Game Theory and Mechanism Design

Y. Narahari

Practice Problems in NBS, Core, Shapley Value

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Problem Set 6

Warm-up

1. Nash Bargaining Problem: (a) Investigate whether it is necessary that the default point should belong to the feasible set. (b) Why should the feasible set be convex? (c) Why should the feasible set be closed?

2. Question 3 on page 414

3. Question 4 on page 414

4. Is the majority voting game superadditive? Convex?

5. Is the minimal spanning tree game superadditive? Convex?

6. Compute the core of the logistics game discussed in Chapter 27. Recall that $N = \{1, 2, 3, 4\}$ and the characteristic function is

$$v(1) = v(2) = v(3) = v(4) = 0$$

$$v(12) = v(13) = v(14) = v(23) = v(24) = v(34) = v(234) = v(123) = 0$$

$$v(134) = 40;$$

$$v(124) = 45;$$

$$v(1234) = 65$$

7. Problem 6 on Page 445
Let us consider a version of divide the dollar problem with 4 players and total worth equal to 400. Suppose that any coalition with three or more players will be able to achieve the total worth. Also, a coalition with two players will be able to achieve the total worth only if player 1 is a part of the two player coalition. Set up a characteristic function for this TU game and compute the Shapley value.

8. Problem 7 on Page 445
   There are four players $\{1, 2, 3, 4\}$ who are interested in a wealth of 400 (real number). Any coalition containing at least two players and having player 1 would be able to achieve the total wealth of 400. Similarly, any coalition containing at least three players and containing player 2 also would be able to achieve the total wealth of 400. Set up a characteristic form game for this situation and compute the Shapley value.

9. Problem 9 on Page 445

Workhorse

1. Problem 1 on page 397
2. Problem 2 page 397 and problem 3 on page 398
3. Problem 4 on page 398
4. Problem 6 on Page 398
5. Question 2 on page 413
6. Question 9 on page 414
7. Problem 3 on Page 427
8. Problem 13 on Page 428
9. Consider the following characteristic form game with three players.

$$v(1) = v(2) = v(3) = 0;$$

$$v(12) = a; v(13) = b; v(23) = c;$$

$$v(123) = 1$$

   Assume that $0 \leq a, b, c \leq 1$.

   (a) Find the conditions under which the core is non-empty.
   (b) Compute the Shapley value.
   (c) Assuming the core is non-empty, does the Shapley value belong to the core? Under what conditions will the Shapley value belong to the core of this game.

   (d) Given a TU game (N, v), define the dual game (N, w) by

$$w(C) = v(N) - v(N \setminus C)\ \forall C \subseteq N$$

   Show that the dual of the dual game is the original game (primal game) itself. Also show that the Shapley values of the primal game and the dual game are identical.

Thought Provoking

1. It has been stated that the core of a TU game is convex and compact. Prove this result.

2. A market game is a TU game that consists of a set B of buyers and a set S of sellers such that $N = B \cup S$ and $B \cap S = \varnothing$, and $v(C) = min(\mid C \cap B \mid, \mid C \cap S \mid);\ \forall C \subseteq N$. Compute the core of a market game.

3. Consider the glove market example. What will be the core of this game if there are $1,000,000$ left glove suppliers and $1,000,000$ right glove suppliers?

4. Give an example of a non-convex game for which the core is non-empty and the Shapley value belongs to the core. Now give an example of another non-convex game for which the core is non-empty and the Shapley value does not belong to the core.

5. Problem 3 on page 444
   Suppose (N, v) is a TU game and we define a unique imputation as follows.

$$\xi_i(N, v) = v(\{1, ..., i-1, i\} - v(\{1, ..., i-1\})\ \forall i \in N$$

   Which of the Shapley axioms does the above satisfy and which of the Shapley axioms does it violate.