Strategic form Game

$$\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

$$S = S_1 \times S_2 \times \cdots \times S_n$$

$$= \bigtimes_{i \in N} S_i$$

$$S_{-i} = \bigtimes_{\substack{j \in N \\ j \neq i}} S_j = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

$s = (s_1, s_2, \ldots, s_n) \in S$ strategy profile

$s = (s_i, s_{-i})$ where $s_i \in S_i$; $s_{-i} \in S_{-i}$

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

Notion of Dominance

s_i' strongly dominates s_i if

$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$

s_i' very weakly dominates s_i if

$u_i(s_i', s_{-i}) \geqslant u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$

s_i' weakly dominates s_i if it very weakly dominates s_i and

$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for some $s_{-i} \in S_{-i}$

Dominant Strategies

A strategy $s_i^* \in S_i$ that dominates every other strategy is called a dominant strategy.

SDS Strongly dominant strategy
WDS Weakly dominant strategy
VWDS Very weakly dominant strategy

Dominant Strategy Equilibrium

$(s_1^*, s_2^*, \ldots, s_n^*)$ where s_i^* $\forall i \in N$

$(s_1^*, s_2^*, \ldots, s_n^*)$ where s_i^* $\forall i \in N$ is a dominant strategy.

SDSE, WDSE, VWDSE

Prisoner's Dilemma

1 \ 2	NC	C
NC	−2, −2	−10, −1
C	−1, −10	−5, −5

For player 1,

$u(C, C) > u_1(NC, C)$

$u_1(C, NC) > u_1(NC, NC)$

Hence C is a SDS

For player 2,

$u_2(C, C) > u_2(C, NC)$

$u_2(NC, C) > u_2(NC, NC)$

Hence C is a SDS

Thus (C, C) is a SDSE

Prisoner's Dilemma Version 2

1 \ 2	NC	C
NC	−2, −2	−10, −2
C	−2, −10	−5, −5

Here (C, C) is a WDSE

Prisoner's Dilemma Version 3

1 \ 2	NC	C

1 \ 2	NC	C
NC	-2, -2	-5, -2
C	-2, -5	-5, -5

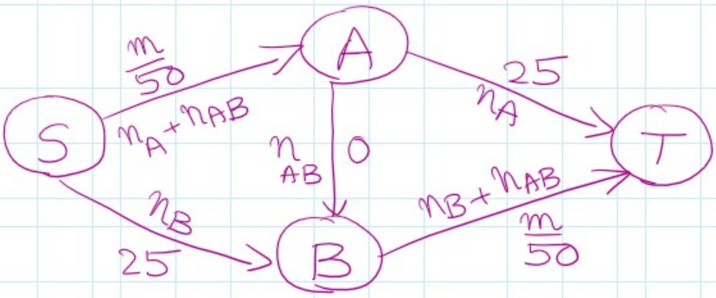
(C, C) is a VWDSE

Coordination Game

1 \ 2	A	B
A	100, 100	0, 0
B	0, 0	10, 10

A does not dominate B
B does not dominate A

Braess Paradox Game with
a high speed bridge



$$u_i(s_1, s_2, \ldots, s_n)$$

$$= -\left(\frac{n_A(s) + n_{AB}(s)}{50}\right) - 25 \quad (s_i = A)$$

$$= -25 - \left(\frac{n_B(s) + n_{AB}(s)}{50}\right) \quad (s_i = B)$$

$$= -\left(\frac{n_A(s) + n_{AB}(s)}{50}\right) - \left(\frac{n_B(s) + n_{AB}(s)}{50}\right) \quad (s_i = AB)$$

$$= -\left(\frac{n_A(s)+n_{AB}(s)}{50}\right) - \left(\frac{n_B(s)+n_{AB}(s)}{50}\right) \quad (s_i = AB)$$

Suppose $\underline{s}_i$ is any strategy profile of players other than player i

$$u_i(AB, \underline{s}_i) = -\left(\frac{n_A(\underline{s}_i)+n_{AB}(\underline{s}_i)+1}{50}\right) - \left(\frac{n_B(\underline{s}_i)+n_{AB}(\underline{s}_i)+1}{50}\right)$$

$$u_i(A, \underline{s}_i) = -\left(\frac{n_A(\underline{s}_i)+1+n_{AB}(\underline{s}_i)}{50}\right) - 25$$

$$u_i(B, \underline{s}_i) = -25 - \left(\frac{n_B(\underline{s}_i)+1+n_{AB}(\underline{s}_i)}{50}\right)$$

Clearly $\forall\, \underline{s}_i \in \underline{S}_i$,

$$u_i(AB, \underline{s}_i) > u_i(A, \underline{s}_i)$$
$$u_i(AB, \underline{s}_i) > u_i(B, \underline{s}_i)$$

This shows that "AB" is a strongly dominant strategy equilibrium for each player i

Thus $(AB, \ldots, AB)$ is a SDSE yielding a delay of

$$\frac{1000}{50} + \frac{1000}{50} = 40$$

Now consider the profile

$$(\underbrace{A, \ldots, A}_{500 \text{ times}}, \underbrace{B, \ldots, B}_{500 \text{ times}})$$

This profile gives a total delay of

$10 + 25 = 35$ for each vehicle!

Vickrey Auction Game

$N = \{1, 2, \ldots, n\}$

Valuations : $v_1, v_2, \ldots, v_n$

Bids : $b_1, b_2, \ldots, b_n$

$$y_i(b_1, \ldots, b_n) = 1 \quad \text{if } b_i > b_j \text{ for } j = 1, \ldots, i-1$$
$$\& \ b_i \geq b_j \text{ for } j = i+1, \ldots, n$$
$$= 0 \quad \text{else}$$

$$t_i(b_1, \ldots, b_n) = \text{highest losing bid}$$

$$u_i(b_1, \ldots, b_n) = y_i(b_1, \ldots, b_n)(v_i - t(b_1, \ldots, b_n))$$

We now show that $(b_1 = v_1 ; b_2 = v_2 ; \ldots ; b_n = v_n)$ is a weakly dominant strategy equilibrium.

TST $\forall i \in N$, $b_i = v_i$ is a WDS

i.e., $\forall b_i' \neq v_i$, the following should hold:

$$u_i(v_i, \underline{b}_i) \geq u_i(b_i', \underline{b}_i) \ \forall \underline{b}_i \in \underline{S}_i$$

and

$$u_i(v_i, \underline{b}_i) > u_i(b_i', \underline{b}_i) \text{ for some } \underline{b}_i \in \underline{S}_i$$

Without loss of generality, we show that

$\forall \underline{b}_1 = (b_2, b_3, \ldots, b_n) \in \underline{S}_1$,

$$u_1(v_1, \underline{b}_1) \geq u_1(b_1, \underline{b}_1) \ \forall b_1 \in S_1$$

There are only two possibilities

$v_1 \geq \max(b_2, \ldots, b_n)$

$v_1 < \max(b_2, \ldots, b_n)$

Case 1: $v_1 \geq \max(b_2, \ldots, b_n)$

$b_1 \geq \max(b_2, \ldots, b_n)$
1 wins

$u_1 = v_1 - \max(b_2, \ldots, b_n)$
≥ 0

Independent of b_1
Any bid satisfying the above will give ~~this~~ utility.

$b_1 < \max(b_2, \ldots, b_n)$
1 loses

$u_1 = 0$

max utility $= v_1 - \max(b_2 \ldots, b_n)$
with $b_1 = v_1$,
utility $= v_1 - \max(b_2, \ldots, b_n)$

Case 2: $v_1 < \max(b_2, \ldots, b_n)$

$b_1 \geq \max(b_2, \ldots, b_n)$
1 Wins

$u_1 = v_1 - \max(b_2, \ldots, b_n)$
< 0

Independent of b_1

$b_1 < \max(b_2, \ldots, b_n)$
1 loses

$u_1 = 0$

max utility $= 0$ obtained when 1 loses!
utility with $b_1 = v_1$ is $= 0$

We have proved only the first part.
Leave it to you show that
$u_i(v_i, \underline{b}_{-i}) > u_i(b_i', \underline{b}_{-i})$ for some $\underline{b}_{-i} \in \underline{S}_{-i}$