A MOTIVIC POISSON FORMULA FOR SPLIT ALGEBRAIC TORI WITH AN APPLICATION TO MOTIVIC HEIGHT ZETA FUNCTIONS

MARGARET BILU AND LOÏS FAISANT

Preliminary version of January 5, 2026.

Abstract. We prove a motivic version of the Poisson formula and apply it to the study of the motivic height zeta function of split projective toric varieties, in the context of the motivic Manin-Peyre principle.

Contents

Introduction 1
1. Grothendieck rings 5
2. Symmetric products and motivic Euler products 11
3. Divisors on curves and motivic functions 17
4. Multiplicative motivic harmonic analysis 20
5. Toric harmonic analysis 24
6. Intermezzo: motivic L-functions associated to $\mathscr{O}^\times$-invariant characters 27
7. Application to the motivic height zeta function 29
8. Analysis of the integral over the Fourier domain 34
References 40

Introduction

Harmonic analysis stands as one of the foundational tools in the number theorist's arsenal for addressing counting problems. Within the framework of the Batyrev-Manin-Peyre conjectures—which predict the distribution of rational points on Fano varieties over global fields—this method has been instrumental since the field's inception.

Classically, the counting problem is encoded into a height zeta function on which harmonic analysis is performed. In that context, an important role is played by the Poisson formula.

The goal of the present work is to state and prove a geometric-motivic analogue of the Poisson formula for split algebraic tori and apply it to the study of the motivic height zeta function of toric varieties over function fields of curves. In particular, we recover a motivic stabilisation result for the class of the moduli space of high degree morphisms from a smooth and geometrically irreducible projective curve to a smooth split projective toric variety that was previously obtained by the second author using a different approach [Fai25b, Fai25a]. This work also extends residue-type results of Bourqui [Bou09] and Bilu–Das–Howe [BDH22].

The classical setting. Let us recall the classical Poisson formula in its form used by Batyrev and Tschinkel in their proof of Manin's conjecture for toric varieties over number fields [BT95, BT98] and function fields of curves over finite fields [Bou11a]. Let $\mathscr{G}$ be a locally compact topological commutative group, endowed with a certain Haar measure $\mathrm{d}x$, and let $\mathscr{H} \subset \mathscr{G}$ be a cocompact discrete subgroup of $\mathscr{G}$. We give ourselves a function $f:\mathscr{H}\to\mathbf{R}$ and assume that f can be extended to an L^1-function on the whole of $\mathscr{G}$. Finally, up to normalising $\mathrm{d}x$, one can assume that $\mathscr{H}$ has covolume 1 in $\mathscr{G}$.

Assume that the Fourier transform $\hat{f}$ of f with respect to $\mathrm{d}x$ is an L^1-function on the group $\mathscr{H}^\perp$ of characters of $\mathscr{G}$ which are trivial on $\mathscr{H}$. Then

$$\sum_{x\in\mathscr{H}} f(x) = \sum_{\chi\in\mathscr{H}^\perp}\hat{f}(\chi). \tag{0.0.1}$$

Given a global field K, Batyrev–Tschinkel and Bourqui apply this formula to the adele group

$$\mathscr{G} = U(\mathbb{A}_K)$$

of an algebraic torus U defined over K, with $\mathscr{H}=U(K)$ being its set of K-rational points diagonally embedded in $U(\mathbb{A}_K)$.

Moduli spaces of curves and motivic Batyrev-Manin-Peyre principle. Assume now that K is the function field of a geometrically irreducible algebraic curve $\mathscr{C}$ defined over an absolute base field k and let X be a projective variety defined over k. By the valuative criterion of properness, the set of K-rational points of X can be identified with the set of k-morphisms $\mathscr{C}\to X$. Such morphisms are parametrised by a k-scheme $\mathrm{Hom}_k(\mathscr{C},X)$. Moreover, any morphism $f:\mathscr{C}\to X$ induces a linear form $\deg(f)\in\mathrm{Pic}(X)^\vee$, called the *multidegree*, which sends the class of a line bundle to the degree of its pull-back to the curve $\mathscr{C}$. For any $\alpha\in\mathrm{Pic}(X)^\vee$, the condition $\deg(f)=\alpha$ defines a subscheme $\mathrm{Hom}_k^\alpha(\mathscr{C},X)$ of finite type. Finally, if U is a non-empty open subset of X, we denote by $\mathrm{Hom}_k^\alpha(\mathscr{C},X)_U$ the submoduli space whose points corresponds to morphisms sending the generic point of $\mathscr{C}$ into U.

Assume that X is a Fano-like variety in the sense of [Fai25b, Definition 1] (for example, a Fano variety or a split smooth projective toric variety). The motivic Batyrev-Manin-Peyre principle is a set of predictions concerning the behaviour of the class of $\mathrm{Hom}_k^\alpha(\mathscr{C},X)_U$ in a suitable localisation of the Grothendieck ring of k-varieties $\mathrm{K}_0\mathbf{Var}_k$. As a group, $\mathrm{K}_0\mathbf{Var}_k$ is defined by generators and relations: generators are isomorphism classes $[Y]$ of k-varieties and relations are of the form

$$[Y]-[Z]-[Y-Z]$$

whenever Z is a closed subscheme of a variety Y. The ring structure of $\mathrm{K}_0\mathbf{Var}_k$ is given by taking Cartesian products of varieties:

$$[Y_1][Y_2]=[Y_1\times_k Y_2].$$

The class of the affine line plays a particular role and is denoted by $\mathbf{L}_k$. It is convenient to invert it by working in the localisation

$$\mathscr{M}_k=\mathrm{K}_0\mathbf{Var}_k[\mathbf{L}_k^{-1}].$$

The ring $\mathscr{M}_k$ admits a filtration by the virtual dimension and a corresponding completion $\widehat{\mathscr{M}_k}^{\dim}$ which allows one to define sums of absolutely convergent series.

The most basic version of the motivic Batyrev-Manin-Peyre principle asks whether the normalised class

$$\left[\mathrm{Hom}_k^\delta(\mathscr{C},X)\right]\mathbf{L}_k^{-\delta\,\cdot\,\omega_X^{-1}}$$

stabilises (if necessary, in a well-chosen completion of $\mathscr{M}_k$) as $\delta\in\mathrm{Pic}(X)^\vee$ lies inside the movable cone of X and goes arbitrarily far away from its boundaries. This question is closely related to the study of the following series.

Definition. The multivariate motivic height zeta function is the formal series

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = \sum_{\delta\in\mathrm{Pic}(X)^\vee} \left[\mathrm{Hom}_k^\delta(\mathscr{C},X)_U\right]\mathbf{T}^\delta.$$

Split toric varieties and main results. In this article we focus on the case of smooth, projective and split toric varieties.

Let $U\simeq\mathbf{G}_m^n$ be a split torus and X_Σ a split smooth projective variety over a field k compactifying U, defined by a certain fan Σ of the lattice of cocharacters $\mathcal{X}_*(U)=\mathrm{Hom}_{\mathrm{gp}}(\mathbf{G}_m,U)$ of U. Let r be the Picard rank of X_Σ and ω_{X_Σ} its canonical line bundle.

Theorem 1 (Meromorphic continuation)**.** *There exists an $\eta>0$ and an integer a_{X_Σ} such that the formal series*

$$(1-(\mathbf{L}T)^{a_{X_\Sigma}})^{\times}\zeta_H^{\mathrm{mot}}\left(T^{\omega_{X_\Sigma}^{-1}}\right)$$

converges for $|T|<\mathbf{L}_k^{-1+\eta}$, with its value at $\mathbf{L}^{-1}$ being a non-zero element of $\widehat{\mathscr{M}_k}^{\dim}$.

The following second statement is compatible with the predictions from [Fai25b].

Theorem 2 (Multi-height motivic stabilisation)**.** *As the class*

$$\delta\in\mathrm{Pic}(X_\Sigma)^\vee\cap\mathrm{Eff}(X_\Sigma)^\vee$$

goes arbitrarily far away from the boundary of $\mathrm{Eff}(X_\Sigma)^\vee$, the normalised class

$$\left[\mathrm{Hom}_k^\delta(\mathscr{C},X_\Sigma)_U\right]\mathbf{L}_k^{-\,\delta\,\cdot\,\omega_{X_\Sigma}^{-1}}$$

tends to the non-zero motivic Euler product

$$\mathbf{L}_k^{(1-g(\mathscr{C}))\dim(X_\Sigma)}\left(\frac{\left[\mathrm{Pic}^0(\mathscr{C})\right]\mathbf{L}_k^{1-g(\mathscr{C})}}{\mathbf{L}_k-1}\right)^{\mathrm{rk}(\mathrm{Pic}(X_\Sigma))}\prod_{p\in\mathscr{C}}\left(1-\mathbf{L}_{\kappa(p)}^{-1}\right)^{\mathrm{rk}(\mathrm{Pic}(X_\Sigma))}\frac{[X_{\kappa(p)}]}{\mathbf{L}_{\kappa(p)}^{\dim(X_\Sigma)}}.$$

Relation with earlier works.

The motivic Poisson formula for additive groups. The first occurrence of harmonic analysis in the motivic setting was the motivic Poisson formula of Hrushovski and Kazhdan [HK09]. While originally formulated in the language of model theory of valued fields, it was translated in [CLL16] into a more geometric context, where it corresponds to a motivic analogue of the formula (0.0.1) in the specific case when $\mathscr{G}=\mathbf{G}_a^n(\mathbf{A}_F)$ are the adelic points of the additive group scheme $\mathbf{G}_a^n$ for $F=k(\mathscr{C})$ the function field of a curve, and $\mathscr{H}=\mathbf{G}_a^n(F)$, diagonally embedded into $\mathscr{G}$. In the motivic setting there is no local compactness, and the classical theory of integration is replaced by a form of motivic integration, with the motivic Poisson formula being an equality between two classes in the so-called *Grothendieck ring of varieties with exponentials.* The first ingredient of the construction is the definition of *motivic Schwartz-Bruhat functions*, which are motivic analogues of locally constant functions with compact support, and which in this setting are elements of appropriate relative Grothendieck rings with exponentials. Integration and Fourier transformation operators are defined as pushforwards between such Grothendieck rings. Thanks to the self-duality properties of the additive group, the objects appearing on both sides of the Poisson formula are of the same nature, given as the image of a summation operator applied to a motivic Schwartz-Bruhat function and its Fourier transform, respectively. The Poisson formula itself is proved by first reducing to (analogues of) characteristic functions of balls, where it follows from the Riemann-Roch theorem and Serre duality of the curve $\mathscr{C}$.

This Poisson formula was used successively in [CLL16], [Bil23] and [Fai23] to study moduli spaces of curves on equivariant compactifications of vector groups.

Previous work for toric varieties in the motivic setting.

Main ideas. As for the motivic Poisson formula in the additive setting, the first step is an appropriate variant of the Grothendieck ring of varieties. Il faudrait aussi une comparaison avec [CLNV24], tu t'y connais mieux donc je te laisse faire To explain our choice, let us first describe the objects that we will need to work with.

Let $F = k(\mathscr{C})$ be the function field of a smooth projective curve, and for every closed point $v \in \mathscr{C}$, denote by F_v the corresponding completion. For the sake of simplicity, we specialize in this paragraph to the case where our torus U is one-dimensional: $U = \mathbf{G}_m$, so that the local height functions which we want to build motivic analogues of are functions $f : F_v^\times \to \mathbf{C}$ which are invariant modulo $\mathcal{O}_v^\times$, and thus may be thought of as functions on $F_v^\times/\mathcal{O}_v^\times \simeq \mathbf{Z}$ (where the isomorphism is given by the valuation), which also turn out to be finitely supported. Thus, our families of local functions will naturally arise as elements of a relative Grothendieck ring over the constant group scheme $\underline{\mathbf{Z}}_{\mathscr{C}}$. For a more general torus U, $\mathbf{Z}$ will be replaced by the lattice of cocharacters $\mathcal{X}_*(U)$.

As opposed to the additive setting described above, there is no self-duality in the multiplicative setting, and Fourier transforms are *a priori* objects of a different nature: they are motivic functions over the Cartier dual $D(\mathbf{Z}_{\mathscr{C}})$, which is isomorphic to $\mathbf{G}_{m,\mathscr{C}}$. As in [Bil23], the passage from local to global is done via the notion of symmetric product.

Organisation of the paper. In Section 1, we start with recalling the classical definitions of rings of varieties before introducing a new variant of these rings which allows us to deal with multiplicative characters on constant group schemes.

We proceed the same way in Section 2 where we first recall the definition of motivic Euler products from [Bil23] before extending the notion to motivic functions on constant group schemes and on their Cartier duals. Then Section 3 is devoted to an algebraic parametrisation of divisors on an algebraic curve $\mathscr{C}$ in order to better understand the coefficients of motivic Euler products.

We define Fourier transforms of motivic functions on constant group schemes in Section 4, where we obtain Fourier inversion and Poisson formulas. This framework is adapted to the toric setting in Section 5.

We define a class of motivic L-functions and study their poles in a devoted Section 6. We believe that this notion will be of independent interest in many settings outside the present context.

We eventually apply our theory in the two final sections: in Section 7 we apply the motivic Poisson formula to the motivic height zeta function and the final analysis of the contributions of the Fourier transforms is performed in Section 8.

Notations and conventions. To avoid confusion, in this article U will always be a split n-dimensional torus over the absolute base field k, while we will use a capital T for indeterminates and its bold version $\mathbf{T}$ for sets of indeterminates. On the other hand, lowercase t will usually denote a local parameter.

The k-curve $\mathscr{C}$ is assumed to be smooth, projective and geometrically irreducible. Moreover, we assume that it admits a k-divisor of degree one (this additional assumption is automatically satisfied if $k = \mathbf{F}_q$ or if k is algebraically closed). Its function field is denoted by F and its set $|\mathscr{C}|$ of closed points is identified with the set M_F of discrete valuations of F.

We use bold letters to differentiate schemes and group schemes from respectively sets and abstract groups, with the following exception: if M is an abstract group and S a base scheme, the constant group scheme above S will be denoted by $\underline{M}_S$.

Acknowledgments. LF acknowledges partial funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 101034413 and from KU Leuven through the project grant number C16/23/010.

1. Grothendieck rings

This section aims to define a new variant of the ring of varieties that will allow us to handle certain families of multiplicative characters. We first start with recalling the classical definition of ring of varieties over an arbitrary base scheme.

1.1. **Rings of varieties.** Let S be a scheme. An S-variety is a finitely presented scheme morphism $X \to S$.

Definition 1.1. The *Grothendieck group of S-varieties*

$$\mathrm{K}_0\mathbf{Var}_S$$

is defined as the abelian group generated by the isomorphism classes of *S-varieties*, with relations given by the *cut-and-paste relations*

$$X - Y - U$$

whenever X is an S-variety, Y is a closed subscheme of X and U is its open complement in X. The class of an S-variety X in this ring is denoted by $[X \to S]$, $[X]_S$ or simply $[X]$ if the structural morphism is clear from the context. Let $\mathbf{L}_S$ be the class of the affine line.

A ring structure on $\mathrm{K}_0\mathbf{Var}_S$ is defined by

$$[X][Y] = [X \times_S Y]$$

for which $[\mathrm{id} : S \to S]$ is the unitary element.

A constructible subset of an S-variety is a finite union of locally closed subsets. By [CLNS18, p. 59], any constructible subset X of an S-variety admits a class $[X]$ in $\mathrm{K}_0\mathbf{Var}_S$.

Lemma 1.2 ([CLL16, Lemma 1.1.8] or [CH22, Theorem 1])**.** *Let φ be a motivic function on S, that is to say an element of $\mathrm{K}_0\mathbf{Var}_S$. If $s^*\varphi = 0$ in $\mathrm{K}_0\mathbf{Var}_{\kappa(s)}$ for all point $s \in S$ then $\varphi = 0$ in $\mathrm{K}_0\mathbf{Var}_S$.*

Definition 1.3. The localised Grothendieck ring of varieties $\mathscr{M}_S$, sometimes called *ring of motivic integration* is the localisation of $\mathrm{K}_0\mathbf{Var}_S$ at $\mathbf{L}_S$.

Definition 1.4. The ring $\mathscr{M}_S$ admits a decreasing filtration by the virtual dimension: for $m \in \mathbf{Z}$, let $\mathcal{F}^m\mathscr{M}_S$ be the subgroup of $\mathscr{M}_S$ generated by elements of the form

$$[X]\mathbf{L}_S^{-i}$$

where X is an S-variety and i an integer such that $\dim_S(X) - i \leqslant -m$. The completion of $\mathscr{M}_S$ with respect to this decreasing dimensional filtration is the projective limit

$$\widehat{\mathscr{M}_S}^{\dim} = \varprojlim \mathscr{M}_S/\mathcal{F}^m\mathscr{M}_S.$$

It comes with a morphism $\mathscr{M}_S \to \widehat{\mathscr{M}_S}^{\dim}$.

1.2. **Biduality for constant commutative group schemes.** A reference for this subsection is [DGA+62, Exp. VIII]. Let S be a scheme. Recall that the Cartier dual of a group scheme G above S is the S-group scheme

$$D(G) = \mathrm{Hom}_{S-\mathrm{gp}}(G, \mathbf{G}_{m,S})$$

of multiplicative characters of G. It is a closed subscheme of $\mathrm{Hom}_{S-\mathrm{sch}}(G, \mathbf{G}_{m,S})$ and comes with a canonical pairing of group schemes

$$\mathrm{ev} : \left\{ \begin{array}{ccc} G \times_S D(G) & \longrightarrow & \mathbf{G}_m \\ (g, \chi) & \longmapsto & \chi(g). \end{array} \right.$$

A constant commutative group scheme Γ above S is an S-group scheme of the form

$$\Gamma = \underline{M}_S = \coprod_M S$$

where M is an abstract commutative group. A group scheme $G \to S$ is said to be *diagonalisable* if it is of the form $G = D(\underline{M}_S)$ where M is an abstract commutative group, and *locally diagonalisable* if every point of S admits an open neighborhood U such that $G_{|U}$ is diagonalisable.

Theorem 1.5 ([DGA+62, Exp. VIII, Théorème 1.2])**.** *Let Γ be a constant commutative group scheme above S.*

Then the canonical morphism

$$\Gamma \longrightarrow D(D(\Gamma))$$

is an isomorphism.

Proposition 1.6 ([DGA+62, Exp. VIII, Prop. 2.1])**.** *Let $G = D(\underline{M}_S)$ be a diagonalisable group scheme over S. Then G is faithfully flat and affine over S. Moreover,*

- *M is of finite type as an abstract group if and only if $G \to S$ is of finite type ;*
- *M is finite if and only if $G \to S$ is finite, and M is torsion if and only if $G \to S$ is integral ;*
- *M is of finite type, with torsion of order coprime to the residual characteristics of S, if and only if $G \to S$ is smooth.*

1.3. **Ring of varieties with multiplicative characters.** Let S be a variety and M be an abstract commutative group.

Definition 1.7. The Grothendieck group of S-varieties with characters on M is the $\mathbf{Z}$-module

$$\mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S$$

generated by isomorphism classes

$$[X \times_S D(\underline{L}_S), h]_{S \times D(\underline{M}_S)}$$

of diagrams

$$(X \times_S D(\underline{L}_S), h : D(\underline{L}_X) \to \mathbf{G}_m) = \begin{array}{c} X \times_S D(\underline{L}_S) = D(\underline{L}_X) \xrightarrow{h} \mathbf{G}_m \\ \big\downarrow{\scriptstyle (f,g)} \qquad\qquad\qquad\qquad \\ S \times_S D(\underline{M}_S) \qquad\qquad\qquad\qquad \end{array}$$

where

- $f : X \to S$ is an S-variety,
- g corresponds by Cartier duality to a morphism $\underline{M}_S \to \underline{L}_S$ of constant S-groups,

- $h \in \mathrm{Hom}_{\mathrm{gp}}(D(\underline{L}_X), \mathbf{G}_{m,X})$

modulo cut-and-paste relations on the first factor

$$\left[\begin{tikzcd} X \times_S D(\underline{L}_S) \arrow[r, equal] \arrow[d, "{(f,g)}"] & D(\underline{L}_X) \arrow[r, "h"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right]$$

$$= \left[\begin{tikzcd} U \times_S D(\underline{L}_S) \arrow[r, equal] \arrow[d, "{(f_{|U},g)}"] & D(\underline{L}_U) \arrow[r, "h_{|U}"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right] + \left[\begin{tikzcd} Z \times_S D(\underline{L}_S) \arrow[r, equal] \arrow[d, "{(f_{|Z},g)}"] & D(\underline{L}_Z) \arrow[r, "h_{|Z}"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right]$$

for every Zariski-closed subscheme Z of X with open complement U. A product law is defined relatively to $S \times D(\underline{M}_S)$:

$$\left[\begin{tikzcd} X_1 \times_S D(\underline{L_1}_S) \arrow[r, equal] \arrow[d] & D(\underline{L_1}_S) \arrow[r, "h_1"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right] \times \left[\begin{tikzcd} X_2 \times_S D(\underline{L_2}_S) \arrow[r, equal] \arrow[d] & D(\underline{L_2}_S) \arrow[r, "h_2"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right]$$

$$= \left[\begin{tikzcd} (X_1 \times_S X_2) \times_S (D(\underline{L_1}_S) \times_{D(\underline{M}_S)} D(\underline{L_2}_S)) \arrow[r, "{(h_1,h_2)}"] \arrow[d] & \mathbf{G}_m^2 \arrow[r, "\text{group law}"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right]$$

In particular, this endows $\mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S$ with a structure of $\mathrm{K}_0\mathbf{Var}_S$-module.

Remark 1.8. When M is the trivial group, then $\mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S = \mathrm{K}_0\mathbf{Var}_S$.

Notation 1.9 (Pullback and pushforward)**.** Let $p : S \to S'$ be a morphism of schemes. Then, as usual with Grothendieck rings, p induces a ring morphism:

$$p^* : \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_{S'} \to \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S$$

given by

$$\left[\begin{tikzcd} X \times_{S'} D(\underline{L}_{S'}) \arrow[r, equal] \arrow[d, "{(f,g)}"] & D(\underline{L}_X) \arrow[r, "h"] & \mathbf{G}_m \\ S' \times_{S'} D(\underline{M}_{S'}) & & \end{tikzcd}\right] \mapsto$$

$$\left[\begin{tikzcd} (X \times_{S'} S) \times_S D(\underline{L}_S) \arrow[r, equal] \arrow[d, "{(p^*f,p^*g)}"] & D(\underline{L}_{X \times_{S'} S}) \arrow[r, "p^*h"] & \mathbf{G}_m \\ S \times_S D(\underline{M}_S) & & \end{tikzcd}\right]$$

and, if p is of finite presentation, a group morphism $p_!$:

$$p_! : \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S \to \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_{S'}$$

given by

$$\left[\begin{array}{c} X\times_S D(\underline{L}_S) = X\times_S (S\times_{S'} D(\underline{L}_{S'})) = D(\underline{L}_X) \xrightarrow{h} \mathbf{G}_m \\ \big\downarrow{\scriptstyle (f,g)} \\ S\times_S D(\underline{M}_S) \end{array}\right] \mapsto$$

$$\left[\begin{array}{c} X\times_{S'} D(\underline{L}_{S'}) = D(\underline{L}_{X_{S'}}) \xrightarrow{h_{S'}} \mathbf{G}_m \\ \big\downarrow{\scriptstyle (p\circ f, g_{S'})} \\ S'\times_{S'} D(\underline{M}_{S'}) \end{array}\right]$$

Remark 1.10. More generally, if $p : S \to S'$ is a morphism of schemes and $\mathfrak{a} \in \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S$ which is supported over a subscheme of S of finite presentation over S', we may make sense of $p_!\mathfrak{a} \in \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_{S'}$. We will use this repeatedly in the special case where p is the projection $\underline{M}_S \to S$ and $\mathfrak{a}$ has finite support.

Notation 1.11 (Evaluation map)**.** For any $m \in M$, we write

$$\mathrm{ev}(m,\cdot) := [S\times_S D(\underline{M}_S), \chi\mapsto \chi(m)]_{S\times_S D(\underline{M}_S)} \in \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S.$$

This should be thought of as the motivic Fourier transform of the elementary motivic function

$$\delta_m : M \to \mathrm{K}_0\mathbf{Var}_S$$

sending m to the class of S and $m' \neq m$ to zero. Note that by definition of the product in $\mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S$, we have $\mathrm{ev}(m,\cdot)\,\mathrm{ev}(n,\cdot) = \mathrm{ev}(m+n,\cdot)$ for all $m,n\in M$.

We may also define the evaluation pairing

$$\mathrm{ev} := [\underline{M}_S\times_S D(\underline{M}_S), (m,\chi)\mapsto \chi(m)] \in \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_{\underline{M}_S}.$$

Orthogonality relations are encoded by the following morphism of integration with respect to the character variables.

Definition 1.12 (Integration with respect to the second factor)**.** Let

$$D(\underline{M}_S) \to D(\underline{N}_S)$$

be a *surjective* morphism of diagonalisable S-group schemes, of finite presentation, corresponding to an injection of abstract groups $N \hookrightarrow M$ with cokernel of finite type. There exists a unique $\mathrm{K}_0\mathbf{Var}_S$-linear group morphism

$$\int_{D(\underline{M}_S)/D(\underline{N}_S)} : \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S \to \mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_S$$

sending any effective class

$$[X\times_S D(\underline{L}_S), \chi\mapsto\chi(l)]_{D(\underline{M}_S)}$$

to

$$[X\times_S D(\underline{L}_S), \chi\mapsto\chi(l)]_{D(\underline{N}_S)}$$

if l belongs to the trivial congruence class of $\underline{L}_X$ modulo the image of $\underline{N}_X$, and to zero otherwise. Moreover, this morphism factors via the injection

$$\mathrm{K}_0\mathbf{Var}_S \hookrightarrow \mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_S$$

sending $[X]_S$ to $[X\times_S D(\underline{M}_S), 1]_{S\times_S D(\underline{M}_S)}$.

Remark 1.13. In particular, if N is trivial, we obtain a morphism

$$\int_{D(\underline{M}_S)/S} : \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S \to \mathrm{K}_0\mathbf{Var}_S$$

sending $[X \times_S D(\underline{L}_S), \chi \mapsto \chi(l)]$ to $[X]_S$ if $l = 0$ and zero otherwise. In particular, we get the formal identity

$$\int_{D(\underline{M}_S)/S} \mathrm{ev}(m,\chi)\mathrm{d}\chi = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.3.2}$$

for all $m \in \underline{M}_S$.

It is also clear from the definition that for any $N' \subset N \subset M$

$$\int_{D(\underline{M}_S)/D(\underline{N'}_S)} = \int_{D(\underline{N}_S)/D(\underline{N'}_S)} \circ \int_{D(\underline{M}_S)/D(\underline{N}_S)}.$$

Definition 1.14. Let

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

be a *split* exact sequence of $\mathbf{Z}$-modules, for which a splitting $M \simeq N \oplus P$ is given. The extension morphism

$$\mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_S \to \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S$$

sends an effective class

$$\left[\begin{array}{ccccc} X \times_S D(\underline{L}_S) & = & D(\underline{L}_X) & \xrightarrow{h} & \mathbf{G}_m \\ \Big\downarrow{\scriptstyle (f,g)} & & & & \\ S \times_S D(\underline{N}_S) & & & & \end{array}\right]$$

to the class

$$\left[\begin{array}{ccccc} X \times_S D(\underline{L}_S) \times_S D(\underline{P}_S) & = & D(\underline{(L \oplus P)}_X) & \xrightarrow{(h,1)} & \mathbf{G}_m \\ \Big\downarrow{\scriptstyle (f,(g,\mathrm{id}))} & & & & \\ S \times_S D(\underline{M}_S) & & & & \end{array}\right].$$

Lemma 1.15. *For any $M_1 \to L_1$ and $M_2 \to L_2$ we have*

$$\int_{D(M_1 \times M_2)/D(N_1 \times N_2)} = \Big(\int_{D(M_1)/D(N_1)}\Big) \boxtimes_{D(N_1 \times N_2)} \Big(\int_{D(M_2)/D(N_2)}\Big).$$

Lemma 1.16. *Let $p : S \to T$ be a morphism of finite presentation and $N \hookrightarrow M$ as before. The diagram*

$$\begin{array}{ccc} \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S & \longrightarrow & \mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_S \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_T & \longrightarrow & \mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_T \end{array}$$

is commutative.

Definition 1.17. We define the localisation

$$\mathcal{C}har_M\mathscr{M}_S = \mathrm{K}_0\mathbf{Char}_M\mathbf{Var}_S\left[\mathbf{L}_S^{-1}, \left(\frac{1}{1 - \mathrm{ev}(m,\cdot)\mathbf{L}_S^{-a}}\right)_{m \in M, a \in \mathbf{Z}_{>0}}\right].$$

1.4. **Dimensional filtration and convergence of series.**

Definition 1.18. For any integer i, we define $\mathrm{Fil}^i \mathscr{C}har_M\mathscr{M}_S$ as the subgroup of $\mathscr{C}har_M\mathscr{M}_S$ generated by classes of the form $[X,f]_S\mathbf{L}_S^{-m}$ where X is a variety over S and m is an integer such that $\dim_S X - m \leq -i$. This defines a decreasing filtration $(\mathrm{Fil}^i \mathscr{C}har_M\mathscr{M}_S)_{i\in\mathbf{Z}}$ on $\mathscr{C}har_M\mathscr{M}_S$, endowing $\mathscr{C}har_M\mathscr{M}_S$ with the structure of a filtered ring.

Definition 1.19. For any class $\mathfrak{a} \in \mathscr{C}har_M\mathscr{M}_S$, we define

$$\dim_S(\mathfrak{a}) = \inf\{i \in \mathbf{Z}, \mathfrak{a} \in \mathrm{Fil}^i \mathscr{C}har_M\mathscr{M}_S\}$$

Definition 1.20. We define the completed Grothendieck ring with characters

$$\widehat{\mathscr{C}har_M\mathscr{M}_S} = \varprojlim \mathscr{C}har_M\mathscr{M}_S/\,\mathrm{Fil}^i \mathscr{C}har_M\mathscr{M}_S.$$

Remark 1.21. The above fitration induces the classical dimensional filtration $\mathrm{Fil}^i \mathscr{M}_S$ on the localised Grothendieck ring. Since inverse limits are left exact, the injective morphism $\mathscr{M}_S \to \mathscr{C}har_M\mathscr{M}_S$ induces an injective morphism $\widehat{\mathscr{M}_S} \to \widehat{\mathscr{C}har_M\mathscr{M}_S}$ between the corresponding completions.

The $\dim_S$ function introduced in Definition 1.19 extends naturally to $\widehat{\mathscr{C}har_M\mathscr{M}_S}$.

Definition 1.22. Let $F(T) = \sum_{i\geq 0} A_i T^i \in \widehat{\mathscr{C}har_M\mathscr{M}_S}[[T]]$ be a power series with coefficients in $\widehat{\mathscr{C}har_M\mathscr{M}_S}$. The radius of convergence of F is defined to be

$$\sigma_F = \limsup_{i\geq 1} \frac{\dim_S(A_i)}{i}.$$

We say that F converges for $|T| < \mathbf{L}^{-r}$ if $r \geq \sigma_F$.

If F converges for $|T| < \mathbf{L}^{-r}$, then for every element $\mathfrak{a} \in \widehat{\mathscr{C}har_M\mathscr{M}_S}$ such that $\dim_S(\mathfrak{a}) < -r$, the value $F(\mathfrak{a})$ exists as an element of $\widehat{\mathscr{C}har_M\mathscr{M}_S}$.

Definition 1.23. Let $F(\mathbf{T}) = \sum_{\mathbf{m}\in\Lambda} A_{\mathbf{m}}\mathbf{T}^{\mathbf{m}} \in \widehat{\mathscr{C}har_M\mathscr{M}_S}[[T]]$ be a power series with coefficients in $\widehat{\mathscr{C}har_M\mathscr{M}_S}$. Let $\lambda \in \mathring{\Lambda}^\vee$. The radius of convergence of F in the direction λ is defined to be

$$\sigma_{F,\lambda} = \limsup_{\mathbf{m}\in\Lambda} \frac{\dim_S(A_{\mathbf{m}})}{\langle \mathbf{m},\lambda\rangle}.$$

We say that F converges for $|T| < \mathbf{L}^{-s\lambda}$ if $s \geq \sigma_{F,\lambda}$.

1.5. **Motivic stabilisation.** The following is a motivic version of [Bou11b, Lemme 12].

Lemma 1.24. *Let $(\mathfrak{a}_{\boldsymbol{d}})_{\boldsymbol{d}\in\mathbf{N}^r}$ be a family of classes in $\mathscr{M}_S$ and*

$$F(\mathbf{T}) = \sum_{\boldsymbol{d}\in\mathbf{N}^r} \mathfrak{a}_{\boldsymbol{d}}\mathbf{T}^{\boldsymbol{d}}.$$

Assume that $F(\mathbf{T})$ converges absolutely for $|T_i| < \mathbf{L}^{-\rho_i+\varepsilon}$ and consider the power series

$$\frac{F(\mathbf{T})}{\prod_{i=1}^r (1-\mathbf{L}^{\rho_i}T_i)} = \sum_{\boldsymbol{d}\in\mathbf{N}^r} \mathfrak{b}_d \mathbf{T}^{\boldsymbol{d}}.$$

Then for all $0 < \eta < \varepsilon$

$$\mathfrak{b}_{\boldsymbol{d}} - F(\mathbf{L}^{-\rho})\mathbf{L}^{\langle \rho,\boldsymbol{d}\rangle}$$

has virtual dimension bounded above by

$$-\eta + \max_{i=1,\ldots,n}\left((1-\eta)\rho_i d_i + \sum_{j\neq i}\rho_j d_j\right).$$

Proof. It goes as for [Bou11b, Lemme 12]. The difference $\mathfrak{b}_{\boldsymbol{d}} - F(\mathbf{L}^{-\boldsymbol{\rho}})\mathbf{L}^{\langle \boldsymbol{\rho}, \boldsymbol{d}\rangle}$ is the coefficient of degree $\boldsymbol{d} \in \mathbf{N}^r$ of $\frac{F(\mathbf{T}) - F(\mathbf{L}^{-\boldsymbol{\rho}})}{(1-\mathbf{L}^{\rho_1}T_1)\cdots(1-\mathbf{L}^{\rho_r}T_r)}$. We can write

$$\frac{F(\mathbf{T}) - F(\mathbf{L}^{-\boldsymbol{\rho}})}{\prod_{i=1}^r (1-\mathbf{L}^{\rho_i}T_i)} = \sum_{j=1}^{r} \underbrace{\frac{F(T_1, \dots, T_j, \mathbf{L}^{-\rho_{j+1}}, \dots, \mathbf{L}^{-\rho_r}) - F(T_1, \dots, T_{j-1}, \mathbf{L}^{-\rho_j}, \dots, \mathbf{L}^{-\rho_r})}{\prod_{i=1}^r (1-\mathbf{L}^{\rho_i}T_i)}}_{=:G_j(\mathbf{T})}$$

where the degree $\boldsymbol{d} \in \mathbf{N}^r$ coefficient of $G_j(\mathbf{T})$ is

$$\mathfrak{b}_{j,\boldsymbol{d}} = -\sum_{\substack{0 \leqslant \delta_1 \leqslant d_1 \\ \vdots \\ 0 \leqslant \delta_{j-1} \leqslant d_{j-1} \\ \delta \in \mathbf{N} \\ \delta_{j+1} \in \mathbf{N} \\ \vdots \\ \delta_r \in \mathbf{N}}} \mathfrak{a}_{\delta_1, \dots, \delta_{j-1}, d_j + \delta + 1, \delta_{j+1}, \dots \delta_r} \mathbf{L}^{-(\delta+1)\rho_j + \sum_{\ell \neq j}(d_\ell - \delta_\ell)\rho_\ell}.$$

By assumption, we have

$$\dim(\mathfrak{a}_{\delta_1, \dots, \delta_{j-1}, d_j+\delta+1, \delta_{j+1}, \dots \delta_r}) < \langle (\delta_1, \dots, \delta_{j-1}, d_j + \delta + 1, \delta_{j+1}, \dots \delta_r), (1-\eta)\boldsymbol{\rho}\rangle$$

for $0 < \eta < \varepsilon$ and $\boldsymbol{d}$ large enough, hence the claim. □

2. Symmetric products and motivic Euler products

The aim of this section is to recall the definitions of symmetric products and motivic Euler products [Bil23], and explain how one can interpret them as functions on the scheme of effective divisors of the curve $\mathscr{C}$.

2.1. Symmetric products. Following for example [BH21, §6.1], we start with a few basic definitions concerning symmetric products of varieties. These products are the first building blocks leading to the notion of motivic Euler product.

If n is a non-negative integer, the n-th symmetric power of the variety $X \to S$ is the quotient

$$\mathbf{Sym}^n_{/S}(X) = \underbrace{X \times_S \dots \times_S X}_{n \text{ times}} / \mathfrak{S}_n$$

and if $Y \to X$ is another variety, one can form the relative symmetric product

$$\mathbf{Sym}^n_{X/S}(Y) = (\mathbf{Sym}^n_{/S}(Y) \longrightarrow \mathbf{Sym}^n_{/S}(X)).$$

From now on, let I be a set. If $\pi = (n_i)_{i \in I} \in \mathbf{N}^{(I)}$ is a *generalised partition*, that is, an element of the free abelian monoid on I, we set

$$\mathbf{Sym}^\pi_{/S}(X) = \prod_{i \in I} \mathbf{Sym}^{n_i}_{/S}(X)$$

and if $\mathscr{Y} = (Y_i \to X)_{i \in I}$ is a family of varieties, we define

$$\begin{aligned} \mathbf{Sym}^\pi_{X/S}(\mathscr{Y}) &= \prod_{i \in I} \mathbf{Sym}^{n_i}_{X/S}(Y_i) \\ &= \left(\prod_{i \in I} \mathbf{Sym}^{n_i}_{/S}(Y_i) \to \prod_{i \in I} \mathbf{Sym}^{n_i}_{/S}(X) \right). \end{aligned}$$

We now explain how to define symmetric products of classes in $\mathrm{K}_0\mathbf{Var}_X$. The product

$$\mathrm{K}_0\mathbf{Var}_{\mathbf{Sym}^\bullet_{/S}(X)} := \prod_{n \geq 0} \mathrm{K}_0\mathbf{Var}_{\mathbf{Sym}^n_{/S}(X)}$$

is a group for the Cauchy product law given by

$$((ab)_n)_{n\in\mathbf{N}} = \left(\sum_{i=0}^{n} a_i b_{n-i}\right)_{n\in\mathbf{N}}$$

where for every $n\in\mathbf{N}$ and $i\in\{0,...,n\}$, the product comes from the exterior product

$$a_i \boxtimes b_{n-i} \in \mathrm{K}_0\mathbf{Var}_{\mathbf{Sym}^i_{/S}(X)\times_k \mathbf{Sym}^{n-i}_{/S}(X)}$$

composed with the pushforward via the natural arrows

$$\mathbf{Sym}^i_{/S}(X)\times_k \mathbf{Sym}^{n-i}_{/S}(X) \longrightarrow \mathbf{Sym}^n_{/S}(X).$$

Then there is a unique group morphism (see [Bil23, Lemma 3.5.1.2])

$$\mathrm{K}_0\mathbf{Var}_X \to \mathrm{K}_0\mathbf{Var}_{\mathbf{Sym}^\bullet_{/S}(X)}$$

which to the class $[Y\to X]$ of an X-variety Y associates the family

$$([\mathbf{Sym}^i_{X/S}(Y)\to \mathbf{Sym}^i_{/S}(X)])_{i\geq 1}$$

of classes of its symmetric products; for every $a\in \mathrm{K}_0\mathbf{Var}_X$ we then denote by $\mathbf{Sym}^i_{X/S}(a)$ the i-th coordinate of its image via this morphism.

Then, if $\mathscr{A}=(a_i)_{i\in I}$ is a family of classes in $\mathrm{K}_0\mathbf{Var}_X$ and $\pi=(n_i)_{i\in I}\in\mathbf{N}^{(I)}$, we set

$$\mathbf{Sym}^\pi_{X/S}(\mathscr{A}) = \boxtimes_{i\in I}\mathbf{Sym}^{n_i}_{X/S}(a_i)$$

in $\mathrm{K}_0\mathbf{Var}_{\mathbf{Sym}^\pi_{/S}(X)}$.

In $\prod_{i\in I} X^{n_i}$ lies the open subset

$$\left(\prod_{i\in I} X^{n_i}\right)_*$$

obtained by removing the big diagonal (elements with at least two equal coordinates). The image of this open subset in $\mathbf{Sym}^\pi_{/S}(X)$ will be denoted by

$$\mathbf{Sym}^\pi_{/S}(X)_*$$

and similarly the restriction of $\mathbf{Sym}^\pi_{X/S}(\mathscr{A})$ to $\mathbf{Sym}^\pi_{/S}(X)_*$ will be denoted by

$$\mathbf{Sym}^\pi_{X/S}(\mathscr{A})_*.$$

2.2. **Mixed symmetric products.** In order to deal with possibly non-effective divisors, we are going to use the slightly more general notion of *mixed* symmetric products. A reference for this subsection is [Bil23, §3.3.2].

As before, let X be a variety over S. All products of varieties in this sections are taken relatively to S.

Let $p\geqslant 1$ be an integer (in our first application, we will take $p=2$) and let $(n_1,\dots,n_p)$ be a tuple of non-negative integers. The product $\mathfrak{S}_{n_1}\times...\times\mathfrak{S}_{n_p}$ acts by permutation on

$$X^{n_1+...+n_p} = X^{n_1}\times_S\dots\times_S X^{n_p}.$$

We define

$$\mathbf{Sym}^{n_1,\dots,n_p}_{/S}(X)_*$$

to be the image of the complement of the diagonal of $X^{n_1+...+n_p}$ in the quotient via this action, that is, in

$$\mathbf{Sym}^{n_1}_{/S}(X)\times\dots\times\mathbf{Sym}^{n_p}_{/S}(X).$$

More generally, consider sets $I_1, \ldots, I_p$ and for every j, a generalised partition $\pi_j = (n_{i,j})_{i\in I_j}$. Then, in the same way, we may define the variety $\mathbf{Sym}^{\pi_1,\ldots,\pi_p}_{/S}(X)_*$ to be the image of the complement of the big diagonal in

$$\prod_{(i_1,\ldots,i_p)\in I_1\times\ldots\times I_p} \mathbf{Sym}^{n_{i_1,1}}_{/S}(X)\times\ldots\times\mathbf{Sym}^{n_{i_p,p}}_{/S}(X).$$

Let, for every $j=1,\ldots,p$, $\mathscr{A}_j = (a_{i,j})_{i\in I_j}$ be a family of classes in $\mathrm{K}_0\mathbf{Var}_X$ indexed by the set I_j. The class

$$\mathbf{Sym}^{\pi_1,\ldots,\pi_p}_{X/S}(\mathscr{A}_1,\ldots,\mathscr{A}_p)_* \in \mathrm{K}_0\mathbf{Var}_{\mathbf{Sym}^{\pi_1,\ldots,\pi_p}_{/S}(X)}$$

is defined as the restriction to $\mathbf{Sym}^{\pi_1,\ldots,\pi_p}_{/S}(X)$ of the class

$$\prod_{(i_1,\ldots,i_p)\in I_1\times\ldots\times I_p} \mathbf{Sym}^{n_{i_1,1}}_{X/S}(a_{i_1,1})\times\ldots\times\mathbf{Sym}^{n_{i_p,p}}_{X/S}(a_{i_p,p})$$

in $\mathrm{K}_0\mathbf{Var}_{\prod_{(i_1,\ldots,i_p)\in I_1\times\ldots\times I_p}\mathbf{Sym}^{n_{i_1,1}}_{/S}(X)\times\ldots\times\mathbf{Sym}^{n_{i_p,p}}_{/S}(X)}$.

Remark 2.1. Let I be a non-empty set. Assume that we are given a partition

$$I = I_1\sqcup\ldots\sqcup I_p$$

into non-empty subsets $I_1,\ldots,I_p$. For every $n_1,\ldots,n_p$, there is a canonical étale morphism

$$\mathbf{Sym}^{n_1,\ldots,n_p}_{/S}(X)\longrightarrow\mathbf{Sym}^{n_1+\ldots+n_p}_{/S}(X).$$

Any family $\mathscr{A}=(a_i)_{i\in I}$ induces families

$$\mathscr{A}_j = (a_{i_j})_{i_j\in I_j}\quad j\in\{1,\ldots,p\}$$

and symmetric products $\mathbf{Sym}^{\pi_1,\ldots,\pi_p}_{X/S}(\mathscr{A}_1,\ldots,\mathscr{A}_p)_*$.

2.3. **Motivic Euler products.** Let us now recall the definition of motivic Euler products introduced in [Bil23, Chap. 3].

Definition 2.2. Let $\mathscr{A}=(a_i)_{i\in I}$ be a family of classes above X. The formal motivic Euler product

$$\prod_{x\in X/S}\left(1+\sum_{i\in I}a_{i,x}T_i\right)$$

associated to $\mathscr{Y}$ is by definition the formal series

$$\sum_{\pi\in\mathbf{N}^{(I)}}\mathbf{Sym}^{\pi}_{X/S}(\mathscr{A})_*T_\pi.$$

More generally, if p is a positive integer, the formal motivic Euler product

$$\prod_{x\in X/S}\left(1+\sum_{j=1}^{p}\sum_{i\in I_j}(a_{i,1,x}T_{i,1}+\ldots.+a_{i,p,x}T_{i,p})\right)$$

associated to a given set $\mathscr{A}_1,\ldots,\mathscr{A}_p$ of families respectively indexed by sets $I_1,\ldots,I_p$, is the formal series

$$\sum_{\pi_1,\ldots,\pi_p\in\mathbf{N}^{(I)}}\mathbf{Sym}^{\pi_1,\ldots,\pi_p}_{X/S}(\mathscr{A}_1,\ldots,\mathscr{A}_p)_*T_{\pi_1,\ldots,\pi_p}.$$

2.4. **Motivic Euler products and characters.** In this section we extend the notion of motivic Euler product from [Bil23] to the Grothendieck ring of varieties with multiplicative characters. In particular, this will allow us to make sense of products of the specific form

$$\prod_{v\in\mathscr{C}}\left(1+\sum_{m\in\mathbf{Z}^n_{\neq 0}} X_{m,v}\chi_v(m)T^m\right)$$

appearing later as certain Fourier transforms. The approach is completely analogous to how one defines motivic Euler products for varieties with exponentials in [Bil23, Section 3.6.1].

Let I be a set. Let S be a scheme, X a variety over S, $(M_i)_{i\in I}$ a family of commutative groups, $(G_i)_{i\in I}$ the corresponding family of diagonalisable groups over X, and

$$(\mathscr{Y},h) = (Y_i\times_X D(L_i), h_i : D(\underline{L_i}_{Y_i})\to\mathbf{G}_m)$$

a family of quasiprojective varieties with characters over X. Let $\pi=(n_i)_{i\in I}\in\mathbf{N}^{(I)}$ be a generalised partition.

We consider the product

$$\prod_{i\in I}(Y_i\times_X D(L_i))^{n_i}$$

(taken relatively to the base scheme S), endowed with the map

$$h^\pi : \prod_{i\in I} D(\underline{L_i}_{Y_i})^{n_i}\to\mathbf{G}_m$$

above $\prod_{i\in I}Y_i^{n_i}$ given by

$$(\chi_{i,1},\ldots,\chi_{i,n_i})\mapsto\prod_{i\in I}\prod_{j=1}^{n_i}h_i(\chi_{i,j}).$$

The aforementioned product comes with a morphism

$$\prod_{i\in I}Y_i^{n_i}\times_X D(L_i)^{n_i}\to\prod_{i\in I}X^{n_i}$$

(all products are taken relatively to S) and we denote by

$$\left(\prod_{i\in I}Y_i^{n_i}\times_X D(L_i)^{n_i}\right)_{*,X/S}$$

the inverse image of the complement of the big diagonal in $\prod_{i\in I}X^{n_i}$. There is a action of $\prod_{i\in I}\mathfrak{S}_{n_i}$ by permutation of the coordinates on this restriction, and we denote by

$$\mathbf{Sym}^\pi_{X/S}((\mathscr{Y},h))$$

the corresponding quotient. On the other hand, the map h^π restricts to $(\prod_{i\in I}Y_i^{n_i})_{*,X/S}$, and is invariant modulo the permutation action of $\prod_{i\in I}\mathfrak{S}_{n_i}$, thus descending to a morphism

$$h^{(\pi)} : \mathbf{Sym}^\pi_{X/S}((\mathscr{Y},h))\to\mathbf{G}_m$$

of group schemes above $\mathbf{Sym}^\pi_{X/S}((Y_i)_{i\in I})$.

Proposition 2.3. *Taking the class defines an element of*

$$\mathrm{K}_0\mathbf{Char}_{\oplus_{i\in I}M_i^{n_i}}\mathbf{Var}_{\mathbf{Sym}^\pi_{/S}(X)}.$$

Definition 2.4. The motivic Euler product

$$\prod_{x\in X/S}\left(1+\sum_{i\in I}[Y_i\times D(M_i),h_i]T_i\right)$$

is a notation for the series

$$\sum_{\pi}\left[\mathbf{Sym}^{\pi}_{X/S}((Y_i\times D(M_i))_{i\in I})_*,h^{(\pi)}\right]T_{\pi}.$$

Since the source of the maps h_i are actually the second factors $D(\underline{M_i})$, this definition is extended to non-effective classes the same way as for the classical definition for [Bil23].

2.5. **Symmetric products and Hilbert 90.** We generalise [Bil23, Proposition 3.4.0.1] to algebraic groups satisfying Hilbert 90. Ajouter une définition ou une réf au sujet de Hilbert 90

Proposition 2.5. *Let X be a variety over S, $\mathscr{Y}=(Y_i)_{i\in I}$ a family of varieties above X and $\mathscr{G}=(G_i)_{i\in I}$ a family of algebraic groups over S satisfying Hilbert 90. Let $\mathscr{Y}\times\mathscr{G}$ be the family $(Y_i\times G_i)_{i\in I}$, each element of this family being viewed above X via the first projection.*

Then for any generalised partition $\pi=(n_i)_{i\in I}$, the symmetric product

$$\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y}\times\mathscr{G})_*$$

is endowed with the structure of a $\prod_i G_i^{n_i}$-torsor (for the Zariski topology) above $\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y})_*$. In particular, $\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y}\times\mathscr{G})_*$ defines a class above $\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y})_*\times_S\prod_i[G_i]^{n_i}$ and

$$\left[\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y}\times\mathscr{G})_*\right]=\left[\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y})_*\right]\prod_i[G_i]^{n_i}$$

relatively to $\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y})_*\times_S\prod_i[G_i]^{n_i}$.

Proof. The argument used to prove [Bil23, Proposition 3.4.0.1] works here almost verbatim. Since

$$\left(\prod_{i\in I}Y_i^{n_i}\times\prod_{i\in I}G_i^{n_i}\right)_*\simeq\left(\prod_{i\in I}Y_i^{n_i}\times\prod_{i\in I}G_i^{n_i}\right)\times_{\prod_{i\in I}X^{n_i}}\left(\prod_{i\in I}X_i^{n_i}\right)_*\simeq\left(\prod_{i\in I}Y_i^{n_i}\right)_*\times\prod_{i\in I}G_i^{n_i}$$

we have the Cartesian square:

$$\begin{tikzcd}
\left(\prod_{i\in I}Y_i^{n_i}\right)_*\times\prod_{i\in I}G_i^{n_i} \arrow[r] \arrow[d, "q'"] & \left(\prod_{i\in I}Y_i^{n_i}\right)_* \arrow[d, "q"] \\
\mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y}\times\mathscr{G})_* \arrow[r, "p"] & \mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y})_*
\end{tikzcd}$$

where q' is an étale-local trivialisation of p and thus the claim follows by Hilbert 90 for G_i, $i\in I$. $\square$

Example 2.6. Let S be the spectrum of a field k. Take $\mathscr{Y}$ to be the constant family $\mathscr{C}\to\mathscr{C}$ and $(G_i)_{i\in I}$ to be the constant family $\mathbf{G}_{m,k}$. Then the previous proposition tells us that

$$\mathbf{Sym}^{\pi}_{\mathscr{C}/k}\left(\mathscr{C}\times_k\mathbf{G}_m\right)_*=\mathbf{Sym}^{\pi}_{\mathscr{C}/k}\left(\mathbf{G}_{m,\mathscr{C}}\right)_*$$

is a $\mathbf{G}_m^{\sum_{i\in I}n_i}$-torsor above $\mathbf{Sym}^{\pi}_{/k}(\mathscr{C})_*$, a Zariski trivialisation being canonically given by the Cartesian square

$$\begin{tikzcd}
\left(\prod_i\mathscr{C}^{n_i}\right)_*\times\mathbf{G}_m^{\sum_{i\in I}n_i} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "q'"] & \left(\prod_i\mathscr{C}^{n_i}\right)_* \arrow[d, "q"] \\
\mathbf{Sym}^{\pi}_{\mathscr{C}/k}\left(\mathbf{G}_{m,\mathscr{C}}\right)_* \arrow[r, "p"] & \mathbf{Sym}^{\pi}_{/k}(\mathscr{C})_*\,.
\end{tikzcd}$$

2.6. Symmetric products of functions on constant group schemes. In what follows, we will need to work with symmetric products in the following setting: let X be a quasi-projective variety over S, M be a set and consider $Y = \underline{M}_X$ the corresponding constant X-scheme (which, by definition, is the coproduct $\bigsqcup_{m \in M} X$, and thus is locally algebraic).

2.6.1. *Domain of definition.* The symmetric group $\mathfrak{S}_n$ acts on the n-fold product

$$\underline{M}_X \times_S \ldots \times_S \underline{M}_X.$$

We want to check that the quotient $\underline{M}_X \times_S \ldots \times_S \underline{M}_X / \mathfrak{S}_n$ exists as a scheme. For this, it suffices to check [Gro71, Exposé V, Proposition 1.8] that the orbit of every point is contained in an affine open. Let $x \in \underline{M}_X \times_S \ldots \times_S \underline{M}_X$. Then the orbit of x is contained in a finite union of products $X \times_S \ldots \times_S X$ of copies of X. Thus, for the sake of proving our statement we may assume that M is finite, and then it follows from quasi-projectivity of X over S.

Thus, the quotient $\mathbf{Sym}^n_{/S}(\underline{M}_X)$ exists as a (locally algebraic) scheme, and it is by construction endowed with a morphism to $\mathbf{Sym}^n_{/S}(X)$. By pulling back via the inclusion $\mathbf{Sym}^n_{X/S}(X)_* \to \mathbf{Sym}^n_{/S}(X)$, we also get a scheme $\mathbf{Sym}^n_{X/S}(\underline{M}_X)_*$.

Assume that M is endowed with a commutative group structure. Then the group scheme structure $\underline{M}_X$ over X canonically induces a group scheme structure over X^n on

$$\underline{M}^n_{X^n} \simeq \underline{M}_X \times_S \ldots \times_S \underline{M}_X,$$

which passes to the quotient and gives $\mathbf{Sym}^n_{X/S}(\underline{M}_X)$, respectively $\mathbf{Sym}^n_{X/S}(\underline{M}_X)_*$, a group scheme structure over $\mathbf{Sym}^n_{/S}(X)$, resp. $\mathbf{Sym}^n_{X/S}(\underline{M}_X)_*$.

The construction extends to mixed symmetric products in the obvious way.

2.6.2. *Symmetric products of functions.* Let again $X \to S$ be a quasi-projective variety above an absolute base scheme S.

Let I and $(M_i)_{i \in I}$ be sets and $(X_{m_i})_{(m_i) \in \prod_{i \in I} M_i}$ a family of varieties above X, which we see as a motivic function on the constant group $\prod_{i \in I} \underline{M_i}_X$ above X. Products and co-products of schemes are taken relatively to S. We consider the family

$$(\varphi_i)_{i \in I} = \left(\coprod_{m_i \in M_i} (X_{m_i} \to X) \right)_{i \in I}.$$

For any generalised partion $\pi = (n_i)_{i \in I}$, we have canonical isomorphisms

$$\begin{aligned} \prod_{i \in I} \left(\coprod_{m_i \in M_i} X_{m_i} \right)^{n_i} &\simeq \prod_{i \in I} \left(\coprod_{m_{i,1} \in M_i} X_{m_{i,1}} \right) \times_S \cdots \times_S \left(\coprod_{m_{i,n_i} \in M_i} X_{m_{i,n_i}} \right) \\ &\simeq \coprod_{(m_{i,j}) \in \prod_{i \in I} M_i^{n_i}} \prod_{i \in I} \left(X_{m_{i,1}} \times_S \ldots \times_S X_{m_{i,n_i}} \right). \end{aligned}$$

For every $i \in I$, the action of $\sigma_i \in \mathfrak{S}_{n_i}$ on X^{n_i} permuting the factors induces an action on $\left(\coprod_{m_{i,1} \in M_i} X_{m_{i,1}} \right) \times_S \cdots \times_S \left(\coprod_{m_{i,n_i} \in M_i} X_{m_{i,n_i}} \right)$ which does the following: for $(m_{i,j})_{j=1}^{n_i} \in M_i^{n_i}$ identifies the component $X_{m_{i,1}} \times_S \ldots \times_S X_{m_{i,n_i}}$ with the component $X_{m_{i,\sigma(1)}} \times_S \ldots \times_S X_{m_{i,\sigma_i(n_i)}}$ given by $(m_{i,\sigma(j)})_{j=1}^{n_i} \in M_i^{n_i}$ (the factors are the same, they are simply reordered) relatively to

X^{n_i}. Taking quotients with respect to this action leads to the commutative diagram

$$\begin{array}{ccc} \coprod_{(m_{i,j})\in\prod_{i\in I} M_i^{n_i}} \prod_{i\in I} X_{m_{i,1}} \times_X \dots \times_X X_{m_{i,n_i}} & \longrightarrow & \prod_{i\in I} X^{n_i} \\ \big\downarrow {\scriptstyle /\prod_{i\in I}\mathfrak{S}_{n_i}} & & \big\downarrow {\scriptstyle /\prod_{i\in I}\mathfrak{S}_{n_i}} \\ \mathbf{Sym}^{\pi}_{X/S}((\varphi_i)_{i\in I}) & \longrightarrow & \mathbf{Sym}^{\pi}_{/S}(X) \end{array}$$

and one restricts to $\mathbf{Sym}^{\pi}_{/S}(X)_*$ to get $\mathbf{Sym}^{\pi}_{X/S}((\varphi_i)_{i\in I})_*$.

Example 2.7. Take $M_i = \mathbf{Z}$ and $X_i = X$ for every $i \in I$. Then

$$\left(\coprod_{m_i\in M_i} X_{m_i}\right)_{i\in I} = \left(\coprod_{\mathbf{Z}} X\right)_{i\in I} = (\underline{\mathbf{Z}}_X)_{i\in I}$$

and the previous diagram becomes

$$\begin{array}{ccccc} \prod_{i\in I} \underbrace{\underline{\mathbf{Z}}_X \times_S \dots \times_S \underline{\mathbf{Z}}_X}_{n_i \text{ times}} & \xrightarrow{\simeq} & \coprod_{\prod_{i\in I}\mathbf{Z}^{n_i}} \prod_{i\in I} X^{n_i} & \longrightarrow & \prod_{i\in I} X^{n_i} \\ & & \big\downarrow {\scriptstyle /\prod_{i\in I}\mathfrak{S}_{n_i}} & & \big\downarrow {\scriptstyle /\prod_{i\in I}\mathfrak{S}_{n_i}} \\ & & \mathbf{Sym}^{\pi}_{X/S}((\underline{\mathbf{Z}}_X)_{i\in I}) & \longrightarrow & \mathbf{Sym}^{\pi}_{/S}(X). \end{array}$$

The fibre above a point $D = \sum_{i\in I} D_i$ where $D_i = [p_{i,1}] + \dots + [p_{i,n_i}]$ is an unordered product

$$\oplus_{i\in I} \oplus_{i=1}^{n_i} \underline{\mathbf{Z}}_{\kappa(p_i)}.$$

3. Divisors on curves and motivic functions

The goals of this section are

- to parameterise (not necessarily effective) divisors on $\mathscr{C}$ by algebraic objects,
- to define a certain class of motivic functions on divisors,
- to explain how one can see motivic Euler products as motivic functions,
- and finally parametrise principal divisors.

3.1. **Parametrising (non-effective) divisors on a curve.** As an abstract group, $\mathrm{Div}(\mathscr{C})$ is filtered by the subgroups $\mathrm{Div}_{\mathsf{S}}(\mathscr{C})$ of divisors with support in S, giving

$$\mathrm{Div}(\mathscr{C}) = \varinjlim_{\mathsf{S}} \mathrm{Div}_{\mathsf{S}}(\mathscr{C}) = \varinjlim_{\mathsf{S}} \bigoplus_{x\in\mathsf{S}} \mathbf{Z}_{\kappa(x)} = \bigoplus_{x\in|\mathscr{C}|} \mathbf{Z}_{\kappa(x)}$$

where $|\mathscr{C}|$ is the set of closed points of $\mathscr{C}$. Of course, this filtration restricts to degree-zero divisors

$$\mathrm{Div}^0(\mathscr{C}) = \varinjlim_{\mathsf{S}} \mathrm{Div}^0_{\mathsf{S}}(\mathscr{C})$$

and to principal divisors

$$\mathrm{PDiv}(\mathscr{C}) = \varinjlim_{\mathsf{S}} \mathrm{PDiv}_{\mathsf{S}}(\mathscr{C}).$$

Hence, a non-zero element D of $\mathrm{Div}(\mathscr{C})$ is determined by two parameters:

- the subset $S \neq \varnothing$, containing the support $\mathrm{Supp}(D) \subset S$;
- a tuple of integers in $\mathbf{Z}^S$, seen as a function $S \to \mathbf{Z}$ supported on $\mathrm{Supp}(D)$; equivalently, an element of $(\mathbf{Z} \setminus \{0\})^{\mathrm{Supp}(D)}$.

One can see the finite sets $\mathrm{Supp}(D)$ and S as étale divisors on $\mathscr{C}$. Moreover, one can fix $n = |S|$ and make $\mathrm{Supp}(D)$ and S vary.

Any generalised partition $\varpi = (n_i)_{i\in\mathbf{Z}_{>0}}$ of an integer n, that is to say a collection such that $\sum_{i\geq 1} in_i = n$, defines a partition of $s = \sum_i n_i$ in the usual sense. The identity morphism

$$\prod_{i\geqslant 1}\mathscr{C}^{n_i} \longrightarrow \mathscr{C}^s$$

induces, after taking quotients by appropriate symmetric groups, an étale morphism at the level of symmetric products

$$\mathbf{Sym}^{\pi}_{/k}(\mathscr{C})_* \longrightarrow \mathbf{Sym}^{s}_{/k}(\mathscr{C})_*.$$

At the level of geometric points, this morphism sends a geometric effective divisor

$$\sum_{i\in\mathbf{Z}_{>0}} i([p_{i,1}] + \ldots + [p_{i,n_i}])$$

to its support divisor

$$\sum_{\substack{i\in\mathbf{N}^*\\ j\in\{1,\ldots,n_i\}}} [p_{ij}].$$

Writing any divisor D as a difference $D = D^+ - D^-$ of two effective divisors with disjoint support, we can extend this construction to get a morphism

$$\mathbf{Sym}^{\pi^+,\pi^-}_{/k}(\mathscr{C})_* \longrightarrow \mathbf{Sym}^{s^+,s^-}_{/k}(\mathscr{C})_*$$

sending a geometric point

$$\begin{aligned} D &= D^+ - D^- \\ &= \sum_{i\in\mathbf{N}^*} i\left([p_{i,1}] + \ldots + [p_{i,n_i^+}] - [q_{i,1}] + \ldots + [q_{i,n_i^-}]\right) \end{aligned}$$

to its support divisor

$$\mathrm{Supp}(D) = \sum_{\substack{i\in\mathbf{N}^*\\ j^+\in\{1,\ldots,n_i^+\}\\ j^-\in\{1,\ldots,n_i^-\}}} \left([p_{ij^+}] + [q_{ij^-}]\right).$$

For every $n\in\mathbf{N}$ we define

$$\mathbf{Div}_{\leqslant n}(\mathscr{C}) = \coprod_{\substack{\pi^+=(n_i^+)_{i\in\mathbf{Z}_{>0}}\\ \pi^-=(n_i^-)_{i\in\mathbf{Z}_{>0}}\\ \sum_i n_i^+ + \sum_i n_i^- \leqslant n}} \mathbf{Sym}^{(\pi^+,\pi^-)}_{/k}(\mathscr{C})_*.$$

where for any $n' \leqslant n$ the part

$$\coprod_{\substack{\pi^+=(n_i^+)_{i\in\mathbf{Z}_{>0}}\\ \pi^-=(n_i^-)_{i\in\mathbf{Z}_{>0}}\\ \sum_i n_i^+ + \sum_i n_i^- = n'}} \mathbf{Sym}^{(\pi^+,\pi^-)}_{/k}(\mathscr{C})_*$$

parametrises divisors of support of size exactly n'. By convention we take

$$\mathbf{Sym}^{(0,0)}_{/k}(\mathscr{C})_* = \mathrm{Spec}(k).$$

This definition comes with canonical inclusion morphisms

$$i_{n,m} : \mathbf{Div}_{\leqslant n}(\mathscr{C}) \longrightarrow \mathbf{Div}_{\leqslant m}(\mathscr{C})$$

for every $n \leqslant m$, from which we get "restriction" ring morphisms

$$i_{n,m}^{*} : \mathrm{K}_0\mathbf{Var}_{\mathbf{Div}_{\leqslant m}(\mathscr{C})} \longrightarrow \mathrm{K}_0\mathbf{Var}_{\mathbf{Div}_{\leqslant n}(\mathscr{C})}.$$

and "extension-by-zero" ring morphisms (which do not preserve the unit)

$$(i_{n,m})_! : \mathrm{K}_0\mathbf{Var}_{\mathbf{Div}_{\leqslant n}(\mathscr{C})} \longrightarrow \mathrm{K}_0\mathbf{Var}_{\mathbf{Div}_{\leqslant m}(\mathscr{C})}.$$

It is then quite natural to introduce the ind-scheme

$$\mathbf{Div}(\mathscr{C}) = \varinjlim_{n} \mathbf{Div}_{\leqslant n}(\mathscr{C}) = \coprod_{\substack{\pi^+=(n_i^+)_{i\in\mathbf{Z}_{>0}}\\ \pi^-=(n_i^-)_{i\in\mathbf{Z}_{>0}}}} \mathbf{Sym}_{/k}^{(\pi^+,\pi^-)}(\mathscr{C})_*$$

as well as the ring of motivic functions (without unit element)

$$\mathrm{Fun}(\mathbf{Div}(\mathscr{C})) = \varinjlim_{n} \mathrm{K}_0\mathbf{Var}_{\mathbf{Div}_{\leqslant n}(\mathscr{C})}.$$

Elements of $\mathrm{Fun}(\mathbf{Div}(\mathscr{C}))$ can be understood as *families* of motivic functions on $\mathrm{Div}_{\mathsf{S}}(\mathscr{C})$ for varying S with bounded length.

3.2. **Euler products and motivic functions on the scheme of divisors.** As in [Fai25a], we adopt the following point of view. When $I = \mathbf{Z}_{>0}$, the coefficients of the formal motivic Euler product

$$\prod_{x\in\mathscr{C}} \left(1 + \sum_{m\in\mathbf{Z}_{>0}} Y_{m,x} T_m\right) = \sum_{\pi\in\mathbf{Z}_{\geqslant 0}^{(\mathbf{Z}_{>0})}} \mathbf{Sym}^{\pi}_{X/S}(\mathscr{Y})_* \mathbf{T}^{\pi}$$

can be understood as functions on the ind-scheme $\mathbf{Div}_+(\mathscr{C})$ of *effective* divisors on $\mathscr{C}$, via the identifications [Del73, p. 437]

$$\mathbf{Div}^d_+(\mathscr{C}) \simeq \mathbf{Sym}^d_{/k}(\mathscr{C})$$

and the stratification of $\mathbf{Sym}^d_{/k}(\mathscr{C})$ as

$$\coprod_{\substack{\pi=(n_i)\\ \sum_i n_i i = d}} \mathbf{Sym}^{\pi}_{/k}(\mathscr{C}).$$

Similarly, the coefficients of the mixed formal motivic Euler product

$$\prod_{x\in\mathscr{C}} \left(1 + \sum_{m\in\mathbf{Z}_{>0}} (Y_{m,1,x} + Y_{m,2,x}) T^m\right) = \sum_{\substack{\pi_1\in\mathbf{Z}_{\geqslant 0}^{(\mathbf{Z}_{>0})}\\ \pi_2\in\mathbf{Z}_{\geqslant 0}^{(\mathbf{Z}_{>0})}}} \mathbf{Sym}^{\pi_1,\pi_2}_{X/S}(\mathscr{Y}_1, \mathscr{Y}_2)_* T^{\pi_1,\pi_2}$$

can be interpreted as motivic functions on the ind-scheme $\mathbf{Div}(\mathscr{C})$ of divisors on $\mathscr{C}$. The first partition corresponds to the positive part, the second one to the negative part.

3.3. **Abel-Jacobi morphism and principal divisors.** Another morphism we are going to heavily use is the one sending

$$D = D_+ - D_- \in \mathbf{Sym}^{\pi^+,\pi^-}_{/k}(\mathscr{C})_*$$

to the isomorphism class

$$[\mathscr{O}_{\mathscr{C}}(D)] = [\mathscr{O}_{\mathscr{C}}(D_+) \otimes \mathscr{O}_{\mathscr{C}}(-D_-)] \in \mathbf{Pic}(\mathscr{C})$$

of the line bundle $\mathscr{O}_{\mathscr{C}}(D)$. The fibre of this morphism over the trivial class $[\mathscr{O}_{\mathscr{C}}]$ consists of principal divisors in $\mathbf{Sym}^{\pi^+,\pi^-}_{/k}(\mathscr{C})_*$. By taking a disjoint union on all the possible lengths $s = (s^+, s^-)$ of support, we get a way to parametrise all such principal divisors.

Then the disjoint union

$$\mathbf{PDiv}(\mathscr{C}) = \coprod_{\pi^+,\pi^-} p^{-1}_{\pi^+,\pi^-}([\mathscr{O}_{\mathscr{C}}]) \hookrightarrow \coprod_{\substack{\pi^+=(n_i^+)_{i\in\mathbf{Z}_{>0}}\\ \pi^-=(n_i^-)_{i\in\mathbf{Z}_{>0}}}} \mathbf{Sym}^{(\pi^+,\pi^-)}_{/k}(\mathscr{C})_*$$

which can be viewed above

$$\coprod_{s^+,s^-} \mathbf{Sym}^{s^+,s^-}_{/k}(\mathscr{C})_*.$$

Note that if $\sum_i in_i^+ \neq \sum_i in_i^-$ then $p^{-1}_{\pi^+,\pi^-}([\mathscr{O}_{\mathscr{C}}])$ is necessarily empty.

Equivalently, if Δ is the diagonal in $\mathrm{Pic}(\mathscr{C})\times\mathrm{Pic}(\mathscr{C})$, the ind-scheme $\mathbf{PDiv}(\mathscr{C})$ parametrising principal divisors on $\mathscr{C}$ is given by the Cartesian square

$$\begin{array}{ccc} \mathbf{PDiv}(\mathscr{C}) & \longrightarrow & \displaystyle\coprod_{\substack{\pi^+=(n_i^+)_{i\in\mathbf{Z}_{>0}}\\ \pi^-=(n_i^-)_{i\in\mathbf{Z}_{>0}}}} \mathbf{Sym}^{(\pi^+,\pi^-)}_{/k}(\mathscr{C})_* \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \mathbf{Pic}(\mathscr{C})\times\mathbf{Pic}(\mathscr{C}) \end{array}$$

where the vertical arrow on the right sends (D_+,D_-) to $(\mathscr{O}_{\mathscr{C}}(D_+),\mathscr{O}_{\mathscr{C}}(D_-))$.

4. Multiplicative motivic harmonic analysis

In this section we develop an elementary version of motivic harmonic analysis which in practice works well with abstract commutative groups and their Cartier duals, classically called *diagonalisable algebraic groups*. The setting we adopt is sometimes more general than needed and we included alternative simpler proofs that work in the specific situation we consider in our application to the height zeta function.

4.1. **Fourier transforms and inversion formulas.** Recall that we have the class

$$\mathrm{ev} = [\underline{M}_S \times_S D(\underline{M}_S), (m,\chi)\mapsto \chi(m)] \in \mathcal{C}har_M\mathscr{M}_{\underline{M}_S}.$$

Definition 4.1. Let M be an abstract commutative group and let $\psi \in \mathcal{C}har_M\mathscr{M}_{\underline{M}_S}$ be a motivic function supported over finitely many points of $\underline{M}_S$. The motivic Fourier transform of ψ is defined as the class

$$\mathscr{F}\psi = \psi\cdot\mathrm{ev} \in \mathcal{C}har_M\mathscr{M}_S$$

where the product is taken in $\mathcal{C}har_M\mathscr{M}_{\underline{M}_S}$ and then viewed in $\mathcal{C}har_M\mathscr{M}_S$. We can see this as the motivic function associating

$$(\mathscr{F}\psi)(\chi) = \sum_{m\in M}\psi(m)\,\mathrm{ev}(m,\chi)$$

to any $\chi\in D(\underline{M}_S)$.

Assume ψ is of the form $[X\to \underline{M}_S, 1]$. Then, explicitly,

$$\mathscr{F}\psi = [X\times_{\underline{M}_S}\underline{M}_S\times_S D(\underline{M}_S), (x,m,\chi)\mapsto\chi(m)]_{S\times_S D(\underline{M}_S)}$$

$$\begin{tikzcd}
X \arrow[d] & \underline{M}_S \times_S D(\underline{M}_S) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[r, "\mathrm{ev}"] & \mathbf{G}_{m,S} \\
\underline{M}_S \arrow[dr] & & D(\underline{M}_S) \arrow[dl] \\
& S &
\end{tikzcd} \tag{4.1.3}$$

Example 4.2. Taking $\psi = [\{m\} \times S \subset \underline{M}_S, 1]$ for some $m \in M$ (interpreted as the characteristic function of m), we obtain

$$\mathscr{F}\psi = [S \times_S D(\underline{M}_S), \chi \mapsto \chi(m)] = \mathrm{ev}(m, \cdot)$$

Proposition 4.3 (Motivic Fourier inversion formula)**.** *Let $\psi \in \mathscr{M}_{\underline{M}_S}$ be a motivic function having finite support. Then the motivic Fourier transform of ψ is well-defined and*

$$\psi(x) = \int_{D(\underline{M}_s)} (\mathscr{F}\psi)(\chi) \cdot \chi(-x)$$

for any $x \in \underline{M}_S$.

Proof. By Lemma 1.2 we can assume $S = \mathrm{Spec}(k)$. The function ψ can be written as a finite linear combination of indicator functions of points of $M = \underline{M}_{\mathrm{Spec}(k)}$, so it is enough to prove the statement for an elementary function of the form

$$\mathbf{1}_{\{x_0\}} = [\{x_0\} \hookrightarrow M]_M$$

for some point x_0. Then,

$$\mathscr{F}\mathbf{1}_{\{x_0\}} = [D(M), \chi \in D(M) \mapsto \chi(x_0)]_{D(M)} = \mathrm{ev}(x_0, \cdot\,)$$

and

$$\int_{D(\underline{M})} \mathscr{F}\mathbf{1}_{\{x_0\}}(\chi) \cdot \chi(-x) = \int_{D(\underline{M})} \chi(x_0 - x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

by (1.3.2), hence the identity. ☐

Our Proposition 4.3 is actually a special case of the following.

Theorem 4.4 (Local motivic Poisson formula)**.** *Let $H \subset G$ be commutative groups. We identify $D(G/H)$ with the kernel of the restriction morphism $\mathrm{res}_{|H} : D(G) \to D(H)$.*

Then for any ψ as before,

$$\sum_{h \in H} \psi(g + h) = \int_{D(G/H)} \mathscr{F}\psi(\chi) e(\chi, -g).$$

In particular, taking $g = 0_G$, one gets

$$\sum_{h \in H} \psi(h) = \int_{D(G/H)} \mathscr{F}\psi(\chi).$$

Proof. By Lemma 1.2 again we can assume $S = \mathrm{Spec}(k)$. We can modify the proof of Proposition 4.3 by replacing the functional

$$\psi \mapsto \psi(x) = \sum_{g \in G} \mathbf{1}_{\{0\}}(g) \psi(x + g)$$

by
$$\psi \mapsto \sum_{h\in H} \psi(x+h) = \sum_{g\in G} \mathbf{1}_H(g)\psi(x+g).$$
Up to replacing ψ by a translate, it is sufficient to prove the second identity. Instead of seeing $\chi \mapsto e(g,\chi) = \chi(g)$ as a character on $\widehat{G} = \{0\}^{\perp}$, we consider its restriction to
$$H^{\perp} = \ker(\mathrm{res}_{|H}) \subset \widehat{G}.$$
It is trivial if and only if $g \in H$. Assume that ψ is supported on $\{g_0\}$. Using
$$\int_{D(G/H)} = \int_{D(G)} \mathbf{1}_{D(G/H)}$$
where
$$\mathbf{1}_{D(G/H)} = [D(G/H) \hookrightarrow D(G), \mathrm{ev}]_{D(G)}$$
we get
$$\int_{D(G/H)} \mathscr{F}\psi(\chi) = \int_{D(G)} \mathbf{1}_{D(G/H)} \sum_{g\in G} \psi(g)\chi(g) = \int_{D(G)} \mathbf{1}_{D(G/H)}\psi(g_0)\chi(g_0).$$
Now if $g_0 \in H$, $\psi(g) = \mathbf{1}_H(g)\psi(g)$ and the integration operator gives 1. If $g_0 \notin H$, the last integral on $D(G)$ equals zero because g_0 is not sent to 0 by $G \to G/H$. Hence the result. $\square$

4.2. **Application to the abstract group of divisors.** The (abstract) group $\mathrm{Div}_{\mathsf{S}}(\mathscr{C})$ of divisors supported on a finite set S of points of $\mathscr{C}$ may be identified with $\mathbf{Z}^{\mathsf{S}}$. If a function ψ is defined on $\mathrm{Div}_{\mathsf{S}}(\mathscr{C})$, it can be seen, via extension by zero, as a function on $\mathrm{Div}_{\mathsf{S}'}(\mathscr{C})$ for every S' containing S. Then for every such S', the associated Fourier transform $\mathscr{F}\psi(\chi)$ will not depend on $\chi_v \in \mathrm{D}(\underline{\mathbf{Z}}_{\kappa(v)})$ for $v \notin \mathsf{S}$.

Under the same hypothesis, if ψ is seen as a function on $\mathrm{Div}(\mathscr{C}) = \varinjlim_{\mathsf{S}} \mathrm{Div}_{\mathsf{S}}(\mathscr{C})$, then for any subgroup $H \subset \mathrm{Div}(\mathscr{C})$, ψ is H-equivariant if and only if ψ is $(H \cap \mathrm{Div}_{\mathsf{S}}(\mathscr{C}))$-equivariant for any S containing the support of ψ.

Typically, in our application we will take
$$H = \mathrm{PDiv}_{\mathsf{S}}(\mathscr{C})$$
to be the group of principal divisors of $\mathscr{C}$ having support contained in a finite set $\mathsf{S} \subset |\mathscr{C}|$ of closed points of the curve $\mathscr{C}$ (or of $\mathscr{C} \otimes k'$ for some extension k'/k). In that case, H is contained in the subgroup
$$H_0 = \mathrm{Div}^0_{\mathsf{S}}(\mathscr{C}) = (1,...,1)^{\perp} \cap \mathrm{Div}_{\mathsf{S}}(\mathscr{C})$$
of degree-zero divisors having support in S. Then the commutative diagram of exact sequences
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H \arrow[d] & H \arrow[d] & & \\
0 \arrow[r] & H_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/H_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_0/H \arrow[r] \arrow[d] & G/H \arrow[r] \arrow[d] & (G/H)/(H_0/H) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$
induces by duality a diagram of exact sequences of diagonalisable algebraic groups.

By choosing once and for all a k-divisor $\mathfrak{D}_1$ of degree one on the curve and enlarging S if necessary, we fix a section of the quotient morphism

$$\deg : G \to G/H_0$$

and a splitting of the first horizontal sequence of $\mathbf{Z}$-modules by writing any divisor D as

$$(D - \deg(D)\mathfrak{D}_1) + \deg(D)\mathfrak{D}_1.$$

This splitting induces a splitting of the second horizontal sequence and by duality, splittings of $D(G)$ and $D(G/H)$.

4.3. **Fourier transforms in families.** Let M be an abstract commutative group, S a scheme and X a quasi-projective variety over S. We consider the constant group scheme $\underline{M}_X$. Recall that Section 2.6 constructs symmetric products of such objects as group schemes over the corresponding symmetric products of X.

The evaluation morphism $\mathrm{ev} : \underline{M}_X \times_X D(\underline{M}_X) \to \mathbf{G}_m$ induces a morphism

$$\mathrm{ev}^n : (\underline{M}_X)^n \times_{X^n} D(\underline{M}_X)^n \to \mathbf{G}_m$$

(where powers are taken relatively to S) given by

$$(g_1, ..., g_n, \chi_1, ..., \chi_n) \mapsto \prod_{i=1}^{n} \chi_i(g_i) \in \mathbf{G}_m.$$

Let $\pi = (n_i)_{i\in I}$ be a generalized partition. Combining the ev^{n_i} for every n_i, passing to the quotient by the permutation action, restricting and using **??** we get a morphism

$$\mathbf{Sym}^{\pi}_{X/S}(\mathrm{ev})_* : \mathbf{Sym}^{\pi}_{X/S}(\underline{M}_X)_* \times_{\mathbf{Sym}^{\pi}_{/S}(X)_*} \mathbf{Sym}^{\pi}_{X/S}(D(\underline{M}_X))_* \to \mathbf{G}_m.$$

We may view $\mathbf{Sym}^{\pi}_{X/S}(\mathrm{ev})_*$ as a class in $\mathcal{C}har_{\mathbf{Sym}^{\pi}(M)}\mathscr{M}_{\mathbf{Sym}^{\pi}_{X/S}(\underline{M}_X)}$ in the sense of Section 2.6.

Definition 4.5. Let $\psi \in \mathcal{C}har_{\prod_i M^{n_i}}\mathscr{M}_{\mathbf{Sym}^{\pi}_{X/S}(\underline{M}_X)}$ be such that in the fibre above every point of $\mathbf{Sym}^{\pi}_{/S}(X)$ it is supported over finitely many points. Then we may define its *global Fourier transform* $\mathscr{F}\psi$ as

$$\mathscr{F}\psi = \psi \cdot \mathrm{ev} \in \mathcal{C}har_{\prod_i M^{n_i}}\mathscr{M}_{\mathbf{Sym}^{\pi}_{/S}(X)}$$

where the product is taken in $\mathcal{C}har_{\prod_i M^{n_i}}\mathscr{M}_{\mathbf{Sym}^{\pi}_{X/S}(\underline{M}_X)}$ and then viewed in $\mathcal{C}har_{\prod_i M^{n_i}}\mathscr{M}_{\mathbf{Sym}^{\pi}_{/S}(X)}$.

Proposition 4.6 (Compatibility between local and global Fourier transform)**.** *Let $(\psi_i)_{i\in I}$ be a family of elements of $\mathcal{C}har_M\mathscr{M}_{\underline{M}_X}$, each supported over a finite number of points of M. Let π be a generalised partition, and consider $\mathbf{Sym}^{\pi}((\psi_i)_{i\in I}) \in \mathcal{C}har_{\prod_i M^{n_i}}\mathscr{M}_{\mathbf{Sym}^{\pi}_{X/S}(\underline{M}_X)}$. Then in the Grothendieck ring $\mathcal{C}har_{\prod_i M^{n_i}}\mathscr{M}_{\mathbf{Sym}^{\pi}(X)}$, we have the equality*

$$\mathscr{F}\,\mathbf{Sym}^{\pi}((\psi_i)_{i\in I}) = \mathbf{Sym}^{\pi}_{X/S}((\mathscr{F}\psi_i)_{i\in I}),$$

where the right-hand side is viewed as an element of this ring via Proposition 2.3.

Proof. To be written up. □

5. Toric harmonic analysis

5.1. **Local invariant functions and their Fourier transforms.** We will now specialize a bit more to the concrete setup that we need for our purposes. Let k be a field and let U be a split algebraic torus of dimension n defined over k. We denote by $\mathcal{X}^*(U) = \mathrm{Hom}(U, \mathbf{G}_m)$ its group of characters and $\mathcal{X}_*(U) = \mathrm{Hom}(\mathbf{G}_m, U)$ its group of cocharacters. There is a natural pairing

$$\langle \cdot, \cdot \rangle : \mathcal{X}^*(U) \times \mathcal{X}_*(U) \to \mathbf{Z}.$$

The aim of this section is to define motivic incarnations of families of local functions $(f_v : U(F_v) \to \mathbf{C})_{v \in \mathscr{C}}$ which are $U(\mathcal{O}_v)$-invariant. The starting point here is that the local degree mapping

$$\deg_{F,v} : U(F_v) \to \mathcal{X}_*(U)$$

induces an isomorphism

$$U(F_v)/U(\mathcal{O}_v) \simeq \mathcal{X}_*(U),$$

so that such a local function may be viewed as a function on the free $\mathbf{Z}$-module $\mathcal{X}_*(U)$ of rank n. Moreover, in our setting, we only need to consider such functions with finite support in $\mathcal{X}_*(U)$. This motivates the following definition:

Definition 5.1. A family of local motivic $U(\mathcal{O})$-invariant functions is an element of the ring $\mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathcal{X}_*(U)_{\mathscr{C}}}$ supported over finitely many points of $\mathcal{X}_*(U)$. These elements form a subring of $\mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathcal{X}_*(U)_{\mathscr{C}}}$ denoted by $\mathrm{Fun}(U(F_v))$.

Notation 5.2. Let $\mathbf{a} \in \mathcal{X}_*(U)$ be a cocharacter of U. We denote by

$$\mathbf{1}_{\mathbf{t}^{\mathbf{a}} U(\mathcal{O})}$$

the family of local motivic $U(\mathcal{O})$-invariant functions given by the class of the subscheme

$$\{\mathbf{a}\} \times \mathscr{C} \hookrightarrow \mathcal{X}_*(U) \times_k \mathscr{C} = \mathcal{X}_*(U)_{\mathscr{C}}$$

viewed in $\mathrm{Fun}(U(F_v))$. When picking identifications $U \simeq \mathbf{G}_m^n$ and $\mathcal{X}_*(U) \simeq \mathbf{Z}^n$, so that $\mathbf{a} = (a_1, \dots, a_n)$ is viewed as a tuple of integers, it is the motivic incarnation of the family of characteristic functions of the subsets

$$t_1^{a_1} \mathcal{O}_v^\times \times \dots \times t_n^{a_n} \mathcal{O}_v^\times \subset (F_v^\times)^n.$$

as v runs over points of $\mathscr{C}$.

Remark 5.3. The forgetful morphism

$$\mathrm{Fun}(U(F_v)) \to \mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathscr{C}}$$

may be interpreted as integration over $U(F_v)$ for each v.

By the theory of Section 4.1 applied to the constant group scheme $\mathcal{X}_*(U)_{\mathscr{C}}$ over $\mathscr{C}$, we then have access to a notion of Fourier transformation for such a family of functions. Recall that in this framework, the Cartier dual of $\mathcal{X}_*(U)_{\mathscr{C}}$ is $U_{\mathscr{C}}$.

Definition 5.4. A family of local motivic functions with characters on U is an element of $\mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathscr{C}}$.

Given $f = (f_v)_v \in \mathrm{Fun}(U(F_v))$ a family of local motivic $U(\mathcal{O})$-invariant functions, its Fourier transform $\mathscr{F}f$ is by definition a family of local motivic functions with characters.

Example 5.5. Let $\mathbf{a} \in \mathcal{X}_*(U)$. We have

$$\begin{aligned}\mathscr{F}\mathbf{1}_{\mathbf{t}^{\mathbf{a}}U(\mathcal{O})} &= [(\{\mathbf{a}\} \times \mathscr{C}) \times_{\mathscr{C}} U_{\mathscr{C}}, \mathrm{ev}]_{\mathbf{G}^n_{m,\mathscr{C}}} \\ &= \left[\mathbf{G}^n_{m,\mathscr{C}}, \chi \mapsto \chi(\mathbf{a})\right] = \mathrm{ev}(\mathbf{a}, \cdot) \in \mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathscr{C}}.\end{aligned}$$

5.2. **Global K-invariant functions and global Fourier transforms.** Denoting by $\mathbf{K}(U)$ the subset $\prod_v U(\mathcal{O}_v) \subset U(\mathbf{A}_F)$, we define motivic incarnations of $\mathbf{K}(U)$-invariant global functions of the form

$$f = \prod_v f_v,$$

where the f_v are local $U(\mathcal{O}_v)$-invariant functions. For this to make sense motivically, we assume that the f_v are chosen among a finite number of different possibilities.

Concretely, let I be a set and let $(f_i)_{i\in I}$ a collection of families of local motivic $U(\mathcal{O})$-invariant functions. Then the number of factors n_i where f_v is given by $f_{i,v}$ is encoded by a partition $\pi = (n_i)_{i\in I}$.

Definition 5.6. Let I be a set and let $\pi = (n_i)_{i\in I}$ be a partition. A family of global motivic $\mathbf{K}(U)$-invariant functions of level π is a finite linear combination, with integral coefficients, of elements of the form

$$\mathbf{Sym}^{\pi}_{\mathscr{C}/k}((f_i)_{i\in I})_* \in \mathcal{C}har_{(\mathcal{X}_*(U))^{\pi}}\mathscr{M}_{\mathbf{Sym}^{\pi}_{/k}(\mathscr{C})_*}$$

(in the sense of Section 2.6.2) where for every $i \in I$, f_i is a family of local motivic $U(\mathcal{O})$-invariant functions (having finite support, by definition).

Remark 5.7. Such a symmetric product has natural maps

$$\mathbf{Sym}^{\pi}((f_i)_{i\in I}) \to \mathbf{Sym}^{\pi}(\mathcal{X}_*(U)_{\mathscr{C}}) \to \mathbf{Sym}^{\pi}(\mathscr{C}).$$

The bottom symmetric product parametrises supports $\mathbf{Sym}^{\pi}(\mathscr{C})$, and the intermediate symmetric product $\mathbf{Sym}^{\pi}(\mathcal{X}_*(U)_{\mathscr{C}})$ parametrises domains of definition.

Concretely, above $D = \sum i(v_{i,1} + \ldots + v_{i,n_i}) \in \mathbf{Sym}^{\pi}(\mathscr{C})$ we have the functions given by f_i at the places $v_{i,1}, \ldots, v_{i,n_i}$, defined on

$$\prod_{i\in I} U(F_{v_{i,1}}) \times \ldots \times U(F_{v_{i,n_i}})$$

which, since the functions are $U(\mathcal{O})$-invariant, is modeled by the fiber $(\mathbf{Sym}^{\pi}(\mathcal{X}_*(U)_{\mathscr{C}}))_D = \prod_{v\in|D|} \mathcal{X}_*(U)_v$.

Example 5.8 (Family of characteristic functions)**.** Assume that there exists $\mathbf{a}_i$ such that

$$f_i = \mathbf{1}_{t^{\mathbf{a}_i}U(\mathcal{O})}$$

for every $i \in I$. Then the fibre of $\mathbf{Sym}^{\pi}((f_i)_{i\in I})$ above $D = \sum i(v_{i,1} + \ldots + v_{i,n_i}) \in \mathbf{Sym}^{\pi}(\mathscr{C})$ is the global function

$$\prod_{i\in I}\prod_{j=1}^{n_i} \mathbf{1}_{t^{\mathbf{a}_i}U(\mathcal{O}_{v_{i,j}})}.$$

5.3. **Summation over rational points.** In this section, we define a motivic analogue of the operation which, to an appropriately integrable function $f : U(\mathbf{A}_F) \to \mathbf{C}$ on the idèles associates

$$\sum_{x \in U(F)} f(x).$$

More specifically, since in the previous paragraph we have defined our functions in families (parametrised by symmetric products), we will explain how to perform this operation simultaneously for all functions in such a family.

Proposition 5.9. *There is a constructible map*

$$\mathbf{Sym}^{\pi}(\mathcal{X}_*(U)_{\mathscr{C}}) \to \mathrm{Hom}_{\mathrm{gp}}(\mathcal{X}^*(U), \mathbf{Pic}(\mathscr{C}))$$

given by

$$\sum_{i \in I} i((\mathbf{b}_{i,1}, v_{i,1}) + \ldots + (\mathbf{b}_{i,n_i}, v_{i,n_i})) \mapsto \left(\mathbf{a} \mapsto \left[\sum_{i,j} \langle \mathbf{a}, \mathbf{b}_{i,j} \rangle v_{i,j} \right] \right).$$

Proof. To be written up. □

We denote by $\mathbf{PDiv}^{\pi}(\mathscr{C})$ the fiber above 0 of this map.

Then given $f \in \mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathbf{Sym}^{\pi}(\mathcal{X}_*(U)_{\mathscr{C}})}$ a family of global motivic functions of level π, we denote by

$$\sum_{D \in \mathbf{PDiv}^{\pi}(\mathscr{C})} f(D)$$

the image in $\mathcal{C}har_{\mathcal{X}_*(U)}\mathscr{M}_{\mathbf{Sym}^{\pi}(\mathscr{C})}$ of the pullback of f to $\mathbf{PDiv}^{\pi}(\mathscr{C})$.

5.4. **Global characters and character functions.** Analogously to Definition 5.6, we define:

Definition 5.10. A family of global character functions of level π is a finite linear combination, with integral coefficients, of elements of the form

$$\mathbf{Sym}^{\pi}_{\mathscr{C}/k}((g_i)_{i \in I})$$

where $(g_i)_{i \in I}$ is a family of local motivic character functions on $D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}})$.

There are natural maps

$$\mathbf{Sym}^{\pi}_{\mathscr{C}/k}((g_i)_{i \in I}) \to \mathbf{Sym}^{\pi}_{\mathscr{C}/k}(D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}})) \to \mathbf{Sym}^{\pi}(\mathscr{C}).$$

We may think of the symmetric product $\mathbf{Sym}^{\pi}_{\mathscr{C}/k}(D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}}))$ as parameterizing families of global characters with support of size $|\pi|$. The fibre of $\mathbf{Sym}^{\pi}_{\mathscr{C}/k}((g_i)_{i \in I})$ above $D = \sum i(v_{i,1} + \ldots + v_{i,n_i}) \in \mathbf{Sym}^{\pi}(\mathscr{C})$ represents, through the local isomorphism $D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}})_{v_{i,j}} \simeq U_{v_{i,j}}$, functions defined on the character domain

$$\prod_{i \in I} U_{v_{i,1}} \times \ldots \times U_{v_{i,n_i}}.$$

The natural evaluation pairing $\underline{\mathcal{X}_*(U)}_{\mathscr{C}} \times_{\mathscr{C}} D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}}) \to \mathbf{G}_m$ induces a pairing

$$\mathbf{Sym}^{\pi}_{\mathscr{C}/k}(\underline{\mathcal{X}_*(U)}_{\mathscr{C}}) \times_{\mathbf{Sym}^{\pi}(\mathscr{C})} \mathbf{Sym}^{\pi}_{\mathscr{C}/k}(D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}})) \to \mathbf{G}_m.$$

We denote by $\mathbf{PDiv}^{\pi}(\mathscr{C})^{\perp} \subset \mathbf{Sym}^{\pi}_{\mathscr{C}/k}(D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}}))$ the orthogonal to $\mathbf{PDiv}^{\pi}(\mathscr{C})$ via this pairing, in the sense of Section 4.2.

5.5. **The motivic multiplicative Poisson formula.**

Theorem 5.11. *Let f be a family of global motivic $\mathbf{K}$-invariant functions of level π. Then*

$$\sum_{D \in \mathbf{PDiv}^{\pi}(\mathscr{C})} f(D) = \int_{\mathbf{PDiv}^{\pi}(\mathscr{C})^{\perp}} \mathscr{F} f$$

in $\mathcal{C}har_{\mathscr{M}_{\mathbf{Sym}^{\pi}(\mathscr{C})}}$.

Proof. We fix a support $E \in \mathbf{Sym}^{\pi}(\mathscr{C}) = \sum_{i \in I} i(v_{i,1} + \ldots + v_{i,n_i})$ and apply Theorem 4.4 to $H = \mathrm{Div}_{\mathsf{S}}(\mathscr{C} \times \kappa(E))$ with $\mathsf{S} = \cup_{i \in I}\{v_{i,1}, \ldots + v_{i,n_i}\}$. □

6. Intermezzo: motivic L-functions associated to $\mathcal{O}^{\times}$-invariant characters

Definition 6.1 (Universal L-function for families of k-invariant characters)**.** The universal L function is the formal series

$$L(T) = \prod_{v \in \mathscr{C}} \left(\sum_{n \geqslant 0} \mathrm{ev}_n T^n \right) \in \mathrm{K}_0\mathbf{Char}_{\mathbf{Sym}^{\bullet}\mathbf{Z}}\mathbf{Var}_{\mathbf{Sym}^{\bullet}\mathscr{C}}[[T]]$$

which should be understood as the motivic function sending χ to the formal series

$$\prod_{v \in \mathscr{C}} (1 - \chi_v T)^{-1}.$$

Proposition 6.2 (To be refined)**.** *Let $\chi : \mathscr{C} \overset{PW}{\to} \mathbf{G}_m$ be invariant with respect to principal divisors. Then, the L-function associated to χ atisfies the functional equation.*

$$L(T, \chi) = \chi(\omega_{\mathscr{C}})\mathbf{L}^{g-1}T^{2g-2}L(1/\mathbf{L}T, \chi^{-1})$$

More precisely, there exists a polynomial of degree at most $2g-2$ such that

$$L(T, \chi) = \frac{P(T, \chi)}{(1 - \chi(1)T)(1 - \mathbf{L}\chi(1)T)}.$$

and satisfying

$$P(T, \chi) = T^{2g-2}\chi(\omega_{\mathscr{C}})P(1/\mathbf{L}T, \chi^{-1}).$$

Proof. The invariance of χ means that, for every $n \geq 0$, it is constant on the fibers of the morphism of schemes

$$p : \mathbf{Sym}^n(\mathscr{C}) \to \mathrm{Pic}^n(\mathscr{C})$$

and thus descends to a class on $\mathrm{Pic}^n(\mathscr{C})$.

Recall that we have fixed a degree 1 divisor $\mathfrak{D}_1$ on $\mathscr{C}$. Then by [CLNS18, Chapter 7, Example 1.1.10] we know moreover that $p : \mathbf{Sym}^n(\mathscr{C}) \to \mathrm{Pic}^n(\mathscr{C})$ is a projective bundle of rank $n - g$ for $n > 2g - 2$.

Thus, similarly to the proof of the rationality of the Kapranov zeta function in [CLNS18, Theorem 1.3.1], setting $\mathbf{P}_k^{n-g} = \varnothing$ whenever $n < g$, we compare the L-function to its approximation

$$\sum_{n \in \mathbf{N}} \left[\mathbf{P}_k^{n-g}\right] \sum_{L \in \mathrm{Pic}^n(\mathscr{C})} \chi(L) T^n.$$

We introduce

$$\begin{aligned} P_1(T,\chi) &= \sum_{n=0}^{2g-2}\left(\sum_{D\in \mathbf{Sym}^n_{/k}(\mathscr{C})}\chi(D) - \sum_{L\in\mathbf{Pic}^n(\mathscr{C})}\left[\mathbf{P}_k^{n-g}\right]\chi(L)\right)T^n \\ &= \sum_{n=0}^{2g-2}\sum_{[L]\in\mathbf{Pic}^n(\mathscr{C})}\chi(L)\left(\sum_{D\in[L]}1-\left[\mathbf{P}_k^{n-g}\right]\right)T^n \end{aligned}$$

so that

$$L(T,\chi) = P_1(T,\chi) + \sum_{n\in\mathbf{N}}\left[\mathbf{P}_k^{n-g}\right]\sum_{L\in\mathrm{Pic}^n(\mathscr{C})}\chi(L)T^n.$$

For any $n\in\mathbf{N}$,

$$\left[\mathbf{P}_k^{n-g}\right]\sum_{L\in\mathbf{Pic}^n(\mathscr{C})}\chi(L) = \begin{cases}\left[\mathbf{P}_k^{n-g}\right][\mathbf{Pic}^n(\mathscr{C})]\chi(n) & \text{if } \chi \text{ is } \mathrm{Div}^0(\mathscr{C})\text{-invariant}\\ 0 & \text{otherwise,}\end{cases}$$

where we write $\chi(n)=\chi(n\mathfrak{D}_1)$. In particular, if χ is non-trivial, then $L(T,\chi)$ is a polynomial of degree at most $2g-2$. Let $(b_n)_{n\in\mathbf{N}}$ be the coefficients of $P_1(T,\chi)$ and $(\overline{b_n})_{n\in\mathbf{N}}$ the ones of $P_1(T,1/\chi)$. It is clear that $P_1(T,\chi)=0$ when $g=0$ so we assume $g>0$. We use the notation of [CLNS18, Chapter 7, Example 1.1.10]: for a given n, let $\mathscr{L}$ be the universal line bundle on $X\times\mathrm{Pic}^n(X)$, and denote by $\mathscr{E}$ (resp. $\mathscr{F}$) the pushforward of $\mathscr{L}$ (resp. $\mathscr{L}^\vee\otimes\mathrm{pr}_{\mathscr{C}}^*\omega_{\mathscr{C}}$) to $\mathrm{Pic}^n(\mathscr{C})$. Then we may consider a stratification Π_n of $\mathrm{Pic}^m(\mathscr{C})$ into integral subvarieties V such that the restrictions of $\mathscr{E}$ and $\mathscr{F}$ to each stratum V are locally free, of ranks $r(V)$ and $s(V)$, respectively. If $L\in\mathbf{Pic}^n(\mathscr{C})$, by Serre duality and Riemann-Roch we have

$$h^0(\mathscr{C},\omega_{\mathscr{C}}\otimes L^\vee) = h^0(\mathscr{C},L)+g-1-n$$

hence $s(V)=r(V)+g-1-n$ for every stratum V.

Therefore,

$$\begin{aligned} \sum_{D\in\mathbf{Sym}^{2g-2-n}(\mathscr{C})}\chi(D) &= \sum_{V\in\Pi_{2g-2-n}}\left(1+\mathbf{L}+\ldots+\mathbf{L}^{r(V)+g-2-n}\right)\sum_{L\in V}\chi(L)\\ &= \mathbf{L}^{g-1-n}\sum_{V\in\Pi_{2g-2-n}}\left(1+\mathbf{L}+\ldots+\mathbf{L}^{r(V)-1}\right)\sum_{L\in V}\chi(L)\\ &\quad+\sum_{V\in\Pi_{2g-2-n}}\left(1+\mathbf{L}+\ldots+\mathbf{L}^{g-2-n}\right)\sum_{L\in V}\chi(L)\\ &= \ldots\\ &= \mathbf{L}^{g-1-n}\chi(\omega_{\mathscr{C}})\sum_{D\in\mathbf{Sym}^n}\chi(-D)+\chi(\omega_{\mathscr{C}})\left[\mathbf{P}_k^{g-2-n}\right]\sum_{L\in\mathrm{Pic}^n(\mathscr{C})}\chi(-L)\end{aligned}$$

hence

$$b_{2g-2-n} = \sum_{D\in\mathbf{Sym}^n_{/k}(\mathscr{C})}\chi(D) - \sum_{L\in\mathbf{Pic}^n(\mathscr{C})}\left[\mathbf{P}_k^{n-g}\right]\chi(L) = \mathbf{L}^{g-1-n}\chi(\omega_{\mathscr{C}})\chi(n)\overline{b_n}.$$

We set

$$P(T,\chi) = (1-\chi(1)T)(1-\chi(1)\mathbf{L}T)P_1(T,\chi)+\chi(g)T^g[\mathrm{Pic}^0(\mathscr{C})]$$

and want to show that $P(T,\chi) = T^{2g-2}\chi(\omega_{\mathscr{C}})P(1/\mathbf{L}T,\chi^{-1})$. The coefficient of degree $2g-n$ of $P(T,\chi^{-1})$ equals

$$\begin{aligned}&\overline{b_{2g-n}} - (\mathbf{L}+1)\chi(-1)\overline{b_{2g-n-1}} + \mathbf{L}\chi(-2)\overline{b_{2g-n-2}}\\&= \mathbf{L}^{g-n}\chi(\omega_{\mathscr{C}})(\mathbf{L}\chi(2)b_{n-2} - \chi(1)(\mathbf{L}+1)b_{n-1} + b_n)\end{aligned}$$

which is $\mathbf{L}^{g-n}\chi(\omega_{\mathscr{C}})$ times the coefficient of degree n of $P(T,\chi)$. □

7. Application to the motivic height zeta function

7.1. **An introductory example: the projective line.** We explain without any proof what are the objects coming into play in the very specific case of $X = \mathbf{P}^1_F$ viewed as a toric variety. Define for every non-zero $m \in \mathbf{Z}$

$$\mathbf{1}_{t^m\mathcal{O}^\times} = \{m\} \times \mathscr{C}$$

as a subscheme of the constant group scheme $\underline{\mathbf{Z}}_{\mathscr{C}}$. Its fibre above $v \in \mathscr{C}$ is denoted $\mathbf{1}_{t^m\mathcal{O}_v^\times}$ and is the motivic incarnation of the $\mathcal{O}_v^\times$-invariant function $\mathbf{1}_{t^m\mathcal{O}_v^\times}$ on the completion F_v of F at v. Thus we get a family

$$(\mathbf{1}_{t^m\mathcal{O}^\times})_{m\in\mathbf{Z}}$$

of subschemes of $\underline{\mathbf{Z}}_{\mathscr{C}}$ which can be used to define a generalised motivic Euler product like those that are described above.

In this notation, denoting by T_-, T_+ our indeterminates, the local height is the motivic function on $\underline{\mathbf{Z}}_{\mathscr{C}}$

$$H_v(\mathbf{T},\cdot) = 1 + \sum_{m<0} \mathbf{1}_{t^m\mathcal{O}_v^\times} T_-^m + \sum_{m>0} \mathbf{1}_{t^m\mathcal{O}_v^\times} T^m \in \mathrm{K}_0\mathbf{Var}_{\underline{\mathbf{Z}}_{\mathscr{C}}}[\mathbf{T}]$$

fitting into a motivic Euler product

$$H(T,\cdot) = \prod_{v\in\mathscr{C}} \left(1 + \sum_{m<0} \mathbf{1}_{t^m\mathcal{O}_v^\times} T_-^m + \sum_{m>0} \mathbf{1}_{t^m\mathcal{O}_v^\times} T_+^m\right).$$

The expansion of this motivic Euler product is indexed by pairs of non-negative integers (n_+, n_-), and the coefficient corresponding to (n_+, n_-) lives in the Grothendieck ring above the variety $\mathbf{Sym}^{(n_+,n_-)}_{/k}(\mathscr{C})_*$ parametrising zero-cycles D on $\mathscr{C}$ of the form

$$D_+ - D_-$$

where D_+, D_- are effective with disjoint supports (hence the subscript $*$), of respective degrees n_+ and n_-. Each coefficient may be viewed as a family of characteristic functions of balls inside the idèles $\mathbb{A}_F^\times$, parametrised by $\mathbf{Sym}^{(n_+,n_-)}\mathscr{C}$, the fibre above $D = \sum_v m_v v$ being the (motivic incarnation of the) function $\mathbf{1}_D := \prod_v \mathbf{1}_{t^{m_v}\mathcal{O}_v^*}$.

In Section 5.3 we defined a summation operator

$$\sum_{P\in\mathrm{PDiv}(\mathscr{C})}$$

which performs a sum over principal divisors.

The relative Cartier dual of $\underline{\mathbf{Z}}_{\mathscr{C}}$ is $\mathbf{G}_{m,\mathscr{C}}$ and the local Fourier transforms are

$$\mathscr{F}\,\mathbf{1}_{t^m\mathcal{O}^\times}(\cdot) = \mathrm{ev}_{\mathscr{C}}(m,\cdot),$$

where

$$\mathrm{ev}_{\mathscr{C}} : \mathbf{Z}_{\mathscr{C}} \times_{\mathscr{C}} \mathbf{G}_{m,\mathscr{C}} \to \mathbf{G}_{m,\mathscr{C}}$$

is the evaluation map (relative to $\mathscr{C}$). Explicitly, $\mathscr{F}\,\mathbf{1}_{t^m\mathcal{O}^\times}(\chi_v) = \chi_v(m)$. Thus,

$$\begin{aligned}\mathscr{F}\,H_v(T,\cdot)(\chi_v) &= 1 + \sum_{m<0}\chi_v(m)T_-^m + \sum_{m>0}\chi_v(m)T_+^m\\ &= \frac{1}{1-\chi_v(1)T_+} + \frac{1}{1-\chi_v^{-1}(1)T_-} - 1\\ &= \frac{1-T_-T_+}{(1-\chi_v(1)T_+)(1-\chi_v^{-1}(1)T_-)}.\end{aligned}$$

Note that this special case has the pecularity that the numerator of the local Fourier transform does not depend on the character. Taking motivic Euler products with characters in the sense of Section 2 provides the motivic Euler product with characters

$$\begin{aligned}\prod_{v\in\mathscr{C}}\mathscr{F}\,H_v(T,\cdot)(\chi_v) &= \prod_{v\in\mathscr{C}}\frac{1-T_-T_+}{(1-\chi_v(1)T_+)(1-\chi_v^{-1}(1)T_-)}\\ &= Z_{\mathscr{C}}^{\mathrm{Kap}}(T_-T_+)^{-1}L(\chi,T_+)L(\chi^{-1},T_-),\end{aligned}$$

where (χ_v) is a family of local characters all living in $\mathbf{G}_m$, and L denotes the motivic L-functions from Section 6.

Applying the motivic Poisson formula to each of the coefficients of this series leads to the expression

$$\sum_{d\in\mathbf{N}}\left[\mathrm{Hom}_k(\mathscr{C},\mathbf{P}^1)_U\right]T^d = (\mathbf{L}-1)\int_{\mathrm{PDiv}(\mathscr{C})^\perp}\mathscr{F}\,H(T,\chi)\mathrm{d}\chi.$$

7.2. **General geometric setting.** Let k be a field and let U be an algebraic torus of dimension n defined over k. We denote by $\mathcal{X}^*(U) = \mathrm{Hom}(U,\mathbf{G}_m)$ its group of characters and $\mathcal{X}_*(U) = \mathrm{Hom}(\mathbf{G}_m,U)$ its group of cocharacters. There is a natural pairing

$$\langle\cdot,\cdot\rangle : \mathcal{X}^*(U)\times\mathcal{X}_*(U)\to\mathbf{Z}. \tag{7.2.4}$$

Let Σ be a projective and regular fan of the $\mathbf{Z}$-module $\mathcal{X}_*(U)$. We denote by $\Sigma(1)$ the set of its rays (that is, its one-dimensional faces), and for every cone $\sigma\in\Sigma$ we denote by $\sigma(1)$ the set of rays of σ. For every $\alpha\in\Sigma(1)$, we denote by ρ_α a generator of the ray α. We denote by $\mathrm{PL}(\Sigma)$ the group of piecewise linear functions on Σ. The pairing (7.2.4) naturally extends to a pairing

$$\langle\cdot,\cdot\rangle_\Sigma : \mathrm{PL}(\Sigma)\times\mathcal{X}_*(U)\to\mathbf{Z}. \tag{7.2.5}$$

given by $\langle\varphi,\mathbf{n}\rangle_\Sigma = \varphi(\mathbf{n})$.

We now assume that $U=\mathbf{G}_m^n$ is split. Then the fan Σ defines a smooth projective split toric variety X_Σ with open orbit U. Each ρ_α gives rise to a U-invariant divisor $\mathfrak{D}_\alpha$ on X_Σ, and the natural map

$$\varphi\mapsto\sum_\alpha\varphi(\rho_\alpha)\mathfrak{D}_\alpha$$

defines an isomorphism $\mathrm{PL}(\Sigma)\simeq\bigoplus_\alpha\mathbf{Z}\mathfrak{D}_\alpha$. We will often use this identification in what follows: in particular, $\mathfrak{D}_\alpha$ represents the piecewise linear function which sends ρ_α to 1 and all other $\rho_\beta,\ \beta\neq\alpha$ to 0.

We then have an exact sequence

$$0\longrightarrow\mathcal{X}^*(U)\xrightarrow{\ \gamma\ }\mathrm{PL}(\Sigma)\longrightarrow\mathrm{Pic}(X_\Sigma)\longrightarrow 0 \tag{7.2.6}$$

where γ sends an element $\boldsymbol{m}\in\mathcal{X}^*(U)$ to the unique φ in $\mathrm{PL}(\Sigma)$ such that $\varphi(\rho_\alpha)=\langle\boldsymbol{m},\rho_\alpha\rangle$ for every $\alpha\in\Sigma(1)$, while the second arrow sends each $\mathfrak{D}_\alpha$ to its class in $\mathrm{Pic}(X_\Sigma)$.

In what follows, we will work with a family of variables $(T_\alpha)_{\alpha\in\Sigma(1)}$ indexed by $\Sigma(1)$. We will write

$$\mathbf{T}^{\langle\cdot,\mathbf{n}\rangle_\Sigma} := \prod_\alpha T_\alpha^{\langle\mathfrak{D}_\alpha,\mathbf{n}\rangle_\Sigma}.$$

Let σ be the cone of Σ containing $\mathbf{n}$. Then, by definition, there is a unique representation $\mathbf{n} = \sum_{\alpha\in\sigma(1)} n_\alpha\rho_\alpha$ for some positive integers $n_\alpha > 0$, and $\mathbf{T}^{\langle\cdot,\mathbf{n}\rangle_\Sigma} = \prod_{\alpha\in\sigma(1)} T_\alpha^{n_\alpha}$.

7.3. **Local and global multiheights.**

7.3.1. *Degree on* $\mathbf{G}_m$. Let $\mathscr{C}$ be a smooth projective geometrically irreducible curve over k, and let $F = k(\mathscr{C})$ its function field. For every closed point $v\in\mathscr{C}$, the completion F_v is a valued field, and the corresponding valuation v induces a degree map:

$$\begin{aligned}\deg_{F,v} : \mathbf{G}_m(F_v) &\to \mathbf{Z}\\ x_v &\mapsto v(x_v)\end{aligned}$$

Combining these for all places v, we get the global degree:

$$\mathbf{div}_F : \begin{cases} \mathbf{G}_m(\mathbb{A}_F) & \longrightarrow \quad \mathrm{Div}(\mathscr{C}) \\ \quad (x_v) & \longmapsto \quad \sum_{v\in|\mathscr{C}|}\deg_{F,v}(x_v)[v] \end{cases}$$

Now, rational points $x\in\mathbf{G}_m(F)$ are in one-to-one correspondence with morphisms $\tilde{x} : \mathscr{C}\to\mathbf{P}^1$ with image intersecting $\mathbf{G}_m$. The divisor $\mathbf{div}_F(x)$ is then entirely determined by the pair $(\tilde{x}^*\{0\},\tilde{x}^*\{\infty\})$ of effective divisors with disjoint supports, and

$$\mathbf{div}_F : \begin{cases} \mathbf{Hom}_k(\mathscr{C},\mathbf{P}^1_k)_{\mathbf{G}_m} & \longrightarrow \quad (\mathbf{Div}_+(\mathscr{C})\times\mathbf{Div}_+(\mathscr{C}))_* \\ \quad (\tilde{x} : \mathscr{C}\to\mathbf{P}^1_k) & \longmapsto \quad (\tilde{x}^*\{0\},\tilde{x}^*\{\infty\}). \end{cases}$$

realises $\mathbf{Hom}_k(\mathscr{C},\mathbf{P}^1_k)_{\mathbf{G}_m}$ as a $\mathbf{G}_m$-torsor.

7.3.2. *Degree on an algebraic torus.* Let U be an algebraic torus of dimension n. One can compose $\mathbf{div}_F$ with any $\boldsymbol{m}\in\mathcal{X}^*(U)$. This provides a new morphism

$$\mathbf{div}_U : \begin{cases} U(\mathbb{A}_F) & \longrightarrow \quad \mathrm{Hom}_{\mathrm{gp}}(\mathcal{X}^*(U),\mathbf{Div}(\mathscr{C})) = \oplus_{v\in|\mathscr{C}|}\mathcal{X}_*(U) \\ \quad (x_v) & \longmapsto \quad [\boldsymbol{m}\mapsto\mathbf{div}_F(\boldsymbol{m}((x_v)))]\,. \end{cases}$$

By composing with the Abel-Jacobi morphism, we also get a map

$$\overline{\mathbf{div}_U} \begin{cases} U(\mathbb{A}_F) & \longrightarrow \quad \mathrm{Hom}_{\mathrm{gp}}(\mathcal{X}^*(U),\mathrm{Pic}(\mathscr{C})) \\ \quad (x_v) & \longmapsto \quad [\boldsymbol{m}\mapsto[\mathbf{div}_F(\boldsymbol{m}((x_v)))]]\,. \end{cases}$$

The fiber above $0\in\mathrm{Hom}_{\mathrm{gp}}(\mathcal{X}^*(U),\mathrm{Pic}(\mathscr{C}))$ is exactly the set $U(F)$.

In case U is split, we can fix an isomorphism $U\simeq\mathbf{G}_m^n$ inducing $\mathcal{X}_*(U)\simeq\mathbf{Z}^n$. Then the previous morphism is easier to describe geometrically: any $x\in U(F)$ induces a canonical lift $\mathscr{C}\to(\mathbf{P}^1_k)^n$ whose i-th coordinate $\tilde{Y}_i$ does not factor through the boundary divisors $\{0_i\},\{\infty_i\}$ of the i-th factor $\mathbf{P}^1_k$, for $i=1,...,n$. The morphism $\mathbf{div}_U$ is then given by

$$\mathbf{div}_{\mathbf{G}_m^n} : \begin{cases} \mathbf{Hom}_k(\mathscr{C},(\mathbf{P}^1_k)^n)_{\mathbf{G}_m^n} & \longrightarrow \quad (\mathbf{Div}_+(\mathscr{C})\times\mathbf{Div}_+(\mathscr{C}))^n_* \\ \quad (\tilde{x} : \mathscr{C}\to(\mathbf{P}^1_k)^n) & \longmapsto \quad (\tilde{Y}_i^*\{0_i\},\tilde{Y}_i^*\{\infty_i\})_{i=1}^n. \end{cases}$$

In any case, the morphism $\mathbf{deg}_U$ is obtained by composing $\mathbf{div}_U$ with the degree morphism

$$\begin{cases} \oplus_{v\in|\mathscr{C}|}\mathcal{X}_*(U) & \longrightarrow \quad \mathcal{X}_*(U) \\ \sum_{v\in|\mathscr{C}|}\boldsymbol{m}_v[v] & \longmapsto \quad \sum_{v\in|\mathscr{C}|}\boldsymbol{m}_v\deg(\kappa(v):k). \end{cases}$$

7.4. **Motivic zeta function.**

$$\begin{array}{ccc} \mathcal{X}^*(U) \times \mathcal{X}_*(U) & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathrm{PL}(\Sigma) \times \mathcal{X}_*(U) & \longrightarrow & \mathbf{Z} \end{array}$$

For every x, we can see $\mathbf{deg}_U(x)$ as an element of $\mathcal{X}(U)^\vee$ and $\mathrm{Pic}(X_\Sigma)^\vee$ as a sublattice of $\mathrm{PL}(\Sigma)^\vee$ via the dual exact sequence

$$0 \to \mathrm{Pic}(X_\Sigma)^\vee \to \mathrm{PL}(\Sigma)^\vee \to \mathcal{X}_*(U) \to 0$$

of (7.2.6).

Definition 7.1. The formal height of a point $x \in U(F)$, corresponding to $\tilde{x} : \mathscr{C} \to (\mathbf{P}^1_k)^n$, is defined as

$$H(x, \mathbf{T}) = \mathbf{T}^{\mathbf{deg}_U(\tilde{x})}.$$

Definition 7.2. The motivic height zeta function is the motivic series

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = \sum_{\tilde{x} \in \mathbf{Hom}_k\left(\mathscr{C}, (\mathbf{P}^1_k)^n\right)_U} H(\tilde{x}, \mathbf{T}).$$

It should be interpreted as $\sum_{x \in U(F)} H(x, \mathbf{T})$.

Proposition 7.3. *The motivic height zeta function can be rewritten*

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = \sum_{\delta \in \mathrm{Pic}(X_\Sigma)^\vee} \left[\mathbf{Hom}_k^\delta(\mathscr{C}, X_\Sigma)_U\right] \mathbf{T}^\delta.$$

Proof. It follows from the definition of the toric degree and the fact that $\mathbf{Hom}_k(\mathscr{C}, X_\Sigma)_U$ and $\mathbf{Hom}_k\left(\mathscr{C}, (\mathbf{P}^1_k)^n\right)_U$ represent two functors that are equivalent. □

7.5. **Adelic height zeta function as a motivic Euler product.**

Definition 7.4. The motivic adelic height function of an adelic point $x \in U(\mathbf{A}_F)$ is the product

$$H(\boldsymbol{x}, \mathbf{T}) = \prod_{v \in \mathscr{C}} \mathbf{T}_v^{\langle \deg_U(\boldsymbol{x}_v), \,\cdot\, \rangle} = \prod_{v \in \mathscr{C}} \left(\sum_{\boldsymbol{m}_v \in \mathcal{X}_*(U)} \mathbf{1}_{t_v^{\boldsymbol{m}_v} U(\mathcal{O}_v)}(\boldsymbol{x}_v) \mathbf{T}_v^{\langle \boldsymbol{m}_v, \,\cdot\, \rangle} \right).$$

Remark 7.5. Since $\boldsymbol{x}_v \in U(\mathcal{O}_v)$ for all but finitely many $v \in \mathscr{C}$, the product makes sense as it is actually finite. It also makes sense as a global motivic Euler product viewed above $\mathbf{Div}(\mathscr{C})^n$ since

$$\sum_{\boldsymbol{m}_v \in \mathcal{X}_*(U)} \mathbf{1}_{t_v^{\boldsymbol{m}_v} U(\mathcal{O}_v)} \mathbf{T}_v^{\langle \boldsymbol{m}_v, \,\cdot\, \rangle} = 1 + \sum_{\substack{\boldsymbol{m}_v \in \mathcal{X}_*(U) \\ \boldsymbol{m}_v \neq \mathbf{0}}} \mathbf{1}_{t_v^{\boldsymbol{m}_v} U(\mathcal{O}_v)} \mathbf{T}_v^{\langle \boldsymbol{m}_v, \,\cdot\, \rangle}.$$

Using the definition of $\mathbf{div}_{\mathbf{G}_m^n}$, we obtain the following.

Proposition 7.6. *The motivic height zeta function $\zeta_H^{\mathrm{mot}}(\mathbf{T})$ coincides with the motivic sum on $\mathbf{PDiv}(\mathscr{C})$ of the motivic height function $H^{\mathrm{mot}}(\,\cdot\,, \mathbf{T})$ times $[U]_k$, that is to say*

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = [U]_k \sum_{\boldsymbol{D} \in \mathbf{PDiv}(\mathscr{C})^n} H^{\mathrm{mot}}(\boldsymbol{D}, \mathbf{T}). \tag{7.5.7}$$

7.6. Application of the Poisson formula.

Lemma 7.7. *For any family $\boldsymbol{\chi}$ of global characters,*

$$(\mathscr{F}\, H)(\boldsymbol{\chi}, \mathbf{T}) = \prod_{v\in\mathscr{C}} (\mathscr{F}\, H_v)(\boldsymbol{\chi}_v, \mathbf{T}_v) = \prod_{v\in\mathscr{C}} \left(\sum_{\boldsymbol{m}_v \in \mathcal{X}_*(U)} \boldsymbol{\chi}_v(\boldsymbol{m}_v) \mathbf{T}_v^{\langle \boldsymbol{m}_v, \cdot \rangle} \right).$$

An application of Theorem 5.11 to each of the coefficients of the series defining $H^{\mathrm{mot}}(\cdot, \mathbf{T})$ above $\mathbf{Div}(\mathscr{C})^n$ provides the following expression for $\zeta_H^{\mathrm{mot}}(\mathbf{T})$. We write

$$\int_{\mathbf{PDiv}(\mathscr{C})^\perp} = \sum_{\pi} \int_{\mathbf{PDiv}(\mathscr{C})^\perp_\pi}.$$

Proposition 7.8. *The multivariate motivic height zeta function $\zeta_H^{\mathrm{mot}}(\mathbf{T})$ can be rewritten*

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = (\mathbf{L}-1)^n \int_{\mathbf{PDiv}(\mathscr{C})^\perp} (\mathscr{F}\, H^{\mathrm{mot}})(\boldsymbol{\chi}, \mathbf{T}) \mathrm{d}\boldsymbol{\chi}. \tag{7.6.8}$$

7.7. The local factors of the Fourier transform.

Definition 7.9. As in [BT98, Definition 3.6] and [Bou11a] we define a polynomial $Q_\Sigma(\mathbf{X})$ with $\mathbf{X} = (X_\alpha)_{\alpha\in\Sigma(1)}$ via the relation

$$\sum_{\sigma\in\Sigma} \prod_{\alpha\in\sigma(1)} \frac{X_\alpha}{1-X_\alpha} = \frac{Q_\Sigma(\mathbf{X})}{\prod_{\alpha\in\Sigma(1)}(1-X_\alpha)}.$$

The following proposition is uniform in v, hence it holds in our rings of varieties over $\mathscr{C}$. It is the motivic analogue of [BT95, Theorem 2.2.6] quoted as Théorème 4.17 in [Bou11a]. Remember that algebraic characters on the group scheme $\mathbf{Z}^n_{\kappa(v)}$ play the role of characters on the quotient $U(F_v)/U(\mathcal{O}_v)$. Recall also that $\rho_\alpha \in \mathcal{X}_*(T)$ is the generator of the ray given by $\alpha \in \Sigma(1)$, so the evaluation by $\boldsymbol{\chi}_v$ makes sense.

Proposition 7.10. *We have*

$$(\mathscr{F}\, H_v(\mathbf{T}_v))(\boldsymbol{\chi}_v) = \frac{Q_\Sigma(\boldsymbol{\chi}_v \mathbf{T})}{\prod_{\alpha\in\Sigma(1)}(1-\boldsymbol{\chi}_v(\rho_\alpha)T_\alpha)}$$

relatively to $\boldsymbol{\chi}_v \in \widehat{\mathcal{X}_*(T)}_{\kappa(v)} = \mathbf{G}^n_{m,\kappa(v)}$ and $v \in \mathscr{C}$.

Proof. Indeed, by definition of the local Fourier transform,

$$\begin{aligned}
(\mathscr{F}\, H_v(\mathbf{T}_v))(\boldsymbol{\chi}_v) &= \sum_{\boldsymbol{m}_v\in\mathcal{X}_*(T)_{\kappa(v)}} H_v(\mathbf{T}_v, \boldsymbol{m}_v)\boldsymbol{\chi}_v(\boldsymbol{m}_v) \\
&= \sum_{\sigma\in\Sigma} \sum_{(n_\alpha)\in\mathbf{Z}^{\sigma(1)}_{>0}} H_v\left(\mathbf{T}_v, \sum_\alpha n_\alpha\rho_\alpha\right) \prod_{\alpha\in\sigma(1)} \boldsymbol{\chi}_v(\rho_\alpha)^{n_\alpha} \\
&= \sum_{\sigma\in\Sigma} \prod_{\alpha\in\sigma(1)} \frac{\boldsymbol{\chi}_v(\rho_\alpha)T_{v,\alpha}}{1-\boldsymbol{\chi}_v(\rho_\alpha)T_{v,\alpha}}
\end{aligned}$$

hence the claim, applying the definition of Q_Σ to $X_\alpha = \boldsymbol{\chi}_v(\rho_\alpha)T_{v,\alpha}$. □

Proposition 7.11. *The multivariate motivic height zeta function $\zeta_H^{\mathrm{mot}}(\mathbf{T})$ can be rewritten*

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = (\mathbf{L}-1)^n \int_{\mathbf{PDiv}(\mathscr{C})^\perp} \prod_{v\in\mathscr{C}} \frac{Q_\Sigma(\boldsymbol{\chi}_v\mathbf{T})}{\prod_{\alpha\in\Sigma(1)}(1-\boldsymbol{\chi}_v(\rho_\alpha)T_\alpha)} \mathrm{d}\boldsymbol{\chi}. \tag{7.7.9}$$

7.8. **Splitting of $\mathbf{PDiv}(\mathscr{C})^\perp$.** We separate the part of characters depending on the degree from the part depending only on the degree-zero linear class. This manipulation is very classical but we have to explain how it fits into our motivic setting.

Recall that we fixed once and for all a k-divisor $\mathfrak{D}_1 = \sum \mathfrak{d}_v[v]$ of degree one on $\mathscr{C}$. Any support S being given as in Section 4.2, we can always enlarge it to include the support of $\mathfrak{D}_1$. Characters $\boldsymbol{z} : \boldsymbol{m} \mapsto \boldsymbol{z}^{\boldsymbol{m}}$ on $M = \mathcal{X}_*(U) \simeq \mathbf{Z}^n$ will be identified with

$$\boldsymbol{m} \mapsto \boldsymbol{\chi}(\boldsymbol{m}\mathfrak{D}_1) = \left(\prod_{v\in|\mathfrak{D}_1|} \chi_v(\mathfrak{d}_v[v])\right)^{\boldsymbol{m}}$$

via

$$\boldsymbol{z} = \prod_{v\in|\mathfrak{D}_1|} \chi_v(\mathfrak{d}_v\cdot).$$

We are able to write any divisor D supported on S as

$$D = (D_+ - \deg(D_+)\mathfrak{D}_1) - (D_- - \deg(D_-)\mathfrak{D}_1) + \deg(D)\mathfrak{D}_1$$

so that for any family of characters $\boldsymbol{\chi}$ we have

$$\boldsymbol{\chi}(D) = \boldsymbol{\chi}(D_+ - \deg(D_+)\mathfrak{D}_1)\boldsymbol{\chi}((D_- - \deg(D_-)\mathfrak{D}_1))^{-1}\boldsymbol{z}^{\deg(D)}.$$

In other words we perform the change of variables

$$\boldsymbol{\chi} \in \mathrm{D}(\mathrm{Div}_{\mathsf{S}}(\mathscr{C})) \mapsto \left(\underbrace{\boldsymbol{\chi}(\,\cdot\, - \deg(\,\cdot\,)\mathfrak{D}_1))}_{=:\boldsymbol{\chi}'}, \underbrace{\boldsymbol{\chi}(\deg(\,\cdot\,)\mathfrak{D}_1)}_{=:\boldsymbol{z}}\right) \in \mathrm{D}(\mathrm{Div}^0_{\mathsf{S}}(\mathscr{C})) \times \mathrm{D}(M).$$

Moreover, Example 2.6 allows us to move $\boldsymbol{z}$ outside the motivic Euler product.

This way we are able to write:

$$\int_{\mathbf{PDiv}(\mathscr{C})^\perp} = \int_{D((\mathrm{Div}(\mathscr{C})/\,\mathrm{Div}^0(\mathscr{C}))^n} \boxtimes \int_{\mathbf{PDiv}^0(\mathscr{C})^\perp}$$

Lemma 7.12. *Under the previous splitting,*

$$(\mathscr{F} H_v(\mathbf{T}_v))(\boldsymbol{\chi}'_v, \boldsymbol{z}) = \frac{Q_\Sigma(\boldsymbol{\chi}'_v \prod_\alpha \langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)}{\prod_{\alpha\in\Sigma(1)}(1 - \boldsymbol{\chi}'_v(\rho_\alpha)\prod_\alpha \langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)}.$$

In the sequel we write $\boldsymbol{\chi}$ for $\boldsymbol{\chi}'$. The previous discussion leads to the following.

Proposition 7.13. *The multivariate motivic height zeta function $\zeta_H^{\mathrm{mot}}(\mathbf{T})$ can be rewritten*

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = (\mathbf{L}-1)^n \int_{D(\mathcal{X}_*(U))} \int_{\mathbf{PDiv}^0(\mathscr{C})^\perp} \prod_{v\in\mathscr{C}} \frac{Q_\Sigma(\boldsymbol{\chi}_v \prod_\alpha \langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)}{\prod_{\alpha\in\Sigma(1)}(1 - \boldsymbol{\chi}_v(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)} \mathrm{d}\boldsymbol{\chi}\mathrm{d}\boldsymbol{z}. \tag{7.8.10}$$

8. Analysis of the integral over the Fourier domain

8.1. **Motivic $\mathfrak{L}$-functions of free Z-modules and indicator functions of cones.** We develop an elementary motivic analogue of the theory from [Bou11a, Chap. 6]. We keep most of Bourqui's notations here. Some of the proofs are very similar to the ones in the classical setting; some of them are included for the reader's convenience.

Conventions are the following: in the whole section, N is a free $\mathbf{Z}$-module, and we denote by

$$\langle\cdot,\cdot\rangle : N^\vee \times N \to \mathbf{Z}$$

the pairing with $N^\vee$. For a cone $\Lambda \subset N_{\mathbf{R}}$, we denote by $\overset{\circ}{\Lambda}$ its relative interior. In what follows, Υ will always denote rational polyhedral cone in $N_{\mathbf{R}}$ (i.e. generated by a finite number of elements of N). Such a cone may be written as the support of a regular fan.

8.1.1. *$\mathfrak{L}$-function associated to a cone.*

Definition 8.1. Let S be a scheme, N a free $\mathbf{Z}$-module of finite rank and N_S its associated constant group scheme over S.

For every strictly convex rational polyhedral cone Υ in $N_{\mathbf{R}}$, we define the $\mathfrak{L}$-function associated to N relative to S to be the formal power series

$$\mathfrak{L}_{N,\Upsilon}(\mathbf{T}) = \sum_{y\in N\cap\Upsilon} \mathbf{T}^y \in \mathrm{K}_0\mathbf{Var}_S[[N\cap\Upsilon]] = \mathrm{K}_0\mathbf{Var}_S[[\mathbf{T}]].$$

It is the formal power series associated to the motivic function $\mathbf{1}_{N\cap\Upsilon} \in \mathrm{K}_0\mathbf{Var}_{N_S}$.

This definition naturally extends to all the variants and localisations of the Grothendieck ring of varieties we use in the present work. In particular,

$$\mathfrak{L}_{N,\Upsilon}(\boldsymbol{\chi}\mathbf{T}) = \sum_{y\in N\cap\Upsilon} \boldsymbol{\chi}(y)\mathbf{T}^y \in \mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_S[[N\cap\Upsilon]].$$

More generally, if

$$\mathfrak{a} \in \mathrm{K}_0\mathbf{Char}_N\mathbf{Var}_{N_S}$$

is a motivic function with characters on N_S and A is a subset of $N_{\mathbf{R}}$, we set

$$\mathfrak{L}_{N,A,\mathfrak{a}} = \sum_{y\in N\cap A} \mathfrak{a}(y)\mathbf{T}^y.$$

If $\mathfrak{a}$ is identically 1, we simply write $\mathfrak{L}_{N,A}$.

Lemma 8.2. *If λ_0 lies in the relative interior of $\Upsilon^\vee$, the series*

$$\mathfrak{L}_{N,\Upsilon}\left(T^{\lambda_0}\right) = \sum_{y\in N\cap\Upsilon} T^{\langle\lambda_0,y\rangle}$$

converges for $|T| < 1$ and admits a pole at $T = 1$ of order at most $\dim(\Upsilon)$*. Moreover, if* $\dim(\Upsilon) = \mathrm{rk}(N)$*, there exists a positive integer a_{λ_0} such that*

$$\left((1 - T^{a_{\lambda_0}})^{\mathrm{rk}(N)}\mathfrak{L}_{N,\Upsilon}\left(T^{\lambda_0}\right)\right)_{T=1} = a_{\lambda_0}^{\mathrm{rk}(N)}\mathfrak{X}_{N,\Upsilon^\vee}(\lambda_0) \in \mathbf{N}$$

where

$$\mathfrak{X}_{N,\Upsilon^\vee}(\lambda) = \int_\Upsilon \exp(-\langle y,\lambda\rangle)\mathrm{d}y$$

with $\mathrm{d}y$ *being the Lebesgue measure on $N_{\mathbf{R}}^\vee$ normalised by $N^\vee$.*

Proof. We can write Υ as the support of a regular fan Δ to get

$$\mathfrak{L}_{N,\Upsilon}\left(T^{\lambda_0}\right) = \sum_{\delta\in\Delta}\prod_{\ell\in\delta(1)}\left(\frac{1}{1 - T^{\langle\lambda_0,\rho_\ell\rangle}} - 1\right)$$

where ρ_ℓ is the generator corresponding to $\ell \in \Delta(1)$. Furthermore, one can show [Bou11a, Remarque 5.3] that

$$\mathfrak{X}_{N,\Upsilon^\vee}(\lambda_0) = \sum_{\substack{\delta\in\Delta\\ \dim(\delta)=\mathrm{rk}(N)}}\prod_{\ell\in\delta(1)}\frac{1}{\langle\lambda_0,\rho_\ell\rangle}.$$

Therefore we take

$$a_{\lambda_0} = \mathrm{lcm}_{\ell\in\delta(1)}\langle\lambda_0,\rho_\ell\rangle$$

and check that

$$(1 - T^{a_{\lambda_0}})^{\mathrm{rk}(N)} \sum_{\substack{\delta\in\Delta\\ \dim(\delta)=\mathrm{rk}(N)}}\prod_{\ell\in\delta(1)}\left(\frac{1}{1 - T^{\langle\lambda_0,\rho_\ell\rangle}} - 1\right)$$

takes value

$$a_{\lambda_0}^{\mathrm{rk}(N)}\mathfrak{X}_{N,\Upsilon^\vee}(\lambda_0)$$

at $T=1$. Moreover it is clear from the definitions that $a_{\lambda_0}^{\mathrm{rk}(N)}\mathfrak{X}_{N,\Upsilon^\vee}(\lambda_0)$ is a non-negative integer. □

We assume that we are given an exact sequence of $\mathbf{Z}$-modules, all free of finite rank,

$$0 \longrightarrow M \xrightarrow{i} N \xrightarrow{j} \Gamma \longrightarrow 0.$$

This exact sequence induces an exact sequence of dual modules

$$0 \longrightarrow \Gamma^\vee \xrightarrow{j^\vee} N^\vee \xrightarrow{i^\vee} M^\vee \longrightarrow 0$$

and an exact sequence of algebraic tori

$$0 \longrightarrow D(\underline{\Gamma}_S) \xrightarrow{D(j_S)} D(\underline{N}_S) \xrightarrow{D(i_S)} D(\underline{M}_S) \longrightarrow 0.$$

This allows us to give a meaning to:

$$\mathfrak{L}_{N,\Lambda\cap M_{\mathbf{R}}}(\mathbf{T}) = \sum_{y\in\Lambda\cap i(M)\cap N}\mathbf{T}^y = \sum_{y\in\Lambda\cap M}\mathbf{T}^{i(y)} = \sum_{y\in\Lambda\cap M} i^\vee(\mathbf{T}^y) = \mathfrak{L}_{M,\Lambda\cap M_{\mathbf{R}}}(i^\vee(\mathbf{T})).$$

The following is a motivic version of Bourqui's "lemme technique : forme jouet" [Bou11a, Lemme 6.14].

Lemma 8.3. *Let S be a scheme. We have*

$$\int_{D(\underline{\Gamma}_S)}\mathfrak{L}_{N,\Lambda}(D(j_S)(\boldsymbol{\chi})\mathbf{T})\mathrm{d}\boldsymbol{\chi} = \mathfrak{L}_{N,\Lambda\cap i(M_{\mathbf{R}})}(\mathbf{T})$$

in $\mathrm{K}_0\mathbf{Var}_S[[N\cap\Upsilon]]$.

Proof. The proof is completely formal and goes exactly as for the one of [Bou11a, Lemma 6.14]. We can directly compute the left hand side to get

$$\begin{aligned}\int_{D(\underline{\Gamma}_S)}\mathfrak{L}_{N,\Lambda}(D(j_S)(\boldsymbol{\chi})\mathbf{T})\mathrm{d}\boldsymbol{\chi} &= \int_{D(\underline{\Gamma}_S)}\sum_{y\in\Lambda\cap N}\boldsymbol{\chi}(j(y))\mathbf{T}^y\mathrm{d}\boldsymbol{\chi}\\ &= \sum_{y\in\Lambda\cap N}\mathbf{T}^y\int_{D(\underline{\Gamma}_S)}\boldsymbol{\chi}(j(y))\mathrm{d}\boldsymbol{\chi}\\ &= \sum_{\substack{y\in\Lambda\cap N\\ j(y)=0}}\mathbf{T}^y \qquad \text{(by def. of } \textstyle\int_{D(\underline{\Gamma}_S)}\text{)}\\ &= \mathfrak{L}_{N,\Lambda\cap i(M_{\mathbf{R}})}(\mathbf{T})\end{aligned}$$

which is what we wanted. □

Notation 8.4. From now on, we fix once and for all a basis $(\lambda_i)_{i\in I}$ of N and take Λ to be the cone generated by this basis.

Definition 8.5. An *elementary admissible motivic function of non-negative multiplicity* is a motivic formal series

$$\mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\mathbf{T}) = \sum_{y\in N\cap\Upsilon}\mathfrak{a}(y)\mathbf{T}^y$$

converging for $|\mathbf{T}|<\mathbf{L}^\varepsilon$ for some $\varepsilon>0$.

Whenever r is a non-negative integer, an *elementary admissible motivic function of multiplicity at least* $-r$ is a motivic function f on Λ admitting a decomposition

$$f(\mathbf{T}) = g(\mathbf{T})\mathfrak{L}_{N',\mathring{\Upsilon},\mathfrak{a}}(\mathbf{T})$$

where g is an elementary admissible function of non-negative multiplicity, Υ is a rational polyhedral subcone of $N_{\mathbf{R}}$ of dimension at most r and N' is a subgroup of N.

Admissible motivic functions of multiplicity at least $-r$ are finite linear combination of such elementary functions.

8.1.2. *Decomposition lemmas and bounds on degrees.* Recall that $\Lambda \subset N_{\mathbf{R}}$ is the simplicial cone generated by $(\lambda_{i\in I})$. Let $(\lambda_i^\vee)_{i\in I}$ be the basis of $N^\vee$ induced by $(\lambda_i)_{i\in I}$ and define $\lambda^\vee = \sum_{i\in I}\lambda_i^\vee$. We also fix a rational polyhedral cone $\Upsilon \subset N_{\mathbf{R}}$ contained in Λ, and write Υ as the support of a regular fan Δ. Following Bourqui [Bou11a, §6.4.1], we define for any subset $K \subset I$, any cone $\delta \in \Delta$ and any element $z \in \Lambda \cap N$,

$$\delta(K,z) = \{y \in \mathring{\delta} \mid \forall\, i \in K, \quad \langle \lambda_i^\vee, y\rangle < \langle \lambda_i^\vee, z\rangle\}$$

as well as

$$\delta(1)_K = \{\ell \in \delta(1) \mid \forall\, i \in K, \quad \langle \lambda_i^\vee, \rho_\ell\rangle = 0\}.$$

We denote by δ_K the cone spanned by rays in $\delta(1)_K$ and by δ^K the cone spanned by rays of δ not in $\delta(1)_K$.

The following lemma is an immediate extension of [Bou11a, Lemme 6.5] to our setting.

Lemma 8.6. *For any $z \in \Lambda \cap N$ we have*

$$\mathfrak{L}_{N,\Upsilon\cap(z+\Lambda)}(\mathbf{T}) = \sum_{\substack{\delta\in\Delta \\ K\subset I}} (-1)^{|K|}\mathfrak{L}_{N,\delta(K,z)}(\mathbf{T}). \tag{8.1.11}$$

We will also reemploy the following (see [Bou11a, p. 123] for a proof).

Lemma 8.7 ([Bou11a, Lemme 6.5])**.** *Let δ be a cone of Δ and $K \subset I$.*

(1) *There is a decomposition*

$$\delta(K,z) = \left(\mathring{\delta}_K \cap N\right) \oplus \left(\delta^K(K,z) \cap N\right).$$

(2) *The set $\delta^K(K,z)\cap N$ is finite, of cardinality bounded by*

$$\langle z, \lambda^\vee\rangle^{\mathrm{rk}(N)}.$$

(3) *Moreover, for any $y \in \delta^K(K,z)\cap N$ we have*

$$\langle y, \lambda^\vee\rangle \leqslant |I| \cdot \langle z, \lambda^\vee\rangle \cdot_{\ell\in\Delta(1)} \langle \rho_\ell, \lambda^\vee\rangle.$$

(4) *If K is empty, then $\delta(1)_K = \delta(1)$ and $\delta^K(K,z)\cap N = \{0\}$.*

Lemma 8.8 ([Bou11a, Lemme 6.6])**.** *Let δ be a cone of Δ. Assume that*

(1) *δ has maximal dimension,*
(2) *K is a non-empty subset of I,*
(3) *and $(N/\langle\Upsilon\rangle)^\vee \cap \lambda^\vee = \{0\}$.*

Then $\delta(1)_K$ is a proper *subset of $\delta(1)$.*

8.1.3. *Convergence lemmas.*

Lemma 8.9. *Let $\mathfrak{a}$ be a motivic function on $\Lambda \cap N$ relatively to a scheme S and $\varepsilon > 0$ be such that $\mathfrak{L}_{N,\Lambda,\mathfrak{a}}$ converges for $|\mathbf{T}| < \mathbf{L}^{\varepsilon}$. In particular, it implies that the sum*

$$\mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\mathbf{1}) = \sum_{y \in \Lambda \cap N} \mathfrak{a}(y)$$

is well-defined in the dimensional completion.

Let

$$f_1(\mathbf{T}) = \int_{D(\underline{\Gamma}_S)} \mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\hat{j}(\boldsymbol{\chi})\mathbf{T})\mathfrak{L}_{N,\Lambda}(\hat{j}(\boldsymbol{\chi})\mathbf{T})\mathrm{d}\boldsymbol{\chi} \in \mathrm{K}_0\mathbf{Var}_S[[\mathbf{T}]].$$

This motivic formal series converges for $|\mathbf{T}| < 1$.

Assume that $\Gamma^{\vee} \cap \Lambda^{\vee} = \{0\}$. Then the series

$$f_1(\mathbf{T}) - \mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\mathbf{1})\mathfrak{L}_{N,\Lambda\cap M_{\mathbf{R}}}(\mathbf{T})$$

is an admissible motivic function of multiplicity at least $-\operatorname{rk}(M) + 1$.

Proof. First, one shows that

$$f_1(\mathbf{T}) = \sum_{y_1 \in \Lambda \cap N} \mathfrak{a}(y_1)\mathfrak{L}_{N,\Lambda\cap M\cap(y_1+\Lambda)}(\mathbf{T}).$$

Indeed, exactly as in the proof of [Bou11a, Proposition 6.15], we can write

$$\begin{aligned}
f_1(\mathbf{T}) &= \int_{D(\underline{\Gamma}_S)} \mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\hat{j}(\boldsymbol{\chi})\mathbf{T})\mathfrak{L}_{N,\Lambda}(\hat{j}(\boldsymbol{\chi})\mathbf{T})\mathrm{d}\boldsymbol{\chi} \\
&= \sum_{(y_0,y_1)\in(\Lambda\cap N)^2} \mathfrak{a}(y_1)\mathbf{T}^{y_0+y_1} \int_{D(\underline{\Gamma}_S)} \boldsymbol{\chi}(j(y_0+y_1))\mathrm{d}\boldsymbol{\chi} \\
&= \sum_{y_1 \in (\Lambda\cap N)} \mathfrak{a}(y_1) \sum_{\substack{y_0 \in \Lambda\cap N \\ j(y_0+y_1)=0}} \mathbf{T}^{y_0+y_1} \\
&= \sum_{y_1 \in (\Lambda\cap N)} \mathfrak{a}(y_1) \sum_{\substack{y \in \Lambda\cap N \\ y \in y_1+\Lambda\cap N}} \mathbf{T}^{y} \\
&= \sum_{y_1 \in \Lambda\cap N} \mathfrak{a}(y_1)\mathfrak{L}_{N,\Lambda\cap M\cap(y_1+\Lambda)}(\mathbf{T}).
\end{aligned}$$

One can write $\Upsilon = \Lambda \cap M_{\mathbf{R}}$ as the support of a regular fan Δ and use Lemma 8.6 to get

$$f_1(\mathbf{T}) = \sum_{\delta\in\Delta}\sum_{J\subset I}(-1)^{|J|}\mathfrak{L}_{N,\overset{\circ}{\delta}_J}(\mathbf{T}) \sum_{y_1\in\Lambda\cap N} \mathfrak{a}(y_1)\mathfrak{L}_{N,\delta^J(J,y_1)}(\mathbf{T}).$$

The main term of this series is given by $J = \varnothing$ and equals

$$\mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\mathbf{1})\mathfrak{L}_{M,\Lambda\cap M_{\mathbf{R}}}(\mathbf{T}).$$

If δ and $J \neq 0$ are given, we know by Lemma 8.7 that for any $y_1 \in \Lambda \cap N$ there are at most $\langle y_1, \lambda^{\vee}\rangle^{\operatorname{rk}(N)}$ elements in $\delta^J(J, y_1)$ and that for any such element $y \in \delta^J(J, y_1)$, the scalar product $\langle y, \lambda^{\vee}\rangle$ is bounded by $|I| \cdot \langle y_1, \lambda^{\vee}\rangle \cdot_{\ell\in\Delta(1)} \langle \rho_\ell, \lambda^{\vee}\rangle$. It means in particular that the series

$$\sum_{y_1\in\Lambda\cap N} a_{y_1}\mathfrak{L}_{N,\delta^J(J,y_1)}(\mathbf{T})$$

is an elementary admissible function of positive multiplicity. Finally, since we assume that $\Gamma^\vee \cap \Lambda^\vee = \{0\}$, Lemma 8.8 ensures that $\delta(1)_J$ is a proper subset of $\delta(1)$. This implies that poles of $\mathfrak{L}_{N,\overset{\circ}{\delta}_K}(T)$ are controlled up to order $-(\mathrm{rk}(M)-1)$ and finally that

$$\mathfrak{L}_{N,\overset{\circ}{\delta}_J}(\mathbf{T}) \sum_{y_1 \in \Lambda \cap N} \mathfrak{a}(y_1) \mathfrak{L}_{N,\delta^J(J,y_1)}(\mathbf{T})$$

is admissible of multiplicity at least $-(\mathrm{rk}(M)-1)$. □

Since the order of the pole of $\mathfrak{L}_{M,\Lambda\cap M_{\mathbf{R}}}(\mathbf{T})$ is exactly $\mathrm{rk}(M)$, we get:

Corollary 8.10. *Let λ_0 be in the interior of $\Lambda^\vee$.*

$$\left((1-T^{a_{\lambda_0}})^{\mathrm{rk}(M)} f_1\left(T^{\lambda_0}\right)\right)_{T=1} = a_{\lambda_0}^{\mathrm{rk}(M)} \mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\mathbf{1}) \mathfrak{X}_{M^\vee, i^\vee(\Lambda^\vee)}(i^\vee(\lambda_0)).$$

8.2. **Application to the motivic height zeta function.** In our application, the short exact sequence

$$0 \longrightarrow M \xrightarrow{i} N \xrightarrow{j} \Gamma \longrightarrow 0$$

is given by the short exact sequence of $\mathbf{Z}$-modules

$$0 \longrightarrow \mathrm{Pic}(X_\Sigma)^\vee \xrightarrow{\pi^\vee} \mathrm{PL}(\Sigma)^\vee \xrightarrow{\gamma^\vee} \mathcal{X}_*(U) \longrightarrow 0.$$

dual to Equation (7.2.6). We also get for every scheme S an exact sequence of diagonalisable algebraic groups (here split tori)

$$1 \longrightarrow D(\underline{\mathcal{X}_*(U)}_S) \xrightarrow{D(\gamma^\vee)} D(\underline{\mathrm{PL}(\Sigma)^\vee}_S) \xrightarrow{D(\pi^\vee)} D(\underline{\mathrm{Pic}(X_\Sigma)^\vee}_S) \longrightarrow 1.$$

Our cone $\Lambda \subset N = \mathrm{PL}(\Sigma)$ will be the cone generated by the dual basis $\mathfrak{D}_\alpha^\vee$, i.e. $\Lambda = \mathrm{Eff}(X_\Sigma)^\vee$.

Recall from (7.8.10) that we have the expression

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = (\mathbf{L}-1)^n \int_{D(\mathcal{X}_*(U))} \int_{\mathbf{PDiv}^0(\mathscr{C})^\perp} \prod_{v\in\mathscr{C}} \frac{Q_\Sigma((\boldsymbol{\chi}_v(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)_{\alpha\in\Sigma(1)})}{\prod_{\alpha\in\Sigma(1)}(1-\boldsymbol{\chi}_v(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)} \mathrm{d}\boldsymbol{\chi}\mathrm{d}\boldsymbol{z}$$

where we write $\boldsymbol{\chi}_v(\rho_\alpha)$ for the fibre above $v \in \mathscr{C}$ of the algebraic cocharacter corresponding to ρ_α, that is to say

$$\boldsymbol{\chi}(\rho_\alpha) = \mathrm{ev}(\rho_\alpha, \cdot\,) = \left[\boldsymbol{\chi} \in D(\underline{\mathcal{X}_*(U)}_{\mathscr{C}}) \mapsto \boldsymbol{\chi}(\rho_\alpha) \in \mathbf{G}_m\right] \in \mathrm{K}_0\mathbf{Char}_{\mathcal{X}_*(U)}\mathbf{Var}_{\mathscr{C}},$$

to which the motivic Euler product is applied, and similarly $\langle \boldsymbol{z}, \rho_\alpha\rangle$ for the motivic class of the algebraic cocharacter

$$\langle \boldsymbol{z}, \rho_\alpha\rangle = \mathrm{ev}(\rho_\alpha, \cdot\,) = \left[\boldsymbol{z} \in D(\underline{\mathcal{X}_*(U)}) \simeq D(\underline{(\mathbf{Z}\mathfrak{D}_1)^n}) \mapsto \boldsymbol{z}(\rho_\alpha)\right] \in \mathrm{K}_0\mathbf{Char}_{\mathcal{X}_*(U)}\mathbf{Var}_k.$$

Using multiplicativity of motivic Euler products, we have

$$\begin{aligned}\prod_{v\in\mathscr{C}} & \frac{Q_\Sigma((\boldsymbol{\chi}_v(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)_{\alpha\in\Sigma(1)})}{\prod_{\alpha\in\Sigma(1)}(1-\boldsymbol{\chi}_v(\rho_\alpha)\prod_\alpha\langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)} \\ &= \left(\prod_{\alpha\in\Sigma(1)} L(\boldsymbol{\chi}(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle, T_\alpha)\right) \prod_{v\in\mathscr{C}} Q_\Sigma((\boldsymbol{\chi}_v(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle T_\alpha)_{\alpha\in\Sigma(1)})\end{aligned}$$

which is our motivic analogue of expression (4.3.22) in [Bou11a, Lemme 4.44].

To apply the results from the previous section, let us formally define

$$P(\mathbf{T}) = \left(\prod_{\alpha\in\Sigma(1)}(1-T_\alpha)\right) \int_{\mathbf{PDiv}^0(\mathscr{C})^\perp} \left(\prod_{\alpha\in\Sigma(1)} L(\boldsymbol{\chi}(\rho_\alpha)\langle \boldsymbol{z}, \rho_\alpha\rangle, \mathbf{L}^{-1}T_\alpha)\right) \left(\prod_{v\in\mathscr{C}} Q_\Sigma(\boldsymbol{\chi}_v(\rho_\alpha)\mathbf{L}^{-1}T_{\alpha,v})\right) \mathrm{d}\boldsymbol{\chi}$$

(note that we normalise by $\mathbf{L}$ each indeterminate T_αPourquoi ?) so that

$$\zeta_H^{\mathrm{mot}}(\mathbf{T}) = (\mathbf{L}-1)^n \int_{D(\underline{\mathcal{X}_*(U)}_k)} \mathfrak{L}_{\mathbf{Z}^{\Sigma(1)},\mathrm{Eff}(X)^\vee}(((\langle \boldsymbol{z},\rho_\alpha\rangle T_\alpha)_{\alpha\in\Sigma(1)})P(((\langle \boldsymbol{z},\rho_\alpha\rangle T_\alpha)_{\alpha\in\Sigma(1)})\mathrm{d}\boldsymbol{z}$$

where we recall that

$$\mathfrak{L}_{\mathbf{Z}^{\Sigma(1)},\mathrm{Eff}(X)^\vee}(\mathbf{T}) = \frac{1}{\prod_{\alpha\in\Sigma(1)}(1-T_\alpha)}.$$

Lemma 8.11. *The motivic Euler product*

$$\prod_{v\in\mathscr{C}} Q_\Sigma((T_\alpha)_{\alpha\in\Sigma(1)})$$

converges for $|\mathbf{T}| < \mathbf{L}^{-\frac{1}{2}}$.

Proof. By [BT95, Proposition 2.2.3], the polynomial $Q_\Sigma - 1$ contains only monomials of degree ≥ 2. □

Now we can apply Lemma 8.9 and Corollary 8.10 to that expression with $\mathfrak{L}_{N,\Lambda,\mathfrak{a}} = P$ Il faudrait vérifier l'hypothèse de convergence dans le Lemme 8.9 and get the following.

Theorem 8.12. *There exists an* $\eta > 0$ *such that the series*

$$(1-(\mathbf{L}T)^{a_{\lambda_0}})^{\mathrm{rk}(M)}\zeta_H^{\mathrm{mot}}\left(T^{\omega_{X_\Sigma}^{-1}}\right)$$

converges for $|T| < \mathbf{L}^{-1+\eta}$. *Moreover,*

$$\left((1-(\mathbf{L}T)^{a_{\lambda_0}})^{\mathrm{rk}(M)}\zeta_H^{\mathrm{mot}}\left(T^{\omega_{X_\Sigma}^{-1}}\right)\right)_{T=\mathbf{L}^{-1}} = a_{\lambda_0}^{\mathrm{rk}(\mathrm{Pic}(X_\Sigma))}\alpha^*(X_\Sigma)\gamma_H^{\mathrm{mot}}(X_\Sigma)$$

in $\widehat{\mathscr{M}_k}^{\dim}$.

Proof. What remains to do is to compute $\mathfrak{L}_{N,\Lambda,\mathfrak{a}}(\mathbf{1}) = P(\mathbf{1})$. □

Combining with Lemma 1.24, we get Theorem 2.

References

[BDH22] Margaret Bilu, Ronno Das, and Sean Howe, *Zeta statistics and Hadamard functions*, Advances in Mathematics **407** (2022), 108556. ↑1

[BH21] Margaret Bilu and Sean Howe, *Motivic Euler products in motivic statistics*, Algebra & Number Theory **15** (2021), no. 9, 2195–2259. ↑11

[Bil23] Margaret Bilu, *Motivic Euler products and motivic height zeta functions*, Memoirs of the American Mathematical Society **282** (2023), no. 1396. ↑4, 11, 12, 13, 14, 15

[Bou09] David Bourqui, *Produit eulérien motivique et courbes rationnelles sur les variétés toriques*, Compositio Mathematica **145** (2009), no. 6, 1360–1400. ↑1

[Bou11a] ———, *Fonction zêta des hauteurs des variétés toriques non déployées*, Memoirs of the American Mathematical Society **211** (2011), no. 994. ↑1, 33, 34, 35, 36, 37, 38, 39

[Bou11b] ———, *La conjecture de Manin géométrique pour une famille de quadriques intrinsèques*, manuscripta mathematica **135** (2011), no. 1, 1–41. ↑10, 11

[BT95] Victor V Batyrev and Yuri Tschinkel, *Rational points of bounded height on compactifications of anisotropic tori*, International Mathematics Research Notices **1995** (1995), no. 12, 591–635. ↑1, 33, 40

[BT98] ———, *Manin's conjecture for toric varieties*, Journal of Algebraic Geometry **7** (1998), no. 1, 15–53. ↑1, 33

[CH22] Raf Cluckers and Immanuel Halupczok, *Evaluation of motivic functions, non-nullity, and integrability in fibers*, Advances in Mathematics **409** (2022), 108635. ↑5

[CLL16] Antoine Chambert-Loir and François Loeser, *Motivic height zeta functions*, American Journal of Mathematics (2016), 1–59. ↑3, 4, 5

[CLNS18] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag, *Motivic integration*, Springer, 2018. ↑5, 27, 28

[CLNV24] Raf Cluckers, François Loeser, Kien Huu Nguyen, and Floris Vermeulen, *Motivic Mellin transforms*, arXiv preprint arXiv:2412.17764 (2024). ↑4

[Del73] Pierre Deligne, *Cohomologie à supports propres*, Théorie des topos et cohomologie étale des schémas. Tomes 1 à 3 (1973), 649. ↑19

[DGA+62] M Demazure, A Grothendieck, M Artin, JE Bertin, P Gabriel, M Raynaud, and JP Serre, *Séminaire de géométrie algébrique du Bois Marie 1962-64. Schémas en groupes (SGA 3). Tome II: Groupes de type multiplicatif, et structure des schemas en groupes generaux*, 1962. ↑6

[Fai23] Loïs Faisant, *Geometric Batyrev-Manin-Peyre for equivariant compactifications of additive groups*, Beitr. Algebra Geom. **64** (2023), no. 3, 783–850. ↑4

[Fai25a] Loïs Faisant, *Motivic counting of rational curves with tangency conditions via universal torsors*, arXiv preprint arXiv:2502.11704 (2025). ↑1, 19

[Fai25b] ———, *Motivic distribution of rational curves and twisted products of toric varieties*, Algebra & Number Theory **19** (2025), no. 5, 883–965. ↑1, 2, 3

[Gro71] Alexander Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, Lecture notes in mathematics, vol. 224, Springer-Verlag, 1971. ↑16

[HK09] Ehud Hrushovski and David Kazhdan, *Motivic Poisson summation*, Mosc. Math. J. **9** (2009), no. 3, 569–623, back matter. MR2562794 ↑3

Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique, France
Email address: `margaret.bilu@polytechnique.edu`

KU Leuven, Department of Mathematics, B-3001 Leuven, Belgium
Email address: `lois.faisant@kuleuven.be`