$\Pi = \langle N, (S_i), (u_i) \rangle$

PSNE

$(s_1^*, s_2^*, \ldots, s_n^*) \in S$ is a PSNE if $\forall i \in N$,

$u_i(s_i^*, \underline{s}_i^*) \geq u_i(s_i, \underline{s}_i^*) \ \forall s_i \in S_i$

or equivalently

$u_i(s_i^*, \underline{s}_i^*) = \max_{s_i \in S_i} u_i(s_i, \underline{s}_i^*)$

Suppose

$b_i(\underline{s}_i) = \{ s_i \in S_i : u_i(s_i, \underline{s}_i) \geq u_i(s_i', \underline{s}_i) \ \forall s_i' \in S_i \}$

Best Response Correspondence

$(s_1^*, \ldots, s_n^*)$ is a PSNE iff

$s_i^* \in b_i(\underline{s}_i^*) \quad \forall i \in N \quad \ldots\ldots (1)$

Define

$b(s_1, s_2, \ldots, s_n) = b_1(\underline{s}_1) \times \ldots \times b_n(\underline{s}_n)$

therefore

$b(s_1^*, s_2^*, \ldots, s_n^*) = b_1(\underline{s}_1^*) \times \ldots \times b_n(\underline{s}_n^*)$

(1) Can be written as

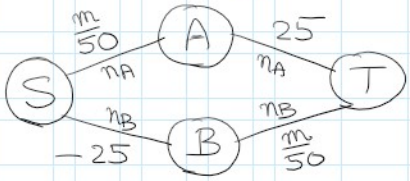
$(s_1^*, \ldots, s_n^*)$ is a PSNE iff

$(s_1^*, \ldots, s_n^*) \in b_1(\underline{s}_1^*) \times \ldots \times b_n(\underline{s}_n^*)$
$= b(s_1^*, \ldots, s_n^*)$

Thus we have $s^* \in b(s^*)$

$\therefore$ PSNE is a fixed point of the best response correspondence "b"

Braess Paradox game without bridge



$$u_i(s_1, s_2, \ldots, s_n) = -\left(\frac{n_A(s)}{50}\right) - 25 \quad (s_i = A)$$
$$= -25 - \left(\frac{n_B(s)}{50}\right) \quad (s_i = B)$$

Consider a strategy profile

$s = (s_i, \underline{s}_i)$ such that

$n_A(s) = n_B(s) = 500$

$s = (s_i, \underline{s}_i)$ such that

$n_A(s) = n_B(s) = 500$

Note that

$$u_i(s_i, \underline{s}_i) = -10 - 25 \quad (s_i = A)$$
$$= -25 - 10 \quad (s_i = B)$$

Suppose $s_i = A$

$$u_i(A, \underline{s}_i) = -35$$

whereas

$$u_i(B, \underline{s}_i) = -25 - \frac{501}{50} < -35$$

Suppose $s_i = B$

then we see that

$$u_i(B, \underline{s}_i) = -35$$

whereas

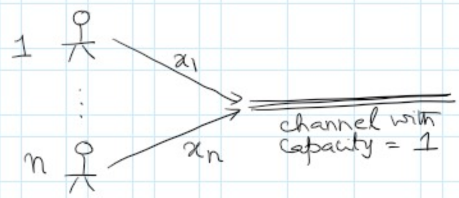
$$u_i(A, \underline{s}_i) = -\frac{501}{50} - 25 < -35$$

Thus $(s_i, \underline{s}_i)$ where $n_A(s) = n_B(s)$ is a PSNE.

Question: Are there any other PSNEs for this example?

Example 2: Bandwidth sharing game

$\exists$ a communication channel of capacity $= 1$



$N = \{1, 2, \dots, n\}$ Transmitters

$S_1 = S_2 = \dots = S_n = [0, 1]$

$x_i \in S_i$ Flow from player i

If $\sum_{j \in N} x_j \geq 1$, then no transmission and $u_i = 0 \ \forall i \in N$

If $\sum_{j \in N} x_j < 1$, then

$$u_i(x_1, x_2, \dots, x_n) = x_i \left(1 - \sum_{j=1}^{n} x_j\right)$$
$$= x_i (1 - x_i - t_i)$$

where $t_i = \sum_{j \neq i} x_j$

Suppose $(x_1^*, \dots, x_n^*)$ is a PSNE. Then

$$u_i(x_i^*, \underline{x}_i^*) = \max_{x_i \in S_i} u_i(x_i, \underline{x}_i^*) \quad \forall i \in N$$

Thus we have

$$x_i^* = \underset{x_i \in [0,1]}{\operatorname{argmax}} \; x_i (1 - x_i - t_i) \quad \forall i \in N \qquad \text{where } t_i = \sum_{j \neq i} x_j^*$$

$$\Rightarrow \quad x_i^* = \frac{1 - t_i}{2} \quad \forall i \in N$$

$$\Rightarrow \quad x_i^* = \frac{1 - \sum_{j \neq i} x_j^*}{2} \quad \forall i \in N$$

one can show that the above system of equations has a unique solution:

$$x_i^* = \frac{1}{n+1} \quad \forall i \in N$$

Thus $\left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$ is a unique PSNE for this system.

$$u_i(x^*) = \left(\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right) = \frac{1}{(n+1)^2}$$

$$\sum u_i(x^*) = \frac{n}{(n+1)^2}$$

Consider the non-PSNE profile

$$x^{\circ} = \left(\frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n}\right)$$

$$\sum u_i(x^{\circ}) = \frac{n}{2n}\left(1 - \frac{n}{2n}\right) = \frac{1}{4}$$

$$> \frac{n}{(n+1)^2} \quad \text{for any } n \geq 2$$

This shows that a PSNE may not maximize the social welfare.

Leads to the notion of price of anarchy.

Interpretations of NE

(1) Prescription

(2) Prediction

(3) Self-enforcing agreement

(3) Self-enforcing agreement
(4) Evolution and steady-state

Mixed Strategies

Consider $\Pi = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

A mixed strategy for player i is a simply a probability distribution on the pure strategies S_i

$\sigma_i : S_i \to [0,1]$ satisfying

$\sigma_i(s_i) \geq 0 \quad \forall s_i \in S_i$

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1$$

$$\Pi_{ME} = \langle N, (\Delta(S_i))_{i \in N}, U_i \rangle$$

where

$\Delta(S_i)$ is the collection of all probability distributions over S_i

$$\Delta(S_i) = \left\{ (\sigma_{i1}, \ldots, \sigma_{i|S_i|}) : \sigma_{ij} \geq 0 \text{ for } j = 1, 2, \ldots, |S_i| ; \sum_{j=1}^{|S_i|} \sigma_{ij} = 1 \right\}$$

We make the independent randomization assumption:

$$\sigma(s_1, s_2, \ldots, s_n) = \sigma_1(s_1)\,\sigma_2(s_2) \cdots \sigma_n(s_n) \quad \forall (s_1, s_2, \ldots, s_n) \in S$$

$U_2 : \bigtimes_{i \in N} \Delta(S_2) \to \mathbb{R}$ defined by

$$U_2(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1, \ldots, s_n) \in S} \sigma(s_1, \ldots, s_n)\, u_i(s_1, \ldots, s_n)$$

$$= \sum_{(s_1, \ldots, s_n) \in S} \left(\prod_{j \in N} \sigma_j(s_j) \right) u_i(s_1, \ldots, s_n)$$

<u>An Example</u> (BOS)

	A	B
A	2, 1	0,0
B	0,0	1, 2

$\sigma_1 = (\sigma_1(A), \sigma_1(B))$

$\sigma_2 = (\sigma_2(A), \sigma_2(B))$

$$u_1(\sigma_1, \sigma_2) = \sigma_1(A)\sigma_2(A)\, u_1(A, A) + \sigma_1(A)\sigma_2(B)\, u_1(A, B) + \sigma_1(B)\sigma_2(A)\, u_1(B, A) + \sigma_1(B)\sigma_2(B)\, u_1(B, B)$$

$$= 2\sigma_1(A)\sigma_2(A) + \sigma_1(B)\sigma_2(B)$$

Similarly

$u_2(\sigma_1, \sigma_2) = \sigma_1(A)\sigma_2(A) + 2\sigma_1(B)\sigma_2(B)$

Suppose $\sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$; $\sigma_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$

then we get

$u_1(\sigma_1, \sigma_2) = \frac{2}{3}$; $u_2(\sigma_1, \sigma_2) = \frac{2}{3}$

Mixed Strategy Nash Equilibrium

$$\Pi = \left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$$

$(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is called a MSNE if

$\forall i \in N,$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta(S_i)$$

Suppose

$$b_i(\sigma_{-i}) = \left\{ \sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \forall \sigma_i' \in \Delta(S_i) \right\}$$

Then $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is MSNE

$$\iff \sigma_i^* \in b_i(\sigma_{-i}^*) \quad \forall i \in N$$

The above can also be written as

$$\sigma^* \in b(\sigma^*)$$

Thus MSNE is a fixed point of the best response correspondence.

Example:

BoS Example

1 \ 2	A	B
A	2,1	0,0
B	0,0	1,2

$\sigma_1 = (\sigma_1(A), \sigma_1(B))$

$\sigma_2 = (\sigma_2(A), \sigma_2(B))$

$$u_1(\sigma_1,\sigma_2) = 1 + 3\sigma_1(A)\sigma_2(A) - \sigma_1(A) - \sigma_2(A)$$

$$u_2(\sigma_1,\sigma_2) = 2 + 3\sigma_1(A)\sigma_2(A) - 2\sigma_1(A) - 2\sigma_2(A)$$

Suppose (σ_1^*, σ_2^*) is a MSNE. Then

$$3\sigma_1^*(A)\sigma_2^*(A) - \sigma_1^*(A) - \sigma_2^*(A)$$
$$\geq 3\sigma_1(A)\sigma_2^*(A) - \sigma_1(A) - \sigma_2^*(A) \quad \forall \sigma_1 \in \Delta(S_1)$$

this leads to

$$3\sigma_1^*(A)\sigma_2^*(A) - \sigma_1^*(A) \geq 3\sigma_1(A)\sigma_2^*(A) - \sigma_1(A)$$
$$\forall \sigma_1 \in \Delta(S_1)$$

Similarly

$$3\sigma_1^*(A)\sigma_2^*(A) - 2\sigma_2^*(A) \geq 3\sigma_1^*(A)\sigma_2(A) - 2\sigma_2(A)$$
$$\forall \sigma_2 \in \Delta(S_2)$$

This leads to

$$\sigma_1^*(A)\left(3\sigma_2^*(A) - 1\right) \geq \sigma_1(A)\left(3\sigma_2^*(A) - 1\right)$$
$$\forall \sigma_1 \in \Delta(S_1)$$

$$\sigma_2^*(A)\left(3\sigma_1^*(A) - 2\right) \geq \sigma_2(A)\left(3\sigma_1^*(A) - 2\right)$$
$$\forall \sigma_2 \in \Delta(S_2)$$

We have 3 cases

$3\sigma_2^*(A) > 1$ $\quad$ $3\sigma_2^*(A) < 1$ $\quad$ $3\sigma_2^*(A) = 1$

Case 1: $3\sigma_2^*(A) > 1$

$$\Longrightarrow \sigma_1^*(A) \geq \sigma_1(A) \quad \forall \sigma_1 \in \Delta(S_1)$$
$$\Longrightarrow \sigma_1^*(A) = 1$$
$$\Longrightarrow \sigma_2^*(A) \geq \sigma_2(A) \quad \forall \sigma_2 \in \Delta(S_2)$$
$$\Longrightarrow \sigma_2^*(A) = 1$$

This leads to a MSNE (PSNE) : (A, A)

Case 2 : $3\sigma_2^*(A) < 1$

this leads to a MSNE (PSNE) : (B, B)

Case 3 : $3\sigma_2^*(A) = 1$

$$\Longrightarrow \sigma_2^*(A) = \frac{1}{3}$$
$$\Longrightarrow \sigma_2^*(B) = \frac{2}{3}$$
$$\Longrightarrow 3\sigma_1^*(A) - 2 = 0$$
$$\Longrightarrow \sigma_1^*(A) = \frac{2}{3}$$

This leads to a MSNE

$$\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$$