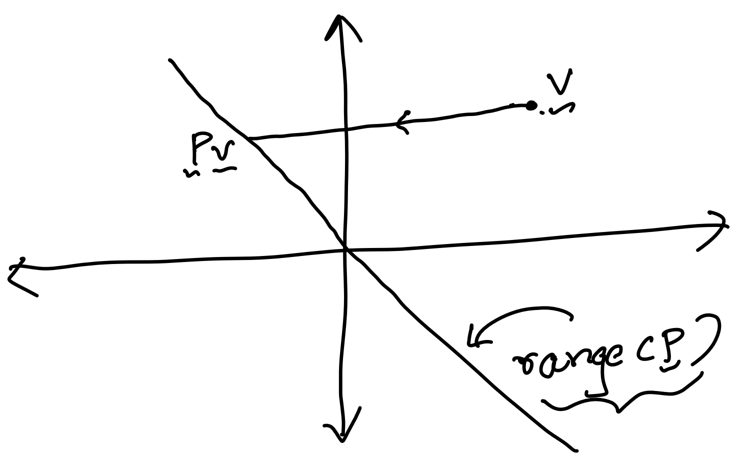
Projectors

A projection on a vector space V is a linear operator $\underset{\sim}{P} : V \to V$ such that $\underset{\sim}{P}^2 = \underset{\sim}{P}$

In the finite - dimensional case, a square matrix $\underset{\sim}{P}$ is called ~~a~~ projector matrix if it is equal to its square i.e $\underset{\sim}{P}^2 = \underset{\sim}{P}$

The condition $\underset{\sim}{P}^2 = \underset{\sim}{P}$ is called idempotent condition!



Geometrically $\underset{\sim}{P}\underset{\sim}{v}$ would be a shadow projected by $\underset{\sim}{v}$ onto range (P)

if one were to shine light onto range (P)!

From what direction does the light shine it is from $\underset{\sim}{v}$ to $\underset{\sim}{P}\underset{\sim}{v}$

So, $\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}$ is the direction of light source

$$\underset{\sim}{P}\left[\underbrace{\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}}\right] = \underset{\sim}{P}^2\underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v}$$

$$= \underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v} = 0$$

i.e $\underbrace{\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v}} \in \text{null}(\underset{\sim}{P})$

Remarks:-

1. $\underset{\sim}{P} \in \mathbb{R}^{m \times m}$, $\underset{\sim}{P}^2 = \underset{\sim}{P}$ (Idempotency) is satisfied by a projector

2. $\underset{\sim}{P}\underset{\sim}{v} - \underset{\sim}{v} \in \text{null}(P)$ and is the direction of projection of $\underset{\sim}{v}$ onto range(P)

3. If $\underset{\sim}{P}$ is a projector and vector $\underset{\sim}{x} \in$ range(P), then $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{x}$

Pf :- If $\underset{\sim}{x} \in \text{range}(P)$, then

$$\underset{\sim}{x} = \underset{\sim}{P}\underset{\sim}{y} \text{ for some } y$$

then $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{P}(\underset{\sim}{P}\underset{\sim}{y})$

$$= \underset{\sim}{P}^2\underset{\sim}{y} = \underset{\sim}{P}\underset{\sim}{y} = \underset{\sim}{x}$$

i.e $\underset{\sim}{x}$ lies exactly in its own shadow.

(4) If $\underset{\sim}{P}$ is a projector, then $(\underset{\sim}{I} - \underset{\sim}{P})$ is also a projector

$$(\underset{\sim}{I}-\underset{\sim}{P})^2 = (\underset{\sim}{I}-\underset{\sim}{P})(\underset{\sim}{I}-\underset{\sim}{P})$$

$$= \underset{\sim}{I} - \underset{\sim}{P} - \underset{\sim}{P} + \underset{\sim}{P}^2 = \underset{\sim}{I} - \underset{\sim}{P}$$

$\underset{\sim}{I} - \underset{\sim}{P}$ is called complementary projector to $\underset{\sim}{P}$!

onto what space does $(I-P)$ project? range $(I-P)$

Consider any vector in range $(I-P)$

$$\rightarrow (I-P)\underset{\sim}{x} = \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}$$

$$\underset{\sim}{P}(\underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}) = 0$$

$\Rightarrow \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x} \in \text{null}(\underset{\sim}{P})$

This means $\underline{\text{range}(\underset{\sim}{I}-\underset{\sim}{P}) \subseteq \text{null}(\underset{\sim}{P})}$ — (1)

Similarly let us consider any vector $\underset{\sim}{x}$ in $\text{null}(P)$ i.e $\underset{\sim}{P}\underset{\sim}{x} = \underset{\sim}{0}$

then $(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{x}$

$= \underset{\sim}{x} - \underset{\sim}{P}\underset{\sim}{x}$

$= \underset{\sim}{x}$

$\underline{\text{null}(\underset{\sim}{P}) \subseteq \text{range}(\underset{\sim}{I}-\underset{\sim}{P})}$ — (2)

From (1) and (2) $\text{range}(\underset{\sim}{I}-\underset{\sim}{P}) = \text{null}(\underset{\sim}{P})$

we can also deduce $\text{range}(\underset{\sim}{P}) = \text{null}(\underset{\sim}{I}-\underset{\sim}{P})$

IV $\text{null}(I-P) \cap \text{null}(P) = \{0\}$

i.e $\text{range}(P) \cap \text{null}(P) = \{0\}$

Pf:- Let $\underset{\sim}{v}$ be in both $\text{null}(\underset{\sim}{P})$ and $\text{null}(\underset{\sim}{I}-\underset{\sim}{P})$

Then $\underset{\sim}{P}\underset{\sim}{v} = (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{v} = \underset{\sim}{0}$

$(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{v} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v} - \underset{\sim}{P}\underset{\sim}{v} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$ $\Rightarrow$ $\text{null}(\underset{\sim}{I}-\underset{\sim}{P}) \cap \text{null}(\underset{\sim}{P}) = \{0\}$

$\Rightarrow \text{range}(P) \cap \text{null}(P) = \{0\}$