forallx

THE MISSISSIPPI STATE EDITION

GREGORY JOHNSON

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Preface

I will begin by quoting E. J. Lemmon.

> It is not easy, and perhaps not even useful, to explain briefly what logic is. Like most subjects, it comprises many different kinds of problem and has no exact boundaries; at one end, it shades off into mathematics, at another, into philosophy. The best way to find out what logic is is to do some. (1965, p. 1)

He then continues, "None the less, a few very general remarks about the subject may help to set the stage for the rest of this book." Following his lead, here are some general remarks. First, formal logic is the study of formal languages. Unlike natural languages (such as English, Spanish, and Mandarin), in a formal language, every part of the language is precisely defined. Using a formal language limits what we can do. Natural languages are extremely flexible and adaptable; the main formal language that we will be studying is not. The trade-off, however, is that the formal languages that we will study are very precise, and they make clear some of the fundamental aspects of human reasoning.

Second, this textbook is designed for a single semester introduction to formal (i.e., deductive) logic. It primarily covers *truth-functional logic*. (For reference, truth-functional logic also goes by other names: *propositional logic* and *sentential logic*.) In truth functional logic, individual statements (e.g., "the cat is on the mat" or "Westerby is talking with Ricardo") are treated as units that can be combined into more complex statements with *or*, *and*, *not*, *if* …*then* …, and *if and only if*. The study of truth functional logic, then, is the study of the properties of these more complex statements and the logical relationships between them.

The final part of the book introduces *first-order logic*, which includes, in addition to the 'or', 'and', 'not', 'if …then …', and 'if and only if', *names*, *variables*, *predicates*, *identity*, and what are called *quantifiers*. These additional elements of the language make it much more expressive than the truth-functional language, and this broadens the logical analysis that can be undertaken.

Incidentally, the title *forallx* (i.e., "for all x") is a reference to *first-order logic*. This is a symbolic expression in first-order logic: $\forall x(Kx \rightarrow Gx)$, and it is read, "for all x, if x is K, then x is G." Hence, the name of the textbook. (If, for instance, K stands for "is a king," and G stands for "is greedy," then $\forall x(Kx \rightarrow Gx)$ means "for all x, if x is a king, then x is greedy," or "everyone who is a king is greedy.")

For instructors

This textbook is for a single semester course in formal logic. It is assumed that the students will have no prior exposure to formal logic (or to informal logic for that matter). It covers truth functional (i.e., propositional) logic and introduces first-order logic. The coverage of the former is designed to guide students as they develop the ability to evaluate arguments using truth tables and proofs. The coverage of the later is only intended to expose students, in the last two or three weeks of the semester, to the syntax of FOL, translation to and from FOL, and simple proofs using the rules of derivation for FOL.

The rules of derivation for TFL given in chapter 12 are similar to those used in Allen and Hand's *Logic Primer*. (See section B.2 in the appendix for the exact list.) Proofs are constructed using Fitch notation, not the Lemmon-style system used by Allen and Hand.

This book is based on a text originally written by P. D. Magnus and revised and expanded by Tim Button, J. Robert Loftis, Aaron Thomas-Bolduc, and Richard Zach. I have made additional revisions, taken out chapters that are not needed for the 1000-level logic course at Mississippi State University, and added instructions for using the logic software Carnap (http://carnap.io/), which can be used in conjunction with Parts 3 - 5.

Part 1

Key concepts of logic

1 Arguments

1.1 We begin here.

In everyday discourse, the word *argument* typically refers to a verbal disagreement between two people. In logic and philosophy, however, it has a different and special meaning (although plenty of people do argue, in the everyday sense of the word, in logic and philosophy). We will use *argument* to refer to a set of sentences like these:

1. It is raining heavily.
2. If you do not take an umbrella, you will get soaked.

---

3. Therefore, you should take an umbrella.

In this set, the first two sentences support – or justify – the third sentence. The sentences providing support are the *premises*. The sentence that is supported by (or justified by) the premises is the *conclusion*. Together, *premises* and a *conclusion* comprise an *argument*.

> **Argument**
>
> An ARGUMENT is a set of sentences. One or more of the sentences provide support for another sentence in the set. The sentences providing support are PREMISES. The sentence being supported is the CONCLUSION.
>
> We can also say that, in an argument, the conclusion *follows from* the premises.

That's the definition of an argument, but a broader analysis must include the idea that arguments can be good or bad – or somewhere inbetween. A good argument is one in which the premises do, in fact, support the conclusion. For such an argument, if the premises are true, then we have good reason to believe that the conclusion is true. On the other hand, a bad (or a weak) argument is still an argument. It is just one in which the premises provide little support for the conclusion.

In the definition of an argument, we said that each premise and the conclusion is a sentence. And, as we saw, both premises and the conclusion in the example are individual sentences. All arguments can be expressed this way and many are, but a single sentence can also contain a complete argument, as is shown here:

> Joan was wearing sunglasses, and so it must have been sunny.

This argument has one premise and a conclusion. The premise and the conclusion could both be individual sentences, but here they are just independent clauses separated by the 'and'. (The premise is before the 'and', and the conclusion is after it.)

Many arguments also start with premises and end with a conclusion. But not all arguments are expressed in this order. For instance, here we have our first argument again, but the conclusion is at the beginning:

> You should take an umbrella. After all, it is raining heavily.
> And if you do not take an umbrella, you will get soaked.

We can also have the conclusion in the middle:

> It is raining heavily. Accordingly, you should take an umbrella,
> given that if you do not take an umbrella, you will get soaked.

When approaching an argument, we want to know whether or not the conclusion is supported by the premises. So, first, we must identify the premise or premises (the sentences providing support) and the conclusion (and the sentence being supported). As a guide, these words are often used to indicate that a sentence or clause is the conclusion of an argument:

> so, therefore, hence, thus, accordingly, consequently

By contrast, these expressions often indicate that we are dealing with a premise, rather than a conclusion:

> since, because, given that

So that we can undertake a more detailed and precise analysis of some kinds of arguments, in chapter 4, we will begin introducing a formal language: truth functional logic. But before we get there, in this chapter and chapter 2, we will cover some basic logical notions that apply to arguments in a natural language like English. Then, in chapter 3, we will examine logical notions that apply to just sentences (not full arguments), and still in a natural language like English.

1.2 Sentences

Only sentences that can be true or false can be the premises or the conclusion of an argument. The following types of sentences cannot be true or false, and so they cannot be part of an argument.

Questions 'Are you sleepy yet?' is, obviously, a question. Although you might be sleepy or you might be alert, the question itself is neither true nor false. For this reason, questions will not count as sentences in logic.

Imperatives Imperative sentences are, essentially, commands (although they can be nicer than what we usually think of as a command). For instance, 'Wake up!', 'Sit up straight', and 'Please, tell me how to set the table' are all imperatives. Although it might be a good idea for you to sit up, and you may or may not do it, the command is neither true nor false. Note, however, that commands are not always phrased as imperatives. As Cartman might say, "You will respect my authority." This is a command, but it is also true or false – either you will or you will not respect Cartman's authority – and so it counts as a sentence in logic.

Exclamations Some exclamatory sentences can be true or false (and so they are also declarative sentences) and some cannot be. 'It's Friday!' is an exclamation, and it is true or false. It can be part of an argument. On the other hand, a sentence such as 'Ouch!' is neither true nor false, and so it cannot be part of an argument.

1.3 Truth values

Going forward, by SENTENCE, we will mean a declarative sentence. We impose this restriction because the premises and conclusion of an argument must be capable of having a TRUTH VALUE. That is, we must be able to assign a value about its truth to each sentence in an argument. Although more advanced "non-classical" logic systems introduce more options, the two truth values that concern us are just 'true' and 'false'.

> truth values
>
> TRUTH VALUES are the logical values that a sentence can have, *true* and *false*.

Practice exercises

Highlight the phrase that expresses the conclusion of each of these arguments:

1. It is sunny. So, I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

2 Validity and other standards

2.1 Validity

Consider this argument:

1. You are reading this book.
2. This is a logic book.
3. Therefore, you are a logic student.

> When we list the premises and the conclusion of an argument this way, the final line is always the conclusion. However many lines there are before the final one are the premises.

If the premises of this argument are true—which, as it turns out, they are—it is very likely that the conclusion is true. But it is possible that someone besides a logic student is reading this book. If, say, the roommate of the book's owner picked it up and began looking through it, he or she would not immediately become a logic student. So, for this argument, we can say that, if the premises are true, then it is *likely*, but not certain, that the conclusion is also true.

Now, take this one:

1. Paris is in France, or it is in Germany.
2. Paris is not in Germany.
3. Therefore, Paris is in France.

For this argument, if the premises are true—which, again, they are—then the conclusion has to be true. There is no way for the premises to be true and the conclusion to be false.

Here is another example,

1. Paris is in Sweden, or it is in Spain.
2. Paris is not in Sweden.
3. Therefore, Paris is in Spain.

Although this argument might strike you as a bit odd, we can say almost the exact same thing about this one as we did for the previous one:

> In this argument, if the premises are true, then the conclusion has to be true. There is no way for the premises to be true and the conclusion to be false.

We have to drop the bit about the premises being true because the first one is false. But nonetheless, *if the premises are true*, then the conclusion has to be true.

This brings us to an important definition as well as an important point about doing logic. First the definition.

Valid

These are two equivalent definitions of VALID (or DEDUCTIVELY VALID):

1. An argument is VALID when, and only when, it is the case that, if the premises are true, then the conclusion has to be true.
2. An argument VALID when, and only when, it is impossible for all of the premises to be true and the conclusion to be false.

Every argument that does not satisfy the definition of *valid* is INVALID (or DEDUCTIVELY INVALID).

Typically, the study of logic focuses on determining when the conclusion of an argument follows from the premises with certainty. From the perspective of logic, whether the premises actually are true is less important. Of course, determining whether or not they are true can be important for many reasons, but this task is normally left to historians, scientists, or the Hardy boys.

We want to know whether, if all the premises *were* true, would the conclusion also have to be true? Consider this argument:

1. Paris is a large city in France, or Paris is a large city on Jupiter.
2. Paris is not a large city in France.
3. Therefore, Paris is a large city on Jupiter.

This argument is valid. *If* both premises are true (they're not, but if they were), then the conclusion has to be true. Now, let's think about this argument:

1. London is in England.
2. Beijing is in China.
3. Therefore, Paris is in France.

The premises and conclusion of this argument are all true, but the argument is invalid. If Paris were, somehow, to become independence from the rest of France, then the conclusion would be false, even though both of the premises would remain true. Thus, it is *possible* for the premises of this argument to be true and the conclusion false. Hence, the argument is *invalid*.

The important point to remember is that validity is not about the actual truth or falsity of the sentences in the argument. It is about whether it is *possible* or *impossible* for all of the premises to be true and the conclusion to be false. (Or, to say the same thing in a different way, whether or not the conclusion has to be true *if* all of the premises are true.)

We can, however, classify the arguments that are valid and have all true premises. We call these SOUND.

Sound

An argument is SOUND when, and only when, it is valid and has all true premises.

The second argument on p. 7 is sound.

2.2 Inductively strong arguments

Many good arguments are invalid. Consider this one:

1. In January 2017, it rained in London.
2. In January 2018, it rained in London.
3. In January 2019, it rained in London.
4. In January 2020, it rained in London.
5. In January 2021, it rained in London.
6. In January 2022, it rained in London.
7. In January 2023, it rained in London.
8. Therefore, next January, it will rain in London.

This argument generalizes from observations about several cases to a conclusion about all cases. This argument could be made stronger by adding additional premises, for instance: 'In January 2016, it rained in London,' 'In January 2015, it rained in London,' and so on. But, however many premises like this we add, the argument will remain invalid. Even if it has rained in London every January for the past 10,000 years, it remains *possible* that it won't rain in London next January. Hence, this argument is invalid. But, at the same time, you might think, "but it's still a good argument!" It is, and we have a way of classifying such arguments.

> Inductively strong
>
> An argument is INDUCTIVELY STRONG when (and only when) [1] it is not valid and [2] it is the case that if the premises are true, then their being true makes it likely that the conclusion is true.

An inductively strong argument is one for which the conclusion has a high probability of being true (if the premises are). Arguments that are invalid, but not inductively strong, can have conclusions with every possible probability of being true (if the premises are true) from very high to zero. To simplify matters, we can say that the options are *inductively strong*, *medium*, or *weak*. Since we are on a continuum, we could be much more fine grained than this. (But we won't. See table 2.1.)

The premises being true,	These are all invalid.
make it very probable that the conclusion will be true.	inductively strong
...	
make it somewhat probable that the conclusion will be true.	inductively medium
...	
do not make it very likely that the conclusion will be true.	inductively weak

Table 2.1: Every argument is valid or invalid. Invalid arguments can have any degree of inductive strength, depending on how likely the conclusion is to be true given the premises.

And finally, we have a concept for inductively strong arguments that serves the role that *sound* does for valid ones.

Reliable

An argument is RELIABLE when (and only when) it is inductively strong and has all true premises.

In this textbook, however, we will set aside the analysis of inductively strong arguments and focus on just valid versus invalid ones.

Practice exercises

A. Determine if each of the following arguments is valid or invalid.

(1)
1. Socrates is a man.
2. All men are carrots.
3. Therefore, Socrates is a carrot.

(2)
1. Either today is Labor Day, or the building is full.
2. The building isn't full.
3. Therefore, today is Labor Day.

(3)
1. If the green van is missing, then Claire is at the beach.
2. The green van is missing.

3. Therefore, Claire is at the beach.

(4) If Jones decided that she is going to get divorced, then she called a lawyer. Jones just called a lawyer. Hence, she has decided that she's going to get divorced.

(5)
1. Jeff is playing basketball, or Mary is watching television.
2. Mary is watching television.
3. Therefore, Jeff is playing basketball.

(6)
1. 160 12th graders at Central High School were asked if they planned to go to college next year.
2. 75 percent said that they were planning to go to college the following year.
3. Therefore, about 75 percent of all the 12th graders at Central High School are probably going to college next year.

(7)
1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is in New Jersey, then Mary stole the painting.

(8)
1. As vacation destinations, Florence and Lisbon have many similarities: nice weather, historical attractions, and great restaurants.
2. Sarah enjoyed visiting Florence.
3. Therefore, Sarah will probably enjoy visiting Lisbon.

(9)
1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is not in New Jersey, then Mary did not steal the painting.

(10)
1. Amy is on campus.
2. Therefore, Amy is on campus, or she is on the moon.

(11)
1. Jack is taking a nap.
2. Therefore, Jack is taking a nap, and Kate is reading.

(12)
1. If Roger is in the bank, then Steven is waiting in the apartment.
2. Roger is not in the bank.
3. Therefore, Steven is not waiting in the apartment.

(13)
1. If Joan is at work, then Kate is sleeping.

2. Therefore, if Kate is not sleeping, then Joan is not at work.

(14)
1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is taking a nap, then Mary is in the library.

(15)
1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Mary is in the library, then Claire is taking a nap.

(16)
1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is not taking a nap, then Mary is not in the library.

(17)
1. George is an architect, or Susan is a lawyer.
2. George is not an architect.
3. Therefore, Susan is a lawyer.

(18)
1. Amy is walking in the park, or Sarah is playing basketball.
2. Amy is walking in the park.
3. Therefore, Sarah is not playing basketball.

(19)
1. George is mowing the lawn.
2. Therefore, George is mowing the lawn, and Fred is looking for his coat.

(20)
1. Almost all sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

(21)
1. All sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

B. For each statement, determine if it is possible or not. If it is possible, given an example as illustration. If it is not possible, then explain why it isn't.

1. A valid argument that has one false premise and one true premise
2. A valid argument that has a false conclusion
3. A valid argument that has only false premises
4. A valid argument with only false premises and a false conclusion
5. An invalid argument that can be made valid by the addition of a new premise
6. A valid argument that can be made invalid by the addition of a new premise

2.3 Answers

A.

(1)
1. Socrates is a man.
2. All men are carrots.
3. Therefore, Socrates is a carrot.

This argument is valid.

(2)
1. Either today is Labor Day, or the building is full.
2. The building isn't full.
3. Therefore, today is Labor Day.

This argument is valid.

(3)
1. If the green van is missing, then Claire is at the beach.
2. The green van is missing.
3. Therefore, Claire is at the beach.

This argument is valid.

(4) If Jones decided that she is going to get divorced, then she called a lawyer. Jones just called a lawyer. Hence, she has decided that she's going to get divorced.

This argument is invalid.

(5)
1. Jeff is playing basketball, or Mary is watching television.
2. Mary is watching television.
3. Therefore, Jeff is playing basketball.

This argument is invalid.

(6)
1. 240 12th graders at Central High School were asked if they planned to go to college next year.
2. 75 percent said that they were planning to go to college the following year.
3. Therefore, about 75 percent of all the 12th graders at Central High School are probably going to college next year.

This argument is invalid.

(7)
1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is in New Jersey, then Mary stole the painting.

This argument is invalid.

(8)
1. As vacation destinations, Florence and Lisbon have many similarities: nice weather, historical attractions, and great restaurants.
2. Sarah enjoyed visiting Florence.
3. Therefore, Sarah will probably enjoy visiting Lisbon.

This argument is invalid.

(9)
1. If Mary stole the painting, then Jeff is in New Jersey.
2. Therefore, if Jeff is not in New Jersey, then Mary did not steal the painting.

This argument is valid.

(10)
1. Amy is on campus.
2. Therefore, Amy is on campus, or she is on the moon.

This argument is valid.

(11)
1. Jack is taking a nap.
2. Therefore, Jack is taking a nap, and Kate is reading.

This argument is invalid.

(12)
1. If Roger is in the bank, then Steven is waiting in the apartment.
2. Roger is not in the bank.
3. Therefore, Steven is not waiting in the apartment.

This argument is invalid.

(13)
1. If Joan is at work, then Kate is sleeping.
2. Therefore, if Kate is not sleeping, then Joan is not at work.

This argument is valid.

(14)
1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is taking a nap, then Mary is in the library.

This argument is invalid.

(15)
1. If Mary is in the library, then Jeff is watching tv.
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3. Therefore, if Mary is in the library, then Claire is taking a nap.

This argument is invalid.

(16)
1. If Mary is in the library, then Jeff is watching tv.
2. If Jeff is watching tv, then Claire is taking a nap.
3. Therefore, if Claire is not taking a nap, then Mary is not in the library.

This argument is valid.

(17)
1. George is an architect, or Susan is a lawyer.
2. George is not an architect.
3. Therefore, Susan is a lawyer.

This argument is valid.

(18)
1. Amy is walking in the park, or Sarah is playing basketball.
2. Amy is walking in the park.
3. Therefore, Sarah is not playing basketball.

For the way that we will define ‘or’ in the logic system that is developed in this textbook, this argument is invalid.

(19)
1. George is mowing the lawn.
2. Therefore, George is mowing the lawn, and Fred is looking for his coat.

This argument is invalid.

(20)
1. Almost all sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

This argument is invalid.

(21)
1. All sea lions live in the Atlantic Ocean around New York and New Jersey.
2. Sammy is a sea lion.
3. Therefore, Sammy lives in the Atlantic Ocean around New York and New Jersey.

This argument is valid.

B.

(1) A valid argument that has one false premise and one true premise
Yes, this is possible.

1. All whales are mammals. (*true*)
2. All mammals are plants. (*false*)
3. Therefore, all whales are plants.

(2) A valid argument that has a false conclusion
Yes, this is possible. See example from previous exercise.

3 Other concepts of logic

The concept of a valid argument is central to logic. In this section, we will introduce some other important concepts that apply just to sentences, not to full arguments.

3.1 Joint possibility

Consider these two sentences:

B1. Jane's only brother is shorter than her.
B2. Jane's only brother is taller than her.

Without knowing Jane and her brother, we have no way of knowing which, if either, of these sentences is true. Yet we can say that *if* B1 is true, *then* B2 must be false. Similarly, if B2 is true, then B1 must be false. It is impossible that both sentences are true at the same time. In other words, these sentences are inconsistent. On the other hand, G1 and G2 can both be true at the same time.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

One of these sentences may be false and the other true, but it is *possible* that they are both true at the same time.

jointly possible and impossible

A set of sentences are JOINTLY POSSIBLE when, and only when, it is possible for them all to be true at the same time.

A set of sentences are JOINTLY IMPOSSIBLE when, and only when, it is *not* possible for them all to be true at the same time.

So, G1 and G2 are *jointly possible* while B1 and B2 are *jointly impossible*.

We can investigate the joint possibility of any number of sentences. For example, let's add two more sentences to G1 and G2:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two extra-terrestrials at the wild animal park.
G4. Every giraffe at the wild animal park is an extra-terrestrial.

Together, G1 and G4 entail that there are at least four extra-terrestrials giraffes at the park. This conflicts with G3, which states that there are no more than two extra-terrestrials there. So, the sentences G1–G4 are jointly impossible. They cannot all be true together. (Notice also that just G1, G3 and G4 are jointly impossible, while G1, G2, and G3 are jointly possible.)

3.2 Necessary equivalence

Sentences G1 and G2 – which we said were jointly possible – can both be true at the same time. They can also both be false, or one false and the other true. A stronger relationship holds between these two sentences:

John went to the store after he washed the dishes.
John washed the dishes before he went to the store.

These two sentences must have the same truth value. That is, they must either both be true or both be false. It is impossible for one to be true and one to be false (at the same time). When two sentences *must* (or *necessarily*) have the same truth value, they are NECESSARILY EQUIVALENT.

> necessarily equivalent
>
> Two sentences are NECESSARILY EQUIVALENT if they must have the same truth value. (I.e., they must both be true or they both must be false.)

3.3 Necessary truths, necessary falsehoods, and contingency

Consider these sentences:

a. It is raining.
b. Either it is raining here, or it is not.
c. It is both raining here and not raining here.

In order to know if sentence (*a*) is true, you would need to look outside or check a weather forcasting app. It might be true, or it might be false. A sentence that is capable of being true and capable of being false (in different circumstances, of course) is CONTINGENT.

Sentence (*b*) is different. You do not need to look outside to know that it is true. Regardless of what the weather is, it is either raining or it is not. Thus, this sentence is a NECESSARY TRUTH.

Similarly, you do not need to check the weather to determine whether or not sentence (*c*) is true. It must be false, simply as a matter of logic. It might be raining here and not raining across town. It might be raining now but stop raining before you finish reading this sentence. It is impossible, however, for it to be both raining and not raining in the same place and at the same time. Therefore, this sentence is a NECESSARY FALSEHOOD.

sentences: necessary and contingent

A NECESSARY TRUTH is a sentence that must be true; it could not possibly be false.
A NECESSARY FALSEHOOD is a sentence that must be false; it could not possibly be true.
A CONTINGENT SENTENCE is neither a necessary truth nor a necessary falsehood. It may be true or it may not.

Finally, a sentence might always be true and still be contingent. For instance, this sentence is true:

d. Mary Todd married Abraham Lincoln in 1842.

And there is no way, now, that it will ever be false. But it could have been false. Todd and Lincoln could have gotten married in a different year, or Todd could have married someone else or no one at all. A full analysis of this (and other) contingent truths would be too lengthy to undertake here, but hopefully you can see that things could have worked out in such a way that (*d*) would be false.

This is in contrast to a sentence like this one: 'Today, in Starkville, Mississippi, it is Thursday, or it is not Thursday'. Or this one: '5 + 7 = 12'. These sentences cannot be false, and there is no way to imagine a possible series of events that would make them false. Hence, they are not contingent. They are necessary truths.

3.4 Practice exercises

A. Determine if each sentence is a necessary truth, a necessary falsehood, or contingent.

1. Caesar crossed the Rubicon.
2. Someone once crossed the Rubicon.
3. No one has ever crossed the Rubicon.
4. If Caesar crossed the Rubicon, then someone has.
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
6. If anyone has ever crossed the Rubicon, it was Caesar.
7. Elephants dissolve in water.
8. Wood is a light, durable substance useful for building things.
9. If wood is a good building material, it is useful for building things.
10. I live in a three story building that is two stories tall.
11. If gerbils are mammals, they nurse their young.

B. Which of the following pairs of sentences are necessarily equivalent?

1. Elephants dissolve in water.
   If you put an elephant in water, it will dissolve.
2. All mammals dissolve in water.
   If you put an elephant in water, it will dissolve.

3. George Bush was the 43rd president.
   Barack Obama was the 44th president.
4. Barack Obama was the 44th president.
   Barack Obama was president immediately after the 43rd president.
5. Elephants dissolve in water.
   All mammals dissolve in water.
6. Thelonious Monk played piano.
   John Coltrane played tenor sax.
7. Thelonious Monk played with John Coltrane.
   John Coltrane played with Thelonious Monk.
8. All professional pianists begin playing as young children.
   The professional pianist Bud Powell began playing as a young child.
9. Bud Powell suffered from severe mental illness.
   All professional pianists suffer from severe mental illness.
10. John Coltrane was deeply religious.
    John Coltrane viewed music as an expression of spirituality.

C.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two Martians at the wild animal park.
G4. Every giraffe at the wild animal park is a Martian.

Determine if each set of sentences is jointly possible or jointly impossible.

1. Sentences G2, G3, and G4
2. Sentences G1, G3, and G4
3. Sentences G1, G2, and G4
4. Sentences G1, G2, and G3

D.

M1. All people are mortal.
M2. Socrates is a person.
M3. Socrates will never die.

M4. Socrates is mortal.

Determine if each set of sentences is jointly possible or jointly impossible.

1. Sentences M1, M2, and M3
2. Sentences M2, M3, and M4
3. Sentences M2 and M3
4. Sentences M1 and M4
5. Sentences M1, M2, M3, and M4

E. For each statement, determine whether or not it is possible. If it is possible, give an example that illustrates the statement. If it is not possible, explain why not.

1. A valid argument, the conclusion of which is a necessary falsehood
2. An invalid argument, the conclusion of which is a necessary truth
3. A necessary truth that is contingent
4. Two necessarily equivalent sentences, both of which are necessary truths
5. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
6. Two necessarily equivalent sentences that together are jointly impossible
7. A jointly possible collection of sentences that contains a necessary falsehood
8. A jointly impossible set of sentences that contains a necessary truth
9. A valid argument with premises that are all necessary truths and with a conclusion that is contingent
10. A valid argument with true premises and a false conclusion
11. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
12. A jointly possible collection of sentences, all of which are contingent
13. A false necessary truth
14. A valid argument with false premises
15. A necessarily equivalent pair of sentences that are not jointly possible
16. A necessary truth that is also a necessary falsehood

17. A jointly possible collection of sentences that are all necessary falsehoods

3.5 Answers

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon.
   Contingent
2. Someone once crossed the Rubicon.
   Contingent
3. No one has ever crossed the Rubicon.
   Contingent
4. If Caesar crossed the Rubicon, then someone has.
   Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
   Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar.
   Contingent
7. Elephants dissolve in water.
   Contingent
8. Wood is a light, durable substance useful for building things.
   Contingent
9. If wood is a good building material, it is useful for building things.
   Necessarily true
10. I live in a three story building that is two stories tall.
    Necessarily false
11. If gerbils are mammals, they nurse their young.
    This sentence is necessarily true. (*Mammalia* is defined as the class of animals wherein the females have mammaries and nurse their young. Hence, 'If gerbils are mammals, they nurse their young' is necessarily true.)

B.

1. Elephants dissolve in water.
   If you put an elephant in water, it will dissolve.
   These sentences are necessarily equivalent.
2. All mammals dissolve in water.
   If you put an elephant in water, it will dissolve.
   These sentences are *not* necessarily equivalent.
3. George Bush was the 43rd president.
   Barack Obama was the 44th president.
   These sentences are *not* necessarily equivalent.
4. Barack Obama was the 44th president.
   Barack Obama was president immediately after the 43rd president.
   These sentences are necessarily equivalent.
5. Elephants dissolve in water.
   All mammals dissolve in water.
   These sentences are *not* necessarily equivalent.
6. Thelonious Monk played piano.
   John Coltrane played tenor sax.
   These sentences are *not* necessarily equivalent.
7. Thelonious Monk played with John Coltrane.
   John Coltrane played with Thelonious Monk.
   These sentences are necessarily equivalent.
8. All professional pianists begin playing as young children.
   The professional pianist Bud Powell began playing as a young child.
   These sentences are *not* necessarily equivalent.
9. Bud Powell suffered from severe mental illness.
   All professional pianists suffer from severe mental illness.
   These sentences are *not* necessarily equivalent.
10. John Coltrane was deeply religious.
    John Coltrane viewed music as an expression of spirituality.
    These sentences are *not* necessarily equivalent.

C.

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.
G3. There are not more than two Martians at the wild animal park.
G4. Every giraffe at the wild animal park is a Martian.

1. Sentences G2, G3, and G4 — Jointly possible
2. Sentences G1, G3, and G4 — Jointly impossible
3. Sentences G1, G2, and G4 — Jointly possible
4. Sentences G1, G2, and G3 — Jointly possible

D.

M1. All people are mortal.
M2. Socrates is a person.
M3. Socrates will never die.
M4. Socrates is mortal.

1. Sentences M1, M2, and M3 — Jointly impossible
2. Sentences M2, M3, and M4 — Jointly impossible
3. Sentences M2 and M3 — Jointly possible
   Person, at least in the philosophical sense, is different than *human being* (although the two concepts generally overlap). *Person* means, basically, *moral agent*, and so, for instance, God, if he exists, is a person. Consequently, just the sentence 'Socrates is a person' doesn't tell us whether or not Socrates will die.
4. Sentences M1 and M4 — Jointly possible
5. Sentences M1, M2, M3, and M4 — Jointly impossible

E.

1. A valid argument, the conclusion of which is a necessary falsehood
   Yes, this is possible. This is a valid argument, and the conclusion is a necessary falsehood:
   P1. If today is Tuesday, then $1 + 1 = 3$.
   P2. Today is Tuesday.
   C. Therefore, $1 + 1 = 3$.
2. An invalid argument, the conclusion of which is a necessary truth

No, this is not possible. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.

3. A necessary truth that is contingent
   No, this is not possible. If a sentence is a necessary truth, it cannot possibly be false, but a contingent sentence can be false.
4. Two necessarily equivalent sentences, both of which are necessary truths
   Yes, this is possible. '4 is even', '4 is divisible by 2'.
5. Two necessarily equivalent sentences, one of which is a necessary truth and one of which is contingent
   No, this is not possible. A necessary truth cannot possibly be false, while a contingent sentence can be false. So in any situation in which the contingent sentence is false, it will have a different truth value from the necessary truth. Thus, they will not necessarily have the same truth value, and so they will not be equivalent.
6. Two necessarily equivalent sentences that together are jointly impossible
   Yes, this is possible. '$1 + 1 = 4$' and '$1 + 1 = 3$'.
7. A jointly possible collection of sentences that contains a necessary falsehood
   No, this is not possible. If a sentence is necessarily false, there is no way to make it true, let alone make it true along with all the other sentences.
8. A jointly impossible set of sentences that contains a necessary truth
   Yes, this is possible. '$1 + 1 = 4$' and '$1 + 1 = 2$'.
9. A valid argument with premises that are all necessary truths and with a conclusion that is contingent *
   This is not possible. In a deductively valid argument, the information contained in the conclusion is information that is in the premises. If all of the premises are necessary truths, then the conclusion will be as well.
10. A valid argument with true premises and a false conclusion
    This is not possible. A valid argument is one where if the premises

are true, then the conclusion has to be true. Thus, if the premises are true, the conclusion has to be as well.

11. A jointly possible collection of sentences that contains two sentences that are not necessarily equivalent
    Yes, this is possible. G1 and G2 on p. 19 are jointly possible, but they are not necessarily equivalent.
12. A jointly possible collection of sentences, all of which are contingent
    Yes, this is possible. G1 and G2 on p. 19 are both contingent.
13. A false necessary truth
    This is not possible. A necessary truth is a sentence that has to be true, and so it could not be false.
14. A valid argument with false premises
    Yes, this is possible. This argument has false premises, and it is valid:
    P1. Mississippi is in Canada.
    P2. New York City is in Mississippi.
    C. Therefore, New York City is in Canada.
15. A necessarily equivalent pair of sentences that are not jointly possible
    Yes, this is possible (although it isn't the standard case). These two sentences are both necessary falsehoods:
    (a) It is November and it is not November.
    (b) Jeff is in Texas and he is not in North America.
    Since they are both always false, they are necessarily equivalent. But, at the same time, since they are both always false, they cannot be jointly possible.
16. A necessary truth that is also a necessary falsehood
    This is not possible. A necessary truth is a sentence that must be (and is always) true. A necessary falsehood is a sentence that must be (and is always) false. Consequently, one sentence cannot be both.
17. A jointly possible collection of sentences that are all necessary falsehoods
    This is not possible. See the answer to question 17.

Part 2

Truth-functional logic

4 Form and symbolization

4.1 Validity in virtue of form

Consider the following two arguments. First this one:

1. If it is raining outside, then Mary is miserable.
2. It is raining outside.
3. Therefore, Mary is miserable.

and then this one:

1. If Mary is a student, then Leiser is a spy.
2. Mary is a student.
3. Therefore, Leiser is a spy.

Both arguments are valid, and, as perhaps you can see, they share a common *form*. We can represent the form by itself this way:

1. If A, then B
2. A
3. Therefore, B

Any argument with this form will be valid. It doesn't matter what English sentences are put in the places of A and B.

Here is another valid argument:

1. Seoul is larger than London.
2. London is larger than Chicago.
3. Therefore, Seoul is larger than Chicago.

This argument also has a particular form that makes it valid, and we can represent its form like this:

1. C is larger than D.

2. D is larger than F.
3. Therefore, C is larger than F.

For the first argument form that we examined, A and B could be any sentences. Here, C, D, and F are names (not full sentences), and we can put any names (for anything) in the places of C, D, and F, and the argument will be valid.

In contrast, this argument is valid, but there is no particular form that makes it so.

1. Juanita is a vixen.
2. Therefore, Juanita is a fox.

Unlike the previous three examples, this argument is valid, not because of the form of the argument, but because of the particular meanings of *fox* and *vixen*.

These examples illustrate *validity in virtue of form*. The arguments about Mary, Leiser, Seoul, London, and Chicago are valid, but—unlike the argument about Juanita—their being valid has nothing to do with the specific meaning of 'Mary is miserable', 'Leiser is a spy', 'Seoul', 'London', or 'Chicago.' (Whether the arguments are sound depends on these meanings, but not whether the arguments are valid.) Instead, these arguments are valid in virtue of the meanings of just these words: *if*, *then* and *is larger than*.

valid in virtue of form

Let us define STRUCTURE WORDS as *if-then*, *and*, *or*, *not*, *if and only if*, and comparative adjectives followed by *than* (e.g., *larger than*, *faster faster than*, *older than*).
An argument is VALID IN VIRTUE ITS FORM when it remains valid under these conditions: (1) structure words are always part of the argument and (2) with those words in place, sentences or names can be freely substituted into the argument and it will remain valid.

This is not a perfect definition. *Form* in formal logic is much broader than just the use of these "structure words." This is a good definition with which to start, however.

And, although valid for reasons *other than* the argument's form is an interesting topic, our focus will be on arguments that are valid because of their form – and in fact, only valid in virtue of some forms. In part 5, we will broaden the analysis a bit, but in parts 2, 3, and 4 of this textbook, we will be interested in arguments where the form is set by the use of *if-then*, *and*, *or*, *not*, and *if and only if*.

> Going forward, we will set aside arguments that are valid because they employ comparative adjectives. To make a final point about them, though, arguments that use comparative adjectives are valid because these terms denote a TRANSATIVE RELATION. Such a relation exists when the relation between two elements in a series applies to any elements in the series as long as the elements are taken in order.

Here are some examples of arguments that are valid in virtue of *or*, *not*, and *and*. This one:

1. Claire is either a student, or she is a spy.
2. Claire is not a student.
3. Therefore, Claire is a spy.

has this form:

1. G or H
2. not G
3. Therefore, H

And this argument:

1. It's not the case that Jeff both studies often and acts in lots of plays.
2. Jeff acts in lots of plays.
3. Therefore, Jeff does not study often.

has a form that we can represent like this:

1. not (K and L)
2. L
3. Therefore, not K

4.2 Atomic sentences and symbolization

Consider this sentence again:

(a) If it is raining outside, then Mary is miserable.

'It is raining outside' and 'Mary is miserable' are *subsentences* of sentence (a). To specify the structure of the first argument in §4.1, we replaced the sentences and subsentences in it with individual letters. 'It is raining outside' was replaced with 'A', and 'Mary is miserable' was replaced with 'B'. This kind of representation—letters standing for sentences or subsentences—is one important part of the formal language that we develop in this book.

Atomic sentences

An ATOMIC SENTENCE is a sentence that (1) can be true or false and (2) no smaller complete sentence can be extracted from it.

In English, an atomic sentence is a declarative sentence that does not contain any subsentences.

In our logic system, an atomic sentence is a single capital letter in this font: $A, B, C, \ldots$.

Atomic sentences are the basic building blocks used to form more complex sentences. We will use uppercase Roman letters for atomic sentences in our logic system. If, by chance, we ever need more than twenty-six different atomic sentences, we can obtain additional ones by adding subscripts to letters. Here, for instance, are five different atomic sentences:

$$M, P, P_1, P_2, M_{17}$$

We will use atomic sentences to represent, or *symbolize*, certain English sentences. To do this, we provide a SYMBOLIZATION KEY, such as the following.

A: It is raining outside
B: Mary is miserable

When we do this, we are not fixing this symbolization once and for all. We are just saying that, for the time being, we will think of the atomic sentence 'A' as symbolizing the English sentence 'It is raining outside', and the atomic sentence, 'B', as symbolizing the English sentence 'Mary is miserable'. Later, when we are dealing with different sentences or different arguments, we can provide a new symbolization key; for instance,

A: Jeff stole the document.
B: Jeff is in the safe house.

5 Logical operators

TRUTH-FUNCTIONAL PROPOSITIONAL LOGIC is a branch of logic that focuses on the relationships between atomic sentences. One part of truth-functional propositional logic (or 'TFL' for short) is a formal language. This formal language consists of atomic sentences of TFL—the sentence letters that were introduced in §4.2—and the LOGICAL OPERATORS 'and', 'or', 'not', 'if ..., then ...' and 'if and only if'. A logical operator is a word or phrase that modifies a sentence or connects two sentences to form a more complex sentence. We call these operators *truth-functional* because the truth of the complex sentences depends entirely on the truth of the atomic sentences of which they are composed. (*Logical operators* are also sometimes referred to as *connectives* because, except in the case of 'not', these operators connect two simpler sentences.)

In addition to symbolizing English sentences with sentence letters, we also want to symbolize the truth-functional logical operators. The symbols that we will use are shown in table 5.1. The operators listed there are not the only ones that we have in English. Others are, for example, 'unless', 'neither ... nor ...', 'necessarily', and 'because'. As we will see, the first two can be expressed with the connectives that are in table 5.1. The last two, however, cannot. Although they are logical operators, 'necessarily' and 'because' are not truth functional.

SYMBOL	THE SENTENCE'S NAME	ITS MEANING
$\neg$	negation	'It is not the case that...'
&	conjunction	'Both... and ...'
$\vee$	disjunction	'Either... or ...'
$\rightarrow$	conditional	'If ... then ...'
$\leftrightarrow$	biconditional	'... if and only if ...'

Table 5.1: The logical operators of truth functional logic

Once we have introduced these logical operators (in this chapter and in chapter 7) and have explained what can and cannot be a sentence in TFL (which we will do in chapter 6), our formal language will be complete. Although the formal language is central, truth-functional propositional logic does not consist only of a formal language. There is also a *deductive system*, which we will explore in part 4.

5.1 Negation

Consider how we might symbolize these sentences:

1. Mary is in Barcelona.
2. It is not the case that Mary is in Barcelona.
3. Mary is not in Barcelona.

To begin, we need an atomic sentence. This will be our symbolization key:

B: Mary is in Barcelona.

B is sentence 1, and so we don't need to do anything else there. The second sentence is partially symbolized as 'It is not the case that B'. In order to complete the symbolization, we need a symbol for 'it is not the case that'. Or, put differently, we need a symbol that, when added to B, will express 'the negation of B'. We will use '$\neg$' and symbolize sentence 2 as '$\neg B$'.

Sentence 3 also contains the word 'not', and it is equivalent to sentence 2. As such, we can also symbolize it as '$\neg B$'.

Negation

A sentence can be symbolized as $\neg$A if it can be paraphrased in English as 'It is not the case that …'

Here are a few more examples:

4. The cog can be replaced.
5. The cog is irreplaceable.
6. The cog is not irreplaceable.

For these, we will use this symbolization key:

R: The cog is replaceable

Sentence 4 is symbolized just by 'R'. Sentence 5 can be reworded as *it is not the case that the cog is replaceable*. So even though sentence 5 does not contain the word 'not', we will symbolize it '$\neg R$'. Sentence 6, you will notice, is the denial of sentence 5. So, we symbolize 6 as '$\neg\neg R$'.

Finally, consider these English sentences:

7. Jane is happy.
8. Jane is unhappy.

If we use'H' stand for 'Jane is happy', then we can symbolize sentence 7 as 'H'. It would be a mistake, however, to symbolize sentence 8 with '$\neg H$'. '$\neg H$' means 'Jane is not happy', but 'Jane is not happy' does not have the same meaning as 'Jane is unhappy'. After all, Jane might be neither happy nor unhappy; her mood might just be neutral. In order to symbolize sentence 8, we would need a different sentence letter.

5.2 Conjunction

Let's start with these sentences:

9. Adam is athletic, and Barbara is also athletic.
10. Barbara and Adam are both athletic.
11. Adam is not athletic, but Barbara is.

We will need separate sentence letters to symbolize sentences 9 and 10, and so we will use this symbolization key:

A: Adam is athletic.
B: Barbara is athletic.

Sentence 9 can be partially symbolized as 'A and B'. To symbolize the 'and'. We will use '&', which is called the *ampersand*. Thus, sentence 9 becomes '$(A \,\&\, B)$'. When two sentences are connected with an '&', the resulting sentence is called a CONJUNCTION. The two sentences that are combined with the '&' are the CONJUNCTS of the conjunction. So, 'A' and 'B' are the conjuncts of the conjunction '$(A \& B)$'.

Although it is worded differently, sentence 10 has the same meaning as sentence 9. Thus, it is also symbolized as '$(A \& B)$'. Notice that we don't symbolize the word 'also' in sentence 9 or 'both' in 10. Words like 'both' and 'also' function to draw our attention to the fact that two sentences are being joined to form a conjunction. They may affect the emphasis of a sentence in English, but we don't (and can't) symbolize such terms in TFL.

For sentence 11, let's first symbolize 'Adam is not athletic' as '$\neg A$'. 'Barbara is' means 'Barbara is athletic', and so that subsentence is symbolized as 'B'. 'But' may have a slightly different meaning in English than 'and', but, grammatically, they serve the same role: to join two sentences to form a conjunction. Putting this altogether, sentence 11 is symbolized as '$(\neg A \& B)$'.

Conjunction

A sentence can be symbolized as $(\mathsf{A} \ \& \ \mathsf{B})$ if it can be paraphrased any of these ways in English:

> 'Both…, and…',
> '…, and…',
> '…, but …',
> '…, although …',
> '…, as well as …'

Parentheses

Although we will relax this requirement later, a conjunction in TFL should be enclosed in parentheses. (A negation—for instance, '$\neg P$'—should not, though.) The purpose of the parentheses is to let us be perfectly explicit about how each logical operator is related to each sentence letter. We will see more illustrations of this once we have introduced all of the logical operators, but consider these two English sentences, and think about whether they have the same or different meanings:

12. Kate is not at school, and Sarah is sleeping.
13. It is not the case that both Kate is at school and Sarah is sleeping.

We will use this symbolization key:

K: Kate is at school.
S: Sarah is sleeping.

Sentences 12 and 13 do not have the same meaning, and so we can't symbolize them in exactly the same way.

In sentence 12, the 'not' only applies to 'Kate is at school.' Thus, this sentence becomes '$(\neg K \mathbin{\&} S)$'. In sentence 13, the 'it is not the case that' applies to the whole 'Kate is at school and Sarah is sleeping'. We can tell this because the 'it is not the case that' is before the 'both', which signals the beginning of the conjunction. Hence, for this sentence, we need to put the '$\neg$' outside the parentheses like this: '$\neg(K \mathbin{\&} S)$'.

5.3 Disjunction

We will start with these sentences:

14. Either Amy is at the train station, or Kate driving to Santa Fe.
15. Amy or Sarah is at the train station.

And we will use this symbolization key:

A: Amy is at the train station.
K: Kate is driving to Santa Fe.
S: Sarah is at the train station.

To represent the 'or' in sentences 14 and 15, we will use the symbol '$\vee$' (which we call the *wedge*, not *v*). Sentence 14, then, is written as '$(A \vee K)$'. When two sentences are connected with an '$\vee$', the resulting sentence is called a DISJUNCTION. 'A' and 'K' are the DISJUNCTS of the disjunction '$(A \vee K)$'.

Sentence 15 is only slightly more complicated. We can paraphrase it as 'Either Amy is at the train station, or Sarah is at the train station', and then we symbolize it as '$(A \vee S)$'.

Disjunction

A sentence can be symbolized as $(\mathsf{A} \vee \mathsf{B})$ if it can be paraphrased in English as 'Either…, or….' Each of the disjuncts must be a sentence.

The inclusive or

Sometimes in English, the word 'or' is used in a way that excludes the possibility that both disjuncts are true. This is called an EXCLUSIVE OR. An *exclusive or* is clearly intended when it says on a restaurant menu "Entrees come with either soup or salad." This means that, with your entree, you may have soup or you may have salad, but you cannot have both.

At other times, the word 'or' allows for the possibility that both disjuncts might be true. For instance, Amy might say, "I am going to get an A in Logic or an A in German III." She probably means that she will get an A in one or both of those courses. (After all, if she did end up getting an A in both, we wouldn't insist that she was wrong when she said, "I am going to get an A in Logic or an A in German III.")

When we mean that *one* or the *other* or *both* of the disjuncts is true, then we are using the INCLUSIVE OR. The TFL symbol '$\vee$' always symbolizes an *inclusive or*.

Negation and disjunction

Think about these sentences:

16. Either Amy is not at the train station, or Sarah is not at the train station.
17. Neither Amy nor Sarah is at the train station.
18. Either Amy is at the train station or Sarah is at the train station, but both are not.

Sentence 16 is symbolized as '$(\neg A \vee \neg S)$'. Sentences 17 and 18 are a little trickier.

According to 17, is either one at the train station? No. So, when when we paraphrase it we get this:

> It is not the case that either Amy is at the train station or Sarah is at the train station.

As our paraphrased sentence shows, we are negating the entire disjunction. Hence, we symbolize sentence 17 as '$\neg(A \vee S)$'.

Sentence 18 expresses the meaning of the *exclusive-or*: one or the other, but not both. The '$\vee$', however, represents the *inclusive-or*: one or the other, or both. Therefore, to represent 18 in TFL, we need to break the sentence into two parts.

The first part, 'Amy is at the train station or Sarah is at the train station', is symbolized as '$(A \vee S)$'. The second part, which states that both won't be there, is paraphrased this way: 'It is not the case that both Amy is at the train station and Sarah is at the train station'. This, we symbolize as '$\neg(A \,\&\, S)$'. We put the two parts together with an 'and', and sentence 18 becomes '$((A \vee S) \,\&\, \neg(A \,\&\, S))$'.

These last two examples demonstrate that we can sometimes symbolize English sentences that, at first, appear not to be using the logical operators of TFL. We can do this as long as we can figure out a way to paraphrase the English sentence so that it is using some combination of '*and*', '*or*' (i.e., the inclusive-or), '*not*', '*if ..., then ...*', and '*if and only if*'.

5.4 Conditional

We will start with this sentence:

19. If Jean is in Paris, then Jean is in France.

And we will use this symbolization key:

> P: Jean is in Paris.
> F: Jean is in France

Sentence 19 has this form: 'if P, then F', and we call this type of sentence a CONDITIONAL. We will use '$\rightarrow$' to symbolize 'if ..., then ...'. Thus, sentence 19 becomes '$(P \rightarrow F)$'.

If Jean is in Paris, then she is in France	If A, then B.
Jean is in France if she is in Paris.	B if A.
Whenever Jean is in Paris, she is in France.	Whenever A, B.
Jean is in France provided that she is in Paris.	B provided that A.
Provided that Jean is in Paris, she is in France.	Provided that A, B.
Jean is in Paris only if she is in France.	A only if B.

Table 5.2: The most common way of expressing a conditional in English is as 'If Jean is in Paris, then she is in France.' This table lists some alternative but equivalent ways of expressing the same sentence.

In a conditional, what goes before the '→' is called the ANTECEDENT, and what comes after the '→' is called the CONSEQUENT. So, in sentence 19, 'Jean is in Paris' is the antecedent, and 'Jean is in France' is the consequent.

Conditional

A sentence can be symbolized as $\mathsf{A} \rightarrow \mathsf{B}$ if it can be paraphrased in English as 'If A, then B'.

Many English expressions can be represented using the conditional, and the most common alternatives to 'if A, then B' are listed in table 5.2. If you think about it, you'll see that all six of the sentences in the table have the same meaning, and so they can all be symbolized as '$(\mathsf{A} \rightarrow \mathsf{B})$'. (Or, in this case, as '$(P \rightarrow F)$'.)

5.5 Biconditional

All of the logical operators that we have discussed so far are ones with which you were already familiar because you are an English speaker. The biconditional, which is mostly commonly expressed as '…*if and only if* …', is one that you might not have really noticed before—even if you have used it on occasion. We'll start with the basic case.

20. The Bearcats won if and only if they scored more points than the Razorbacks.

And this will be our symbolization key:

B: The Bearcats won.
R: The Bearcats scored more points than the Razorbacks.

The symbol '$\leftrightarrow$' will stand for 'if and only if', and so we can symbolize sentence 20 as '$B \leftrightarrow R$'.

Now, let's probe a little further into the meaning of 'if and only if' with a different example.

21. If Mary has a sunburn, then she went to the beach.
22. If she went to the beach, then Mary has a sunburn.
23. If Mary has a sunburn, then she went to the beach, and if she went to the beach, then Mary has a sunburn.
24. Mary has a sunburn if and only if she went to the beach.

We will use this symbolization key:

S: Mary has a sunburn.
B: Mary went to the beach.

From the previous section, you know that we symbolize sentences 21 and 22 like this:

21. $(S \rightarrow B)$
22. $(B \rightarrow S)$

Sentence 23, then, is a conjunction created by combining 21 and 22:

23. $((S \rightarrow B) \,\&\, (B \rightarrow S))$

And sentence 24, is symbolized with the '$\leftrightarrow$':

24. $(S \leftrightarrow B)$

Sentence 23, it turns out, has the same meaning as sentence 24. '$(S \rightarrow B) \,\&\, (B \rightarrow S)$' is equivalent to '$(S \leftrightarrow B)$'. We call '$(S \leftrightarrow B)$' a BICONDITIONAL, because it is equivalent to the two conditionals that have their antecedent and the consequent switched.

The expression 'if and only if' occurs frequently in philosophy, mathematics, and logic, and sometimes you will see it abbreviated 'iff'. (Although even when 'iff' is written, we still say 'if and only if.')

> Biconditional
>
> A sentence can be symbolized as $\mathsf{A} \leftrightarrow \mathsf{B}$ if it can be paraphrased in English as 'A iff B'—that is, as 'A if and only if B'.

5.6 Unless

We have now introduced all of the logical operators of TFL. We can use them together to symbolize many kinds of sentences. An especially difficult case is when we use the English-language connective 'unless'. Take this sentence:

25. You will catch a cold unless you wear a jacket.

To symbolize 25, we will use this symbolization key:

J: You will wear a jacket.
D: You will catch a cold.

One meaning of sentence 25 is that if you do not wear a jacket, then you will catch a cold. This we symbolize as '$(\neg J \rightarrow D)$'. Alternatively, the sentence can mean that if you do not catch a cold, then you must have worn a jacket. This is symbolized as '$(\neg D \rightarrow J)$'. And, finally, it can also mean that either you will wear a jacket or you will catch a cold. This, we symbolize as '$(J \vee D)$'.

All three ways of symbolizing sentence 25 are correct. Indeed, in chapter 9 we will see that all three symbolizations are equivalent in TFL. Following the somewhat standard practice, however, we will define *unless* as a disjunction.

> Unless
>
> If a sentence can be paraphrased as 'A unless B,' then it can be symbolized as '$\mathsf{A} \vee \mathsf{B}$'.

There is a complication with treating 'unless' as a disjunction, however. As we said earlier, 'or' has an inclusive and an exclusive meaning, but in TFL, 'or' is always inclusive. Speakers of English, however, often use 'unless' to mean something more like the exclusive-or. Suppose someone says: 'I will go running unless it snows'. They probably mean 'either I will go running or it will snow, but not both'. So, it can be argued that the conditional—i.e., 'if it does not snow, then I will go running' ($\neg S \rightarrow R$)—captures the meaning of 'unless' better than does the disjunction.

5.7 The turnstile

The final symbol that we need is, technically, not a symbol of TFL, but it is useful to have when displaying arguments in TFL. The symbol '$\vdash$' is called the *turnstile*. The purpose of the turnstile is to separate the sentences that are the premises of an argument from the sentence that is the conclusion, and it can be read as *therefore*. Here is an example,

$$(P \rightarrow C), (P \vee D) \vdash (\neg C \rightarrow D)$$

In this argument, the premises are '$(P \rightarrow C)$' and '$(P \vee D)$', and the conclusion is '$(\neg C \rightarrow D)$'.

5.8 Practice exercises

A. Using the symbolization key given, translate each English sentence into TFL.

A: Those creatures are aliens.
C: Those creatures are centaurs.
V: Those creatures are vampires.

Always use capital letters for the atomic sentences, and, in this case, be especially careful to distinguish between V and $\vee$.

1. Those creatures are not aliens.
2. Those creatures are aliens, or they are not.
3. Those creatures are either vampires or centaurs.

4. Those creatures are neither vampires nor centaurs.
5. If those creatures are centaurs, then it is not the case that they are vampires or aliens.
6. Either those creatures are aliens, or they are both centaurs and vampires.

B. Using the symbolization key given, translate each English sentence into TFL.

A: Mr. Adams was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mr. Edwards was murdered.
F: The murder weapon was a frying pan.

1. Either Mr. Adams or Mr. Edwards was murdered.
2. If Mr. Adams was murdered, then the cook did it.
3. If Mr. Edwards was murdered, then the cook did not do it.
4. Either the butler did it, or the Duchess is lying.
5. The cook did it only if the Duchess is lying.
6. If the murder weapon was not a frying pan, then the cook did not do it.
7. If the murder weapon was not a frying pan, then either the cook or the butler did it.
8. Mr. Adams was murdered if and only if Mr. Edwards was not murdered.
9. It is not the case that either the Duchess is lying or Mr. Edwards was not murdered.
10. If Mr. Adams was murdered, he was killed with a frying pan.
11. The cook did it, and the butler did not.
12. Of course the Duchess is lying!

C. Using the symbolization key given, translate each English sentence into TFL.

E_1: Ava is an electrician.

E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
2. If Ava is a firefighter, then she is satisfied with her career.
3. Ava is a firefighter, unless she is an electrician.
4. Harrison is an unsatisfied electrician.
5. Neither Ava nor Harrison is an electrician.
6. Both Ava and Harrison are electricians, but Ava is satisfied with her career and Harrison is not satisfied with his career.
7. Harrison is satisfied with his career only if he is a firefighter.
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
10. If Harrison is both an electrician and a firefighter, then he is satisfied with his career.
11. It is not the case that Harrison is both an electrician and a firefighter.
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.

D. Using the symbolization key given, translate each English-language sentence into TFL.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.
2. Neither Miles Davis nor John Coltrane played tuba.
3. John Coltrane did not play both tenor sax and tuba.

4. John Coltrane did not play tenor sax unless he also played soprano sax.
5. John Coltrane did not play tuba, but Miles Davis did.
6. Miles Davis played trumpet only if he also played tuba.
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
8. It is not the case that if John Coltrane played tuba then Miles Davis played trumpet or tuba.
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.

E. Give a symbolization key, and then translate the following English sentences into TFL.

1. It is not the case that Alice and Bob are both spies.
2. If either Alice or Bob is a spy, then the code has been broken.
3. If neither Alice nor Bob is a spy, then the code has not been unbroken.
4. The letter is in the German embassy, unless someone has broken the code.
5. Either the code has been broken or it has not, but the letter is in German embassy regardless.
6. Either Alice or Bob is a spy, but not both.

F. For each argument, first, make a symbolization key, and then translate all of the sentences of the argument into TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cross. Dorothy plays piano in the morning unless she is distracted. So, if Roger does not wake up cross, then Dorothy must be distracted.
2. It will either rain or snow on Tuesday. If it rains, Neville will be gloomy. If it snows, Neville will be cold. Therefore, Neville will either be gloomy or cold on Tuesday.
3. If Zoey remembered to do her chores, then the house is clean but not neat. If she forgot, then the house is neat but not clean. Therefore, the house is either neat or clean; but not both.

5.9 Answers

A.

A: Those creatures are aliens.
C: Those creatures are centaurs.
V: Those creatures are vampires.

1. Those creatures are not aliens.
   $\neg A$
2. Those creatures are aliens, or they are not.
   $(A \lor \neg A)$
3. Those creatures are either vampires or centaurs.
   $(V \lor C)$
4. Those creatures are neither vampires nor centaurs.
   $\neg(C \lor V)$
5. If those creatures are centaurs, then it is not the case that they are vampires or aliens.
   $(C \to \neg(V \lor A))$
6. Either those creatures are aliens, or they are both centaurs and vampires.
   $(A \lor (C \,\&\, V))$

B.

A: Mr. Adams was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mr. Edwards was murdered.
F: The murder weapon was a frying pan.

1. Either Mr. Adams or Mr. Edwards was murdered.
   $(A \lor E)$
2. If Mr. Adams was murdered, then the cook did it.
   $(A \to C)$
3. If Mr. Edwards was murdered, then the cook did not do it.

$(E \rightarrow \neg C)$

4. Either the butler did it, or the Duchess is lying.
   $(B \vee D)$
5. The cook did it only if the Duchess is lying. (See table 5.2.)
   $(C \rightarrow D)$
6. If the murder weapon was not a frying pan, then the cook did not do it.
   $(\neg F \rightarrow \neg C)$
7. If the murder weapon was not a frying pan, then either the cook or the butler did it.
   $(\neg F \rightarrow (C \vee B))$
8. Mr. Adams was murdered if and only if Mr. Edwards was not murdered.
   $(A \leftrightarrow \neg E)$
9. It is not the case that either the Duchess is lying or Mr. Edwards was not murdered.
   $\neg(D \vee \neg E)$
10. If Mr. Adams was murdered, he was killed with a frying pan.
   $(A \rightarrow F)$
11. The cook did it, and the butler did not.
   $(C \,\&\, \neg B)$
12. Of course the Duchess is lying!
   D

C.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
   $(E_1 \,\&\, E_2)$
2. If Ava is a firefighter, then she is satisfied with her career.

$(F_1 \rightarrow S_1)$

3. Ava is a firefighter, unless she is an electrician.
$(F_1 \vee E_1)$
4. Harrison is an unsatisfied electrician.
$(E_2 \mathbin{\&} \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
$\neg(E_1 \vee E_2)$
6. Both Ava and Harrison are electricians, but Ava is satisfied with her career and Harrison is not satisfied with his career.
$((E_1 \mathbin{\&} E_2) \mathbin{\&} (S_1 \mathbin{\&} \neg S_2))$
7. Harrison is satisfied with his career only if he is a firefighter.
$(S_2 \rightarrow F_2)$
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
$((\neg E_1 \rightarrow \neg E_2) \mathbin{\&} (E_1 \rightarrow E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
$(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he is satisfied with his career.
$((E_2 \mathbin{\&} F_2) \rightarrow S_2)$
11. It is not the case that Harrison is both an electrician and a firefighter.
$\neg(E_2 \mathbin{\&} F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
$((F_2 \mathbin{\&} F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D.

J_1: John Coltrane played tenor sax.
J_2: John Coltrane played soprano sax.
J_3: John Coltrane played tuba
M_1: Miles Davis played trumpet
M_2: Miles Davis played tuba

1. John Coltrane played tenor and soprano sax.

$J_1 \& J_2$

2. Neither Miles Davis nor John Coltrane played tuba.
   $\neg(M_2 \vee J_3)$ or $\neg M_2 \& \neg J_3$
3. John Coltrane did not play both tenor sax and tuba.
   $\neg(J_1 \& J_3)$ or $\neg J_1 \vee \neg J_3$
4. If John Coltrane did not play tenor sax, then he played soprano sax.
   $\neg J_1 \rightarrow J_2$
5. John Coltrane did not play tuba, but Miles Davis did.
   $\neg J_3 \& M_2$
6. Miles Davis played trumpet only if he also played tuba.
   $M_1 \rightarrow M_2$
7. If Miles Davis played trumpet, then John Coltrane played at least one of these three instruments: tenor sax, soprano sax, or tuba.
   $M_1 \rightarrow (J_1 \vee (J_2 \vee J_3))$ or $M_1 \rightarrow ((J_1 \vee J_2) \vee J_3)$
8. It is not the case that if John Coltrane played tuba, then Miles Davis played trumpet or tuba.
   $\neg(J_3 \rightarrow (M_1 \vee M_2))$
9. Miles Davis and John Coltrane both played tuba if and only if Coltrane did not play tenor sax and Miles Davis did not play trumpet.
   $(J_3 \& M_2) \leftrightarrow (\neg J_1 \& \neg M_1)$ or $(J_3 \& M_2) \leftrightarrow \neg(J_1 \vee M_1)$

E.

A: Alice is a spy.
B: Bob is a spy.
C: The code has been broken.
L: The letter is in German embassy.

1. It is not the case that Alice and Bob are both spies.
   $\neg(A \& B)$
2. If either Alice or Bob is a spy, then the code has been broken.
   $((A \vee B) \rightarrow C)$
3. If neither Alice nor Bob is a spy, then the code has not been unbroken.

$\neg(A \lor B) \to \neg C$

4. The letter is in the German embassy, unless someone has broken the code.
$(L \lor C)$
5. Either the code has been broken or it has not, but the letter is in German embassy regardless.
$((C \lor \neg C) \,\&\, L)$
6. Either Alice or Bob is a spy, but not both.
$((A \lor B) \,\&\, \neg(A \,\&\, B))$

F. For each argument, write a symbolization key and symbolize all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cross. Dorothy plays piano in the morning unless she is distracted. So, if Roger does not wake up cross, then Dorothy must be distracted.

P: Dorothy plays the piano in the morning.
C: Roger wakes up cross.
D: Dorothy is distracted.

$(P \to C), (P \lor D) \vdash (\neg C \to D)$

2. It will either rain or snow on Tuesday. If it rains, Neville will be gloomy. If it snows, Neville will be cold. Therefore, Neville will either be gloomy or cold on Tuesday.

T_1: It rains on Tuesday
T_2: It snows on Tuesday
G: Neville is gloomy on Tuesday
C: Neville is cold on Tuesday

$(T_1 \lor T_2), (T_1 \to G), (T_2 \to C) \vdash (G \lor C)$

3. If Zoey remembered to do her chores, then the house is clean but not neat. If she forgot, then the house is neat but not clean. Therefore, the house is either neat or clean; but not both.

Z: Zoey remembered to do her chores

C: The house is clean.

N: The house is neat.

$$(Z \to (C \mathbin{\&} \neg N)), (\neg Z \to (N \mathbin{\&} \neg C)) \vdash ((N \vee C) \mathbin{\&} \neg(N \mathbin{\&} C)).$$

6 Sentences of TFL

'Bring with thee airs from heaven or blasts from hell' is a sentence of English. '$(P \vee Q)$' is a sentence of TFL. Oddly, although we can identify sentences of English when we encounter them, there is not a formal definition of *sentence of English* that will tell us, for any possible combination of words and punctuation, whether or not it is a sentence of English. It is possible, however, to provide such a definition for sentences of TFL, and we will examine that definition in this chapter.

6.1 Expressions

You have been introduced to the symbols of TFL in the previous two chapters. They are also summarized in table 6.1. We define an EXPRESSION OF TFL as any string of symbols of TFL. Take any of the symbols of TFL and write them down, in any order, and you have an expression of TFL.

6.2 Sentences

Many expressions of TFL will be total gibberish. We want to know when an expression of TFL amounts to a *sentence*. To that end, we have the following seven rules, which are one part of the grammar of TFL.

atomic sentences	$A, B, C, \ldots, Z$
with subscripts if needed	$A_1, A_2, A_3, A_4, \ldots, J_{10}, J_{11}, \ldots$
logical operators	$\neg, \&, \vee, \rightarrow, \leftrightarrow$
parentheses	(,)

Table 6.1: The three types of symbols of TFL

Sentences of TFL

1. Every atomic sentence is a sentence.
2. If A is a sentence, then $\neg\mathsf{A}$ is a sentence.
3. If A and B are sentences, then $(\mathsf{A} \,\&\, \mathsf{B})$ is a sentence.
4. If A and B are sentences, then $(\mathsf{A} \vee \mathsf{B})$ is a sentence.
5. If A and B are sentences, then $(\mathsf{A} \rightarrow \mathsf{B})$ is a sentence.
6. If A and B are sentences, then $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a sentence.
7. Nothing else is a sentence.

Notice that A and A are different fonts. A is an atomic sentence in TFL. A is not, actually, part of TFL. Rather, it stands for any sentence in TFL. That sentence could be A or it could be $(B \rightarrow D)$ or anything else. This use of *metavariables* is explained more fully in §6.6.

From the previous chapter, you have the basic idea about how the logical operators are used. The simplest cases are when '&', '∨', '→', or '↔' only connect two atomic sentences or when '¬' is before a single atomic sentence, as we find here:

$(P \leftrightarrow Q)$
$\neg R$

But our logic system must be more complex than this to be useful. To see how we can create more complex sentences in TFL, we will start with these two sentences:

$(P \,\&\, S)$
$(R \rightarrow T)$

Now, let's see how we would express "$(P \,\&\, S)$ *or* $(R \rightarrow T)$." Although '$(P \,\&\, S)$' and '$(R \rightarrow T)$' are both composed of two atomic sentences and a logical operator, we can treat each as a unit and combine them with a '∨' like this:

$((P \& S) \vee (R \rightarrow T))$

Similarly, we can express "*not* $(P \& S)$" by treating the '$(P \& S)$' as a unit and adding a '$\neg$':

$\neg(P \& S)$

We can even, if we need to do so, express "*not* $\neg(P \& S)$" by implementing the same procedure again. Now, '$\neg(P \& S)$' is the unit and we add a '$\neg$' to it:

$\neg\neg(P \& S)$

These procedures can be used to form an infinite number of sentences of TFL. We just have to be sure that we are following rules 1 – 7, given above. And when following these rules, we must remember that A and B are units that can stand for either single atomic sentences or longer sentences. In fact, they can stand for sentences of any length.

Ultimately, you want to be able to just look at an expression and tell whether or not it is a correctly formed sentence of TFL, and with time you will be able to do so. Here are some examples of sentences of TFL:

1. $((P \& R) \rightarrow (S \rightarrow T))$
2. $(P \& (R \rightarrow (S \rightarrow T)))$
3. $((P \leftrightarrow \neg S) \vee \neg(T \leftrightarrow R))$
4. $((R \leftrightarrow T) \& \neg(P \vee (Q \vee \neg T)))$
5. $\neg(P \rightarrow \neg(R \vee (S \leftrightarrow T)))$
6. $((S \vee P) \& \neg(R \vee \neg\neg R))$

These, on the other hand, are **not** sentences of TFL because each violates one or more of 1 – 7:

1. $(P\neg \& R)$
2. $((R \& \vee S) \rightarrow Q)$
3. $(R \& Q \rightarrow)$
4. $((P\neg Q) \& (R \leftrightarrow T))$
5. $(P, Q \leftrightarrow T)$
6. $(P \& Q \& R)$

You will learn to recognize sentences of TFL more quickly if you write neatly and space the atomic sentences, logical operators, and parentheses as is shown in this textbook. Spaces are not actually part of TFL, and so technically, you don't need to use them. But just as you would never add or drop spaces when writing sentences in English, you should not add or drop spaces when using TFL.

6.3 The main logical operator

Setting aside the '¬' for a moment, each of the other logical operators, as you know, combine two sentences (which may, themselves, be composed of multiple sentences). The two sentences that are being combined by a logical operator comprise that operator's SCOPE.

Scope

For '&', '∨', '→', and '↔', the SCOPE of the logical operator is the two sentences that the logical operator is conjoining.

For '¬', the SCOPE is the sentence being negated by this operator.

So, the scope of the '&' in '$(P \,\&\, Q)$' is the 'P' and the 'Q'. And the scope of the '&' in '$((P \,\&\, Q) \rightarrow T)$' is still just the '$P$' and the '$Q$'. The scope of the '→' in '$((P \,\&\, Q) \rightarrow T)$', meanwhile, is '$((P \,\&\, Q)$' and '$T$'.

Here is a more complex example:

$$(((T \rightarrow P) \,\&\, R) \vee (S \leftrightarrow Q))$$

The scope of the '→' is 'T' and 'P'.
The scope of the '&' is '$(T \rightarrow P)$' and 'R'.
The scope of the '↔' is 'S' and 'Q'.
The scope of the '∨' is '$((T \rightarrow P) \,\&\, R)$' and '$(S \leftrightarrow Q)$'.

When the scope of a logical operator is the whole sentence (besides that operator itself), then that logical operator is the MAIN LOGICAL OPERATOR for the sentence. So, in the previous example, the '∨' is the main logical operator. This means that the sentence is a disjunction. (The subsentences are different kinds of sentences, but the whole sentence is a disjunction.)

Being able to identify the main logical operator is very important for what you will be learning in parts 3 and 4 of this textbook, and so when you see a sentence of TFL, you always want to determine the scope of each logical operator and the identify the main logical operator.

Here are some more examples:

1. $((P \,\&\, R) \rightarrow (\neg Q \,\&\, S))$ The main logical operator is the '$\rightarrow$'.
2. $(((S \vee N) \leftrightarrow Q) \,\&\, (T \rightarrow R))$ The main logical operator is the '&'.
3. $(((\neg D \vee N) \rightarrow R) \leftrightarrow T)$ The main logical operator is the'$\leftrightarrow$'.
4. $(P \,\&\, (T \leftrightarrow (Q \vee R)))$ The main logical operator is the '&'.
5. $((\neg E \vee F) \leftrightarrow G)$ The main logical operator is the '$\leftrightarrow$'.

Unlike the other operators, the '$\neg$', doesn't connect two sentences; it just negates a sentence (which may be composed of multiple subsentences). Hence, the scope of the '$\neg$' is the sentence that is being negated. For each of these examples, the scope of the '$\neg$' is the whole sentence:

6. $\neg Q$
7. $\neg(P \vee T)$
8. $\neg(R \vee (S \rightarrow N))$
9. $\neg((P \,\&\, R) \vee (S \rightarrow T))$

And since, in each of these examples, the scope of the '$\neg$' is the whole sentence, the '$\neg$' is the main logical operator in each case.

Once the main logical operator has been identified, we know what kind of sentence we have and what its components are. (This will be super important when we get to chapter 12.) If it is a conjunction, then part of the sentence will be one conjunct and the rest will be the other conjunct (and nothing will be left over). If it's a disjunction, then part of the sentence will be one disjunct and the rest will the other disjunct, again with nothing left over. If it's a conditional, then part of the sentence will be the antecedent and the rest will be the consequent. And if it's a negation, then the whole sentence (minus the 'not' itself) is being negated.

another way of indentifying the main logical operator

Alternatively, when the sentence includes the outermost brackets, you can find the main logical operator by using this method:

(1) If the first symbol in the sentence is '$\neg$', then that is the main logical operator.

(2) Otherwise, start counting the brackets by following one of these two procedures. (The open-bracket is '(' and the closed bracket is ')'.)

  (2a) Start from the left, and begin counting. For each open-bracket add 1, and for each closing-bracket, subtract 1. When your count is at exactly 1, the next operator you come to (*apart* from a '$\neg$') is the main logical operator.

  (2b) If starting at the left-side of the sentence doesn't seem to work, follow the same procedure, but begin at the far right and work left.

As we will discuss in the next section, in some cases, it is acceptable to omit the outermost brackets in a sentence of TFL. For instance, although it is not strictly allowable according to the rules given in §6.2, because it will not introduce any confusion or ambiguity, we can write '$(P \,\&\, R) \to Q$' instead of '$((P \,\&\, R) \to Q)$'. When the outermost brackets are dropped, we add (3) and (4) to our method.

(3) When the outermost brackes are omitted, (2a) and (2b) can still be used, but stop when your count gets to zero instead of 1.

(4) For sentences that contain two or more atomic sentences, if '$\neg$' is the main logical operator, then the outermost brackets have to be used. (When '$\neg$' is the main logical operator—as it is in this example: $\neg((P \,\&\, Q) \vee R)$—the '$\neg$' will be outside the outermost brackets.)

(5) For sentences that contain two or more atomic sentences, when the outermost brackets are omitted, (1) no longer applies and '$\neg$' won't be the main logical operator.

6.4 Using parentheses

Parentheses are required for any sentence of TFL containing two or more atomic sentences. For instance, even in a simple sentence such as '$(Q \,\&\, R)$', they are required. One reason for this is because the rules given in §6.2 require it. Those rules don't make a distinction between sentences containing only two atomic sentences and sentences containing more than two. They just tell us to use parentheses in any sentence containing a '&', '$\vee$', '$\rightarrow$', or '$\leftrightarrow$'. Another reason for using parentheses is that we might make '$(Q \,\&\, R)$' a sub-sentence in a more complex sentence. For example, we might want to negate '$(Q \,\&\, R)$', which would give us '$\neg(Q \,\&\, R)$'. If we just had '$Q \,\&\, R$' without the parentheses and put a '$\neg$' in front of it, we would have '$\neg Q \,\&\, R$', which has a different meaning than '$\neg(Q \,\&\, R)$'.

That said, there are some convenient conventions that we can use as long as we are careful. First, as long as the entire sentence is not about to become a sub-sentence, we can omit the sentence's *outermost* parentheses. Thus, we allow ourselves to write '$Q \,\&\, R$' instead of '$(Q \,\&\, R)$' when '$Q \,\&\, R$' is the whole sentence. We must remember, however, to put parentheses around it when we want to embed the sentence into a more complex one.

Second, it can be a bit difficult to stare at long sentences with many nested pairs of parentheses. To make things a bit easier on the eyes, we will allow ourselves to use square brackets, '[' and ']', in addition to rounded ones. So, there is no logical difference, for example, between '$(P \vee Q)$' and '$[P \vee Q]$'.

Combining these two conventions, we can rewrite this sentence:

$$(((H \rightarrow I) \vee (I \rightarrow H)) \,\&\, (J \vee K))$$

like this:

$$[(H \rightarrow I) \vee (I \rightarrow H)] \,\&\, (J \vee K)$$

The scope of each logical operator is now much easier to identify.

6.5 Object language and metalanguage

Consider these two sentences:

10. Justin Trudeau is a Prime Minister.
11. *Justin Trudeau* is the Canadian Prime Minister's name.

When we want to talk about the Prime Minister of Canada, which we are doing in sentence 10, we *use* his name. When we want to talk about the Prime Minister's name, as we are in sentence 11 we *mention* the name.

Since we are describing a formal language, truth-functional logic, we often *mention* sentences of TFL. For example, we are mentioning '$P \to Q$' in sentence 12:

12. '$P \to Q$' is a conditional.

And likewise, we are mentioning TFL sentences in 13 and 14.

13. 'D' is an atomic sentence of TFL.
14. '$\neg(\neg Q \lor R)$' is a sentence of TFL if '$(\neg Q \lor R)$' is a sentence of TFL.

When we talk about a language (i.e., mention it), the language about which we are talking is called the OBJECT LANGUAGE. The language that we use to talk about the object language is called the METALANGUAGE.

The object language that concerns us right now is TFL, but any language can be an object language. If, for instance, we want to talk about German, then German is the object language, and English—the language we are using to talk about German—is the metalanguage. Here is an example:

15. 'Schnee ist weiß' is a German sentence.

In sentence 15, we are saying that the clause at the beginning of the sentence is a German sentence.

Whenever we want to talk, in English, about some specific expression of TFL, we need to indicate that we are *mentioning* the expression, rather than *using* it. We can do this by using single quotation marks or italics (although italics are also sometimes used simply for emphasis and not to indicate that we are mentioning a term or sentence). In this textbook, we will also indicate that we are mentioning an expression by placing it centered on the page like this:

$$\neg(\neg Q \lor R)$$

6.6 Metavariables

Sometimes we refer to specific expressions of TFL like 'D' and '$\neg(\neg Q \lor R)$'. Other times, however, we want to say something about an arbitrary expression of TFL, not a specific one. To do this, we use these uppercase letters:

$$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$$

You probably noticed that we used these letters in our definition of a sentence of TFL in §6.2. For instance, this is one rule in that definition:

3. If A and B are sentences of TFL, then $(\mathsf{A} \,\&\, \mathsf{B})$ is a sentence of TFL.

We use 'A' and 'B' (and other capital letters in this font) when we want the letter to stand for any possible sentence of TFL. Hence, 'A' can stand for 'A' (or 'B') or for '$(P \lor Q)$' or '$((R \to T) \,\&\, \neg Q)$' or anything else.

'$\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$' are not actually part of TFL. Rather, they are part of the metalanguage—that is, English—that we use to talk about expressions of TFL.

> A METAVARIABLE is a variable in the metalanguage (i.e., English) that represents any sentence in our formal language of TFL. The symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \ldots$ are used for the metavariables.

6.7 Practice exercises

A. For each of the following, (a) is it a sentence of TFL, strictly speaking, and (b) is it a sentence of TFL, allowing for our relaxed bracketing conventions? If, by either of those standards, it is a sentence of TFL, then (c) what is the main logical operator?

1. (A)
2. $J_{374} \lor \neg J_{374}$

3. $\neg\neg\neg\neg F$
4. $\neg \,\&\, S$
5. $(G \,\&\, \neg G)$
6. $(A \to (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$
7. $[(Z \leftrightarrow S) \to W] \,\&\, [J \vee X]$
8. $(F \leftrightarrow \neg D \to J) \vee (C \,\&\, D)$

B. What is the scope of each connective in this sentence?

$$[(H \to I) \vee (I \to H)] \,\&\, (J \vee K)$$

6.8 Answers

A. For each of the following, (a) is it a sentence of TFL, strictly speaking, and (b) is it a sentence of TFL, allowing for our relaxed bracketing conventions? If, by either of those standards, it is a sentence of TFL, then (c) what is the main logical operator?

1. (A) — (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ — (a) no (b) yes (c) the '$\vee$'
3. $\neg\neg\neg\neg F$ — (a) yes (b) yes (c) the first '$\neg$'
4. $\neg \,\&\, S$ — (a) no (b) no
5. $(G \,\&\, \neg G)$ — (a) yes (b) yes (c) the '&'
6. $(A \to (A \,\&\, \neg F)) \vee (D \leftrightarrow E)$ — (a) no (b) yes (c) the '$\vee$'
7. $[(Z \leftrightarrow S) \to W] \,\&\, [J \vee X]$ — (a) no (b) yes (c) the '&'
8. $(F \leftrightarrow \neg D \to J) \vee (C \,\&\, D)$ — (a) no (b) no

B. $[(H \to I) \vee (I \to H)] \,\&\, (J \vee K)$

The scope of the left-most '$\to$' is '$(H \to I)$'.
The scope of the right-most '$\to$' is '$(I \to H)$'.
The scope of the left-most '$\vee$ is '$[(H \to I) \vee (I \to H)]$'.
The scope of the right-most '$\vee$' is '$(J \vee K)$'.
The scope of the '&' is the entire sentence, and so the '&' is the main logical operator and the sentence is a conjunction.

Part 3

Truth tables

7 Characteristic truth tables

7.1 A quick introduction to truth tables

Consider this sentence:

> Either the key is on the table, or Jane is on the train and the key is not on the table.

Now, ask yourself, when is this sentence true and when is it false?

(a) If *the key is on the table* is true, then the sentence will be true regardless of whether *Jane is on the train* is true or false.
(b) If *the key is on the table* is false, then the sentence will be true as long as *Jane is on the train* is true.
(c) But if both *the key is on the table* is false and *Jane is on the train* is false, then the sentence will be false.

We worked that out by thinking about the different possible scenarios, and the logic of 'or' and 'and'. An alternative, and a somewhat easier method, is to create a truth table. Truth tables tell us when a sentence is true or false, and, as we will see in chapters 9 and 10, they allow us to perform other analyses as well.

This is a truth table – in fact, it is the truth table for the above sentence:

J	K	K	$\vee$	$(J$	$\&$	$\neg$	$K)$
T	T	T	**T**	T	F	F	T
T	F	F	**T**	T	T	T	F
F	T	T	**T**	F	F	F	T
F	F	F	**F**	F	F	T	F

To begin thinking about truth tables, notice the following features of this table.

J	K	K	$\vee$	$(J$	$\&$	$\neg$	$K)$
T	T	T	**T**	T	F	F	T
T	F	F	**T**	T	T	T	F
F	T	T	**T**	F	F	F	T
F	F	F	**F**	F	F	T	F

Figure 7.1: In a truth table, the columns under the atomic sentences on the right are copied from the columns on the far left.

1. The sentence for which we are creating the truth table, in this case, '$K \vee (J \& \neg K)$', is at the top of the truth table, to the right of the vertical line.
2. The atomic sentences that are in '$K \vee (J \& \neg K)$' are at the top of the truth table to the left of the vertical line, and they are in alphabetical order there.
3. The 'T's and 'F's below the horizontal line stand for 'true' and 'false'. (See the definition of TRUTH VALUES at the beginning of the next section.)
4. Below the 'J' and 'K' on the left are the different possible combinations of true and false. Below the 'J', each 'T' is a scenario where J is true, and each 'F' is a scenario where J is false. (And likewise for K.) So, on the first line (right below the horizontal line), J is true, and K is true. On the second line, J is true, and K is false. On the third line, J is false, and K is true. And on the fourth line, J is false, and K is false.
5. The columns of Ts and Fs that are to the left of the vertical line are repeated on the right side of the table under each 'J' and 'K', respectively. (See figure 7.1.)

These are the basic features of every truth table. The task when creating a truth table is knowing how, and in what order, to fill in the columns under the logical operators. In this chapter, we will examine simple sentences containing only one logic operator. These are the "characteristic truth tables" for each logical operator. In the next chapter, we will explore how to create truth tables for more complex sentences.

7.2 The characteristic truth tables

You were introduced to five logical operators in chapter 5, and now we need to explain when sentences using each one are true and when they are false.

Truth values

Truth values are the logical values that a sentence can have: *true* and *false*.

Conjunction For any sentences A and B, the conjunction (A & B) is true if and only if both A and B are true. If one or both of A and B are false, then the sentence (A & B) is false. We can summarize this in the *characteristic truth table for conjunction*:

A	B	A	&	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	F	F

Looking at line 1, we see that, when A is true and B is true, there is a 'T' under the '&', which indicates that (A & B) is true. On line 2, meanwhile, A is true and B is false, which means that (A & B) is false.

Lines 3 and 4 represent the final two combinations of 'true' and 'false' for A and B. On line 3, A is false and B is true. When that is the case, (A & B) is false. And then on line 4, A and B are both false. In that scenario, (A & B) is, again, false.

Note that conjunction is *symmetrical*. The truth value for (A & B) is always the same as the truth value for (B & A).

Negation For any sentence A, if A is true, then ¬A is false. And likewise, if A is false, then ¬A is true. This is represented in the *characteristic truth table for negation*:

A	$\neg$	A
T	F	T
F	T	F

Disjunction Recall that '$\vee$' always represents the inclusive-or. So, for any sentences A and B, the disjunction $(\mathsf{A} \vee \mathsf{B})$ is true when A is true or B is true or both are true. The only instance when $(\mathsf{A} \vee \mathsf{B})$ is false is when both A and B are false. This is represented in the *characteristic truth table for disjunction*:

A	B	A	$\vee$	B
T	T	T	T	T
T	F	T	T	F
F	T	F	T	T
F	F	F	F	F

This is a good time to explain another point. We are, in this chapter, simply stipulating when each of these types of sentences are true and false. This amounts to a definition for each logical operator in TFL. (Thus, the meaning of '$\vee$' is what is given in the above truth table.) We have reasons for defining them these ways, and there is a consensus that these are the best definitions. But, in the end, these are the correct truth tables for each logical operator because these are the ways that we have chosen to set them.

Conceivably, we could say that $(\mathsf{A} \vee \mathsf{B})$ is false when A and B are both false *and* when A and B both are true. That would agree with the way that we, at least some of the time, use *or* in English. But that's not what we've chosen to do, and so the way that $(\mathsf{A} \vee \mathsf{B})$ is defined in the truth table above is going to apply from this point forward (and similarly for all of the other connectives).

Conditional The conditional is interesting and, for some, philosophically contentious. One way to think about the conditional is as rule: if the antecedent happens, then the consequent has to happen. So, for instance, take this conditional:

> If it is Wednesday, then I am on campus by 10:00 am.

This sentence is obviously true when (1) it is Wednesday and I am on campus by 10:00 am. Conversely, this sentence is false when (2) it is Wednesday, but I am not on campus by 10:00 am. (If that happens, the rule has been broken.) Those two scenarios are represented by lines 1 and 2 in the characteristic truth table for the conditional, which is as follows.

A	B	A	→	B
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

For the other two scenarios, we have to concentrate a bit.

(3) Our conditional is also true when it is not Wednesday (let's say it's Tuesday), but I'm on campus by 10:00 am. In this case, the rule *if it is Wednesday, then I am on campus by 10:00 am* hasn't been broken; it just doesn't apply. So, when the antecedent is false and the consequent is true, the conditional is true. That's represented by line 3 of the characteristic truth table for the conditional.

(4) Similarly, when it is not Wednesday and I am not on campus by 10:00 am, the rule hasn't been broken. It is still in force. It just hasn't been invoked at all. So even though the antecedent didn't happen and the consequent didn't happen, the conditional is still true. (In other words, let's say, it's Saturday and, at 10:00 am, I am at home in bed. It's false that 'it is Wednesday' and it's false that 'I am on campus by 10:00 am', but it's still true that *if it is Wednesday, then I am on campus by 10:00 am*.) This scenario is represented on line 4 of the characteristic truth table.

Those four scenarios are pretty straightforward. The conditional is philosophically contentious, however, because every conditional is not as simple as 'if it is Wednesday, then I am on campus by 10:00 am'. Take a conditional where the antecedent is always false: 'if the King of England is on the moon, then Mississippi State University is in Starkville.' This isn't much of a rule, but it is a conditional. And since, in our current world, the antecedent is false and the consequent is true, the sentence is true. Even stranger, consider this conditional: 'if the King of England is on the moon, then pigs can fly.' Now, the antecedent is always false and the consequent is always false (at least in our world), but, as is shown on line 4 of the characteristic truth table, the sentence is true.

Sometimes the truth values for the antecedent, the consequent, and the whole conditional make sense (as in our first example) and sometimes they seem odd. That has generated philosophical debate, but it actually does not present a problem for us. The conditional is precisely defined by its characteristic truth table. We, then, simply use that definition, and we don't have to make any decisions about whether a particular conditional is odd or should really be considered true or false.

Finally, notice that, unlike the conjunction and the disjunction, the conditional is *asymmetrical*. You cannot switch the antecedent and consequent without changing the meaning of the sentence. This is because $A \rightarrow B$ (e.g., 'if it is Wednesday, then I am on campus by 10:00 am') has a different truth table than $B \rightarrow A$ (e.g., 'if I am on campus by 10:00 am, then it is Wednesday').

Biconditional As we said in §5.5, the biconditional is equivalent to the conjunction of a conditional running in each direction—that is, to $(A \rightarrow B) \& (A \rightarrow B)$. Consequently, on every line where both $A \rightarrow B$ is true and $B \rightarrow A$ is true, $A \leftrightarrow B$ is true. On every line where either $A \rightarrow B$ is false or $B \rightarrow A$ is false, $A \leftrightarrow B$ is false. That yields the following characteristic truth table for the biconditional.

A	B	A	↔	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	T	F

7.3 The counterfactual conditional

As we have seen, in English, *or* can be used as the *inclusive or* or as the *exclusive or*, but in TFL we have only the *inclusive or* (pp. 44, 74). The conditional is in a somewhat similar situation. In the section on the conditional in this chapter, I used the example 'If it is Wednesday, then I am on campus by 10:00 am' to illustrate that the characteristic truth table for the conditional fits with the way that we treat conditional statements in English. This is only partially true. 'If it is Wednesday, then I am on campus by 10:00 am' is an *indicative conditional*, which is the type of conditional that is often, although not always, used in English.

> indicative conditional
>
> An INDICATIVE CONDITIONAL has indicative sentences as the antecedent and the consequent. Truth-functional logic uses the indicative conditional.
> An *indicative sentence* is one that expresses a fact – or expresses what seems to be the case, e.g., 'today is Wednesday' or 'Jones is at the train station'.

By way of introducing the other type of conditional (which is not a part of TFL), consider these two sentences:

1. If Theodore Roosevelt had won the 1912 presidential election, he would have become the first person to serve a third term as president of the United States.
2. If Theodore Roosevelt had won the 1912 presidential election, he would have transformed into an adult male moose.

One would assume (correctly) that the first sentence is true and the second one is false. But if we evaluate them using the characteristic truth table for the conditional (p. 75), we find that both are true. Both have false antecedents – Theodore Roosevelt did not win the 1912 presidential election – and both have false consequents. (He didn't become the first person to serve a third term, and he also didn't transform into a moose.)

So, what is going on here? These are *counterfactual conditionals* (also called *subjunctive conditionals*), not indicative conditionals. They ask us to treat a statement that is false (because it didn't happen) as if it had happened (i.e., to treat something as counter to the facts). Then, once we've made this move – in this case, we are imagining that Teddy Roosevelt did win the 1912 election—we can evaluate whether the consequent would have happened or not. For sentence 1, it would have, and so this sentence is true. (But notice that the reason why this sentence is true is different than the reason that we get from consulting the characteristic truth table for the conditional.) Sentence 2, on the other hand, is false because if Roosevelt had won, he would not have transformed into a moose.

8 Complete truth tables

In chapter 7, we examined the characteristic truth tables for the logical operators of TFL. The characteristic truth tables show us when a sentence with only one of the logical operators is true and when it is false. That, in effect, is a definition for each logical operator. Now that we have those definitions, we can investigate when other, more complex sentences are true and false – for instance, ones like '$(H \& I) \rightarrow H$' and '$(M \& (N \vee P))$', which we will go through in this chapter. Once we understand how to create truth tables, we can investigate other properties of sentences of TFL, which we will do in chapters 9 and 10.

Before we begin, we will define TRUTH-VALUE ASSIGNMENT.

> Truth-value assignment
>
> A *truth-value assignment* is any assignment of truth values to particular atomic sentences of TFL. **Each row of a truth table represents a possible truth-value assignment.** The entire truth table represents all possible truth-value assignments.

Thus, the truth table provides us with a way of finding the truth values of complex sentences on each possible truth-value assignment – that is, for every combination of 'true' and 'false' for every atomic sentence.

8.1 An example

Consider the sentence '$(H \& I) \rightarrow H$', which contains three atomic sentences, although only two different ones. We set up the truth table for this sentence by putting H and I on the left side of the vertical line and '$(H \& I) \rightarrow H$' on the right. (Although H appears twice in '$(H \& I) \rightarrow H$', we only need one H on the left.) Below the H and I on the left side, we put every combination of 'T' and 'F'.

Since we have two atomic sentences on the left, there are four combinations of true and false. For consistency, the Ts and Fs should always be listed this way: (a) in the column next to the vertical line, they alternate T, F, T, F; (b) in the next column (to the left), they alternate in pairs: T, T, F, F; and (c) if there are more than two atomic sentences, then more columns and more rows are needed, but the pattern remains the same. (See table 8.1 on p. 85.)

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T					
T	F					
F	T					
F	F					

Once the left side of the truth table is completed, we begin filling in the right side. First, we copy the truth values for the atomic sentences. For the H, that gives us this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T				T
T	F	T				T
F	T	F				F
F	F	F				F

Adding the truth values for I, we have this:

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T		T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

8.2 Truth tables and scope

Now , there are two columns that remain. The one under the '&' and the one under the '→'. You may have thought that a logical operator's scope was of only passing importance when we discussed it in §6.3. It is, however, extremely important for understanding how to complete truth tables for sentences that contain more than one logical operator.

To determine the order in which we complete a truth table, we follow this rule:

1. We can only fill in the column under a logical operator when the columns for everything in that logical operator's scope are complete.

In '$(H \,\&\, I) \to H$', the scope of the '&' is the 'H' and the 'I' inside the parentheses. Since the columns under the 'H' and the 'I' are complete, we can fill in the column under the '&'. To do this, we turn to the characteristic truth table for the conjunction. As is shown on p. 73, when 'H' and 'I' are both true, we put a 'T' below the '&'.

H	I	(H	&	I)	$\to$	H
T	T	T	**T**	T		T
T	F	T		F		T
F	T	F		T		F
F	F	F		F		F

On the second line, 'H' is true and 'I' is false. That means that '$(H \,\&\, I)$' is false, and so we put 'F' on the second line below the '&'.

H	I	(H	&	I)	$\to$	H
T	T	T	**T**	T		T
T	F	T	**F**	F		T
F	T	F		T		F
F	F	F		F		F

Following the characteristic truth table for conjunction, we fill in the truth values for the third and fourth lines, and that completes the column under the '&'.

H	I	(H	&	I)	$\to$	H
T	T	T	**T**	T		T
T	F	T	**F**	F		T
F	T	F	**F**	T		F
F	F	F	**F**	F		F

The scope of the '→' is the '(H & I)' and the 'H'. Since the columns under both are complete, we can finish the truth table by filling in the column under the '→'. But first, we need a second rule.

2. For any sentence or sub-sentence (other than an atomic sentence), the column under its main logical operator represents the truth value for the sentence. (For an atomic sentence, the column under the letter represents its truth value.)

Since the '&' is the main logical operator in '(H & I)', the column below the '&' is one of the columns that we must consult to fill in the column under the '→'. The other is the column below the 'H' after the '→'. Using the truth values in those two columns, we then refer to the characteristic truth table for the conditional on p. 75.

On the first line, '(H & I)' is true and 'H' is true, and so we put a 'T' beneath the '→'.

H	I	(H	&	I)	→	H
T	T	T	T	T	**T**	T
T	F	T	F	F		T
F	T	F	F	T		F
F	F	F	F	F		F

On the second row, '(H & I)' is false and 'H' is true. (That's the truth-value assignment given on the third line of the characteristic truth table for the conditional, not the second.) A conditional is true when the antecedent is false and the consequent is true, and so we put a 'T' in the second row beneath the '→'.

H	I	(H	&	I)	→	H
T	T	T	T	T	T	T
T	F	T	F	F	**T**	T
F	T	F	F	T		F
F	F	F	F	F		F

On the third and fourth rows, '(H & I)' is false and 'H' is false, and so again, we put 'T' below the '→' on each line. (On both of these lines, the antecedent is false and the consequent is false, and so these correspond to line four in the characteristic truth table for the conditional.)

H	I	(H	&	I)	$\rightarrow$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Since the '$\rightarrow$' is the main logical operator for '$(H \,\&\, I) \rightarrow H$', we've now determine the truth values for this sentence. The column of 'T's beneath the '$\rightarrow$' tells us that '$(H \,\&\, I) \rightarrow H$' is true regardless of the truth values of 'H' and 'I'. Those atomic sentences can be true or false in any combination, and the full sentence, '$(H \,\&\, I) \rightarrow H$', remains true. Since we have considered all four possible assignments of truth and falsity to 'H' and 'I', we can say that '$(H \,\&\, I) \rightarrow H$' is true on every *truth-value assignment*.

In the truth table for any sentence, the most important column is the one beneath the main logical operator for the sentence because this column tells us the truth value of the entire sentence. We have emphasized it in the last truth table above by putting this column in bold. When you work through truth tables yourself, you should similarly emphasize it with underlining or circling.

8.3 Building complete truth tables

A COMPLETE TRUTH TABLE has a line for every possible combination of *true* and *false* for the atomic sentences that compose the full sentence. Each line represents a *truth-value assignment*, and a complete truth table has a line for all of the different truth-value assignments.

The size of the complete truth table depends on the number of *different* atomic sentences in the table. The truth table for a TFL sentence that contains only one atomic sentence, perhaps repeated multiple times, needs only two rows. Here is an example of such a sentence:

$$((P \,\&\, \neg P) \rightarrow P)$$

The truth table for this—or any sentence containing only one atomic sentence—is only two lines because there are only two possibilities: 'P' can be true or it can be false.

P	((P	&	$\neg$	P)	$\rightarrow$	P)
T	T	F	F	T	**T**	T
F	F	F	T	F	**T**	F

As we have seen, for a sentence that contains two different atomic sentences, we need four lines for a complete truth table.

The complete truth table for a sentence that contains three different atomic sentences, meanwhile, needs eight lines, as shown right below. Notice that the 'T's and 'F's in the columns below 'N' and 'P' (on the left side) follow the same pattern as the example in the previous section. The column under the 'M', meanwhile, has four 'T's and then four 'F's.

M	N	P	M	&	(N	$\vee$	P)
T	T	T	T	**T**	T	T	T
T	T	F	T	**T**	T	T	F
T	F	T	T	**T**	F	T	T
T	F	F	T	**F**	F	F	F
F	T	T	F	**F**	T	T	T
F	T	F	F	**F**	T	T	F
F	F	T	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F

A complete truth table for a sentence that contains four different atomic sentences has to have 16 lines. If the sentence has five different letters, the truth table must have 32 lines. If it has six letters, it will have 64 lines, and so on. This is the rule for the number of rows that a truth table must have: for n different atomic sentences, the truth table for the sentence must have 2^n rows.

8.4 Some more examples

1. To create a truth table for '$(P \leftrightarrow Q) \rightarrow (P \vee Q)$', first, we fill in the columns below each P and Q. Next, since the truth values for

THE LEFT SIDE OF THE TRUTH TABLE

COLUMN	PATTERN
first (next to the vertical line)	T, F, T, F, ...
second	T, T, F, F, ...
third	T, T, T, T, F, F, F, F, ...
fourth	8 Ts, 8 Fs, ...
fifth	16 Ts, 16 Fs, ...

Table 8.1: Every truth table for the same sentence should be the same. To ensure that they are, the columns on the left side of the truth table should be filled in using the patterns given in this table. The first column is the one closest to the vertical line.

everything in the scope of the '↔' and the '∨' are complete, we can fill in the columns below those two logical operators (in either order).

P	Q	$($P$	$\leftrightarrow$	$Q)$	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T	T	T		T	T	T
T	F	T	F	F		T	T	F
F	T	F	F	T		F	T	T
F	F	F	T	F		F	F	F

Once we have those columns complete, we finish the truth table by filling in the column under the '→', which we do by looking at the column under the '↔' and the column under the '∨'.

P	Q	$(P$	$\leftrightarrow$	$Q)$	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	T	T	F
F	T	F	F	T	**T**	F	T	T
F	F	F	T	F	**F**	F	F	F

2. To make a truth table for '$P \& \neg Q$', after we have filled in the columns below the P and Q, we fill in the column under the ¬. To do that, we look at the column under the Q.

P	Q	$(P$	$\&$	$\neg$	$Q)$
T	T	T		**F**	T
T	F	T		**T**	F
F	T	F		**F**	T
F	F	F		**T**	F

Then, to complete the truth table, we fill in the column under the '&'—which we do by looking at the column under the P and the column under the '¬'.

P	Q	$(P$	$\&$	$\neg$	$Q)$
T	T	T	**F**	F	T
T	F	T	**T**	T	F
F	T	F	**F**	F	T
F	F	F	**F**	T	F

3. For '$\neg(P \rightarrow Q)$', after we have filled in the columns under the 'P' and the 'Q', we fill in the column under the '→'.

P	Q	$\neg$	$(P$	$\rightarrow$	$Q)$
T	T		T	T	T
T	F		T	F	F
F	T		F	T	T
F	F		F	T	F

Then, to complete the table, we fill in the column under the '¬'. To fill in that column, we look at the column under the '→'.

P	Q	$\neg$	$(P$	$\rightarrow$	$Q)$
T	T	**F**	T	T	T
T	F	**T**	T	F	F
F	T	**F**	F	T	T
F	F	**F**	F	T	F

4. In '$(P \& \neg Q) \lor Q$', the '$\lor$' is the main logical operator, and so we will fill in the column under it last. First (after we have filled in the columns under the 'P' and the 'Q'), we fill in the column under the '$\neg$'.

P	Q	(P	&	$\neg$	Q)	$\lor$	Q
T	T	T		F	T		T
T	F	T		T	F		F
F	T	F		F	T		T
F	F	F		T	F		F

Next, while looking at the column under the 'P' and under the '$\neg$', we fill in the column under the '&'.

P	Q	(P	&	$\neg$	Q)	$\lor$	Q
T	T	T	**F**	F	T		T
T	F	T	**T**	T	F		F
F	T	F	**F**	F	T		T
F	F	F	**F**	T	F		F

Then last, we fill in the column under the '$\lor$' while looking at the column under the '&' and under the 'Q'.

P	Q	(P	&	$\neg$	Q)	$\lor$	Q
T	T	T	F	F	T	**T**	T
T	F	T	T	T	F	**T**	F
F	T	F	F	F	T	**T**	T
F	F	F	F	T	F	**F**	F

8.5 Truth tables in Carnap

You should practice making truth tables on paper, but you also need to make them on the online site Carnap (https://carnap.io/). Using Carnap is pretty straightforward, and it's made easier because the left side of the truth table is completed for you. (See figure 8.1.) On the right side, below each atomic sentence and connective, you have the option of selecting a 'T' or an 'F'. (See figure 8.2.)

Most often (although not always), the problems in Carnap will be set up so that you will only be able to submit your answers when they are

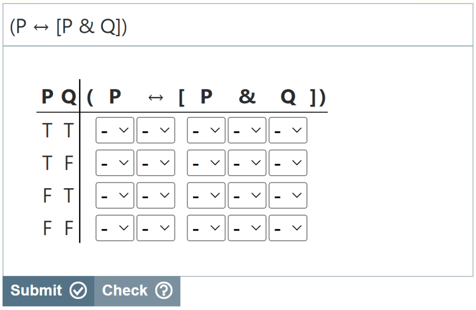


Figure 8.1

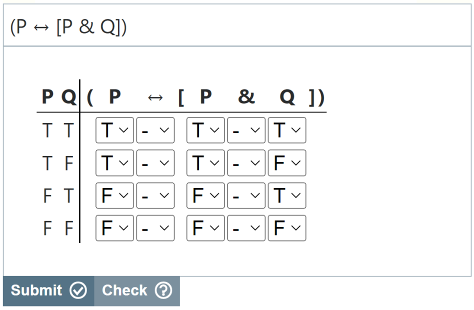


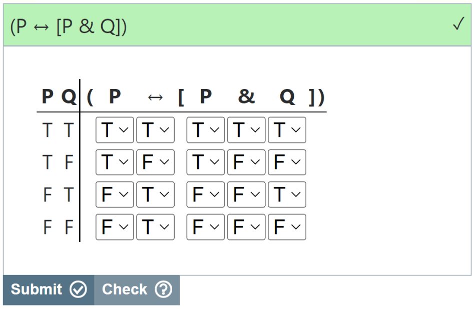
Figure 8.2

correct. At those times, once the truth table is complete, you will select 'Check'. Carnap will tell you "Success!" or "Something's not quite right." It is easy to make a mistake when filling in a truth table, and so if something is not quite right, then you have to inspect every truth value until you find the mistake. Then select 'Check' again. Once Carnap confirms that the truth table is correct, select 'Submit'. **Don't forget to submit after you complete every truth table correctly.**

8.6 Practice exercises

A. Make a complete truth table for each sentence.

1. $A \to A$
2. $C \to \neg C$
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$



P	Q	(P	↔	[P	&	Q])
T	T	T	T	T	T	T
T	F	T	F	T	F	F
F	T	F	T	F	F	T
F	F	F	T	F	F	F

Figure 8.3: A completed and verified truth table in Carnap.

4. $(A \to B) \vee (B \to A)$
5. $(A \& B) \to (B \vee A)$
6. $\neg(A \vee B) \leftrightarrow (\neg A \& \neg B)$
7. $\big[(A \& B) \& \neg(A \& B)\big] \& C$
8. $[(A \& B) \& C] \to B$
9. $\neg\big[(C \vee A) \vee B\big]$

B. Check whether each of these statements is true.

1. '$((A \& B) \& C)$' and '$(A \& (B \& C))$' have the same truth table
2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table
3. '$((A \vee B) \& C)$' and '$(A \vee (B \& C))$' do not have the same truth table
4. '$((A \to B) \to C)$' and '$(A \to (B \to C))$' do not have the same truth table
5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table

C. Make truth tables for the following sentences, and mark the column that represents the possible truth values for the whole sentence.

1. $\neg(N \leftrightarrow (P \to N))$
2. $\neg[(X \& Y) \vee (X \vee Y)]$
3. $(A \to B) \leftrightarrow (\neg B \leftrightarrow \neg A)$
4. $[C \leftrightarrow (D \vee E)] \& \neg C$
5. $\neg(C \& (B \& H)) \leftrightarrow (C \vee (B \vee H))$
6. $(D \& \neg D) \to G$
7. $(\neg P \vee \neg R) \leftrightarrow R$

8. $\neg\neg(\neg A \,\&\, \neg B)$
9. $[(D \,\&\, H) \rightarrow J] \rightarrow \neg(D \vee H)$
10. $\neg[(D \leftrightarrow F) \leftrightarrow G] \rightarrow (\neg D \,\&\, F)$

8.7 Answers

A.

1. $A \rightarrow A$

A	A	$\rightarrow$	A
T	T	**T**	T
F	F	**T**	F

2. $C \rightarrow \neg C$

C	C	$\rightarrow$	$\neg$	C
T	T	**F**	F	T
F	F	**T**	T	F

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	$\neg$	$(A$	$\leftrightarrow$	$\neg$	$B)$
T	T	T	T	T	**T**	T	T	F	F	T
T	F	T	F	F	**T**	F	T	T	T	F
F	T	F	F	T	**T**	F	F	T	F	T
F	F	F	T	F	**T**	T	F	F	T	F

4. $(A \rightarrow B) \vee (B \rightarrow A)$

A	B	$(A$	$\rightarrow$	$B)$	$\vee$	$(B$	$\rightarrow$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	T	T	**T**	T	F	F
F	F	F	T	F	**T**	F	T	F

5. $(A \,\&\, B) \to (B \vee A)$

A	B	$(A$	$\&$	$B)$	$\to$	$(B$	$\vee$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	F	T	**T**	T	T	F
F	F	F	F	F	**T**	F	F	F

6. $\neg(A \vee B) \leftrightarrow (\neg A \,\&\, \neg B)$

A	B	$\neg$	$(A$	$\vee$	$B)$	$\leftrightarrow$	$(\neg$	A	$\&$	$\neg$	$B)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

7. $[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)] \,\&\, C$

A	B	C	$[(A$	$\&$	$B)$	$\&$	$\neg$	$(A$	$\&$	$B)]$	$\&$	C
T	T	T	T	T	T	F	F	T	T	T	**F**	T
T	T	F	T	T	T	F	F	T	T	T	**F**	F
T	F	T	T	F	F	F	T	T	F	F	**F**	T
T	F	F	T	F	F	F	T	T	F	F	**F**	F
F	T	T	F	F	T	F	T	F	F	T	**F**	T
F	T	F	F	F	T	F	T	F	F	T	**F**	F
F	F	T	F	F	F	F	T	F	F	F	**F**	T
F	F	F	F	F	F	F	T	F	F	F	**F**	F

8. $[(A \,\&\, B) \,\&\, C] \rightarrow B$

A	B	C	$[(A$	$\&$	$B)$	$\&$	$C]$	$\rightarrow$	B
T	T	T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	F	F	**T**	T
T	F	T	T	F	F	F	T	**T**	F
T	F	F	T	F	F	F	F	**T**	F
F	T	T	F	F	T	F	T	**T**	T
F	T	F	F	F	T	F	F	**T**	T
F	F	T	F	F	F	F	T	**T**	F
F	F	F	F	F	F	F	F	**T**	F

9. $\neg[(C \vee A) \vee B]$

A	B	C	$\neg[($	C	$\vee$	$A)$	$\vee$	$B]$
T	T	T	**F**	T	T	T	T	T
T	T	F	**F**	F	T	T	T	T
T	F	T	**F**	T	T	T	T	F
T	F	F	**F**	F	T	T	T	F
F	T	T	**F**	T	T	F	T	T
F	T	F	**F**	F	F	F	T	T
F	F	T	**F**	T	T	F	T	F
F	F	F	**T**	F	F	F	F	F

B.

1. '$((A \& B) \& C)$' and '$(A \& (B \& C))$' have the same truth table

A	B	C	$(A$	$\&$	$B)$	$\&$	C	A	$\&$	$(B$	$\&$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**F**	F	T	**F**	F	F	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table

A	B	C	$(A$	$\vee$	$B)$	$\vee$	C	A	$\vee$	$(B$	$\vee$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**T**	F	T	**T**	T	T	F
T	F	T	T	T	F	**T**	T	T	**T**	F	T	T
T	F	F	T	T	F	**T**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**T**	F	F	**T**	T	T	F
F	F	T	F	F	F	**T**	T	F	**T**	F	T	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

3. '$((A \vee B) \,\&\, C)$' and '$(A \vee (B \,\&\, C))$' do not have the same truth table

A	B	C	$(A$	$\vee$	$B)$	$\&$	C	A	$\vee$	$(B$	$\&$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**T**	T	F	F
T	F	T	T	T	F	**T**	T	T	**T**	F	F	T
T	F	F	T	T	F	**F**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**F**	T	F	F
F	F	T	F	F	F	**F**	T	F	**F**	F	F	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

4. '$((A \rightarrow B) \rightarrow C)$' and '$(A \rightarrow (B \rightarrow C))$' do not have the same truth table

A	B	C	$(A$	$\rightarrow$	$B)$	$\rightarrow$	C	A	$\rightarrow$	$(B$	$\rightarrow$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**T**	T	T	**T**	F	T	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**F**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	T	T
F	F	F	F	T	F	**F**	F	F	**T**	F	T	F

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table.

A	B	C	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	C	A	$\leftrightarrow$	$(B$	$\leftrightarrow$	$C)$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**F**	F	T	**F**	T	F	F
T	F	T	T	F	F	**F**	T	T	**F**	F	F	T
T	F	F	T	F	F	**T**	F	T	**T**	F	T	F
F	T	T	F	F	T	**F**	T	F	**F**	T	T	T
F	T	F	F	F	T	**T**	F	F	**T**	T	F	F
F	F	T	F	T	F	**T**	T	F	**T**	F	F	T
F	F	F	F	T	F	**F**	F	F	**F**	F	T	F

C.

1. $\neg(N \leftrightarrow (P \rightarrow N))$

N	P	$\neg$	$(N$	$\leftrightarrow$	$(P$	$\rightarrow$	$N))$
T	T	**F**	T	T	T	T	T
T	F	**F**	T	T	F	T	T
F	T	**F**	F	T	T	F	F
F	F	**T**	F	F	F	T	F

2. $\neg[(X \,\&\, Y) \vee (X \vee Y)]$

X	Y	$\neg$	[(X	&	Y)	$\vee$	(X	$\vee$	Y)]
T	T	**F**	T	T	T	T	T	T	T
T	F	**F**	T	F	F	T	T	T	F
F	T	**F**	F	F	T	T	F	T	T
F	F	**T**	F	F	F	F	F	F	F

3. $(A \rightarrow B) \leftrightarrow (\neg B \leftrightarrow \neg A)$

A	B	(A	$\rightarrow$	B)	$\leftrightarrow$	($\neg$	B	$\leftrightarrow$	$\neg$	A)
T	T	T	T	T	**T**	F	T	T	F	T
T	F	T	F	F	**T**	T	F	F	F	T
F	T	F	T	T	**F**	F	T	F	T	F
F	F	F	T	F	**T**	T	F	T	T	F

4. $[C \leftrightarrow (D \vee E)] \,\&\, \neg C$

C	D	E	[C	$\leftrightarrow$	(D	$\vee$	E)]	&	$\neg$	C
T	T	T	T	T	T	T	T	**F**	F	T
T	T	F	T	T	T	T	F	**F**	F	T
T	F	T	T	T	F	T	T	**F**	F	T
T	F	F	T	F	F	F	F	**F**	F	T
F	T	T	F	F	T	T	T	**F**	T	F
F	T	F	F	F	T	T	F	**F**	T	F
F	F	T	F	F	F	T	T	**F**	T	F
F	F	F	F	T	F	F	F	**T**	T	F

5. $\neg(C \,\&\, (B \,\&\, H)) \leftrightarrow (C \vee (B \vee H))$

C	B	H	¬	(C	&	(B	&	H))	↔	(C	∨	(B	∨	H))
T	T	T	F	T	T	T	T	T	**F**	T	T	T	T	T
T	T	F	T	T	F	T	F	F	**T**	T	T	T	T	F
T	F	T	T	T	F	F	F	T	**T**	T	T	F	T	T
T	F	F	T	T	F	F	F	F	**T**	T	T	F	F	F
F	T	T	T	F	F	T	T	T	**T**	F	T	T	T	T
F	T	F	T	F	F	T	F	F	**T**	F	T	T	T	F
F	F	T	T	F	F	F	F	T	**T**	F	T	F	T	T
F	F	F	T	F	F	F	F	F	**F**	F	F	F	F	F

6. $(D \,\&\, \neg D) \rightarrow G$

D	G	(D	&	¬	D)	→	G
T	T	T	F	F	T	**T**	T
T	F	T	F	F	T	**T**	F
F	T	F	F	T	F	**T**	T
F	F	F	F	T	F	**T**	F

7. $(\neg P \vee \neg R) \leftrightarrow R$

P	R	(¬	P	∨	¬	R)	↔	R
T	T	F	T	F	F	T	**F**	T
T	F	F	T	T	T	F	**F**	F
F	T	T	F	T	F	T	**T**	T
F	F	T	F	T	T	F	**F**	F

8. $\neg\neg(\neg A \,\&\, \neg B)$

A	B	¬	¬	(¬	A	&	¬	B)
T	T	**F**	T	F	T	F	F	T
T	F	**F**	T	F	T	F	T	F
F	T	**F**	T	T	F	F	F	T
F	F	**T**	F	T	F	T	T	F

Figure 8.4: Mistakes happen. Try to minimize them by not rushing and always consulting the characteristic truth tables in chapter 7. Peanuts © 1969

9. $[(D \& H) \rightarrow J] \rightarrow \neg(D \vee H)$

D	*H*	*J*	[(D	&	H)	→	J]	→	¬	(D	∨	H)
T	T	T	T	T	T	T	T	**F**	F	T	T	T
T	T	F	T	T	T	F	F	**T**	F	T	T	T
T	F	T	T	F	F	T	T	**F**	F	T	T	F
T	F	F	T	F	F	T	F	**F**	F	T	T	F
F	T	T	F	F	T	T	T	**F**	F	F	T	T
F	T	F	F	F	T	T	F	**F**	F	F	T	T
F	F	T	F	F	F	T	T	**T**	T	F	F	F
F	F	F	F	F	F	T	F	**T**	T	F	F	F

10. $\neg[(D \leftrightarrow F) \leftrightarrow G] \rightarrow (\neg D \& F)$

D	*F*	*G*	¬	[(D	↔	F)	↔	G]	→	(¬	D	&	F)
T	T	T	F	T	T	T	T	T	**T**	F	T	F	T
T	T	F	T	T	T	T	F	F	**F**	F	T	F	T
T	F	T	T	T	F	F	F	T	**F**	F	T	F	F
T	F	F	F	T	F	F	T	F	**T**	F	T	F	F
F	T	T	T	F	F	T	F	T	**T**	T	F	T	T
F	T	F	F	F	F	T	T	F	**T**	T	F	T	T
F	F	T	F	F	T	F	T	T	**T**	T	F	F	F
F	F	F	T	F	T	F	F	F	**F**	T	F	F	F

9 Six concepts

As we did in the previous chapter, we begin with the definition of TRUTH-VALUE ASSIGNMENT.

truth-value assignment

A *truth-value assignment* is any assignment of truth values to particular atomic sentences of TFL. Each row of a truth table represents a possible truth-value assignment. The entire truth table represents all possible truth-value assignments.

Let's say that we are going to create a truth table for '$P \lor \neg Q$'. (See figure 9.1.) On the first line of the truth table, you may recall, we make P = 'true' and Q = 'true'. That is one truth-value assignment. On the second line, we make P = 'true' and Q = 'false'. That is another truth-value assignment. The assignments of 'true' and 'false' to 'P' and 'Q' on lines 3 and 4, then, are the remaining possible truth-value assignments, when we have a sentence containing only two atomic sentences.

In the previous chapter, we used truth tables to determine—for each possible truth-value assignment—the truth value of any TFL sentence.

This is one truth-value assignment.

P	Q	P	$\lor$	$\neg$	Q
T	T	T	**T**	F	T
T	F	T	**T**	T	F
F	T	F	**F**	F	T
F	F	F	**T**	T	F

Figure 9.1: P = 'true' and Q = 'true' is one truth-value assignment. On this truth-value assignment, '$P \lor \neg Q$' is true. P = 'true' and Q = 'false' is another truth-value assignment. '$P \lor \neg Q$' is also true on this truth-value assignment. P = 'false' and Q = 'true' (line 3), and P = 'false' and Q = 'false' (line 4) are the other two truth-value assignments in this truth table.

In this chapter, we will extend this type of analysis. We will examine six properties that apply (or may apply) to either single TFL sentences (*tautology*, *contradiction*, and *contingent*) or sets of TFL sentences (*equivalent*, *jointly consistent*, and *jointly inconsistent*). For each, we use a truth table to determine which property the sentence or set of sentences has.

9.1 Tautologies and contradictions

In §3.3, we said that a *necessary truth* is a sentence that must be true, a *necessary falsehood* is a sentence that must be false, and a sentence that is neither a necessary truth or a necessary falsehood is *contingent*. The first two, *necessary truth* and *necessary falsehood*, have surrogates in TFL. We will start with the surrogate for necessary truth.

Tautology

A sentence of TFL is a TAUTOLOGY if and only if it is true on every truth-value assignment.

We can determine whether a sentence is a tautology using a truth table. If the sentence is true on every line of a complete truth table (that is, if there is a 'T' on every line under the main connective), then it is true on every truth-value assignment. And if it is true on every truth-value assignment, it is a tautology. The example from §8.1, '$(H \& I) \rightarrow H$', for instance, is a tautology.

H	I	$(H$	$\&$	$I)$	$\rightarrow$	H
T	T	T	T	T	**T**	T
T	F	T	F	F	**T**	T
F	T	F	F	T	**T**	F
F	F	F	F	F	**T**	F

Tautology is only a surrogate, however, for *necessary truth*. There are some necessary truths that we cannot adequately symbolize in TFL. An example is '$2 + 2 = 4$'. This *must* be true, but if we try to symbolize it in TFL, the best we can offer is an atomic sentence, perhaps,

F: $2 + 2 = 4$

But an atomic sentence by itself cannot be a tautology. (To see this, try making a truth table for just 'F'.) Still, if we can adequately symbolize some English sentence as a TFL sentence, and that TFL sentence is a tautology, then the English sentence expresses a necessary truth.

We have a similar surrogate for *necessary falsehood*.

Contradiction

A sentence of TFL is a CONTRADICTION if and only if it is false on every truth-value assignment.

Again, we can determine whether a sentence is a contradiction with a truth table. If the sentence is false on every line of a complete truth table, then it is false on every truth-value assignment, and so it is a contradiction. The standard example of a contradiction is '$P \,\&\, \neg P$'. Since we have only one letter in this sentence, it is only a two line truth table, but on each line, the sentence is false.

P	P	$\&$	$\neg$	P
T	T	**F**	F	T
F	F	**F**	T	F

Similarly, although its truth table has four lines, '$(P \vee Q) \leftrightarrow (\neg P \,\&\, \neg Q)$' is a contradiction.

P	Q	$(P$	$\vee$	$Q)$	$\leftrightarrow$	$(\neg$	P	$\&$	$\neg$	$Q)$
T	T	T	T	T	**F**	F	T	F	F	T
T	F	T	T	F	**F**	F	T	F	T	F
F	T	F	T	T	**F**	T	F	F	F	T
F	F	F	F	F	**F**	T	F	T	T	F

In §3.3, we defined CONTINGENT as "a sentence that is capable of being true and capable of being false (in different circumstances, of course)." A truth table, then, provides us with those different circumstances.

Contingent

A sentence that is true on at least one truth-value assignment and false on at least one truth-value assignment is contingent.

Or, we can also say: any sentence that is neither a tautology nor a contradiction is contingent.

'$\neg(P \vee Q)$', for instance, is contingent.

P	Q	$\neg$	$(P$	$\vee$	$Q)$
T	T	**F**	T	T	T
T	F	**F**	T	T	F
F	T	**F**	F	T	T
F	F	**T**	F	F	F

9.2 Equivalence

There are several possible logical relationships that can exist between two or more sentences of TFL. We we examine three relationships, and we will focus on pairs of sentences. The first logical relationship is EQUIVALENCE.

Equivalent

A and B are EQUIVALENT if and only if, for every truth-value assignment, their truth values agree (that is, if and only if there is no truth-value assignment for which they have opposite truth values).

Equivalently, if $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology, then A and B are EQUIVALENT.

Recall from 6.6, that A stands for any possible sentence of TFL (as do B, C, D, etc.). Hence, 'A' can stand for '$P \vee Q$' or '$(P \leftrightarrow \neg R)\ \&\ T$' or anything else.

Consider the sentences '$\neg(P \vee Q)$' and '$\neg P\ \&\ \neg Q$'. Are they equivalent? To find out, we construct a truth table containing both sentences.

P	Q	$\neg$	$(P$	$\vee$	$Q)$	$\neg$	P	$\&$	$\neg$	Q
T	T	**F**	T	T	T	F	T	**F**	F	T
T	F	**F**	T	T	F	F	T	**F**	T	F
F	T	**F**	F	T	T	T	F	**F**	F	T
F	F	**T**	F	F	F	T	F	**T**	T	F

Looking at the columns for the main logical operators ('¬' for the first sentence, '&' for the second), we see that on the first three rows, both sentences are false. On the final row, both are true. Since they match on every row—that is, on every truth-value assignment for '*P*' and '*Q*'—the two sentences are equivalent.

9.3 Consistency

In §3.1, we said that sentences are *jointly possible* if and only if it is possible for all of them to be true at once. The surrogate for this concept in TFL is JOINTLY CONSISTENT.

Jointly consistent

A and B are JOINTLY CONSISTENT if and only if there is some truth-value assignment that makes them both true *and* they are not equivalent.

Equivalently, if

(1) there is at least one truth-value assignment that makes (A&B) true, and
(2) (A ↔ B) is *not* a tautology,

then A and B are JOINTLY CONSISTENT.

The requirement that the two sentences not be equivalent is not always included, but we will distinguish between sentences that are jointly consistent from those that are equivalent.

This was one of the examples in §3.1:

G1. There are at least four giraffes at the wild animal park.
G2. There are exactly seven gorillas at the wild animal park.

These are jointly possible because it is possible for them both to be true at the same time. It takes nothing away from their joint possibility that they can also be false at the same time or one can be false while the other is true. Applying that same observation to *jointly consistent*, all we need is one line where both sentences are true. (More than one line is fine also, although the truth values for the two sentences shouldn't match on every line. If they do, then the sentences are equivalent.) '$(P \vee Q)$' and '$(P \,\&\, \neg Q)$' have one line where they are both true, and so they are jointly consistent.

P	Q	P	$\vee$	Q	P	$\&$	$\neg$	Q
T	T	T	**T**	T	T	**F**	F	T
T	F	T	**T**	F	T	**T**	T	F
F	T	F	**T**	T	F	**F**	F	T
F	F	F	**F**	F	F	**F**	T	F

And finally, in §3.1, we also said that sentences are *jointly impossible* if and only if it is *not* possible for all of them to be true at once. The surrogate for this concept in TFL is JOINTLY INCONSISTENT.

Jointly inconsistent

A and B are JOINTLY INCONSISTENT if and only if there is no truth-value assignment that makes them both true.

There are three ways that two sentences can be jointly inconsistent.

(1) One each line, the truth value for one sentence is 'T' and the truth value for the other sentence is 'F'. For instance, the truth values for '$P \vee Q$' and '$\neg P \,\&\, \neg Q$' never match. On each line, one is true and the other is false. Hence, two sentences are jointly inconsistent in this way when $\neg(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology.

P	Q	P	$\vee$	Q	$\neg$	P	$\&$	$\neg$	Q
T	T	T	**T**	T	F	T	**F**	F	T
T	F	T	**T**	F	F	T	**F**	T	F
F	T	F	**T**	T	T	F	**F**	F	T
F	F	F	**F**	F	T	F	**T**	T	F

(2) When the truth value for one sentence is 'T', then the truth value for the other sentence is 'F', but both sentences can be false at the same time. For example, '$\neg(\neg P \vee Q)$' and '$(\neg P \,\&\, \neg Q)$' are never both true on the same line, but they are false on the same line. For two sentences that are jointly inconsistent in this way, this criterion is satisfied: $\neg(\mathsf{A} \,\&\, \mathsf{B})$ is a tautology.

P	Q	$\neg$	($\neg$	P	$\vee$	Q)	$\neg$	P	&	$\neg$	Q
T	T	**F**	F	T	T	T	F	T	**F**	F	T
T	F	**T**	F	T	F	F	F	T	**F**	T	F
F	T	**F**	T	F	T	T	T	F	**F**	F	T
F	F	**F**	T	F	T	F	T	F	**T**	T	F

(3) Both sentences are false on every line. For example, the truth values for '$\neg P \,\&\, P$' and '$\neg Q \,\&\, Q$' are always the same. On each line, both sentences are false. So, for sentences that are jointly inconsistent in this way, both of these criteria must be satisfied: $\neg(\mathsf{A} \,\&\, \mathsf{B})$ is a tautology and $(\mathsf{A} \leftrightarrow \mathsf{B})$ is a tautology. (And the latter, recall, means that these sentences are equivalent, and so here *jointly inconsistent* and *equivalent* overlap.)

P	Q	$\neg$	P	&	P	$\neg$	Q	&	Q
T	T	F	T	**F**	T	F	T	**F**	T
T	F	F	T	**F**	T	T	F	**F**	F
F	T	T	F	**F**	F	F	T	**F**	T
F	F	T	F	**F**	F	T	F	**F**	F

9.4 Practice exercises

A. Revisit your answers to the exercises in part A of chapter 8, and determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

B. Create a truth table for each sentence, and then determine whether the sentence is a **tautology**, a **contradiction**, or is **contingent**.

1. $\neg B \,\&\, B$
2. $\neg D \vee D$
3. $(A \,\&\, B) \vee (B \,\&\, A)$
4. $\neg[A \rightarrow (B \rightarrow A)]$
5. $A \leftrightarrow [A \rightarrow (B \,\&\, \neg B)]$
6. $[(A \,\&\, B) \leftrightarrow B] \rightarrow (A \rightarrow B)$

C. For each set of sentences, create a truth table and then determine whether the sentences are **jointly consistent** or **jointly inconsistent**.

1. $A \rightarrow A, \neg A \rightarrow \neg A, A \,\&\, A, A \vee A$
2. $A \vee B, A \rightarrow C, B \rightarrow C$
3. $B \,\&\, (C \vee A), A \rightarrow B, \neg(B \vee C)$
4. $A \leftrightarrow (B \vee C), C \rightarrow \neg A, A \rightarrow \neg B$
5. $A \,\&\, \neg B, \neg(A \rightarrow B), B \rightarrow A$
6. $A \vee B, A \rightarrow \neg A, B \rightarrow \neg B$
7. $\neg(\neg A \vee B), A \rightarrow \neg C, A \rightarrow (B \rightarrow C)$
8. $A \rightarrow B, A \,\&\, \neg B$
9. $A \rightarrow (B \rightarrow C), (A \rightarrow B) \rightarrow C, A \rightarrow C$
10. $\neg B, A \rightarrow B, A$
11. $\neg(A \vee B), A \leftrightarrow B, B \rightarrow A$
12. $A \vee B, \neg B, \neg B \rightarrow \neg A$
13. $A \leftrightarrow B, \neg B \vee \neg A, A \rightarrow B$
14. $(A \vee B) \vee C, \neg A \vee \neg B, \neg C \vee \neg B$

D. For each pair of sentences, create a truth table and then determine whether the sentences are **equivalent** or are not.

1. A and $\neg A$
2. $A \& \neg A$ and $\neg B \leftrightarrow B$
3. $[(A \lor B) \lor C]$ and $[A \lor (B \lor C)]$
4. $A \lor (B \& C)$ and $(A \lor B) \& (A \lor C)$
5. $[A \& (A \lor B)] \rightarrow B$ and $A \rightarrow B$
6. $A \rightarrow A$ and $A \leftrightarrow A$
7. $\neg(A \rightarrow B)$ and $\neg A \rightarrow \neg B$
8. $A \lor B$ and $\neg A \rightarrow B$
9. $(A \rightarrow B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$
10. $A \leftrightarrow (B \leftrightarrow C)$ and $A \& (B \& C)$

E.

1. Suppose that A and B are equivalent. What can you say about $\mathsf{A} \leftrightarrow \mathsf{B}$?
2. Suppose that A and B are jointly inconsistent. What can you say about $(\mathsf{A} \& \mathsf{B})$?
3. Suppose that A and B are equivalent. What can you say about $(\mathsf{A} \lor \mathsf{B})$?
4. Suppose that A and B are *not* equivalent. What can you say about $(\mathsf{A} \lor \mathsf{B})$?
5. Consider this principle:

   > Suppose A and B are equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

   Is this principle correct? Explain your answer.

9.5 Answers

A. From chapter 8

1. $A \rightarrow A$ tautology
2. $C \rightarrow \neg C$ contingent
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ tautology
4. $(A \rightarrow B) \vee (B \rightarrow A)$ tautology
5. $(A \,\&\, B) \rightarrow (B \vee A)$ tautology
6. $\neg(A \vee B) \leftrightarrow (\neg A \,\&\, \neg B)$ tautology
7. $\big[(A \,\&\, B) \,\&\, \neg(A \,\&\, B)\big] \,\&\, C$ contradiction
8. $[(A \,\&\, B) \,\&\, C] \rightarrow B$ tautology
9. $\neg\big[(C \vee A) \vee B\big]$ contingent

B. Use a truth table to determine whether each sentence is a tautology, a contradiction, or contingent.

1. $\neg B \,\&\, B$ is a contradiction.

B	$\neg$	B	$\&$	B
T	F	T	**F**	T
F	T	F	**F**	F

2. $\neg D \vee D$ is a tautology.

D	$\neg$	D	$\vee$	D
T	F	T	**T**	T
F	T	F	**T**	F

3. $(A \,\&\, B) \vee (B \,\&\, A)$ is contingent.

A	B	$(A$	$\&$	$B)$	$\vee$	$(B$	$\&$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**F**	F	F	T
F	T	F	F	T	**F**	T	F	F
F	F	F	F	F	**F**	F	F	F

4. $\neg[A \to (B \to A)]$ is a contradiction.

A	B	$\neg$	$[A$	$\to$	$(B$	$\to$	$A)]$
T	T	**F**	T	T	T	T	T
T	F	**F**	T	T	F	T	T
F	T	**F**	F	T	T	F	F
F	F	**F**	F	T	F	T	F

5. $A \leftrightarrow [A \to (B \,\&\, \neg B)]$ is a contradiction.

A	B	A	$\leftrightarrow$	$[A$	$\to$	$(B$	$\&$	$\neg$	$B)]$
T	T	T	**F**	T	F	T	F	F	T
T	F	T	**F**	T	F	F	F	T	F
F	T	F	**F**	F	T	T	F	F	T
F	F	F	**F**	F	T	F	F	T	F

6. $[(A \,\&\, B) \leftrightarrow B] \to (A \to B)$ is contingent.

A	B	$[(A$	$\&$	$B)$	$\leftrightarrow$	$B]$	$\to$	$(A$	$\to$	$B)$
T	T	T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	T	F	**F**	T	F	F
F	T	F	F	T	F	T	**T**	F	T	T
F	F	F	F	F	T	F	**T**	F	T	F

C.

Use a truth table to determine whether the sentences in each set are **jointly consistent** or **jointly inconsistent**.

1. $A \to A, \neg A \to \neg A, A \,\&\, A, A \vee A$
   These sentences are jointly consistent. (See line 1.)

A	A	$\to$	A	$\neg$	A	$\to$	$\neg$	A	A	$\&$	A	A	$\vee$	A
T	T	**T**	T	F	T	**T**	F	T	T	**T**	T	T	**T**	T
F	F	**T**	F	T	F	**T**	T	F	F	**F**	F	F	**F**	F

2. $A \vee B, A \rightarrow C, B \rightarrow C$
   These sentences are jointly consistent. (See line 1.)

A	B	C	A	$\vee$	B	A	$\rightarrow$	C	B	$\rightarrow$	C
T	T	T	T	**T**	T	T	**T**	T	T	**T**	T
T	T	F	T	**T**	T	T	**F**	F	T	**F**	F
T	F	T	T	**T**	T	T	**T**	T	F	**T**	T
T	F	F	T	**T**	F	T	**F**	F	F	**T**	F
F	T	T	F	**T**	F	F	**T**	T	T	**T**	T
F	T	F	F	**T**	T	F	**T**	F	T	**F**	F
F	F	T	F	**F**	F	F	**T**	T	F	**T**	T
F	F	F	F	**F**	F	F	**T**	F	F	**T**	F

3. $B \,\&\, (C \vee A), A \rightarrow B, \neg(B \vee C)$
   These sentences are jointly inconsistent.

A	B	C	B	&	$(C$	$\vee$	$A)$	A	$\rightarrow$	B	$\neg$	$(B$	$\vee$	$C)$
T	T	T	T	**T**	T	T	T	T	**T**	T	**F**	T	T	T
T	T	F	T	**T**	F	T	T	T	**T**	T	**F**	T	T	F
T	F	T	F	**F**	T	T	T	T	**F**	F	**F**	F	T	T
T	F	F	F	**F**	F	T	T	T	**F**	F	**T**	F	F	F
F	T	T	T	**T**	T	T	F	F	**T**	T	**F**	T	T	T
F	T	F	T	**F**	F	F	F	F	**T**	T	**F**	T	T	F
F	F	T	F	**F**	T	T	F	F	**T**	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F	F	**T**	F	**T**	F	F	F

4. $A \leftrightarrow (B \vee C), C \rightarrow \neg A, A \rightarrow \neg B$
These sentences are jointly consistent. (See line 8.)

A	B	C	A	$\leftrightarrow$	$(B$	$\vee$	$C)$	C	$\rightarrow$	$\neg$	A	A	$\rightarrow$	$\neg$	B
T	T	T	T	**T**	T	T	T	T	**F**	F	T	T	**F**	F	T
T	T	F	T	**T**	T	T	F	F	**T**	F	T	T	**F**	F	T
T	F	T	T	**T**	F	T	T	T	**F**	F	T	T	**T**	T	F
T	F	F	T	**F**	F	F	F	F	**T**	F	T	T	**T**	T	F
F	T	T	F	**F**	T	T	T	T	**T**	T	F	F	**T**	F	T
F	T	F	F	**F**	T	T	F	F	**T**	T	F	F	**T**	F	T
F	F	T	F	**F**	F	T	T	T	**T**	T	F	F	**T**	T	F
F	F	F	F	**T**	F	F	F	F	**T**	T	F	F	**T**	T	F

5. $A \,\&\, \neg B, \neg(A \rightarrow B), B \rightarrow A$
These sentences are jointly consistent. (See line 2.)

A	B	A	&	¬	B	¬	(A	→	B)	B	→	A
T	T	T	F	F	T	F	T	T	T	T	T	T
T	F	T	**T**	T	F	**T**	T	F	F	F	**T**	T
F	T	F	F	F	T	F	F	T	T	T	F	F
F	F	F	F	T	F	F	F	T	F	F	T	F

6. $A \vee B, A \rightarrow \neg A, B \rightarrow \neg B$
These sentences are jointly inconsistent.

A	B	A	∨	B	A	→	¬	A	B	→	¬	B
T	T	T	T	T	T	F	F	T	T	F	F	T
T	F	T	T	F	T	F	F	T	F	T	T	F
F	T	F	T	T	F	T	T	F	T	F	F	T
F	F	F	F	F	F	T	T	F	F	T	T	F

7. $\neg(\neg A \vee B), A \to \neg C, A \to (B \to C)$
   These sentences are jointly consistent.

A	B	C	$\neg$	$(\neg$	A	$\vee$	B)	A	$\to$	$\neg$	C	A	$\to$	(B	$\to$	C)
T	T	T	F	F	T	T	T	T	F	F	T	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T	T	F	T	F	T	F	F
T	F	T	T	F	T	F	F	T	F	F	T	T	T	F	T	T
T	F	F	**T**	F	T	F	F	T	**T**	T	F	T	**T**	F	T	F
F	T	T	F	T	F	T	T	F	T	F	T	F	F	T	T	T
F	T	F	F	T	F	T	T	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	F	T	F	T	F	T	F	T	T
F	F	F	F	T	F	T	F	F	T	T	F	F	T	F	T	F

8. $A \to B, A \,\&\, \neg B$
   These sentences are jointly inconsistent.

A	B	A	$\to$	B	A	$\&$	$\neg$	B
T	T	T	T	T	T	F	F	T
T	F	T	F	F	T	T	T	F
F	T	F	T	T	F	F	F	T
F	F	F	T	F	F	F	T	F

9. $A \to (B \to C), (A \to B) \to C, A \to C$
   These sentences are jointly consistent.

A	B	C	A	$\to$	$(B$	$\to$	$C)$	$(A$	$\to$	$B)$	$\to$	C	A	$\to$	C
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	F	F	T	F	F
T	F	T	T	T	F	T	T	T	F	F	T	T	T	T	T
T	F	F	T	T	F	T	F	T	F	F	T	F	T	F	F
F	T	T	F	T	T	T	T	F	T	T	T	T	F	T	T
F	T	F	F	T	T	F	F	F	T	T	F	F	F	T	F
F	F	T	F	T	F	T	T	F	T	F	T	T	F	T	T
F	F	F	F	T	F	T	F	F	T	F	F	F	F	T	F

10. $\neg B, A \to B, A$
    These sentences are jointly inconsistent.

A	B	$\neg$	B	A	$\rightarrow$	B	A
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	T
F	T	F	T	F	T	T	F
F	F	T	F	F	T	F	F

11. $\neg(A \vee B), A \leftrightarrow B, B \rightarrow A$
These sentences are jointly consistent.

A	B	$\neg$	$(A$	$\vee$	$B)$	A	$\leftrightarrow$	B	B	$\rightarrow$	A
T	T	F	T	T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	F	F	F	T	T
F	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	F	F	F	T	F	F	T	F

12. $A \vee B, \neg B, \neg B \rightarrow \neg A$
These sentences are jointly inconsistent.

A	B	A	$\vee$	B	$\neg$	B	$\neg$	B	$\rightarrow$	$\neg$	A
T	T	T	T	T	F	T	F	T	T	F	T
T	F	T	T	F	T	F	T	F	F	F	T
F	T	F	T	T	F	T	F	T	T	T	F
F	F	F	F	F	T	F	T	F	T	T	F

13. $A \leftrightarrow B, \neg B \vee \neg A, A \rightarrow B$
These sentences are jointly consistent.

A	B	A	$\leftrightarrow$	B	$\neg$	B	$\vee$	$\neg$	A	A	$\rightarrow$	B
T	T	T	T	T	F	T	F	F	T	T	T	T
T	F	T	F	F	T	F	T	F	T	T	F	F
F	T	F	F	T	F	T	T	T	F	F	T	T
F	F	F	T	F	T	F	T	T	F	F	T	F

14. $(A \vee B) \vee C, \neg A \vee \neg B, \neg C \vee \neg B$
These sentences are jointly consistent.

A	B	C	$(A$	$\vee$	$B)$	$\vee$	C	$\neg$	A	$\vee$	$\neg$	B	$\neg$	C	$\vee$	$\neg$	B
T	T	T	T	T	T	T	T	F	T	F	F	T	F	T	F	F	T
T	T	F	T	T	T	T	F	F	T	F	F	T	T	F	T	F	T
T	F	T	T	T	F	T	T	F	T	T	T	F	F	T	T	T	F
T	F	F	T	T	F	T	F	F	T	T	T	F	T	F	T	T	F
F	T	T	F	T	T	T	T	T	F	T	F	T	F	T	F	F	T
F	T	F	F	T	T	T	F	T	F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T	F	T	T	F	F	T	T	T	F
F	F	F	F	F	F	F	F	T	F	T	T	F	T	F	T	T	F

D. Use a truth table to determine whether the sentences in each set are **equivalent** or not.

1. A and $\neg A$ are not equivalent.

A	A	$\neg$	A
T	**T**	**F**	T
F	**F**	**T**	F

2. A & $\neg A$ and $\neg B \leftrightarrow B$ are equivalent.

A	B	A	&	$\neg$	A	$\neg$	B	$\leftrightarrow$	B
T	T	T	**F**	F	T	F	T	**F**	T
T	F	T	**F**	F	T	T	F	**F**	F
F	T	F	**F**	T	F	F	T	**F**	T
F	F	F	**F**	T	F	T	F	**F**	F

3. $[(A \vee B) \vee C]$ and $[A \vee (B \vee C)]$ are equivalent.

A	B	C	$[(A$	$\vee$	$B)$	$\vee$	$C]$	$[A$	$\vee$	$(B$	$\vee$	$C)]$
T	T	T	T	T	T	**T**	T	T	**T**	T	T	T
T	T	F	T	T	T	**T**	F	T	**T**	T	T	F
T	F	T	T	T	F	**T**	T	T	**T**	F	T	T
T	F	F	T	T	F	**T**	F	T	**T**	F	F	F
F	T	T	F	T	T	**T**	T	F	**T**	T	T	T
F	T	F	F	T	T	**T**	F	F	**T**	T	T	F
F	F	T	F	F	F	**T**	T	F	**T**	F	T	T
F	F	F	F	F	F	**F**	F	F	**F**	F	F	F

4. $A \vee (B \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C)$ are equivalent.

A	B	C	A	$\vee$	$(B$	$\&$	$C)$	$(A$	$\vee$	$B)$	$\&$	$(A$	$\vee$	$C)$
T	T	T	T	**T**	T	T	T	T	T	T	**T**	T	T	T
T	T	F	T	**T**	T	F	F	T	T	T	**T**	T	T	F
T	F	T	T	**T**	F	F	T	T	T	F	**T**	T	T	T
T	F	F	T	**T**	F	F	F	T	T	F	**T**	T	T	F
F	T	T	F	**T**	T	T	T	F	T	T	**T**	F	T	T
F	T	F	F	**F**	T	F	F	F	T	T	**F**	F	F	F
F	F	T	F	**F**	F	F	T	F	F	F	**F**	F	T	T
F	F	F	F	**F**	F	F	F	F	F	F	**F**	F	F	F

5. $[A \,\&\, (A \vee B)] \rightarrow B$ and $A \rightarrow B$ are equivalent

A	B	$[A$	$\&$	$(A$	$\vee$	$B)]$	$\rightarrow$	B	A	$\rightarrow$	B
T	T	T	T	T	T	T	**T**	T	T	**T**	T
T	F	T	T	T	T	F	**F**	F	T	**F**	F
F	T	F	F	F	T	T	**T**	T	F	**T**	T
F	F	F	F	F	F	F	**T**	F	F	**T**	F

6. $A \rightarrow A$ and $A \leftrightarrow A$ are equivalent.

A	A	$\rightarrow$	A	A	$\leftrightarrow$	A
T	T	T	T	T	T	T
F	F	T	F	F	T	F

7. $\neg(A \to B)$ and $\neg A \to \neg B$ are not equivalent.

A	B	$\neg$	$(A$	$\to$	$B)$	$\neg$	A	$\to$	$\neg$	B
T	T	F	T	T	T	F	T	T	F	T
T	F	T	T	F	F	F	T	T	T	F
F	T	F	F	T	T	T	F	F	F	T
F	F	F	F	T	F	T	F	T	T	F

8. $A \vee B$ and $\neg A \to B$ are equivalent.

A	B	A	$\vee$	B	$\neg$	A	$\to$	B
T	T	T	T	T	F	T	T	T
T	F	T	T	F	F	T	T	F
F	T	F	T	T	T	F	T	T
F	F	F	F	F	T	F	F	F

9. $(A \to B) \to C$ and $A \to (B \to C)$ are not equivalent.

A	B	C	$(A$	$\to$	$B)$	$\to$	C	A	$\to$	$(B$	$\to$	$C)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	T	F	T	F	F
T	F	T	T	F	F	T	T	T	T	F	T	T
T	F	F	T	F	F	T	F	T	T	F	T	F
F	T	T	F	T	T	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F	F	T	T	F	F
F	F	T	F	T	F	T	T	F	T	F	T	T
F	F	F	F	T	F	F	F	F	T	F	T	F

10. $A \leftrightarrow (B \leftrightarrow C)$ and $A \,\&\, (B \,\&\, C)$ are not equivalent.

A	B	C	A	$\leftrightarrow$	$(B$	$\leftrightarrow$	$C)$	A	$\&$	$(B$	$\&$	$C)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	F	T	F	F
T	F	T	T	F	F	F	T	T	F	F	F	T
T	F	F	T	T	F	T	F	T	F	F	F	F
F	T	T	F	F	T	T	T	F	F	T	T	T
F	T	F	F	T	T	F	F	F	F	T	F	F
F	F	T	F	T	F	F	T	F	F	F	F	T
F	F	F	F	F	F	T	F	F	F	F	F	F

E.

1. Suppose that A and B are equivalent. What can you say about A ↔ B?

   A and B have the same truth value on every line of a complete truth table, so A ↔ B is true on every line. It is a tautology.

2. Suppose that A and B are jointly inconsistent. What can you say about (A & B)?

   Since the sentences are jointly inconsistent, there is no truth-value assignment on which they are both true. So their conjunction is false on every truth-value assignment. It is a contradiction

3. Suppose that A and B are equivalent. What can you say about (A ∨ B)?

   Not much. Since A and B are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is equivalent to them.

4. Suppose that A and B are *not* equivalent. What can you say about (A ∨ B)?

   A and B have different truth values on at least one line of a complete truth table, and (A ∨ B) will be true on that line. On other lines, it might be true or false. So $(A \vee B)$ is either a tautology or it is contingent; it is *not* a contradiction.

5. Consider this principle:

   > Suppose A and B are logically equivalent. Suppose an argument contains A (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced A with B.

   Is this principle correct? Explain your answer.

   > The principle is correct. Since A and B are logically equivalent, they have the same truth table. So every truth-value assignment that makes A true also makes B true, and every truth-value assignment that makes A false also makes B false. So if no truth-value assignment makes all the premises true and the conclusion false, when A was among the premises or the conclusion, then no truth-value assignment makes all the premises true and the conclusion false, when we replace A with B.

10 Truth tables and validity

10.1 Validity

Having examined the logical relations between two sentences in §9.2 and §9.3, we can now go a step further and consider the relationship between the premises and the conclusion of an argument. Recall the definition of VALID.

valid

An argument is VALID when (and only when) it is the case that if the premises are true, then the conclusion has to be true.

When using a truth table to determine if an argument is valid, we list the premise or premises first, then, the turnstile symbol ('$\vdash$'), and, finally, the conclusion. We will use '$\neg L \rightarrow (M \vee L), \neg L \vdash M$' as our example.

The symbol '$\vdash$' is used to separate the premises from the conclusion in arguments in TFL. It can be read as *therefore*.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T		**T**
T	F	T	F	**T**	T	T	F	**T**	F		**T**
F	T	F	T	**T**	F	T	T	**F**	T		**F**
F	F	T	F	**F**	F	F	F	**T**	F		**F**

Once the truth table is completed for '$\neg L \rightarrow (M \vee L), \neg L \vdash M$', we investigate whether this argument satisfies (or violates) the definition of *valid*. Ask yourself, "When both premises are true, is the conclusion true?" And "Is there any line (that is, any truth-value assignment) where both premises are true and the conclusion is false?" If the answer to the first question is always "yes," then the argument is valid. If the answer to the second question is ever "no," then the argument is invalid.

As you can see, there is only one row where both '$\neg L \rightarrow (M \vee L)$' and '$\neg L$' are true, and so that is the row that mainly concerns us. On that row, the conclusion is also true. Hence, '$\neg L \rightarrow (M \vee L), \neg L \vdash M$' is valid.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T		**T**
T	F	T	F	**T**	T	T	F	**T**	F	✓	**T**
F	T	F	T	**T**	F	T	T	**F**	T		**F**
F	F	T	F	**F**	F	F	F	**T**	F		**F**

When using truth tables to determine if an argument is valid, we will put '✓' and '✗' in the column under the turnstile. As just shown, when all of the premises are true and the conclusion is true, we put a '✓' on that line beneath the turnstile. If, on a line, all of the premise are true and the conclusion is false, then we put a '✗' beneath the turnstile.

Also (and **importantly!**), when there is a line where one or more of the premises are false, we put a '✓' beneath the turnstile—whether the conclusion is true or false. An argument is valid when it is the case that *if the premises are true*, then the conclusion has to be true. It doesn't matter if there are truth-value assignments (i.e., lines) where both premises are not true. Such lines don't violate our definition of *valid*, and so they get a '✓'.

Completing our truth table we have this:

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	M
T	T	F	T	**T**	T	T	T	**F**	T	✓	**T**
T	F	T	F	**T**	T	T	F	**T**	F	✓	**T**
F	T	F	T	**T**	F	T	T	**F**	T	✓	**F**
F	F	T	F	**F**	F	F	F	**T**	F	✓	**F**

Now, let's make one small (but significant) change to the argument: $\neg L \rightarrow (M \vee L), \neg L \vdash \neg M$. The premises are the same, but now the conclusion is $\neg M$ instead of M.

The truth values for the premises are the same, and the truth values for the conclusion have, on each line, flipped from T to F or vice versa. Now, when we evaluate each line, what do we find? As before, on lines

1, 3, and 4, one of the premises is false, and so they get a '✓'. On line 2, the premises are true and the conclusion is false. That line gets a '✕'! Because there is a line where the premises are true and the conclusion is false, '$\neg L \rightarrow (M \vee L), \neg L \vdash \neg M$' is not valid.

M	L	$\neg$	L	$\rightarrow$	$(M$	$\vee$	$L)$	$\neg$	L	$\vdash$	$\neg M$
T	T	F	T	**T**	T	T	T	**F**	T	✓	**F**
T	F	T	F	**T**	T	T	F	**T**	F	✕	**F**
F	T	F	T	**T**	F	T	T	**F**	T	✓	**T**
F	F	T	F	**F**	F	F	F	**T**	F	✓	**T**

good and bad lines

Let's call lines that violate the definition of VALID *bad lines* and the lines that do not *good lines*.

(1) Any line where all of the premises are true and the conclusion is false **is a bad line**. Put an '✕' on that line.

(2) Any line where all of the premises are true and the conclusion is true **is a good line**. Put a '✓' on that line.

(3) Any line where the conclusion is true cannot be a bad line. (So, whatever the case may be with the premises, **it's a good line**.) Put a '✓' on that line.

(4) Any line where at least one premise is false cannot be a bad line. So, **it's a good line**. Put a '✓' on that line.

10.2 Some examples

Here are some examples using truth tables to determine whether an argument is valid. As a reminder, the definition of valid is given in §10.1, and we can also use 1 – 4 on p. 121 (which are consequences of the definition). We will begin with arguments that have only one premise and then do some with multiple premises.

1. First we will determine if '$P \& Q \vdash Q$' is valid. The premise, '$P \& Q$', is only true on line 1. Since it is false on lines 2 – 4, we know that those are good lines. (See guideline 4.) On line 1, 'P & Q' is true and the conclusion, 'Q', is true, and so that is also a good line. (See guideline 2.) Since every line is a good line, this argument is valid.

P	Q	P	&	Q	$\vdash$	Q
T	T	T	**T**	T	✓	**T**
T	F	T	**F**	F	✓	**F**
F	T	F	**F**	T	✓	**T**
F	F	F	**F**	F	✓	**F**

2. In '$\neg(P \vee Q) \vdash \neg P \& Q$', the premise is false on lines 1 – 3, and so we know that those are good lines. On line 4, the premise is true and the conclusion is false, which means that line 4 is a bad line. (See guideline 1.) Since it has at least one bad line, this argument is not valid.

P	Q	$\neg$	$(P$	$\vee$	$Q)$	$\vdash$	$\neg$	P	&	Q
T	T	**F**	T	T	T	✓	F	T	**F**	T
T	F	**F**	T	T	F	✓	F	T	**F**	F
F	T	**F**	F	T	T	✓	T	F	**T**	T
F	F	**T**	F	F	F	×	T	F	**F**	F

3. Now an argument with two premises: '$P \rightarrow Q, \neg Q \vdash \neg P$'. Since both premise are not true on lines 1, 2, and 3, those are all good lines. On line 4, both premises are true and the conclusion is true, and so that is a good line. Since every line is a good line, this argument is valid.

P	Q	P	$\rightarrow$	Q	$\neg$	Q	$\vdash$	$\neg$	P
T	T	T	**T**	T	**F**	T	✓	**F**	T
T	F	T	**F**	F	**T**	F	✓	**F**	T
F	T	F	**T**	T	**F**	T	✓	**T**	F
F	F	F	**T**	F	**T**	F	✓	**T**	F

4. Next, consider '$P \rightarrow Q, P \rightarrow \neg Q \vdash P$'. Since the second premise is false on line 1 and the first premise is false on line 2, those are good

lines. On line 3, both of the premises are true and the conclusion is false. That's a bad line. And then the same is also the case on line 4, and so that is a bad line also. Since two of the lines in this truth table are bad lines, the argument is invalid.

P	Q	P	$\rightarrow$	$Q,$	P	$\rightarrow$	$\neg$	Q	$\vdash$	P
T	T	T	**T**	T	T	**F**	F	T	✓	**T**
T	F	T	**F**	F	T	**T**	T	F	✓	**T**
F	T	F	**T**	T	F	**T**	F	T	×	**F**
F	F	F	**T**	F	F	**T**	T	F	×	**F**

5. In the last argument, we have three premises. One of the premises is false on each of lines 1, 2, 4, 5, 7, and 8, and so those are all good lines. On line 3, all of the premises are true and the conclusion is true, and so that is a good line. On line 6, all of the premises are true but the conclusion is false, and so that is a bad line. Since one of the lines is a bad line, this argument is invalid.

P	Q	R	P	$\vee$	$Q,$	P	$\rightarrow$	$R,$	Q	$\rightarrow$	$\neg$	R	$\vdash$	R
T	T	T	T	**T**	T	T	**T**	T	T	**F**	F	T	✓	**T**
T	T	F	T	**T**	T	T	**F**	F	T	**T**	T	F	✓	**F**
T	F	T	T	**T**	F	T	**T**	T	F	**T**	F	T	✓	**T**
T	F	F	T	**T**	F	T	**F**	F	F	**T**	T	F	✓	**F**
F	T	T	F	**T**	T	F	**T**	T	T	**F**	F	T	✓	**T**
F	T	F	F	**T**	T	F	**T**	F	T	**T**	T	F	×	**F**
F	F	T	F	**F**	F	F	**T**	T	F	**T**	F	T	✓	**T**
F	F	F	F	**F**	F	F	**T**	F	F	**T**	T	F	✓	**F**

10.3 '$\vdash$' versus '$\rightarrow$'

When using truth tables to determine whether an argument is valid, it may help you to notice a similarity between '$\vdash$' and '$\rightarrow$'. As you know, a conditional is true under every circumstance except when the antecedent is true and the consequent if false. (So, when we have a 'T' under the antecedent and an 'F' under the consequent, we put an 'F' under the '$\rightarrow$'.) Meanwhile, in an argument, when all of the premises are true and the conclusion is false, the argument is invalid. (So, for a specific line, when we have a 'T' under every premise and an 'F' under the conclusion, we put a '✕' under the '$\vdash$'.)

The reasoning here is similar. In both cases, we are violating the principle – of either the conditional or of a valid argument – when we have a false sentence that follows from a sentence or a set of sentences that are all true. Thus, if $\mathsf{A} \rightarrow \mathsf{C}$ is false, then $\mathsf{A} \vdash \mathsf{C}$ is invalid (and if $\mathsf{A} \vdash \mathsf{C}$ is invalid, then $\mathsf{A} \rightarrow \mathsf{C}$ is false). Conversely, whenever $\mathsf{A} \rightarrow \mathsf{C}$ is true, then $\mathsf{A} \vdash \mathsf{C}$ is valid (and vice versa).

10.4 The limits of this type of analysis

We have seen in chapters 9 and 10 that truth tables are a useful tool for analyzing sentences – whether those are individual sentences, pairs of sentences, or arguments. There are limitations to this type of analysis, however, and it worth understanding some of those limitations.

First, consider this argument:

1. Daisy has four legs.
   Therefore, Daisy has more than two legs.

To symbolize this argument in TFL, we would have to use two different atomic sentences – perhaps 'F' for the premise and 'T' for the conclusion. The English version of this argument is clearly valid, but '$F \vdash T$' is just as clearly invalid.

F	T	T	$\vdash$	F
T	T	T	✓	T
T	F	F	✓	T
F	T	T	✕	F
F	F	F	✓	F

Hence, we should keep in mind that while some English sentences can be effectively translated into TFL, not all can be.

Next, consider this sentence:

2. John is neither bald nor not-bald.

This is symbolized in TFL as '$\neg(B \vee \neg B)$', and, as you can see from the truth table, it is a contradiction.

B	$\neg$	$(B$	$\vee$	$\neg$	$B)$
T	**F**	T	T	F	T
F	**F**	F	T	T	F

But sentence 2 does not seem like a contradiction. After all, someone could very well add "John is on the borderline of baldness," which would (it seems) mean that sentence 2 is true. But since TFL cannot represent a place between 'B' and '$\neg B$', it cannot treat *John is neither bald nor not-bald* as true.

Third, let's think about this statement:

3. It's not the case that, if God exists, he answers evil prayers.

Symbolizing this in TFL, we have '$\neg(G \rightarrow E)$'. As we can see from the truth table, '$\neg(G \rightarrow E)$' $\vdash$'G' is valid.

E	G	$\neg$	$(G$	$\rightarrow$	$E)$	$\vdash$	G
T	T	**F**	T	T	T	✓	**T**
T	F	**F**	F	T	T	✓	**F**
F	T	**T**	T	F	F	✓	**T**
F	F	**F**	F	T	F	✓	**F**

So sentence 3 seems to entail that God exists. But that's not what we expect. An atheist could believe that 'It's not the case that, if God exists, he answers evil prayers' without accepting that God does, in fact, exist.

It might be that sentence 3, despite appearances, does not express what we mean. We can try rephrasing it this way:

4. If God exists, he does not answer evil prayers.

This we symbolize as '$G \rightarrow \neg E$'. Now, as shown in the truth table on the left, 'G' does not follow from this premise. (That is, the argument '$G \rightarrow \neg E \vdash G$' is invalid.) But, at the same time, from the premise '$\neg G$' (i.e., 'God does not exist'), it follows that 'if God exists, he answers evil prayers'.

E	G	(G	$\to$	$\neg$	E)	$\vdash$	G
T	T	T	**F**	F	T	✓	**T**
T	F	F	**T**	F	T	×	**F**
F	T	T	**T**	T	F	✓	**T**
F	F	F	**T**	T	F	×	**F**

E	G	$\neg$	G	$\vdash$	(G	$\to$	E)
T	T	**F**	T	✓	T	**T**	T
T	F	**T**	F	✓	F	**T**	T
F	T	**F**	T	✓	T	**F**	F
F	F	**T**	F	✓	F	**T**	F

(We can also put these final two points as follows. When 'G' is false, '$G \to \neg E$' is true, and so we don't have to be committed to the existence of God to accept that 'If God exists, he does not answer evil prayers'. But if 'G' is false, then '$G \to E$'—i.e., 'If God exists, he answers evil prayers'—is true.)

In different ways, these three examples illustrate some of the limitations of using a language like TFL that can only handle truth-functional connectives. These limitations give rise to some interesting questions in philosophical logic, however. The case of John's baldness (or non-baldness) raises the general question of what logic we should use when dealing with *vague* discourse. The case of God answering evil prayers illustrates some of the *paradoxes of material implication*.

Part of the purpose of studying truth-functional propositional logic is to equip ourselves with the tools to explore these questions of philosophical logic. But we have to walk before we can run; and so we have to become proficient using TFL before we can adequately discuss its limits and consider alternatives.

10.5 Practice exercises

A. Create a truth table for each argument and then determine if the argument is valid or invalid.

1. $A \rightarrow A \vdash A$
2. $A \rightarrow (A \,\&\, \neg A) \vdash \neg A$
3. $A \vee (B \rightarrow A) \vdash \neg A \rightarrow \neg B$
4. $A \vee B, B \vee C, \neg A \vdash B \,\&\, C$
5. $(B \,\&\, A) \rightarrow C, (C \,\&\, A) \rightarrow B \vdash (C \,\&\, B) \rightarrow A$
6. $A \rightarrow B, B \vdash A$
7. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$
8. $A \rightarrow B, A \rightarrow C \vdash B \rightarrow C$
9. $A \rightarrow B, B \rightarrow A \vdash A \leftrightarrow B$
10. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \vdash A$
11. $A \vee B, B \vee C, \neg B \vdash A \,\&\, C$
12. $A \rightarrow B, \neg A \vdash \neg B$
13. $A, B \vdash \neg(A \rightarrow \neg B)$
14. $\neg(A \,\&\, B), A \vee B, A \leftrightarrow B \vdash C$

B.

1. Suppose that $(\mathsf{A} \,\&\, \mathsf{B}) \rightarrow \mathsf{C}$ is neither a tautology nor a contradiction. Is it possible to determine if $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ is valid or not? Explain.
2. Suppose that A is a contradiction. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid? Explain.
3. Suppose that C is a tautology. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid? Explain.

10.6 Answers

A.

1. $A \to A \vdash A$
   This argument is invalid.

A	A	$\to$	A	$\vdash$	A
T	T	**T**	T	✓	T
F	F	**T**	F	×	F

2. $A \to (A \,\&\, \neg A) \vdash \neg A$
   This argument is valid.

A	A	$\to$	$(A$	$\&$	$\neg$	$A)$	$\vdash$	$\neg$	A
T	T	**F**	T	F	F	T	✓	**F**	T
F	F	**T**	F	F	T	F	✓	**T**	F

3. $A \vee (B \to A) \vdash \neg A \to \neg B$
   This argument is valid.

A	B	A	$\vee$	$(B$	$\to$	$A)$	$\vdash$	$\neg$	A	$\to$	$\neg$	B
T	T	T	**T**	T	T	T	✓	F	T	**T**	F	T
T	F	T	**T**	F	T	T	✓	F	T	**T**	T	F
F	T	F	**F**	T	F	F	✓	T	F	**F**	F	T
F	F	F	**T**	F	T	F	✓	T	F	**T**	T	F

4. $A \vee B, B \vee C, \neg A \vdash B \,\&\, C$
   This argument is invalid.

A	B	C	A	$\vee$	$B,$	B	$\vee$	$C,$	$\neg$	A	$\vdash$	B	$\&$	C
T	T	T	T	**T**	T	T	**T**	T	**F**	T	$\checkmark$	T	**T**	T
T	T	F	T	**T**	T	T	**T**	F	**F**	T	$\checkmark$	T	**F**	F
T	F	T	T	**T**	F	F	**T**	T	**F**	T	$\checkmark$	F	**F**	T
T	F	F	T	**T**	F	F	**F**	F	**F**	T	$\checkmark$	F	**F**	F
T	T	T	F	**T**	T	T	**T**	T	**T**	F	$\checkmark$	T	**T**	T
T	T	F	F	**T**	T	T	**T**	F	**T**	F	$\times$	T	**F**	F
T	F	T	F	**F**	F	F	**T**	T	**T**	F	$\checkmark$	F	**F**	T
T	F	F	F	**F**	F	F	**F**	F	**T**	F	$\checkmark$	F	**F**	F

5. $(B \,\&\, A) \to C, (C \,\&\, A) \to B \vdash (C \,\&\, B) \to A$
   This argument is invalid.

$(B$	$\&$	$A)$	$\to$	$C,$	$(C$	$\&$	$A)$	$\to$	B	$\vdash$	$(C$	$\&$	$B)$	$\to$	A
T	T	T	**T**	T	T	T	T	**T**	T	$\checkmark$	T	T	T	**T**	T
T	T	T	**F**	F	F	F	T	**T**	T	$\checkmark$	F	F	T	**T**	T
F	F	T	**T**	T	T	T	T	**F**	F	$\checkmark$	T	F	F	**T**	T
F	F	T	**T**	F	F	F	T	**T**	F	$\checkmark$	F	F	F	**T**	T
T	F	F	**T**	T	T	F	F	**T**	T	$\times$	T	T	T	**F**	F
T	F	F	**T**	F	F	F	F	**T**	T	$\checkmark$	F	F	T	**T**	F
F	F	F	**T**	T	T	F	F	**T**	F	$\checkmark$	T	F	F	**T**	F
F	F	F	**T**	F	F	F	F	**T**	F	$\checkmark$	F	F	F	**T**	F

6. $A \to B, B \vdash A$
   This argument is invalid.

A	B	A	$\to$	$B,$	B	$\vdash$	A
T	T	T	T	T	T	$\checkmark$	T
T	F	T	F	F	F	$\checkmark$	T
F	T	F	T	T	T	$\times$	F
F	F	F	T	F	F	$\checkmark$	F

7. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$
This argument is valid.

A	B	C	A	$\leftrightarrow$	$B,$	B	$\leftrightarrow$	C	$\vdash$	A	$\leftrightarrow$	C
T	T	T	T	T	T	T	T	T	✓	T	T	T
T	T	F	T	T	T	T	F	F	✓	T	F	F
T	F	T	T	F	F	F	F	T	✓	T	T	T
T	F	F	T	F	F	F	T	F	✓	T	F	F
F	T	T	F	F	T	T	T	T	✓	F	F	T
F	T	F	F	F	T	T	F	F	✓	F	T	F
F	F	T	F	T	F	F	F	T	✓	F	F	T
F	F	F	F	T	F	F	T	F	✓	F	T	F

8. $A \rightarrow B, A \rightarrow C \vdash B \rightarrow C$
This argument is invalid.

A	B	C	A	$\rightarrow$	$B,$	A	$\rightarrow$	C	$\vdash$	B	$\rightarrow$	C
T	T	T	T	T	T	T	T	T	✓	T	T	T
T	T	F	T	T	T	T	F	F	✓	T	F	F
T	F	T	T	F	F	T	T	T	✓	F	T	T
T	F	F	T	F	F	T	F	F	✓	F	T	F
F	T	T	F	T	T	F	T	T	✓	T	T	T
F	T	F	F	T	T	F	T	F	×	T	F	F
F	F	T	F	T	F	F	T	T	✓	F	T	T
F	F	F	F	T	F	F	T	F	✓	F	T	F

9. $A \rightarrow B, B \rightarrow A \vdash A \leftrightarrow B$
This argument is valid.

A	B	A	$\rightarrow$	$B,$	B	$\rightarrow$	A	$\vdash$	A	$\leftrightarrow$	B
T	T	T	T	T	T	T	T	✓	T	T	T
T	F	T	F	F	F	T	T	✓	T	F	F
F	T	F	T	T	T	F	F	✓	F	F	T
F	F	F	T	F	F	T	F	✓	F	T	F

10. $A \vee \big[A \rightarrow (A \leftrightarrow A)\big] \vdash A$
This argument is invalid.

A	A	$\vee$	$\big[A$	$\rightarrow$	$(A$	$\leftrightarrow$	$A)\big]$	$\vdash$	A
T	T	T	T	T	T	T	T	✓	T
F	F	T	F	T	F	T	F	×	F

11. $A \vee B, B \vee C, \neg B \vdash A \,\&\, C$
This argument is valid.

A	B	C	A	$\vee$	$B,$	B	$\vee$	$C,$	$\neg$	B	$\vdash$	A	$\&$	C
T	T	T	T	T	T	T	T	T	F	T	✓	T	T	T
T	T	F	T	T	T	T	T	F	F	T	✓	T	F	F
T	F	T	T	T	F	F	T	T	T	F	✓	T	T	T
T	F	F	T	T	F	F	F	F	T	F	✓	T	F	F
F	T	T	F	T	T	T	T	T	F	T	✓	F	F	T
F	T	F	F	T	T	T	T	F	F	T	✓	F	F	F
F	F	T	F	F	F	F	T	T	T	F	✓	F	F	T
F	F	F	F	F	F	F	F	F	T	F	✓	F	F	F

12. $A \rightarrow B, \neg A \vdash \neg B$
This argument is invalid.

A	B	A	$\rightarrow$	$B,$	$\neg$	A	$\vdash$	$\neg$	B
T	T	T	T	T	F	T	✓	F	T
T	F	T	F	F	F	T	✓	T	F
F	T	F	T	T	T	F	×	F	T
F	F	F	T	F	T	F	✓	T	F

13. $A, B \vdash \neg(A \rightarrow \neg B)$
This argument is valid.

A	B	$A,$	B	$\vdash$	$\neg$	$(A$	$\rightarrow$	$\neg$	$B)$
T	T	T	T	✓	T	T	F	F	T
T	F	T	F	✓	F	T	T	T	F
F	T	F	T	✓	F	F	T	F	T
F	F	F	F	✓	F	F	T	T	F

14. $\neg(A \,\&\, B), A \vee B, A \leftrightarrow B \vdash C$
This argument is valid.

A	B	C	$\neg$	$(A$	$\&$	$B),$	A	$\vee$	$B,$	A	$\leftrightarrow$	B	$\vdash$	C
T	T	T	F	T	T	T	T	T	T	T	T	T	✓	T
T	T	F	F	T	T	T	T	T	T	T	T	T	✓	F
T	F	T	T	T	F	F	T	T	F	T	F	F	✓	T
T	F	F	T	T	F	F	T	T	F	T	F	F	✓	F
F	T	T	T	F	F	T	F	T	T	F	F	T	✓	T
F	T	F	T	F	F	T	F	T	T	F	F	T	✓	F
F	F	T	T	F	F	F	F	F	F	F	T	F	✓	T
F	F	F	T	F	F	F	F	F	F	F	T	F	✓	F

B.

1. Suppose that $(\mathsf{A} \,\&\, \mathsf{B}) \rightarrow \mathsf{C}$ is neither a tautology nor a contradiction. Is it possible to determine if $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ is valid or not?

   Since the sentence $(\mathsf{A} \,\&\, \mathsf{B}) \rightarrow \mathsf{C}$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, A and B are true and C is false. Hence, the argument, '$\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$', is invalid.

2. Suppose that A is a contradiction. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid?

   Since A is false on every line of a truth table, there is no line on which A and B are true and C is false. Hence, the argument is valid. (Although that would be kind of an odd argument since we know that one of the premises is a contradiction.)

3. Suppose that C is a tautology. Is $\mathsf{A}, \mathsf{B} \vdash \mathsf{C}$ valid or invalid?

   Since C is true on every line of a complete truth table, there is no line on which A and B are true and C is false. Hence, the argument is valid.

Part 4

Natural deduction for TFL

11 Natural deduction

11.1 Natural deduction versus truth tables

An argument is valid when (and only when) it is impossible for all of the premises to be true and the conclusion to be false. And we have seen that truth tables can be used to determine whether an argument is valid. In the next chapter, you will learn another method for verifying that an argument is valid. Before we turn to this new method, however, let's review the strengths and weakness of truth tables.

1. The truth table method for determining if an argument is valid focuses directly on the definition of *valid*. Each line of a complete truth table corresponds to a truth-value assignment. Thus, given an argument in TFL, truth tables reveal whether or not the conclusion is true when all of the premises true.
2. Truth tables also allow us to easily and rigorously set the meaning for each logical operator. As we discussed in §5.3, in English, 'or' can take the inclusive-or meaning (one or the other, or both) or the exclusive-or meaning (one or the other, but not both), and, at different times, both meanings are used in English. We can discuss which English meaning is closest to the meaning of '$\lor$' in TFL (it's the inclusive-or), but, in the end, we just set the meaning of the symbol '$\lor$' with this truth table:

A	B	$\mathsf{A} \lor \mathsf{B}$
T	T	T
T	F	T
F	T	T
F	F	F

Hence, the definition for the '∨' is simple this: the operator that connects A and B to yield the truth values shown in this truth table. And, then, the same goes for the other logical operators.

3. To create a truth table, the number of lines needed is 2^n, where n is the number of different letters in the argument. So, an argument with four different sentence letters will require a 16 line truth table, one with five letters will require 32 lines, one with six different letters will require 64 lines, and so on. Hence, while a truth table can be used to determine if any argument is valid or invalid, one of the weakness of this method is that it is difficult to use when the argument contains more than four different sentence letters.

4. But what is typically seen as the biggest weakness of using truth tables to determine if an argument is valid is that it doesn't reveal to us *why* the argument is valid. It doesn't, in other words, lay out the reasoning that demonstrates why (and how) the conclusion follows from the premises.

As an alternative to truth tables, we have a *natural deduction system*. Such a system allows us to verify that an argument is valid and to see why it is valid. We do this by making explicit the reasoning process that takes us from the premises to the conclusion. We begin with twelve basic rules—which we call *rules of derivation*. (For instance, this is one of the rules: if we know that '$P \vee Q$' is true; and we also know that '¬P' is true, then we can assert that 'Q' is true.) The rules can be combined, and with just these twelve, we hope to be able to show how we get from the premises to the conclusion for all of the valid arguments that can be represented in TFL.

There are different natural deduction systems that can be used with TFL. But all, for the most part, reflect the ways that we naturally reason—at least insofar as the reasoning involves 'and', 'or', 'not', 'if …, then …', and 'if and only if'.

11.2 Truth functional propositional logic

We have reached a point where it is useful to summarize what TFL is. As you know, the symbols of TFL are the sentence letters that represent atomic sentences, the logical operators $\neg$, &, $\vee$, $\rightarrow$, and $\leftrightarrow$, and parentheses. These, then, can be combined into sentences using the rules given in chapter 6. And, then, in chapter 7, truth tables were used to set the meaning of the logical operators.

Truth tables also give us a method for verifying that an argument satisfies the definition of *valid*. *Valid* is a concept and is not, strictly speaking a part of TFL. Rather it is a property of some arguments that can, to an extent, be studied and explicated using TFL. Similarly, as you have seen, *tautology*, *contradiction*, *contingent*, *equivalent*, *jointly consistent*, and *jointly inconsistent* are concepts that can be explained using TFL.

The final part of TFL is the system of natural deduction, which sets the rules for how sentences containing the logical operators can be combined or taken apart.

11.3 Fitch

The modern development of natural deduction dates from simultaneous but unrelated papers by Gerhard Gentzen and Stanisław Ja□kowski that were published in 1934. The natural deduction system that we will use, however, is based largely on work by Frederic Fitch that was first published in 1952. Consequently, the format that is used in the next chapter for writing proofs is called *Fitch notation*.

12 The rules of derivation

12.1 Proofs

As was explained in the previous chapter, creating a PROOF is one way of demonstrating that an argument is valid. (And, as you know, using a truth table is the other way.) A proof is a list of sentences. The sentence or sentences at the beginning of the list are the premises of the argument. Every other sentence in the list follows from earlier sentences by a specific rule (with one exception, which we will get to in §12.7). The final sentence is the conclusion of the argument.

As an illustration, consider this argument:

$$\neg(A \vee B) \vdash \neg A \,\&\, \neg B$$

We start the proof by numbering the line and writing the premise:

$$\begin{array}{r|ll} 1 & \underline{\neg(A \vee B)} & \quad :\text{PR} \end{array}$$

Every line in a proof is numbered so that we can refer to it later if we need to do so. We have also indicated that this is a premise by putting 'PR' at the end of the line. And we have drawn a line underneath the premise. Everything written below the line will either be a sentence that can be derived from that premise, or it will be a new assumption that we introduce. The colon that is right before 'PR' is, technically, optional, but it has to be used in Carnap to separate the TFL sentence from the 'PR' (or the rule) that is written at the end of each line.

The conclusion of this argument is '$\neg A \,\&\, \neg B$'; and so we want our proof to end – on some line, we'll call it n – with that sentence:

1	$\neg(A \vee B)$	:PR
2	...	
	...	
	...	
n	$\neg A \,\&\, \neg B$	

It doesn't matter how many lines it takes to arrive at the conclusion, although, generally, we prefer a shorter proof over a longer one.

Now, suppose we have this argument:

$$A \vee B, \neg(A \,\&\, C), \neg(B \,\&\, \neg D) \vdash \neg C \vee D$$

This argument has three premises, and so we start by listing them, numbering each line, and drawing a line under the final premise:

1	$A \vee B$	:PR
2	$\neg(A \,\&\, C)$	:PR
3	$\neg(B \,\&\, \neg D)$	:PR

This, meanwhile, will be the final line of the proof:

n	$\neg C \vee D$

Setting up the premises and the conclusion is, however, the easy part. The real task—and the interesting part—is determining each of the steps that get us from the premise or premises to the conclusion.

To do that, we will use a NATURAL DEDUCTION system. In this system, there are two rules for each logical operator: an INTRODUCTION rule, which allows us to derive a new sentence that has the logical operator as the main connective, and an ELIMINATION rule, which allows us to extract a sub-sentence from a sentence that has that logical operator as the main connective. (Table 12.1 contains a list of the rules.) These rules can then be combined to demonstrate each step that must be taken to get from the premises to the conclusion. All of the rules introduced in this chapter are also summarized on pp. 236 - 237.

THE RULES OF DERIVATION	
conjunction introduction rule	conjuction elimination rule
disjunction introduction rule	disjunction elimination rule
conditional introduction rule	conditional elimination rule
biconditional introduction rule	biconditional elimination rule
negation introduction rule	negation elimination rule
reiteration rule	
double negation rule	

Table 12.1

12.2 Conjunction introduction and elimination

Let's say that we know that Sarah is swimming. We also, as it happens, know that Amy is reading. We are, therefore, justified in stating, "Sarah is swimming and Amy is reading." This reasoning process, which we all do naturally, is part of our natural deduction system. It is called the CONJUNCTION INTRODUCTION RULE.

conjunction introduction rule

m	A	
n	B	
	A & B	:&I m, n

If we have A on a line and B on a line, then we can put A & B on a new line. The 'A' and 'B' can occur in either order, and the conjunction can be 'A & B' or 'B & A'.

The 'm' and 'n' will never appear in an actual proof. In a proof, the lines are numbered 1, 2, 3, etc. The 'm' and 'n' are used in the statement of the rule to indicate that A and B can be on any lines in the proof. If you look ahead, you will see that some of rules given in §12.2 - 12.5 consist of three lines and some have two.

(a) For each of the rules that consist of three lines, you can add what is on the last line (of the rule) to your proof when, and only when, you have what is given on the first two lines (of the rule).

(b) For each of the rules that consist of two lines, you can add what is on the last line (of the rule) to your proof when, and only when, you have what is given on the first line (of the rule).

Returning to the example about Sarah and Amy, we can use this symbolization key:

S: Sarah is swimming.
R: Amy is reading.

Let's say that 'S' and 'R' are our premises (although they don't have to be to use the conjunction introduction rule), and so they are on lines 1 and 2. Then on any subsequent line – but, in this case, it will be line 3 – we can get 'S & R' by using the conjunction introduction rule.

1	S	:PR
2	R	:PR
3	S & R	:&I 1, 2

To show that this application of the conjunction introduction rule is our justification for the new 'S & R' on line 3, we put '&I 1, 2' on the far right. This indicate that 'S & R' was obtained by applying the conjunction introduction rule to the 'S' on line 1 and the 'R' on line 2.

The conjunction introduction rule introduces a sentence with '&' as the main connective. We also have a rule that let's us extract what is on one side of a conjunction and put it on a new line. Suppose someone tells you that *Jeff is eating and Mary is sleeping*. Assuming that whoever told you this is reliable, you are entitled to infer simply that *Jeff is eating*. You are also entitled to infer that *Mary is sleeping*. These are applications of the CONJUNCTION ELIMINATION RULE (which is actually two similar rules).

conjunction elimination rule

m	A & B	
	A	:&E m

m	A & B	
	B	:&E m

If we have A & B on a line, then, on a new line, we can put either A by itself or B by itself.

When you have a conjunction on one line of a proof, you can use the conjunction elimination rule to obtain either of the conjuncts on a new line. You can only, however, apply this rule when the '&' is the main logical operator. So, for instance, you cannot use the conjunction elimination rule to obtain 'D' from '$C \vee (D \,\&\, E)$' (because '$\vee$' is the main logical operator). The same holds for all of the other rules. **Each of the rules of derivation can only be applied to the main logical operator of a sentence.**

We will now construct a couple of proofs using the two rules just introduced. First, one for this argument: '$P \,\&\, Q, \neg R \vdash Q \,\&\, \neg R$'.

1	$P \,\&\, Q$	:PR
2	$\neg R$	:PR

After we have listed the premises, we use the conjunction elimination rule to get 'Q' on a line by itself.

1	$P \,\&\, Q$	:PR
2	$\neg R$	:PR
3	Q	:&E 1

And then, to finish the proof, we use the conjunction introduction rule to get '$Q \,\&\, \neg R$'.

1	$P \& Q$	:PR
2	$\neg R$	:PR
3	Q	:&E 1
4	$Q \& \neg R$	:&I 2, 3

Notice that there is nothing in this representation of the proof to indicate that the last line is the conclusion. It's only because we began with '$P \& Q, \neg R \vdash Q \& \neg R$' that we know that, on line 4, we have arrived at the conclusion that we want.

Next, we will take up the proof for this argument: '$(A \vee B) \& (G \& H) \vdash (A \vee B) \& H$'. After listing the premise, we can use the conjunction elimination rule twice to get '$A \vee B$' and '$G \& H$' on lines by themselves.

1	$(A \vee B) \& (G \& H)$	:PR
2	$(A \vee B)$	:&E 1
3	$(G \& H)$	:&E 1

Now that '$G \& H$' is on its own line, we can use the conjunction elimination rule again to get H on a line by itself.

1	$(A \vee B) \& (G \& H)$	:PR
2	$(A \vee B)$	:&E 1
3	$(G \& H)$	:&E 1
4	H	:&E 3

In our final step, we use the conjunction introduction rule to get the conclusion, '$(A \vee B) \& H$'.

1	$(A \vee B) \,\&\, (G \,\&\, H)$	:PR
2	$(A \vee B)$	:&E 1
3	$(G \,\&\, H)$	:&E 1
4	H	:&E 3
5	$(A \vee B) \,\&\, H$	:&I 2, 4

12.3 Disjunction intro and elim

Disjunction elimination For the disjunction rules, let's start with this example:

> *Sarah is swimming or Jeff is eating a burrito.*

When is this true? Recall from §5.3 that we are using the inclusive-or. So, *Sarah is swimming or Jeff is eating a burrito* is true when either

(a) *Sarah is swimming* is true, or
(b) *Jeff is eating a burrito* is true, or
(c) both are true.

That's a lot of options, but if all we know is that Sarah is swimming *or* Jeff is eating a burrito, then we don't know precisely what either one of them is doing. But let's say that someone (whom we trust completely) tells us that, actually, Sarah is *not* swimming. This piece of information about Sarah, then, let's us safely infer that Jeff is eating a burrito. This reasoning process is an example of the DISJUNCTION ELIMINATION RULE.

disjunction elimination rule

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\neg\mathsf{B}$	
	A	$:\vee\mathrm{E}\ m, n$

m	$\mathsf{A} \vee \mathsf{B}$	
n	$\neg\mathsf{A}$	
	B	$:\vee\mathrm{E}\ m, n$

If we have '$\mathsf{A} \vee \mathsf{B}$' on a line and, on another line, we have what is either before or after the '$\vee$' with a '$\neg$' before it (i.e., '$\neg\mathsf{B}$' or '$\neg\mathsf{A}$'), then, on a new line, we can put what is on the other side of the '$\vee$'.

Disjunction introduction *Sarah is swimming or Jeff is eating a burrito* is true when *Sarah is swimming* is false (she isn't swimming) and *Jeff is eating a burrito* is true. In other words, the disjunction will be true as long as one of the disjuncts is true. This feature of disjunctions lets us make an inference that we don't use often in our everyday lives. It is a very simple inference, however. Take any sentence. We'll use *you are studying logic*. That's true. Since *you are studying logic* is true, each one of these sentences is also true:

> You are studying logic, or you are studying German.
> You are studying logic, or your mother is in Fiji.
> You are studying logic, or a dragon is on the moon.

The idea is that, if we know that a sentence is true, we can create a longer sentence by adding 'or *any sentence whatsoever*' and the disjunction will also be true. This feature of the disjunction underlies the DISJUNCTION INTRODUCTION RULE (which, again, is two similar rules).

disjunction introduction rule

m	A	
	A ∨ B	:∨I m

m	A	
	B ∨ A	:∨I m

If we have 'A' on a line, then on a new line we can repeat the 'A' and add '∨' and anything else.

Now, we will construct a proof for '$P \vee Q, R \,\&\, \neg Q \vdash P \vee S$'. In this proof, we will use the conjunction elimination rule, disjunction introduction rule, and the disjunction elimination rule.

After listing the premises, we use the conjunction elimination rule to get '$\neg Q$' on a line by itself.

1	$P \vee Q$	:PR
2	$R \,\&\, \neg Q$	:PR
3	$\neg Q$	:&E 2

Once we have the '$\neg Q$' on a line, we can (with the '$P \vee Q$' on line 1) use the disjunction elimination rule to get 'P' on line 4.

1	$P \vee Q$	:PR
2	$R \,\&\, \neg Q$	:PR
3	$\neg Q$	:&E 2
4	P	:∨E 1, 3

And then last, we use the disjunction introduction rule to get the conclusion.

1	$P \lor Q$	:PR
2	$R \,\&\, \neg Q$	:PR
3	$\neg Q$	:&E 2
4	P	:∨E 1, 3
5	$P \lor S$	:∨I 4

Introduction and elimination

When you use the rules that are in this chapter, you are applying the *patterns* given in the definition of each rule. Every pattern is different, and so you have to make sure that you understand each one. But you don't, actually, have to think beyond the patterns—although you can, and some people find it helpful to think about how the rules of derivation conform to what you learned about each logical operator in the chapter on the characteristic truth tables (chapter 7).

It can also be useful to understand why we call these *introduction* and *elimination* rules. (Or, at least, it's useful not to misunderstand why we use these terms.) The introduction rules are given this name because, in each case, we introduce a logical operator. For instance, if a proof begins this way:

1	P	:PR
2	Q	:PR

then, clearly, there is not an '&' in the proof yet. The conjunction introduction rule, however, lets us introduce one:

3	$P \,\&\, Q$	:&I 1, 2

The elimination rules, meanwhile, all, in a way, eliminate a logical operator. It might, however, be more useful to think of these as *extraction* rules because, really, what they do is allow us to extract a part of a sentence and put it on a new line. For instance, if we have this:

$$\begin{array}{l|ll} 1 & \underline{R \,\&\, T} & :\text{PR} \end{array}$$

then we can use the &E rule to take the '*R*' and put it on a line by itself:

$$\begin{array}{l|ll} 2 & R & :\&\text{E } 1 \end{array}$$

We might think of this as having eliminated the '&', and, in a way, that's what we have done. But the '*R* & *T*' is still on line 1 and can be used again in the proof. Hence, *extraction rules* is a little more accurate than *elimination rules*, but *elimination* is the commonly used term, and so we will stick with it.

Double negation

The DOUBLE NEGATION RULE is a rule of convenience that sometimes compliments the disjunction-elimination rule. (There are also times when it will be used in the proofs that are discussed in 15.1, but, for the material in this chapter, it will only be used with the disjunction-elimination rule.) First, notice that the disjunction-elimination rule is very specific. To use it, we need, on one line of our proof, a sentence with the form '$\mathsf{A} \vee \mathsf{B}$', and on another line of our proof, we need one side of the disjunction (either A or B) with a '¬' in front of it—that is, either $\neg\mathsf{A}$ or $\neg\mathsf{B}$.

This presents a problem if we have these two sentences somewhere in a proof:

$$\begin{array}{l|l} m & \neg P \vee Q \\ n & P \end{array}$$

You might think that, given those two lines, we can put '*Q*' on a new line like this:

$$\begin{array}{l|ll} m & \neg P \vee Q & \\ n & P & \\ & Q & :\vee\text{E } m, n \end{array}$$

In a sense, this is the right idea for the disjunction elimination rule—one side of the disjunction '$\neg P \lor Q$' has to be true and the 'P' on line n means that '$\neg P$' is false. Hence, we should be allowed to put 'Q' on a new line. The disjunction elimination rule, however, does not permit this. To see why, let's distinguish between NEGATION and DENIAL.

negation and denial

The NEGATION of a sentence is the sentence with a 'not' added to it.
The DENIAL of a sentence is the sentence with either a 'not' added or a 'not' removed.

For example,

1. the negation of 'today is Tuesday' is 'today is not Tuesday'.

In TFL,

2. the negation of A is $\neg\mathsf{A}$. The negation of $\neg\mathsf{A}$ is $\neg\neg\mathsf{A}$.

Meanwhile,

3. the denial of 'it is not raining' is either (a) 'it is raining' or (b) 'it is not not raining'.

In TFL,

4. the denial of $\neg\mathsf{A}$ is either A or $\neg\neg\mathsf{A}$.

To use the disjunction elimination rule, we must have the **negation** of one side of the disjunction on another line. In the example above, we have the denial of $\neg P$, not its negation on line n. The DOUBLE NEGATION RULE helps us correct this so that we can use the disjunction elimination rule more often.

Before we see how the double negation rule can help us with our derivation, let's introduce the rule. The first version of the double negation rule allows us to add two *not*s (i.e., *not not*) to a sentence in TFL—which, of course, will not change the sentence's truth value. The second version of the double negation rule allows us to remove two *nots*, although needing to do this is less common.

double negation rule

m	A	
n	¬¬A	:DN m

m	¬¬A	
n	A	:DN m

Let's say that this is the argument for which we need to provide a proof: '$\neg P \lor Q, P \vdash Q$'. After the premises, we use the double negation rule to get '$\neg\neg P$' from line 2.

1	$\neg P \lor Q$	:PR
2	P	:PR
3	$\neg\neg P$	:DN 2

The 'P' on line 2 and the '$\neg\neg P$' on line 3 have exactly the same meaning. The only difference between 'P' and '$\neg\neg P$' is their form. But now that we have '$\neg\neg P$', we have the negation of what is on the left side of the disjunction (which is '$\neg P$'). That allows us to use the disjunction elimination rule, and we can get the conclusion.

1	$\neg P \lor Q$	:PR
2	P	:PR
3	$\neg\neg P$	:DN 2
4	Q	:∨E 1, 3

Remember that, in this chapter, you will only use the double negation rule right before you use the disjunction elimination rule, and you will only use it some of the time with the disjunction elimination rule.

when not to use (left) and when to use (right) the double negation rule

m	A ∨ B	
n	¬B	
	A	:∨E m, n

m	A ∨ ¬B	
n	B	
p	¬¬B	:DN n
	A	:∨E m, p

12.4 Conditional elimination

For the conditional, we will cover the elimination rule now and the introduction rule in §12.7. Consider the following argument:

1. If the envelope is on the table, then Aleksander is in the safe house.
2. The envelope is on the table.
3. Therefore, Aleksander is in the safe house.

In this argument—which is valid—we have a conditional and then, on a separate line, the antecedent of that conditional ('the envelope is on the table'). This allows us to safely infer the consequent ('Aleksander is in the safe house'). In short, if we have a conditional and we know that the antecedent of the conditional is true, then we know that the consequent has to be true. (See also the discussion of the conditional on p. 75.) Deriving the consequent of the conditional in this way is an application of the CONDITIONAL ELIMINATION RULE. This rule is also sometimes called *modus ponens*. When we use the rule, the conditional and the antecedent of the conditional can be separated from one another, and they can appear in any order.

conditional elimination rule

$$\begin{array}{r|ll} m & A \to B & \\ n & A & \\ & B & :\!\to\!\mathrm{E}\ m, n \end{array}$$

If we have '$A \to B$' on a line and, on a separate line, we have just 'A' (i.e., the antecedent of the conditional), then, on a new line, we can put 'B' (i.e., the consequent of the conditional).

biconditional elimination rule

If we have '$A \leftrightarrow B$' on a line and, on a separate line, we have 'A', then, on a new line, we can put 'B'.

$$\begin{array}{r|ll} m & A \leftrightarrow B & \\ n & A & \\ & B & :\!\leftrightarrow\!\mathrm{E}\ m, n \end{array}$$

Or, if we have '$A \leftrightarrow B$' on a line and, on a separate line, we have 'B', then, on a new line, we can put 'A'.

$$\begin{array}{r|ll} m & A \leftrightarrow B & \\ n & B & \\ & A & :\!\leftrightarrow\!\mathrm{E}\ m, n \end{array}$$

12.5 Biconditional intro and elim

The BICONDITIONAL ELIMINATION RULE is similar to the conditional elimination rule but a bit more flexible. If you have a biconditional on one line and the left side of the biconditional on another line, you can put the right side on a new line. Or, if you have the right side, you can put the left side on a new line. Notice the difference between the conditional elimination rule and the biconditional elimination rule. There are two ways to use the biconditional elimination rule. There is only one way to use the conditional elimination rule.

In chapter 7, we said that the biconditional is "the conjunction of a conditional running in each direction." This is the basis for the BICONDITIONAL INTRODUCTION RULE. If we have both conditionals, $\mathsf{A} \rightarrow \mathsf{B}$ and $\mathsf{B} \rightarrow \mathsf{A}$ on separate lines in our proof, then we can put $\mathsf{A} \leftrightarrow \mathsf{B}$ on a new line.

biconditional introduction rule

m	$\mathsf{A} \rightarrow \mathsf{B}$	
n	$\mathsf{B} \rightarrow \mathsf{A}$	
	$\mathsf{A} \leftrightarrow \mathsf{B}$	$:\leftrightarrow\text{I } m, n$

If we have '$\mathsf{A} \rightarrow \mathsf{B}$' and '$\mathsf{B} \rightarrow \mathsf{A}$' on two lines of our proof, then, on a new line, we can put '$\mathsf{A} \leftrightarrow \mathsf{B}$'.

12.6 Some examples

We will now look at some proofs that use the rules that are covered in §§12.2 – 12.5. There are also practice exercises using these rules on p. 169.

1. For a proof of '$P \rightarrow Q, R \& P \vdash Q \& R$', we use the conjunction introduction rule, the conjunction elimination rule, and the conditional elimination rule.

1	$P \rightarrow Q$	:PR
2	$R \& P$	:PR
3	P	:&E 2
4	R	:&E 2
5	Q	:→E 1, 3
6	$Q \& R$	:&I 4, 5

2. For a proof of '$R \leftrightarrow T, P \vee T, \neg P \vdash R$', we use the disjunction elimination rule and the biconditional elimination rule.

1	$R \leftrightarrow T$	:PR
2	$P \vee T$	:PR
3	$\neg P$	:PR
4	T	:$\vee$E 2, 3
5	R	:$\leftrightarrow$E 1, 4

3. For '$C \,\&\, (D \vee \neg F), F \,\&\, G \vdash C \,\&\, (D \vee H)$', we use all five of the rules introduced in §§12.2 and 12.3.

1	$C \,\&\, (D \vee \neg F)$	:PR
2	$F \,\&\, G$	:PR
3	C	:&E 1
4	$D \vee \neg F$	:&E 1
5	F	:&E 2
6	$\neg\neg F$	:DN 5
7	D	:$\vee$E 4, 6
8	$D \vee H$	:$\vee$I 7
9	$C \,\&\, (D \vee H)$	:&I 3, 8

4. For a proof of '$(R \& T) \to Q, T \& S, R \vdash Q$', we use the conjunction elimination, conjunction introduction, and conditional elimination rules.

1	$(R \& T) \to Q$	:PR
2	$T \& S$	:PR
3	R	:PR
4	T	:&E 2
5	$R \& T$	:&I 3, 4
6	Q	:$\to$E 1, 5

5. For a proof of '$P \leftrightarrow (R \vee S), T \to R, Q \& T \vdash P$', we use the conjunction elimination, conditional elimination, disjunction introduction, and biconditional introduction rules.

1	$P \leftrightarrow (R \vee S)$	:PR
2	$T \to R$	:PR
3	$Q \& T$	:PR
4	T	:&E 3
5	R	:$\to$E 2, 4
6	$R \vee S$	:$\vee$I 5
7	P	:$\to$E 1, 5

6. And last, a proof for this argument:

$$(S \to T) \vee \neg R, (T \to S) \vee Q, R \& \neg Q \vdash T \leftrightarrow S$$

requires the conjunction elimination rule, the double negation rule, the disjunction elimination rule, and the biconditional introduction rule.

1	$(S \to T) \lor \neg R$	:PR
2	$(T \to S) \lor Q$	:PR
3	$R \,\&\, \neg Q$	:PR
4	R	:&E 3
5	$\neg Q$	:&E 3
6	$\neg\neg R$	:DN 4
7	$S \to T$	:∨E 1, 6
8	$T \to S$	:∨E 2, 5
9	$T \leftrightarrow S$	:↔I 7, 8

12.7 Conditional introduction

The CONDITIONAL INTRODUCTION RULE is a little bit more complicated than the conditional elimination rule, but, with some thought (and some practice), it is easily grasped. We'll start with this symbolization key for the sentence letters G and L:

G: Kate's German class meets today.
L: Kate's logic class meets today.

And this is our argument:

$$G \lor L \vdash \neg G \to L$$

We will go through the proof for this argument, and in the process explain the conditional introduction rule. We start by listing the premise.

1	$G \lor L$	:PR

Next, we need to make a new assumption: 'Kate's German class is *not* meeting today'. We might say that we're making this assumption "for the sake of argument" or to see where it leads. To indicate that this is an assumption that we have supplied, we put '$\neg G$' on line 2 this way:

$$\begin{array}{l|ll}
1 & G \vee L & :\text{PR} \\ \hline
2 & \quad \neg G & :\text{AS}
\end{array}$$

You will notice right away that the '$\neg G$' is indented. Whenever we make an assumption ourselves, we must indent it and the lines that follow. This creates a SUBPROOF that is set off from the rest of the proof. The assumption is cited with 'AS', and we put a line under the assumption just as we do with the final premise. With this assumption in place, we next use the disjunction elimination rule to get L on line 3.

$$\begin{array}{l|ll}
1 & G \vee L & :\text{PR} \\ \hline
2 & \quad \neg G & :\text{AS} \\
3 & \quad L & :\vee\text{E } 1, 2
\end{array}$$

The idea for the first three lines of this proof are, first, we know that *Kate's German class meets today or her logic class meets today*. (Or, at least, we are assuming that '$G \vee L$' is true because that is the premise that we were given). Next, on line 2, we are, in effect, asking, "What if her German class is not meeting today?" That is, what will follow if we make this assumption? Well, one thing that will follow is that Kate's logic class must be meeting today.

So, on line 2, we have asked, What if *Kate's German class is not meeting today*? On line 3, we have one answer: *Kate's logic class is meeting today*. Therefore, on line 4, we can use the conditional introduction rule to put these two together as *if Kate's German class is not meeting today, then Kate's logic class is meeting today*.

$$\begin{array}{l|ll}
1 & G \vee L & :\text{PR} \\ \hline
2 & \quad \neg G & :\text{AS} \\
3 & \quad L & :\vee\text{E } 1, 2 \\
4 & \neg G \rightarrow L & :\rightarrow\text{I } 2\text{–}3
\end{array}$$

For this final step, we have gone back to the original vertical line of the proof.

When we use the conditional introduction rule, the assumption that we make will always be the antecedent of the conditional. The last line of the subproof, meanwhile, will always be the consequent of the conditional.

conditional introduction rule

i	| | A	:AS
j	| | B	
	| A → B	:→I i–j

We begin by making an assumption: 'A'. We then derive 'B'. Once that is done, we know that *if* A, *then* B, and we can put the conditional on the line after the subproof.
There can be as many or as few lines as needed between lines i and j.
The lines cited are the range for the subproof, beginning with the line where the assumption is.

To simplify matters at the beginning of section 12.1, I only use the term *premises* to refer to the premises of an argument. A premise, however, is just a type of assumption. It is an assumption because we are taking it as given, and so it requires no justification – just like the assumption that we make at the beginning of a subproof.

Subproofs Lines i through j are called a SUBPROOF. These are the rules for subproofs:

1. Once a subproof has been closed, none of the lines in the subproof can be used again. (The conditional A → B can be used later in the proof because it is outside of the subproof.)
2. A subproof is closed by the application of the conditional introduction rule – or, as you will see shortly, the negation introduction or the negation elimination rules.
3. When we close a subproof, the assumption made at the beginning of the subproof has been *discharged*.
4. A proof is not complete until every assumption that we have made (and so not counting the premises) is discharged.

12.8 Some more examples

Each of these examples uses the conditional introduction rule.

1. We will start with a proof for this argument:

   $$P \to Q, Q \to R \vdash P \to R$$

   We start by listing both of our premises. Next, since we want '$(P \to R)$', we assume the antecedent of that conditional.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS

   Now, even though it is an assumption that we've introduced, since 'P' is on a line by itself (and the subproof has not yet been closed), we can use it for our next step. With 'P' and the '$P \to Q$' on line 1, we can use the conditional elimination rule to get 'Q'.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:$\to$E 1, 3

   With the 'Q' on line 4 and '$Q \to R$' on line 2, we can use the conditional elimination rule again; this time to get 'R'. So, by assuming 'P', we were able to get 'R'. Last, we apply the conditional introduction rule, which discharges our assumption and completes the proof.

1	$P \rightarrow Q$	:PR
2	$Q \rightarrow R$	:PR
3	$\quad P$	:AS
4	$\quad Q$	:→E 1, 3
5	$\quad R$	:→E 2, 4
6	$P \rightarrow R$	:→I 3–5

2. Next, let's construct a proof for this argument: '$F \rightarrow (G \,\&\, H) \vdash F \rightarrow G$'. We proceed this way:

1	$F \rightarrow (G \,\&\, H)$	:PR
2	$\quad F$	:AS
3	$\quad G \,\&\, H$	:→E 1, 2
4	$\quad G$	:&E 3
5	$F \rightarrow G$	:→I 2–4

3. As you know, the biconditional elimination rule is similar to the conditional elimination rule. (But they are not the same. See p. 153 to compare them.) We should also, however, be able to start with a biconditional, say '$B \leftrightarrow C$' and derive either of the conditionals: '$B \rightarrow C$' or '$C \rightarrow B$'. This is easily done with the conditional introduction rule.

1	$B \leftrightarrow C$	:PR
2	$\quad B$	:AS
3	$\quad C$	:↔E 1, 2
4	$B \rightarrow C$	:→I 2–3

And, with a similar proof, we can also derive '$C \rightarrow B$'.

4. In the proof for '$\neg P \vee (R \& Q) \vdash P \rightarrow Q$', we will use the conditional introduction rule as well as double negation rule and disjunction-elimination rule.

1	$\neg P \vee (R \,\&\, Q)$	:PR
2	$\quad P$	:AS
3	$\quad \neg\neg P$	:DN 2
4	$\quad R \,\&\, Q$	:$\vee$E 1, 3
5	$\quad Q$	:&E 4
6	$P \rightarrow Q$	:$\rightarrow$I 2–5

12.9 Negation introduction and elimination

Here is a simple mathematical argument in English:

1. Assume that there is some greatest natural number. Call it G.
2. That number plus one is also a natural number.
3. $G + 1$ is greater than G.
4. Thus, G is the greatest natural number (accordingt to 1), and there is a natural number greater than G (according to 3).
5. The previous line is a contradiction.
6. Therefore, the assumption that we made on line 1 is false. There is no greatest natural number.

This type of argument is traditionally called a *reductio*. Its full Latin name is *reductio ad absurdum*, which means 'reduction to absurdity' (although *absurdity* in the sense that we generally use the word today isn't part of this). In a reductio, we assume something for the sake of argument—for example, that there is a greatest natural number. Then we show that the assumption leads to two contradictory sentences—for example, 'G is the greatest natural number' and 'G is not the greatest natural number.' In this way, we have shown that the original assumption must be false, which means that the denial of the assumption is true.

Our two negation rules (which are basically the same rule) formalize this reasoning process.

negation introduction rule

m	| | A	:AS
n	| | B	
p	| | ¬B	
	| ¬A	:¬I m–p

Assume A. Derive a contradiction. (That is, get B and ¬B on the last two lines of the subproof). Exit the subproof, and put ¬A on the first line after the subproof.

There can be as many or as few lines as needed between lines m and n, but n and p have to be consecutive lines. B and ¬B can be any contradiction that it is possible to derive, and A can be one half of the contradiction.

negation elimination rule

m	| | ¬A	:AS
n	| | B	
p	| | ¬B	
	| A	:¬E m–p

Assume ¬A. Derive a contradiction. (That is, get B and ¬B on the last two lines of the subproof). Exit the subproof, and put A on the first line after the subproof.

Notice that, just as we do when using the conditional introduction rule, we begin by making an assumption. The subproof that follows is indented, and the assumption that we made must be discharged by applying either the negation introduction rule or the negation elimination rule.

When using either of the negation rules, the last two lines of the subproof must be an explicit contradiction: B on one line and its negation, ¬B, on the next line (or vice versa). Those two lines cannot be separated. When you cite the rule, however, the lines that you give are the lines for the whole subproof (starting with the assumption), not just the two lines containing the contradiction.

Reiteration

To get a contradiction on the last two lines of a subproof, we will usually have to move a sentence that is on an earlier line to the last or second-to-last line of the subproof. This is done with the REITERATION RULE. Just as the double negation rule is a rule of convenience that sometimes compliments the disjunction elimination rule, the reiteration rule is a rule of convenience that compliments the negation elimination and negation introduction rules.

reiteration rule

m	A	
n	A	:R m

To demonstrate both the negation elimination rule and the reiteration rule, we will go through the proof for this argument: '$\neg P \rightarrow \neg Q, Q \vdash P$'. Looking at the argument, you'll notice that our conclusion is 'P', but we cannot get 'P' by using &E, ∨E, →E, or ↔E. This means that we will need to use one of our negation rules.

After the premises, we make the assumption that we need for negation elimination. Since, ultimately, we want 'P', we will assume '$\neg P$' so that, once we discharge that assumption (and close the subproof), we will have the 'P' that we are after.

1	$\neg P \rightarrow \neg Q$	:PR
2	Q	:PR
3	$\neg P$	:AS

We then use the conditional elimination rule to get $\neg Q$ on line 4.

1	$\neg P \to \neg Q$	:PR
2	Q	:PR
3	$\quad \neg P$	:AS
4	$\quad \neg Q$	:$\to$E 1, 3

The Q on line 2 and $\neg Q$ on line 4 are a contradiction, but to use the negation elimination rule we need to have 'Q' on line 5. To get it there, we use the reiteration rule.

1	$\neg P \to \neg Q$	:PR
2	Q	:PR
3	$\quad \neg P$	:AS
4	$\quad \neg Q$	:$\to$E 1, 3
5	$\quad Q$	:R 2

Now that '$\neg Q$' and 'Q' are on consecutive lines, we can use the negation elimination rule to discharge the assumption that we made, and that gives us the conclusion we are after: 'P'.

1	$\neg P \to \neg Q$	:PR
2	Q	:PR
3	$\quad \neg P$	:AS
4	$\quad \neg Q$	:$\to$E 1, 3
5	$\quad Q$	:R 2
6	P	:$\neg$E 3–5

We just used the negation elimination rule. The negation introduction rule is, essentially, the same. Whether you use the negation introduction rule or negation elimination rule is just a function of whether you want '¬A' or 'A' on the line after the subproof.

12.10 Even more examples

The negation introduction rule or the negation elimination rule is used in each of these proofs.

1. $P \to Q, \neg Q \vdash \neg P$

$$
\begin{array}{r|lll}
1 & P \to Q & & :\text{PR} \\
2 & \underline{\neg Q} & & :\text{PR} \\
3 & & \vert\ \underline{P} & :\text{AS} \\
4 & & \vert\ Q & :{\to}\text{E } 1, 3 \\
5 & & \vert\ \neg Q & :\text{R } 2 \\
6 & \neg P & & :\neg\text{I } 3\text{–}5
\end{array}
$$

2. $P \to \neg Q \vdash \neg(P \,\&\, Q)$

$$
\begin{array}{r|lll}
1 & \underline{P \to \neg Q} & & :\text{PR} \\
2 & & \vert\ \underline{P \,\&\, Q} & :\text{AS} \\
3 & & \vert\ P & :\&\text{E } 2 \\
4 & & \vert\ \neg Q & :{\to}\text{E } 1, 3 \\
5 & & \vert\ Q & :\&\text{E } 2 \\
6 & \neg(P \,\&\, Q) & & :\neg\text{I } 2\text{–}5
\end{array}
$$

3. $B \,\&\, C, \neg(B \,\&\, D) \vdash \neg D$

1	$B \,\&\, C$	:PR
2	$\neg(B \,\&\, D)$	:PR
3	$\quad D$	:AS
4	$\quad B$	:&E 1
5	$\quad B \,\&\, D$	:&I 3, 4
6	$\quad \neg(B \,\&\, D)$	:R 2
7	$\neg D$	:¬I 3–6

4. The proof for '$\neg P \vdash P \to Q$' requires two subproofs. First, we assume 'P' so that we can use the conditional introduction rule at the end of the proof. Then, we assume '$\neg Q$' so that we can use the negation elimination rule and get 'Q' on the last line of the first subproof.

1	$\neg P$	:PR
2	$\quad P$	:AS
3	$\quad\quad \neg Q$	:AS
4	$\quad\quad P$	:R 2
5	$\quad\quad \neg P$	:R 1
6	$\quad Q$	:¬E 3–5
7	$P \to Q$	:→I 2–6

5. The proof for '$Q \lor S, Q \to T, S \to T \vdash T$' also requires two subproofs.

1	$Q \lor S$	:PR
2	$Q \to T$	:PR
3	$S \to T$	:PR
4	$\quad \neg T$	:AS
5	$\quad\quad \neg Q$	:AS
6	$\quad\quad S$	:$\lor$E 1, 5
7	$\quad\quad T$	:$\to$E 3, 6
8	$\quad\quad \neg T$	:R 4
9	$\quad Q$	:$\neg$E 5–8
10	$\quad T$	:$\to$E 2, 9
11	$\quad \neg T$	:R 4
12	T	:$\neg$E 4–11

12.11 Invalid arguments

In this chapter, we have taken it for granted that each argument that we have encountered has been valid. The purpose of providing a proof is (1) to confirm that it is valid and (2) to show why it is valid – that is, to lay out each step that takes us from the premises to the conclusion. If an argument is invalid, however, we are stuck. It is impossible to provide a correct proof of an invalid argument using the rules given in this chapter. At the same time, not being able to provide a proof for an argument doesn't mean that the argument is invalid. Perhaps the proof is just too complicated for us to figure out.

In chapter 11, we discussed some reasons to prefer natural deduction to truth tables for checking that an argument is valid. To show that an argument is invalid, however, creating a truth table is not merely a superior method, it is our only option.

12.12 Practice exercises

A. Give a proof for each argument using the rules from §§12.2 – 12.5.

1. $\neg P \to (Q \lor P), \neg P \vdash Q$
   Note: After you list the premises on lines 1 and 2, notice that $\neg P$ is the antecedent of the sentence on line 1 and $\neg P$ is by itself on line 2. This means using the conditional elimination rule is an option.
2. $P \to (Q \lor \neg P), P \vdash Q$
   Note: This problem is similar to the one right above, but it's not exactly the same. If you're not sure how to do this one, look at the subsection on the double negation rule (pp. 149 - 152).
3. $D \,\&\, H, H \leftrightarrow J \vdash J \lor N$
4. $R \,\&\, S, (S \lor Q) \to T \vdash T$
   Note: Since $(S \lor Q)$ is the antecedent of the conditional on line 2, to use the conditional elimination rule, you need to have $(S \lor Q)$ on a line by itself. Since $(S \lor Q)$ is a disjunction, you will need to use the disjunction introduction rule to get it. (Check how that rule works if you don't remember: p. 147.)
5. $G \,\&\, (H \,\&\, J), (H \lor M) \to K \vdash K$
6. $P \,\&\, (Q \lor R), P \to \neg R \vdash Q$
7. $(P \lor \neg Q) \leftrightarrow R, R \,\&\, Q \vdash P \,\&\, R$
8. $(R \,\&\, T) \to Q, R \lor \neg P, P \,\&\, T \vdash Q$
   Note: After you list the premises for this proof, you'll see that the conclusion is the consequent of the conditional on line 1. To use the conditional elimination rule, you need to use the two other sentences (on lines 2 and 3) to get the antecedent on a line by itself. That will take several steps. Think about whether you need to use the double negation rule for one of those steps.
9. $S \to T, Q \,\&\, \neg R, \neg R \leftrightarrow (T \to S) \vdash S \leftrightarrow T$
10. $(L \lor M) \to N, P \leftrightarrow N, L \vdash L \,\&\, P$
11. $(P \,\&\, R) \leftrightarrow (S \lor T), P \to Q, T \vdash Q$
12. $S \to T, T \to S, T \leftrightarrow (T \leftrightarrow S) \vdash S$

B. Give a proof for each argument.

1. $P \rightarrow (Q \rightarrow R) \vdash (P \,\&\, Q) \rightarrow R$
2. $Q \rightarrow R \vdash (Q \,\&\, S) \rightarrow (R \vee T)$
3. $M \,\&\, (\neg N \rightarrow \neg M) \vdash (N \,\&\, M) \vee \neg M$
4. $(Z \,\&\, K) \leftrightarrow (Y \,\&\, M), D \,\&\, (D \rightarrow M) \vdash Y \rightarrow Z$
5. $C \rightarrow (E \,\&\, G), \neg C \rightarrow G \vdash G$
6. $\neg(P \rightarrow Q) \vdash \neg Q$
7. $S \leftrightarrow T \vdash S \leftrightarrow (T \vee S)$
8. $D \vee F, D \rightarrow G, F \rightarrow H \vdash G \vee H$

12.13 Answers

A.

1. $\neg P \rightarrow (Q \vee P), \neg P \vdash Q$

1	$\neg P \rightarrow (Q \vee P)$	:PR
2	$\neg P$	:PR
3	$Q \vee P$	:$\rightarrow$E 1, 2
4	Q	:$\vee$E 2, 3

2. $P \rightarrow (Q \vee \neg P), P \vdash Q$

1	$P \rightarrow (Q \vee \neg P)$	:PR
2	P	:PR
3	$Q \vee \neg P$	:$\rightarrow$E 1, 2
4	$\neg\neg P$	:DN 2
5	Q	:$\vee$E 3, 4

3. $D \,\&\, H, H \leftrightarrow J \vdash J \vee N$

1	$D \,\&\, H$	:PR
2	$H \leftrightarrow J$	:PR
3	H	:&E 1
4	J	:$\leftrightarrow$E 2, 3
5	$J \vee N$	:$\vee$I 4

4. $R \,\&\, S, (S \vee Q) \rightarrow T \vdash T$

1	$R \,\&\, S$	:PR
2	$(S \vee Q) \rightarrow T$	:PR
3	S	:&E 1
4	$S \vee Q$	:$\vee$I 3
5	T	:$\rightarrow$E 2, 4

5. $G \,\&\, (H \,\&\, J), (H \vee M) \rightarrow K \vdash K$
Note: In this proof, just like in the previous one, you need to use the disjunction introduction rule to get an antecedent—in this case, '$(H \vee M)$'—on a line by itself (so that you can then use the conditional elimination rule). This is a good trick to remember.

1	$G \,\&\, (H \,\&\, J)$	:PR
2	$(H \vee M) \rightarrow K$	:PR
3	$H \,\&\, J$	:&E 1
4	H	:&E 3
5	$H \vee M$	:$\vee$I 4
6	K	:$\rightarrow$I 2, 5

6. $P \,\&\, (Q \vee R), P \rightarrow \neg R \vdash Q$

1	$P \,\&\, (Q \vee R)$	:PR
2	$P \rightarrow \neg R$	:PR
3	P	:&E 1
4	$\neg R$	:$\rightarrow$I 2, 3
5	$Q \vee R$	:&E 1
6	Q	:$\vee$E 4, 5

7. $(P \vee \neg Q) \leftrightarrow R, R \,\&\, Q \vdash P \,\&\, R$

1	$(P \vee \neg Q) \leftrightarrow R$	:PR
2	$R \,\&\, Q$	:PR
3	R	:&E 2
4	$P \vee \neg Q$	:↔E 1, 3
5	Q	:&E 2
6	$\neg\neg Q$	:DN 5
7	P	:∨E 4, 6
8	$P \,\&\, R$	:&I 3, 7

8. $(R \,\&\, T) \rightarrow Q, R \vee \neg P, P \,\&\, T \vdash Q$

1	$(R \,\&\, T) \rightarrow Q$	:PR
2	$R \vee \neg P$	:PR
3	$P \,\&\, T$	:PR
4	P	:&E 3
5	$\neg\neg P$	:DN 4
6	R	:∨E 2, 5
7	T	:&E 3
8	$R \,\&\, T$	:&I 6, 7
9	Q	:→E 1, 8

9. $S \to T, Q \,\&\, \neg R, \neg R \leftrightarrow (T \to S) \vdash S \leftrightarrow T$

1	$S \to T$	:PR
2	$Q \,\&\, \neg R$	:PR
3	$\neg R \leftrightarrow (T \to S)$	:PR
4	$\neg R$	:&E 2
5	$T \to S$	:↔E 3, 4
6	$S \leftrightarrow T$	:↔I 1, 5

10. $(L \vee M) \to N, P \leftrightarrow N, L \vdash L \,\&\, P$

1	$(L \vee M) \to N$	:PR
2	$P \leftrightarrow N$	:PR
3	L	:PR
4	$L \vee M$	:∨I 3
5	N	:→E 1, 4
6	P	:↔E 2, 5
7	$L \,\&\, P$	:&I 3, 6

11. $(P \,\&\, R) \leftrightarrow (S \vee T), P \to Q, T \vdash Q$

1	$(P \,\&\, R) \leftrightarrow (S \vee T)$	:PR
2	$P \to Q$	:PR
3	T	:PR
4	$S \vee T$	:∨I 3
5	$P \,\&\, R$	:↔E 1, 4
6	P	:&E 5
7	Q	:→E 2, 6

12. $S \rightarrow T, T \rightarrow S, T \leftrightarrow (T \leftrightarrow S) \vdash S$

1	$S \rightarrow T$	:PR
2	$T \rightarrow S$	:PR
3	$T \leftrightarrow (T \leftrightarrow S)$	:PR
4	$T \leftrightarrow S$	:$\leftrightarrow$I 1, 2
5	T	:$\leftrightarrow$E 3, 4
6	S	:$\rightarrow$E 2, 5

B.

1. $P \rightarrow (Q \rightarrow R) \vdash (P \,\&\, Q) \rightarrow R$

1	$P \rightarrow (Q \rightarrow R)$	:PR
2	$\quad P \,\&\, Q$	:AS
3	$\quad P$	:&E 2
4	$\quad Q \rightarrow R$	:$\rightarrow$E 1, 3
5	$\quad Q$	:&E 2
6	$\quad R$	:$\rightarrow$E 4, 5
7	$(P \,\&\, Q) \rightarrow R$	:$\rightarrow$I 2–6

2. $Q \rightarrow R \vdash (Q \,\&\, S) \rightarrow (R \vee T)$

1	$Q \rightarrow R$	:PR
2	$\quad Q \,\&\, S$	:AS
3	$\quad Q$	:&E 2
4	$\quad R$	:$\rightarrow$E 1, 3
5	$\quad R \vee T$	:$\vee$I 4
6	$(Q \,\&\, S) \rightarrow (R \vee T)$	:$\rightarrow$I 2–5

3. $M \mathbin{\&} (\neg N \to \neg M) \vdash (N \mathbin{\&} M) \vee \neg M$

1	$M \mathbin{\&} (\neg N \to \neg M)$	:PR
2	M	:&E 1
3	$\neg N \to \neg M$	:&E 1
4	$\mid\ \neg N$	:AS
5	$\mid\ \neg M$	:→E 3, 4
6	$\mid\ M$	:R 2
7	N	:¬E 4–6
8	$N \mathbin{\&} M$	:&I 2, 7
9	$(N \mathbin{\&} M) \vee \neg M$	:∨I 8

4. $(Z \mathbin{\&} K) \leftrightarrow (Y \mathbin{\&} M), D \mathbin{\&} (D \to M) \vdash Y \to Z$

1	$(Z \mathbin{\&} K) \leftrightarrow (Y \mathbin{\&} M)$	:PR
2	$D \mathbin{\&} (D \to M)$	:PR
3	D	:&E 2
4	$D \to M$	:&E 2
5	M	:→E 3, 4
6	$\mid\ Y$	:AS
7	$\mid\ Y \mathbin{\&} M$	:&I 5, 6
8	$\mid\ Z \mathbin{\&} K$	:↔E 1, 7
9	$\mid\ Z$	:&E 8
10	$Y \to Z$	:→I 6–9

5. $C \rightarrow (E \mathbin{\&} G), \neg C \rightarrow G \vdash G$

1	$C \rightarrow (E \mathbin{\&} G)$	:PR
2	$\neg C \rightarrow G$	:PR
3	$\quad \neg G$	:AS
4	$\quad\quad C$	:AS
5	$\quad\quad E \mathbin{\&} G$	:$\rightarrow$E 1, 4
6	$\quad\quad G$	:&E 5
7	$\quad\quad \neg G$	:R 3
8	$\quad \neg C$	:$\neg$I 4–7
9	$\quad G$	:$\rightarrow$E 2, 8
10	$\quad \neg G$	:R 3
11	G	:$\neg$E 3–10

6. $\neg(P \rightarrow Q) \vdash \neg Q$

1	$\neg(P \rightarrow Q)$	:PR
2	$\quad Q$	:AS
3	$\quad\quad P$	:AS
4	$\quad\quad Q$	:R 2
5	$\quad P \rightarrow Q$	:$\rightarrow$I 3–4
6	$\quad \neg(P \rightarrow Q)$	:R 1
7	$\neg Q$	:$\neg$I 2–6

7. $S \leftrightarrow T \vdash S \leftrightarrow (T \vee S)$

1	$S \leftrightarrow T$	:PR
2	$\quad S$	:AS
3	$\quad T$	:$\leftrightarrow$E 1, 2
4	$\quad T \vee S$	:$\vee$I 3
5	$S \rightarrow (T \vee S)$	:$\rightarrow$I 2–4
6	$\quad T \vee S$	:AS
7	$\quad\quad \neg S$	:AS
8	$\quad\quad T$	:$\vee$E 6, 7
9	$\quad\quad S$	:$\leftrightarrow$E 1, 8
10	$\quad\quad \neg S$	:R 7
11	$\quad S$	:$\neg$E 7–10
12	$(T \vee S) \rightarrow S$	:$\rightarrow$I 6–11
13	$S \leftrightarrow (T \vee S)$	:$\leftrightarrow$I 5, 12

8. $D \vee F, D \to G, F \to H \vdash G \vee H$

1	$D \vee F$	:PR
2	$D \to G$	:PR
3	$F \to H$	:PR
4	$\quad \neg(G \vee H)$	:AS
5	$\quad\quad \neg D$	:AS
6	$\quad\quad F$	:$\vee$E 1, 5
7	$\quad\quad H$	:$\to$E 3, 6
8	$\quad\quad G \vee H$	:$\vee$I 7
9	$\quad\quad \neg(G \vee H)$	:R 4
10	$\quad D$	:$\neg$E 5–9
11	$\quad G$	:$\to$E 2, 10
12	$\quad G \vee H$	:$\vee$I 11
13	$\quad \neg(G \vee H)$	:R 4
14	$G \vee H$	:$\neg$E 4–13

13 Proofs in Carnap

Creating proofs in Carnap is not difficult. To type the connectives, use the symbols on the right in table 13.1.

Carnap will number the lines automatically. After the TFL sentence on each line, there has to be a colon (':') before the 'PR', 'AS', or the rule. Carnap is flexible with the spacing on a line, but as a guideline, put a tab space between the sentence and ':PR', ':AS', or the rule (:→E, :∨I, etc.). Also indent subproofs with a tab space. (Carnap will let you use more or fewer spaces, but a subproof has to be indented some amount.)

To create a proof, you are given an interface like the one shown in figure 13.1. As you can see, the argument is given at the top. In this case, the premises are $P \to \neg Q$ and $R \,\&\, P$, and the conclusion is $\neg Q$. (The premises are separated by commas. The premises and the conclusion are separated by the turnstile ('⊢').)

Begin by listing the premises, and don't forget to put ':PR' after each one. If there is a problem with a line — the sentence isn't formed correctly, the rule you've cited isn't being used correctly, or there's some other mistake — Carnap will put ? or ⚠ at the end of the line. When the line is ok, you will get a '+'. We finish this proof using the &E and →E rules (figure 13.3). When the proof is correct, the box containing the argument will turn green, and the proof can be submitted.

TFL OPERATOR	IN CARNAP
¬	~
&	&
∨	v (lowercase v)
→	-> (dash, greater than sign)
↔	<->

Table 13.1

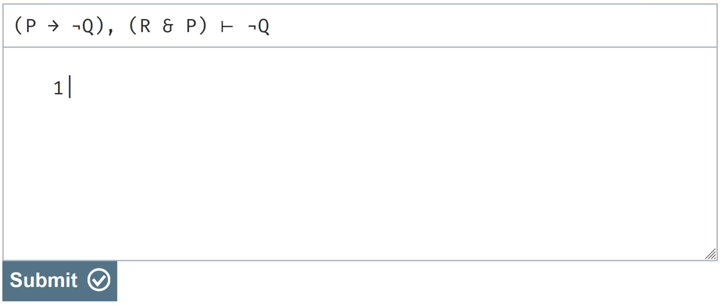


Figure 13.1

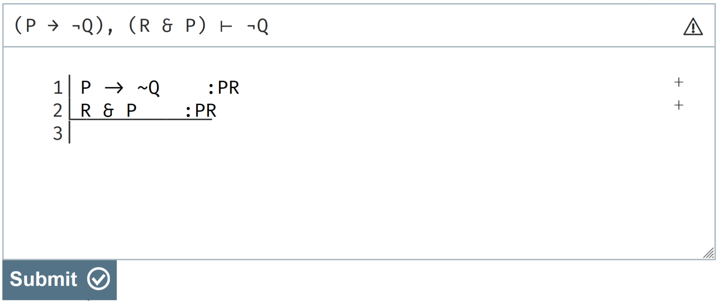


Figure 13.2

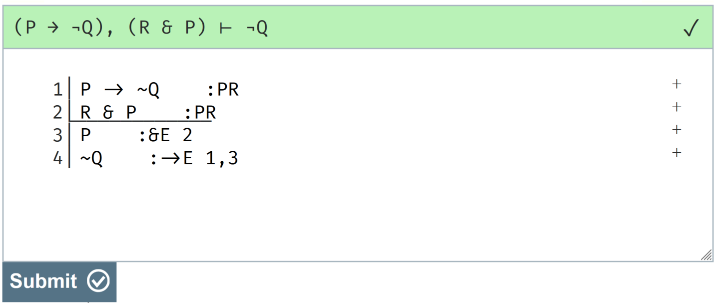


Figure 13.3

Our next example, $(P \vee Q) \vdash (\neg P \rightarrow Q)$, requires a subproof. We begin as before. To create the subproof, put a tab space before $\neg P$ and put ':AS' at the end of the line (figure 13.4). Since the next line is also part of the subproof, we again need a tab before the Q. We end the subproof (and discharge the assumption) with the $\rightarrow$I rule. $\neg P \rightarrow Q$ is not indented (so no tabs or spaces before the $\neg P$). That's the conclusion, and so if everything is correct, Carnap will give you the green bar and you can submit the proof (figure 13.5).

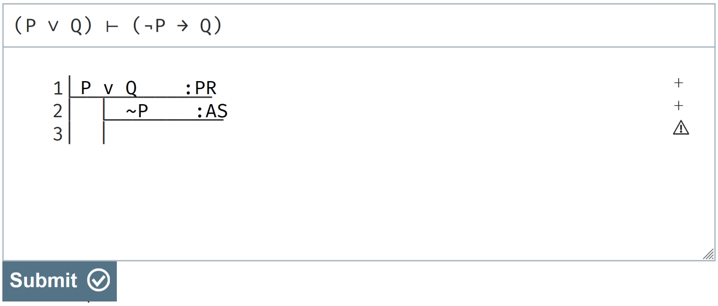


Figure 13.4

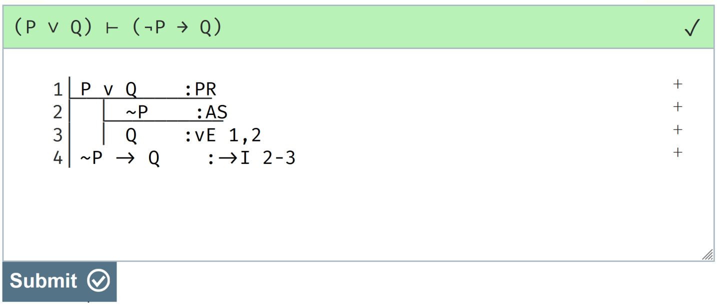


Figure 13.5

Although creating proofs in Carnap is not difficult, you do have to be careful. Creating a program that can verify proofs that use only the rules of derivation given in chapter 12 is relatively simple because there are only a small number of rules and, to produce proofs of valid arguments, we follow those rules very strictly. But, as a consequence, Carnap is not designed to understand what you are trying to do if you deviate from the rules, even if it is a minor deviation or an innocent mistake. So, some reminders:

1. As long as '¬' is not the main logical operator, you can drop the outermost parentheses. All other parentheses have to be used.
2. Capitalize 'PR', 'AS','E', 'I' (in the rules), and all atomic sentences.

3. Don't forget the ':' right before PR, AS, or the rule that you are citing.
4. There is no space between the &, ∨, →, ↔, or ¬ and the 'E' or 'I'.
5. There is a space (and no punctuation) after the 'E' or 'I'.
6. There is a comma between the two lines that have to be cited for &I, ∨E, →E, and ↔E (e.g., '→E 2,4').
7. There is a dash between the two lines that have to be cited for →I, ¬I, and ¬E (e.g., '¬E 4-6').

14 Some strategies

There is no simple recipe for constructing proofs, and there is no substitute for practice. Here, however, are some questions to ask yourself and some strategies to keep in mind.

1. Do you know all of the rules? **If you don't have them memorized yet, then they should be written on a sheet of paper that you have next to you while you're working.**
2. Are there steps that you can take without making an assumption? If yes, is it worth taking those steps?
3. If you're not sure how to proceed, but you can do conjunction elimination, conditional elimination, disjunction elimination, or biconditional elimination, then do them just to see what happens.

The theme for 4 – 7 is "think ahead." Some amount of trial and error is often necessary, but, especially when you are constructing a proof that will contain a subproof, it's important to think about how each step that you take will affect the later parts of your proof.

4. If an assumption is needed, is it for →I, ¬I, or ¬E? **Don't make an assumption if you don't know which of these rules you plan to use when you close the subproof.**
5. If an assumption is needed, what should it be? (If you want to get $P \to Q$, then you're going to use →I and your assumption should be P.)
6. If you make an assumption, then you should know what you want on either the last line or the last two lines of your subproof.
    a. If you're using →I, then you will need the consequent of the conditional on the last line of the subproof.

b. If you're using ¬I or ¬E, then you need a contradiction on the last two lines of your subproof, although that can be any contradiction. It doesn't have to be related to the assumption.

7. Sometimes it is useful to work backwards from the conclusion. The conclusion, of course, will be the last line of your proof, and you can, if you wish, put it at the bottom of the proof anytime. For example, let's say that you need to provide a proof for this argument: $P \to (\neg Q \to R) \vdash (P \,\&\, \neg Q) \to R$. You can begin this way:

$$
\begin{array}{l|l l}
1 & \underline{P \to (\neg Q \to R)} & :\text{PR} \\
 & & \\
 & & \\
 & & \\
 & (P \,\&\, \neg Q) \to R &
\end{array}
$$

Knowing that you need to arrive at a conditional, you also know these three things: (1) you need to use the conditional-introduction rule, (2) what your assumption should be, and (3) what will be on the last line of your subproof.

$$
\begin{array}{l|l l}
1 & \underline{P \to (\neg Q \to R)} & :\text{PR} \\
2 & \quad | \; \underline{P \,\&\, \neg Q} & :\text{AS} \\
 & \quad | & \\
 & \quad | & \\
 & \quad | \; R & \\
 & (P \,\&\, \neg Q) \to R & :\to\text{I}
\end{array}
$$

Sketching out a proof in this way is easy to do when you are writing on paper. If you are doing it in Carnap, be careful of the spacing that you put on each blank line.

8. The negation introduction and negation elimination rules are a last resort. Use them when you can't use any of the other rules. When you do use them, always have in mind that, when you complete the subproof, you will have the opposite of the assumption. Hence, a good guideline is to make the assumption the opposite of the conclusion.
   (If you have to make two assumptions – and both assumptions will be discharged with one of these rules – this guideline only applies to the first assumption. Determining the best choice for a second assumption sometimes takes a little trial and error.)
9. **Persist**. Try different things. If one approach fails, then try something else.

15 Proof-theoretic concepts

15.1 Theorems

You are familiar with arguments that have this form:

$$\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash \mathsf{C}$$

We may also, however, have a sentence for which it is possible to give a proof with no premises: $\vdash \mathsf{C}$. In this case, we say that C is a THEOREM.

> Theorem
>
> C is a THEOREM if and only if $\vdash \mathsf{C}$

One such sentence is '$\neg(P \& \neg P)$'. To show that this sentence is a theorem, we give a proof that has no premises and no undischarged assumptions. To get started, we do, however, have to make an assumption. We will assume '$P \& \neg P$'. Once we show that this assumption leads to contradiction, we can discharge it and we will have '$\neg(P \& \neg P)$'. This is the proof:

1	| | $P \& \neg P$	
2	| | P	:&E 1
3	| | $\neg P$	:&E 1
4	| $\neg(P \& \neg P)$	:¬I 1–3

This theorem, '$\vdash \neg(P \& \neg P)$' is an instance of what is sometimes called *the law of non-contradiction*.

To show that a sentence is a theorem, we just have to find a suitable proof. On the other hand, it is not possible to show that a sentence is *not* a theorem this same way. To show that a sentence is not a theorem with our natural deduction system, we would have to demonstrate, not just that certain proof strategies fail, but that *no* proof is possible. Even if we fail in trying to give a proof for a sentence in a thousand different ways, perhaps the proof is just too long and complex for us to figure out.

15.2 Equivalent, consistent, and inconsistent

In §9.2, we defined *equivalent* in terms of truth tables, namely, if two sentences have the same truth value on every line of a truth table, then they are equivalent. We can also show that two sentences are equivalent using our natural deduction system. To indicate that we have shown that the two sentences are equivalent with a derivation (or actually with two derivations), we will call this equivalence PROVABLY EQUIVALENT.

Provably equivalent

Two sentences A and B are PROVABLY EQUIVALENT iff each can be derived from the other. I.e., $A \vdash B$ and $B \vdash A$.
(Equivalently, A and B are PROVABLY EQUIVALENT if $\vdash A \leftrightarrow B$.)

As in the case of showing that a sentence is a theorem, it is relatively easy to show that two sentences are provably equivalent: it just requires a pair of proofs. Showing that sentences are *not* provably equivalent is not possible for t he same reason that it isn't possible to show that a sentence is not a theorem. Even if we fail to produce two proofs showing that two sentences are provably equivalent, that doesn't mean that the proofs don't exist. It just means that we've failed to figure out what they are.

We also, in §9.3, defined *jointly inconsistent* using truth tables: sentences are jointly inconsistent if there is no line on a truth table where they are all true. Again, we can show that two or more sentences are jointly inconsistent with our natural deduction system.

Provably inconsistent

The sentences $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are PROVABLY INCONSISTENT iff, from them, a contradiction can be derived. I.e. $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \vdash (\mathsf{B} \,\&\, \neg\mathsf{B})$.

To show that a set of sentences are provably inconsistent, we use the sentences as premises and then derive a contradiction. (Any contradiction will do.) For instance, this proof demonstrates that $P \,\&\, Q$ and $\neg P \vee \neg Q$ are provably inconsistent.

1	$P \,\&\, Q$	:PR
2	$\neg P \vee \neg Q$	:PR
3	P	:&E 1
4	$\neg\neg P$	:DN 3
5	$\neg Q$	:∨E 2, 4
6	Q	:&E 1

Showing that some set of sentences are *not* provably inconsistent is, as you might guess at this point, not possible. Doing so would require showing, not just that we have failed to derive a contradiction from a set a sentences, but that no such derivation is possible.

Table 15.1 summarizes what we have covered in this chapter. As we will discuss in the next chapter, when the presence (or the absence) of a logical property cannot be demonstrated using our natural deduction system, we have to resort to using a truth table.

TO CHECK	THAT IT IS	THAT IT IS NOT
theorem	one proof	not possible with proofs
equivalent	two proofs	not possible with proofs
inconsistent	one proof	not possible with proofs
consistent	not possible with proofs	one proof

Table 15.1: This table summarizes what is required to check each of these logical notions.

15.3 Practice exercises

A. Give a proof for each of these theorems.

1. $\vdash O \to O$
2. $\vdash S \to (S \vee R)$
3. $\vdash N \vee \neg N$
4. $\vdash \neg((R \vee T) \mathbin{\&} (\neg R \mathbin{\&} \neg T))$
5. $\vdash (R \leftrightarrow M) \to (M \to R)$
6. $\vdash J \leftrightarrow [J \vee (L \mathbin{\&} \neg L)]$
7. $\vdash (P \to Q) \vee (Q \to P)$

B. Show that each of the following pairs of sentences are provably equivalent. (To indicate that the inference from the premise to the conclusion goes from the first sentence to the second and vice versa, we use the symbols $\dashv\vdash$.)

1. $T \to S \dashv\vdash \neg S \to \neg T$
2. $R \to Q \dashv\vdash \neg(R \mathbin{\&} \neg Q)$

15.4 Answers

A.

1. $\vdash O \to O$

1	$\mid\ \mid O$	:AS
2	$\mid\ \mid O$	:R 1
3	$\mid O \to O$	:→I 1–2

2. $\vdash S \to (S \lor R)$

1	$\mid\ \mid S$	:AS
2	$\mid\ \mid S$	:R 1
3	$\mid\ \mid S \lor R$	:∨I 2
4	$\mid S \to (S \lor R)$	:→I 1–3

3. $\vdash N \lor \neg N$

1	$\mid\ \mid \neg(N \lor \neg N)$	:AS
2	$\mid\ \mid\ \mid N$	:AS
3	$\mid\ \mid\ \mid N \lor \neg N$	:∨I 2
4	$\mid\ \mid\ \mid \neg(N \lor \neg N)$	:R 1
5	$\mid\ \mid \neg N$	:¬I 2–4
6	$\mid\ \mid N \lor \neg N$	:∨I 5
7	$\mid\ \mid \neg(N \lor \neg N)$	:R 1
8	$\mid N \lor \neg N$	:¬E 1–7

4. $\vdash \neg((R \vee T) \;\&\; (\neg R \;\&\; \neg T))$

1	$\mid\ \mid\ (R \vee T) \;\&\; (\neg R \;\&\; \neg T)$	:AS
2	$\mid\ \mid\ R \vee T$	:&E 1
3	$\mid\ \mid\ \neg R \;\&\; \neg T$	:&E 1
4	$\mid\ \mid\ \neg R$	:&E 3
5	$\mid\ \mid\ \neg T$	:&E 3
6	$\mid\ \mid\ T$	:∨E 2, 4
7	$\mid\ \neg((R \vee T) \;\&\; (\neg R \;\&\; \neg T))$	:¬I 1–6

5. $\vdash (R \leftrightarrow M) \rightarrow (M \rightarrow R)$

1	$\mid\ \mid\ R \leftrightarrow M$	:AS
2	$\mid\ \mid\ \mid\ M$	:AS
3	$\mid\ \mid\ \mid\ R$	:↔E 1, 2
4	$\mid\ \mid\ M \rightarrow R$	:→I 2–3
5	$\mid\ (R \leftrightarrow M) \rightarrow (M \rightarrow R)$	:→I 1–4

6. $\vdash J \leftrightarrow (J \vee (L \,\&\, \neg L))$

1	J	:AS
2	$J \vee (L \,\&\, \neg L)$	:$\vee$I 1
3	$J \rightarrow (J \vee (L \,\&\, \neg L))$	:$\rightarrow$I 1–2
4	$J \vee (L \,\&\, \neg L)$	:AS
5	$L \,\&\, \neg L$	:AS
6	L	:&E 5
7	$\neg L$	:&E 5
8	$\neg(L \,\&\, \neg L)$	:$\neg$E 5–7
9	J	:$\vee$E 4, 8
10	$(J \vee (L \,\&\, \neg L)) \rightarrow J$	:$\rightarrow$I 4–9
11	$J \leftrightarrow (J \vee (L \,\&\, \neg L))$	:$\leftrightarrow$I 3, 10

7. $\vdash (P \rightarrow Q) \vee (Q \rightarrow P)$

1	$\mid \; \mid \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	:AS
2	$\mid \; \mid \; \mid P$	:AS
3	$\mid \; \mid \; \mid \; \mid \neg Q$	:AS
4	$\mid \; \mid \; \mid \; \mid \; \mid Q$	:AS
5	$\mid \; \mid \; \mid \; \mid \; \mid P$	:R 2
6	$\mid \; \mid \; \mid \; \mid Q \rightarrow P$	:$\rightarrow$I 4–5
7	$\mid \; \mid \; \mid \; \mid (P \rightarrow Q) \vee (Q \rightarrow P)$	:$\vee$I 6
8	$\mid \; \mid \; \mid \; \mid \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	:R 1
9	$\mid \; \mid \; \mid Q$	:$\neg$E 3–8
10	$\mid \; \mid P \rightarrow Q$	:$\rightarrow$I 2–9
11	$\mid \; \mid (P \rightarrow Q) \vee (Q \rightarrow P)$	:$\vee$I 10
12	$\mid \; \mid \neg((P \rightarrow Q) \vee (Q \rightarrow P))$	:R 1
13	$\mid (P \rightarrow Q) \vee (Q \rightarrow P)$	:$\neg$E 1–12

B.

1. $T \rightarrow S \dashv\vdash \neg S \rightarrow \neg T$

1	$\mid T \rightarrow S$	:PR
2	$\mid \; \mid \neg S$	:AS
3	$\mid \; \mid \; \mid T$	:AS
4	$\mid \; \mid \; \mid S$	:$\rightarrow$E 1, 3
5	$\mid \; \mid \; \mid \neg S$	:R 2
6	$\mid \; \mid \neg T$	:$\neg$I 2–5
7	$\mid \neg S \rightarrow \neg T$	:$\rightarrow$I 2–6

1	$\neg S \rightarrow \neg T$	:PR
2	$\quad T$	:AS
3	$\quad\quad \neg S$	:AS
4	$\quad\quad \neg T$	:→E 1, 3
5	$\quad\quad T$	:R 2
6	$\quad S$	:¬E 3–5
7	$T \rightarrow S$	:→I 2–6

2. $R \rightarrow Q \dashv\vdash \neg(R \,\&\, \neg Q)$

1	$R \rightarrow Q$	:PR
2	$\quad R \,\&\, \neg Q$	:AS
3	$\quad R$	:&E 2
4	$\quad \neg Q$	:&E 2
5	$\quad Q$	:→E 1, 3
6	$\neg(R \,\&\, \neg Q)$	:¬I 2–5

1	$\neg(R \,\&\, \neg Q)$	:PR
2	$\quad R$	:AS
3	$\quad\quad \neg Q$	:AS
4	$\quad\quad R \,\&\, \neg Q$	:&I 2, 3
5	$\quad\quad \neg(R \,\&\, \neg Q)$	:R 1
6	$\quad Q$	:¬E 3–5
7	$R \rightarrow Q$	:→I 2–6

16 Soundness and completeness

We have two ways of checking or verifying that an argument is valid: (1) using truth tables and (2) using the natural deduction system to provide a proof. Consequently, we also have two ways of characterizing the concept of *validity*. (See table 16.1.) You might think that we can take it for granted that, with respect to determining if an argument is valid, both methods will always give us the same result, but that is not exactly the case. (We, right now, can take it for granted, but that's only because the requisite work to show that the two methods will always agree has already been done.) If you think about it for a moment, you'll notice that the two methods don't have anything in common, and so, it is not intuitively obvious that they will always produce the same result. But they do.

How do we know that the truth table method and the natural deduction method will always agree? Demonstrating that they will goes beyond the scope of this book. But we will review the two properties that a logic system (like TFL) must have for the two methods to always be in agreement. To begin, let us define two new terms.

> *p-valid*: being valid because a proof can be given using the rules in our natural deduction system. (*p-valid* is short for *proof-valid*. This is also sometimes called *syntactically valid*).
> *tt-valid*: being valid because there is no line in a truth table where the premises are true and the conclusion is false. (This is also sometimes called *semantically valid*).

First, it must be the case that every argument that is p-valid is tt-valid. This property is called SOUNDNESS.

	TRUTH TABLE (SEMANTIC) DEFINITION	PROOF-THEORETIC (SYNTACTIC) DEFINITION
Tautology	A sentence whose truth table has a T on every line under the main connective	A sentence that can be derived without any premises. I.e., a theorem.
Contradiction	A sentence whose truth table has an F on every line under the main connective	A sentence whose negation can be derived without any premises
Contingent sentence	A sentence whose truth table has both T and F (in any combination) under the main connective	A sentence that is not a theorem or contradiction
Equivalent sentences	The columns under the main connective for both sentences are identical.	The sentences can be derived from each other
Inconsistent sentences	Sentences that do not have a single line in their truth tables where, in the column under the main connective, they all have a T.	Sentences from which one can derive a contradiction
Consistent sentences	Sentences that have at least one line in their truth tables where, in the column under the main connective, they all have a T.	Sentences that are not inconsistent
Valid argument	An argument whose truth table has no lines where there is a T under each main connective for the premises and an F under the main connective for the conclusion.	An argument where one can derive the conclusion from the premises

Table 16.1: The two ways of defining each of these logical concepts in TFL.

Soundness

SOUNDNESS is a property of a logic system iff, for any argument, if the argument is p-valid, then the argument tt-valid.

Equivalently, SOUNDNESS is a property of a logic system iff, for any sentence, if a sentence is a theorem, then it is a tautology.

Soundness is a property of TFL because every argument for which we can give a proof (and hence show that it is valid that way) will also be valid by the truth table method.

> *Soundness*, the property of logical systems that we are discussing here, is different than *sound*, the property of individual arguments, that is defined on p. 9.

Soundness is the property that goes in this direction: p-valid ⇒ tt-valid. The other direction, tt-valid ⇒ p-valid, is called COMPLETENESS.

> Like '→', '⇒' can be read as 'if …, then …'.
>
> Since 'p-valid ⇒ tt-valid' is not an expression in TFL, we shouldn't use the '→' symbol in it. Instead, we are using the *metalogical arrow* to express the relationship between *p-valid* and *tt-valid*.

Completeness

COMPLETENESS is a property of a logic system iff, for any argument, if the argument is tt-valid, the the argument is p-valid.

Equivalently, COMPLETENESS is a property of a logic system iff, for any sentence, if the sentence is a tautology, then it is a theorem.

Proving that a logic system is complete is generally harder than proving soundness. Proving soundness for a logic system amounts to showing that all of the rules of the deduction system work the way they are supposed to work. Showing that a logic system is complete means showing that all of the rules that are needed have been included, and none have been left out. Again, showing this is beyond the scope of this book. The important point is that, happily, TFL is both sound and complete. This is not the case for all formal languages (or all logical systems). Because it is true of TFL, we can choose to give proofs or give truth tables – whichever is easier for the task at hand.

Some people are naturally drawn to truth tables because they can be produced mechanically, and that seems easier. But, as we mentioned in chapter 11, when arguments contain more than three letters, their truth table become quite large. Also, providing a proof informs us of the steps that must be taken to get from the premises to the conclusion. It illustrates *why* an argument is valid in a way that a truth table cannot. Comparing proofs also gives us insight into how arguments are similar or different, and that, in turn, informs us about the similarities and differences between various reasoning strategies. Truth tables, meanwhile, tell us nothing but whether an argument is valid or invalid.

It also bears mentioning that TFL is the standard first step into formal logic, but more complex systems of logic cannot employ truth tables and so derivations must be used. It is wise, therefore, to master derivations in TFL before moving onto to other branches of logic.

At the same time, there are some logical properties, the presence (or really the absence) of which, can only only be established with truth tables. In each of these cases, we might surmise from our failure to find a proof that the property is present, but our failure might just be a consequence of not trying hard enough. This is true for showing that (1) an argument is invalid, (2) a sentence is *not* a theorem, (3) a sentence is *not* a contradiction, (4) a sentence is contingent (which is to say that it's *not* a theorem and *not* a contradiction), (4) two sentences are *not* equivalent, and (5) two or more sentences are consistent (which is to say that they are *not* inconsistent). If we wish to show that any of those properties apply, then we have to resort to truth tables.

16.1 Practice exercises

A. For each of the following, if the argument is valid, give a proof. If it is not valid, make a truth table showing that it is not.

TO VERIFY	THAT IT IS	THAT IT IS NOT
Tautology	proof or a truth table	truth table
Contradiction	proof or a truth table	truth table
Contingent	truth table	proof or a truth table
Equivalent	proof or a truth table	truth table
Consistent	truth table	proof or a truth table
Valid	proof or a truth table	truth table

Table 16.2: This table summarizes what is required to check each of these logical properties.

1. $\neg(P \vee Q) \vdash P \,\&\, Q$
2. $\neg(A \vee B) \vdash C \rightarrow \neg A$
3. $\neg(A \vee B) \vdash B \rightarrow A$
4. $\neg(A \vee (B \vee C)) \vdash \neg C$
5. $\neg(A \,\&\, (B \vee C)) \vdash \neg C$
6. $((\neg X \leftrightarrow X) \vee X) \vdash X$
7. $F \,\&\, (K \,\&\, R) \vdash F \leftrightarrow (K \leftrightarrow R)$
8. $\neg L, K \rightarrow \neg L \vdash \neg K$
9. $L, K \rightarrow \neg L \vdash \neg K$
10. $\vdash A \rightarrow [((B \,\&\, C) \vee D) \rightarrow A]$
11. $\vdash A \rightarrow (A \rightarrow B)$

16.2 Answers

A.

1. $\neg(P \vee Q) \vdash P \,\&\, Q$ is invalid.

P	Q	$\neg$	(P	$\vee$	Q)	$\vdash$	(P	&	Q)
T	T	**F**	T	T	T	✓	T	**T**	T
T	F	**F**	T	T	F	✓	T	**F**	F
F	T	**F**	F	T	T	✓	F	**F**	T
F	F	**T**	F	F	F	×	F	**F**	F

2. $\neg(A \vee B) \vdash C \rightarrow \neg A$ is valid.
3. $\neg(A \vee B) \vdash B \rightarrow A$ is valid.
4. $\neg(A \vee (B \vee C)) \vdash \neg C$ is valid.
5. $\neg(A \,\&\, (B \vee C)) \vdash \neg C$ is invalid.

A	B	C	$\neg$	(A	&	(B	$\vee$	C))	$\vdash$	$\neg$	C
T	T	T	**F**	T	T	T	T	T	✓	**F**	T
T	T	F	**F**	T	T	T	T	F	✓	**T**	F
T	F	T	**F**	T	T	F	T	T	✓	**F**	T
T	F	F	**T**	T	F	F	F	F	✓	**T**	F
F	T	T	**T**	F	F	T	T	T	×	**F**	T
F	T	F	**T**	F	F	T	T	F	✓	**T**	F
F	F	T	**T**	F	F	F	T	T	×	**F**	T
F	F	F	**T**	F	F	F	F	F	✓	**T**	F

6. $(\neg X \leftrightarrow X) \vee X \vdash X$ is valid.
7. $F \,\&\, (K \,\&\, R) \vdash F \leftrightarrow (K \leftrightarrow R)$ is valid.
8. $\neg L, K \rightarrow \neg L \vdash \neg K$ is invalid.

K	L	$\neg$	L,	$(K$	$\rightarrow$	$\neg$	$L)$	$\vdash$	$\neg$	K
T	T	**F**	T	T	**F**	F	T	✓	**F**	T
T	F	**T**	F	T	**T**	T	F	×	**F**	T
F	T	**F**	T	F	**T**	F	T	✓	**T**	F
F	F	**T**	F	F	**T**	T	F	✓	**T**	F

9. $L, K \rightarrow \neg L \vdash \neg K$ is valid.

10. $\vdash A \rightarrow ([(B \,\&\, C) \vee D] \rightarrow A)$ is theorem.
11. $\vdash A \rightarrow (A \rightarrow B)$ is not a theorem, and so it is not a tautology.

A	B	$\vdash$	A	$\rightarrow$	$(A$	$\rightarrow$	$B)$
T	T	✓	T	**T**	T	T	T
T	F	✗	T	**F**	T	F	F
F	T	✓	F	**T**	F	T	T
F	F	✓	F	**T**	F	T	F

Part 5

First-order logic

17 The basics of first-order logic

17.1 Introduction to first-order logic

We have mastered truth functional logic. There may be difficult proofs that still confound us, but we have covered everything that there is to cover in this branch of logic. We now move on to the logic system that is typically studied immediately after truth functional logic: first-order logic (which is also called *predicate logic*).

As you know, the study of truth functional logic is based on the relationships between atomic sentences and sentences that contain *and*, *or*, *if …then …*, *if and only if* and the negation.

Hence, if we know this: *if Mary was arrested, then Peter when to the police*, and we know that *Mary was arrested*, then we know that it must follow that *Peter went to the police*. If we chose to symbolize this in TFL, we can easily do so:

1. $M \rightarrow P$
2. M
3. P

Here, however, is a valid argument that cannot be represented using TFL.

1. All humans are mortal.
2. Peter is a human.
3. Therefore, Peter is mortal.

Since there are no logical operators in any of these three sentences, if we tried to symbolize the argument, we would just have three seemingly unrelated atomic sentences:

1. H
2. P
3. M

So represented, this argument is invalid. First-order logic, however, will give us the means to represent it—and many others—as a valid argument.

17.2 The content of first-order logic

In first-order logic (FOL), we still use the logical operators from TFL, the rules of derivation given in chapter 12, and parentheses. We will, however, replace the atomic sentences of TFL with formulas composed of the following.

(*a*) predicates
(*b*) names for specifical individuals or things
(*c*) variables
(*d*) quantifiers

We also add the identity symbol, '=', to the symbols from TFL.

Names Names are simple. They designate specific individuals (or any specific entity). We represent them with the lowercase letters a – r.

Predicates Predicates—or properties—are attributes that individuals can have. These, for instance are predicates:

______ is tall.
______ is a mammal.
______ ate dinner.

As you can see, we need to add a name to the predicate to form a complete English sentence. If we use the name 'Carol', then we will have 'Carol is tall', 'Carol is a mammal', and 'Carol ate dinner'. Predicates are symbolized with uppercase letters, and we can use any letter A through

Z. Symbolizing 'is tall' as T, 'is a mammal' as M, 'ate dinner' as D, and 'Carol' as c, we combine this name and these predicate in FOL this way:

$T(c)$
$M(c)$
$D(c)$

Those are *one-place predicates* because each takes only one name. These are *two-place predicates* because each takes two names:

______ is the sister of ______ .
______ is in love with ______ .
______ is taller than ______ .

With names added, we can symbolize these sentences as is shown here. Notice that the order of the names matters.

Abby is the sister of Carol: $S(a, c)$
Carol is the sister of Abby: $S(c, a)$
David is in love with Carol: $L(d, c)$
Carol is not in love with David: $\neg L(c, d)$

And likewise, we can have three- or four- (or more) place predicates.

Quantifiers and variables The most notable feature of first-order logic is the use of what are called *quantifiers*. There are two in this logic system:

(a) The universal quantifier, which is symbolized with '$\forall$' and can be translated as "every" or "all."
(b) The existential quantifier, which is symbolized with '$\exists$' and can be translated as "some" or "there exists."

When we use these quantifiers, we are not designating a specific individual and so, with them, we must use variables. Variables work with predicates the same way as names do, but when we have a variable, we generally (although not always) have a quantifier. We represent variables with the lowercase letters s – z.

Here are two English sentences that are each translated to FOL with a quantifier, a predicate, and a variable:

Everyone ate dinner: $\forall x D(x)$.
Someone ate dinner: $\exists x D(x)$.

As we did with TFL, there will be a point where we won't be as concerned with the English sentence that is represented by an expression in FOL. So, reading the first one, as an expression in FOL (and without a specific English translation for D), we say, "for all x, $D(x)$ or "all x are D." And for the second one: "there exists an x such that $D(x)$" or "some x is D."

Identity The identity symbol, '=', is used to signify that two things are the same; that is, not just similar or equivalent, but the same. So, while two graduate students who both teach German I might—as far as the students care—be equivalent, we wouldn't use the identity symbol to indicate that they are. Rather, we use the identity symbol for cases such as 'Peter Parker is identical to Spider-Man.'

Domains A logic system must be precise. But 'all' or 'every', when taken literally and without any qualification, refer to a vast set of individuals or things. For instance, if 'H' represents 'is happy', then '$\forall x H(x)$' represents 'Everyone is happy'. But when we say this in English, we are not referring to everyone now alive on Earth or everyone who ever was alive or who ever will be alive. Rather, we mean something like 'everyone now in the room is happy,' or 'everyone enrolled in this course is happy'.

So, to be precise when we use quantified expressions in FOL, we need to specify a DOMAIN. The domain is the collection of things about which we want to refer. If we want to talk about people in Starkville, we define the domain as *people in Starkville*. We write this at the beginning of the symbolization key, like this:

domain: people in Starkville

The quantifiers, then, apply to (or range over) the domain. Given this domain, '$\forall x$' is to be read roughly as 'Every person in Starkville ...', and '$\exists x$' is to be read roughly as 'Some person in Starkville, ...'.

In FOL, the domain must always include at least one thing. Moreover, in English we can legitimately infer ‘someone is angry’ from ‘David is angry’. Likewise in FOL, we will want to be able to infer ‘$\exists x A(x)$’ from ‘$A(d)$’. So, when we are using a name (and not a quantifier), it still must be the case that each name picks out exactly one thing in the domain. (Although there can be members of a domain that don’t have names or have more than one name.)

18 FOL: Translations and scope

18.1 Translations

Here is a translation scheme.

domain: people in Starkville
B: ______ ate a burrito
N: ______ took a nap
L: ______ loves ______
a: Abigail
c: Carol
d: David

We can then translate the following sentences from first-order logic to English as is shown. First, here are some sentences composed of predicates and names.

$B(a)$: Abigail ate a burrito.

$N(d)$: David took a nap.

$\neg N(c)$: Carol did not take a nap.

$N(d)$ & $B(a)$: David took a nap and Abigail ate a burrito.

$B(a) \rightarrow N(a)$: If Abigail ate a burrito, then Abigail took a nap.

$L(d, c)$ & $B(c)$: David loves Carol, and Carol ate a burrito.

Here are some quantified expressions.

$\forall y N(y)$: Everyone took a nap.

$\exists x B(x)$: Someone ate a burrito.

$\neg \forall x N(x)$: It is not the case that everyone took a nap.

$\forall x \neg N(x)$: Everyone did not take a nap.

$\neg\exists y B(y)$: It is not the case that someone ate a burrito.

$\exists y \neg B(y)$: Someone did not eat a burrito.

$\exists x[B(x) \mathbin{\&} N(x)]$: Someone ate a burrito and took a nap.

If you are stuck on a translation, you can always just translate $\forall x$ and $\exists x$ like this:

For all x, x is …
There exists an x such that x is …

So, to translate the expression, proceed this way. (1) Begin reading from the left. (2) Read the universal or existential quantifier and variable as "For all …" or "There exists …". (3) Read the predicates as is given in the translation scheme. (4) Read the logical operators as we did in TFL. (5) If there is a 'not' sign somewhere, translate it right where it is as 'it is not the case that'. Once you've translated the sentence this way, you may want to word it more naturally using 'someone', 'everyone', and just 'not'.

These sentences contain either both a name and a variable or both of the quantifiers (and two different variables).

$\exists x[B(x) \mathbin{\&} N(c)]$: Someone ate a burrito, and Carol took a nap.

$\exists x L(x, c)$: Someone loves Carol.

$\exists x \forall y[B(x) \mathbin{\&} N(y)]$: Someone ate a burrito and everyone took a nap.

$\forall x \exists y L(x, y)$: Everyone loves someone.

$\exists x \exists y L(x, y)$: Someone loves someone.

And these are the formats for common expressions that are translated into first-order logic as conditionals or conjunctions:

$\forall x[F(x) \to G(x)]$: For all x, if x is an F, then x is a G' **or** 'Every F is G.'

$\exists x[F(x) \mathbin{\&} G(x)]$:'There exists an x, such that x is an F and a G' **or** Some F is G'.

'No F is G': $\neg\exists x[F(x) \mathbin{\&} G(x)]$ **or** $\forall x[F(x) \to \neg G(x)]$

'Only Fs are Gs': $\neg\exists x[G(x) \mathbin{\&} \neg F(x)]$ **or** $\forall x[G(x) \to F(x)]$

Finally, these existentially and universally quanitfied sentences are equivalent.

> equivalent quantified expressions
>
> $\exists x \neg B(x)$ is equivalent to $\neg \forall x B(x)$.
>
> $\forall x \neg D(x)$ is equivalent to $\neg \exists x D(x)$.

18.2 Scope

Just like the logical operators, the universal quantifier ('$\forall$') and the existential quantifier ('$\exists$') have a scope. For both, their scope is the part of the expression to which the quantifier applies – or, as we often say, the part of the sentence that it *ranges over*. Basically, the scope of the quantifiers works like the scope of the negation (i.e., the '$\neg$').

(1) If there is not a parenthesis between the quantifier and a predicate – e.g., '$\forall x F(x)$' or '$\forall x \neg F(x)$' – then the scope of the quantifier is that predicate (or the predicate and a $\neg$) and its variable or variables.

(2) If there is a parenthesis between the quantifier and the first predicate – e.g., '$\forall x[F(x) \vee G(x)]$' – then the scope of the quantifier is everything inside the parentheses.

Notice that this is all the same as the scope of the '$\neg$'.

19 The rules of derivation for FOL

19.1 New rules for FOL

We retain the TFL rules that are given in chapter 14, and now we add introduction and elimination rules for the universal and existential quantifiers and the identity elimination rule.

19.2 Universal elimination

If we know, for instance, that everyone in some domain likes chocolate, then we know that a specific individual in this domain—let's say, Carol—likes chocolate. The universal elimination rule captures this reasoning process.

universal elimination rule

m	$\forall x A(x)$	
	$A(c)$	$:\forall E\ m$

If we have $\forall x A(x)$ on a line, then we can put $A(c)$ on a new line.
Any predicate can be used in the place of 'A', any variable can occur in place of 'x', and any name can be used in place of 'c'.

19.3 Existential introduction

If we know, for instance, that David is on the train, then we know that someone is on the train. The existential introduction rule is based on this simple reasoning process.

existential introduction rule

m	$\mathsf{A(c)}$	
	$\exists \mathsf{xA(x)}$	:$\exists$I m

If we have $\mathsf{A(c)}$ on a line, then we can put $\exists \mathsf{xA(x)}$ on a new line.

The universal elimination and existential introduction rules are straightforward. It's obvious that if *everyone* has some property, then any particular individual (in that group) has it. It's equally clear that if a particular individual has a property, then *someone* has it.

The universal introduction rule and existential elimination rule are less intuitive—although when we think about them, we'll see that they "logically" make sense.

19.4 Universal introduction

We'll take up the universal introduction rule first. What would it take to introduce the claim that everyone likes chocolate [i.e., $\forall x C(x)$]? One method would be to check that every single individual in the domain likes chocolate. This, however, isn't practical for our purposes since a domain can have an infinite number of members. We need a different way of introducing a universal quantifier.

To begin thinking about the method that we will use, consider this argument:

$$\forall x(F(x) \,\&\, G(x)) \vdash \forall x F(x)$$

This argument is valid. If everything is both F *and* G, then everything is F. But how do we show this? We begin the proof this way:

1	$\forall x(F(x) \,\&\, G(x))$	:PR
2	$F(a) \,\&\, G(a)$	:$\forall$E 1
3	$F(a)$	:&E 2

We have derived '$F(a)$'. This is an *instance* of the conclusion that we are after: '$\forall x F(x)$'. (For example, 'Albert is fast' is one *instance* of 'everyone is fast'.) Alternatively, on lines 2 and 3 (and using the universal elimination and conjunction elimination rules), we could have put '$F(b)$', '$F(c)$', '$F(m_2)$', '$F(r_{791})$'), or anything else until we run out of space, time, or patience.

So, from the premise $\forall x(F(x) \;\&\; G(x))$, we could, in principle, get $F(\ldots)$ for any name. That is, we could, in principle, use the name of every individual in the domain. (In reality, we can't do this, however, because our proof might never end.) Therefore, because we just arbitrarily chose 'a' for the $F(a) \;\&\; G(a)$—and then derived $F(a)$—we should be allowed to infer $\forall x F(x)$ from the $F(a)$.

This brings us to the following idea. We can use the UNIVERSAL INTRODUCTION RULE to get '$\forall x F(x)$' when the 'c' in 'F(c)' is arbitrarily chosen from the names of everyone or everything in the domain. And therefore, in this situation, we can complete the proof with this rule.

1	$\forall x(F(x) \;\&\; G(x))$	:PR
2	$F(a) \;\&\; G(a)$	:$\forall$E 1
3	$F(a)$	:&E 2
4	$\forall x F(x)$	:$\forall$I 3

universal introduction rule

m	A(c)	
	∀xA(x)	:$\forall$I m

If we have A(c) on a line, then we can put ∀xA(x) on a new line provided these conditions are met:

1. c must not occur in any premise or undischarged assumption.
2. x must not occur in A(c, …).

19.5 Existential elimination

The first thing to note about the existential elimination rule is that when we use it, we begin with and usually end with an existentially quantified sentence. A typical way to use the rule is as follows.

(1) Begin with an existentially quantified sentence.
(2) As an assumption, state a possible instance of this existentially quantified sentence.
(3) Inside the sub-proof, derive another existentially quantified sentence.
(4) Close the sub-proof and put the sentence from 3 on the next line.

Hence, we shouldn't get hung up on the word *elimination*. We do eliminate the existential quantifier for the second step, but, in the end, we're right back to having an existentially quantified sentence – albeit a different one than the one with which we began.

Now, let's think about how this rule works a little more carefully, suppose that we know that *something* is F. The problem is that simply knowing this does not tell us which particular thing is F. So from '$\exists x F(x)$' we cannot immediately infer '$F(a)$', or '$F(d)$', or any other instance of the sentence. What can we do? How can we derive anything from an existentially quantified premise?

Suppose we know that something is F. Furthermore, we know that everything that is F is G. In English, we might pursue the following line of reasoning:

> Since something is F, there is some particular thing that is F. We do not know anything about it, other than that it's F, but for convenience, let's call it "Oby". So, Oby is F. Since everything that is F is G, it follows that "Oby" is G. And since Oby is G, it follows that *something* is G. Nothing depends on who or what, exactly, our "Oby" is. But something is G.

We can capture this reasoning pattern in a proof as follows:

1	$\exists x F(x)$	:PR
2	$\forall x(F(x) \to G(x))$	:PR
3	$\quad F(o)$	:AS
4	$\quad F(o) \to G(o)$	:$\forall$E 2
5	$\quad G(o)$	:$\to$E 3, 4
6	$\quad \exists x G(x)$	:$\exists$I 5
7	$\exists x G(x)$	:$\exists$E 1, 3–6

$\exists x F(x)$ is one of the premises. After the premises, on line 3, we made an assumption: '$F(o)$'. The idea here is that premise 1 tell us that *something* is an F. So, on line 3 we introduce some arbitrary name for it: 'o'. (Other than removing the existential quantifier and replacing the variable with a name, our assumption must match the sentence in premise 1.) The name we picked is arbitrary. We've assumed nothing about the object named by 'o' other than that the predicate 'F' is true of it. On the basis of the assumption $F(o)$, we can, in a few steps, get '$\exists x G(x)$'. Since nothing depended on which specific object 'o' names, our reasoning pattern is perfectly general. We could equally well have arrived at '$\exists x G(x)$' by using any other name on line 3. We can therefore discharge the assumption '$F(o)$' on line 3 and put '$\exists x G(x)$' on line 7 using the EXISTENTIAL ELIMINATION RULE.

existential elimination rule

m	$\exists$xA(x ...)	
i	$\quad$ A(c)	:AS
j	$\quad$ B	
	B	:$\exists$E m, i–j

The name c may not occur outside the subproof (including in the original existential $\exists$xA(x) or in B).

The name that is in the assumption cannot occur outside the sub-proof.

This means that we could not have ended the sub-proof with line 5, where we have '$G(o)$'. We can, however, easily get from '$G(o)$' to '$\exists G(x)$' with the existential introduction rule. And since the name that is in the assumption doesn't occur in '$\exists G(x)$', this can be the last line of the sub-proof.

So, the B in the rule will often be an existentially quantified sentence. But B can any sentence as long as it doesn't contain the name introduced in the assumption that begins the subproof.

One more thing to note about this rule is that the only time when we are really eliminating an existential quantifier is when we make our assumption. And that sentence, $\mathsf{A}(\mathsf{c})$, cannot appear outside of the subproof. This elimination step is still significant, however, because it provides a sentence that can be used with the rules that were introduced for TFL.

> The constraint that we have on the existential elimination rule is more restrictive than strictly necessary. The name c that we assumed can occur outside the subproof, as long as it doesn't occur in $\exists\mathsf{x}\mathsf{A}(\mathsf{x}\ \ldots)$, in an earlier undischarged assumption, or in B.

19.6 Identity rules

Here's a deep thought: everything is identical to itself. The IDENTITY INTRODUCTION RULE allows us to state this fact.

identity introduction rule

$$\begin{array}{|ll} \mathsf{c} = \mathsf{c} & \quad =\mathrm{I} \end{array}$$

For any name, state that it is identical to itself. No line number is given with the rule.

When thinking about identities, however, the more interesting assertion is one like 'Bruce Wayne *is* Batman'. A sentence with the form $\mathsf{a} = \mathsf{b}$, however, must be given as a premise or an assumption. It cannot be introduced with the identity introduction rule. If it is a premise or assumption, though, then we can use the IDENTITY ELIMINATION RULE.

identity elimination rule

m	a = b	
n	A(a)	
	A(b)	=E m, n

If you have a = b on one line and A(a) on another line, you can put A(b) on a new line.

19.7 Some examples

Although the quantifiers aren't logical operators, when thinking about how we use these rules, it is useful to treat them as if they are. If we count them as logical operators, then we can continue to follow this guideline (from p. 143): **Each of the rules of derivation can only be applied to the main logical operator of a sentence**. And as we said in §6.3, the main logical operator is the one that's scope is the entire sentence.

So, consider a sentence like this one:

$$\forall x[B(x) \,\&\, N(x)]$$

The scope of the universal quantifier is the whole sentence. Therefore, we treat it as the main logical operator. As such, we can use the universal elimination rule on this sentence, and we can't use the conjunction elimination rule. If we apply the universal elimination rule, then we have a sentence like this one:

$$B(c) \,\&\, N(c)$$

Now, we can use the conjunction elimination rule to get either '$B(c)$' or '$N(c)$' on a new line.

Here are some examples of proofs that use the rules introduced in this chapter, along with the rules of derivation from chapter 12.

1. $\forall x F(x) \vdash \forall y F(y)$

1	$\forall x F(x)$	:PR
2	$F(a)$	:$\forall$E 1
3	$\forall y F(y)$	:$\forall$I 2

2. $\forall x[G(x) \rightarrow H(x)], \forall x G(x) \vdash \forall x H(x)$

1	$\forall x[G(x) \rightarrow H(x)]$	:PR
2	$\forall x G(x)$	:PR
3	$G(a) \rightarrow H(a)$	:$\forall$E 1
4	$G(a)$	:$\forall$E 2
5	$H(a)$	:$\rightarrow$E 3, 4
6	$\forall x H(x)$	:$\forall$I 5

3. $\forall x[F(x) \rightarrow G(x)], F(a) \vdash \exists x[F(x) \mathbin{\&} G(x)]$

1	$\forall x[F(x) \rightarrow G(x)]$	:PR
2	$F(a)$	:PR
3	$F(a) \rightarrow G(a)$	:$\forall$E 1
4	$G(a)$	:$\rightarrow$E 2, 3
5	$F(a) \mathbin{\&} G(a)$	:&I 2, 4
6	$\exists x[F(x) \mathbin{\&} G(x)]$	:$\exists$I 5

4. $\exists x M(x) \vdash \exists x[M(x) \lor N(x)]$

1	$\exists x M(x)$	:PR
2	$\quad M(a)$	:AS
3	$\quad M(a) \lor N(a)$	:$\lor$I 2
4	$\quad \exists x[M(x) \lor N(x)]$	:$\exists$I 3
5	$\exists x[M(x) \lor N(x)]$	:$\exists$E 1, 2–4

5. $\exists x \neg F(x), \forall x[F(x) \lor G(x)] \vdash \exists x G(x)$

1	$\exists x \neg F(x)$	:PR
2	$\forall x[F(x) \lor G(x)]$	:PR
3	$\quad \neg F(a)$	:AS
4	$\quad F(a) \lor G(a)$	:$\forall$E 2
5	$\quad G(a)$	:$\lor$E 3, 4
6	$\quad \exists G(x)$	:$\exists$I 5
7	$\exists G(x)$	:$\exists$E 1, 3–6

6. $\exists x[B(x) \to D(x)], \forall x B(x) \vdash \exists x D(x)$

1	$\exists x[B(x) \to D(x)]$	:PR
2	$\forall x D(x)$	:PR
3	$\quad B(a) \to D(a)$	:AS
4	$\quad B(a)$	:$\forall$E 2
5	$\quad D(a)$	:$\to$E 3, 4
6	$\quad \exists x D(x)$	:$\exists$I 5
7	$\exists x D(x)$	:$\exists$E 1, 3–6

7. $G(a) \leftrightarrow H(a), a = d \vdash G(d) \leftrightarrow H(d)$

1	$G(a) \leftrightarrow H(a)$	:PR
2	$a = d$	:PR
3	$G(d) \leftrightarrow H(d)$	=E 1, 2

8. $a = b, M(b, a) \vdash \exists x M(x, x)$

1	$a = b$	:PR
2	$M(b, a)$	:PR
3	$M(a, a)$	=E 1, 2
4	$\exists x M(x, x)$	:∃I 3

9. $M(a) \vee N(b), N(b) \rightarrow b = d, \neg M(a) \vdash N(d)$

1	$M(a) \vee N(b)$	:PR
2	$N(b) \rightarrow b = d$	:PR
3	$\neg M(a)$	:PR
4	$N(b)$	:∨E 1, 3
5	$b = d$	:→E 2, 4
6	$N(d)$	=E 4, 5

10. $\forall x \forall y \forall z([L(x, y) \,\&\, L(y, z)] \rightarrow L(x, z)), L(a, b), L(b, c) \vdash L(a, c)$

This is one of the examples of an arguments is that is valid in virtue of its from chapter 4:

1. Seoul is larger than London.
2. London is larger than Chicago.
3. Therefore, Seoul is larger than Chicago.

We couldn't, however, represent such an argument in TFL. Now, we can do so with FOL. We will use this two-place predicate:

L: ______ is larger than ______

For this argument, this predicate must have the *transitive relation*. This relation is defined with this sentence, which will be a premise in the argument:

$$\forall x \forall y \forall z([L(x,y) \,\&\, L(y,z)] \rightarrow L(x,z))$$

In English, this is 'for all x, for all y, and for all z, if x is larger than y and y is larger than z, then x is larger than z'. (Not all two-place predicates are transitive, which is why we need this premise. You can see this by thinking about a different two-place that 'L' might represent: '______ loves ______'.)

This is the argument:

$$\forall x \forall y \forall z([L(x,y) \,\&\, L(y,z)] \rightarrow L(x,z)), L(a,b), L(b,c) \vdash L(a,c)$$

We use the universal elimination rule three times to remove each of the universal quantifiers. Then, it is a simple matter of using the conjunction introduction and conditional elimination rules to get the conclusion 'a is larger than c'.

1	$\forall x \forall y \forall z([L(x,y) \,\&\, L(y,z)] \rightarrow L(x,z))$	:PR
2	$L(a,b)$	:PR
3	$L(b,c)$	:PR
4	$\forall y \forall z([L(a,y) \,\&\, L(y,z)] \rightarrow L(a,z))$	:$\forall$E 1
5	$\forall z([L(a,b) \,\&\, L(b,z)] \rightarrow L(a,z))$	:$\forall$E 4
6	$(L(a,b) \,\&\, L(b,c)) \rightarrow L(a,c)$	:$\forall$E 5
7	$L(a,b) \,\&\, L(b,c)$	:&I 2, 3
8	$L(a,c)$	:$\rightarrow$E 6, 7

Appendices

A Symbolic notation

A.1 Alternative nomenclature

Truth-functional logic. TFL goes by other names. Sometimes it is called *sentential logic*, because this branch of logic deals fundamentally with sentences. Sometimes it is called *propositional logic* because it might also be thought to deal fundamentally with propositions. We have used with *truth-functional logic* to emphasize that it deals only with assignments of truth and falsity to sentences and that its connectives are all truth-functional.

Formulas. In §6, we defined *sentences* of TFL. These are also sometimes called 'formulas' (or 'well-formed formulas') since in TFL there is no distinction between a formula and a sentence.

Truth-value assignments. *Truth-value assignments* may also be called *truth-assignments* or *valuations*.

A.2 Alternative symbols

In the history of formal logic, different symbols have been used at different times and by different authors. Often, authors were forced to use notation that their printers could typeset. This appendix presents some common symbols, so that you can recognize them if you encounter them in an article or in another book.

Negation. Two commonly used symbols are the *hoe*, '$\neg$', and the *swung dash* or *tilda*, '$\sim$.' In some more advanced formal systems it is necessary to distinguish between two kinds of negation; the distinction is sometimes represented by using both '$\neg$' and '$\sim$'. Older texts sometimes indicate negation by a line over the formula being negated, e.g., $\overline{A \mathbin{\&} B}$.

SYMBOLS OF FORMAL LOGIC	
negation	¬, ∼
conjunction	∧, &, ▪
disjunction	∨
conditional	→, ⊃
biconditional	↔, ≡

Table A.1

Disjunction. The symbol '∨' is typically used to symbolize inclusive disjunction. One etymology is from the Latin word 'vel', meaning 'or'.

Conjunction. Conjunction is often symbolized with the *ampersand*, '&'. The ampersand is a decorative form of the Latin word 'et', which means 'and'. (Its etymology still lingers in certain fonts, particularly in italic fonts; thus an italic ampersand might appear as '*&*'.) This symbol is commonly used in natural English writing (e.g. 'Smith & Sons'), and so even though it is a natural choice, many logicians use a different symbol to avoid confusion between the object and metalanguage—as a symbol in a formal system, the ampersand is not the English word '&'. The most common choice now is '∧', which is a counterpart to the symbol used for disjunction. Sometimes a single dot, '·', is used. In some older texts, there is no symbol for conjunction at all; 'A and B' is simply written 'AB'.

Conditional. There are two common symbols for the conditional (which can also be called the *material conditional*): the *arrow*, '→', and the *hook*, '⊃'.

Biconditional. The *double-headed arrow*, '↔', is used in systems that use the arrow to represent the biconditional. Systems that use the hook for the conditional typically use the *triple bar*, '≡', for the biconditional.

B Quick reference

B.1 Characteristic Truth Tables

A	$\neg A$
T	F
F	T

A	B	$A \mathbin{\&} B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

A	$\neg A$
$\top$	$\bot$
$\bot$	$\top$

A	B	$A \mathbin{\&} B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$

A	$\neg A$
1	0
0	1

A	B	$A \mathbin{\&} B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

B.2 Rules of derivation for TFL

When you have what is in blue, then, on a new line, you can put what is in red. *m*, *n*, *p*, and *q* stand for lines numbers. *m* and *n* don't have to be consecutive line numbers. The *p* and *q* in the negation-introduction and negation-elimination rules are consecutive line numbers.

Conjunction Intro

m	A	
n	B	
	A & B	:&I *m*, *n*

m	A	
n	B	
	B & A	:&I *m*, *n*

Conjunction Elim

m	A & B	
	A	:&E *m*

m	A & B	
	B	:&E *m*

Disjunction Intro

m	A	
	A ∨ B	:∨I *m*

m	A	
	B ∨ A	:∨I *m*

Disjunction Elim

m	A ∨ B	
n	¬B	
	A	:∨E *m*, *n*

m	A ∨ B	
n	¬A	
	B	:∨E *m*, *n*

Double Negation

m	A	
	¬¬A	:DN *m*

Conditional Elim

$$\begin{array}{r|ll}
m & A \to B & \\
n & A & \\
 & B & :\to\mathrm{E}\ m, n
\end{array}$$

Conditional Intro

$$\begin{array}{r|l|ll}
m & & A & :\mathrm{AS} \\ \cline{3-3}
n & & B & \\
 & \multicolumn{2}{l}{A \to B} & :\to\mathrm{I}\ m\text{–}n
\end{array}$$

Biconditional Intro

$$\begin{array}{r|ll}
m & A \to B & \\
n & B \to A & \\
 & A \leftrightarrow B & :\leftrightarrow\mathrm{I}\ m, n
\end{array}$$

Negation Intro

$$\begin{array}{r|l|ll}
m & & A & :\mathrm{AS} \\ \cline{3-3}
p & & B & \\
q & & \neg B & \\
 & \multicolumn{2}{l}{\neg A} & :\neg\mathrm{I}\ m\text{–}q
\end{array}$$

Biconditional Elim

$$\begin{array}{r|ll}
m & A \leftrightarrow B & \\
n & B & \\
 & A & :\leftrightarrow\mathrm{E}\ m, n
\end{array}$$

$$\begin{array}{r|ll}
m & A \leftrightarrow B & \\
n & A & \\
 & B & :\leftrightarrow\mathrm{E}\ m, n
\end{array}$$

Negation Elim

$$\begin{array}{r|l|ll}
m & & \neg A & :\mathrm{AS} \\ \cline{3-3}
p & & B & \\
q & & \neg B & \\
 & \multicolumn{2}{l}{A} & :\neg\mathrm{E}\ m\text{–}q
\end{array}$$

Reiteration

$$\begin{array}{r|ll}
m & A & \\
 & A & :\mathrm{R}\ m
\end{array}$$