Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Quiz 4

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Notations: (i) Vectors $\mathbf{v}$ and matrices $\mathbf{M}$ are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$. (iv) $\mathbf{I}_n$ denotes the identity matrix of order n. (v) 0_n denotes the null matrix of order $n \times n$

1. Which of the following is/are true? (2 Points)
   (a) $\mathbf{QR}$ factorization and $\mathbf{LU}$ factorization can turn out to be identical for certain full rank matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$.
   (b) $\mathbf{LU}$ factorization using Gaussian elimination without partial pivoting if it exists, is always unique for any full rank matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$.
   (c) Gaussian elimination algorithm with partial pivoting will not have row interchanges for row-diagonally dominant full rank matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$.
   (d) $\mathbf{LU}$ factorization using Gaussian elimination with partial pivoting algorithm results in $\mathbf{L}$ having $\|\mathbf{L}\| = O(1)$.

2. Which of the following is/are true? (2.5 Points)
   (a) Recall that a given matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ can always be written as $\mathbf{A} = \mathbf{A}_{\mathrm{s}} + \mathbf{A}_{\mathrm{ss}}$, such that $\mathbf{A}_{\mathrm{s}} = \mathbf{A}_{\mathrm{s}}^T$ and $\mathbf{A}_{\mathrm{ss}} = -\mathbf{A}_{\mathrm{ss}}^T$. If $\mathbf{A}_{\mathrm{s}}$ is symmetric positive definite, then $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
   (b) Let $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & -4 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & -8 \\ 0 & 0 & -67 \end{bmatrix}$ be the $\mathbf{LU}$ factorization of a given matrix $\mathbf{A}$. Then this matrix $\mathbf{A}$ has to be a symmetric positive definite matrix.
   (c) If $\mathbf{R}$ is an exact Cholesky factor of a symmetric positive definite matrix $\mathbf{A}$ and $\tilde{\mathbf{R}}$ is the Cholesky factor obtained on a finite precision computer for the same matrix $\mathbf{A}$. Then $\frac{\|\tilde{\mathbf{R}}^T\tilde{\mathbf{R}} - \mathbf{A}\|}{\|\mathbf{A}\|} \leq \frac{\|\tilde{\mathbf{R}} - \mathbf{R}\|}{\|\mathbf{R}\|}$
   (d) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix and $\mathbf{X} \in \mathbb{R}^{m\times m}$ is any matrix, then $\mathbf{X}^T\mathbf{A}\mathbf{X}$ is always symmetric positive definite.

3. Let, $\mathbf{M} \in \mathbb{R}^{n\times n}$ be a non-zero matrix with $n \geq 2$ and $\mathbf{M}^2 = \mathbf{0}$ where, $\mathbf{0}$ is a null matrix. Let, $\alpha \in \mathbb{R}\backslash\{0\}$. Then, (2 Points)

   (a) α is an eigenvalue of both $(\mathbf{M} + \alpha\mathbf{I})$ and $(\mathbf{M} - \alpha\mathbf{I})$

   (b) The algebraic multiplicity of the eigenvalue α for $(\alpha\mathbf{I} - \mathbf{M})$ is n

   (c) $-\alpha$ is one of the k distinct eigenvalues of $(\mathbf{M} - \alpha\mathbf{I})$ with $k > 1$

   (d) None of the above

4. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$ be any two non-zero matrices, then which of the following is true ? (2 Points)

   (a) If $\mathbf{A}$ is non-singular then $(\mathbf{B})$ and $(\mathbf{ABA}^{-1})$ have the same eigenvalues.

   (b) If $\mathbf{A}$ is real skew-symmeiric matrix with λ as eigenvalue then $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$.

   (c) Two diagonalizable matrices $\mathbf{A}$ and $\mathbf{B}$ with the same eigenvalues and eigenvectors must be the same matrix.

   (d) If λ is an eigenvalue with both algebraic multiplicity and geometric multiplicity to be 3 then the dimension of column space of $\mathbf{A} - \lambda\mathbf{I}_m$ is 3.

5. For any real symmetric matrix $\mathbf{A}$, the Rayleigh quotient is given by $r(\mathbf{x}) = \dfrac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$. Which of the following is/are correct. (3 Points)

   (a) For any $\mathbf{x}$, $r(\mathbf{x})$ is between the minimum and maximum eigenvalues of $\mathbf{A}$.

   (b) If the distance between a unit vector $\mathbf{x}$ and the eigenvector $\mathbf{q}_J$ of a real symmetric matrix $\mathbf{A}$, in the sense of 2 norm is Δ, i.e. $||\mathbf{x} - \mathbf{q}_J||_2 = \Delta$. Then $|\mathbf{x}^T\mathbf{A}\mathbf{x} - \lambda_J| = O(\Delta)$, where λ_J is the eigenvalue corresponding to eigenvector $\mathbf{q}_J$

   (c) Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. The value $\beta = 5$ minimizes the $||\mathbf{A}\mathbf{x} - \beta\mathbf{x}||$

   (d) Let $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{b} \\ \mathbf{b}^T & \gamma \end{bmatrix}$ (for some symmetric matrix $\mathbf{B}$, some vector $\mathbf{b}$, and some real number γ). Then, the smallest eigenvalue of $\mathbf{B} \leq$ smallest eigenvalue of $\mathbf{A}$.

6. Which of the following is/are true: (2 Points)

   (a) An eigenvalue solver can be designed to compute eigenvalues and eigenvectors of a given matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ in a finite number of steps, using exact arithmetic.

   (b) An eigenvalue solver designed to compute all eigenvalues and eigenvectors of a symmetric dense matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ requires at most $O(m^3)$ work, if it is not initially reduced to tri-diagonal form in Phase 1.

   (c) Power iteration produces a sequence of vectors $\mathbf{v}^{(i)}$ that converges to the eigenvector corresponding to the largest eigenvalue of $\mathbf{A} \in \mathbb{R}^{m\times m}$ starting with any initial guess vector $\mathbf{v}^{(0)} \neq \mathbf{0}$.

   (d) Let $\mathbf{F} \in \mathbb{R}^{m\times m}$ denote the Householder reflector that introduces zeros below the diagonal entry of symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ in the 1st column when pre-multiplied with $\mathbf{A}$. Then eigenvalues of $\mathbf{FAF}^T$ and $\mathbf{A}$ are the same.

7. Which of the following is/are true: (2 Points)

   (a) Suppose that matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ and λ be an eigenvalue of A, then so is $\bar{\lambda}$ (complex conjugate).

   (b) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be any non-zero diagonalizable matrix then rank of $\mathbf{A}$ = number of non-zero eigenvalues of $\mathbf{A}$.

   (c) All orthogonal matrices are normal matrices.

   (d) Consider the matrix $\mathbf{A} = \mathbf{I}_9 - 2\mathbf{u}\mathbf{u}^T$, with $\mathbf{u} = \frac{1}{3}[1,1,1,1,1,1,1,1,1]^T$ where $\mathbf{I}_9$ is a 9×9 identity matrix. If λ and μ are two distinct eigenvalues of $\mathbf{A}$, then $|\lambda - \mu| = 0$.

8. Given matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 0 & 8 & 0 & 4 & 2 \\ 0 & 0 & 0 & 8 & 4 \end{bmatrix}$, which of the following is/are true: (2 Points)

   (a) The sum of eigenvalues of $\mathbf{A}$ is 16.

   (b) The sum of eigenvalues of $\mathbf{AA}^T$ is 194.

   (c) The product of eigenvalues of $\mathbf{AA}^T$ is 9216.

   (d) Eigenvalues of $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are the same.

   (e) The eigenvalues of $\mathbf{AA}^T$ are always non-zero and positive.