



DIPLOMA PROJECT

Author: Loke Gustafsson

Date: May 12, 2020

Supervisor: Mats Gustafsson

Simulating gas with
self-gravitational interaction using
smoothed-particle hydrodynamics

Abstract

This is a very deep and insightful abstract to my diploma project. This is an example reference!.¹ Oh, look!² Another reference.³ And again!.⁴ Lol⁵

¹Joe J Monaghan. “Smoothed particle hydrodynamics”. In: *Annual review of astronomy and astrophysics* 30.1 (1992), pp. 543–574.

²Ibid.

³Loke Gustafsson. “This work”. In: *INception whoa* 30.1 (2020), pp. 543–574.

⁴Monaghan, “Smoothed particle hydrodynamics”, op. cit.

⁵Gustafsson, “This work”, op. cit.

Contents

1	Introduction	3
2	Theory	4
2.1	Gravity	4
2.2	Ideal gas	4
2.3	Smoothed-particle Hydrodynamics	4
2.3.1	Introduction	4
2.3.2	Fundamentals	4
2.3.3	Operators	5
2.3.4	Governing equations	6
2.4	Computing particle neighborhoods	7
2.4.1	Cluster dissimilarity	7
3	Method	8
4	Result	9
5	Discussion	10
	References	11

Introduction

Theory

2.1 Gravity

2.2 Ideal gas

2.3 Smoothed-particle Hydrodynamics

2.3.1 Introduction

Smoothed-particle hydrodynamics (SPH) is a method in computational fluid dynamics used to numerically solve a system of differential equations. Insert history.

SPH is a particle-based, Lagrangian method. This confers several advantages over mesh-based, Eulerian methods in astrophysical problems such as those studied in this paper:

1. The lack of boundary conditions.
2. Large regions of low density
3. Adaptive resolutions

2.3.2 Fundamentals

For a given vector or scalar field A we define the integral interpolant $A_I \approx A$ as

$$A_I(\mathbf{r}) = \int A(\mathbf{r}') W(\mathbf{r} - \mathbf{r}', h) d\mathbf{r}' \quad (2.1)$$

where \mathbf{r} and \mathbf{r}' are position vectors with units m and h is a scalar “smoothing length” with units m . The (scalar) kernel function $W(\mathbf{r}, h)$ has units m^{-3} and satisfies

$$\int W(\mathbf{r}, h) d\mathbf{r} = 1 \quad (2.2)$$

and

$$\lim_{h \rightarrow 0} W(\mathbf{r}, h) d\mathbf{r} = \delta(x) = \begin{cases} +\infty & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

The field A_I is a smoothed version of A , with a granularity dependent on h . The core idea in SPH is to approximate the integral interpolant with a finite sum over the particles, and so we define the summation interpolant $A_S \approx A_I$ as

$$A_S(\mathbf{r}) = \sum_b V_b A(b) W(\mathbf{r} - \mathbf{r}_b, h) \quad (2.4)$$

where the sum is over all the SPH particles. The particles each have a position \mathbf{r} , velocity \mathbf{v} , mass m and thermal energy u . All other fields have to be computed and interpolated from these, the most fundamental of which is the density.

The density field is particularly important due to the particles having known masses, and the summation interpolant using the volumes of the particles. Writing the volume V_b as $\frac{m_b}{\rho(\mathbf{r}_b)}$ allows us to talk about the density rather than the volume:

$$A_S(\mathbf{r}) = \sum_b \frac{m_b}{\rho(\mathbf{r}_b)} A(\mathbf{r}_b) W(\mathbf{r} - \mathbf{r}_b, h) \quad (2.5)$$

which also gives us the density interpolant

$$\rho_S(\mathbf{r}) = \sum_b \frac{m_b}{\rho(\mathbf{r}_b)} \rho(\mathbf{r}_b) W(\mathbf{r} - \mathbf{r}_b, h) = \sum_b m_b W(\mathbf{r} - \mathbf{r}_b, h) \quad (2.6)$$

At this point we drop the I and S subscripts for the interpolants since this would be unnecessarily verbose and intent will be clear from context. This allows us to shorten $A_S(\mathbf{r}_b)$ to A_b . We will also write $W(\mathbf{r}_a - \mathbf{r}_b, h)$ as W_{ab} and let ∇_a denote the gradient with respect to the space coordinates of particle a .

2.3.3 Operators

In order to solve the differential equations we need a way to compute gradients, divergence and curls within the SPH framework. More specifically, we need a way to compute ∇A , $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$, where A and \mathbf{F} are arbitrary scalar and vector fields respectively.

Using (2.5) we can write $(\nabla A)_a$ as

$$(\nabla A)_a = (\nabla_a A)_a = \sum_b \frac{m_b}{\rho_b} A_b \nabla_a W_{ab} = \sum_b \frac{m_b A_b \nabla_a W_{ab}}{\sum_c m_c W_{bc}} \quad (2.7)$$

giving us the gradient of any field for which we can compute A_b .

However, it turns out¹ that this nested sum decreases precision. Instead, the usual approach is to calculate the interpolated $(\nabla(\rho A))_a$ and then retrieve A through $\rho \nabla A = \nabla(\rho A) - \nabla \rho A$.

¹Monaghan, "Smoothed particle hydrodynamics", op. cit.

ρA is a field like any other, but the fact that the density cancels in its interpolant makes the computation simpler:

$$\nabla_a(\rho A) = \nabla_a \sum_b \frac{m_b}{\rho_b} \rho_b A_b W_{ab} = \sum_b m_b A_b \nabla_a W_{ab} \quad (2.8)$$

∇A is then computed as

$$\rho_a(\nabla A)_a = \nabla_a(\rho A) - (\nabla \rho)_a A_a = \sum_b m_b (A_b - A_a) \nabla_a W_{ab} \quad (2.9)$$

Similarly using the product rules of the divergence and curl and simplifying gives us the two other operators:

$$\rho_a(\nabla \cdot \mathbf{F})_a = \sum_b m_b (\mathbf{F}_b - \mathbf{F}_a) \cdot \nabla_a W_{ab} \quad (2.10)$$

$$\rho_a(\nabla \times \mathbf{F})_a = - \sum_b m_b (\mathbf{F}_b - \mathbf{F}_a) \times \nabla_a W_{ab} \quad (2.11)$$

However, this is not the entire story as SPH allows most fields to be interpolated in different ways. In particular, RETURN WHEN I ACTUALLY UNDERSTAND THE SYMMETRIC PRESSURE INTERPOLANT

ACTUALLY I AM STUPID AND THIS IS TRIVIAL: $(P_b - P_a)(\mathbf{r}_a - \mathbf{r}_b)$ is not symmetric!

2.3.4 Governing equations

This work uses the Euler equation for momentum balance of a compressible inviscid fluid:

$$\frac{d\mathbf{v}_a}{dt} = -\frac{1}{\rho_a}(\nabla P)_a + \mathbf{g}_a \quad (2.12)$$

A mass balance equation is not necessary as the density is interpolated from the particle masses in each iteration.

The only external force considered is gravity, as magnetism, radiative transfer and special relativity are all ignored in this work: (The gravitational field is the negative gradient of the gravitational potential field)

$$\mathbf{g}_a = -\nabla_a \int \frac{-G\rho(\mathbf{r}')}{|\mathbf{r}_a - \mathbf{r}'|} d\mathbf{r}' = \int \frac{G\rho(\mathbf{r}')(\mathbf{r}_a - \mathbf{r}')}{|\mathbf{r}_a - \mathbf{r}'|^3} d\mathbf{r}' \quad (2.13)$$

This integral can be computed by summing over all the particles. Note that this is not an interpolation integral, and computing it naively for all the particles would take quadratic time regardless of the kernel function. For this reason, we use a more sophisticated tree structure to compute the gravitational acceleration.

Calculating the gradient of the pressure requires us to compute the pressure at the particles using the ideal gas law.

$$\frac{2E}{3} = nRT = PV = \frac{Pm}{\rho} \quad (2.14)$$

Giving us the pressure at a particle

$$P_a = \frac{2E_a}{3m_a} \rho_a = \tau_a \rho_a \quad (2.15)$$

The pressure factor $\tau_a = \frac{2E_a}{3m_a}$ is constant in this paper's isothermal simulation.

2.4 Computing particle neighborhoods

- Gravity can be computed using mesh methods and tree methods.
- SPH requires a tree to find a particle's neighbors.
- We might as well use tree-based gravity.
- This means we need to construct a tree in $O(n)$
- We prefer mesh-independent trees for numerical reasons.
- Let's use binary hierarchical clustering!
- Possible in $O(n \log n)$ through the nearest-neighbor chain algorithm combined with finding nearest neighbors in $O(\log n)$
- The nearest-neighbor chain algorithm requires insertions into our find-neighbors data structure
- Let's use a R*-tree!

Actually let's not, because my tree-gravity is really dumb and causes numerical artifacts

- Let's use FMM for gravity and Vaydia's algorithm for SPH!
- (Read my bookmark: "life-saving paper")

2.4.1 Cluster dissimilarity

Computing gravity through FMM in a binary space partitioning tree introduces errors compared to naive $O(n^2)$ gravity. We want to design our algorithm to minimize this errors, in this case through choosing the correct cluster distance function.

The function should be equivalent to

Method

Result

Discussion

References

- Gustafsson, Loke. “This work”. In: *INception whoa* 30.1 (2020), pp. 543–574.
- Monaghan, Joe J. “Smoothed particle hydrodynamics”. In: *Annual review of astronomy and astrophysics* 30.1 (1992), pp. 543–574.