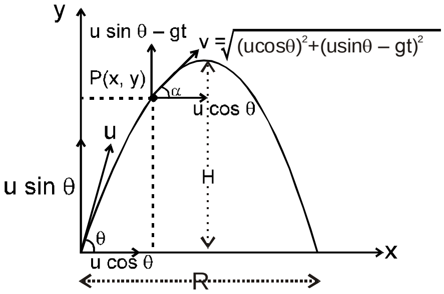
PROJECTILE MOTION

1. PROJECTILE THROWN AT AN ANGLE WITH HORIZONTAL



- Consider a projectile thrown with a velocity u making an angle θ with the horizontal.
- Initial velocity u is resolved in components in a coordinate system in which horizontal direction is taken as x-axis, vertical direction as y-axis and point of projection as origin.

  $u_x = u \cos\theta$ $\qquad$ $u_y = u \sin\theta$

- Again this projectile motion can be considered as the combination of horizontal and vertical motion. Therefore,

	Horizontal direction	Vertical direction
(a)	Initial velocity $u_x = u \cos\theta$	Initial velocity $u_y = u \sin\theta$
(b)	Acceleration $a_x = 0$	Acceleration $a_y = g$
(c)	Velocity after time t, $v_x = u \cos\theta$	Velocity after time t,$v_y = u \sin\theta - gt$

1.1 Time of flight :

The displacement along vertical direction is zero for the complete flight.
Hence, along vertical direction net displacement = 0

$\Rightarrow \quad (u \sin\theta)\, T - \frac{1}{2} gT^2 = 0 \quad \Rightarrow \quad$ $\mathbf{T = \frac{2u\sin\theta}{g}}$

1.2 Horizontal range :

$R = u_x .T \quad \Rightarrow \quad R = u\cos\theta . \frac{2u\sin\theta}{g}$

$$\mathbf{R = \frac{u^2 \sin 2\theta}{g}}$$

1.3 Maximum height :

At the highest point of its trajectory, particle moves horizontally, and hence vertical component of velocity is zero.

Using 3rd equation of motion i.e.

$v^2 = u^2 + 2as$

we have for vertical direction

$0 = u^2 \sin^2\theta - 2gH \quad \Rightarrow \quad$ $\mathbf{H = \frac{u^2 \sin^2\theta}{2g}}$

1.4 Resultant velocity :

$\vec{v} = v_x\hat{i} + v_y\hat{j}$ $= u\cos\theta\ \hat{i} + (u\sin\theta - gt)\ \hat{j}$

Where, $|\ \vec{v}\ | = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$ and $\tan\alpha = v_y / v_x$.

Also, $v\cos\alpha = u\cos\theta$ $\Rightarrow v = \dfrac{u\cos\theta}{\cos\alpha}$

2. EQUATION OF TRAJECTORY

The path followed by a particle (here projectile) during its motion is called its **Trajectory**. Equation of trajectory is the relation between instantaneous coordinates (Here x & y coordinate) of the particle.

If we consider the horizontal direction,

$x = u_x.t$

$x = u\cos\theta.\ t$ ...(1)

For vertical direction :

$y = u_y\ .\ t - 1/2\ gt^2$

$= u\sin\theta\ .\ t - 1/2\ gt^2$ ...(2)

Eliminating 't' from equation (1) & (2)

$$y = u\sin\theta\ .\ \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\Rightarrow \quad \boldsymbol{y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}}$$

This is an equation of parabola called as trajectory equation of projectile motion.

Other forms of trajectory equation :

- $$\boldsymbol{y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}}$$

- $$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

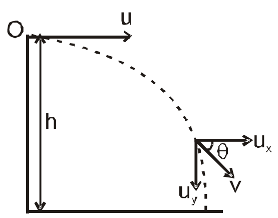
$$\Rightarrow \quad y = x\tan\theta\left[1 - \frac{gx}{2u^2\cos^2\theta\tan\theta}\right]$$

$$\Rightarrow \quad y = x\tan\theta\left[1 - \frac{gx}{2u^2\sin\theta\cos\theta}\right]$$

$$\Rightarrow \quad \boldsymbol{y = x\tan\theta\left[1 - \frac{x}{R}\right]}$$

3. PROJECTILE THROWN PARALLEL TO THE HORIZONTAL FROM SOME HEIGHT

Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall study the characteristics of projectile motion by resolving the motion along horizontal and vertical directions.



	Horizontal direction	**Vertical direction**
(i)	Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

3.1 Time of flight :

This is equal to the time taken by the projectile to return to ground. From equation of motion

$S = ut + \frac{1}{2}at^2$, along vertical direction, we get

$-h = u_y t + \frac{1}{2}(-g)t^2$

$\Rightarrow \quad h = \frac{1}{2}gt^2 \quad \Rightarrow \quad \mathbf{t = \sqrt{\frac{2h}{g}}}$

3.2 Horizontal range :

Distance covered by the projectile along the horizontal direction between the point of projection to the point on the ground.

$R = u_x \, . \, t$

$R = u\sqrt{\frac{2h}{g}}$

3.3 Velocity at a general point P(x, y) :

$v = \sqrt{u_x^2 + u_y^2}$

Here horizontal velocity of the projectile after time t

$v_x = u$

velocity of projectile in vertical direction after time t

$v_y = 0 + (-g)t = -gt = gt$ (downward)

$\therefore \quad v = \sqrt{u^2 + g^2t^2}$ and $\tan\theta = v_y/v_x$

3.4 Velocity with which the projectile hits the ground :

$V_x = u$

$V_y^2 = 0^2 - 2g(-h)$

$V_y = \sqrt{2gh}$

$V = \sqrt{V_x^2 + V_y^2} \qquad \Rightarrow \mathbf{V = \sqrt{u^2 + 2gh}}$

3.5 Trajectory equation :

The path traced by projectile is called the trajectory.

After time t,

$x = ut$ ...(1)

$y = \frac{-1}{2}gt^2$ ...(2)

From equation (1)

$t = x/u$

Put the value of t in equation (2)

$\mathbf{y = \frac{-1}{2}g \, . \, \frac{x^2}{u^2}}$

This is trajectory equation of the particle projected horizontally from some height.

4. PROJECTION FROM A TOWER

Case (i) : Horizontal projection

$u_x = u$; $u_y = 0$; $a_y = -g$

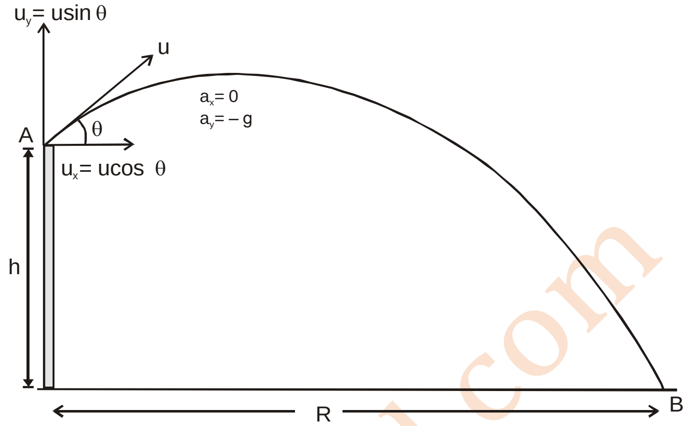
This is same as previous section (section 4)

Case (ii) : Projection at an angle θ above horizontal

$u_x = u\cos\theta$;

$u_y = u\sin\theta$;

$a_y = -g$



Equation of motion between A & B (in Y direction)

$S_y = -h$, $u_y = u\sin\theta$, $a_y = -g$, $t = T$

$S_y = u_y t + \frac{1}{2} a_y t^2 \quad \Rightarrow \quad -h = u\sin\theta\, t - \frac{1}{2} g t^2$

Solving this equation we will get time of flight, T.

And range, $R = u_x T = u\cos\theta\, T$

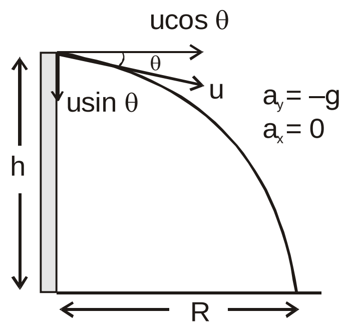
Also, $v_y^2 = u_y^2 + 2a_y S_y$

$= u^2 \sin^2\theta + 2gh$

$v_x = u\cos\theta$

$v_B = \sqrt{v_y^2 + v_x^2} \quad \Rightarrow \quad \mathbf{v_B = \sqrt{u^2 + 2gh}}$

Case (iii) : Projection at an angle θ below horizontal



$u_x = u\cos\theta$;

$u_y = -u\sin\theta$;

$a_y = -g$

$S_y = u_y t + \frac{1}{2} a_y t^2$

$S_y = -h$, $u_y = -u\sin\theta$, $t = T$, $a_y = -g$

$\Rightarrow -h = -u\sin\theta\, T - \frac{1}{2} g T^2 \quad \Rightarrow \quad h = u\sin\theta\, T + \frac{1}{2} g T^2$.

Solving this equation we will get time of flight, T.

And range, $R = u_x T = u\cos\theta\, T$

$v_x = u\cos\theta$

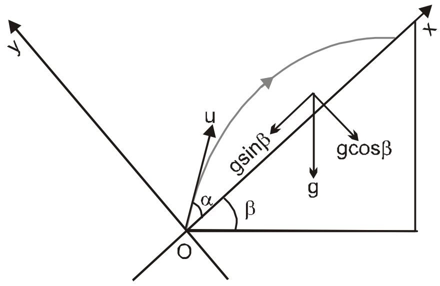
$v_y^2 = u_y^2 + 2a_y S_y = u^2\sin^2\theta + 2(-g)(-h)$

$v_y^2 = u^2\sin^2\theta + 2gh$

$v_B = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$

5. PROJECTION ON AN INCLINED PLANE

Case (i) : Particle is projected up the incline



Here α is angle of projection w.r.t. the inclined plane. x and y axis are taken along and perpendicular to the incline as shown in the diagram.

In this case: $a_x = -g\sin\beta$

$u_x = u\cos\alpha$

$a_y = -g\cos\beta$

$u_y = u\sin\alpha$

Time of flight (T) :

When the particle strikes the inclined plane y becomes zero

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$\Rightarrow \quad 0 = u\sin\alpha T - \frac{1}{2} g\cos\beta T^2$$

$$\Rightarrow \quad T = \frac{2u\sin\alpha}{g\cos\beta} = \frac{2u_\perp}{g_\perp}$$

Where $u_\perp$ and $g_\perp$ are component of u and g perpendicular to the incline.

Maximum height (H) :

When half of the time is elapsed y coordinate is equal to maximum distance from the inclined plane of the projectile

$$H = u\sin\alpha\left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}g\cos\beta\left(\frac{u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2_\perp}{2g_\perp}$$

Range along the inclined plane (R):

When the particle strikes the inclined plane x coordinate is equal to range of the particle

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad R = u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}g\sin\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad R = \frac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$$

Case (ii) : Particle is projected down the incline

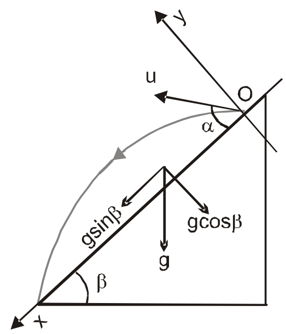
In this case :

$a_x = g\sin\beta$

$u_x = u\cos\alpha$

$a_y = -g\cos\beta$

$u_y = u\sin\alpha$



Time of flight (T) :

When the particle strikes the inclined plane y coordinate becomes zero

$$y = u_y t + \frac{1}{2} a_y t^2 \qquad \Rightarrow \qquad 0 = u\sin\alpha T - \frac{1}{2} g\cos\beta T^2$$

$$\Rightarrow \qquad T = \frac{2u\sin\alpha}{g\cos\beta} = \frac{2u_\perp}{g_\perp}$$

Maximum height (H) :

When half of the time is elapsed y coordinate is equal to maximum height of the projectile

$$H = u\sin\alpha\left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2} g\sin\beta\left(\frac{u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \qquad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2{}_\perp}{2g_\perp}$$

Range along the inclined plane (R):

When the particle strikes the inclined plane x coordinate is equal to range of the particle

$$x = u_x t + \frac{1}{2} a_x\ t^2$$

$$\Rightarrow \qquad R = u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\beta}\right) + \frac{1}{2} g\sin\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \qquad R = \frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$$

Standard results for projectile motion on an inclined plane :

	Up the Incline	Down the Incline
Range	$\frac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$	$\frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$
Time of flight	$\frac{2u\sin\alpha}{g\cos\beta}$	$\frac{2u\sin\alpha}{g\cos\beta}$
Angle of projection for maximum range	$\frac{\pi}{4} - \frac{\beta}{2}$	$\frac{\pi}{4} + \frac{\beta}{2}$
Maximum Range	$\frac{u^2}{g(1+\sin\beta)}$	$\frac{u^2}{g(1-\sin\beta)}$

Here α is the angle of projection with the incline and β is the angle of incline.