Уравнения математической физики
БДЗ-2

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1 Найти решение на бесконечной струне методом Даламбера

$$U_{tt} = 4U_{xx} + 4x$$

$$\begin{cases} U(x,0) = 6x^3 + 5 \\ U_t(x,0) = 7x^2 + x + 2 \end{cases}$$

Решение:

$$U(x,0) = \phi(x)$$
$$U_t(x,0) = \psi(x)$$

Формула Даламбера:

$$U(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(\xi)d\xi + \frac{1}{2c}\int\limits_0^t\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,\xi)d\xi d\tau$$

Где:

$$c = \sqrt{4} = 2$$
$$\phi(x) = 6x^3 + 5$$
$$\psi(x) = 7x^3 + x + 2$$
$$f(x,t) = 4x$$

Подставим в формулу:

$$U(x,t) = \frac{(6(x+2t)^3+5+6(x-2t)^3+5)}{2} + \frac{1}{4}\int\limits_{x-2t}^{x+2t}(7\xi^2+\xi+2)d\xi + \frac{1}{4}\int\limits_0^t\int\limits_{x+2(t-\tau)}^{x-2(t-\tau)} 4\xi d\xi d\tau$$

$$U(x,t) = 3(x^3+6x^2t+12xt^2+8t^3) + 3(x^3-6x^2t+12xt^2-8t^3) + 5 + (\frac{7\xi^3}{3}+\frac{\xi^2}{2}+2\xi)\Big|_{x-2t}^{x+2t} + \frac{1}{4}\int\limits_0^t 2\xi^2\Big|_{x-2(t-\tau)}^{x+2(t-\tau)}d\xi$$

$$U(x,t) =$$
$$6x^3+72xt^2+5+\frac{7}{3}(x+2t)^3+\frac{(x+2t)^2}{2}+2(x+2t)-\frac{7}{3}(x-2t)^3-\frac{(x-2t)^2}{2}-2(x-2t)+\frac{1}{2}\int\limits_0^t((x+2(t-\tau))^2-(x-2(t-\tau))^2)d\tau$$

$$U(x,t) = 6x^3 + 72xt^2 + 5 + 14x^2t + \frac{56}{3}t^3 + 2xt + 4t + 14x^2t + \frac{56}{3}t^3 + 2xt + 4t + \int\limits_0^t(4x(t-\tau))d\tau$$

$$U(x,t) = 6x^3 + 72xt^2 + 28x^2t + \frac{112}{3}t^3 + 4xt + 8t + 5 + 4x(t\tau - \frac{\tau^2}{2}\Big|_0^t)$$
$$U(x,t) = 6x^3 + 72xt^2 + 28x^2t + \frac{112}{3}t^3 + 4xt + 8t + 5 + 4x(t^2 - \frac{t^2}{2})$$
$$U(x,t) = 6x^3 + 72xt^2 + 28x^2t + \frac{112}{3}t^3 + 4xt + 8t + 5 + 4x(t^2 - \frac{t^2}{2})$$
$$U(x,t) = 6x^3 + 74xt^2 + 28x^2t + \frac{112}{3}t^3 + 4xt + 8t + 5$$

Ответ:

$$U(x,t) = 6x^3 + 74xt^2 + 28x^2t + \frac{112}{3}t^3 + 4xt + 8t + 5$$

2 Найти решение краевой задачи колебаний струны

$$U_{tt} = 9U_{xx} + 6$$

$$\begin{cases} U(x,0) = x^2 \\ U_t(x,0) = x^3 \\ U(0,t) = 0 \\ U(l,t) = 0 \end{cases}$$

Решение:
Разбиваем на 2 подзадачи:
1. Свободные колебания струны с закрепленными концами.

$$U'_{tt} = 9U'_{xx}$$

$$\begin{cases} U'(x,0) = x^2 = \phi(x) \\ U'_t(x,0) = x^3 = \psi(x) \\ U'(0,t) = 0 \\ U'(l,t) = 0 \end{cases}$$

2. Вынужденные колебания струны с закрепленными концами при ненулевых начальных условиях.

$$U''_{tt} = 9U''_{xx} + 6$$

$$\begin{cases} U''(x,0) = 0 \\ U''_t(x,0) = 0 \\ U''(0,t) = 0 \\ U''(l,t) = 0 \end{cases}$$

Решаем 1 подзадачу:

$$U'_{tt} = 9U'_{xx}$$

$$\begin{cases} U'(x,0) = x^2 = \phi(x) \\ U'_t(x,0) = x^3 = \psi(x) \\ U'(0,t) = 0 \\ U'(l,t) = 0 \end{cases}$$

$$c = \sqrt{9} = 3$$

Будем искать решение в виде:

$$U' = \sum_{n=1}^{\infty} (A_n \cos \frac{3\pi n}{l} t + B_n \sin \frac{3\pi n}{l} t) \sin \frac{3\pi n}{l} x$$

$$A_n = \tfrac{2}{l} \int\limits_0^l \phi(x) \sin \tfrac{\pi n}{l} x dx$$

$$B_n = \tfrac{1}{\pi n} \int\limits_0^l \psi(x) \sin \tfrac{\pi n}{l} x dx$$

Подставим функции и посчитаем коэфиценты A_n и B_n.
$A_n = \frac{2}{l}\int\limits_0^l x^2 \sin\frac{\pi n}{l}xdx = \left[\begin{array}{cc} U = x^2 & dU = 2xdx \\ dV = \sin\frac{\pi n}{l}xdx & V = -\frac{l}{\pi n}\cos\frac{\pi n}{l}x \end{array}\right] =$
$= \frac{2}{l}(-\frac{x^2 l}{\pi n}\cos\frac{\pi n}{l}x\Big|_0^l + \int\limits_0^l \frac{2lx}{\pi n}\cos\frac{\pi n}{l}xdx) =$
$= -\frac{2l^2}{\pi n}\cos\pi n + \int\limits_0^l \frac{4x}{\pi n}\cos\frac{\pi n}{l}xdx =$
$= \left[\begin{array}{cc} U = x & dU = dx \\ dV = \cos\frac{\pi n}{l}xdx & V = \frac{l}{\pi n}\sin\frac{\pi n}{l}x \end{array}\right] =$
$= -\frac{2l^2}{\pi n}\cos\pi n + \frac{4}{\pi n}(\frac{xl}{\pi n}\sin\frac{\pi n}{l}x\Big|_0^l - \frac{l}{\pi n}\int\limits_0^l \sin\frac{\pi n}{l}xdx) =$
$= -\frac{2l^2}{\pi n}\cos\pi n - \frac{4l}{\pi^2 n^2}\int\limits_0^l \sin\frac{\pi n}{l}xdx =$
$= -\frac{2l^2}{\pi n}\cos\pi n + \frac{4l^2}{\pi^3 n^3}\cos\frac{\pi n}{l}x\Big|_0^l =$
$= -\frac{2l^2}{\pi n}\cos\pi n + \frac{4l^2}{\pi^3 n^3}\cos\pi n + \frac{4l^2}{\pi^3 n^3}$

$B_n = \frac{1}{\pi n}\int\limits_0^l x^3 \sin\frac{\pi n}{l}xdx =$
$= \left[\begin{array}{cc} U = x^3 & dU = 3x^2dx \\ dV = \sin\frac{\pi n}{l}xdx & V = -\frac{l}{\pi n}\cos\frac{\pi n}{l}x \end{array}\right] =$
$= \frac{1}{\pi n}(-\frac{x^3 l}{\pi n}\cos\frac{\pi n}{l}x\Big|_0^l + \int\limits_0^l \frac{3lx^2}{\pi n}\cos\frac{\pi n}{l}xdx) =$
$= -\frac{l^4}{\pi^2 n^2}\cos\pi n + \frac{3l}{\pi^2 n^2}\int\limits_0^l x^2\cos\frac{\pi n}{l}xdx =$
$= \left[\begin{array}{cc} U = x^2 & dU = 2xdx \\ dV = \cos\frac{\pi n}{l}xdx & V = \frac{l}{\pi n}\sin\frac{\pi n}{l}x \end{array}\right] =$
$= -\frac{l^4}{\pi^2 n^2}\cos\pi n + \frac{3l}{\pi^2 n^2}(\frac{x^2 l}{\pi n}\sin\frac{\pi n}{l}x\Big|_0^l - \int\limits_0^l x\sin\frac{\pi n}{l}xdx) =$
$\left[\begin{array}{cc} U = x & dU = dx \\ dV = \sin\frac{\pi n}{l}xdx & V = -\frac{l}{\pi n}\cos\frac{\pi n}{l}x \end{array}\right] =$
$= -\frac{l^4}{\pi^2 n^2}\cos\pi n + \frac{6l^2}{\pi^3 n^3}(\frac{lx}{\pi n}\cos\frac{\pi n}{l}\Big|_0^l - \frac{l}{\pi n}\int\limits_0^l \cos\frac{\pi n}{l}xdx) =$
$= -\frac{l^4}{\pi^2 n^2}\cos\pi n + \frac{6l^4}{\pi^4 n^4}\cos\pi n$
Подставим :

$$U' = \sum_{n=1}^{\infty}((-\frac{2l^2}{\pi n}\cos\pi n + \frac{4l^2}{\pi^3 n^3}\cos\pi n + \frac{4l^2}{\pi^3 n^3})\cos\frac{3\pi n}{l}t + (-\frac{l^4}{\pi^2 n^2}\cos\pi n + \frac{6l^4}{\pi^4 n^4}\cos\pi n)\sin\frac{3\pi n}{l}t)\sin\frac{3\pi n}{l}x$$

Решаем 2 подзадачу:

$$U''_{tt} = 9U''_{xx} + 6$$

$$\begin{cases} U''(x,0) = 0 \\ U''_t(x,0) = 0 \\ U''(0,t) = 0 \\ U''(l,t) = 0 \end{cases}$$

$$c = \sqrt{9} = 3$$
$$f = 6$$

Будем искать решение в виде:

$$U'' = \frac{l}{3\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big[\int\limits_0^t F_n(\tau)\sin\frac{3\pi n}{l}(t-\tau)d\tau\Big]\sin\frac{\pi n}{l}x$$

$F_n = \frac{2}{l}\int\limits_0^l 6\sin\frac{\pi n}{l}xdx = -\frac{12}{\pi n}\cos\frac{\pi n}{l}x\Big|_0^l = -\frac{12}{\pi n}+\frac{12}{\pi n}$

Подставим: $U'' = \dfrac{l}{3\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\Big[\int\limits_0^t(-\frac{12}{\pi n}+\frac{12}{\pi n})\sin\frac{3\pi n}{l}(t-\tau)d\tau\Big]\sin\frac{\pi n}{l}x =$

$= \dfrac{4l}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}(1-\cos\pi n)(\frac{l}{3\pi n}\cos\frac{3\pi n}{l}(t-\tau)\Big|_0^t)\sin\frac{\pi n}{l}x =$

$= \dfrac{4l}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}(1-\cos\pi n)\sin\frac{\pi n}{l}x(\frac{l}{3\pi n}\cos\frac{3\pi n}{l}t+\frac{l}{3\pi n}) =$

$= \dfrac{4l^2}{3\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos\pi n)\sin\frac{\pi n}{l}x(\cos\frac{3\pi n}{l}t+1)$

Тогда решение:

$$U(x,t) = U' + U''$$

$$U(x,t) = \sum_{n=1}^{\infty}((-\frac{2l^2}{\pi n}\cos\pi n+\frac{4l^2}{\pi^3n^3}\cos\pi n+\frac{4l^2}{\pi^3n^3})\cos\frac{3\pi n}{l}t+(-\frac{l^4}{\pi^2n^2}\cos\pi n+\frac{6l^4}{\pi^4n^4}\cos\pi n)\sin\frac{3\pi n}{l}t)\sin\frac{3\pi n}{l}x + \sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos\pi n)\sin\frac{\pi n}{l}x(\cos\frac{3\pi n}{l}t+1)$$

Ответ: $U(x,t) = \displaystyle\sum_{n=1}^{\infty}((-\frac{2l^2}{\pi n}\cos\pi n+\frac{4l^2}{\pi^3n^3}\cos\pi n+\frac{4l^2}{\pi^3n^3})\cos\frac{3\pi n}{l}t+(-\frac{l^4}{\pi^2n^2}\cos\pi n+\frac{6l^4}{\pi^4n^4}\cos\pi n)\sin\frac{3\pi n}{l}t)\sin\frac{3\pi n}{l}x+$
$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos\pi n)\sin\frac{\pi n}{l}x(\cos\frac{3\pi n}{l}t+1)$

3 Найти решение краевой задачи колебаний струны

$$U_{tt} = 9U_{xx}$$

$$\begin{cases} U(x,0) = 0 \\ U_t(x,0) = 0 \\ U(0,t) = 7t + 10 = \mu(t) \\ U(l,t) = e^{-t} = \nu(t) \end{cases}$$

Решение:
Введем функцию $W(x,t) = \mu(t) + \frac{x}{l}[\mu(t) - \mu(t)]$
Будем искать решение в виде $U(x,t) = y(x,t) + W(x,t)$
$U(x,t) = y(x,t) + \mu(t) + \frac{x}{l}[\mu(t) - \mu(t)] = y(x,t) + 7t + 10 + \frac{x}{l}(e^{-t} - 7t - 10) = y(x,t) + 7t + 10 + \frac{xe^{-t}}{l} - \frac{7tx}{l} + \frac{10x}{l}$
$U_{xx} = y_{xx}$
$U_{tt} = y_{tt} + \frac{xe^{-t}}{l}$
Получаем новое уравнение:
$y_{tt} = 9y_{xx} - \frac{xe^{-t}}{l}$
И получаем:
$y(x,t) = U(x,t) - 7t - 10 - \frac{xe^{-t}}{l} + \frac{7tx}{l} + \frac{10x}{l}$
$y_t(x,t) = U_t(x,t) - 7 + \frac{xe^{-t}}{l} + \frac{7x}{l}$
Тогда перейдем к новому уравнению с новыми условиями:

$$y_{tt} = 9y_{xx} - \frac{xe^{-t}}{l}$$

$$\begin{cases} y(x,0) = \frac{9x}{l} - 10 = \bar{\phi}(x) \\ y_t(x,0) = \frac{8x}{l} - 7 = \bar{\psi}(x) \\ y(0,t) = 0 \\ y(l,t) = 0 \end{cases}$$

Разбиваем на 2 подзадачи:
1. Свободные колебания струны с закрепленными концами.

$$y'_{tt} = 9y'_{xx}$$

$$\begin{cases} y'(x,0) = \frac{9x}{l} - 10 = \bar{\phi}(x) \\ y'_t(x,0) = \frac{8x}{l} - 7 = \bar{\psi}(x) \\ y'(0,t) = 0 \\ y'(l,t) = 0 \end{cases}$$

2. Вынужденные колебания струны с закрепленными концами при ненулевых начальных условиях.

$$y''_{tt} = 9y''_{xx} - \frac{xe^{-t}}{l}$$

$$\begin{cases} y''(x,0) = 0 \\ y''_t(x,0) = 0 \\ y''(0,t) = 0 \\ y''(l,t) = 0 \end{cases}$$

Решаем 1 подзадачу:

$$y'_{tt} = 9y'_{xx}$$

$$\begin{cases} y'(x,0) = \frac{9x}{l} - 10 = \bar{\phi}(x) \\ y'_t(x,0) = \frac{8x}{l} - 7 = \bar{\psi}(x) \\ y'(0,t) = 0 \\ y'(l,t) = 0 \end{cases}$$

$$c = \sqrt{9} = 3$$

Будем искать решение в виде:

$$y' = \sum_{n=1}^{\infty} (A_n \cos \frac{3\pi n}{l} t + B_n \sin \frac{3\pi n}{l} t) \sin \frac{3\pi n}{l} x$$

$$A_n = \tfrac{2}{l} \int\limits_0^l \bar{\phi}(x) \sin \tfrac{\pi n}{l} x dx$$

$$B_n = \tfrac{1}{\pi n} \int\limits_0^l \bar{\psi}(x) \sin \tfrac{\pi n}{l} x dx$$

Подставим функции и посчитаем коэфиценты A_n и B_n.

$A_n = \frac{2}{l} \int\limits_0^l \bar{\phi}(x) \sin \frac{\pi n}{l} x dx =$

$= \frac{2}{l} \int\limits_0^l (\frac{9x}{l} - 10) \sin \frac{\pi n}{l} x dx =$

$= \frac{18}{l^2} \int\limits_0^l x \sin \frac{\pi n}{l} x dx - \frac{20}{l} \int\limits_0^l \sin \frac{\pi n}{l} x dx =$

$= \begin{bmatrix} U = x & dU = dx \\ dV = \sin \frac{\pi n}{l} x dx & V = -\frac{l}{\pi n} \cos \frac{\pi n}{l} x \end{bmatrix} =$

$= \frac{18}{l^2} (\frac{lx}{\pi n} \cos \frac{\pi n}{l} x \Big|_0^l + \frac{l}{\pi n} \int\limits_0^l \cos \frac{\pi n}{l} x dx) + \frac{20}{\pi n} \cos \frac{\pi n}{l} x \Big|_0^l = \frac{2}{\pi n} \cos \pi n - \frac{20}{\pi n}$

$B_n = \frac{1}{\pi n} \int\limits_0^l \bar{\psi}(x) \sin \frac{\pi n}{l} x dx =$

$= \frac{1}{\pi n} \int\limits_0^l (\frac{8x}{l} - 7) \sin \frac{\pi n}{l} x dx =$

$= \frac{8}{\pi n l} \int\limits_0^l x \sin \frac{\pi n}{l} x dx - \frac{7}{\pi n} \int\limits_0^l \sin \frac{\pi n}{l} x dx =$

$= \begin{bmatrix} U = x & dU = dx \\ dV = \sin \frac{\pi n}{l} x dx & V = -\frac{l}{\pi n} \cos \frac{\pi n}{l} x \end{bmatrix} =$

$= \frac{8}{\pi n l} (\frac{lx}{\pi n} \cos \frac{\pi n}{l} x \Big|_0^l + \frac{l}{\pi n} \int\limits_0^l \cos \frac{\pi n}{l} x dx) + \frac{7l}{\pi^2 n^2} \cos \frac{\pi n}{l} x \Big|_0^l = \frac{l}{\pi^2 n^2} \cos \pi n - \frac{7l}{\pi^2 n^2}$

Подставим :

$$y' = \sum_{n=1}^{\infty} ((\frac{2}{\pi n} \cos \pi n - \frac{20}{\pi n}) \cos \frac{3\pi n}{l} t + (\frac{l}{\pi^2 n^2} \cos \pi n - \frac{7l}{\pi^2 n^2}) \sin \frac{3\pi n}{l} t) \sin \frac{3\pi n}{l} x$$

Решаем 2 подзадачу:

$$y''_{tt} = 9y''_{xx} - \tfrac{xe^{-t}}{l}$$

$$\begin{cases} y''(x,0)=0 \\ y''_t(x,0)=0 \\ y''(0,t)=0 \\ y''(l,t)=0 \end{cases}$$

Будем искать решение в виде:

$$y''=\frac{l}{3\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big[\int\limits_0^t F_n(\tau)\sin\frac{3\pi n}{l}(t-\tau)d\tau\Big]\sin\frac{\pi n}{l}x$$

$F_n(t)=\frac{2}{l}\int\limits_0^l(-\frac{xe^{-t}}{l})\sin\frac{\pi n}{l}xdx=$

$=-\frac{2e^{-t}}{l^2}\int\limits_0^l x\sin\frac{\pi n}{l}xdx=$

$=\begin{bmatrix} U=x & dU=dx \\ dV=\sin\frac{\pi n}{l}xdx & V=-\frac{l}{\pi n}\cos\frac{\pi n}{l}x \end{bmatrix}=$

$=-\frac{2e^{-t}}{l^2}(\frac{lx}{\pi n}\cos\frac{\pi n}{l}x\Big|_0^l+\frac{l}{\pi n}\int\limits_0^l\cos\frac{\pi n}{l}xdx)=\frac{2e^{-t}}{\pi n}cos(\pi n)$

Подставим: $y''=\dfrac{l}{3\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\Big[\int\limits_0^t(\frac{2e^{-\tau}}{\pi n}cos(\pi n))\sin\frac{3\pi n}{l}(t-\tau)d\tau\Big]\sin\frac{\pi n}{l}x=$

$=\dfrac{l}{3\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\frac{2\cos(\pi n)\left(l^2\sin\left(\frac{3\pi nt}{l}\right)+lne^{-t}\left(3\pi-3\pi e^t\cos\left(\frac{3\pi nt}{l}\right)\right)\right)}{n\pi\left(l^2+9\pi^2n^2\right)}\sin\frac{\pi n}{l}x$

Тогда решение для y:

$$y(x,t)=y'+y''$$

$$y(x,t)=\sum_{n=1}^{\infty}((\frac{2}{\pi n}\cos\pi n-\frac{20}{\pi n})\cos\frac{3\pi n}{l}t+(\frac{l}{\pi^2n^2}\cos\pi n-\frac{7l}{\pi^2n^2})\sin\frac{3\pi n}{l}t)\sin\frac{3\pi n}{l}x+$$
$$\frac{l}{3\pi}\sum_{n=1}^{\infty}\frac{1}{n}\frac{2\cos(\pi n)\left(l^2\sin\left(\frac{3\pi nt}{l}\right)+lne^{-t}\left(3\pi-3\pi e^t\cos\left(\frac{3\pi nt}{l}\right)\right)\right)}{n\pi\left(l^2+9\pi^2n^2\right)}\sin\frac{\pi n}{l}x$$

Вернемся к $U(x,t)$

$$U(x,t)=y(x,t)+7t+10+\tfrac{xe^{-t}}{l}-\tfrac{7tx}{l}+\tfrac{10x}{l}$$

Подставим $y(x,t)$ и Получим ответ:

$$U(x,t)=\sum_{n=1}^{\infty}((\frac{2}{\pi n}\cos\pi n-\frac{20}{\pi n})\cos\frac{3\pi n}{l}t+(\frac{l}{\pi^2n^2}\cos\pi n-\frac{7l}{\pi^2n^2})\sin\frac{3\pi n}{l}t)\sin\frac{3\pi n}{l}x+$$
$$\frac{l}{3\pi}\sum_{n=1}^{\infty}\frac{1}{n}\frac{2\cos(\pi n)\left(l^2\sin\left(\frac{3\pi nt}{l}\right)+lne^{-t}\left(3\pi-3\pi e^t\cos\left(\frac{3\pi nt}{l}\right)\right)\right)}{n\pi\left(l^2+9\pi^2n^2\right)}\sin\frac{\pi n}{l}x+7t+10+\frac{xe^{-t}}{l}-\frac{7tx}{l}+\frac{10x}{l}$$

4 Найти решение краевой задачи колебаний струны

$$U_t = 16U_{xx} + t^2 \cos x$$

$$\begin{cases} U(x,0) = e^{x(l-x)} - 1 = \phi(x) \\ U(0,t) = 0 \\ U(l,t) = 0 \end{cases}$$

Решение:

$$c = \sqrt{16} = 4$$
$$f(x,t) = t^2 \cos x$$

Разбиваем на 2 подзадачи:
1.

$$U'_t = 16U'_{xx}$$

$$\begin{cases} U'(x,0) = e^{x(l-x)} - 1 = \phi(x) \\ U'(0,t) = 0 \\ U'(l,t) = 0 \end{cases}$$

Будем искать решение в виде:

$$U'(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\tfrac{\pi n}{l} x\right) \exp\left(-\left(\tfrac{4\pi n}{l}\right)^2 t\right).$$
$$C_n = \frac{2}{l} \int\limits_0^l (e^{\xi(l-\xi)} - 1) \sin \frac{\pi n}{l} \xi \, d\xi.$$

c_n неберущийся интеграл тогда запишем $U'(x,t)$:

$$U'(x,t) = \sum_{n=1}^{\infty} (\frac{2}{l} \int\limits_0^l (e^{\xi(l-\xi)} - 1) \sin \frac{\pi n}{l} \xi \, d\xi) \sin\left(\frac{\pi n}{l} x\right) \exp\left(-\left(\frac{4\pi n}{l}\right)^2 t\right)$$

2.

$$U''_t = 16U''_{xx} + t^2 cos(x)$$

$$\begin{cases} U''(x,0) = 0 \\ U''(0,t) = 0 \\ U''(l,t) = 0 \end{cases}$$

Будем искать решение в виде:

$$U''(x,t) = u(x,\ t) = \sum_{n=1}^{\infty} \left[\int\limits_0^t \exp\left(-\left(\frac{\pi n a}{l}\right)^2 (t-\tau)\right) F_n(t) d\tau \right] \sin\left(\frac{\pi n}{l} x\right)$$
$$F_n(t) = \tfrac{2}{l} \int\limits_0^l f(\xi,\ t) \sin\left(\tfrac{\pi n}{l} \xi\right) \, d\xi$$

Найдем F_n:

$F_n(t) = \frac{2}{l}\int\limits_0^l f(\xi,t)\sin\left(\frac{\pi n}{l}\xi\right)d\xi = \frac{2}{l}\int\limits_0^l t^2 cos(x)\sin\left(\frac{\pi n}{l}\xi\right)d\xi = \frac{2t^2\pi n}{\pi^2 n^2-1}(1-\cos l\cos\pi n)$

Подставим F_n в $U''(x,t)$

$U''(x,t) = u(x,t) = \sum\limits_{n=1}^{\infty}\left[\int\limits_0^t \exp\left(-\left(\frac{\pi n4}{l}\right)^2(t-\tau)\right)\frac{2t^2\pi n}{\pi^2n^2-1}(1-\cos l\cos\pi n)d\tau\right]\sin\left(\frac{\pi n}{l}x\right)$

$= \sum\limits_{n=1}^{\infty}\frac{2t^2\pi n}{\pi^2n^2-1}(1-\cos l\cos\pi n)(\frac{l^2}{16\pi^2n^2}(\exp(\frac{-4\pi n}{l}+\frac{2n\pi t}{l}-1)))\sin\left(\frac{\pi n}{l}x\right)$

Вернемся к $U(x,t) = U'(x,t) + U''(x,t)$

Подставим и получаем ответ:

$$U(x,t) = \sum_{n=1}^{\infty}(\frac{2}{l}\int\limits_0^l(e^{\xi(l-\xi)}-1)\sin\frac{\pi n}{l}\xi\,d\xi)\sin\left(\frac{\pi n}{l}x\right)\exp\left(-\left(\frac{4\pi n}{l}\right)^2 t\right)+$$
$$\sum_{n=1}^{\infty}\frac{2t^2\pi n}{\pi^2n^2-1}(1-\cos l\cos\pi n)(\frac{l^2}{16\pi^2n^2}(\exp(\frac{-4\pi n}{l}+\frac{2n\pi t}{l}-1)))\sin\left(\frac{\pi n}{l}x\right)$$