Knygos David Tall - Knowing and Teaching Elementary Mathematics santrauka

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How Humans Learn to Think Mathematically

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How do we learn to think mathematically?

Input through our senses
Output through our actions
Making mental links to form **thinkable concepts**
Linking them together in **knowledge structures**

Over time humans build increasingly sophisticated knowledge structures based on their experiences.

There are some aspects which we all share as human beings.
There are other aspects that depend on how we think about our previous experiences.

Set-befores & Met-befores

A '**set-before**' is a mental ability that we are all born with, which make take a little time to mature as our brains make connections in early life.

Three major set-befores in mathematical thinking relate to

Input-perception	Output-action	Formulating concepts
Recognition	**Repetition**	**Language**

A '**met-before**' is a personal mental structure in our brain *now* as a result of experiences met before.
Many different met-befores are possible, depending on experiences available in our society at the time.

2+2 is 4 **after 2 comes 3** **addition makes bigger**
take-away makes smaller **multiplication makes bigger**
all expressions (such as 2+3, 22/7, 3.48x23.4) have answers.

Set-befores & Met-befores

A '**set-before**' is a mental ability that we are all born with, which make take a little time to mature as our brains make connections in early life.

The three set-befores give a basic framework for learning to think mathematically.

A '**met-before**' is a personal mental structure in our brain *now* as a result of experiences met before.

Met-befores affect the way that we vary as individuals, both in terms of how our learning progresses and also in terms of our attitudes to mathematics.

Set-befores & Met-befores

The terms 'set-before' and 'met-before' which work better in English than in some other languages started out as a joke.

The term 'metaphor' is often used to represent how we interpret one knowledge structure in terms of another.

I wanted a simple word to use when talking to children.

When they use their earlier knowledge to interpret new ideas I could ask them how their thinking related to what they had **met before**.

It was a joke: the word play met**A**phor, met**B**efore.

The joke *worked* well with teachers and children: What have you **met before** that causes you to think like this?

Set-Befores

The three major set-befores in mathematical thinking are
Recognition, Repetition, Language

Recognition + *language* allows classifying categories such as 'cat' and 'dog', triangle, square, rectangle, circle.

Repetition + *language* allows practising sequences of actions
… used in counting …
… column arithmetic …
… adding fractions …
… learning algorithms …
… solving equations …

May be performed automatically without meaning
procedural thinking

Language (& Symbols) flexible **proceptual thinking**

Symbols e.g. 3+2 may be used as a process (addition) or concept (sum) to enable *flexible* thinking (**procept**)

Building on Set-befores

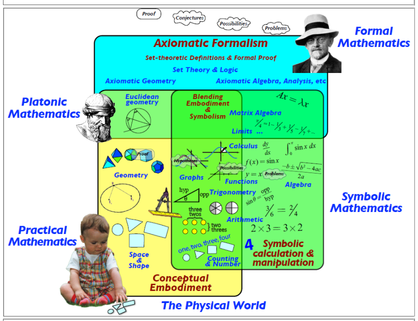
Each set-before leads to a different form of ***compression of knowledge*** into thinkable concepts.

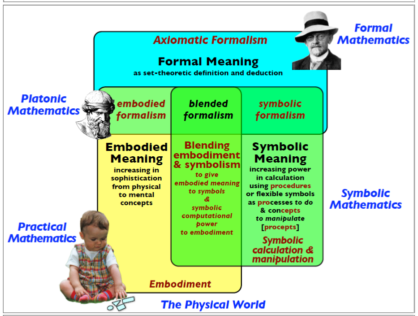
- ***recognition*** leads to **categorization**
  'this is a triangle, that is a graph'
- ***repetition*** leads to **encapsulation**
  the process of addition becomes sum $3+4$
  integration is conceived as integral $\int \sin x\, dx$
- ***language*** leads to **definition**
  first defining properties of figures & operations $\mathbb{R}$
  later as set-theoretic definitions. $\aleph_0$

Building on Set-befores

Each set-before leads to ***a different world of mathematical thinking***.

- ***recognition*** leads to **conceptual embodiment** (in which we categorize and build knowledge structures about things we perceive and think about);
- ***repetition*** leads to **procedural symbolism** through action (such as counting) that may remain **procedural** or may may be symbolized as thinkable concepts such as number, with symbols that function both as processes to do and concepts to think about (**proceptual symbolism**); $3+4$ $\int \sin x\, dx$
- ***language*** leads to specifying properties of objects as definitions in geometry and properties of actions formulated as 'rules of arithmetic', and much later in university mathematics to **axiomatic-formalism** (based on formal definitions and proof). $\mathbb{R}$ $\aleph_0$





Personal Development

The three worlds of mathematics is a framework for the broad development of mathematical thinking based on the three set-befores which we all share.

However, each of us builds in our own personal way, and a significant factor relates to how we build and use met-befores that arise through successive experiences.

Met-Befores

Current ideas based on experiences met before.
Examples:
Two and two makes four.
Addition makes bigger.
Multiplication makes bigger.
Take away makes smaller.
Every arithmetic expression 2+2, 3x4, 27÷9 has an answer.
Squares and Rectangles are different.
Different symbols eg △ and □ represent different things.

Met-Befores

Current ideas based on experiences met before.
Examples:
Two and two makes four. … works in later situations.
Addition makes bigger. … fails for negative numbers.
Multiplication makes bigger. … fails for fractions.
Take away makes smaller. … fails for negative numbers.
An algebraic expression 2x+1 does not have an 'answer'.
Later, *by definition*, a square is a rectangle.
Different symbols can represent the same thing.

Met-Befores

Sometimes met-befores are *supportive* in a new context.
Two and two makes four… works in most cases.
Addition makes bigger … works for positive fractions.
Sometimes met-befores are *problematic* in a new context.
Addition makes bigger … fails for negative quantities
Multiplication makes bigger … fails for proper fractions
Take away makes smaller … fails for negatives and for infinite sets.
Arithmetic expressions 2+2, 3x4, 27÷9 have an answer.
Algebraic expressions such as 2+3x do not have an answer.

Blending different knowledge structures

When we encounter new ideas, we try to make sense of them with our met-befores.

These may come from **blending** together different knowledge structures, with some aspects in common and others clashing.

Mathematical concepts are often a blend with aspects that are common, but with others that cause confusion.

Certain aspects give new power and pleasure to some, but clashing aspects can others confusion and fear....

Blending different conceptions of number

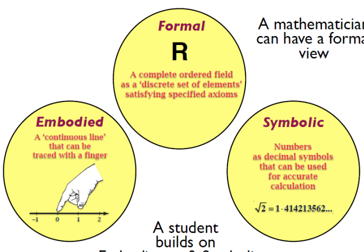
Successive number systems have properties that conflict

Counting Numbers
each number has a next with none between,
starts counting at 1, then 2, 3, ...
addition makes bigger, take-away smaller, multiplication bigger.

Fractions
a fraction has many names: $^1/_3, ^4/_{12}, ^7/_{21}$ …
there is no 'next' fraction.
addition and multiplication involve new techniques,
addition makes bigger, take-away smaller,
multiplication may be smaller.

Integers
each number has a next with none between,
numbers can be positive or negative,
addition may get smaller, take-away may get larger,
multiplication of negatives gives a positive.

The Real Numbers as a Multi-blend



A mathematician can have a formal view

A student builds on Embodiment & Symbolism

Increasing sophistication of number systems

New knowledge **blends** together new perceptions with existing met-befores.

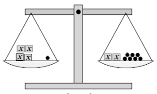
Blending occurs *between* and *within* different aspects of embodiment, symbolism and formalism.

Mathematicians usually view the number systems as an expanding system:

$$\mathbf{N} \subset \begin{matrix}\mathbf{F} \\ \mathbf{Z}\end{matrix} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$$

Cognitively the development is more usefully expressed in terms of *blends*.

Equations as a blend

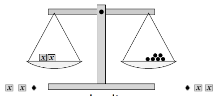


embodiment

$4x+1 = 2x+7$

symbolism

Equations as a blend



embodiment

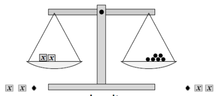
$2x = 6$

symbolism

What happens now???

What is the meaning of taking half of each side?

Equations as a blend



embodiment

$x = 3$

symbolism

What happens if the equation is $2x\text{-}1 = 5\text{-}3x$ **?**

How can we have $2x$ on the left hand side and take off 1?
More general equations do not have an easy embodiment.

How do we help children make sense of the general?

From Arithmetic to Algebra

The transition from arithmetic to algebra is difficult for many.

The conceptual blend between a linear algebra equation and a physical balance works in simple cases for many children (Vlassis, 2002, Ed. Studies).

The blend breaks down with negatives and subtraction (Lima & Tall 2007, Ed. Studies).

Conjecture: there is no *single* embodiment that matches the flexibility of algebraic notation.

Students conceiving algebra flexibly as generalised arithmetic are likely to find it simple.

Those who remain with inappropriate blends of met-befores may find it distressing and complicated.

From Arithmetic to Algebra

In a study in Brazil, the students were struggling with algebra and teachers tried to help them to answer problems on tests.

Method: 'do the same thing to both sides.'

Result: Students remembered 'what to do'.

An embodiment shifting symbols around, with extra rules.

Change sides *change signs*, so $2x+1 = 6$ becomes $2 = 6 - 1$.

Move 2 in $2x=6$ to the other side and put it under $x = \frac{6}{2}$.

Quadratics solved later by the formula.

'John says $(x\text{-}2)(x\text{-}3)=0$ has solutions 2 and 3. Is he right?'

Students tried to use the formula and most could not cope.

Jim Kaput was well aware of the problems of learning algebra.
Can his legacy be aided by reflecting on the role of met-befores?

From Arithmetic & Algebra to Calculus

The research at the Kaput Centre is particularly focused on SimCalc and the communal use of software to construct and share concepts.

Does an awareness of the relationship between embodiment and symbolism and of the role of met-before give new insight?

Implications for teaching

Many transitions involve problematic met-befores:

from counting to whole numbers
from whole numbers to fractions
from whole numbers to signed numbers
from arithmetic to algebra
from repeated powers to fractional and negative powers
From practical arithmetic to the limit concept
from description to deductive definition
at many other transitions in development of concepts such as the function concept. (linear, quadratic, trig., log., exponential ...)

In each case, conflict between old knowledge (met-before) and new knowledge, can lead to confusion and lack of meaning, so that fragile rote-learning may become the only option...