CHAPTER 1

Basics of p-adic integration towards Batyrev's theorem

ABSTRACT. The guiding thread of this chapter is a theorem, whose proof using p-adic integration is due to Batyrev, stating that two birationally equivalent Calabi-Yau varieties share the same Betti numbers.

We start with recalling general definitions and facts about local and global fields, before coming to a general definition of p-adic integrals on non-Archimedean analytic manifolds. We conclude the chapter with the proof of Batyrev's theorem.

1. Local fields

1.1. First definitions. We start with a few definitions and examples.

DEFINITION 1.1. An *absolute value* on K is a map

$$|\cdot| : K \to \mathbf{R}^+$$

such that

- $(|0|, |1|) = (0, 1)$,
- $|a + b| \leqslant |a| + |b|$ for all $a, b \in K$,
- $|ab| = |a||b|$ for all $a, b \in K$.

The datum of a field together with an absolute value is called *a valued field*.

DEFINITION 1.2. A *local field* is a valued field

- whose absolute value is non-trivial (!)
- and whose associated metric topology is locally compact.

Equivalently and more concretely, a local field is either $\mathbf{R}$, $\mathbf{C}$, a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers, or a field of formal Laurent series $\mathbf{F}_q((t))$, for some power q of a prime p.

EXAMPLE 1.1. Given a prime p, one can define on $\mathbf{Q}$ the p-adic absolute value by

$$|a|_p = p^{-\nu_p(a)}$$

where $\nu_p(a)$ (for $a \neq 0$) is the p-adic valuation of a, that it is to say, the unique integer ν such that $a = p^\nu m/n$ with $\gcd(p, m) = \gcd(p, n) = 1$.

EXAMPLE 1.2. In a similar way, one can define a t-adic absolute value on $\mathbf{F}_q(t)$.

PROPOSITION 1.1. *Every valued field K admits a completion $\widehat{K}$ containing K as a dense subset on which the absolute value of $\widehat{K}$ coincides with the original one of K.*

REMARK 1.1. A local field is automatically complete.

EXAMPLE 1.3. The completion of $\mathbf{Q}$ with respect to the p-adic absolute value is the field of p-adic numbers. The closed unit disk in $\mathbf{Q}_p$ is the ring of p-adic integers.

1.2. Non-Archimedean valued fields.

DEFINITION 1.3. A valued field K is said to be *non-Archimedean* if its absolute value $|\cdot|$ satisfies the ultrametric inequality

$$|x+y| \leqslant \max(|x|,|y|) \quad x,y \in K.$$

DEFINITION 1.4. Let K be a non-Archimedean valued field. The closed unit disk

$$R = \{x \in K \mid |x| \leqslant 1\}$$

is a subgring of K, called the *valuation ring of* K. This terminology means that for any $x \in K^\times$, either x or x^{-1} lies in R.

The ring R is a local ring whose unique maximal ideal is the open unit disk

$$\mathfrak{m} = \{x \in K \mid |x| < 1\}.$$

The *residue field of* K is the quotient of R by $\mathfrak{m}$. In these notes it will generally be denoted by the greek letter κ.

EXAMPLE 1.4. In $\mathbf{F}_q(t)$, respectively $\mathbf{F}_q((t))$, both endowed with the t-adic absolute value

$$|x| = q^{-v_t(x)},$$

the valuation ring is the polynomial ring $\mathbf{F}_q[t]$, respectively the ring of power series $\mathbf{F}_q[[t]]$.

1.3. Hensel's lemma. Henselianity is a crucial property that we are going to use a lot later, without really thinking about it.

DEFINITION 1.5. A local ring R, with maximal ideal $\mathfrak{m}$ and residue field k, is called *henselian* if for every polynomial $f \in R[T]$ and every $a \in R$ such that

$$f(a) \in \mathfrak{m} \text{ and } f'(a) \notin \mathfrak{m}$$

there *exists* a *unique* $b \in R$ such that $f(b) = 0$ and $b - a \in \mathfrak{m}$.

Geometrically, it means that for every $f \in R[T]$, any smooth k-point of the special fibre of the R-suhscheme of $\mathbf{A}^1_R$ defined by f lifts to a unique R-point of $\{f = 0\}$.

LEMMA 1.1 (Hensel's lemma, one variable). *Let R be a complete discrete valuation ring and $f \in R[T]$. Assume that there exists integers $n \geqslant e \geqslant 0$ and $a \in R$ such that*

$$f(a) \in \mathfrak{m}^{n+e+1} \textit{ and } f'(a) \notin \mathfrak{m}^{e+1}.$$

Then there exists a unique lift $b \in R$ of a modulo $\mathfrak{m}^{n+1}$, that is to say such that

$$f(b) = 0 \textit{ and } b - a \in \mathfrak{m}^{n+1}.$$

REMARK 1.2. The previous lemma says that in particular, complete discrete valuation rings are henselian.

PROOF. This is a non-Archimedean application of Newton's algorithm. One has to find a solution of the equation in u

$$f(a + \varpi^{n+1}u) = 0$$

where ϖ is a generator of $\mathfrak{m}$. We see a as the initial guess for a root of f and we must define the next terms of Newton's iteration and control their size. In what follows, when we write $x \in \mathfrak{m}^m$ for some integer m, just think about the non-Archimedean valuation being at least equal to m.

First, applying the Taylor expansion of f

$$f(a+T) = f(a) + f'(a)T + T^2 \underbrace{g(T)}_{\in R[T]}$$

to our equation gives

$$f(a) + f'(a)\varpi^{n+1}u + \varpi^{2n+2}u^2 g(\varpi^{n+1}u) = 0.$$

Then, by assumption, one can write

$$f(a) = u_1 \varpi^{n+e+1}$$

where $u_1 \in R$ and the ratio

$$u_2 = \frac{\varpi^e}{f'(a)}$$

actually lies in R, since $f'(a)$ is not divisible by ϖ^{e+1}. Dividing everywhere by $f'(a)\varpi^{n+1}$ our equation becomes

$$\begin{aligned} u &= -\frac{f(a)}{f'(a)\varpi^{n+1}} - \frac{\varpi^{2n+2}}{f'(a)\varpi^{n+1}} u^2 g(\varpi^{n+1}u) \\ &= -u_1u_2 - \varpi^{n+1-e} u_2 u^2 g(\varpi^{n+1}u). \end{aligned}$$

Since we assume that $n \geqslant e$, we have that necessarily $u + u_1u_2 \in \mathfrak{m}$. Now one puts

$$\begin{aligned} a_1 &= a - \varpi^{n+1}u_1u_2 \\ &= a - \frac{f(a)}{f'(a)}. \end{aligned}$$

The reduction classes of $a_0 = a$ and a_1 agree modulo $\mathfrak{m}^{n+1}$ and $f(a_1) \in \mathfrak{m}^{n+e+1+1}$, getting one first step closer to a root of f. Moreover, it is important to check that $f'(a_1) \notin \mathfrak{m}^{e+1}$ by using the Taylor expansion of f' and the fact that $n \geqslant e$:

$$f'(a_1) = \underbrace{f'(a)}_{\notin \mathfrak{m}^{e+1}} + \underbrace{f''(a)(a_1 - a)}_{\in \mathfrak{m}^{n+1}} + \underbrace{(a_1 - a)^2 h(a_1 - a)}_{\in \mathfrak{m}^{2n+2}} \qquad \text{hence } f'(a_1) \notin \mathfrak{m}^{e+1}.$$

This a_1 we've just constructed is unique modulo $\mathfrak{m}^{n+2}$. Indeed, if a' is another element of R satisfying $a' - a_0 \in \mathfrak{m}^{n+1}$ and $f(a') \in \mathfrak{m}^{n+e+1+1}$, then $a' - a_1 \in \mathfrak{m}^{n+1+1}$. This can be seen by writing

$$f(a') = f(a_1) + f'(a_1)(a' - a) + (a' - a_1)^2 g(a' - a_1)$$

and then the fact that $f'(a_1) \notin \mathfrak{m}^{e+1}$ forces $a' - a_1 \in \mathfrak{m}^{n+1+1}$.

The Newton sequence $(a_m)_{m \in \mathbf{N}}$ is now defined by setting

$$\begin{aligned} a_0 &= a \\ a_{m+1} &= a_m - \frac{f(a_m)}{f'(a_m)} \quad m \in \mathbf{N}. \end{aligned}$$

Repeating the previous argument, one gets that $a_{m+1} - a_m \in \mathfrak{m}^{n+m+1}$ for every $m \in \mathbf{N}$ and $f(a_m) \in \mathfrak{m}^{n+m+e+1}$. By completeness of R, this sequence converges to an element $b \in R$ such that

$$f(b) = 0 \text{ and } b - a \in \mathfrak{m}^{n+1}.$$

This lift b is unique: indeed, if $b' \in R$ is another element such that $f(b') = 0$ and $b' - a \in \mathfrak{m}^{n+1}$, then necessarily $b' - a_m \in \mathfrak{m}^{n+m+1}$ for every $m \in \mathbf{N}$ (by induction on m, and we already did the case $m = 1$ above), so that $b' = b$. □

With a little more effort, one can show a multivariate version of Hensel's lemma with formal power series. The proof is left as an exercise for very brave students.

LEMMA 1.2 (Hensel's lemma). *Let R be a complete discrete valuation ring and $\mathfrak{m}$ its maximal ideal. Fix integers $r \geq \ell \geq 0$. Consider $f_1, \ldots, f_\ell \in R[\![T_1, \ldots, T_r]\!]$ and $a_1, \ldots, a_r \in \mathfrak{m}$ such that*

$$f_i(a) \in \mathfrak{m} \text{ for all } i \in \{1, ..., \ell\}.$$

Assume moreover that the minor

$$\Delta = \det \left(\frac{\partial f_i}{\partial T_j} \right)_{1 \leq i,j \leq \ell}.$$

of the Jacobian matrix is invertible in R.

Then there exist $b_1, \ldots, b_r \in R$ such that

$$f_i(b) = 0 \text{ for all } i \in \{1, ..., \ell\}$$

and

$$a_j \equiv b_j \mod \mathfrak{m} \text{ for all } j \in \{1, ..., r\}.$$

2. Analytic manifolds and integration

We work above a local field K, which can be Archimedean or not. The goal of this section is to define K-analytic manifolds and integration on them.

In practice, in this course we will consider manifolds that are realised as the set

$$M = X(K)$$

of K-point of a smooth K-scheme X of finite type and pure dimension. To fix ideas, the reader can think about $K = \mathbf{Q}_p$ or $K = \mathbf{F}_q((t))$ but the definitions given above also apply to $K = \mathbf{R}$ or $\mathbf{C}$.

2.1. Analytic functions. In this subsection it is sufficient to assume that K is a complete valued field in the sense of Definition 1.1 page 1.

Let $T_1, ..., T_d$ be indeterminates. We use the convenient notation $\mathbf{T}^{\boldsymbol{m}} = T_1^{m_1} ... T_d^{m_d}$ for every $\boldsymbol{m} \in \mathbf{N}^d$.

DEFINITION 2.1. A *convergent* power series in d variable is an element

$$f(\mathbf{T}) \in K[[\mathbf{T}]] = K[[T_1, ..., T_d]]$$

such that the radius of convergence

$$\rho(f) = \sup \left\{ r \in \mathbf{R}_+ \;\middle|\; \text{the sequence } \left(|f_{\boldsymbol{n}}| r^{|\boldsymbol{n}|}\right)_{\boldsymbol{n} \in \mathbf{N}^d} \text{ converges to } 0 \right\}$$

is positive.

Analytic functions are precisely the functions coming locally from a convergent power series.

DEFINITION 2.2. Let U be an open subset of K^d. A function

$$f : U \to K$$

is said to be K-analytic if for every point $a \in U$ there exists a convergent power series

$$f_a \in K[[\mathbf{T}]]$$

such that

$$f(x) = f_a(x - a)$$

for all $x \in D_a(a, \rho(f_a))$.

REMARK 2.1.
- From the definition, one sees that a K-analytic function is automatically continuous.
- Hence, inside the sheaf of continuous functions on U taking values in K, one can define the subsheaf of K-analytic functions.
- The set of K-analytic functions on U admits a structure of K-algebras.

THEOREM 2.1 (Implicit function theorem). *Let K be a complete valued field. Let m and n be two positive integers and let $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two sets of indeterminates.*

Assume that we are given n formal power series

$$F_1, \ldots, F_n \in K[\![\mathbf{X}, \mathbf{Y}]\!]$$

such that

$$F_i(0, 0) = 0 \text{ for all } i \in \{1, ..., n\}$$

and

$$\det \left(\frac{\partial F_i}{\partial Y_j}(0, 0) \right)_{1 \leq i,j \leq n} \neq 0.$$

Then, there exists a unique n-tuple of elements

$$f_1, \ldots, f_n \in K[\![\mathbf{X}]\!]$$

such that

$$F_i(\mathbf{X}, f_1(\mathbf{X}), ..., f_n(\mathbf{X})) = 0$$

for all $i \in \{1, ..., n\}$.

***In case** F **converges** on a neighbourhood of $\mathbf{0} \in K^{m+n}$, **then** f **converges as well** on a neighbourhood of $\mathbf{0} \in K^m$, and on a small enough neighbourhood of $\mathbf{0} \in K^{m+n}$, the vanishing locus of F coincides with the graph of f.*

Moreover, if the valuation on K is discrete and R is the valuation ring of K, the F_i's are all in $R[\![\mathbf{X}, \mathbf{Y}]\!]$ and

$$\det \left(\frac{\partial F_i}{\partial Y_j}(0, 0) \right)_{1 \leq i,j \leq n} \notin \mathfrak{m},$$

then

$$f_1, \ldots, f_n \in R[\![\mathbf{X}]\!].$$

PROOF. After an initial simplification, the proof is still as brutal as one can imagine.

First, one remarks that our problem is invariant under base change for the coefficients of the F_i's. Write

$$F_i(\mathbf{X},\mathbf{Y}) = \sum_{j=1}^{n} a_{ij}Y_j - \underbrace{\sum_{|\boldsymbol{j}|+|\boldsymbol{k}|>0} c_{i\boldsymbol{j}\boldsymbol{k}}\mathbf{X}^{\boldsymbol{i}}\mathbf{Y}^{\boldsymbol{k}}}_{=G_i(\mathbf{X},\mathbf{Y})}$$

so that

$$A = (a_{ij})_{i,j\in\{1,\dots,n\}} = \left(\frac{\partial F_i}{\partial Y_j}(0,0)\right)_{i,j\in\{1,\dots,n\}}$$

is invertible by assumption. Multiplying by A^{-1} the coefficients of $(F_1,\dots,F_n)$, we can assume that A is the identity matrix without changing our problem.

Now we have to solve the system

$$f_i(\mathbf{X}) = \sum_{|\boldsymbol{j}|+|\boldsymbol{k}|>0} c_{i\boldsymbol{j}\boldsymbol{k}}\mathbf{X}^{\boldsymbol{j}}(f_1(\mathbf{X}),\dots,f_n(\mathbf{X}))^{\boldsymbol{k}} \qquad i\in\{1,\dots,n\}.$$

Let us write

$$f_i(\mathbf{X}) = \sum_{d\in\mathbf{N}} f_{i,d}(\mathbf{X})$$

where $f_{i,d}(\mathbf{X})$ is the homogeneous part of degree d of $f_i(\mathbf{X})$, so that we want to check that the $f_{i,d}(\mathbf{X})$'s are uniquely determined. The previous system is equivalent to

$$f_{i,d}(\mathbf{X}) = \sum_{\substack{|\boldsymbol{j}|+|\boldsymbol{k}|>0\\ k_1+\dots+k_n=|\boldsymbol{k}|\\ d_{i',k'}\in\mathbf{Z}_{>0}\\ |\boldsymbol{j}|+\sum_{i',k'} d_{i',k'}=d}} c_{i\boldsymbol{j}\boldsymbol{k}}\mathbf{X}^{\boldsymbol{j}} \prod_{i'\in\{1,\dots,n\}}\prod_{k'\in\{1,\dots,k_i\}} f_{i',d_{i',k'}}(\mathbf{X})$$

In particular

$$f_{i,1}(\mathbf{X}) = \sum_{|\boldsymbol{j}|=1} c_{i\boldsymbol{j}0}\mathbf{X}^{\boldsymbol{j}}.$$

To be completed: one eventually shows that all the other coefficients are uniquely determined. $\square$

Partial derivatives of K-analytic functions $f: U\to K$ are defined as usual by the formula:

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t\to 0}\frac{f(a+t\epsilon_i)-f(a)}{t}$$

where $\varepsilon_1,\dots,\varepsilon_d$ is the canonical basis of K^d. They are automatically K-analytic, because they coincide locally with the formal derivatives of the convergent power series defining f in a neighborhood of $a\in U$.

Definition 2.3. Let $f: U\to K^d$ be a K-analytic function on an open subset $U\subset K^d$. The Jacobian determinant of f is defined at $a\in U$ by

$$\mathrm{Jac}(f)(a) = \det\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{1\le i,j\le n}.$$

Applying Theorem 2.1, we are able to locally invert K-analytic functions whose Jacobian does not vanish in a neighborhood of a point.

THEOREM 2.2 (Local inversion). *Let $f : U \to K^d$ be a K-analytic function on an open subset $U \subset K^d$. Let $a \in U$ such that*

$$\mathrm{Jac}(f)(a) \neq 0.$$

Then, there exist

- *an open neighborhood U_a of a such that $f(U_a)$ is also an open neighborhood of $f(a)$ in K^d ;*
- *a K-analytic function*
$$g_a : f(U_a) \to U_a$$
such that
$$g \circ f = \mathrm{id}_{U_a} \text{ and } f \circ g = \mathrm{id}_{f(U_a)}.$$

2.2. Analytic manifolds.

DEFINITION 2.4. A K-analytic manifold of dimension d can be defined in two ways.

(1) (Concrete) It is a topological space M together with a d-dimensional K-analytic atlas on it: a set of mutually compatible charts (U_i, φ_i) such that the union of the sets U_i covers M (called an atlas), where compatible means that for all i, j, the homeomorphism
$$\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j)$$
is K-analytic.

(2) (Abstract) It is a locally K-ringed space $(M, \mathscr{O}_M)$ which is locally isomorphic to the polydisk
$$E^d(0,1) = \{x \in K^d \mid |x_i| \leqslant 1 \text{ for all } i \in \{1, ..., d\}\}$$
endowed with its sheaf of K-analytic functions.

2.3. Change of variables and gauge forms. From now on we assume that K is a local field in the sense of Definition 1.2.

DEFINITION 2.5. Assume that μ is a Haar measure on $(K, +)$. The locally compact group $(K^d, +)$ can be endowed with an induced Haar measure

$$\mathrm{d}\mu(\boldsymbol{x}) = \mathrm{d}\mu(x_1) \otimes ... \otimes \mathrm{d}\mu(x_d).$$

DEFINITION 2.6 (Modulus of $(K, +, \mu)$). The modulus

$$\mathrm{mod}_K : K \to \mathbf{R}_+$$

is defined by the formula

$$\mu(a\Omega) = \mathrm{mod}_K(a)\mu(\Omega)$$

for every $a \in K$ and every bounded measurable subset Ω of K.

Definition 2.7. Let U be an open subset of K^d and ω be a differential form of degree d on U. In other words, there exists a unique analytic function h on U such that

$$\omega = h(\boldsymbol{x})\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d.$$

The measure

$$\mathrm{mod}_K(\omega)$$

is defined by

$$\int_U \varphi\, \mathrm{mod}_K(\omega) = \int_U \varphi(x)\mathrm{mod}_K(h(\boldsymbol{x}))\mathrm{d}\mu(\boldsymbol{x})$$

for every continuous function φ on U having compact support.

Theorem 2.3 (Local change of variables). *Let U be an open set in K^d and let*

$$f : U \to K^d$$

be an injective K-analytic map. Assume moreover that the Jacobian of f does not vanish on U.

Then, for every integrable function $\varphi : f(U) \to \mathbf{R}$

$$\int_{f(U)} \varphi(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) = \int_U \varphi(f(\boldsymbol{x}))\mathrm{mod}_K(\mathrm{Jac}(f)(\boldsymbol{x}))\mathrm{d}\mu(\boldsymbol{x}).$$

Proof. It is enough to prove the formula on a small open neighborhood of every point of U, prove it for some elementary functions, and then use the chain rule of the Jacobian.

Let ϖ be a generator of $\mathfrak{m}$.

(1) First, one proves it for a linear change of variable $y = Ax + a$ where $A \in \mathrm{GL}_d(K)$ and $a \in K^d$. This comes from the formula

$$\mu(A\Omega) = \mathrm{mod}_K(\det(A))\mu(\Omega).$$

Remark: in most concrete situations this is already enough!

(2) Then one proves it for *special restricted power series*, that is to say for $f \in K[\![\mathbf{X}]\!]$ such that

$$f(0) = 0$$

and $c_{\boldsymbol{i}} \in \mathfrak{m}^{|\boldsymbol{i}|-1}$ for every $\boldsymbol{i} \in \mathbf{N}^d \setminus \{\mathbf{0}\}$, in particular $f \in R[\![\mathbf{X}]\!]$. In that case, $y = f(x)$ is measure-preserving. Indeed, the image of $a + \varpi^e R^d$ under f is $f(a) + \varpi^e R^d$ for every $e \in \mathbf{Z}_{>0}$.

(3) If $f \in R[\![\boldsymbol{X}]\!]$ is convergent in a neighborhood of some a, then for every $e \in \mathbf{Z}_{>0}$

$$g(\mathbf{X}) = \varpi^{-e}(f(a + \varpi^e \mathbf{X}) - f(a))$$

is a special restricted power series.

(4) Hence one can always assume that $f(a) = \mathbf{0}$ and that $f_i(\mathbf{X})$ is of the form

$$f_i(\mathbf{X}) = X_i + \sum_{|\boldsymbol{j}|>2} c_{i,\boldsymbol{j}}\mathbf{X}^{\boldsymbol{j}}$$

and since f_i converges locally, there are some well-chosen $e_0, e_1 \in \mathbf{N}$ such that $\varpi^{e_1} c_{i,\boldsymbol{j}} \varpi^{e_0|\boldsymbol{j}|}$ is in R for every $i \in \{1, \ldots, d\}$ and $\boldsymbol{j}$ (choose e_0 such that $c_{i,\boldsymbol{j}}\varpi^{e_0|\boldsymbol{j}|} \to 0$ as $|\boldsymbol{j}| \to \infty$ for every i and then choose e_1). One can form a special restricted power series $g_i(\mathbf{X})$

$$g_i(\mathbf{X}) = \varpi^{-e} f_i(\varpi^e \mathbf{X})$$

for $e \geqslant 2e_0 + e_1 + 1$.

Then one concludes arguing by composition. $\square$

Definition 2.8 (Metric on a line bundle). A *metric* on a line bundle $L \to M$ is the datum for every open subset U of M of

$$\|\cdot\| : \ell \in \Gamma(U, L) \longmapsto (\|\ell\| : U \to \mathbf{R}_+)$$

such that

(1) $\|\cdot\|$ is continuous;
(2) for every $\ell \in \Gamma(U, L)$ and $x \in U$, $\|\ell\|$ is positive if and only if $\ell(x) \neq 0$;
(3) it is compatible with restriction to any open subset V inside U;
(4) $\|f\ell\|(x) = |f(x)|\|\ell\|(x)$ for every K-analytic function on U.

The datum of a line bundle together with a metric on it is called a *metrized line bundle*.

The measure associated to a top-degree global differential form – also called *gauge form* – on an analytic manifold is defined locally using charts. Naturally, it depends on the existence of such a global form and on the choice of the differential form.

Proposition 2.1. *If M is a K-analytic manifold of dimension d and ω is a differential form of degree d on M, then there exists a unique measure*

$$\mathrm{mod}_K(\omega)$$

on M which locally coincides with the measure associated to differential forms on open subsets of K^d, that is to say such that for every chart (U, f) of M and every integrable function φ having support in U,

$$\int_M \varphi \, \mathrm{mod}_K(\omega) = \int_{f(U)} (\varphi \circ f^{-1}) \, \mathrm{mod}_K((f^{-1})^* \omega).$$

Proof. First, assume that $M = U$ is a open subset of K^d. Then,

$$\omega = h \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$$

for a unique analytic function h on U. As we already saw, the measure $\mathrm{mod}_K(\omega)$ is then given by

$$\int_U \varphi \, \mathrm{mod}_K(\omega) = \int_U \varphi(x) \mathrm{mod}_K(h(x)) \mathrm{d}\mu(x)$$

for every compactly supported continuous function on U. This definition is invariant by K-analytic differomorphism: if $g : V \to U$ is such a change of coordinates,

$$\begin{aligned}\int_U \varphi \, \mathrm{mod}_K(h(x)) \mathrm{d}\mu(x) &= \int_V \varphi \circ g(y) \, \mathrm{mod}_K(h \circ g(y)) \, \mathrm{mod}_K(\mathrm{Jac}(g)(y)) \mathrm{d}\mu(y) \\ &= \int_V \varphi \circ g(y) \, \mathrm{mod}_K(g^* \omega).\end{aligned}$$

Now, in general, if M is an arbitrary K-analytic manifold of dimension d and ω a top-degree differential form on it, let us consider a *finite* family of charts (U_i, f_i) covering the support of $\varphi : M \to K$ together with a partition of unity

$$\sum_i \lambda_i \equiv 1$$

where the support of λ_i is contained in U_i. Then we set

$$\int_M \varphi \operatorname{mod}_K(\omega) = \sum_i \int_{f_i(U_i)} (\lambda_i \circ f_i^{-1}) \cdot (\varphi \circ f_i^{-1}) \cdot \operatorname{mod}_K((f_i^{-1})^*\omega).$$

Using the change of variable formula again, one checks that the right hand side (which is a finite sum) does not depend on the choices of charts and partitions of unity. $\square$

2.4. Analytification of smooth schemes. Given a K-scheme of finite type, there exists a canonical way to endow its set of K-points with a topology satisfying two natural conditions.

Definition 2.9. Let X be a K-scheme of finite type. The *analytic topology* on $X(K)$ is the coarsest topology satisfying the following properties:

- for any Zariski-open subset $U \subset X$, its set $U(K)$ of K-points is open in $X(K)$;
- for every Zariski-open subset $U \subset X$ and any regular function $\varphi \in \mathscr{O}_X(U)$ the map $U(K) \to K$ induced by φ is continuous.

Defining a structure of a K-analytic manifold on a smooth K-scheme of finite type boils down to defining a subsheaf of the sheaf of continuous functions with values in K.

Definition 2.10. Let X be a *smooth* K-scheme of finite type.
Let U be an open subset of $X(K)$. We say that a function

$$f : U \longrightarrow K$$

is analytic at a point $x \in U$ if there exist a Zariski-open neighborhood $V \ni x$ in X, an immersion of K-schemes

$$i : V \hookrightarrow \mathbf{A}^n_K,$$

an open neighborhood $W \ni i(x)$ in $\mathbf{A}^n_K(K) = K^n$ together with an analytic function

$$g : W \longrightarrow K$$

such that

$$f = g \circ i$$

on an analytic neighborhood of $x \in X(K)$.

Proposition 2.2. *Via the previous definitions, the following holds.*

- *Any morphisms of smooth K-schemes induces a morphism of K-analytic manifolds; in particular,*
  - *open immersions induce open immersions of K-analytic manifolds;*
  - *closed immersions induce closed immersions of K-analytic manifolds.*
- *The structure of a K-analytic manifold on $\mathbf{A}^n_K(K) = K^n$ is the natural one: it coincides with the one from the previous sections.*
- *Any étale morphism of smooth K-schemes induces an étale morphisms of K-analytic manifolds (local isomorphisms).*

Proof. The first and second point are easy and left as an exercise.
The third point in an application of the local inversion Theorem 2.2. $\square$

The following proposition says that in the local non-Archimedean setting, rational points of closed subschemes are negligible.

PROPOSITION 2.3. *Let K be a non-Archimedean local field and X a smooth K-scheme. Suppose that X is endowed with a measure μ associated to a gauge form, thanks to Proposition 2.1.*

Let

$$Z \subset X$$

be a closed subscheme of codimension at least 1. Then

$$\mu(Z(K)) = 0.$$

PROOF. The question is local and it is sufficient to prove the following statement: if M is a submanifold of an open subscheme of K^d, of codimension $c \geqslant 1$ everywhere, then M has measure zero in K^d.

Using the implicit function Theorem 2.1 and the change of variable formula Theorem 2.3 we reduce to the case

$$M = \{\underbrace{0, ..., 0}_{c \text{ times}}\} \times E^{d-c}(0,1)$$

inside $E^d(0,1)$, in particular observe that every polydisk

$$E^d(a,r) = \{x \in K^d \mid |x - a| \leqslant r\}$$

is isomorphic to the unit polydisk $E^d(0,1)$. Then the claim follows from Definition 2.5 page 7 and the fact that $\{0\}$ has measure zero in K. $\square$

2.5. Models and associated measures. In this paragraph we explain a very important construction of a certain metric that is going to be used in every chapter of this course.

Construction 2.1 (Metric induced by a model). Assume that K is a non-Archimedean local field (not necessarily complete), with valuation field R, and that $\mathscr{X}$ is a flat separated R-scheme of finite type, with smooth generic fibre $X = \mathscr{X} \otimes_R K$, so that $\mathscr{X}(R)$ injects into $X(K)$ as a compact subset.

Consider $\mathscr{L}$ a coherent sheaf on $\mathscr{X}$ whose generic fibre L is a line bundle (we say that $\mathscr{L}$ is a model of L). This line bundle induces a line bundle on the K-analytic manifold $M = \mathscr{X}(R)$, again denoted by L.

If $x \in M = \mathscr{X}(R)$, the fibre $x^*\mathscr{L}$ is an R-module of finite type, which possibly has a non-empty torsion part $x^*\mathscr{L}_{\mathrm{tors}}$, and $x_L^* L$ is a one dimensional K-vector space.

$$\begin{tikzcd}
x_K^* L \arrow[r] \arrow[dd] & x^*\mathscr{L} \arrow[dd] & \mathscr{L} \arrow[d, dash] \\
 & & \mathscr{X} \arrow[d] \\
\mathrm{Spec}(K) \arrow[r] \arrow[urr, "x_K"] & \mathrm{Spec}(R) \arrow[r, equal] \arrow[ur, "x"'] & \mathrm{Spec}(R)
\end{tikzcd}$$

Then, the R module

$$\mathscr{L}(x) = x^*\mathscr{L}/x^*\mathscr{L}_{\mathrm{tors}}$$

can be seen as a lattice inside $x_K^* L$: indeed, since the square is Cartesian, a point of $x^*\mathscr{L}$, seen as a section $\mathrm{Spec}(R) \to x^*\mathscr{L}$, then composed with $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ induces a unique K-point of $x_K^* L$.

Given any generator y_0 of this lattice, we obtain a norm on the K-vector space $x_K^* L$ by setting $\|ay_0\| = |a|$ for all $a \in K$ (this does not depend on the choice of y_0 since two generators differ by an invertible element).

Now given any section s of L on an open subset U of M, we set $\|s\|(x) = \|s(x)\|$ for all $x \in U$.

Proposition 2.4. *The previous construction defines a metric on the line bundle L in the sense of Definition 2.8.*

Proof. **Postponed.** ☐

Preliminary version of lecture notes for the course:

Some applications of p-adic integration to geometry and arithmetics

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Spring 1, 2025

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