

CHAPTER 1

Basics of p -adic integration towards Batyrev's theorem

ABSTRACT. The guiding thread of this chapter is a theorem, whose proof using p -adic integration is due to Batyrev, stating that two birationally equivalent Calabi-Yau varieties share the same Betti numbers.

We start with recalling general definitions and facts about local and global fields, before coming to a general definition of p -adic integrals on non-Archimedean analytic manifolds. We conclude the chapter with the proof of Batyrev's theorem.

1. Local fields

1.1. First definitions. We start with a few definitions and examples.

DEFINITION 1.1. An *absolute value* on K is a map

$$|\cdot| : K \rightarrow \mathbf{R}^+$$

such that

- $(|0|, |1|) = (0, 1)$,
- $|a + b| \leq |a| + |b|$ for all $a, b \in K$,
- $|ab| = |a||b|$ for all $a, b \in K$.

The datum of a field together with an absolute value is called a *valued field*.

DEFINITION 1.2. A *local field* is a valued field

- whose absolute value is non-trivial (!)
- and whose associated metric topology is locally compact.

Equivalently and more concretely, a local field is either \mathbf{R} , \mathbf{C} , a finite extension of the field \mathbf{Q}_p of p -adic numbers, or a field of formal Laurent series $\mathbf{F}_q((t))$, for some power q of a prime p .

EXAMPLE 1.1. Given a prime p , one can define on \mathbf{Q} the p -adic absolute value by

$$|a|_p = p^{-\nu_p(a)}$$

where $\nu_p(a)$ (for $a \neq 0$) is the p -adic valuation of a , that is to say, the unique integer ν such that $a = p^\nu m/n$ with $\gcd(p, m) = \gcd(p, n) = 1$.

EXAMPLE 1.2. In a similar way, one can define a t -adic absolute value on $\mathbf{F}_q(t)$.

PROPOSITION 1.1. Every valued field K admits a completion \widehat{K} containing K as a dense subset on which the absolute value of \widehat{K} coincides with the original one of K .

REMARK 1.1. A local field is automatically complete.

EXAMPLE 1.3. The completion of \mathbf{Q} with respect to the p -adic absolute value is the field of p -adic numbers. The closed unit disk in \mathbf{Q}_p is the ring of p -adic integers.

1.2. Non-Archimedean valued fields.

DEFINITION 1.3. A valued field K is said to be *non-Archimedean* if its absolute value $|\cdot|$ satisfies the ultrametric inequality

$$|x + y| \leq \max(|x|, |y|) \quad x, y \in K.$$

DEFINITION 1.4. Let K be a non-Archimedean valued field. The closed unit disk

$$R = \{x \in K \mid |x| \leq 1\}$$

is a subring of K , called the *valuation ring* of K . This terminology means that for any $x \in K^\times$, either x or x^{-1} lies in R .

The ring R is a local ring whose unique maximal ideal is the open unit disk

$$\mathfrak{m} = \{x \in K \mid |x| < 1\}.$$

The *residue field* of K is the quotient of R by \mathfrak{m} . In these notes it will generally be denoted by the greek letter κ .

EXAMPLE 1.4. In $\mathbf{F}_q(t)$, respectively $\mathbf{F}_q((t))$, both endowed with the t -adic absolute value

$$|x| = q^{-v_t(x)},$$

the valuation ring is the polynomial ring $\mathbf{F}_q[t]$, respectively the ring of power series $\mathbf{F}_q[[t]]$.

1.3. Hensel's lemma. Henselianity is a crucial property that we are going to use a lot later, without really thinking about it.

DEFINITION 1.5. A local ring R , with maximal ideal \mathfrak{m} and residue field k , is called *henselian* if for every polynomial $f \in R[T]$ and every $a \in R$ such that

$$f(a) \in \mathfrak{m} \text{ and } f'(a) \notin \mathfrak{m}$$

there *exists* a *unique* $b \in R$ such that $f(b) = 0$ and $b - a \in \mathfrak{m}$.

Geometrically, it means that for every $f \in R[T]$, any smooth k -point of the special fibre of the R -subscheme of \mathbf{A}_R^1 defined by f lifts to a unique R -point of $\{f = 0\}$.

LEMMA 1.1 (Hensel's lemma, one variable). *Let R be a complete discrete valuation ring and $f \in R[T]$. Assume that there exists integers $n \geq e \geq 0$ and $a \in R$ such that*

$$f(a) \in \mathfrak{m}^{n+e+1} \text{ and } f'(a) \notin \mathfrak{m}^{e+1}.$$

Then there exists a unique lift $b \in R$ of a modulo \mathfrak{m}^{n+1} , that is to say such that

$$f(b) = 0 \text{ and } b - a \in \mathfrak{m}^{n+1}.$$

REMARK 1.2. The previous lemma says that in particular, complete discrete valuation rings are henselian.

PROOF. This is a non-Archimedean application of Newton's algorithm. One has to find a solution of the equation in u

$$f(a + \varpi^{n+1}u) = 0$$

where ϖ is a generator of \mathfrak{m} . We see a as the initial guess for a root of f and we must define the next terms of Newton's iteration and control their size. In what follows, when we write $x \in \mathfrak{m}^m$ for some integer m , just think about the non-Archimedean valuation being at least equal to m .

First, applying the Taylor expansion of f

$$f(a + T) = f(a) + f'(a)T + T^2 \underbrace{g(T)}_{\in R[T]}$$

to our equation gives

$$f(a) + f'(a)\varpi^{n+1}u + \varpi^{2n+2}u^2g(\varpi^{n+1}u) = 0.$$

Then, by assumption, one can write

$$f(a) = u_1\varpi^{n+e+1}$$

where $u_1 \in R$ and the ratio

$$u_2 = \frac{\varpi^e}{f'(a)}$$

actually lies in R , since $f'(a)$ is not divisible by ϖ^{e+1} . Dividing everywhere by $f'(a)\varpi^{n+1}$ our equation becomes

$$\begin{aligned} u &= -\frac{f(a)}{f'(a)\varpi^{n+1}} - \frac{\varpi^{2n+2}}{f'(a)\varpi^{n+1}}u^2g(\varpi^{n+1}u) \\ &= -u_1u_2 - \varpi^{n+1-e}u_2u^2g(\varpi^{n+1}u). \end{aligned}$$

Since we assume that $n \geq e$, we have that necessarily $u + u_1u_2 \in \mathfrak{m}$. Now one puts

$$\begin{aligned} a_1 &= a - \varpi^{n+1}u_1u_2 \\ &= a - \frac{f(a)}{f'(a)}. \end{aligned}$$

The reduction classes of $a_0 = a$ and a_1 agree modulo \mathfrak{m}^{n+1} and $f(a_1) \in \mathfrak{m}^{n+e+1+1}$, getting one first step closer to a root of f . Moreover, it is important to check that $f'(a_1) \notin \mathfrak{m}^{e+1}$ by using the Taylor expansion of f' and the fact that $n \geq e$:

$$f'(a_1) = \underbrace{f'(a)}_{\notin \mathfrak{m}^{e+1}} + \underbrace{f''(a)(a_1 - a)}_{\in \mathfrak{m}^{n+1}} + \underbrace{(a_1 - a)^2h(a_1 - a)}_{\in \mathfrak{m}^{2n+2}} \quad \text{hence } f'(a_1) \notin \mathfrak{m}^{e+1}.$$

This a_1 we've just constructed is unique modulo \mathfrak{m}^{n+2} . Indeed, if a' is another element of R satisfying $a' - a_0 \in \mathfrak{m}^{n+1}$ and $f(a') \in \mathfrak{m}^{n+e+1+1}$, then $a' - a_1 \in \mathfrak{m}^{n+1+1}$. This can be seen by writing

$$f(a') = f(a_1) + f'(a_1)(a' - a_1) + (a' - a_1)^2g(a' - a_1)$$

and then the fact that $f'(a_1) \notin \mathfrak{m}^{e+1}$ forces $a' - a_1 \in \mathfrak{m}^{n+1+1}$.

The Newton sequence $(a_m)_{m \in \mathbf{N}}$ is now defined by setting

$$\begin{aligned} a_0 &= a \\ a_{m+1} &= a_m - \frac{f(a_m)}{f'(a_m)} \quad m \in \mathbf{N}. \end{aligned}$$

Repeating the previous argument, one gets that $a_{m+1} - a_m \in \mathfrak{m}^{n+m+1}$ for every $m \in \mathbf{N}$ and $f(a_m) \in \mathfrak{m}^{n+m+e+1}$. By completeness of R , this sequence converges to an element $b \in R$ such that

$$f(b) = 0 \text{ and } b - a \in \mathfrak{m}^{n+1}.$$

This lift b is unique: indeed, if $b' \in R$ is another element such that $f(b') = 0$ and $b' - a \in \mathfrak{m}^{n+1}$, then necessarily $b' - a_m \in \mathfrak{m}^{n+m+1}$ for every $m \in \mathbf{N}$ (by induction on m , and we already did the case $m = 1$ above), so that $b' = b$. \square

With a little more effort, one can show a multivariate version of Hensel's lemma with formal power series. The proof is left as an exercise for very brave students.

LEMMA 1.2 (Hensel's lemma). *Let R be a complete discrete valuation ring and \mathfrak{m} its maximal ideal. Fix integers $r \geq \ell \geq 0$. Consider $f_1, \dots, f_\ell \in R[[T_1, \dots, T_r]]$ and $a_1, \dots, a_r \in \mathfrak{m}$ such that*

$$f_i(a) \in \mathfrak{m} \text{ for all } i \in \{1, \dots, \ell\}.$$

Assume moreover that the minor

$$\Delta = \det \left(\frac{\partial f_i}{\partial T_j} \right)_{1 \leq i, j \leq \ell}.$$

of the Jacobian matrix is invertible in R .

Then there exist $b_1, \dots, b_r \in R$ such that

$$f_i(b) = 0 \text{ for all } i \in \{1, \dots, \ell\}$$

and

$$a_j \equiv b_j \pmod{\mathfrak{m}} \text{ for all } j \in \{1, \dots, r\}.$$

2. Analytic manifolds and integration

We work above a local field K , which can be Archimedean or not. The goal of this section is to define K -analytic manifolds and integration on them.

In practice, in this course we will consider manifolds that are realised as the set

$$M = X(K)$$

of K -point of a smooth K -scheme X of finite type and pure dimension. To fix ideas, the reader can think about $K = \mathbf{Q}_p$ or $K = \mathbf{F}_q((t))$ but the definitions given above also apply to $K = \mathbf{R}$ or \mathbf{C} .

2.1. Analytic functions. In this subsection it is sufficient to assume that K is a complete valued field in the sense of [Definition 1.1 page 1](#).

Let T_1, \dots, T_d be indeterminates. We use the convenient notation $\mathbf{T}^{\mathbf{m}} = T_1^{m_1} \dots T_d^{m_d}$ for every $\mathbf{m} \in \mathbf{N}^d$.

DEFINITION 2.1. A *convergent* power series in d variable is an element

$$f(\mathbf{T}) \in K[[\mathbf{T}]] = K[[T_1, \dots, T_d]]$$

such that the radius of convergence

$$\rho(f) = \sup \left\{ r \in \mathbf{R}_+ \mid \text{the sequence } (|f_{\mathbf{n}}| r^{|\mathbf{n}|})_{\mathbf{n} \in \mathbf{N}^d} \text{ converges to } 0 \right\}$$

is positive.

Analytic functions are precisely the functions coming locally from a convergent power series.

DEFINITION 2.2. Let U be an open subset of K^d . A function

$$f : U \rightarrow K$$

is said to be K -analytic if for every point $a \in U$ there exists a convergent power series

$$f_a \in K[[\mathbf{T}]]$$

such that

$$f(x) = f_a(x - a)$$

for all $x \in D_a(a, \rho(f_a))$.

REMARK 2.1. • From the definition, one sees that a K -analytic function is automatically continuous.

- Hence, inside the sheaf of continuous functions on U taking values in K , one can define the subsheaf of K -analytic functions.
- The set of K -analytic functions on U admits a structure of K -algebras.

THEOREM 2.1 (Implicit function theorem). *Let K be a complete valued field. Let m and n be two positive integers and let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two sets of indeterminates.*

Assume that we are given n formal power series

$$F_1, \dots, F_n \in K[[\mathbf{X}, \mathbf{Y}]]$$

such that

$$F_i(0, 0) = 0 \text{ for all } i \in \{1, \dots, n\}$$

and

$$\det \left(\frac{\partial F_i}{\partial Y_j}(0, 0) \right)_{1 \leq i, j \leq n} \neq 0.$$

Then, there exists a unique n -tuple of elements

$$f_1, \dots, f_n \in K[[\mathbf{X}]]$$

such that

$$F_i(\mathbf{X}, f_1(\mathbf{X}), \dots, f_n(\mathbf{X})) = 0$$

for all $i \in \{1, \dots, n\}$.

In case F converges on a neighbourhood of $\mathbf{0} \in K^{m+n}$, then f converges as well on a neighbourhood of $\mathbf{0} \in K^m$, and on a small enough neighbourhood of $\mathbf{0} \in K^{m+n}$, the vanishing locus of F coincides with the graph of f .

Moreover, if the valuation on K is discrete and R is the valuation ring of K , the F_i 's are all in $R[[\mathbf{X}, \mathbf{Y}]]$ and

$$\det \left(\frac{\partial F_i}{\partial Y_j}(0, 0) \right)_{1 \leq i, j \leq n} \notin \mathfrak{m},$$

then

$$f_1, \dots, f_n \in R[[\mathbf{X}]].$$

PROOF. After an initial simplification, the proof is still as brutal as one can imagine.

First, one remarks that our problem is invariant under base change for the coefficients of the F_i 's. Write

$$F_i(\mathbf{X}, \mathbf{Y}) = \sum_{j=1}^n a_{ij} Y_j - \underbrace{\sum_{|j|+|\mathbf{k}|>0} c_{ij\mathbf{k}} \mathbf{X}^i \mathbf{Y}^{\mathbf{k}}}_{=G_i(\mathbf{X}, \mathbf{Y})}$$

so that

$$A = (a_{ij})_{i,j \in \{1, \dots, n\}} = \left(\frac{\partial F_i}{\partial Y_j}(0, 0) \right)_{i,j \in \{1, \dots, n\}}$$

is invertible by assumption. Multiplying by A^{-1} the coefficients of (F_1, \dots, F_n) , we can assume that A is the identity matrix without changing our problem.

Now we have to solve the system

$$f_i(\mathbf{X}) = \sum_{|j|+|\mathbf{k}|>0} c_{ij\mathbf{k}} \mathbf{X}^j (f_1(\mathbf{X}), \dots, f_n(\mathbf{X}))^{\mathbf{k}} \quad i \in \{1, \dots, n\}.$$

Let us write

$$f_i(\mathbf{X}) = \sum_{d \in \mathbf{N}} f_{i,d}(\mathbf{X})$$

where $f_{i,d}(\mathbf{X})$ is the homogeneous part of degree d of $f_i(\mathbf{X})$, so that we want to check that the $f_{i,d}(\mathbf{X})$'s are uniquely determined. The previous system is equivalent to

$$f_{i,d}(\mathbf{X}) = \sum_{\substack{|j|+|\mathbf{k}|>0 \\ k_1+\dots+k_n=|\mathbf{k}| \\ d_{i',k'} \in \mathbf{Z}_{>0} \\ |j|+\sum_{i',k'} d_{i',k'}=d}} c_{ij\mathbf{k}} \mathbf{X}^j \prod_{i' \in \{1, \dots, n\}} \prod_{k' \in \{1, \dots, k_i\}} f_{i',d_{i',k'}}(\mathbf{X})$$

In particular

$$f_{i,1}(\mathbf{X}) = \sum_{|j|=1} c_{ij0} \mathbf{X}^j.$$

To be completed: one eventually shows that all the other coefficients are uniquely determined. \square

Partial derivatives of K -analytic functions $f : U \rightarrow K$ are defined as usual by the formula:

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t \rightarrow 0} \frac{f(a + t\epsilon_i) - f(a)}{t}$$

where $\epsilon_1, \dots, \epsilon_d$ is the canonical basis of K^d . They are automatically K -analytic, because they coincide locally with the formal derivatives of the convergent power series defining f in a neighborhood of $a \in U$.

DEFINITION 2.3. Let $f : U \rightarrow K^d$ be a K -analytic function on an open subset $U \subset K^d$. The Jacobian determinant of f is defined at $a \in U$ by

$$\text{Jac}(f)(a) = \det \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{1 \leq i, j \leq n}.$$

Applying [Theorem 2.1](#), we are able to locally invert K -analytic functions whose Jacobian does not vanish in a neighborhood of a point.

THEOREM 2.2 (Local inversion). *Let $f : U \rightarrow K^d$ be a K -analytic function on an open subset $U \subset K^d$. Let $a \in U$ such that*

$$\text{Jac}(f)(a) \neq 0.$$

Then, there exist

- *an open neighborhood U_a of a such that $f(U_a)$ is also an open neighborhood of $f(a)$ in K^d ;*
- *a K -analytic function*

$$g_a : f(U_a) \rightarrow U_a$$

such that

$$g \circ f = \text{id}_{U_a} \text{ and } f \circ g = \text{id}_{f(U_a)}.$$

2.2. Analytic manifolds.

DEFINITION 2.4. A K -analytic manifold of dimension d can be defined in two ways.

- (1) (Concrete) It is a topological space M together with a d -dimensional K -analytic atlas on it: a set of mutually compatible charts (U_i, φ_i) such that the union of the sets U_i covers M (called an atlas), where compatible means that for all i, j , the homeomorphism

$$\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j)$$

is K -analytic.

- (2) (Abstract) It is a locally K -ringed space (M, \mathcal{O}_M) which is locally isomorphic to the polydisk

$$E^d(0, 1) = \{x \in K^d \mid |x_i| \leq 1 \text{ for all } i \in \{1, \dots, d\}\}$$

endowed with its sheaf of K -analytic functions.

2.3. Change of variables and gauge forms. From now on we assume that K is a local field in the sense of [Definition 1.2](#).

DEFINITION 2.5. Assume that μ is a Haar measure on $(K, +)$. The locally compact group $(K^d, +)$ can be endowed with an induced Haar measure

$$d\mu(\mathbf{x}) = d\mu(x_1) \otimes \dots \otimes d\mu(x_d).$$

DEFINITION 2.6 (Modulus of $(K, +, \mu)$). The modulus

$$\text{mod}_K : K \rightarrow \mathbf{R}_+$$

is defined by the formula

$$\mu(a\Omega) = \text{mod}_K(a)\mu(\Omega)$$

for every $a \in K$ and every bounded measurable subset Ω of K .

DEFINITION 2.7. Let U be an open subset of K^d and ω be a differential form of degree d on U . In other words, there exists a unique analytic function h on U such that

$$\omega = h(\mathbf{x})dx_1 \wedge \dots \wedge dx_d.$$

The measure

$$\text{mod}_K(\omega)$$

is defined by

$$\int_U \varphi \text{mod}_K(\omega) = \int_U \varphi(x) \text{mod}_K(h(\mathbf{x})) d\mu(\mathbf{x})$$

for every continuous function φ on U having compact support.

THEOREM 2.3 (Local change of variables). *Let U be an open set in K^d and let*

$$f : U \rightarrow K^d$$

be an injective K -analytic map. Assume moreover that the Jacobian of f does not vanish on U .

Then, for every integrable function $\varphi : f(U) \rightarrow \mathbf{R}$

$$\int_{f(U)} \varphi(\mathbf{y}) d\mu(\mathbf{y}) = \int_U \varphi(f(\mathbf{x})) \text{mod}_K(\text{Jac}(f)(\mathbf{x})) d\mu(\mathbf{x}).$$

PROOF. It is enough to prove the formula on a small open neighborhood of every point of U , prove it for some elementary functions, and then use the chain rule of the Jacobian.

Let ϖ be a generator of \mathfrak{m} .

- (1) First, one proves it for a linear change of variable $y = Ax + a$ where $A \in \text{GL}_d(K)$ and $a \in K^d$. This comes from the formula

$$\mu(A\Omega) = \text{mod}_K(\det(A))\mu(\Omega).$$

Remark: in most concrete situations this is already enough!

- (2) Then one proves it for *special restricted power series*, that is to say for $f \in K[[\mathbf{X}]]$ such that

$$f(0) = 0$$

and $c_i \in \mathfrak{m}^{|i|-1}$ for every $\mathbf{i} \in \mathbf{N}^d \setminus \{\mathbf{0}\}$, in particular $f \in R[[\mathbf{X}]]$. In that case, $y = f(x)$ is measure-preserving. Indeed, the image of $a + \varpi^e R^d$ under f is $f(a) + \varpi^e R^d$ for every $e \in \mathbf{Z}_{>0}$.

- (3) If $f \in R[[\mathbf{X}]]$ is convergent in a neighborhood of some a , then for every $e \in \mathbf{Z}_{>0}$

$$g(\mathbf{X}) = \varpi^{-e}(f(a + \varpi^e \mathbf{X}) - f(a))$$

is a special restricted power series.

- (4) Hence one can always assume that $f(a) = \mathbf{0}$ and that $f_i(\mathbf{X})$ is of the form

$$f_i(\mathbf{X}) = X_i + \sum_{|\mathbf{j}| > 2} c_{i,\mathbf{j}} \mathbf{X}^{\mathbf{j}}$$

and since f_i converges locally, there are some well-chosen $e_0, e_1 \in \mathbf{N}$ such that $\varpi^{e_1} c_{i,\mathbf{j}} \varpi^{e_0|\mathbf{j}|}$ is in R for every $i \in \{1, \dots, d\}$ and \mathbf{j} (choose e_0 such that $c_{i,\mathbf{j}} \varpi^{e_0|\mathbf{j}|} \rightarrow 0$ as $|\mathbf{j}| \rightarrow \infty$ for every i and then choose e_1). One can form a special restricted power series $g_i(\mathbf{X})$

$$g_i(\mathbf{X}) = \varpi^{-e} f_i(\varpi^e \mathbf{X})$$

for $e \geq 2e_0 + e_1 + 1$.

Then one concludes arguing by composition. \square

DEFINITION 2.8 (Metric on a line bundle). A *metric* on a line bundle $L \rightarrow M$ is the datum for every open subset U of M of

$$\|\cdot\| : \ell \in \Gamma(U, L) \longmapsto (\|\ell\| : U \rightarrow \mathbf{R}_+)$$

such that

- (1) $\|\cdot\|$ is continuous;
- (2) for every $\ell \in \Gamma(U, L)$ and $x \in U$, $\|\ell\|$ is positive if and only if $\ell(x) \neq 0$;
- (3) it is compatible with restriction to any open subset V inside U ;
- (4) $\|f\ell\|(x) = |f(x)|\|\ell\|(x)$ for every K -analytic function on U .

The datum of a line bundle together with a metric on it is called a *metrized line bundle*.

The measure associated to a top-degree global differential form – also called *gauge form* – on an analytic manifold is defined locally using charts. Naturally, it depends on the existence of such a global form and on the choice of the differential form.

PROPOSITION 2.1. *If M is a K -analytic manifold of dimension d and ω is a differential form of degree d on M , then there exists a unique measure*

$$\text{mod}_K(\omega)$$

on M which locally coincides with the measure associated to differential forms on open subsets of K^d , that is to say such that for every chart (U, f) of M and every integrable function φ having support in U ,

$$\int_M \varphi \text{mod}_K(\omega) = \int_{f(U)} (\varphi \circ f^{-1}) \text{mod}_K((f^{-1})^*\omega).$$

PROOF. First, assume that $M = U$ is a open subset of K^d . Then,

$$\omega = h dx_1 \wedge \dots \wedge dx_d$$

for a unique analytic function h on U . As we already saw, the measure $\text{mod}_K(\omega)$ is then given by

$$\int_U \varphi \text{mod}_K(\omega) = \int_U \varphi(x) \text{mod}_K(h(x)) d\mu(x)$$

for every compactly supported continuous function on U . This definition is invariant by K -analytic diffeomorphism: if $g : V \rightarrow U$ is such a change of coordinates,

$$\begin{aligned} \int_U \varphi \text{mod}_K(h(x)) d\mu(x) &= \int_V \varphi \circ g(y) \text{mod}_K(h \circ g(y)) \text{mod}_K(\text{Jac}(g)(y)) d\mu(y) \\ &= \int_V \varphi \circ g(y) \text{mod}_K(g^*\omega). \end{aligned}$$

Now, in general, if M is an arbitrary K -analytic manifold of dimension d and ω a top-degree differential form on it, let us consider a *finite* family of charts (U_i, f_i) covering the support of $\varphi : M \rightarrow K$ together with a partition of unity

$$\sum_i \lambda_i \equiv 1$$

where the support of λ_i is contained in U_i . Then we set

$$\int_M \varphi \bmod_K(\omega) = \sum_i \int_{f_i(U_i)} (\lambda_i \circ f_i^{-1}) \cdot (\varphi \circ f_i^{-1}) \cdot \bmod_K((f_i^{-1})^* \omega).$$

Using the change of variable formula again, one checks that the right hand side (which is a finite sum) does not depend on the choices of charts and partitions of unity. \square

2.4. Analytification of smooth schemes. Given a K -scheme of finite type, there exists a canonical way to endow its set of K -points with a topology satisfying two natural conditions.

DEFINITION 2.9. Let X be a K -scheme of finite type. The *analytic topology* on $X(K)$ is the coarsest topology satisfying the following properties:

- for any Zariski-open subset $U \subset X$, its set $U(K)$ of K -points is open in $X(K)$;
- for every Zariski-open subset $U \subset X$ and any regular function $\varphi \in \mathcal{O}_X(U)$ the map $U(K) \rightarrow K$ induced by φ is continuous.

Defining a structure of a K -analytic manifold on a smooth K -scheme of finite type boils down to defining a subsheaf of the sheaf of continuous functions with values in K .

DEFINITION 2.10. Let X be a *smooth* K -scheme of finite type.

Let U be an open subset of $X(K)$. We say that a function

$$f : U \longrightarrow K$$

is analytic at a point $x \in U$ if there exist a Zariski-open neighborhood $V \ni x$ in X , an immersion of K -schemes

$$i : V \hookrightarrow \mathbf{A}_K^n,$$

an open neighborhood $W \ni i(x)$ in $\mathbf{A}_K^n(K) = K^n$ together with an analytic function

$$g : W \longrightarrow K$$

such that

$$f = g \circ i$$

on an analytic neighborhood of $x \in X(K)$.

PROPOSITION 2.2. *Via the previous definitions, the following holds.*

- Any morphisms of smooth K -schemes induces a morphism of K -analytic manifolds; in particular,
 - open immersions induce open immersions of K -analytic manifolds;
 - closed immersions induce closed immersions of K -analytic manifolds.
- The structure of a K -analytic manifold on $\mathbf{A}_K^n(K) = K^n$ is the natural one: it coincides with the one from the previous sections.
- Any étale morphism of smooth K -schemes induces an étale morphisms of K -analytic manifolds (local isomorphisms).

PROOF. The first and second point are easy and left as an exercise.

The third point is an application of the local inversion [Theorem 2.2](#). \square

The following proposition says that in the local non-Archimedean setting, rational points of closed subschemes are negligible.

PROPOSITION 2.3. *Let K be a non-Archimedean local field and X a smooth K -scheme. Suppose that X is endowed with a measure μ associated to a gauge form, thanks to [Proposition 2.1](#).*

Let

$$Z \subset X$$

be a closed subscheme of codimension at least 1. Then

$$\mu(Z(K)) = 0.$$

PROOF. The question is local and it is sufficient to prove the following statement: if M is a submanifold of an open subscheme of K^d , of codimension $c \geq 1$ everywhere, then M has measure zero in K^d .

Using the implicit function [Theorem 2.1](#) and the change of variable formula [Theorem 2.3](#) we reduce to the case

$$M = \underbrace{\{0, \dots, 0\}}_{c \text{ times}} \times E^{d-c}(0, 1)$$

inside $E^d(0, 1)$, in particular observe that every polydisk

$$E^d(a, r) = \{x \in K^d \mid |x - a| \leq r\}$$

is isomorphic to the unit polydisk $E^d(0, 1)$. Then the claim follows from [Definition 2.5](#) [page 7](#) and the fact that $\{0\}$ has measure zero in K . \square

2.5. Models and associated measures. In this paragraph we explain a very important construction of a certain metric that is going to be used in every chapter of this course.

CONSTRUCTION 2.1 (Metric induced by a model). Assume that K is a non-Archimedean local field (not necessarily complete), with valuation field R , and that \mathcal{X} is a flat separated R -scheme of finite type, with smooth generic fibre $X = \mathcal{X} \otimes_R K$, so that $\mathcal{X}(R)$ injects into $X(K)$ as a compact subset.

Consider \mathcal{L} a coherent sheaf on \mathcal{X} whose generic fibre L is a line bundle (we say that \mathcal{L} is a model of L). This line bundle induces a line bundle on the K -analytic manifold $M = \mathcal{X}(R)$, again denoted by L .

If $x \in M = \mathcal{X}(R)$, the fibre $x^*\mathcal{L}$ is an R -module of finite type, which possibly has a non-empty torsion part $x^*\mathcal{L}_{\text{tors}}$, and x_L^*L is a one dimensional K -vector space.

$$\begin{array}{ccccc}
 x_K^*L & \longrightarrow & x^*\mathcal{L} & & \mathcal{L} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Spec}(K) & \longrightarrow & \text{Spec}(R) & \xlongequal{\quad} & \text{Spec}(R) \\
 & \nearrow x_K & \nearrow x & & \\
 & & & & \mathcal{X}
 \end{array}$$

Then, the R module

$$\mathcal{L}(x) = x^*\mathcal{L} / x^*\mathcal{L}_{\text{tors}}$$

can be seen as a lattice inside x_K^*L : indeed, since the square is Cartesian, a point of $x^*\mathcal{L}$, seen as a section $\text{Spec}(R) \rightarrow x^*\mathcal{L}$, then composed with $\text{Spec}(K) \rightarrow \text{Spec}(R)$ induces a unique K -point of x_K^*L .

Given any generator y_0 of this lattice, we obtain a norm on the K -vector space x_K^*L by setting $\|ay_0\| = |a|$ for all $a \in K$ (this does not depend on the choice of y_0 since two generators differ by an invertible element).

Now given any section s of L on an open subset U of M , we set $\|s\|(x) = \|s(x)\|$ for all $x \in U$.

PROPOSITION 2.4. *The previous construction defines a metric on the line bundle L in the sense of [Definition 2.8](#).*

PROOF. Postponed.

□

Preliminary version of lecture notes for the course:

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to geometry and arithmetics**

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ISTA

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