## Computation of MSNE

Given $\Gamma = \langle N, (S_i), (u_i) \rangle$, how do we determine mixed strategy Nash Equilibria?

## Existence of MSNE

John Nash proved the fundamental result that every finite strategic form game is guaranteed to have a MSNE.

## NASC for MSNE

$\Gamma = \langle N, (S_i), (u_i) \rangle$. $(\sigma_1^*, \ldots, \sigma_n^*)$ is a MSNE iff $\forall i \in N$.

(1) $u_i(s_i, \underline{\sigma}_{-i}^*)$ is the same $\forall s_i \in \delta(\sigma_i^*)$

(2) $u_i(s_i, \underline{\sigma}_{-i}^*) \geq u_i(s_i', \underline{\sigma}_{-i}^*)$
$\forall s_i \in \delta(\sigma_i^*)$ $\forall s_i' \in S_i \setminus \delta(\sigma_i^*)$

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Given a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$, the support is defined by

$$\delta(\sigma_1, \ldots, \sigma_n) = \delta(\sigma_1) \times \cdots \times \delta(\sigma_n)$$

Suppose $k_i = |S_i|$ for $i = 1, 2, \ldots, n$

\# of possible supports for $\sigma_1 = 2^{k_1} - 1$

$\vdots$

\# of possible supports for $\sigma_n = 2^{k_n} - 1$

\# of possible supports for $(\sigma_1, \ldots, \sigma_n)$

$$= (2^{k_1} - 1)(2^{k_2} - 1) \cdots (2^{k_n} - 1)$$

This can be a really huge number!

Suppose we want to explore

$$X_1 \times X_2 \times \cdots \times X_n$$

as a possible support for a MSNE, where

$$X_i \subseteq S_i \quad i = 1, 2, \ldots, n$$

If $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a MSNE with the above support, then there must exist real numbers $w_1, w_2, \ldots, w_n$, such that

$$w_i = u_i(s_i, \underline{\sigma}_{-i}) \quad \forall s_i \in X_i \quad \forall i \in N$$

$$= \sum_{\underline{s}_{-i} \in \underline{S}_{-i}} \left( \prod_{\substack{j \neq i \\ j \in N}} \sigma_j(s_j) \right) u_i(s_i, \underline{s}_{-i}) \quad (1)$$

$$\forall s_i \in X_i \ \forall i \in N$$

$$w_i \geq \sum \left( \prod \sigma_j(s_j) \right) u_i(s_i, \underline{s}_{-i}) \quad (2)$$

$$w_i \geq \sum_{\underline{s}_i \in \underline{S}_i} \left( \prod_{\substack{j \neq i \\ j \in N}} \sigma_j(s_j) \right) u_i(s_i, \underline{s}_i) \quad \forall s_i \in S_i \setminus X_i \quad \forall i \in N \qquad (2)$$

$$\sigma_i(s_i) > 0 \quad \forall s_i \in X_i \quad \forall i \in N \qquad (3)$$

$$\sigma_i(s_i) = 0 \quad \forall s_i \in S_i \setminus X_i \quad \forall i \in N \qquad (4)$$

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1 \quad \forall i \in N \qquad (5)$$

In order to find a MSNE $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we then need to find

$w_1, w_2, \ldots, w_n$

$\sigma_1(s_1) \ \forall s_1 \in S_1$

$\sigma_2(s_2) \ \forall s_2 \in S_2$

$\vdots$

$\sigma_n(s_n) \ \forall s_n \in S_n$

satisfying (1), (2), (3), (4), (5).

We have

$n + \sum_{i \in N} k_i$ variables and

$n + 2\sum_{i \in N} k_i$ equations

for each support $X_1 \times X_2 \times \cdots \times X_n$.

this is a tall order indeed!

The above set of inequalities constitutes so called Nonlinear Complementarity Problem (NLCP). For two player games, this becomes an LCP.

The GTMD book contains an interesting discussion on this topic. Also see the interesting example (original appears in Myerson)

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## Some Results on the Existence of Nash Equilibria

### Existence of Pure Strategy NE (Debreu 1952)

$\Gamma = \langle N, (S_i), (u_i) \rangle$

A PSNE exists if $\forall i \in N$,

(1) $S_i$ is non-empty, convex, compact subset of some Euclidean space

(2) $u_i(s_1, \ldots, s_n)$ is continuous in $(s_1, \ldots, s_n)$

(3) $u_i(s_i, \underline{s}_i)$ is quasi-concave in $s_i$

Note: The above theorem does not apply to finite games.

## Von Neumann – Oskar Morgenstern (1928)

Every two player zero sum game has a MSNE.

- Brouwer's fixed point theorem
- Also using an LP formulation

## John Nash (1950)

Every finite strategic form game has a MSNE

- Uses Kakutani's Fixed Point theorem
- A proof based on Sperner's Lemma is very popular

## Glicksberg theorem (1952)

Consider an infinite strategic form game $\langle N, (S_i), (u_i) \rangle$ such that $\forall i \in N$:

(1) $S_i$ is non-empty and compact

(2) $u_i(s_i, \underline{s}_i)$ is continuous in $s_i$ and $\underline{s}_i$

Then the game has a MSNE.

Please go through Chapter 10 of GTMD book.

## Brouwer's Fixed Point Theorem (1912)

Suppose $X \subset \mathbb{R}^n$ is non-empty, compact, and convex. If $f: X \to X$ is continuous, then $\exists\, x \in X \ni f(x) = x$.

## Kakutani Fixed Point Theorem (1941)

Suppose $X \subset \mathbb{R}^n$ is non-empty, compact, and convex. If $f: X \to X$ is a correspondence (mapping elements of $X$ to subsets of $X$) such that

(a) $f$ is upper hemicontinuous

(b) $f(x)\ \forall x$ is non-empty and convex

Then $\exists\, x \in X \ni x \in f(x)$

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## Maxmin and Minmax Values and Strategies

Motivation: To analyze worstcase behaviour.

### Maxmin Value

Best possible payoff that can be guaranteed to a player in the worstcase when the other players are free to choose any strategies. (Lowerbound on the payoff to player $i$)

$$\underline{v_i} = \max_{s_i \in S_i} \min_{\underline{s}_i \in \underline{S}_i} u_i(s_i, \underline{s}_i)$$

Note that

$$\min_{\underline{s}_i \in \underline{S}_i} u_i(s_i, \underline{s}_i)$$

is the minimum payoff to player $i$ when he plays $s_i$, when other players are free to play whatever they wish to.

| 1 \ 2 | A | B |
|---|---|---|
| A | 4, 1 | 0, 4 |
| B | 1, 5 | 1, 1 |

Note that the above game does not have a PSNE.

Player 1
By playing A, min payoff = 0
By playing B, min payoff = 1
Player 1 is guaranteed 1 by playing B
max (0, 1) is called maxmin value
strategy B is called maxmin strategy.

| | A | B |
|---|---|---|
| A | 4, 1 | 0, 4 |
| B | 1, 5 | 1, 1 |

Player 2
By playing A, min payoff = 1
By playing B, min payoff = 1
max (1, 1) = 1 is the maxmin value
Both strategies A, B are maxmin strategies.

Maxmin strategy is a no-regret strategy — player $i$ is guaranteed to receive at least $\underline{v_i}$ whatever the other players choose to do.

Nash equilibrium strategy is not a no-regret strategy. other players can choose their strategies to bring down the utility of this player.

| | A | B |
|---|---|---|
| A | 10, 10 | 0, 0 |
| B | 0, 0 | 1, 1 |

Suppose $(s_1^*, s_2^*, \ldots, s_n^*)$ is a PSNE.
Then

$$
\begin{aligned}
u_i(s_i^*, \underline{s}_i^*) &= \max_{s_i \in S_i} u_i(s_i, \underline{s}_i^*) \\
&\geq \max_{s_i \in S_i} \left( \min_{\underline{s}_i \in \underline{S}_i} u_i(s_i, \underline{s}_i) \right) \\
&= \max_{s_i \in S_i} \min_{\underline{s}_i \in \underline{S}_i} u_i(s_i, \underline{s}_i) \\
&= \underline{v_i}
\end{aligned}
$$

## Minmax values and minmax strategies

Minmax value of player $i$
Maximum payoff value to which player $i$ can be restricted to by the other players. (Upperbound on the payoff to player $i$)

$$\overline{v_i} = \min_{\underline{s}_i \in \underline{S}_i} \max_{s_i \in S_i} u_i(s_i, \underline{s}_i)$$

| 1 \ 2 | A | B |
|---|---|---|
| A | 4, 1 | 0, 4 |
| B | 1, 5 | 1, 1 |

Max payoff for player 1 when player 2 plays A

plays A
$= \max(4,1) = 4$

Max payoff to player 1 when player 2 plays B
$= \max(0,1) = 1$

By playing B, player 2 can restrict player 1 to get at most 1.

$\min(4,1)$ is called minmax value of player 1.

Strategy B of player 2 is called the minmax strategy against player 1.

| | A | B |
|---|---|---|
| A | 4, 1 | 0, 4 |
| B | 1, 5 | 1, 1 |

Similarly, minmax value of player 2
$= \min(4,5) = 4$

Strategy A of player 1 is called the minmax strategy against player 2

Suppose $\underline{s}_{i}^{minmax}$ is a minmax strategy against $i$. Then

$$\overline{v_i} = \max_{s_i \in S_i} u_i\left(s_i, \underline{s}_{i}^{minmax}\right)$$
$$\geqslant \max_{s_i \in S_i} \left( \min_{\underline{s}_i \in \underline{S}_i} u_i(s_i, \underline{s}_i) \right)$$
$$= \underline{v_i}$$

$\therefore \overline{v_i} \geqslant \underline{v_i}$

How does minmax value compare to a Nash equilibrium utility.

Suppose $(s_i^*, \underline{s}_i^*)$ is a PSNE.

$$u_i(s_i^*, \underline{s}_i^*) = \max_{s_i \in S_i} u_i(s_i, \underline{s}_i^*)$$

$\underline{s}_i^*$ is one particular value of $\underline{s}_i$

$$\geqslant \min_{\underline{s}_i \in \underline{S}_i} \left( \max_{s_i \in S_i} u_i(s_i, \underline{s}_i) \right)$$
$$= \overline{v_i}$$

Thus if $(s_i^*, \underline{s}_i^*)$ is a PSNE, then

$$u_i(s_i^*, \underline{s}_i^*) \geqslant \overline{v_i} \geqslant \underline{v_i}$$

The same discussion can be generalized to maxmin and minmax in mixed strategies.

Question: Are there games for which

$$u_i(s_i^*, \underline{s}_i^*) = \overline{v_i} = \underline{v_i}$$

Yes: Two player zerosum games.

Happens under some conditions for pure strategies

Always happens for mixed strategies.