Chapter 4
DECISIONS UNDER RISK: UTILITY

4-1. Interval Utility Scales

Utilities are just as critical to our account of decisions under risk as probabilities since the rule of maximizing expected utility operates on them. But what are utilities? And what do utility scales measure? In discussing decisions under ignorance I hinted at answers to these questions. But such allusions will not suffice for a full and proper understanding of decisions under risk. So let us begin with a more thorough and systematic examination of the concept of utility.

The first point we should observe is that ordinal utility scales do not suffice for making decisions under risk. Tables 4-1 and 4-2 illustrate why this is so. The

4-1

A_1	6 ¼	1 ¾
A_2	5 ¼	2 ¾

expected utilities of A_1 and A_2 are 9/4 and 11/4, respectively, and so A_2 would be picked. But if we transform table 4-1 ordinally to table 4-2 by simply raising

4-2

A_1	20 ¼	1 ¾
A_2	5 ¼	2 ¾

the utility number 6 to 20, the expected utilities are now 23/4 and 11/4, which results in A_1 being picked. Thus two scales that are ordinal transformations of each other might fail to be equivalent with respect to decisions under risk.

This stands to reason anyway. Ordinal scales represent only the relative standings of the outcomes; they tell us what is ranked first and second, above and below, but no more. In a decision under risk it is often not enough to know that you prefer one outcome to another; you might also need to know whether you prefer an outcome *enough* to take the risks involved in obtaining it. This is reflected in our disposition to require a much greater return on an investment of \$1,000,000 than on one of \$10—even when the probabilities of losing the investment are the same.

Happily, interval scales are all we require for decisions under risk. In addition to recording an agent's ranking of outcomes, we need measure only the relative lengths of the "preference intervals" between them. To understand what is at stake, suppose I have represented my preferences for cola (C), ice cream (I), apples (A), and popcorn (P) on the following line.

least preferred — most preferred

P — A — C — I

If I now form a scale by assigning numbers to the line, using an ordinal scale would require me only to use numbers in ascending order as I go from left to right. But if I use an interval scale, I must be sure that the *relative* lengths of the intervals on the line are reflected as well. Thus I could not assign 0 to P, 1 to A, 2 to C, and 3 to I, for this would falsely equate the interval between, say P and A with that between C and I. More generally, if items x, y, z, and w are assigned utility numbers $u(x)$, $u(y)$, $u(z)$, and $u(w)$ on an interval scale, these numbers must satisfy the following conditions:

a. xPy if and only if $u(x) > u(y)$.
b. xIy if and only if $u(x) = u(y)$.
the preference interval between x and y is greater than or equal to that between z and w if and only if
$|u(x) - u(y)| \geq |u(z) - u(w)|$.

More than one assignment of numbers will satisfy these two conditions but every assignment that does is a positive linear transformation of every other one that does. To illustrate this point, suppose I assign numbers to my preferences as indicated here.

least preferred — most preferred

1 — 2 — 7 — 10

P — A — C — I

Then I have properly represented both the ordinal and interval information. But I could have used other numbers, such as the next set.

least preferred — most preferred

15 — 25 — 75 — 105

P — A — C — I

These are obtained from the first set by a positive linear transformation. (What is it?)

Two ordinal scales count as equivalent if and only if they can be obtained

PROBLEMS

1. Suppose you can bet on one of two horses—Ace or Jack—in a match race. If Ace wins you are paid \$5; if he loses you must pay \$2 to the track. If you bet on Jack and he loses, you pay the track \$10. You judge each horse to be as likely to win as the other. Assuming you make your decisions on the basis of expected monetary values, how much would a winning bet on Jack have to pay before you would be willing to risk \$10?
2. Suppose the interval scale u may be transformed into u' by means of the transformation

$$u' = au + b \qquad (a > 0).$$

   Give the transformation that converts u' back into u.
3. Suppose the u' of the last problem can be transformed into u'' by means of the transformation

$$u'' = cu' + d \qquad (c > 0).$$

   Give the transformation for converting u into u''.
4. Suppose s and s' are equivalent ratio scales. Show that if $s(x) = 2s(y)$, then $s'(x) = 2s'(y)$.
5. Suppose you have a table for a decision under risk in which the probabilities are independent of the acts. Show that if you transform your utility numbers by adding the number b_i to each utility in column i (and assume that the numbers used in different columns are not necessarily the same), the new table will yield the same ordering of the acts.

4-2. Monetary Values vs. Utilities

A popular and often convenient method for determining how strongly a person prefers something is to find out how much money he or she will pay for it. As a general rule people pay more for what they want more; so a monetary scale can be expected to be at least an ordinal scale. But it often works as an interval scale too—at least over a limited range of items. A rough test of this is the agent's being indifferent between the same increase (or reduction) in prices over a range of prices, since the intervals remain the same though the prices change. Thus if I sense no difference between \$5 increases (e.g., from \$100 to \$105) for prices between \$100 and \$200, it is likely that a monetary scale can adequately function as an interval scale for my preferences for items in that price range. Within this range it would make sense for me to make decisions under risk on the basis of expected monetary values (EMVs).

Since we are so used to valuing things in terms of money—we even price intangibles, such as our own labor and time or a beautiful sunset, as well as necessities, such as food and clothing—it is no surprise that EMVs are often used as a basis for decisions under risk. My earlier insurance and car purchase examples typify this approach. Perhaps this is the easiest and most appropriate method for making business decisions, for here the profit motive is paramount.

It is both surprising and disquieting that a large number of nonbusiness decisions are made on the basis of EMVs. For example, it is not unusual for government policy analysts to use monetary values to scale nonmonetary outcomes such as an increase or decrease in highway deaths or pollution-induced cancers. To make a hypothetical example of a historical case, consider the federal government's decision in 1976 to vaccinate the population against an expected epidemic of swine flu. The *actual relevant outcomes* might have been specified in terms of the number of people contracting the disease, the number of deaths and permanent disabilities, and the cost and inconvenience of giving or receiving the vaccine, but instead of assigning values to these outcomes, I will suppose (as is probably the case) that policy analysts turned to their economic consequences. This meant evaluating alternatives first in terms of lost working days and then in terms of a reduced gross national product (GNP), and so on until the cost of administering the vaccine could be compared to the expected benefit to be derived from it. To illustrate this in a simplified form using made-up figures, suppose a flu epidemic will remove five million people from the work force for a period of five days. (Some will be sick, others will care for the sick. I am also talking about absences over and above those expected in normal times.) Further, suppose the average worker contributes $200 per five days of work to the GNP. Finally, suppose the GNP cost of the vaccine program is $40,000,000, that without it there is a 90% chance of an epidemic and with it only a 10% chance. Then we can set up decision table 4-4, which values out-

4-4	Epidemic	No Epidemic
Have Vaccine Program	− $1,040 million .1	− $40 million .9
No Program	− $1,000 million .9	0 .1

comes in terms of dollar costs to the GNP. The EMV of having the program is − $140 million and that of not having it is − $900 million; thus, under this approach at least, the program should be initiated since it minimizes the cost to the GNP. (This example also illustrates an equivalent approach to decisions under risk: Instead of maximizing expected gains one minimizes expected losses. See exercises 1–3 in the next Problems section.)

Perhaps the EMV approach to large-scale decisions is the only practical alternative available to policy analysts. After all, they should make some attempt to factor risks into an analysis of costs and benefits, and that will require at least an approximation to an interval scale. Monetary values provide an accessible and publicly understandable basis for such a scale.

But few people would find EMVs a satisfactory basis for every decision; they will not even suffice for certain business decisions. Sometimes making greater (after-tax) profits is not worth the effort. Just as we are rapidly reaching the point where it is not worth bending over to pick up a single penny, it might not be worth a company's effort to make an extra $20,000. More money would actually have less utility; so money could not even function as an ordinal scale. Of course, this completely contravenes the way an EMVer sees things.

Furthermore, some apparently rational businesspeople gladly sacrifice profits for humanitarian, moral, or aesthetic considerations—even when those considerations cannot be justified in greater profits in the long run. Many companies sponsor scholarships for college students, knowing full well that the associated tax benefits, improvements in corporate image, and recruitment have but a small probability of producing profits in excess of the costs. These people cannot base their decisions solely on EMVs.

On a personal level, too, the "true value" of an alternative is often above or below its EMV. Thus I might pay more for a house in the mountains than its EMV (calculated, say, by real estate investment counselors) because the beauty and solitude of its setting make up the extra value *to me*, or I might continue to drive the old family car out of sentiment long after it has stopped being economical to do so. To see the divergence between EMVs and true values in simple decision problems, consider these examples.

Example 1. True value below EMV. After ten years of work you have saved $15,000 as a down payment on your dream house. You know the house you want and need only turn over your money to have it. Before you can do that your stockbroker calls with a "very hot tip." If you can invest your $15,000 for one month, he can assure you an 80% chance at a $50,000 return. Unfortunately, if the investment fails, you lose everything. Now the EMV of this investment is $37,000—well above the $15,000 you now have in hand. Your broker points out that you can still buy the house a month from now and urges you to make the investment. But you do not, because you feel you cannot afford to risk the $15,000. For you, making the investment is worth less than having $15,000 in hand, and thus it is worth less than its EMV.

Example 2. True value above EMV. Suppose you have been trying to purchase a ticket for a championship basketball game. Tickets are available at $20 but you have only $10 on you. A fellow comes along and offers to match your $10 on a single roll of the dice. If you roll snake eyes, you get the total pot; otherwise he takes it. This means that your chances are 1 in 12 of ending up with the $20 you need and 11 in 12 of losing all you have. The EMV of this is −$7.50—definitely below the $10 you have in hand. But you take the bet, since having the $20 is worth the risk to you. Thus the EMV of this bet is below its true value to you.

In addition to practical problems with EMVs there is an important philosophical difficulty. Even if you are guided solely by the profit motive, there is no logical connection between the EMV of a risk and its monetary value. Consider this example. You alone have been given a ticket for the one and only lottery your state will have. (Although the lottery is in its very first year, the legislature has already passed a bill repealing it.) The ticket gives you one chance in a million of winning $1,000,000. Since you lose nothing if you fail to win, the EMV of the ticket is $1. After the lottery is drawn you win nothing or

\$1,000,000 – but never \$1. Thus how can we connect this figure with a cash value for the bet? Why – assuming money is all that counts – would it be rational for you to sell your ticket for \$2? One is tempted to answer in terms of averages or long runs: If there were many people in your situation, their winnings would average \$1; if this happened to you year after year, your winnings would average \$1. But this will not work for the case at hand, since, by hypothesis, you alone have a free ticket and there will be only one lottery. With a one-shot decision there is nothing to average; so we have still failed to connect EMVs with cash values.

PROBLEMS

1. Given a utility scale u, we can formulate a disutility scale, $d(u)$, by multiplying each entry on the u-scale by -1. The expected disutility of an act is calculated in the same way as its expected utility except that every utility is replaced by its corresponding disutility. Reformulate the rule for maximizing expected utilities as a rule involving expected disutilities.
2. Show that the expected disutility of an act is equal to -1 times its expected utility.
3. Show that using a disutility scale and the rule you formulated in problem 1 yields the same rankings of acts as maximizing expected utilities does.
4. The St. Petersburg game is played as follows. There is one player and a "bank." The bank tosses a fair coin once. If it comes up heads, the player is paid \$2; otherwise the coin is tossed again with the player being paid \$4 if it lands heads. The game continues in this way with the bank continuing to double the amount set. The game stops when the coin lands heads.

   Consider a modified version of this game. The coin will be tossed no more than two times. If heads comes up on neither toss, the player is paid nothing. What is the EMV of this game?

   Suppose the coin will be tossed no more than n times. What is the EMV of the game?

   Explain why an EMVer should be willing to pay any amount to play the unrestricted St. Petersburg game.
5. Consider the following answer to the one-shot lottery objection to EMVs: True, there is only one lottery and only one person has a free ticket. But in a hypothetical case in which there were many such persons or many lotteries, we would find that the average winnings would be \$1. Let us identify the cash value of the ticket with the average winnings in such hypothetical cases. It follows immediately that the cash value equals the EMV.

   Do you think this approach is an adequate solution to the problem of relating EMVs to cash values?

4-3. Von Neumann-Morgenstern Utility Theory

John Von Neumann, a mathematician, and Oskar Morgenstern, an economist, developed an approach to utility that avoids the objections we raised to EMVs.

Although Ramsey's approach to utility antedates theirs, today theirs is better known and more entrenched among social scientists. I present it here because it separates utility from probability, whereas Ramsey's approach generates utility and subjective probability functions simultaneously.

Von Neumann and Morgenstern base their theory on the idea of measuring the strength of a person's preference for a thing by the risks he or she is willing to take to receive it. To illustrate that idea, suppose that we know that you prefer a trip to Washington to one to New York to one to Los Angeles. We still do not know how *much more* you prefer going to Washington to going to New York, but we can measure that by asking you the following question: Suppose you were offered a choice between a trip to New York and a lottery that will give you a trip to Washington if you "win" and one to Los Angeles if you "lose." How great a chance of winning would you need to have in order to be indifferent between these two choices? Presumably, if you prefer New York quite a bit more than Los Angeles, you will demand a high chance at Washington before giving up a guaranteed trip to New York. On the other hand, if you only slightly prefer New York to Los Angeles, a small chance will suffice. Let us suppose that you reply that you would need a 75% chance at Washington—no more and no less. Then, according to Von Neumann and Morgenstern, we should conclude that the New York trip occurs 3/4 of the way between Washington and Los Angeles on your scale.

Another way of representing this is to think of you as supplying a ranking not only of the three trips but also of a lottery (or gamble) involving the best and worst trips. You must be indifferent between this lottery and the middle-ranked trip. Suppose we let the expression

$$L(a, x, y)$$

stand for the lottery that gives you a chance equal to a at the prize x and a chance equal to $1-a$ at the prize y. Then your ranking can be represented as

Washington
New York, $L(3/4$, Washington, Los Angeles$)$
Los Angeles.

We can use this to construct a utility scale for these alternatives by assigning one number to Los Angeles, a greater one to Washington, and the number 3/4 of the way between them to New York. Using a 0 to 1 scale, we would assign 3/4 to New York—but any other scale obtained from this by a positive linear transformation will do as well.

Notice that since the New York trip and the lottery are indifferent they are ranked together on your scale. Thus the utility of the lottery itself is 3/4 on a 0 to 1 scale. But since on that scale the utilities of its two "prizes" (the trips) are 0 and 1, the expected utility of the lottery is also 3/4. We seem to have forged a link between utilities and expected utilities. Indeed, Von Neumann and Morgenstern showed that if an agent ranks lotteries in the manner of our example, their utilities will equal their expected utilities. Let us also note that the Von

Neumann-Morgenstern approach can be applied to any kind of item—whether or not we can sensibly set a price for it—and that it yields an agent's personal utilities rather than monetary values generated by the marketplace. This permits us to avoid our previous problems with monetary values and EMVs.

(You might have noticed the resemblance between the Von Neumann-Morgenstern approach and our earlier approach to subjective probability, where we measured degrees of belief by the amount of a valued quantity the agent was willing to stake. Ramsey's trick consisted in using these two insights together without generating the obvious circle of defining probability in terms of utility and utility in turn in terms of probability.)

The Von Neumann-Morgenstern approach to utility places much stronger demands on agents' abilities to fix their preferences than do our previous conditions of rationality. Not only must agents be able to order the outcomes relevant to their decision problems, they must also be able to order all lotteries involving these outcomes, all compound lotteries involving those initial lotteries, all lotteries compounded from those lotteries, and so on. Furthermore, this ordering of lotteries and outcomes (I will start calling these *prizes*) is subject to constraints in addition to the ordering condition. Put in brief and rough form, these are: (1) Agents must evaluate compound lotteries in agreement with the probability calculus (reduction-of-compound-lotteries condition); (2) given three alternatives A, B, C with B ranked between A and C, agents must be indifferent between B and some lottery yielding A and C as prizes (continuity condition); (3) given two other lotteries agents will prefer the one giving the better "first" prize—if everything else is equal (better-prizes condition); (4) given two otherwise identical lotteries, agents will prefer the one that gives them the best chance at the "first" prize (better-chances condition). If agents can satisfy these four conditions plus the ordering condition of chapter 2, we can construct an interval utility function u with the following properties:

(1) $u(x) > u(y)$ if and only if xPy
(2) $u(x) = u(y)$ if and only if xIy
(3) $u[L(a, x, y)] = au(x) + (1 - a)u(y)$
(4) Any u' also satisfying (1)–(3) is a positive linear transformation of u.

You should recognize (1) and (2) from our discussion of decisions under ignorance (chapter 2). They imply that u is at least an ordinal utility function. But (3) is new. It states that the utility of a lottery is equal to its expected utility. We can also express this by saying that u has the *expected utility property*. The entire result given by (1)–(4) is known as the *expected utility theorem*. Let us now turn to a rigorous proof of it.

First we must specify lotteries more precisely than we have. The agent is concerned with determining the utilities for some set of outcomes, alternatives, or prizes. Let us call these *basic prizes*. Let us also assume that the number of basic prizes is a finite number greater than 1 and that the agent is not indifferent between all of them. Since we can assume that the agent has ranked the prizes, some will be ranked at the top and others at the bottom. For future reference,