Lecture 21 The Shapley Value

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The Shapley value

~~ne of the many brilliant~~ contributions of Lloyd Shapley. PhD work in the late 1940s and 1950.

Given a TU game, the Shapley value $\phi(N, v)$ provides a reasonable and fair way of dividing gains from cooperation given the strategic dynamics captured by the characteristic function v.

Uses "marginal contribution" as the guiding principle. Based on three axioms:

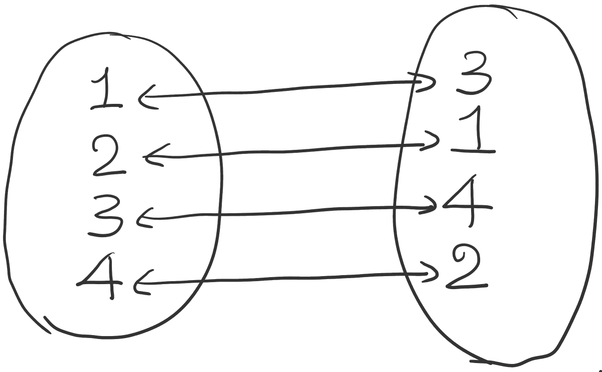
1. Symmetry
2. Linearity
3. Carrier

$$\phi(N, v) = (\phi_1(N, v), \dots, \phi_n(N, v))$$

We use shorthand notation:

$$\phi(v) = (\phi_1(v), \dots, \phi_n(v))$$

Axiom 1 : Symmetry



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Suppose $\pi : N \to N$ is a permutation of $(1, 2, \ldots, n)$. Define the game $(N, \pi v)$ as follows:

$$\pi v(C) = v(\pi^{-1}(C))$$

For example,

$$\pi v(\{1,4\}) = v(\{2,3\})$$
$$\pi v(\{1,2\}) = v(\{2,4\})$$

The symmetry axiom states:

$$\varphi_{\pi(i)}(\pi v) = \varphi_i(v)$$

Another way would be:

$$\varphi_i(\pi v) = \varphi_{\pi^{-1}(i)}(v)$$

This basically asserts that "renaming" does not affect the Shapley value of a player.

Axiom 2: Linearity

Given two games (N, v) and (N, w), consider the convex combination game with char. function

$$pv + (1-p)w \quad \text{where } p \in [0,1]$$

The axiom states:

$$\varphi_i\big(pv + (1-p)w\big) = p\,\varphi_i(v) + (1-p)\,\varphi_i(w)$$
$$\forall i \in N \quad \forall p \in [0,1]$$

Axiom 3: Carrier

$T \subset N$ is called a "carrier" if

$D \subseteq N$ is called a "carrier" if

$$v(C \cap D) = v(C) \quad \forall C \subseteq N$$

Suppose D is a carrier. The following are true:

observation 1:

Suppose $i \notin D$. Consider

$$v(\{i\}) = v(\{i\} \cap D) = v(\phi) = 0$$

Thus an element not belonging to D is a "dummy" player. This means all influential players are contained in D. Of course, D may contain non-influential players also.

observation 2:

If D is a carrier, $D \cup \{i\}$ is also a carrier for any $i \in N$.

$$\begin{aligned} v(C \cap (D \cup \{i\})) &= v((C \cap D) \cup (C \cap \{i\})) \\ &= v(C \cap D) \\ &= v(C) \end{aligned}$$

observation 3:

N is always a carrier since

$$v(C \cap N) = v(C) \quad \forall C \subseteq N$$

observation 4:

Suppose D is a carrier. Then

Observation 4.

Suppose D is a carrier. Then
$v(N \cap D) = v(N)$. This means
$v(D) = v(N)$ for any carrier D.

Carrier Axiom:

If D is a carrier, then

$$\sum_{i \in D} \varphi_i(v) = v(D) = v(N)$$

This means the value of the grand coalition is the sum of Shapley values of elements in any carrier D.

Since N is a carrier,

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

The Shapley Theorem

There exists exactly one mapping

$$\varphi : \mathbb{R}^{2^n - 1} \longrightarrow \mathbb{R}^n$$

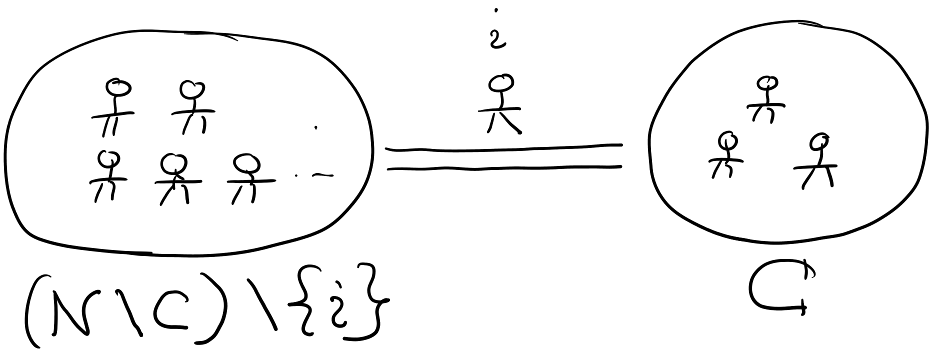
that satisfies the symmetry, linearity, and carrier axioms and

$$\varphi_i(v) = \sum_{C \subseteq N-i} \frac{|C|(n-|C|-1)!}{n!} \left\{ v(C \cup \{i\}) - v(C) \right\}$$

expected marginal contribution of player i to the worth of any coalition.

$$\frac{|C|\,(n-|C|-1)!}{n!}$$

Probability that in any permutation of $(1, 2, \ldots, n)$, the members of C are ahead of player i



Example for computation of Shapley Value

Divide-the-Dollar Version 3

$$v(1) = v(2) = v(3) = v(23) = 0$$

$$v(12) = v(13) = v(123) = 300$$

$$\begin{aligned}
\varphi_1(v) = \; & \frac{2}{6}\big(v(1) - v(\phi)\big) && C = \phi \\
+ \; & \frac{1}{6}\big(v(12) - v(2)\big) && C = \{2\} \\
+ \; & \frac{1}{6}\big(v(13) - v(3)\big) && C = \{3\} \\
+ \; & \frac{2}{6}\big(v(123) - v(23)\big) && C = \{2, 3\}
\end{aligned}$$

$$= 200$$

we can also compute

$$\varphi_2(v) = \varphi_3(v) = 50$$

Alternate Formulation for Shapley Value

Suppose $\pi : N \to N$ is a permutation. Define

$P(\pi, i)$ = Set of players preceding i in permutation π

$m(\pi, i)$ = marginal contribution of i to the set of predecessors in π

$$= v\left(P(\pi, i) \cup \{i\}\right) - v\left(P(\pi, i)\right)$$

It can be shown that

$$\varphi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m(\pi, i) \qquad i = 1, 2, \ldots, n$$

expected marginal contribution of i to its predecessors with expectation taken over all permutations.

Shapley value in a Convex Game

Suppose (N, v) is a convex game. We have seen for any permutation π that

$$\left(m(\pi, 1), m(\pi, 2), \ldots, m(\pi, n) \right) \in C(N, v)$$

Since the core is a convex set, we therefore know that any convex combination of core elements is in the core. Consider the following convex combination:

$$\sum \frac{1}{n!} m(\pi, i)$$

$$\sum_{\pi \in \Pi(N)} \frac{}{n!} \cdots$$

The above is precisely the Shapley value.
Hence in convex games,

$$\varphi_i(v) \in C(N, v)$$

Applications of Shapley Value

(1) Influence maximisation in social networks, influence limitation in rumour networks, etc.

(2) Vaccination planning

(3) Cost sharing, surplus sharing

⋮

Other Solution concepts

- Bargaining Set
- Kernel
- Nucleolus
- Shapley – Shubik index

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