not* a theorem and *not* a contradiction), (4) two sentences are *not* equivalent, and (5) two or more sentences are consistent (which is to say that they are *not* inconsistent). If we wish to show that any of those properties apply, then we have to resort to truth tables.

16.1 Practice exercises

A. For each of the following, if the argument is valid, give a proof. If it is not valid, make a truth table showing that it is not.

1. $\neg(P \vee Q) \vdash P \,\&\, Q$

TO VERIFY	THAT IT IS	THAT IT IS NOT
Tautology	proof or a truth table	truth table
Contradiction	proof or a truth table	truth table
Contingent	truth table	proof or a truth table
Equivalent	proof or a truth table	truth table
Consistent	truth table	proof or a truth table
Valid	proof or a truth table	truth table

Table 16.2: This table summarizes what is required to check each of these logical properties.

2. $\neg(M \vee P) \vdash Q \rightarrow \neg M$
3. $\neg(M \vee P) \vdash P \rightarrow M$
4. $\neg(M \vee (P \vee Q)) \vdash \neg Q$
5. $\neg(M \,\&\, (P \vee Q)) \vdash \neg Q$
6. $((\neg R \leftrightarrow R) \vee R) \vdash R$
7. $F \,\&\, (K \,\&\, R) \vdash F \leftrightarrow (K \leftrightarrow R)$
8. $\neg L, K \rightarrow \neg L \vdash \neg K$
9. $L, K \rightarrow \neg L \vdash \neg K$
10. $\vdash M \rightarrow [((P \,\&\, Q) \vee S) \rightarrow M]$
11. $\vdash M \rightarrow (M \rightarrow P)$

16.2 Answers

A.

1. $\neg(P \vee Q) \vdash P \,\&\, Q$ is invalid.

P	Q	¬	(P	∨	Q)	⊢	(P	&	Q)
T	T	**F**	T	T	T	✓	T	**T**	T
T	F	**F**	T	T	F	✓	T	**F**	F
F	T	**F**	F	T	T	✓	F	**F**	T
F	F	**T**	F	F	F	×	F	**F**	F

2. $\neg(M \lor P) \vdash Q \to \neg M$ is valid.

1	$\neg(M \lor P)$	:PR
2	$\quad Q$	:AS
3	$\qquad M$	:AS
4	$\qquad M \lor P$	:$\lor$I 3
5	$\qquad \neg(M \lor P)$	:R 1
6	$\quad \neg M$	:$\neg$I 3–5
7	$Q \to \neg M$	:$\to$I 2–6

3. $\neg(M \lor P) \vdash P \to M$ is valid.

1	$\neg(M \lor P)$	:PR
2	$\quad P$	:AS
3	$\qquad \neg M$	:AS
4	$\qquad M \lor P$	:$\lor$I 2
5	$\qquad \neg(M \lor P)$	:R 1
6	$\quad M$	:$\neg$E 3–5
7	$Q \to \neg M$	:$\to$I 2–6

4. $\neg(M \lor (P \lor Q)) \vdash \neg Q$ is valid.
5. $\neg(M \& (P \lor Q)) \vdash \neg Q$ is invalid.

M	P	Q	¬	(M	&	(P	∨	Q))	⊢	¬	Q
T	T	T	**F**	T	T	T	T	T	✓	**F**	T
T	T	F	**F**	T	T	T	T	F	✓	**T**	F
T	F	T	**F**	T	T	F	T	T	✓	**F**	T
T	F	F	**T**	T	F	F	F	F	✓	**T**	F
F	T	T	**T**	F	F	T	T	T	×	**F**	T
F	T	F	**T**	F	F	T	T	F	✓	**T**	F
F	F	T	**T**	F	F	F	T	T	×	**F**	T
F	F	F	**T**	F	F	F	F	F	✓	**T**	F

6. $(\neg R \leftrightarrow R) \lor R \vdash R$ is valid.
7. $F \& (K \& R) \vdash F \leftrightarrow (K \leftrightarrow R)$ is valid.

8. $\neg L, K \rightarrow \neg L \vdash \neg K$ is invalid.

K	L	$\neg$	L,	$(K$	$\rightarrow$	$\neg$	$L)$	$\vdash$	$\neg$	K
T	T	**F**	T	T	**F**	F	T	✓	**F**	T
T	F	**T**	F	T	**T**	T	F	×	**F**	T
F	T	**F**	T	F	**T**	F	T	✓	**T**	F
F	F	**T**	F	F	**T**	T	F	✓	**T**	F

9. $L, K \rightarrow \neg L \vdash \neg K$ is valid.

10. $\vdash M \rightarrow ([(P \,\&\, Q) \vee S] \rightarrow M)$ is a theorem.

1	M	:AS
2	$(P \,\&\, Q) \vee S$	:AS
3	M	:R 1
4	$((P \,\&\, Q) \vee S) \rightarrow M$	:→I 2–3
5	$M \rightarrow (((P \,\&\, Q) \vee S) \rightarrow M)$	:→I 1–4

11. $\vdash M \rightarrow (M \rightarrow P)$ is not a theorem, and so it is not a tautology.

M	P	$\vdash$	M	$\rightarrow$	$(M$	$\rightarrow$	$P)$
T	T	✓	T	**T**	T	T	T
T	F	×	T	**F**	T	F	F
F	T	✓	F	**T**	F	T	T
F	F	✓	F	**T**	F	T	F

Part 5

First-order logic

17 The basics of first-order logic

17.1 Introduction to first-order logic

We have mastered truth functional logic. There may be difficult proofs that still confound us, but we have covered everything that there is to cover in this branch of logic. We now move on to the logic system that is typically studied immediately after truth functional logic: first-order logic (which is also called *predicate logic*).

As you know, the language of truth functional logic consists of atomic sentences, the logical operators *and*, *or*, *if* ...*then* ..., *if and only if*, and *not*, and parentheses. First-order logic has all of those logical operators plus three more. It also has parentheses. Instead of atomic sentences, however, these two parts of a sentence are represented separately:

(*a*) the subject of a sentence
(*b*) the verb and direct object of the sentence

This allows us to undertake a much more sophisticated analysis of different kinds of logical relationships (although we will only cover the basics in this textbook). Let's see what it looks like.

17.2 The content of first-order logic

In first-order logic (FOL), we still use the logical operators from TFL, the rules of derivation given in chapter , and parentheses. We will, however, replace the atomic sentences of TFL with sentences composed of the following.

(*a*) predicates
(*b*) names for specific individuals or things
(*c*) variables
(*d*) quantifiers

We also add the identity symbol, '=', to the symbols from TFL.

Names Names are simple. They designate specific individuals (or any specific object). We represent them with the lowercase letters a – t.

Predicates Predicates are attributes or properties that individuals or objects can have. These, for instance, are predicates:

_____ is tall.
_____ is a mammal.
_____ ate dinner.

As you can see, we need to add a name to the predicate to form a complete English sentence. If we use the name 'Carol', then we will have 'Carol is tall', 'Carol is a mammal', and 'Carol ate dinner'. Predicates are symbolized with uppercase letters, and we can use any letter A through Z. Here is a symbolization key:

T: _____ is tall.
M: _____ is a mammal.
D: _____ ate dinner.
c: Carol

Now, we combine the name and the predicates this way:

Tc
Mc
Dc

Notice that, although 'Carol' begins each of the English sentences, in FOL, we put the predicate first and then the name.

Those are *one-place predicates* because each takes only one name. *Two-place predicates* take two names:

S: _____ is the sister of _____ .
L: _____ is in love with _____ .
T: _____ is taller than _____ .

With names added (and the order of the names matters!), we can symbolize these sentences as is shown here.

Abby is the sister of Carol: Sac
Carol is the sister of Abby: Sca
David is in love with Carol: Ldc
Carol is not in love with David: $\neg Lcd$

And likewise, we can have three- or four- (or more) place predicates.

Quantifiers and variables The most notable feature of first-order logic is the use of what are called *quantifiers*. There are two in this logic system:

(*a*) The universal quantifier, which is symbolized with '$\forall$' and can be translated as "every" or "for all."
(*b*) The existential quantifier, which is symbolized with '$\exists$' and can be translated as "some," "for some," or "there exists."

When we use these quantifiers, we are not designating a specific individual and so, with them, we must use VARIABLES. Whereas names identify specific individuals (or objects), variables stand for an individual (or individuals), but not any specific one. Variables work with predicates the same way as names do. When we have a variable, however, we must also have a quantifier. (It is possible to have a variable that is not associated with a quantifier, but that won't be covered here.) We represent variables with the lowercase letters u – z.

Here are two English sentences that are each translated to FOL with a quantifier, a predicate, and a variable:

Everyone ate dinner: $\forall xDx$.
Someone ate dinner: $\exists xDx$.

Notice that we are doing two things in each of these sentences. First, we are specifying if the variable represents everyone (or everything) or someone (or something). That's the purpose of the $\forall x$ and the $\exists x$. Then we indicate the property that the individual or individuals represented by the variable have. That's the Dx.

As we did with TFL, there will be a point where we won't be as concerned with the English sentence that is represented by an expression in FOL. So, reading the first one, as an expression in FOL (and without a specific English translation for D), we say, "for all x, Dx or "all x are D." And for the second one: "for some x, Dx" or "some x is D."

Identity The identity symbol, '=', is used to signify that two things are the same; that is, not just similar or equivalent, but the same. So, while two objects might be what we would call "the same" – two of the same sized wrenches, for instance – we wouldn't use the identity symbol to indicate that they are. Rather, we use the identity symbol for cases such as 'Peter Parker *is* Spider-Man.'

Domains A logic system must be precise. But 'all' or 'every', when taken literally and without any qualification, refer to a vast set of individuals or things. For instance, if H represents 'is happy', then $\forall xHx$ represents 'Everyone is happy'. But when we say this in English, we are not referring to everyone now alive or everyone who ever was alive or who ever will be alive. Rather, we mean something like 'everyone now in the room is happy,' or 'everyone enrolled in this course is happy' ☺.

So, to be precise when we use quantified expressions in FOL, we need to specify a DOMAIN. The domain is the collection of things about which we want to refer. If we want to talk about people in Starkville, we define the domain as *people in Starkville*. We write this at the beginning of the symbolization key, like this:

domain: people in Starkville

The quantifiers, then, apply to (or range over) the domain. Given this domain, '$\forall x$' is to be read roughly as 'Every person in Starkville …', and '$\exists x$' is to be read roughly as 'Some person in Starkville, …'.

In FOL, the domain must always include at least one object. Moreover, in English we can legitimately infer 'someone is happy' from 'David is happy'. Likewise in FOL, we will want to be able to infer $\exists xHx$ from Hd. So, when we use a name, it must be the case that each name picks out exactly one object in the domain (for instance, one person). There can, however, be members of a domain that don't have names or have more than one name.

18 FOL: Translations and scope

18.1 Translations

Here is a symbolization key.

domain: people in Starkville
B: ______ ate a burrito.
N: ______ took a nap.
L: ______ loves ______ .
a: Abigail
c: Carol
d: David

We can then translate the following sentences from first-order logic to English as is shown. First, here are some sentences composed of predicates and names.

Ba: Abigail ate a burrito.

Nd: David took a nap.

$\neg Nc$: Carol did not take a nap.

$Nd \& Ba$: David took a nap and Abigail ate a burrito.

$Ba \rightarrow Na$: If Abigail ate a burrito, then Abigail took a nap.

$Ldc \& Bc$: David loves Carol, and Carol ate a burrito.

Here are some sentences that include quantifiers.

$\forall yNy$: Everyone took a nap.

$\exists xBx$: Someone ate a burrito.

$\neg\forall xNx$: Not everyone took a nap.

$\forall x\neg Nx$: Everyone did not take a nap.

$\neg\exists yBy$: It is not the case that someone ate a burrito.

$\exists y\neg By$: Someone did not eat a burrito.

$\exists x(Bx \,\&\, Nx)$: Someone ate a burrito and took a nap.

If you are stuck on a translation, you can always just translate $\forall x$ and $\exists x$ like this:

For all x, x is …
For some x, x is …

These sentences contain either both a name and a variable or both of the quantifiers (and two different variables).

1. $\exists xBx \,\&\, Nc$: Someone ate a burrito, and Carol took a nap.
2. $\exists xBx \,\&\, \exists yNy$: Someone ate a burrito and someone took a nap.
3. $\exists x(Bx \,\&\, Nx)$: Someone ate a burrito and took a nap.
4. $\exists xLxc$: Someone loves Carol.
5. $\forall x\exists yLxy$: Everyone loves someone.
6. $\exists y\forall xLxy$: Someone is loved by everyone.

According to the second sentence, maybe different people ate the burrito and took the nap, or maybe the same person did both. The third sentence, however, means that the same person did both.

In the final two sentences, the Lxy is the same, but the order of the '$\forall x$' and '$\exists y$' is switched. And, although it's not exactly apparent in the English translations given there, the two FOL sentences have different meanings. $\forall x\exists yLxy$ means that everyone loves someone, but the someone who is loved can be different for different people. Meanwhile, $\exists y\forall xLxy$ means that there is one particular someone who is loved by everyone.

equivalent quantified expressions

$\exists x\neg Bx$ is equivalent to $\neg\forall xBx$.

$\forall x\neg Dx$ is equivalent to $\neg\exists xDx$.

The formats for common expressions that are translated into first-order logic as conditionals or conjunctions

$\forall x(Fx \rightarrow Gx)$: For all x, if x is an F, then x is a G.
Or Every F is G.

$\exists x(Fx \& Gx)$: For some x, x is an F and a G. **Or** Some F is G.

No F is G: $\neg\exists x(Fx \& Gx)$ **or** $\forall x(Fx \rightarrow \neg Gx)$

Only Fs are Gs: $\neg\exists x(Gx \& \neg Fx)$ **or** $\forall x(Gx \rightarrow Fx)$

18.2 Scope

Just like the '&', '∨', '→', '↔', and '¬', the universal quantifier ('∀') and the existential quantifier ('∃') have a scope. For both, their scope is the part of the expression to which the quantifier applies – or, as we often say, the part of the sentence that it *ranges over*. Basically, the scope of the quantifiers works like the scope of the negation operator (i.e., the '¬').

(1) When there is not a parenthesis between the quantifier and a predicate, then the scope of the quantifier is that predicate (or the predicate and a ¬) and its variable or variables. For instance, here the scope of the $\forall x$ is in red:

$\forall xFx$ or $\forall x\neg Fx$

(2) When there is a parenthesis between the quantifier and the first predicate, then the scope of the quantifier is everything inside the parentheses.

$\forall x(Fx \vee Gx)$

(3) When there is a second quantifier between the quantifier and a predicate or a parenthesis, the scope of the first quantifier includes the second quantifier. And likewise if there are more than two quantifiers.

$\forall x\exists yLxy$ or $\forall x\exists y(Fx \rightarrow Gy)$

Notice that this is all the same as the scope of the '¬'.

19 The rules of derivation for FOL

19.1 New rules for FOL

We retain the TFL rules that are given in chapter . We also have introduction and elimination rules for the universal and existential quantifiers and the identity operator.

19.2 Universal elimination

If we know that everyone in a domain likes chocolate, then we know that a specific individual in this domain—let's say, Carol—likes chocolate. The universal elimination rule captures this basic inference.

universal elimination rule

m	$\forall x Dx$	
	Dc	$:\forall E\ m$

If we have $\forall x Dx$ on a line, then we can put Dc on a new line.

Any predicate can be used in the place of 'D', any variable can occur in place of 'x', and any name can be used in place of 'c'.

Similarly:

m	$\forall x(Fx \rightarrow Gx)$	
	$Fc \rightarrow Gc$	$:\forall E\ m$

If a sentence of FOL contains multiple instances of the same variable, the same name must be used to replace every instance of the variable.

19.3 Existential introduction

If we know that Alice is a spy, then we know that someone is is a spy. This kind of inference is the basis for the existential introduction rule.

existential introduction rule

m	Dc	
	∃xDx	:∃I m

If we have **Dc** on a line, then we can put **∃xDx** on a new line.

If a sentence of FOL contains multiple instances of the same name, replace one or more (or all) instances of the name.

m	Fc → Gc	
	∃x(Fx → Gx)	:∃I m

m	Fc → Gc	
	∃x(Fx → Gc)	:∃I m

19.4 A couple of examples

The existential quantifier and the universal quantifier are logical operators, and so we have to continue to follow this guideline from p. 117: Each of the rules of derivation can only be applied to the main logical operator of a sentence. So, consider a sentence like this one:

$$\forall x(Bx \,\&\, Nx)$$

The scope of the universal quantifier is the whole sentence. Therefore, it is the main logical operator. As such, we can use the universal elimination rule, and we can't use the conjunction elimination rule. If we use the universal elimination rule, we get a sentence like this one:

$$(Ba \,\&\, Na)$$

Now, we can use the conjunction elimination rule to get either *Ba* or *Na* on a new line.

Here are some examples of proofs that use the first two rules introduced in this chapter along with the rules from chapter .

1. $\forall x(Fx \rightarrow Gx), \forall zFz \vdash \exists x(Fx \,\&\, Gx)$

1	$\forall x(Fx \rightarrow Gx)$	:PR
2	$\forall zFz$	:PR
3	$Fa \rightarrow Ga$	:∀E 1
4	Fa	:∀E 2
5	Ga	:→E 3, 4
6	$Fa \,\&\, Ga$	:&I 4, 5
7	$\exists x(Fx \,\&\, Gx)$	:∃I 6

2. $\exists yQy \rightarrow \forall zTz, \forall x(Px \,\&\, Qx) \vdash Te$

The main logical operator in the first premise is the '→'. Therefore, first, we have to get '$\exists yQy$' on a line by itself, and then we can use the conditional elimination rule to get '$\forall zTz$'.

1	$\exists yQy \rightarrow \forall zTz$	:PR
2	$\forall x(Px \,\&\, Qx)$	:PR
3	$Pa \,\&\, Qa$	:∀E 2
4	Qa	:&E 3
5	$\exists yQy$	:∃I 4
6	$\forall zTz$	:→E 1, 5
7	Te	:∀E 4

19.5 Universal introduction

The universal elimination and existential introduction rules are straightforward. It's obvious that if *everyone* has some property, then any particular individual (in that group) has it. It's equally clear that if a particular individual has a property, then *someone* has it.

Matters aren't quite that simple for the universal introduction and existential elimination rules, and consequently, these rules are a little more complex.

We'll take up the universal introduction rule first. What would it take to introduce the claim that everyone likes chocolate (i.e., $\forall xCx$)? One method would be to check that every single individual in the domain likes chocolate. This, however, isn't practical for our purposes since a domain can have an infinite number of members. We need a different way of introducing a universal quantifier.

To begin thinking about the method that we will use, consider this argument:

$$\forall x(Fx \;\&\; Gx) \vdash \forall xFx$$

This argument is valid. If everything is both F *and* G, then everything is F. But how do we show this? We begin the proof this way:

1	$\forall x(Fx \;\&\; Gx)$	:PR
2	$Fa \;\&\; Ga$	:$\forall$E 1
3	Fa	:&E 2

We have derived Fa. This is an *instance* of the conclusion that we are after: $\forall xFx$. (For example, 'Albert is fast' is one *instance* of 'everyone is fast'.) Alternatively, on lines 2 and 3 (and using the universal elimination and conjunction elimination rules), we could have put Fb, Fc, Fm_2, Fg_{791}, or anything else until we run out of space, time, or patience.

So, from the premise $\forall x(Fx \;\&\; Gx)$, we could, in principle, use the name of any and every individual in the domain for ($F\_ \;\&\; G\_$). Since we arbitrarily chose a for the ($Fa \;\&\; Ga$) – and then derived Fa – we are allowed to infer $\forall xFx$ from the Fa.

This brings us to the following idea. We can use the UNIVERSAL INTRODUCTION RULE to get $\forall xFx$ from Fa as long as the a in Fa was arbitrarily chosen from the names of everyone or everything in the domain. Since it was, we complete the proof with this rule.

1	$\forall x(Fx \,\&\, Gx)$	:PR
2	$Fa \,\&\, Ga$	:$\forall$E 1
3	Fa	:&E 2
4	$\forall xFx$	:$\forall$I 3

universal introduction rule

m	Dc	
	$\forall \mathsf{xDx}$	:$\forall$I m

If we have Dc on a line, then we can put $\forall\mathsf{xDx}$ on a new line provided these conditions are met:

1. c must not occur in any premise or undischarged assumption.
2. When D is a multi-place predicate, x must not occur in $\mathsf{D}(\ldots\ \mathsf{c}\ \ldots)$.

Similarly:

m	$\mathsf{Fc} \rightarrow \mathsf{Gc}$	
	$\forall\mathsf{x}(\mathsf{Fx} \rightarrow \mathsf{Gx})$	:$\forall$I m

If a sentence of FOL contains multiple instances of the same name, the same variable must be used to replace every instance of the name.

19.6 Existential elimination

The first thing to note about the EXISTENTIAL ELIMINATION RULE is that when we use it, we begin with and usually end with an existentially quantified sentence. A typical way to use the rule is as follows.

(1) Begin with an existentially quantified sentence.

(2) As an assumption, state a possible instance of this existentially quantified sentence.
(3) Inside the subproof, derive another existentially quantified sentence.
(4) Close the subproof and put the sentence from 3 on the next line.

Hence, we shouldn't get hung up on the word *elimination*. We do eliminate the existential quantifier for the second step, but, in the end, we're right back to having an existentially quantified sentence—albeit a different one than the one with which we began.

Now, let's think about how this rule works in a little more detail. Suppose that we know that *someone* is F (say "fast"). The problem is that simply knowing this doesn't tell us which particular person is F. (Abigail? Carol? David? Who?) So, from $\exists x Fx$, we cannot immediately infer Fa, or Fc, or Fd, or any other instance of the sentence. How, then, can we derive anything from an existentially quantified premise?

We will examine how our existential elimination rule works with this argument:

$$\exists x Fx, \forall x(Fx \rightarrow Gx) \vdash \exists x Gx$$

After the premises, we make an assumption.

1	$\exists x Fx$	:PR
2	$\forall x(Fx \rightarrow Gx)$	:PR
3	$\quad Fe$	:AS

Premise 1 tell us that *someone* is an F. So, on line 3, we introduce an arbitrary name for this someone: e. Essentially, we are saying, "let's call this individual who is fast *Eddie*." And we are assuming nothing about "Eddie" other than that the predicate F is true of true of him or her. (I.e., he or she is fast.) Now that we have Fe, we proceed as follows.

1	$\exists x Fx$	:PR
2	$\forall x(Fx \rightarrow Gx)$	:PR
3	$\quad Fe$	:AS
4	$\quad Fe \rightarrow Ge$	:$\forall$E 2
5	$\quad Ge$	:$\rightarrow$E 3, 4

So far, so good. Because e was just a randomly chosen name and we don't know if there is, in fact, an e that is F, we cannot take the e outside the subproof. But we don't need to do so anyway. We can use the existential introduction rule to get $\exists xGx$ on line 6. Then, we can exit the subproof and, using the EXISTENTIAL ELIMINATION RULE, repeat $\exists xGx$ on the next line.

$$\begin{array}{r|l l}
1 & \exists xFx & :\text{PR} \\
2 & \forall x(Fx \rightarrow Gx) & :\text{PR} \\ \hline
3 & \quad \vert\ Fo & :\text{AS} \\
4 & \quad \vert\ Fo \rightarrow Go & :\forall\text{E } 2 \\
5 & \quad \vert\ Go & :\rightarrow\text{E } 3, 4 \\
6 & \quad \vert\ \exists xGx & :\exists\text{I } 5 \\
7 & \exists xGx & :\exists\text{E } 1, 3\text{–}6
\end{array}$$

Notice that, when we cite the rule on line 7, we give the range for the subproof and the number for the line containing the existentially quantified sentence that was the basis for the assumption.

existential elimination rule

$$\begin{array}{r|l l}
m & \exists x\mathsf{D}x & \\
i & \quad \vert\ \mathsf{Dc} & :\text{AS} \\
j & \quad \vert\ \mathsf{B} & \\
 & \mathsf{B} & :\exists\text{E } m, i\text{–}j
\end{array}$$

The name c may not occur outside the subproof.

Given an existentially quantified sentence as a premise, assume an instance of that sentence. Proceed until the desired sentence B is reached. B cannot contain the name that was used in the assumption. Exit the subproof, and repeat B on the next line.

The constraint that we have on the existential elimination rule is more restrictive than strictly necessary. The name **c** that we assumed can occur outside the subproof, as long as it doesn't occur in $\exists x Dx$ (when **D** it is a multi-place predicate), in an earlier undischarged assumption, or in **B**.

19.7 Identity rules

Here's a deep thought: everything is identical to itself. The IDENTITY INTRODUCTION RULE allows us to state this fact.

identity introduction rule

$\mathsf{c} = \mathsf{c}$	$=$I

For any name, state that it is identical to itself. No line number is given with the rule.

When thinking about identities, however, the more interesting assertion is one like 'Bruce Wayne *is* Batman'. A sentence with the form $\mathsf{a} = \mathsf{b}$, however, must be given as a premise or an assumption. It cannot be introduced with the identity introduction rule. If it is a premise or assumption, though, then we can use the IDENTITY ELIMINATION RULE.

identity elimination rule

m	$\mathsf{a} = \mathsf{b}$	
n	Da	
	Db	$=$E m, n

If you have $\mathsf{a} = \mathsf{b}$ on one line and Da on another line, you can put Db on a new line.

19.8 Some more examples

1. $\forall x(Gx \rightarrow Hx), \forall xGx \vdash \forall xHx$

1	$\forall x(Gx \rightarrow Hx)$	:PR
2	$\forall xGx$	:PR
3	$Ga \rightarrow Ha$	:$\forall$E 1
4	Ga	:$\forall$E 2
5	Ha	:$\rightarrow$E 3, 4
6	$\forall xHx$	:$\forall$I 5

2. $\forall x(Mx \leftrightarrow Wx), \exists yPy \rightarrow \forall xWx, Pa \vdash \forall xMx$

1	$\forall x(Mx \leftrightarrow Wx)$	:PR
2	$\exists yPy \rightarrow \forall xWx$	:PR
3	Pa	:PR
4	$\exists yPy$	:$\exists$I 3
5	$\forall xWx$	:$\rightarrow$E 2, 4
6	Wc	:$\forall$E 5
7	$Mc \leftrightarrow Wc$	:$\forall$E 1
8	Mc	:$\leftrightarrow$E 6, 7
9	$\forall xMx$	:$\forall$I 8

3. $\neg Fe \leftrightarrow \forall x Gx, \forall y \neg Hy, \forall x(Hx \vee \neg Fx) \vdash Gb$

1	$\neg Fe \leftrightarrow \forall x Gx$	:PR
2	$\forall y \neg Hy$	:PR
3	$\forall x(Hx \vee \neg Fx)$	:PR
4	$\neg He$	:$\forall$E 2
5	$He \vee \neg Fe$	:$\forall$E 3
6	$\neg Fe$	:$\vee$E 4, 5
7	$\forall x Gx$	:$\leftrightarrow$E 1, 6
8	Gb	:$\forall$E 7

4. $\exists y \forall x Lxy \vdash \forall x \exists y Lxy$

1	$\exists y \forall x Lxy$	:PR
2	$\quad \forall x Lxc$	:AS
3	$\quad Lac$	:$\forall$E 2
4	$\quad \exists y Lay$	:$\exists$I 3
5	$\quad \forall x \exists y Lxy$	:$\forall$I 4
6	$\forall x \exists y Lxy$	:$\exists$E 1, 2–5

On p. 186, we said that $\forall x \exists y Lxy$ and $\exists y \forall x Lxy$ have different meanings. One consequence of this is that $\exists y \forall x Lxy \vdash \forall x \exists y Lxy$ is valid (as just shown), but $\forall x \exists y Lxy \vdash \exists y \forall x Lxy$ is not.

5. $\exists x Mx \vdash \exists x(Mx \vee Nx)$

$$
\begin{array}{r|l l}
1 & \exists x Mx & :\text{PR} \\
\hline
2 & \quad \vert\ Ma & :\text{AS} \\
3 & \quad \vert\ Ma \vee Na & :\vee\text{I } 2 \\
4 & \quad \vert\ \exists x(Mx \vee Nx) & :\exists\text{I } 3 \\
5 & \exists x(Mx \vee Nx) & :\exists\text{E } 1, 2\text{–}4
\end{array}
$$

6. $\exists x \neg Fx, \forall x(Fx \vee Gx) \vdash \exists x Gx$

$$
\begin{array}{r|l l}
1 & \exists x \neg Fx & :\text{PR} \\
2 & \forall x(Fx \vee Gx) & :\text{PR} \\
\hline
3 & \quad \vert\ \neg Fa & :\text{AS} \\
4 & \quad \vert\ Fa \vee Ga & :\forall\text{E } 2 \\
5 & \quad \vert\ Ga & :\vee\text{E } 3, 4 \\
6 & \quad \vert\ \exists Gx & :\exists\text{I } 5 \\
7 & \exists Gx & :\exists\text{E } 1, 3\text{–}6
\end{array}
$$

7. $\exists x(Bx \rightarrow Dx), \forall x Bx \vdash \exists x Dx$

$$
\begin{array}{r|l l}
1 & \exists x(Bx \rightarrow Dx) & :\text{PR} \\
2 & \forall x Dx & :\text{PR} \\
\hline
3 & \quad \vert\ Ba \rightarrow Da & :\text{AS} \\
4 & \quad \vert\ Ba & :\forall\text{E } 2 \\
5 & \quad \vert\ Da & :\rightarrow\text{E } 3, 4 \\
6 & \quad \vert\ \exists x Dx & :\exists\text{I } 5 \\
7 & \exists x Dx & :\exists\text{E } 1, 3\text{–}6
\end{array}
$$

8. $\forall xPx \to \exists y(Qy \,\&\, Ty), \forall zPz \vdash \exists xQx$

1	$\forall xPx \to \exists y(Qy \,\&\, Ty)$	:PR
2	$\forall zPz$	:PR
3	Pa	:$\forall$E 2
4	$\forall xPx$	:$\forall$I 3
5	$\exists y(Qy \,\&\, Ty)$	:$\to$E 1, 4
6	$\quad Qc \,\&\, Tc$	:AS
7	$\quad Qc$	:&E 6
8	$\quad \exists xQx$	:$\exists$I 7
9	$\exists xQx$	:$\exists$E 5, 6–8

9. $Ga \leftrightarrow Ha, a = d \vdash Gd \leftrightarrow Hd$

1	$Ga \leftrightarrow Ha$	:PR
2	$a = d$	:PR
3	$Gd \leftrightarrow Hd$	=E 1, 2

10. $a = b, Mba \vdash \exists xMxx$

1	$a = b$	:PR
2	Mba	:PR
3	Maa	=E 1, 2
4	$\exists xMxx$	:$\exists$I 3

11. $Ma \vee Nb, Nb \rightarrow b = d, \neg Ma \vdash Nd$

$$
\begin{array}{r|ll}
1 & Ma \vee Nb & :\text{PR} \\
2 & Nb \rightarrow b = d & :\text{PR} \\
3 & \neg Ma & :\text{PR} \\
\cline{2-2}
4 & Nb & :\vee\text{E } 1, 3 \\
5 & b = d & :\rightarrow\text{E } 2, 4 \\
6 & Nd & =\text{E } 4, 5
\end{array}
$$

12. $\forall x \forall y \forall z((Lxy \,\&\, Lyz) \rightarrow Lxz), Lab, Lbc \vdash Lac$

This is one of the examples from chapter of an argument that is valid in virtue of its form:

1. Seoul is larger than London.
2. London is larger than Chicago.
3. Therefore, Seoul is larger than Chicago.

We couldn't, however, represent such an argument in TFL. Now, we can with this two-place predicate:

L: ______ is larger than ______ .

Comparative adjectives like *larger than* are transitive. In FOL, the *transitive relation* is defined with a sentence like this one, which will be a premise in the argument:

$$\forall x \forall y \forall z((Lxy \,\&\, Lyz) \rightarrow Lxz)$$

In English, this is read, 'for all x, for all y, and for all z, if x is larger than y and y is larger than z, then x is larger than z'. (Not all two-place predicates are transitive, which is why we need this premise. You can see this by thinking about a different two-place that 'L' might represent: '______ loves ______'.)

In the proof, we use the universal elimination rule three times to remove each of the universal quantifiers. Then, it is a simple matter of using the conjunction introduction and conditional elimination rules to get the conclusion '*a* is larger than *c*'.

1	$\forall x \forall y \forall z((Lxy \,\&\, Lyz) \rightarrow Lxz)$	:PR
2	Lab	:PR
3	Lbc	:PR
4	$\forall y \forall z((Lay \,\&\, Lyz) \rightarrow Laz)$	:$\forall$E 1
5	$\forall z((Lab \,\&\, Lbz) \rightarrow Laz)$	:$\forall$E 4
6	$(Lab \,\&\, Lbc) \rightarrow Lac$	:$\forall$E 5
7	$Lab \,\&\, Lbc$	:&I 2, 3
8	Lac	:$\rightarrow$E 6, 7

Appendices

A Symbolic notation

A.1 Alternative nomenclature

Truth-functional logic. TFL goes by other names. Sometimes it is called *sentential logic*, because this branch of logic deals fundamentally with sentences. Sometimes it is called *propositional logic* because it might also be thought to deal fundamentally with propositions. We have used with *truth-functional logic* to emphasize that it deals only with assignments of truth and falsity to sentences and that its connectives are all truth-functional.

Formulas. In §, we defined *sentences* of TFL. These are also sometimes called 'formulas' (or 'well-formed formulas') since in TFL there is no distinction between a formula and a sentence.

Truth-value assignments. *Truth-value assignments* may also be called *truth-assignments* or *valuations*.

A.2 Alternative symbols

In the history of formal logic, different symbols have been used at different times and by different authors. Often, authors were forced to use notation that their printers could typeset. This appendix presents some common symbols, so that you can recognize them if you encounter them in an article or in another book.

Negation. Two commonly used symbols are the *hoe*, '$\neg$', and the *swung dash* or *tilda*, '$\sim$.' In some more advanced formal systems it is necessary to distinguish between two kinds of negation; the distinction is sometimes represented by using both '$\neg$' and '$\sim$'. Older texts sometimes indicate negation by a line over the formula being negated, e.g., $\overline{A \,\&\, B}$.

SYMBOLS OF FORMAL LOGIC	
negation	$\neg$, $\sim$
conjunction	$\wedge$, &, ▪
disjunction	$\vee$
conditional	$\rightarrow$, $\supset$
biconditional	$\leftrightarrow$, $\equiv$

Table A.1

Disjunction. The symbol '$\vee$' is typically used to symbolize inclusive disjunction. One etymology is from the Latin word 'vel', meaning 'or'.

Conjunction. Conjunction is often symbolized with the *ampersand*, '&'. The ampersand is a decorative form of the Latin word 'et', which means 'and'. (Its etymology still lingers in certain fonts, particularly in italic fonts; thus an italic ampersand might appear as '*&*'.) This symbol is commonly used in natural English writing (e.g. 'Smith & Sons'), and so even though it is a natural choice, many logicians use a different symbol to avoid confusion between the object and metalanguage—as a symbol in a formal system, the ampersand is not the English word '&'. The most common choice now is '$\wedge$', which is a counterpart to the symbol used for disjunction. Sometimes a single dot, '$\cdot$', is used. In some older texts, there is no symbol for conjunction at all; '*A* and *B*' is simply written '*AB*'.

Conditional. There are two common symbols for the conditional (which can also be called the *material conditional*): the *arrow*, '$\rightarrow$', and the *hook*, '$\supset$'.

Biconditional. The *double-headed arrow*, '$\leftrightarrow$', is used in systems that use the arrow to represent the biconditional. Systems that use the hook for the conditional typically use the *triple bar*, '$\equiv$', for the biconditional.

B Quick reference

B.1 Characteristic Truth Tables

$\mathcal{A}$	$\neg\mathcal{A}$
T	F
F	T

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}\,\&\,\mathcal{B}$	$\mathcal{A} \vee \mathcal{B}$	$\mathcal{A} \to \mathcal{B}$	$\mathcal{A} \leftrightarrow \mathcal{B}$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

$\mathcal{A}$	$\neg\mathcal{A}$
$\top$	$\bot$
$\bot$	$\top$

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}\,\&\,\mathcal{B}$	$\mathcal{A} \vee \mathcal{B}$	$\mathcal{A} \to \mathcal{B}$	$\mathcal{A} \leftrightarrow \mathcal{B}$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$

$\mathcal{A}$	$\neg\mathcal{A}$
1	0
0	1

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}\,\&\,\mathcal{B}$	$\mathcal{A} \vee \mathcal{B}$	$\mathcal{A} \to \mathcal{B}$	$\mathcal{A} \leftrightarrow \mathcal{B}$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

B.2 Rules of derivation for TFL

When you have what is in blue, then, on a new line, you can put what is in red. m, n, p, and q stand for lines numbers. m and n don't have to be consecutive line numbers. The p and q in the negation-introduction and negation-elimination rules are consecutive line numbers.

Conjunction Intro

$$\begin{array}{r|ll} m & \mathsf{A} & \\ n & \mathsf{B} & \\ & \mathsf{A} \,\&\, \mathsf{B} & :\&\mathrm{I}\ m, n \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} & \\ n & \mathsf{B} & \\ & \mathsf{B} \,\&\, \mathsf{A} & :\&\mathrm{I}\ m, n \end{array}$$

Conjunction Elim

$$\begin{array}{r|ll} m & \mathsf{A} \,\&\, \mathsf{B} & \\ & \mathsf{A} & :\&\mathrm{E}\ m \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} \,\&\, \mathsf{B} & \\ & \mathsf{B} & :\&\mathrm{E}\ m \end{array}$$

Disjunction Intro

$$\begin{array}{r|ll} m & \mathsf{A} & \\ & \mathsf{A} \vee \mathsf{B} & :\vee\mathrm{I}\ m \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} & \\ & \mathsf{B} \vee \mathsf{A} & :\vee\mathrm{I}\ m \end{array}$$

Disjunction Elim

$$\begin{array}{r|ll} m & \mathsf{A} \vee \mathsf{B} & \\ n & \neg\mathsf{B} & \\ & \mathsf{A} & :\vee\mathrm{E}\ m, n \end{array}$$

$$\begin{array}{r|ll} m & \mathsf{A} \vee \mathsf{B} & \\ n & \neg\mathsf{A} & \\ & \mathsf{B} & :\vee\mathrm{E}\ m, n \end{array}$$

Double Negation

$$\begin{array}{r|ll} m & \mathsf{A} & \\ & \neg\neg\mathsf{A} & :\mathrm{DN}\ m \end{array}$$

Conditional Elim

$$\begin{array}{r|ll} m & A \to B & \\ n & A & \\ & B & :\to\text{E } m, n \end{array}$$

Biconditional Intro

$$\begin{array}{r|ll} m & A \to B & \\ n & B \to A & \\ & A \leftrightarrow B & :\leftrightarrow\text{I } m, n \end{array}$$

$$\begin{array}{r|ll} m & A \to B & \\ n & B \to A & \\ & B \leftrightarrow A & :\leftrightarrow\text{I } m, n \end{array}$$

Biconditional Elim

$$\begin{array}{r|ll} m & A \leftrightarrow B & \\ n & B & \\ & A & :\leftrightarrow\text{E } m, n \end{array}$$

$$\begin{array}{r|ll} m & A \leftrightarrow B & \\ n & A & \\ & B & :\leftrightarrow\text{E } m, n \end{array}$$

Conditional Intro

$$\begin{array}{r|ll} m & \quad\vert\ \underline{A} & :\text{AS} \\ n & \quad\vert\ B & \\ & A \to B & :\to\text{I } m\text{–}n \end{array}$$

Negation Intro

$$\begin{array}{r|ll} m & \quad\vert\ \underline{A} & :\text{AS} \\ p & \quad\vert\ B & \\ q & \quad\vert\ \neg B & \\ & \neg A & :\neg\text{I } m\text{–}q \end{array}$$

Negation Elim

$$\begin{array}{r|ll} m & \quad\vert\ \underline{\neg A} & :\text{AS} \\ p & \quad\vert\ B & \\ q & \quad\vert\ \neg B & \\ & A & :\neg\text{E } m\text{–}q \end{array}$$

Reiteration

$$\begin{array}{r|ll} m & A & \\ & A & :\text{R } m \end{array}$$