Assignment 3

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October 2022

Problem 1

(a) Show that $\mathbf{I} - 2\mathbf{P}$ is orthogonal if $\mathbf{P}$ is a orthogonal projector

Proof: $\mathbf{P}$ is an orthogonal projector which implies that $P^T = P$ and $P = P^2$.
For $\mathbf{I} - 2\mathbf{P}$ to be orthogonal, $(\mathbf{I} - 2\mathbf{P})^T(\mathbf{I} - 2\mathbf{P}) = \mathbf{I}$ and $(\mathbf{I} - 2\mathbf{P})(\mathbf{I} - 2\mathbf{P})^T = \mathbf{I}$

$$\begin{aligned}(\mathbf{I} - 2\mathbf{P})(\mathbf{I} - 2\mathbf{P})^T &= (\mathbf{I} - 2\mathbf{P})^2 \\ &= \mathbf{I} - 4\mathbf{P} + 4\mathbf{P}^2 \\ &= \mathbf{I} - 4\mathbf{P} + 4\mathbf{P} \\ &= \mathbf{I}\end{aligned}$$

$$\begin{aligned}(\mathbf{I} - 2\mathbf{P})^T(\mathbf{I} - 2\mathbf{P}) &= (\mathbf{I} - 2\mathbf{P})^2 \\ &= \mathbf{I} - 4\mathbf{P} + 4\mathbf{P}^2 \\ &= \mathbf{I} - 4\mathbf{P} + 4\mathbf{P} \\ &= \mathbf{I}\end{aligned}$$

$\therefore$ $\boxed{\mathbf{I} - 2\mathbf{P} \text{ is orthogonal if } \mathbf{P} \text{ is a orthogonal projector}}$

(b) Answers:

- $\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$
- If $\mathbf{Pv} = \mathbf{v}$, then $\mathbf{v}$ is in the range of column space of $\mathbf{A}$ else it is in the null space of $\mathbf{A}$. This means that if $\mathbf{v}$ is in the null space of $\mathbf{I} - \mathbf{P}$, then $\mathbf{v}$ is in the range of $\mathbf{A}$. Which means that if $\mathbf{v}$ is in the range of $\mathbf{I} - \mathbf{P}$, then $\mathbf{v}$ is in the null space of $\mathbf{A}$. Hence $range(\mathbf{I} - \mathbf{P}) = null(\mathbf{A}^T)$

- Eigen values of $\mathbf{P}$ can only be 0 or 1.

  If λ is a eigen value of $\mathbf{P}$, then $\mathbf{Pv} = \lambda\mathbf{v}$

$$\begin{aligned} \mathbf{P}(\mathbf{Pv}) &= \mathbf{P}^2\mathbf{v} \\ &= \mathbf{Pv} \\ &= \lambda\mathbf{v} \end{aligned}$$

$$\begin{aligned} \mathbf{P}(\mathbf{Pv}) &= \mathbf{P}(\lambda\mathbf{v}) \\ &= \lambda(\mathbf{Pv}) \\ &= \lambda(\lambda\mathbf{v}) \\ &= \lambda^2\mathbf{v} \end{aligned}$$

$$\begin{aligned} &\implies \lambda\mathbf{v} = \lambda^2\mathbf{v} \\ &\implies \lambda(\lambda - 1) = 0 \\ &\implies \lambda = 0, 1 \end{aligned}$$

(c) Show that $||\mathbf{P}||_2 \geqslant 1$ if $\mathbf{P}$ is a non-zero projection

Proof: We know that $\mathbf{P}^2 = \mathbf{P}$

$$\begin{aligned} &\implies ||\mathbf{P}^2||_2 = ||\mathbf{P}||_2 \\ &\implies ||\mathbf{P}||_2||\mathbf{P}||_2 \geqslant ||\mathbf{P}||_2 \\ &\implies ||\mathbf{P}||_2 \geqslant 1 \end{aligned}$$

Problem 2

By Classical Gram-Schmidt Orthogonalization,
$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}}$, $r_{11} = ||\mathbf{a}_1||$

$\mathbf{q}_2 = \frac{\mathbf{a}_2 - r_{12}\mathbf{q}_1}{r_{22}}$, $r_{12} = \mathbf{q}_1^T\mathbf{a}_2$, $r_{22} = ||\mathbf{a}_2 - r_{12}\mathbf{q}_1||$

$\mathbf{q}_3 = \frac{\mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2}{r_{33}}$, $r_{13} = \mathbf{q}_1^T\mathbf{a}_3$, $r_{23} = \mathbf{q}_2^T\mathbf{a}_3$, $r_{33} = ||\mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2||$

Where $\mathbf{Q} = (\mathbf{q}_1\,\mathbf{q}_2\,\mathbf{q}_3)$, $\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$

Hence,
$\mathbf{q}_1 = \frac{1}{\sqrt{89}}\begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.8480 \\ 0.3180 \\ 0.4240 \end{pmatrix}$, $r_{11} = \sqrt{8^2 + 3^2 + 4^2} = \sqrt{89} = 9.4340$

$\mathbf{q}_2 = \frac{1}{r_{22}}\begin{pmatrix} 1 - (59/89)8 \\ 5 - (59/89)3 \\ 9 - (59/89)4 \end{pmatrix} = \frac{1}{8.2394}\begin{pmatrix} -4.3034 \\ 3.0112 \\ 6.3483 \end{pmatrix} = \begin{pmatrix} -0.5223 \\ 0.3655 \\ 0.7705 \end{pmatrix}$,
$r_{12} = \frac{8+15+36}{\sqrt{89}} = \frac{59}{\sqrt{89}} = 6.2540$, $r_{22} = \sqrt{18.5190 + 9.0675 + 40.3011} = 8.2394$

$\mathbf{q}_3 = \frac{1}{21.4497}\begin{pmatrix} 6 - 6.9214 + 0.5044 \\ 7 - 2.5955 - 0.3530 \\ 2 - 3.4607 - 0.7441 \end{pmatrix} = \begin{pmatrix} -0.0194 \\ 0.1889 \\ -0.1028 \end{pmatrix}$,
$r_{13} = 8.1620$, $r_{23} = 0.9657$, $r_{33} = 21.4497$

Therefore, $\mathbf{Q} = \begin{pmatrix} 0.8480 & -0.5223 & -0.0194 \\ 0.3180 & 0.3655 & 0.1889 \\ 0.4240 & 0.7705 & -0.1028 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 9.4340 & 6.2540 & 8.1620 \\ 0 & 8.2394 & 0.9657 \\ 0 & 0 & 21.4497 \end{pmatrix}$

Python Code:

```
import numpy as np
from numpy.linalg import qr

A = np.array([[8,1,6], [3,5,7], [4,9,2]])
Q, R = qr(A)

print("Q: ", Q)
print("R: ", R)
```

From code, we get

$$\mathbf{Q} = \begin{pmatrix} -0.8479983 & 0.52229204 & 0.09005497 \\ -0.31799936 & -0.36546806 & -0.8748197 \\ -0.42399915 & -0.77048304 & 0.47600483 \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} -9.43398113 & -6.25398749 & -8.16198368 \\ 0 & -8.23939564 & -0.96549025 \\ 0 & 0 & -4.63139839 \end{pmatrix}$$

From the above results we can see that we can make the factorization unique by making sure that the diagonal elements in $\mathbf{R}$ are all positive(or all negative).

Problem 3

(a) n = 4, By Gram Schmidt orthogonalization procedure:

$$q_0(x) = \frac{a_1}{r_{11}},\ r_{11} = ||q_0(x)|| = \sqrt{q_0^T q_0}$$
$$q_1(x) = \frac{a_2 - r_{12} q_0(x)}{r_{22}},\ r_{12} = q_0(x)^T a_2,\ r_{22} = ||a_2 - r_{12} q_0(x)||, ...$$

$$\therefore q_0(x) = \frac{1}{\sqrt{2}}\left[1\right]$$
$$r_{11} = \sqrt{\int_{-1}^{1} 1 \times 1 dx} = \sqrt{2}$$

$$q_1(x) = \sqrt{\frac{3}{2}}\left[x\right]$$
$$r_{12} = \int_{-1}^{1} \frac{1}{\sqrt{2}} \times x dx = 0$$
$$r_{22} = \sqrt{\int_{-1}^{1} x \times x dx} = \sqrt{\frac{2}{3}}$$

$$q_2(x) = \sqrt{\frac{5}{2}}\left[x^2 - \tfrac{1}{3}\right]$$
$$r_{13} = \int_{-1}^{1} \frac{1}{\sqrt{2}} \times x^2 dx = \frac{\sqrt{2}}{3}$$
$$r_{23} = \int_{-1}^{1} \frac{x\sqrt{3}}{\sqrt{2}} \times x^2 dx = 0$$
$$r_{33} = \sqrt{\int_{-1}^{1} x^2 \times x^2 dx} = \sqrt{\frac{2}{5}}$$

$$q_3(x) = \sqrt{\frac{7}{2}}\left[x^3 - \tfrac{3}{5}x\right]$$

$$r_{14} = \int_{-1}^{1} \frac{1}{\sqrt{2}} \times x^3 dx = 0$$

$$r_{24} = \int_{-1}^{1} \frac{x\sqrt{3}}{\sqrt{2}} \times x^3 dx = \frac{\sqrt{6}}{5}$$

$$r_{34} = \int_{-1}^{1} \frac{(3x^2-1)\sqrt{5}}{3\sqrt{2}} \times x^3 dx = 0$$

$$r_{44} = \sqrt{\int_{-1}^{1} x^3 \times x^3 dx} = \sqrt{\frac{2}{7}}$$

Hence,

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}x}{\sqrt{2}} & \frac{3\sqrt{5}x^2-\sqrt{5}}{3\sqrt{2}} & \frac{5\sqrt{7}x^3-3\sqrt{7}}{5\sqrt{2}} \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} \sqrt{2} & 0 & \frac{\sqrt{2}}{3} & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{\sqrt{6}}{5} \\ 0 & 0 & \sqrt{\frac{2}{5}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{7}} \end{bmatrix}$$

(b) For $n \geqslant 2$, $\int_{-1}^{1} q_{n-1}(x)dx = 0$

Proof: We know that the matrix $\mathbf{Q}$ is orthogonal i.e., $q_i(x), q_j(x)$ are orthogonal to each other for $i \neq j$. That is inner product of $q_i(x), q_j(x), (i \neq j)$ is 0.
And as per the definition of inner product, $\int_{-1}^{1} q_i(x)q_j(x)dx = 0, \forall i \neq j$

Let i = 0 and j = n - 1 $\implies \int_{-1}^{1} q_0(x)q_{n-1}(x)dx = 0 \implies \frac{1}{\sqrt{2}}\int_{-1}^{1} q_{n-1}(x)dx = 0.$
Hence proved.

Problem 4

$$\mathbf{A} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

(a) Using CGS,

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}},\ r_{11} = ||\mathbf{a}_1||$$
$$\mathbf{q}_2 = \frac{\mathbf{a}_2 - r_{12}\mathbf{q}_1}{r_{22}},\ r_{12} = \mathbf{q}_1^T\mathbf{a}_2,\ r_{22} = ||\mathbf{a}_2 - r_{12}\mathbf{q}_1||$$

$$\implies \mathbf{q}_1 = \frac{1}{0.99002}\begin{bmatrix} 0.70000 \\ 0.70001 \end{bmatrix} = \begin{bmatrix} 0.70706 \\ 0.70707 \end{bmatrix}, r_{11} = \sqrt{0.49000 + 0.49014} = 0.99002$$
$$\implies \mathbf{q}_2 = \frac{1}{r_{22}}\begin{bmatrix} 0.70711 - 0.70702 \\ 0.70711 - 0.70703 \end{bmatrix} = \frac{1}{0.00012}\begin{bmatrix} 0.00009 \\ 0.00008 \end{bmatrix} = \begin{bmatrix} 0.75000 \\ 0.66667 \end{bmatrix},$$
$$r_{12} = 0.70706 \times 0.70711 + 0.70707 \times 0.70711 = 0.99995,$$
$$r_{22} = \left\| \begin{bmatrix} 0.00009 \\ 0.00008 \end{bmatrix} \right\| = 0.00012$$

$$\therefore \mathbf{Q} = \begin{bmatrix} 0.70706 & 0.75000 \\ 0.70707 & 0.66667 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 0.99002 & 0.99995 \\ 0.00000 & 0.00012 \end{bmatrix}$$

(b) Using Householder's method,
We know that, $\mathbf{Q}_2\mathbf{Q}_1\mathbf{A} = \mathbf{R}$, $\mathbf{Q} = (\mathbf{Q}_1^{-1}\mathbf{Q}_2^{-1})$

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{a}_1 = \begin{bmatrix} 0.70000 \\ 0.70001 \end{bmatrix} \\
\mathbf{v} &= sign(v_{11})||\mathbf{v}_1||\mathbf{e}_1 + \mathbf{v}_1 = \begin{bmatrix} 1.68996 \\ 0.70001 \end{bmatrix} \\
\mathbf{Q}_1 &= \mathbf{I} - \frac{2\mathbf{v}\mathbf{v}^T}{||\mathbf{v}||^2} \\
\implies \mathbf{Q}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{3.34597}\begin{bmatrix} 2.85596 & 1.18299 \\ 1.18299 & 0.49001 \end{bmatrix} \qquad = \begin{bmatrix} -0.70709 & -0.70711 \\ -0.70711 & -0.70711 \end{bmatrix} \\
\implies \mathbf{Q}_1\mathbf{A} &= \begin{bmatrix} -0.70709 & -0.70711 \\ -0.70711 & -0.70711 \end{bmatrix}\begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix} \\
&= \begin{bmatrix} -0.98995 & 0.99999 \\ 0.00000 & 0.00000 \end{bmatrix} \\
\mathbf{Q}_2 &= \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{F} \end{bmatrix} \\
\mathbf{F} &= \mathbf{I} - \frac{2\mathbf{v}\mathbf{v}^T}{||\mathbf{v}||^2} \\
\mathbf{v} &= sign(0)||0||\mathbf{e}_1 + 0 = [0] \\
\implies \mathbf{F} &= [1] \\
\implies \mathbf{Q}_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{R} = \mathbf{Q}_2\mathbf{Q}_1\mathbf{A} &= \begin{bmatrix} -0.98995 & 0.99999 \\ 0.00000 & 0.00000 \end{bmatrix}
\end{aligned}
$$

Since $\mathbf{Q}_1, \mathbf{Q}_2$ are orthogonal, $\mathbf{Q} = \mathbf{Q}_1^T\mathbf{Q}_2^T$

$$
\begin{aligned}
\implies \mathbf{Q} &= \mathbf{Q}_1^T\mathbf{Q}_2^T \\
&= \begin{bmatrix} -0.70709 & -0.70711 \\ -0.70711 & 0.70711 \end{bmatrix}
\end{aligned}
$$

Check for orthogonality: In case of CGS, $\mathbf{q}_1^T\mathbf{q}_2 = 0.70706 * 0.75000 + 0.70707 * 0.66667 \approx 1$ i.e. very large loss of orthogonality

In case of Householder, $\mathbf{q}_1^T\mathbf{q}_2 = 0.70709 * 0.70711 - 0.70711 * 0.70711 \approx 0$ i.e. almost no loss of orthogonality

Problem 5

(a) $r_{12} = q_1^T a_2 = q_{11}a_{21} + q_{12}a_{22} + ... + q_{1m}a_{1m}$

$$
\begin{aligned}
\tilde{r}_{12} &= fl(q_1^T a_2) \\
&= (...(q_{11}(1+\epsilon_{11})a_{21}(1+\epsilon_{21}) + q_{12}(1+\epsilon_{12})a_{22}(1+\epsilon_{22}))(1+\epsilon_{32}) \\
&+ ... + q_{1m}(1+\epsilon_{1m})a_{2m}(1+\epsilon_{2m}))(1+\epsilon_{3m}) \\
&= q_{11}a_{21}(1+\epsilon_{11})(1+\epsilon_{21})(1+\epsilon_{32})...(1+\epsilon_{3m}) \\
&+ ... + q_{1m}a_{2m}(1+\epsilon_{1m})(1+\epsilon_{2m})(1+\epsilon_{3m}) \\
\implies \tilde{r}_{12} - r_{12} &= q_{11}a_{21}(1+\epsilon_{11})(1+\epsilon_{21})(1+\epsilon_{32})...(1+\epsilon_{3m}) \\
&+ ... + q_{1m}a_{2m}(1+\epsilon_{1m})(1+\epsilon_{2m})(1+\epsilon_{3m}) \\
&- (q_{11}a_{21} + q_{12}a_{22} + ... + q_{1m}a_{2m}) \\
&= q_{11}a_{21}(\epsilon_{11} + \epsilon_{21} + \epsilon_{32} + ... + \epsilon_{3m} + O(\epsilon_{machine}^2)) \\
&+ ... + q_{1m}a_{2m}(\epsilon_{1m} + \epsilon_{2m} + \epsilon_{3m} + O(\epsilon_{machine}^2)) \\
&\leqslant q_{11}a_{21}(m\epsilon_{machine} + O(\epsilon_{machine}^2)) \\
&+ ... + q_{1m}a_{2m}(m\epsilon_{machine} + O(\epsilon_{machine}^2)) \\
&\leqslant (q_{11}a_{21} + ... + q_{1m}a_{2m})(m\epsilon_{machine} + O(\epsilon_{machine}^2)) \\
&\leqslant m\epsilon_{machine} + O(\epsilon_{machine}^2)
\end{aligned}
$$

$\therefore \boxed{|\tilde{r}_{12} - r_{12}| \leqslant m\epsilon_{machine} + O(\epsilon_{machine}^2)}$

(b) $\tilde{\mathbf{w}}_2 = fl(\mathbf{a}_2 - fl(\tilde{r}_{12}\mathbf{q}_1))$

$$
\begin{aligned}
\tilde{w}_{2i} &= (a_{2i}(1+\epsilon_{1i}) - \tilde{r}_{12}q_{1i}(1+\epsilon_{2i}))(1+\epsilon_{3i}),\ \forall i \in [1, m] \\
\implies \tilde{w}_{2i} - w_{2i} &= (a_{2i}(1+\epsilon_{1i}) - \tilde{r}_{12}q_{1i}(1+\epsilon_{2i}))(1+\epsilon_{3i}) - (a_{2i} - r_{12}q_{1i}) \\
&= a_{2i}(1 + \epsilon_{1i} + \epsilon_{3i} + O(\epsilon_{machine}^2)) - \tilde{r}_{12}q_{1i}(1 + \epsilon_{2i} + \epsilon_{3i} + O(\epsilon_{machine}^2)) \\
&- (a_{2i} - r_{12}q_{1i}) \\
&\leqslant (a_{2i} - r_{12}q_{1i})(1 + \epsilon_{1i} + \epsilon_{2i} + \epsilon_{3i} + m\epsilon_{machine} + O(\epsilon_{machine}^2) - 1) \\
&\leqslant (m+3)\epsilon_{machine} + O(\epsilon_{machine}^2)
\end{aligned}
$$

$\therefore \boxed{|\tilde{\mathbf{w}}_2 - \mathbf{w}_2| \leqslant (m+3)\epsilon_{machine} + O(\epsilon_{machine}^2)}$

(c) Given that $\tilde{\mathbf{q}}_2 = \frac{\tilde{\mathbf{w}}_2}{\tilde{r}_{22}}$, $\tilde{r}_{22} = ||\tilde{\mathbf{w}}_2||$

$$
\begin{aligned}
q_{1i}\tilde{q}_{2i} &= q_{1i}\frac{\tilde{w}_{2i}}{\tilde{r}_{22}} \\
&\leqslant q_{1i}\left(\frac{w_{2i} + (m+3)\epsilon_{machine} + O(\epsilon_{machine}^2)}{\tilde{r}_{22}}\right) \\
\implies |\mathbf{q}_1^T\tilde{\mathbf{q}}_2| &\leqslant \frac{\sum q_{1i}w_{2i} + (\sum q_{1i})((m+3)\epsilon_{machine} + O(\epsilon_{machine}^2))}{\tilde{r}_{22}} \\
&\leqslant \frac{(m+3)\epsilon_{machine}}{\tilde{r}_{22}}
\end{aligned}
$$

$$
\therefore \boxed{|\mathbf{q}_1^T\tilde{\mathbf{q}}_2| \leqslant \frac{(m+3)\epsilon_{machine}}{\tilde{r}_{22}}}
$$

Problem 6

Code:

```
# NLA, Assignment 3, Problem 6

from math import sin
from random import random

import numpy as np
from numpy.linalg import svd, pinv, norm

import matplotlib.pyplot as plt

def generateData(m,n):
    '''
    Generate A, b where Ax = b is a polynomial model for the function sin(10t)
    '''
    # data = [random() for _ in range(m)]
    data = np.linspace(0, 1, m)
    b = np.array([sin(10 * x) for x in data])
    A = np.array([[
            1 if i == 0 else x**i for x in data
        ] for i in range(n)]).transpose()
    return A, b

def qr_mgs(A):
    '''
    Reduced QR factorization using Modified Gram Schmidt method
    '''
    _, n = A.shape
    R = np.empty((n, n))
    A_t = A.transpose()
    for i in range(n):
        R[i][i] = norm(A_t[i])
        A_t[i] = A_t[i] / R[i][i]
        for j in range(i+1, n):
            R[i][j] = A_t[i].dot(A_t[j])
            A_t[j] = A_t[j] - R[i][j] * A_t[i]
    return A_t.transpose(), R
```

```
def qr_hh(A, b):
    '''
    QR factorization using Householder triangularization method
    Returns R and Q_t * b
    '''
    m, n = A.shape
    for i in range(n):
        x = A[i:m, i:i+1]
        x[0] += np.sign(x[0]) * norm(x)
        x = x / norm(x)
        A[i:m, i:n] -= np.matmul(2*x, np.matmul(x.reshape(1, m - i), A[i:m, i:
        b[i:m] -= 2 * x.reshape(m - i).dot(b[i:m]) * x.reshape(m - i)
    return A, b

def backSubstitution(U, b):
    '''
    Given a upper triangular matrix and the result matrix, returns x
    '''
    n = U.shape[1]
    x = np.empty(n)
    for i in range(n-1, -1, -1):
        slag = sum([U[i][j] * x[j] for j in range(n-1, i-1, -1)])
        x[i] = (b[i] - slag) / U[i][i]
    return x

def getXA(A, b):
    '''
    Returns coefficients of the polymonial using Modified Gram-Schmidt method
    '''
    Q, R = qr_mgs(A)
    return backSubstitution(R, Q.transpose().dot(b))

def getXB(A, b):
    '''
    Returns coefficients of the polymonial using Householder's method
    '''
    R, b = qr_hh(A, b)
    return backSubstitution(R, b)

def getXC(A, b):
```

```
    '''
    Returns coefficients of the polymonial using SVD decomposition
    '''
    U, Sigma, V = svd(A)
    SigmaInverse = np.pad(pinv(np.diag(Sigma)), pad_width=((0, 0), (0, 85)))
    return np.matmul(np.matmul(np.matmul(V, SigmaInverse), U.transpose()), b)

def getXD(A, b):
    '''
    Returns coefficients of the polymonial using pseudo inverse
    '''
    return np.matmul(pinv(A.transpose().dot(A)), A.transpose().dot(b))

# ------------------------------------------------------------------------------
# m = 100, n = 15
A, b = generateData(100, 15)

print(getXA(A, b))

# input = np.sort(np.array([random() for _ in range(100)]))
input = np.linspace(0, 1, 100)
fig, ((axA, axB), (axC, axD)) = plt.subplots(2, 2)

axA.set_title("Using Modified Gram-Schmidt")
axA.plot(input, A.dot(getXA(np.copy(A), b)))
axA.scatter(input, A.dot(getXA(np.copy(A), b)))
axA.set_ylim([-1, 1])

axB.set_title("Using Householder triangularization")
axB.plot(input, A.dot(getXB(np.copy(A), b)))
axB.scatter(input, A.dot(getXB(np.copy(A), b)))
axB.set_ylim([-1, 1])

axC.set_title("Using SVD Decomposition")
axC.plot(input, A.dot(getXC(np.copy(A), b)))
axC.scatter(input, A.dot(getXC(np.copy(A), b)))
axC.set_ylim([-1, 1])

axD.set_title("Using Pseudo Inverse")
axD.plot(input, A.dot(getXD(np.copy(A), b)))
```

```
axD.scatter(input, A.dot(getXD(np.copy(A), b)))
axD.set_ylim([-1, 1])

plt.show()
```

From the plots we can see that using QR Factorization by Householder method and using SVD gives us closer solution due to them being numerically stable.

Problem 7

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 8 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

We know that least square solution of $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$

$$\begin{aligned}
\implies \mathbf{x} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \\
&= \left(\begin{bmatrix} 22 & 40 & 34 \\ 40 & 97 & 58 \\ 34 & 58 & 58 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 2 & 5 & 2 & 8 \\ 3 & 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 0.872 & -0.134 & -0.377 \\ -0.134 & 0.046 & 0.032 \\ -0.377 & 0.032 & 0.206 \end{bmatrix} \begin{bmatrix} 20 \\ 26 \\ 32 \end{bmatrix} \\
&= \begin{bmatrix} -1.333 \\ 0.667 \\ 0.667 \end{bmatrix}
\end{aligned}$$

We were able to find the least square solution as $(\mathbf{A}^T\mathbf{A})^{-1}$ exists as $\mathbf{A}$ is full rank. Hence to compute least square solution by the above method, we need the matrix $\mathbf{A}$ to be full rank which can be done by remove dependent colums from the matrix.