# Lecture 4 Extensive Form Games +PSNE

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In the previous class.

Notion of Dominance

- Strongly Dominant
- Weakly dominant
- Very weakly dominant

Prisoner's Dilemma : SDSE, WDSE, VWDSE

Braess Paradox : SDSE

Vickrey Auction : WDSE

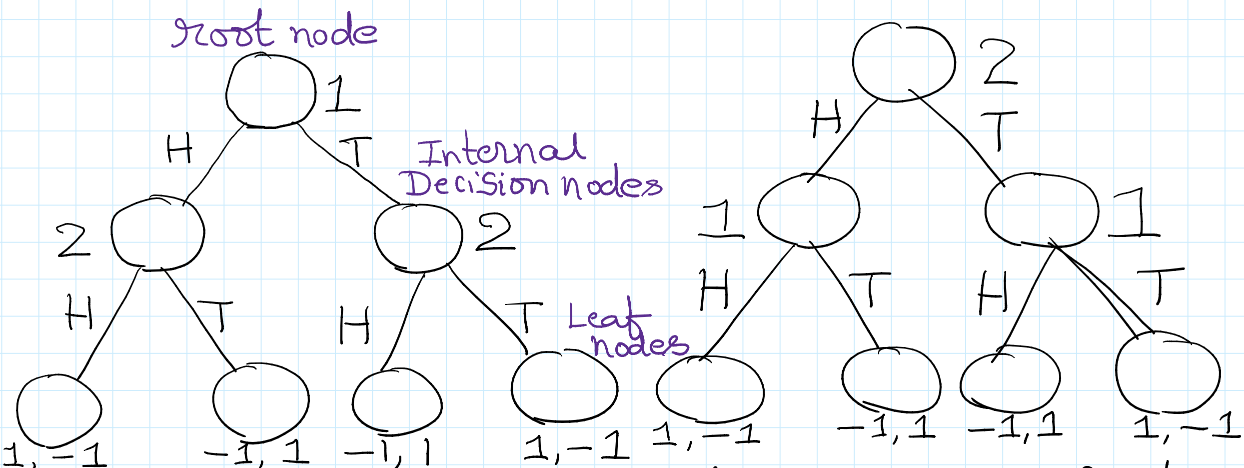
We are currently studying strategic form games.

What is the genesis and motivation for studying strategic form games ?

We start with a detailed representation of games called

Extensive Form Games.

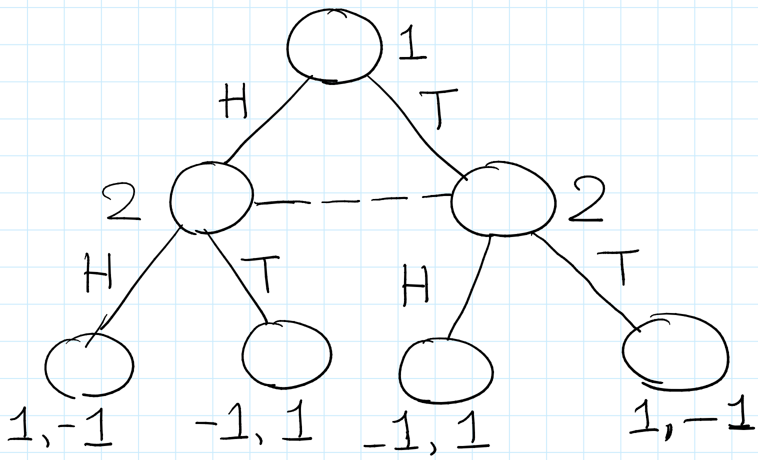
Matching Pennies Game




Player 1 moves first with observation

Player 2 moves first with observation

Without observation




This also represents simultaneous play

## Information Set of a Player

A set of decision nodes of a player which are indistinguishable to the player.

Describes a collection of all possible distinguishable situations in which the player is called upon to choose an action.

The player has the same set of possible actions in each node of the information set.

## Definition of an Extensive Form Game

$$\Gamma = \langle N, (A_i)_{i\in N}, \mathbb{H}, P, (\mathcal{I}_i)_{i\in N}, (u_i)_{i\in N} \rangle$$

$N = \{1, 2, \ldots, n\}$ Players

$A_1, A_2, \ldots, A_n$ Action Sets

$\mathbb{H}$ Set of all terminal histories

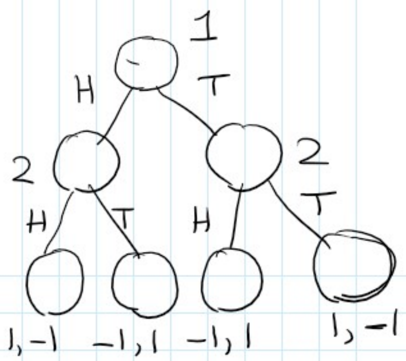
$S_{\mathbb{H}}$ Set of all proper subhistories

$P : S_{\mathbb{H}} \to N$ Player function

$P : S_{\mathbb{H}} \longrightarrow N$ Player function

$\mathbb{I}_i$ = Set of all information sets of player $i$

$u_i : \mathbb{H} \longrightarrow \mathbb{R}$ utility function of player 1




$N = \{1, 2\}$

$A_1 = A_2 = \{H, T\}$

$\mathbb{H} = \{HH, HT, TH, TT\}$

$S_{\mathbb{H}} = \{\varepsilon, H, T\}$

$P(\varepsilon) = 1 \quad P(H) = 2 \quad P(T) = 2$

$\mathbb{I}_1 = \{\{\varepsilon\}\}$

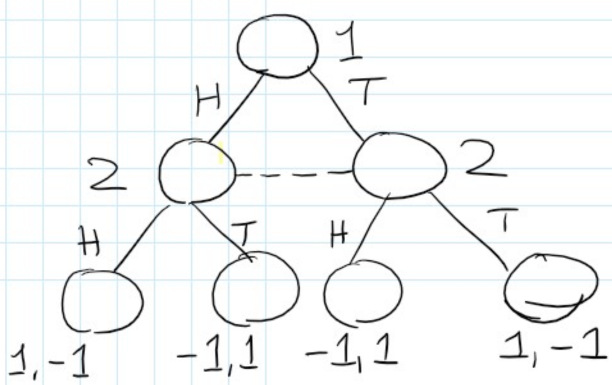
$\mathbb{I}_2 = \{\{H\}, \{T\}\}$

$u_1(HH) = 1 \quad u_2(HH) = -1$

$\vdots$

$u_1(TT) = 1 \quad u_2(TT) = -1$

Now consider




The only change here is $\mathbb{I}_2 = \{\{H, T\}\}$ instead of $\mathbb{I}_2 = \{\{H\}, \{T\}\}$

## The Notion of a Strategy

A complete action plan that specifies what a player will do at each of the ... sets where the player what a player will do at each of the information sets where the player is called upon to play.

$$s_i : \mathbb{I}_i \rightarrow A_i$$

for every information set of player $i$, an action is idenfied.

Suppose $\mathbb{I}_i = \{x, y\}$ $\quad A_i = \{a, b\}$

Then the strategy set of player $i$ would be

$$s_I = \begin{pmatrix} x \rightarrow a \\ y \rightarrow a \end{pmatrix} \quad s_{II} = \begin{pmatrix} x \rightarrow a \\ y \rightarrow b \end{pmatrix}$$

$$s_{III} = \begin{pmatrix} x \rightarrow b \\ y \rightarrow a \end{pmatrix} \quad s_{IV} = \begin{pmatrix} x \rightarrow b \\ y \rightarrow b \end{pmatrix}$$

MP with observation

$$\mathbb{I}_1 = \{\{\varepsilon\}\}$$

$$\mathbb{I}_2 = \{\{H\}, \{T\}\}$$

Strategies of Player 1:

$s_{11} : \{\varepsilon\} \rightarrow H$

$s_{12} : \{\varepsilon\} \rightarrow T$

Strategies of Player 2:

$s_{21} : \{H\} \rightarrow H \; ; \; \{T\} \rightarrow H$

$s_{22} : \{H\} \rightarrow H \; ; \; \{T\} \rightarrow T$

$s_{23} : \{H\} \rightarrow T \; ; \; \{T\} \rightarrow H$

$s_{24} : \{H\} \rightarrow T \; ; \; \{T\} \rightarrow T$

| 1 \ 2 | $s_{21}$ | $s_{22}$ | $s_{23}$ | $s_{24}$ |
|---|---|---|---|---|
| $s_{11}$ | 1, −1 | 1, −1 | −1, 1 | −1, 1 |
| $s_{12}$ | −1, 1 | 1, −1 | −1, 1 | 1, −1 |

Strategic (Normal) Form Game

MP without observation

Here $\mathbb{I}_1 = \{\{\varepsilon\}\}$ $\mathbb{I}_2 = \{\{H, T\}\}$

$s_{11} : \{\varepsilon\} \to H$

$s_{12} : \{\varepsilon\} \to T$

$s_{21} : \{H, T\} \to H$

$s_{22} : \{H, T\} \to T$

| 1 \ 2 | $s_{21}$ | $s_{22}$ |
|---|---|---|
| $s_{11}$ | 1, −1 | −1, 1 |
| $s_{12}$ | −1, 1 | 1, −1 |

Strategic (Normal) Form Game

This is how strategic form games are derived out of extensive form games.

Strategic form games are succinct representations of extensive form games and capture important strategic (equilibrium) behaviour of the underlying extensive form games.

## Notion of Pure Strategy Nash Equilibrium

$$\Gamma = \left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$$

A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*) \in S$ is called a PSNE if $\forall i \in N$,

$$u_i(s_i^*, \underline{s}_{-i}^*) \geq u_i(s_i, \underline{s}_{-i}^*) \quad \forall s_i \in S_i$$

or equivalently

$$u_i(s_i^*, \underline{s}_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \underline{s}_{-i}^*)$$

For each player $i$, $s_i^*$ is an optimal (best response) strategy when the other players are playing their Nash equilibrium strategies $\underline{s}_{-i}^*$.

## Best Response Strategy

Given $\underline{s}_{-i} \in \underline{S}_{-i}$, $s_i$ is called a best response strategy for player $i$ if

$$u_i(s_i, \underline{s}_{-i}) \geq u_i(s_i', \underline{s}_{-i}) \quad \forall s_i' \in S_i$$

$$b_i : \underline{S}_{-i} \longrightarrow 2^{S_i}$$

Define

$$b_i(\underline{s}_{-i}) = \left\{ s_i \in S_i : s_i \text{ is a BRS against } \underline{s}_{-i} \right\}$$

$(s_1^*, s_2^*, \ldots, s_n^*)$ is a PSNE iff

$(s_1^*, s_2^*, \ldots, s_n^*)$ is a PSNE iff

$$s_i^* \in b_i(\underline{s}_{-i}^*) \quad \forall\, i \in N$$

Examples

| 1 \ 2 | A | B |
|---|---|---|
| A | 100, 100 | 0, 0 |
| B | 0, 0 | 10, 10 |

(A, A) is a PSNE
(B, B) is a PSNE
(A, B) is not a PSNE
(B, A) is not a PSNE

| 1 \ 2 | A | B |
|---|---|---|
| A | −2, −2 | −10, −1 |
| B | −1, −10 | −5, −5 |

(B, B) is a PSNE
other strategy profiles are not PSNE

| 1 \ 2 | A | B |
|---|---|---|
| A | 1, −1 | −1, 1 |
| B | −1, 1 | 1, −1 |

No PSNE!

PSNE takes into account only unilateral deviations

PSNE need not produce a socially optimal solution

A dominant strategy equilibrium is always a PSNE but not vice-versa.

Suppose $(s_1^*, s_2^*, \ldots, s_n^*)$ is a VWDSE.

Suppose $(s_1^*, s_2^*, \dots, s_n^*)$ is a VWDSE.

So, we have, $\forall i \in N$, $\forall s_i \in S_i \setminus \{s_i^*\}$,

$$u_i(s_i^*, \underline{s}_i) \geq u_i(s_i, \underline{s}_i) \quad \forall \underline{s}_i \in \underline{S}_i$$

We just choose $\underline{s}_i = \underline{s}_i^*$ and get

$$u_i(s_i^*, \underline{s}_i^*) \geq u_i(s_i^*, \underline{s}_i^*)$$ ... PSNE condition