Computational & Data Sciences
Indian Institute of Science, Bangalore

AUGUST-DECEMBER SEMESTER, 2022
DS290 MODELING AND SIMULATION
final examination

(Time allowed: 180 minutes)

NOTE: Release: 01 Dec 2022, 1400 hours. Due by: 01 Dec 2022, 1700 hours.
Please use the "Turn-in" feature of the "Assignments" on the "MS Teams" for submitting your answer script. Please do not use email to turn in unless all attempts to use the Assignments fail, since there is a substantial probability of losing it in a cluttered mail box.
Points are uniformly distributed over the questions. There are 6 questions, each worth $8\frac{1}{3}$ points. Full points: 50.
IMPORTANT: Please write your name, SR No. and approximate number of lectures/classes out of the total 25 lectures/classes that you estimate to have attended. Please upload your answer script as a single PDF file.

1. Consider the two real n-dimensional vector valued Itô SDEs: $dY_t = A\sqrt{D(Y_t)}dW_t$ and $dX_t = U\Sigma(X_t)dB_t$ where $A \in \mathbb{R}^{n\times n}$ is a matrix of rank $l < n$; matrix $U \in \mathbb{R}^{n\times l}$'s columns are l different columns of a unitary matrix; $D : \mathbb{R}^n \to \mathbb{R}_+^{n\times n}$ is a non-negative diagonal matrix; W_t is an n-dimensional Wiener process in which components are pair-wise independent of each other and are identically distributed (i.e., i.i.d) and B_t is an l-dimensional i.i.d Wiener process. Given that both SDEs have identical initial conditions with probability 1, and that the solutions of the SDEs have the same transition probability density, find a U and an $l \times l$ matrix function $\Sigma : \mathbb{R}^n \to \mathbb{R}^{l\times l}$ in terms of A and $D(.)$.

2. Let W_t, V_t be two n-dimensional (n being a natural number) real Brownian motions (i.e., Wiener Process) which are independent of each other and each of them have independent components. The processes start at zero and let $U_1, U_2 \in \mathbb{R}^{n\times n}$ be two constant unitary matrices. Prove that $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} W_t \\ V_t \end{pmatrix}$ is also a Brownian motion. Use the matrix notation only. Do not use components/entries of the matrices in your proof.

3. Consider the numerical method: $\mathsf{X}_{n+1} = \mathsf{X}_n + \frac{1}{\sqrt{2}}\mathsf{X}_n(\sqrt{2} + \frac{1}{\sqrt{2}}\Delta W_n)\Delta W_n$ used to solve approximately the one dimensional Itô Stochastic Differential Equation (SDE): $dX_t = X_t dW_t$ where W is a real one dimensional Wiener Process and ΔW_n is its increment at the nth time step of magnitude h, i.e., $t_{n+1} - t_n = h$; and X_n is the numerical solution of the SDE at the nth time point along a given sample path.
(a) What multi-indices from $\{0, 1\}$ are effectively used in the above numerical method ? Identify the hierarchical set and remainder set of the multi-indices involved.
(b) What is the expected drift term from the numerical method ?
(c) Does the method converge to the solution of the SDE which it supposedly approximates and discretizes ? If yes, give the strong and/or weak order of convergence as applicable. If no, identify the term(s) that make(s) it non-convergent.

4. If we applied the Milstein method to solve the one dimensional Itô SDE $dX_t = \sqrt{X_t}dW_t,\ X_t > 0$ (where W is a real valued 1-dimensional Wiener Process), what is largest step size one can take so that, with probability 1, $\mathbf{E}(\delta \mathsf{X}_1)^2 \le \mathbf{E}(\delta \mathsf{X}_0)^2$, where, X_1 is the solution of the above SDE at the first time point after taking a single time step and $X_0 = \mathsf{X}_0$ is the initial condition with probability 1; $\delta \mathsf{X}_1$ is the perturbation in the solution X_1 by the Milstein method due to a perturbation $\delta \mathsf{X}_0$ in the initial condition ?

5. Let

$$dY_t = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{pmatrix} \quad (1)$$

be an Itô stochastic differential equation on the complete probability space $(\Omega, \mathcal{A}, P)$. Find a probability measure Q on $\mathcal{F}_t$ such that the following is satisfied on $(\Omega, \mathcal{F}, Q)$:

$$d\tilde{Y}_t = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} d\tilde{W}_t^{(1)} \\ d\tilde{W}_t^{(2)} \end{pmatrix} \quad (2)$$

where

$$\begin{pmatrix} \tilde{W}_t^{(1)} \\ \tilde{W}_t^{(2)} \end{pmatrix} = \begin{pmatrix} -3t \\ t \end{pmatrix} + \begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}. \quad (3)$$

6. A system takes discrete states labelled as $i = 1, \dots, N$ which are unit distance apart from each other (that is distance(i,j)=1 for any $i \ne j$). If the system output $x(t) \in \mathbb{R}$ at time t is in the interval $(x_i^{(L)}, x_i^{(U)}]$ then the system is said to in state i. The state change from i to j, $i \ne j$, happens as per the stochastic differential equation (SDE) $d_{(i,j)}x(t) = a_{ij}dt + \sigma_{ij}dW_t^{ij}$ where $d_{(i,j)}x(t)$ indicates the infinitesimal change of the output, dx, at time t along the transition from state i to state j; a_{ij} and σ_{ij} are constants indpendent of time, output and state values; and W^{ij} is one dimensional real-valued Wiener process such that for two distinct (non-coinciding) transitions (i,j) and (k,l), W^{ij} is independent of W^{kl}. Let $V_i(t)$ be a non-negative real function associated with the state i at time t.
Given $V_i(t)$ with probability 1 and a time interval $0 < \delta t \ll 1$, find (accurate up to $O(\delta t)$) the expectation $\mathsf{E}(\delta V_i(t))$ and the variance $\mathsf{Var}(\delta V_i(t))$ of the total change $\delta V_i(t)$ of $V_i(t)$ over δt at time t. State clearly any assumptions made.
Hint: Use the closed form exponential solution of a constant coefficient SDE and expand it. Also use the first order finite difference $(V_j - V_i)/1$ to evaluate the gradient which is a function of time and does not vary along the transition (i,j). Then use these in Ito Lemma applied to V_i.