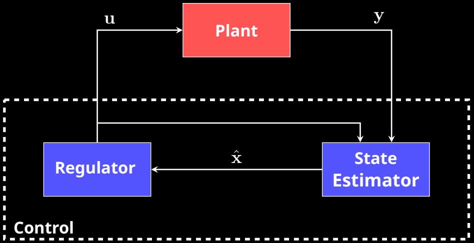
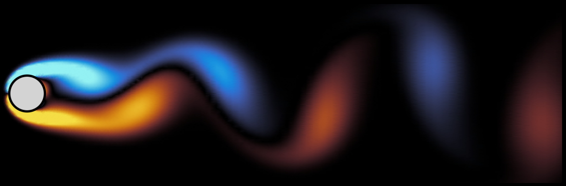
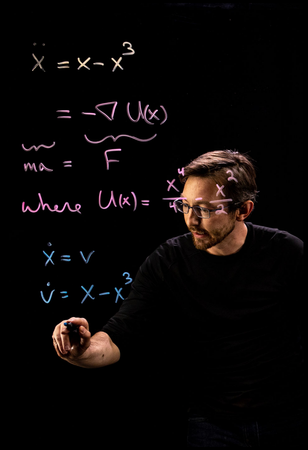




Tips and tricks for Koopman analysis

Jean-Christophe Loiseau



Steve Brunton

Jean-Christophe Loiseau

Bernard Koopman (1900-1981)

Overview of Koopman theory

Poincaré point of view

$$\mathbf{x}_{i+1} = \mathbf{F}(\mathbf{x}_i, \tau)$$
$$\mathbf{y}_{i+1} = \mathbf{g}(\mathbf{x}_i)$$

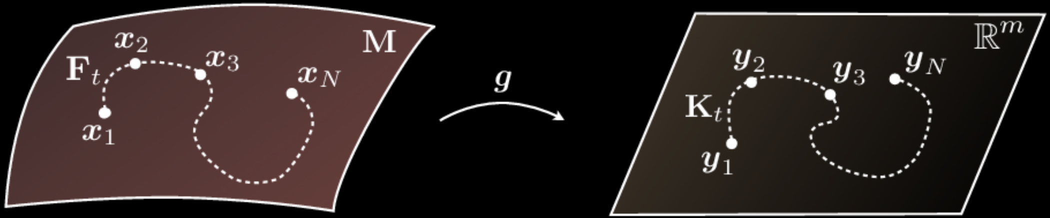
Koopman point of view

$$\mathcal{K}_\tau g = g \circ \mathbf{F}$$
$$\mathcal{K}_\tau g(\mathbf{x}_i) = g(\mathbf{x}_{i+1})$$

$$\mathcal{K}_\tau \left(\alpha g_1 + \beta g_2\right) = \left(\alpha g_1 + \beta g_2\right) \circ \mathbf{F}$$

$$\Updownarrow$$

$$\alpha \mathcal{K}_\tau g_1 + \beta \mathcal{K} g_2 = \alpha g_1 \circ \mathbf{F} + \beta g_2 \circ \mathbf{F}$$



$$
\begin{tikzcd}
y_1 \arrow[r, "\mathbf{K}_t"] \arrow[d, dashed, shift left] & y_2 \arrow[r, "\mathbf{K}_t"] \arrow[d, dashed, shift left] & y_3 \arrow[r, "\mathbf{K}_t"] \arrow[d, dashed, shift left] & \cdots \arrow[r, "\mathbf{K}_t"] \arrow[d, dashed, shift left] & y_N \arrow[d, dashed, shift left] \\
x_1 \arrow[r, "\mathbf{F}_t"'] \arrow[u, "g", shift left] & x_2 \arrow[r, "\mathbf{F}_t"'] \arrow[u, "g", shift left] & x_3 \arrow[r, "\mathbf{F}_t"'] \arrow[u, "g", shift left] & \cdots \arrow[r, "\mathbf{F}_t"'] \arrow[u, "g", shift left] & x_N \arrow[u, "g", shift left]
\end{tikzcd}
$$

$$\mathbf{F}_t : x_k \mapsto x_{k+1}$$

$$g : x_k \mapsto y_k$$

$$\mathbf{K}_t : y_k \mapsto y_{k+1}$$

$$\left\|\hat{\mathcal{K}} - \mathcal{K}\right\|_{HS}^2 = \left\|\hat{\mathcal{K}}\right\|_{HS}^2 - 2\langle\hat{\mathcal{K}}|\mathcal{K}\rangle_{HS} + \|\mathcal{K}\|_{HS}^2$$

Let f_i, g_i and $\hat{\sigma}_i$ be the i[th] singular triplet of $\hat{\mathcal{K}}$. We can then write

$$\left\|\hat{\mathcal{K}} - \mathcal{K}\right\|_{HS}^2 = \sum_{i=1}^{r} \hat{\sigma}_i^2 \langle f_i | f_i \rangle_{\rho_1} \langle g_i | g_i \rangle_{\rho_0} - 2 \sum_{i=1}^{r} \hat{\sigma}_i \langle f_i | \mathcal{K} g_i \rangle + \sum_{j=1}^{\infty} \sigma_j^2$$

Maximize this score based on data

$$\left\|\hat{\mathcal{K}} - \mathcal{K}\right\|_{HS}^2 = -\boxed{\mathcal{R}_E(\hat{\sigma}, f, g)} + \|\mathcal{K}\|_{HS}^2$$

Let $\boldsymbol{\chi}_0 = \begin{bmatrix} \chi_0^{(1)} & \chi_0^{(2)} & \chi_0^{(3)} & \dots \end{bmatrix}^T$ be an arbitrary basis for the input space $\mathcal{L}_0^2$, and let $\mathbf{g} = \mathbf{V}^T \boldsymbol{\chi}_0$ be an orthonormal basis for this space. It must then satisfy

$$\langle g_i | g_j \rangle_{\rho_0} = \mathbf{v}_i^T \mathbf{C}_{00} \mathbf{v}_j = \delta_{ij}$$

with $\mathbf{C}_{00} = \mathbb{E}\left[\boldsymbol{\chi}_0(\mathbf{x}_t)\boldsymbol{\chi}_\mathbf{0}(\mathbf{x}_t)^T\right]$. A similar expression applies for f with basis $\boldsymbol{\chi}_1$ and density ρ_1.

After some algebraic manipulations, we arrive at

$$\begin{aligned} \underset{\mathbf{U},\mathbf{V}}{\text{maximize}} \quad & \mathsf{Tr}\left(\mathbf{U}^T\mathbf{C}_{10}\mathbf{V}\right) \\ \text{subject to} \quad & \mathbf{U}^T\mathbf{C}_{11}\mathbf{U} = \mathbf{I}_r \\ & \mathbf{V}^T\mathbf{C}_{00}\mathbf{V} = \mathbf{I}_r \end{aligned}$$

with $\mathbf{C}_{10} = \mathbb{E}\left[\boldsymbol{\chi}_1(\mathbf{x}_t)\boldsymbol{\chi}_0(\mathbf{x}_{t+\tau})\right]$.

1. Compute the covariance matrices $\mathbf{C}_{00}$, $\mathbf{C}_{11}$, and $\mathbf{C}_{10}$.

2. Perform the truncated SVD

$$\mathbf{C}_{11}^{-\frac{1}{2}}\mathbf{C}_{10}\mathbf{C}_{00}^{-\frac{1}{2}} = \mathbf{U}'\mathbf{K}\mathbf{V}'^{T}$$

3. Compute $\mathbf{U} = \mathbf{C}_{11}^{-\frac{1}{2}}\mathbf{U}'$ and $\mathbf{V} = \mathbf{C}_{00}^{-\frac{1}{2}}\mathbf{V}'$.

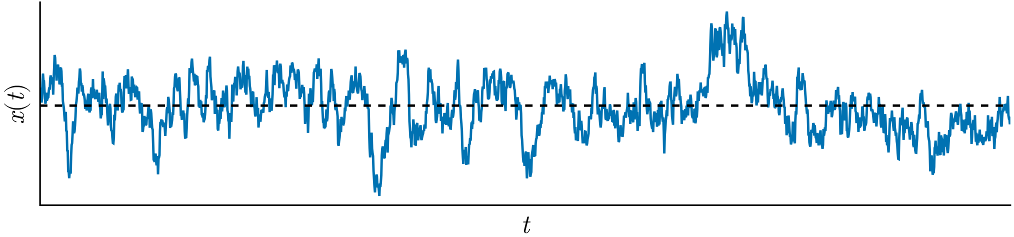
4. Output the singular values K_{ii} and singular functions $f_i = \mathbf{u}_i^T\boldsymbol{\chi}_1$ and $g_i = \mathbf{v}_i^T\boldsymbol{\chi}_0$.

Orstein-Uhlenbeck process

$$\mathrm{d}\mathbf{x}_t = \mathbf{A}\mathbf{x}_t \, \mathrm{d}t + \mathbf{B} \, \mathrm{d}\mathbf{w}_t$$

Orstein-Uhlenbeck process

Typical time-series of the Orstein-Uhlenbeck process with a = -1 and b = 0.2.



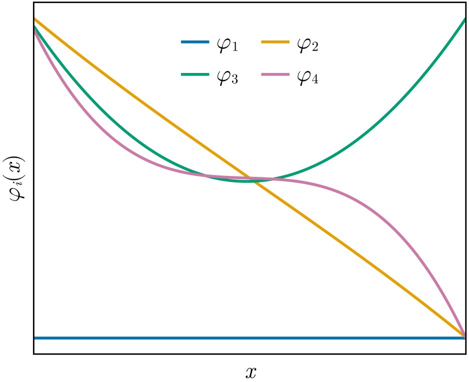
Given an ergodic time-series, use

$$\chi_0 = \chi_1 = \begin{bmatrix} 1 & x & x^2 & x^3 & \cdots \end{bmatrix}^T$$

to approximate the Koopman singular functions.

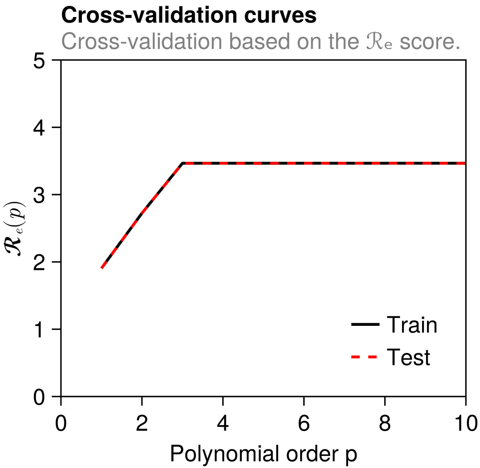
Orstein-Uhlenbeck process

Leading Koopman eigenfunctions consist in simple polynomials in x.



Use the VAMP-E score $\mathcal{R}_e(p)$ for cross-validation purposes.

For the OU process, setting $p > 3$ is unecessary.

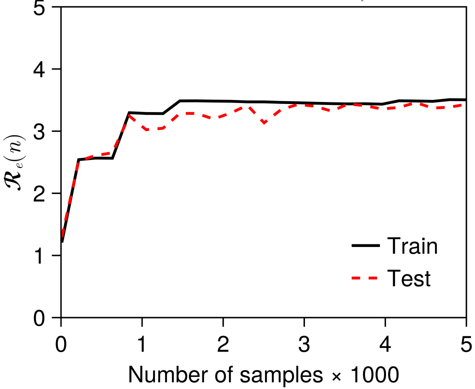


$\mathcal{R}_e$ can also be used to check convergence w.r.t the size of the dataset.

For the OU process, a few thousands samples are sufficient.

Learning curves

Evolution of the VAMP-E $\mathcal{R}_e$ score as a function of the number of samples.

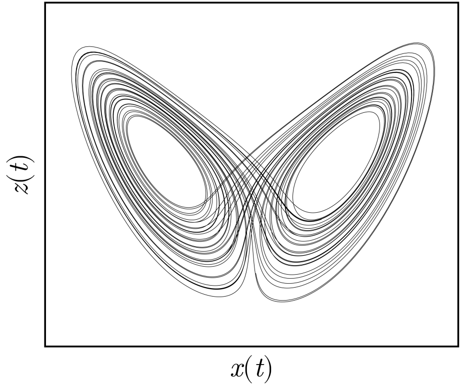


Lorenz system

$$\dot{x} = \sigma\left(y - x\right)$$
$$\dot{y} = x\left(\rho - z\right) - y$$
$$\dot{z} = xy - \beta z$$

Lorenz attractor

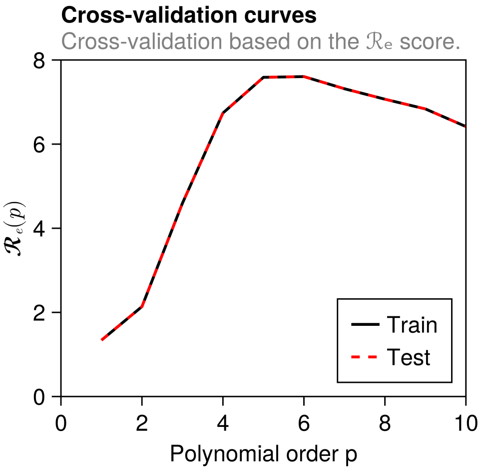
The poster child of chaos theory



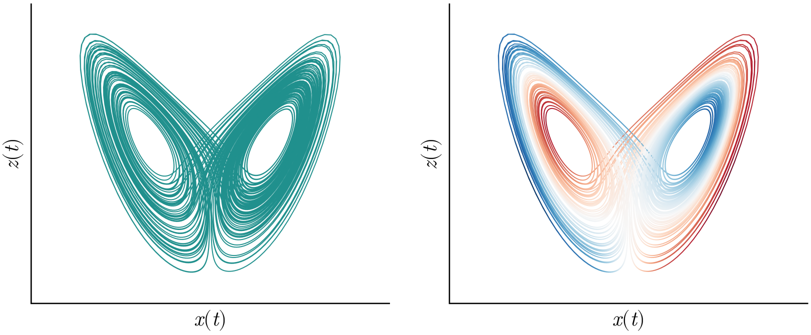
Given an ergodic time-series, use

$$\chi = \begin{bmatrix} x & y & z & x^2 & xy & xz & y^2 & \cdots \end{bmatrix}^T$$

to approximate the Koopman singular functions.



Leading Koopman singular functions of the Lorenz system

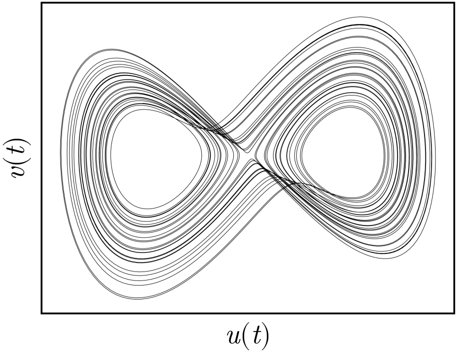


The leading non-trivial Koopman singular function partitions the state space into metastable (almost-invariants) sets.

If a single time-series is available, use Takens theorem and time-delay embedding to reconstruct the attractor.

Lorenz attractor

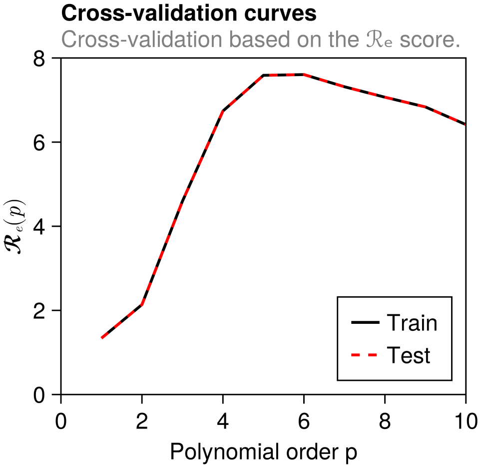
Broomhead-King reconstruction of the attractor from x(t).



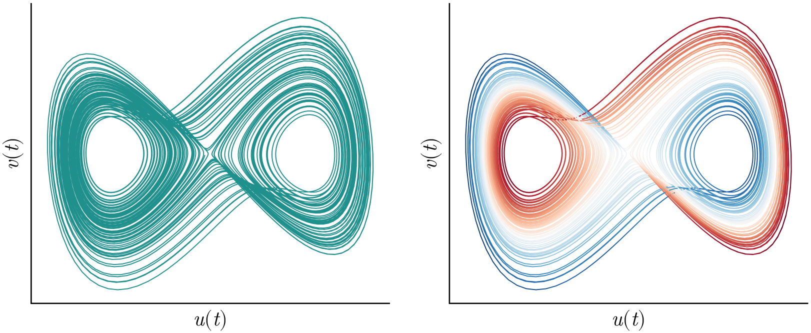
Given the reconstructed attractor, use

$$\chi = \begin{bmatrix} u & v & w & u^2 & uv & uw & v^2 & \cdots \end{bmatrix}$$

to approximate the Koopman singular functions.

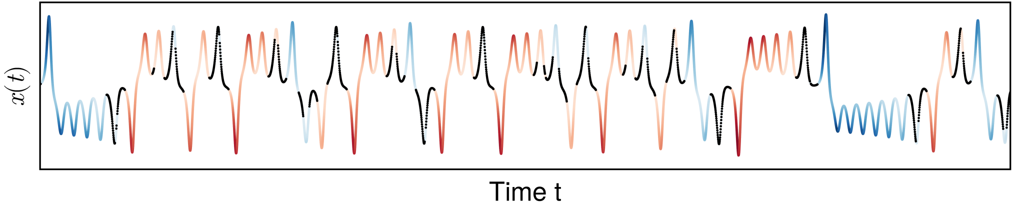


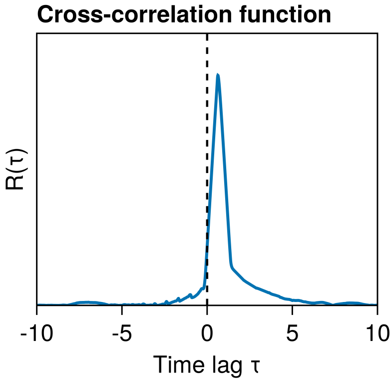
Leading Koopman singular functions of the Lorenz system with time-delay coordinates



Koopman leading singular function evaluated along a trajectory

Points in black correspond to $|\varphi(x)| < \varepsilon$, highlighting that this Koopman function can be used as a predictor of incoming switching event.





Let $f(t) = \mathrm{sign}(x_t)$ and $g(t) = \mathrm{sign}(\varphi_1(x_t))$. On average, $g(t)$ predicts a switching event **a whole period** before it occurs.

Application to a flow example

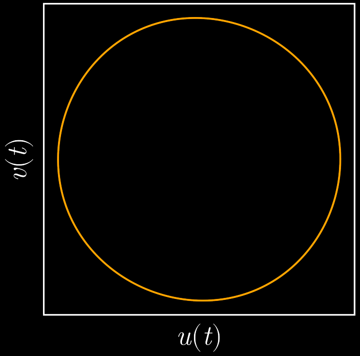
Low-Re cylinder flow : Canonical fluid example of a self-sustained weakly nonlinear oscillator.

In the rest, we'll assume that we only have access to the instantaneous lift and drag coefficients.

The dynamics being periodic, the attractor is a simple limit cycle.

Low-Re Cylinder flow

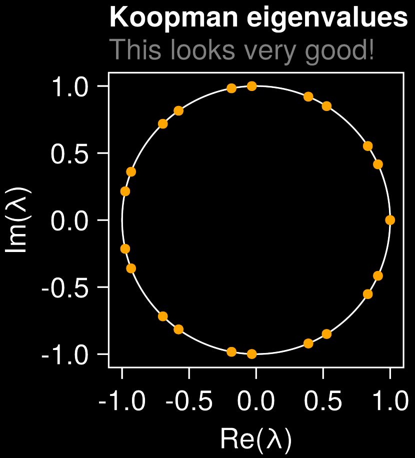
Reconstruction of the attractor from lift and drag.



Given $\mathbf{U}$, $\mathbf{K}$, and $\mathbf{V}$, compute the approximate Koopman eigenvalues

$$\lambda = \mathrm{eig}\left(\mathbf{K}\mathbf{V}^T\mathbf{U}\right)$$

and corresponding residual.



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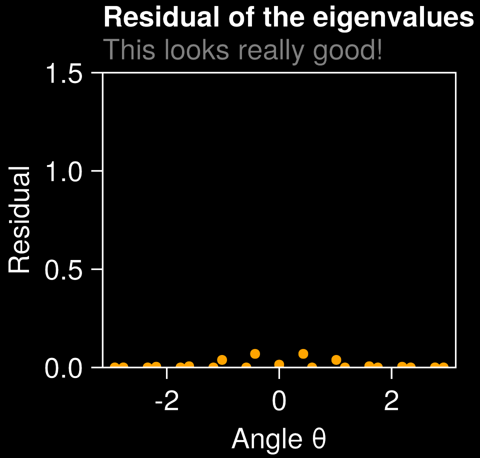

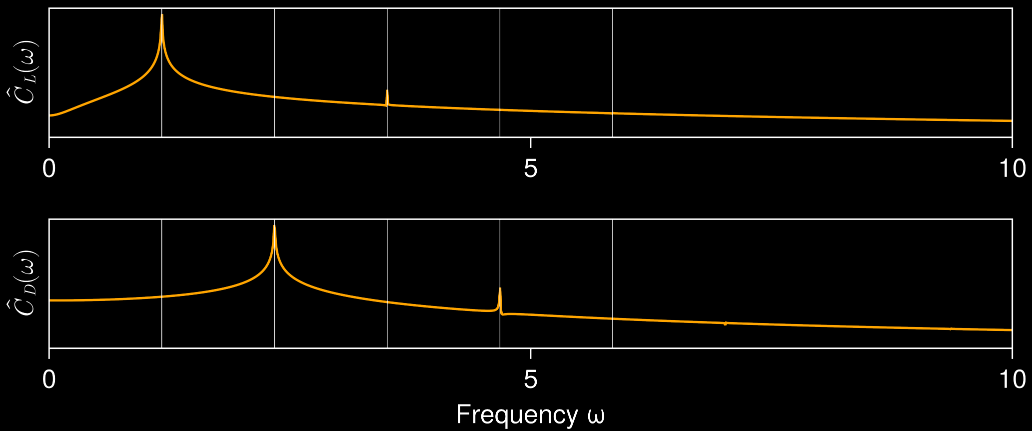
$\hat{C}_L(\omega)$
0
5
10
$\hat{C}_D(\omega)$
0
5
10
Frequency ω

Shear-driven cavity flow : Canonical fluid example for reduced-order modelling and flow control.

In the rest, we'll assume that we only have access to the fluctuation's kinetic energy.

The dynamics being quasi-periodic, the attractor is a T_2-torus.

Given $\mathbf{U}$, $\mathbf{K}$, and $\mathbf{V}$, compute the approximate Koopman eigenvalues

$$\lambda = \operatorname{eig}\left(\mathbf{K}\mathbf{V}^T\mathbf{U}\right)$$

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$$\lambda = \mathrm{eig}\left(\mathbf{K}\mathbf{V}^T\mathbf{U}\right)$$

and corresponding residual.

Thermosyphon : Example of chaotic thermal convection with Lorenz-like dynamics.

In the rest, we'll assume that we only have access to the instantaneous flow rate.

The dynamics being chaotic, it evolves on a strange attractor.

Given $\mathbf{U}$, $\mathbf{K}$, and $\mathbf{V}$, compute the approximate Koopman eigenvalues

$$\lambda = \operatorname{eig}\left(\mathbf{K}\mathbf{V}^T\mathbf{U}\right)$$

and corresponding residual.

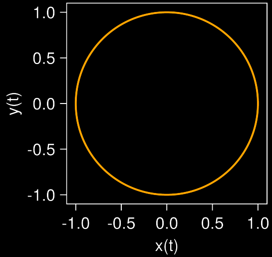
Given $\mathbf{U}$, $\mathbf{K}$, and $\mathbf{V}$, compute the approximate Koopman eigenvalues

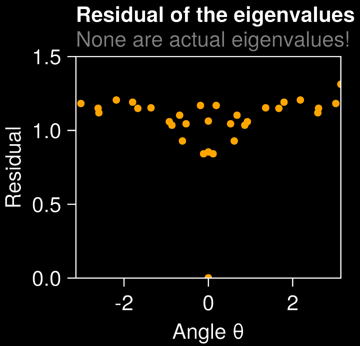
$$\lambda = \mathrm{eig}\left(\mathbf{K}\mathbf{V}^T\mathbf{U}\right)$$

and corresponding residual.

Additional tips and tricks and conclusion

If the dynamics are periodic/quasi-periodic, the Koop-man operator has a point spectrum. Use **FFT** to com-pute them more efficiently !





It is not because you can compute eigenvalues of your approximation that they necessarily are eigenvalues of the true Koopman operator !

Stay up-to-date with the current literature! A lot has been done since the pioneering work of Peter on DMD.

https://loiseaujc.github.io/

https://loiseau-jc.medium.com/

@loiseau_jc

Thank you for your attention!

Any question?