Lecture 2: Myerson's Lemma

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1 Last Time

- Single Item Auction in sealed bid format
- Two decisions by the auction designer: (i) Allocation Rule - who gets the good (ii) Payment – the price charged to the winner
- Allocation Rule x : Bids $\rightarrow$ Winner. Formally, $x : \mathbb{R}_+^n \mapsto \{0,1\}^n$.
- Payment Rule p : Bides $\rightarrow$ Sale price. Formally, $p : \mathbb{R}_+^n \mapsto \mathbb{R}_+$.

Second Price (Vickery) Auction

- Bids submitted $b \in \mathbb{R}_+^n$. Write $i^* \in \arg\max_{i \in [n]} b_i$ and $B := \max_{i \neq i^*} b_i$.
- Highest bidder wins the item and is charged the second highest bid.
- Allocation rule $x_{i^*}(b) = 1$ and $x_i(b) = 0$ for all $i \neq i^*$. Also, $p_{i^*}(b) = B$ and $p_i(b) = 0$ for all $i \neq i^*$.

Theorem 1. *The Vickery auction simultaneously satisfies the following three distinct and desirable properties:*

1. *Strong incentive guarantees. DSIC (dominant strategy incentive compatible): truthful bidding is a* dominant strategy *and never leads to negative utility. (recall why this is a desirable property).*
2. *Strong performance guarantee. Social surplus maximization. Assuming truthful bidding (which is justified by (1)), the allocation maximizes $\sum_i v_i x_i$ where x_i is the amount of stuff allocated to agent i.*
3. *Computational efficiency. Can be implemented in linear time.*

Describe (i) identical items auctions auctions (ii) sponsored search auctions (model of Edelman et al. and Varian)

To obtain such a result for more complicated settings, e.g., Identical items auction and sponsored search auctions, we consider the following two step approach, which decouples the allocation rule and the payment rule.

Mechanism design to Algorithm design:

- Assume, without justification, that the bidders are truthful. Then, construct an allocation rule such that Properties (2) and (3) (performance and computational efficiency) are satisfied.
- Given an answer to the previous step, come up with a payment rule such that Property (1) (i.e., DSIC) holds.

Myerson's lemma provides a powerful and general tool for implementing Step 2.

2 Single Parameter Environments

This framework provides an abstraction to state Myerson's lemma, and captures the three settings (single item auction, multi-item auction, sponsored search auction) we have seen so far.

Here, we have n agents each with a private valuation $v_i \in \mathbb{R}_+$ which denotes agent i's "value per unit stuff" that it gets. Finally, in this environment there is a *feasible set* $X \subset \mathbb{R}^n$. Each element of X is an n-vector $(x_1, \ldots, x_n) \in X$, where x_i represents the "amount of stuff" given to agent i. Therefore, $x : \mathbb{R}_+^n \mapsto X \subset \mathbb{R}^n$.

Examples X in case of single item auction, identical items auctions, and sponsored search auctions.

Quasilinear utility of agent i in single parameter environments is expressed as

$$u_i(b) := v_i x_i(b) - p_i(b).$$

To ensure individual rationality we will focus on payment rules which satisfy $p_i(b) \in [0, b_i x_i(b)]$ for every i and bid profile (bid vector) b.

3 Myerson's Lemma

3.1 Statement of Myerson's Lemma

We define two properties of allocation rules.

Definition 1 (Implementable Allocation Rule). *An allocation rule $x : \mathbb{R}_+^n \mapsto X$ of a single-parameter environment is said to be* implementable *iff there exists a payment rule $p : \mathbb{R}_+^n \mapsto \mathbb{R}_+^n$ such that the sealed-bid auction (x, p) is DSIC.*

Implementable allocation rules are those that extend to DSIC mechanisms. Examples from (i) single item case — it is not clear if awarding the good to the second highest bidder is an implementable rule or not?

In light of the two step approach (of designing auctions) the design question can be restated as: Is the allocation rule obtained in step (1) implementable?

Note that the definition of implementability is "nonoperational," i.e., it is not obvious if, given an allocation rule, we can test whether this property is satisfied or not. By contrast, the next definition encapsulates a property which—at least analytically—seems easier to test.

Definition 2 (Monotone Allocation Rule)**.** *An allocation rule x for a single parameter environment is said to be monotone if for every bidder i and bids b_{-i} of the other agents, the allocation $x_i(z, b_{-i})$ to i is a nondecreasing function of the bid z.*

Myersons Lemma in stated in three parts; each is conceptually interesting and will be useful in later applications.

Theorem 2 (Myerson 1981)**.** *For a single-parameter environment*

1. *An allocation rule is implementable iff it is monotone.*
2. *If an allocation rule x is monotone, then there exists a unique payment rule p such that the auction (x, p) is DSIC (assuming the normalization, $b_i = 0$ implies $p_i(b) = 0$).*
3. *The payment rule in (2) is given by an explicit formula.*

3.2 Proof of Myerson's Lemma

3.3 Application to Sponsored Search Auctions

The unique payment rule which, when coupled with a monotone allocation rule x (assumed to be piecewise constant), leads to a DSIC auction is

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \text{ jump in } x_i(\cdot, b_{-i}) \text{ at } z_j$$

In the sponsored search context the allocation to an agent, $x_i(b_i, b_{-i})$, is either α_j for some slot $j \in [k]$, or it is zero. Say, without loss of generality, that the bids are sorted $b_1 \geq b_2 \geq \ldots \geq b_n$. Then, bidder i gets the ith slot, for all $1 \leq i \leq k$ and $x_i(b_i, b_{-i}) = \alpha_i$. The payment formula now reduces to

$$p_i(b_i, b_{-i}) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1})$$