The Core

The core is a set solution concept for coalitional games. The notion of "core" is motivated by the consideration to keep all coalitions "happy" and "stable."

(N, v) TU game

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ – an allocation is called coalitionally rational if

$$\sum_{i \in C} x_i \geq v(C) \quad \forall C \subseteq N$$

Special case of coalitional rationality is individual rationality which means

$$x_i \geq v(\{i\}) \quad \forall i \in N$$

The allocation is called collectively rational if

$$\sum_{i \in N} x_i = v(N)$$

The core of (N, v) is the collection of all collectively rational and coalitionally rational allocations.

$$C(N, v) = \Big\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in C} x_i \geq v(C) \ \forall C \subseteq N; \ \sum x_i = v(N) \Big\}$$

✓

$i \in C$

$$\sum_{i \in N} x_i = v(N) \Bigg\}$$

Implications of "Core"

1. A coalition C is said to "block" an allocation $(x_1, \ldots, x_n)$ if

$$\sum_{i \in C} x_i < v(C)$$

Note that we can "improve upon" this allocation with (uncountably infinite number of) allocations $(y_1, \ldots, y_n)$ such that

$$\sum_{i \in C} y_i \leq v(C) \quad \text{and} \quad y_i > x_i \quad \forall i \in N$$

The core $C(N, v)$ is the set of allocations $x \in \mathbb{R}^n$ such that no coalition can block x.

2. A core allocation is stable and arises out of "effective" negotiations. If a "referee" recommends a core allocation, no coalition will be unhappy about it

3. $C(N, v)$ is convex, closed, bounded.

4. $C(N, v)$ can be empty, singleton, or may consist of uncountably infinite # of elements.

Example 1 : Version 2 of Divide-the-Dollar

$C(N, v) = \{ (x_1, x_2, 0) : x_1 + x_2 = 300 ; x_1 \geq 0 ; x_2 \geq 0 \}$

$(300, 0, 0) \in C(N, v)$
$(0, 300, 0) \in C(N, v)$
$(150, 150, 0) \in C(N, v)$
$(149, 149, 2) \notin C(N, v)$ since $\{1, 2\}$ blocks this.

Example 2 : Version 3 of Divide-the-Dollar

$C(N, v) = \{ (300, 0, 0) \}$

$(298, 1, 1) \notin C(N, v)$ since $\{1, 2\}$ blocks this
$\{1, 3\}$ also blocks this

$(150, 150, 0) \notin C(N, v)$ since $\{1, 3\}$ blocks this

Example 3 : Version 4 (Majority Voting Game)

$C(N, v) = \phi$

The allocation $(100, 100, 100)$ is blocked by $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$.

$(150, 150, 0)$ is blocked by $\{1, 3\}$ and $\{2, 3\}$.

Example 4 : House Allocation

$C(N, v) = \{ (2, 2, 2) \}$

This corresponds to seller selling the house at max price of 2 to player 2 or player 3.

$(1, 3, 2) \notin C(N, v)$ since $\{1, 3\}$ will block

... since $\{1, 3\}$ can block

$(1,3,2) \notin \mathcal{C}(N,v)$ since $\{1,3\}$ will block

$(1.99, 2.01, 2) \notin \mathcal{C}(N,v)$ since $\{1,3\}$ can block

Shapley – Bondereva characterization

Consider the following minimization problem.

minimize $x_1 + \cdots + x_n$ s.t.

$$\sum_{i \in C} x_i \geq v(C) \quad \forall C \subseteq N$$

$$x_i \in \mathbb{R} \quad \forall i \in N$$

This LP certainly has a feasible solution. An optimal solution will give the least amount of transferrable utility that is necessary for no coalition to block it.

Suppose $(x_1^*, \ldots, x_n^*)$ is an optimal solution. Since it is feasible:

$$x_1^* + \cdots + x_n^* \geq v(N)$$

$x_1^* + \cdots + x_n^* > v(N)$
Then the core is empty

$x_1^* + \cdots + x_n^* = v(N)$
The core is non-empty and each optimal solution is an element of the core

Example: Version 3

min $x_1 + x_2 + x_3$ s.t.

$$\min\ x_1 + x_2 + x_3 \quad \text{s.t.}$$

$$x_1 + x_2 \geq 300$$
$$x_1 + x_3 \geq 300$$
$$x_2 + x_3 \geq 0$$
$$x_1 + x_2 + x_3 \geq 300$$
$$x_1 \geq 0\ ;\ x_2 \geq 0\ ;\ x_3 \geq 0$$

optimal solution is $x_1^* = 300\ ;\ x_2^* = x_3^* = 0$

Example: Majority Voting Game

$$\min\ x_1 + x_2 + x_3 \quad \text{s.t.}$$

$$x_1 + x_2 \geq 300$$
$$x_1 + x_3 \geq 300$$
$$x_2 + x_3 \geq 300$$
$$x_1 + x_2 + x_3 \geq 300$$
$$x_1 \geq 0\ ;\ x_2 \geq 0\ ;\ x_3 \geq 0$$

optimal soln: $x_1^* = x_2^* = x_3^* = 150$

$x_1^* + x_2^* + x_3^* = 450 > 300 = v(N)$

Hence $C(N, v)$ is empty.

Core of a Convex Game

Suppose (N, v) is convex. Then $x = (x_1, \ldots, x_n) \in C(N, v)$ where

$$x_1 = v(1)$$
$$x_2 = v(12) - v(1)$$
$$x_3 = v(123) - v(12)$$
$$\vdots$$
$$x_n = v(N) - v(N \setminus \{n\})$$

$\vdots$

$x_n = v(N) - v(N \setminus \{n\})$

If we allocate marginal contributions to players in a convex game, the allocation is in the core.

Suppose Π is any permutation of N and let

$P(\Pi, i)$ = set of players preceding i in Π

Define

$m(\Pi, i) = v\left(P(\Pi, i) \cup \{i\}\right) - v\left(P(\Pi, i)\right)$

Then $(x_1, \ldots, x_n) \in C(N, v)$ where

$x_1 = m(\Pi, 1)$

$x_2 = m(\Pi, 2)$

$\vdots$

$x_n = m(\Pi, n)$