Q 1 The geometric distribution is memoryless if

$$P(X > s + t | X > s) = P(X > t)$$

where s and t are integers and X is a geometrically distributed random variable. The probability of a failure is denoted by q and

$$\begin{aligned}
&P(X > s) = \sum_{j=s+1}^{\infty} q^{j-1}p = q^s, \\
&P(X > t) = q^t, \text{and} \\
&P(X > s + t) = q^{s+t}; \text{so,} \\
&P[(X > s + t)|X > s] = \left(q^{s+t}/q^s\right) = q^t
\end{aligned}$$

which is equal to $P(X > t)$.

Q2 Two results that are useful to solve this problem are

$$(c + d) \bmod m = c \bmod m + d \bmod m$$

and that if $g = h \bmod m$, then we can write $g = h - km$ for some integer $k \geq 0$. The last result is true because, by definition, g is the remainder after subtracting the largest integer multiple of m that is $\leq h$.

(a) Notice that

$$\begin{aligned}
X_{i+2} &= aX_{i+1} \bmod m \\
&= a[aX_i \bmod m] \bmod m \\
&= a[aX_i - km] \bmod m \qquad \text{(for some integer } k \geq 0) \\
&= a^2X_i \bmod m - akm \bmod m \\
&= a^2X_i \bmod m \qquad \text{(since } akm \bmod m = 0).
\end{aligned}$$

(b) Notice that

$$\begin{aligned}
(a^nX_i) \bmod m &= \{(a^n \bmod m) + [a^n - (a^n \bmod m)]\}\, X_i \bmod m \\
&= \{(a^n \bmod m)X_i \bmod m\} + \{[a^n - (a^n \bmod m)]X_i \bmod m\} \\
&= \{(a^n \bmod m)X_i \bmod m\} + \{kmX_i \bmod m\} \qquad \text{(for some integer } k \geq 0) \\
&= (a^n \bmod m)X_i \bmod m.
\end{aligned}$$

Q3

The answer has the same computation from an example in U N Bhat's book.

(i) Assuming that the two counters operate independently of each other determine the expected number of waiting customers and their mean waiting time at each counter.

	Commercial	Personal	
λ	6/h	12/h	
μ	12/h	24/h	
$\rho = \frac{\lambda}{\mu}$	0.5	0.5	
$L_q = \frac{\rho^2}{1-\rho}$	0.5	0.5	**Answer.**
$W_q = \frac{\rho}{\mu(1-\rho)}$	5 min	2.5 min	**Answer.**

(ii) What is the effect of operating the two queues as a two-server queue with arrival rate 18/h and service rate 18/h? What conclusion can you draw from this operation?

	Two-server queue	
λ	18/h	
μ	18/h	
Number of servers (s)	2	
$\rho = \frac{\lambda}{s\mu}$	0.5	
$\alpha = \frac{\lambda}{\mu}$	1	
$p_0 = [\sum_0^1 \frac{\alpha^r}{r!} + \frac{\alpha^2}{2(1-\rho)}]^{-1}$	0.33	
$L_q = \frac{\rho\alpha^2 p_0}{2(1-\rho)^2}$	0.33	**Answer.**
$W_q = \frac{\alpha^2 p_0}{(2)\ 2\mu(1-\rho)^2}$	1.33 min	**Answer.**

Conclusion: The two-server queue operation is more efficient than the two single-server operations.

Q4. (the answer involves computation same as an example from U N Bhat's book)

(a) [redacted]

Assuming that the arrivals are in a Poisson process with rate 1 per minute (λ) and the service times are exponential with mean 2.5 minute ($1/\mu$). We have $\rho = 2.5$. Also $K = 3$. [redacted]

$$
\begin{aligned}
L_q &= \frac{2.5}{1-2.5} - \frac{(2.5)[1+3(2.5)^3]}{1-(2.5)^4} \\
&= 1.4778 \qquad \textbf{Answer.}
\end{aligned}
$$

Since $\lambda = 1$, the mean waiting time in queue

$$W_q = 3.7271 \text{ min.} \qquad \textbf{Answer.}$$

(b) [redacted]

We use the formula for $1 - F_q(t)$ [redacted] with $t = 1.5$, $1/\mu = 2.5$ and $\rho = 2.5$. We get,

$$
\begin{aligned}
&\text{P(Wait in queue} > 1.5 \text{ min)} \\
&= \frac{1-2.5}{1-(2.5)^3} \sum_{n=1}^{3-1} (2.5)^n \sum_{r=0}^{n-1} e^{-\frac{1.5}{2.5}} \frac{(1.5/2.5)^r}{r!} \\
&= 0.7036 \qquad \textbf{Answer.}
\end{aligned}
$$

(c) [redacted]

With two lines, now $s = 2$ and we have an $M/M/2/3$ system. Accordingly [redacted] we have $\alpha = 2.5$, $\rho = 1.25$, $s = 2$ and $K = 3$. We get

$$
\begin{aligned}
p_0 &= 0.0950, \qquad p_1 = 0.2374 \\
p_2 &= 0.2969, \qquad p_3 = 0.3711
\end{aligned}
$$

Using these results [redacted] we get

$$
\begin{aligned}
W_q &= 0.5902 \text{ min} && \textbf{Answer.} \\
L_q &= \lambda(1-p_3)W_q = 0.3712 && \textbf{Answer.}
\end{aligned}
$$

P(wait in queue > 1.5 min):

$$1 - F_q(1.5) = 0.1422. \qquad \textbf{Answer.}$$

The formula for the probability that the waiting time exceeds t is:

$$\frac{1}{1-p_K} \sum_{n=s}^{K-1} p_n \sum_{r=0}^{n-s} e^{-s\mu t} \frac{(s\mu t)^r}{r!}$$

Q5 (a) Generate x distributed as $|x|$ on [-1,1] by evaluating
$\text{sign}(u-0.5)*\sqrt{(2*|u-0.5|)}$ where u is generated as $U(0,1)$
by a pseudo-random number generator.

(b) 0. Let x be generated by computing $\text{sign}(u-0.5)*\sqrt{(2*|u-0.5|)}$
where u is generated as $U(0,1)$ by a pseudo-random number generator.

1. Generate a random number v distributed as $U(0,1)$ using a pseudo-random number generator.
2. If $v \le (x*x)/|x|$, that is, if $v \le |x|$ (so that there is no division by 0), then accept x as a random variate distributed as $(3/2)*x*x$ else reject and go back to step 1.

The maximum efficiency of the method can be no more than 2/3 since the area under the curve $(3/2)*x*x$ on [-1,1] cover only 2/3rd of the area under the straight line segments $(3/2)*|x|$ on [-1,1].
Any coefficient less than 3/2 multiplied with $|x|$ will not cover the whole area under $(3/2)*x*x$ on [-1,1].