Assignment 0

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Why do you want to join this course?
I know the basics of linear algebra. I want to take this course to learn to use features of linear algebra computationally as it is a necessity for data science and since I am from CDS, it's a basic requirement for any research that I do.

1. Vector Space

Verify if the following form a vector space.

(a) Given that,

$$\mathbb{V} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \, w - x - y + z = 0;\, w, x, y, z \in \mathbb{R} \right\} \tag{1}$$

For $\mathbb{V}$ to be a vector space, it should have atleast one element.

Lets consider an element $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in $\mathbb{V}$ where w = x = y = z = 0,

it satisfies the condition $w - x - y + z = 0$ and $w, x, y, z \in \mathbb{R}$
Hence, $\mathbf{v} \in \mathbb{V} \implies \mathbb{V}$ is not empty

Let v_1, v_2, v_3 be vectors in the set $\mathbb{V}$, then

$$\begin{aligned} v_1 &= \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}, w_1 - x_1 - y_1 + z_1 = 0 \\ v_2 &= \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}, w_2 - x_2 - y_2 + z_2 = 0 \\ v_3 &= \begin{pmatrix} w_3 \\ x_3 \\ y_3 \\ z_3 \end{pmatrix}, w_3 - x_3 - y_3 + z_3 = 0 \end{aligned} \tag{2}$$

1. **Closure under addition property** $(u + v \in \mathbb{V},\, \forall\, u, v \in \mathbb{V})$

   <u>*Proof:*</u>

   let $v_1 + v_2 = l$; where $l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} \implies w = \begin{pmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}$

   As per equation 2,
   $w_1 + x_1 + y_1 + z_1 = 0$ and $w_2 + x_2 + y_2 + z_2 = 0$

   $$\begin{aligned} &\implies (w_1 - x_1 - y_1 + z_1) + (w_2 - x_2 - y_2 + z_2) = 0 \\ &\implies (w_1 + w_2) - (x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = 0 \\ &\implies l_1 - l_2 - l_3 + l_4 = 0 \end{aligned}$$

   $\implies l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}$ and $l_1 - l_2 - l_3 + l_4 = 0 \implies l \in \mathbb{V}$

   $\implies \boxed{v_1 + v_2 \in \mathbb{V},\, \forall\, v_1, v_2 \in \mathbb{V}}$

2. **Commutative property** $(u + v = v + u,\, \forall\, u, v \in \mathbb{V})$

   <u>*Proof:*</u>

   $$v_1 + v_2 = \begin{pmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}, v_2 + v_1 = \begin{pmatrix} w_2 + w_1 \\ x_2 + x_1 \\ y_2 + y_1 \\ z_2 + z_1 \end{pmatrix} \tag{3}$$

   Since the set of $\mathbb{R}$ is commutative under addition,

   $$\begin{aligned} w_1 + w_2 &= w_2 + w_1, \\ x_1 + x_2 &= x_2 + x_1, \\ y_1 + y_2 &= y_2 + y_1, \\ z_1 + z_2 &= z_2 + z_1, \end{aligned} \tag{4}$$

   $\therefore$ from equation 3 and 4, $\boxed{v_1 + v_2 = v_2 + v_1}$

3. **Associative property** $(u + (v + w) = (u + v) + w,\, \forall\, u, v \in \mathbb{V})$

   <u>*Proof:*</u>

   $$v_1 + (v_2 + v_3) = \begin{pmatrix} w_1 + (w_2 + w_3) \\ x_1 + (x_2 + x_3) \\ y_1 + (y_2 + y_3) \\ z_1 + (z_2 + z_3) \end{pmatrix}, (v_1 + v_2) + v_3 = \begin{pmatrix} (w_1 + w_2) + w_3 \\ (x_1 + x_2) + x_3 \\ (y_1 + y_2) + y_3 \\ (z_1 + z_2) + z_3 \end{pmatrix}, \tag{5}$$

   Since the set of $\mathbb{R}$ is associative under addition,

   $$\begin{aligned} w_1 + (w_2 + w_3) &= (w_1 + w_2) + w_3, \\ x_1 + (x_2 + x_3) &= (x_1 + x_2) + x_3, \\ y_1 + (y_2 + y_3) &= (y_1 + y_2) + y_3, \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3, \end{aligned} \tag{6}$$

$\therefore$ from equation 5 and 6, $\boxed{v_1 + v_2 = v_2 + v_1}$

4. **Zero vector** ($0 \in \mathbb{V}$, such that $v + 0 = v$, $\forall\, v \in \mathbb{V}$)

   Proof:

   Let Zero vector (0) be a vector with all elements as 0

$$i.e., \quad 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \implies 0 - 0 - 0 + 0 = 0$$

   Hence as per equation 1, $\boxed{0 \in \mathbb{V}}$

$$v_1 + 0 = \begin{pmatrix} w_1 + 0 \\ x_1 + 0 \\ y_1 + 0 \\ z_1 + 0 \end{pmatrix} = \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = v_1 \implies \boxed{v_1 + 0 = v_1}$$

5. **Additive inverse** (Each $v \in \mathbb{V}$ has a $w \in \mathbb{V}$ such that $v + w = 0$)

   Proof:

   Let $\exists\, w$ such that $v + w = 0$

$$\implies w = -v = -\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -w \\ -x \\ -y \\ -z \end{pmatrix}$$

   As per equation 1,

$$(-w) - (-x) - (-y) + (-z) = -(w - x - y + z) = 0$$
$$\implies w \in \mathbb{V}$$

$$\boxed{\therefore \text{ there exists an additive inverse in } \mathbb{V} \text{ for every } v \in \mathbb{V}}$$

6. **Closure under scalar Multiplication** ($cu \in \mathbb{V}, \forall u \in \mathbb{V}$ and $c \in \mathbb{R}$)

   Proof:

$$\forall c \in \mathbb{R} \text{ and } \forall v \in \mathbb{V},\; cv = c\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cw \\ cx \\ cy \\ cz \end{pmatrix}$$

   As per equation 1,

$$(cw) - (cx) - (cy) + (cz) = c(w - x - y + z) = 0$$
$$\implies \boxed{cv \in \mathbb{V}}$$

   Hence $\mathbb{V}$ is closed under scalar multiplication

7. **Distributive property for scalar addition** ((c + d)v = cv + dv, $\forall v \in \mathbb{V}$ and $\forall c, d \in \mathbb{R}$)

   Proof:

   LHS $\rightarrow (c+d)v$; RHS $\rightarrow cv + dv$;

   Solving LHS:

$$(c+d)v = (c+d)\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (c+d)w \\ (c+d)x \\ (c+d)y \\ (c+d)z \end{pmatrix} = \begin{pmatrix} cw+dw \\ cx+dx \\ cy+dy \\ cz+dz \end{pmatrix} \tag{7}$$

   Solving RHS:

$$cv + dv = c\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} + d\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cw \\ cx \\ cy \\ cz \end{pmatrix} + \begin{pmatrix} dw \\ dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} cw+dw \\ cx+dx \\ cy+dy \\ cz+dz \end{pmatrix} \tag{8}$$

   From equations 7 and 8, $\boxed{LHS = RHS}$
   Hence proved

8. **Distributive property for vector addition** (c(v + w) = cv + cw, $\forall v, w \in \mathbb{V}$ and $\forall c \in \mathbb{R}$)

   Proof:

   LHS $\rightarrow c(v_1 + v_2)$; RHS $\rightarrow cv_1 + cv_2$;

   Solving LHS:

$$c(v_1+v_2) = c\begin{pmatrix} w_1+w_2 \\ x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} c(w_1+w_2) \\ c(x_1+x_2) \\ c(y_1+y_2) \\ c(z_1+z_2) \end{pmatrix} = \begin{pmatrix} cw_1+cw_2 \\ cx_1+cx_2 \\ cy_1+cy_2 \\ cz_1+cz_2 \end{pmatrix} \tag{9}$$

   Solving RHS:

$$cv_1 + cv_2 = \begin{pmatrix} cw_1 \\ cx_1 \\ cy_1 \\ cz_1 \end{pmatrix} + \begin{pmatrix} cw_2 \\ cx_2 \\ cy_2 \\ cz_2 \end{pmatrix} = \begin{pmatrix} cw_1+cw_2 \\ cx_1+cx_2 \\ cy_1+cy_2 \\ cz_1+cz_2 \end{pmatrix} \tag{10}$$

   From equations 9 and 10, $\boxed{LHS = RHS}$
   Hence proved

9. **Identity operation** (Multiplication by the scalar 1, such that $1\,v = v$, $\forall\, v \in \mathbb{V}$)

   Proof:

   We know that,

$$cv = c\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cw \\ cx \\ cy \\ cz \end{pmatrix} \tag{11}$$

   Let $c = 1$,

$$\implies \begin{pmatrix} cw \\ cx \\ cy \\ cz \end{pmatrix} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \tag{12}$$

   Hence as per equations 10 and 11, $\boxed{1\,v = v,\ \forall\, v \in \mathbb{V}}$

$\boxed{\therefore \text{ As } \mathbb{V} \text{ satisfies all the conditions of a vector space, } \mathbb{V} \text{ is a vector space.}}$

(b) Given that,

$$\mathbb{M}^{2x2} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

For $\mathbb{M}^{2x2}$ to be a vector space, it should have atleast one element.
Lets consider an element $\mathbf{m} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathbb{M}^{2x2}$, where $a = b = c = 0 \implies a, b, c \in \mathbb{R}$
Hence, $\mathbf{m} \in \mathbb{M}^{2x2} \implies \mathbb{M}^{2x2}$ is not empty

Let m_1, m_2, m_3 be matrices in the set $\mathbb{M}^{2x2}$, then

$$m_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}, \, a_1, b_1, c_1 \in \mathbb{R}$$
$$m_2 = \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}, \, a_2, b_2, c_2 \in \mathbb{R}$$
$$m_3 = \begin{pmatrix} a_3 & 0 \\ b_3 & c_3 \end{pmatrix}, \, a_3, b_3, c_3 \in \mathbb{R}$$

1. **Closure under addition property** $(u + v \in \mathbb{M}^{2x2}, \, \forall \, u, v \in \mathbb{M}^{2x2})$

   Proof:

   let $m_1 + m_2 = l; \implies m_1 + m_2 = \begin{pmatrix} a_1 + a_2 & 0 + 0 \\ b_1 + b_2 & c_1 + c_2 \end{pmatrix} = l$

   Since, $\mathbb{R}$ is closed under addition, $a_1 + a_2, b_1 + b_2, c_1 + c_2 \in \mathbb{R}$

   $\implies \boxed{m_1 + m_2 \in \mathbb{M}^{2x2}, \, \forall \, m_1, m_2 \in \mathbb{M}^{2x2}}$

2. **Commutative property** $(u + v = v + u, \, \forall \, u, v \in \mathbb{M}^{2x2})$

   Proof:

$$m_1 + m_2 = \begin{pmatrix} a_1 + a_2 & 0 + 0 \\ b_1 + b_2 & c_1 + c_2 \end{pmatrix}, m_2 + m_1 = \begin{pmatrix} a_2 + a_1 & 0 + 0 \\ b_2 + b_1 & c_2 + c_1 \end{pmatrix} \tag{13}$$

   Since the set of $\mathbb{R}$ is commutative under addition,

$$\begin{aligned} a_1 + a_2 &= a_2 + a_1, \\ b_1 + b_2 &= b_2 + b_1, \\ c_1 + c_2 &= c_2 + c_1, \end{aligned} \tag{14}$$

   $\therefore$ from equation 13 and 14, $\boxed{m_1 + m_2 = m_2 + m_1}$

3. **Associative property** $(u + (v + w) = (u + v) + w, \, \forall \, u, v, w \in \mathbb{M}^{2x2})$

   Proof:

$$\begin{aligned} m_1 + (m_2 + m_3) &= \begin{pmatrix} a_1 + (a_2 + a_3) & 0 + (0 + 0) \\ b_1 + (b_2 + b_3) & c_1 + (c_2 + c_3) \end{pmatrix}, \\ (m_1 + m_2) + m_3 &= \begin{pmatrix} (a_1 + a_2) + a_3 & (0 + 0) + 0 \\ (b_1 + b_2) + b_3 & (c_1 + c_2) + c_3 \end{pmatrix}, \end{aligned} \tag{15}$$

Since the set of $\mathbb{R}$ is associative under addition,

$$\begin{aligned} a_1 + (a_2 + a_3) &= (a_1 + a_2) + a_3, \\ b_1 + (b_2 + b_3) &= (b_1 + b_2) + b_3, \\ c_1 + (c_2 + c_3) &= (c_1 + c_2) + c_3, \end{aligned} \tag{16}$$

$\therefore$ from equation 15 and 16, $\boxed{m_1 + m_2 = m_2 + m_1}$

4. **Zero matrix** ($0 \in \mathbb{M}^{2x2}$, such that $v + 0 = v$, $\forall\, v \in \mathbb{M}^{2x2}$)

   Proof:

   Let Zero vector (0) be a vector with all elements as 0

   $$i.e., \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   Hence as per equation , $\boxed{0 \in \mathbb{M}^{2x2}}$

   $$m_1 + 0 = \begin{pmatrix} a_1 + 0 & 0 + 0 \\ b_1 + 0 & c_1 + 0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} = m_1 \implies \boxed{m_1 + 0 = m_1}$$

5. **Additive inverse** (Each $v \in \mathbb{M}^{2x2}$ has a $w \in \mathbb{M}^{2x2}$ such that $v + w = 0$)

   Proof:

   Let $\exists\, w$ such that $m + w = 0$

   $$\implies w = -m = -\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} -a & 0 \\ -b & -c \end{pmatrix}$$

   Since $a, b, c \in \mathbb{R} \implies -a, -b, -c \in \mathbb{R} \implies w \in \mathbb{M}^{2x2}$

   $$\boxed{\therefore \text{there exists an additive inverse in } \mathbb{M}^{2x2} \text{ for every } v \in \mathbb{M}^{2x2}}$$

6. **Closure under scalar Multiplication** ($cu \in \mathbb{M}^{2x2}, \forall u \in \mathbb{M}^{2x2}$ and $c \in \mathbb{R}$)

   Proof:

   $$\forall k \in \mathbb{R} \text{ and } \forall m \in \mathbb{M}^{2x2},\; km = k \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ka & 0 \\ kb & kc \end{pmatrix}$$

   Since $a, b, c \in \mathbb{R} \implies ka, kb, kc \in \mathbb{R} \implies \boxed{km \in \mathbb{M}^{2x2}}$
   Hence $\mathbb{M}^{2x2}$ is closed under scalar multiplication

7. **Distributive property for scalar addition** ((c + d)v = cv + dv, $\forall v \in \mathbb{M}^{2x2}$ and $\forall c, d \in \mathbb{R}$)

   Proof:

   LHS $\rightarrow (k + l)m$; RHS $\rightarrow km + lm$;

   Solving LHS:

   $$(k + l)m = (k + l) \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} (k + l)a & (k + l)0 \\ (k + l)b & (k + l)c \end{pmatrix} = \begin{pmatrix} ka + la & 0 \\ kb + lb & kc + lc \end{pmatrix} \tag{17}$$

Solving RHS:

$$km + lm = k\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + l\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ka & 0 \\ kb & kc \end{pmatrix} + \begin{pmatrix} la & 0 \\ lb & lc \end{pmatrix} = \begin{pmatrix} ka+la & 0 \\ kb+lb & kc+lc \end{pmatrix} \tag{18}$$

From equations 17 and 18, $\boxed{LHS = RHS}$
Hence proved

8. **Distributive property for vector addition** (c(v + w) = cv + cw, $\forall v, w \in \mathbb{M}^{2x2}$ and $\forall c \in \mathbb{R}$)

   Proof:

   LHS $\rightarrow k(m_1 + m_2)$; RHS $\rightarrow km_1 + km_2$;

   Solving LHS:

$$\begin{aligned} k(m_1+m_2) &= k\begin{pmatrix} a_1+a_2 & 0+0 \\ b_1+b_2 & c_1+c_2 \end{pmatrix} \\ &= \begin{pmatrix} k(a_1+a_2) & 0 \\ k(b_1+b_2) & k(c_1+c_2) \end{pmatrix} \\ &= \begin{pmatrix} ka_1+ka_2 & 0 \\ kb_1+kb_2 & kc_1+kc_2 \end{pmatrix} \end{aligned} \tag{19}$$

   Solving RHS:

$$km_1 + km_2 = \begin{pmatrix} ka_1 & 0 \\ kb_1 & kc_1 \end{pmatrix} + \begin{pmatrix} ka_2 & 0 \\ kb_2 & kc_2 \end{pmatrix} = \begin{pmatrix} ka_1+ka_2 & 0 \\ kb_1+kb_2 & kc_1+kc_2 \end{pmatrix} \tag{20}$$

   From equations 19 and 20, $\boxed{LHS = RHS}$
   Hence proved

9. **Identity operation** (Multiplication by the scalar 1, such that $1\,v = v$, $\forall\, v \in \mathbb{M}^{2x2}$)

   Proof:

   We know that,

$$km = k\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ka & 0 \\ kb & kc \end{pmatrix} \tag{21}$$

   Let $k = 1$,

$$\implies \begin{pmatrix} ka & 0 \\ kb & kc \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \tag{22}$$

   Hence as per equations 20 and 21, $\boxed{1\,m = m, \forall\, m \in \mathbb{M}^{2x2}}$

$\boxed{\therefore \text{ As } \mathbb{M}^{2x2} \text{ satisfies all the conditions of a vector space, } \mathbb{M}^{2x2} \text{ is a vector space.}}$

(c) Given that,

$$\mathbb{N} = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} \,\middle|\, \frac{df}{dx} + 2f = 1 \right\}$$

For $\mathbb{N}$ to be a vector space, it should have atleast one element.
Lets consider an element g such that $g(x) = \frac{1-e^{-2x}}{2} \implies \frac{dg}{dx} + 2g = 1$
Hence, $g \in \mathbb{N} \implies \mathbb{N}$ is not empty

Let f_1, f_2, f_3 be functions in the set $\mathbb{N}$, then

$$\frac{df_1}{dx} + 2f_1 = 1,\ \frac{df_2}{dx} + 2f_2 = 1 \tag{23}$$

1. **Closure under addition property** $(u + v \in \mathbb{N}, \forall\, u, v \in \mathbb{N})$

   <u>*Proof:*</u>

$$\text{let } f_1 + f_2 = g; \implies \frac{dg}{dx} = \frac{df_1}{dx} + \frac{df_1}{dx} \tag{24}$$

   Adding equations in 23 and from equation 24,

$$\begin{aligned}
&\frac{df_1}{dx} + 2f_1 + \frac{df_2}{dx} + 2f_2 = 1 + 1 \\
\implies &(\frac{df_1}{dx} + \frac{df_2}{dx}) + 2(f_1 + f_2) = 2 \\
\implies &\frac{dg}{dx} + 2g = 2 \neq 1 \\
\implies &g \notin \mathbb{N}
\end{aligned}$$

   $\implies \boxed{f_1 + f_2 \notin \mathbb{N}, \forall\, f_1, f_2 \in \mathbb{N}}$

$\therefore$ As $\mathbb{N}$ doesn't satisfy the closure under addition property of a vector space, $\mathbb{N}$ is not a vector space.

2. Subspace: A non-empty subset $\mathbb{W}$ of a vector space $\mathbb{V}$ over a field $\mathbb{R}$ is a subspace of $\mathbb{V}$ if and only if

$$\begin{aligned} \mathbf{a} \in \mathbb{W}, \mathbf{b} \in \mathbb{W} &\implies \mathbf{a} + \mathbf{b} \in \mathbb{W} \\ \mathbf{a} \in \mathbb{W}, \alpha \in \mathbb{R} &\implies \alpha\mathbf{a} \in \mathbb{W} \end{aligned} \tag{25}$$

(a) Does $\mathbb{V}$ form a subspace of $\mathbb{R}^3$, where

$$\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x, y, z \geq 0 \right\}$$

1. $\mathbb{V}$ is not empty as $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{V}$

2. Let $v_1, v_2 \in \mathbb{V}$, such that

$$\begin{aligned} v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \; v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}&, \text{ where } x_1, y_1, z_1, x_2, y_2, z_2 \geq 0 \\ \implies v_1 + v_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \end{aligned} \tag{26}$$

From equation 26,

$$\begin{aligned} x_1 + x_2 &\geq 0 \\ y_1 + y_2 &\geq 0 \\ z_1 + z_2 &\geq 0 \\ \implies &\boxed{v_1 + v_2 \in \mathbb{W}} \end{aligned}$$

3. Let $\mathbf{v} \in \mathbb{V}$ and $\alpha \in \mathbb{R}$

$$\alpha\mathbf{v} = \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} \implies \boxed{\mathbf{v} \notin \mathbf{V} \text{ if } \alpha \leq 0}$$

$$\boxed{\therefore \text{ As } \mathbb{V} \text{ doesn't satisfy the equation 25, } \mathbb{V} \text{ is not a subspace.}}$$

(b) Does $\mathbb{V}$ form a subspace of $\mathbb{R}^3$?, where

$$\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x, y, z \in \mathbb{R}, \; x + z = 0 \right\}$$

1. $\mathbb{V}$ is not empty as $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{V}$

2. Let $v_1, v_2 \in \mathbb{V}$, such that

$$\begin{aligned} v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \text{ where } x_1 + z_1 = 0 \text{ and } x_2 + z_2 = 0 \\ \implies v_1 + v_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \end{aligned} \tag{27}$$

From equation 27,

$$(x_1 + x_2) + (z_1 + z_2) = (x_1 + z_1) + (x_2 + z_2) = 0 + 0 = 0$$

$$\implies \boxed{v_1 + v_2 \in \mathbb{W}}$$

3. Let $\mathbf{v} \in \mathbb{V}$ and $\alpha \in \mathbb{R}$

$$\alpha\mathbf{v} = \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$$

We know that $x + z = 0$,

$$\alpha x + \alpha z = \alpha(x + z) = \alpha(0) = 0 \implies \boxed{\mathbf{v} \in \mathbf{V}, \forall \alpha \in \mathbb{R}}$$

$\therefore$ As $\mathbb{V}$ satisfies the equation 25, $\mathbb{V}$ is a subspace.

(c) Does $\mathbb{V}$ form a subspace of $\mathbb{R}^{2x2}$, where

$$\mathbb{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$$

1. $\mathbb{V}$ is not empty as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{V}$

2. Let $v_1, v_2 \in \mathbb{V}$, such that

$$v_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, v_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \text{ where}$$

$$det \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = 0, \implies a_1 d_1 - b_1 c_1 = 0,$$

$$det \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = 0, \implies a_2 d_2 - b_2 c_2 = 0$$

$$\implies v_1 + v_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$\begin{aligned} det \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} &= (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)(c_1 + c_2) \\ &= (a_1 d_1 - b_1 c_1) + (a_2 d_2 - b_2 c_2) + a_1 d_2 + a_2 d_1 - b_1 c_2 - b_2 c_1 \\ &= 0 + a_1 d_2 + a_2 d_1 - b_1 c_2 - b_2 c_1 \end{aligned} \tag{28}$$

$\exists v_1, v_2 \in \mathbb{V}$ such that equation 28 is not equal to zero.

Hence $\boxed{\exists v_1, v_2 \in \mathbb{V} \text{ such that } v_1 + v_2 \notin \mathbb{V}}$

$\therefore$ As $\mathbb{V}$ doesn't satisfy the equation 25, $\mathbb{V}$ is not a subspace.

(d) Intersection of two subspaces of a vector space $\mathbb{V}$ over a field F is a subspace of $\mathbb{V}$
Proof:

Let $\mathbb{V}_1, \mathbb{V}_2$ be two subspaces of a vector space $\mathbb{V}$ over a field F

1. If $\mathbb{V}_1 \cap \mathbb{V}_2 \subseteq \varnothing$, then $\mathbb{V}_1 \cap \mathbb{V}_2$ is not a subspace of $\mathbb{V}$

2. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1 \cap \mathbb{V}_2$, then (assuming $\mathbb{V}_1 \cap \mathbb{V}_2 \nsubseteq \varnothing$)

$$\begin{aligned} &\implies \mathbf{a}, \mathbf{b} \in \mathbb{V}_1,\ \mathbf{a}, \mathbf{b} \in \mathbb{V}_2, \\ &\implies a + b \in \mathbb{V}_1 \text{ and } a + b \in \mathbb{V}_2 \\ &\implies \boxed{a + b \in \mathbb{V}_1 \cap \mathbb{V}_2} \end{aligned}$$

3. Let $\mathbf{a} \in \mathbb{V}_1 \cap \mathbb{V}_2, \alpha \in F$, then (assuming $\mathbb{V}_1 \cap \mathbb{V}_2 \nsubseteq \varnothing$)

$$\begin{aligned} &\implies \mathbf{a} \in \mathbb{V}_1,\ \mathbf{a} \in \mathbb{V}_2, \\ &\implies \alpha\mathbf{a} \in \mathbb{V}_1 \text{ and } \alpha\mathbf{a} \in \mathbb{V}_2 \\ &\implies \boxed{\alpha\mathbf{a} \in \mathbb{V}_1 \cap \mathbb{V}_2} \end{aligned}$$

$\therefore$ As $\mathbb{V}_1 \cap \mathbb{V}_2$ satisfies the equation 25, $\mathbb{V}_1 \cap \mathbb{V}_2$ is a subspace of $\mathbb{V}$ if, $\mathbb{V}_1$ and $\mathbb{V}_2$ are subspaces of $\mathbb{V}$ and $\mathbb{V}_1 \cap \mathbb{V}_2$ is not empty. If $\mathbb{V}_1 \cap \mathbb{V}_2$ is empty, then it is not a subspace of $\mathbb{V}$

3. Linear Dependence and Linear Independence

(a) Let $\mathbf{a} = (5, 3, 7)$, $\mathbf{b} = (2, -4, 1)$, $\mathbf{c} = (0, -26, -9)$ and $\mathbf{d} = (1, 3, 5)$

1. Let $\exists k, l \in \mathbb{R}$ such that $c = ka + lb$
(i.e $\mathbf{c}$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$),

$$\begin{aligned} &\implies (0, -26, -9) = k(5, 3, 7) + l(2, -4, 1) \\ &\implies 5k + 2l = 0 \\ &\quad\;\;\; 3k - 4l = -26 \\ &\quad\;\;\; 7k + l = -9 \\ &\implies \boxed{k = -2,\ l = 5} \end{aligned}$$

Since there is a solution, $\mathbf{c}$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$

2. Let $\exists k, l \in \mathbb{R}$ such that $d = ka + lb$
(i.e $\mathbf{d}$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$),

$$\begin{aligned} &\implies (1, 3, 5) = k(5, 3, 7) + l(2, -4, 1) \\ &\implies 5k + 2l = 1 \\ &\quad\;\;\; 3k - 4l = 3 \\ &\quad\;\;\; 7k + l = 5 \\ &\rightarrow \boxed{\text{no solution}} \end{aligned}$$

Since there is no solution, $\mathbf{d}$ cannot be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$

(b)

$$\mathbf{U} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4 \,|\, 3w + x - 7z = 0 \right\}$$

Let $\mathbb{S}$ be a basis of $\mathbf{U}$ containing n vectors $(\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_n)$

$\implies$ $\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_n$ are linearly independent and every other vector in $\mathbf{U}$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_n$ i.e., $\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_n$
Let $\alpha_1, \alpha_2, ... \alpha_n \in \mathbb{R}$

$$\implies \mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + ... + \alpha_n \mathbf{v}_n, \text{ where } \alpha_1, \alpha_2, ... \alpha_n \in \mathbb{R}$$

$$\implies \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \alpha_1 \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} + ... + \alpha_n \begin{pmatrix} w_n \\ x_n \\ y_n \\ z_n \end{pmatrix}$$

$$\implies \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n \\ \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \\ \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n \\ \alpha_1 z_1 + \alpha_2 z_2 + ... + \alpha_n z_n \end{pmatrix}$$

We know that $\forall$ vectors $\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right)$ in $\mathbf{U}$, $3w + x - 7z = 0 \implies x = 7z - 3w$

$$\implies \begin{pmatrix} w \\ 7z-3w \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n \\ \alpha_1(7z_1 - 3w_1) + \alpha_2(7z_2 - 3w_2) + ... + \alpha_n(7z_n - 3w_n) \\ \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n \\ \alpha_1 z_1 + \alpha_2 z_2 + ... + \alpha_n z_n \end{pmatrix}$$

$$w = \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n \tag{29}$$

$$\implies 7z - 3w = \alpha_1(7z_1 - 3w_1) + \alpha_2(7z_2 - 3w_2) + ... + \alpha_n(7z_n - 3w_n) \tag{30}$$

$$y = \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n \tag{31}$$

$$z = \alpha_1 z_1 + \alpha_2 z_2 + ... + \alpha_n z_n \tag{32}$$

Since 30 is a combination of 29 and 32, we can simplify it to the following three equations

$$\begin{aligned} w &= \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n \\ y &= \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n \\ z &= \alpha_1 z_1 + \alpha_2 z_2 + ... + \alpha_n z_n \end{aligned} \tag{33}$$

We can say that for a given vector $\mathbf{u}$ and the basis $\mathbb{S}$, all the coefficients of the basis vectors can be uniquely identified. (i.e., for a given set of $\alpha_1, \alpha_2, ...\alpha_n$, the $\mathbf{u}$ generated is unique)

From equation 33, to uniquely identify $\alpha_1, \alpha_2, ...\alpha_n$, n should be ≤ 3 and if n is < 3, then there will be cases with no solutions (i.e., all the vectors in $\mathbf{U}$ cannot be expressed as a linear combination of $\mathbb{S}$)

$$\implies n = 3 \text{ i.e., there are exactly 3 vectors in } \mathbb{S}$$

Let (w_1, y_1, z_1) be $(1, 0, 0)$, to maintain linear independence (w_2, y_2, z_2) shouldn't be a scalar multiple of (w_1, y_1, z_1). Let (w_2, y_2, z_2) be $(0, 1, 0)$ and similarly (w_3, y_3, z_3) be $(0, 0, 1)$

Hence a basis for $\mathbf{U}$ is $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, i.e.,

$$\begin{pmatrix} 1 \\ 7(0)-3(1) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7(0)-3(0) \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7(1)-3(0) \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \boxed{\begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 0 \\ 1 \end{pmatrix}}$$

4. Solving linear system of equations multiple ways

(a)

- Set 1

$$
\begin{aligned}
x + 2y &= 3 \\
4x + 5y &= 6
\end{aligned}
$$

- Set 2

$$
\begin{aligned}
x + 2y &= 3 \\
4x + 8y &= 6
\end{aligned}
$$

- Set 3

$$
\begin{aligned}
x + 2y &= 3 \\
4x + 5y &= 12
\end{aligned}
$$

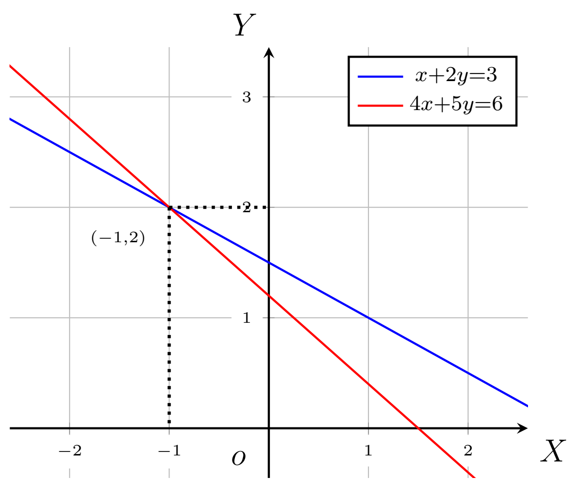

Figure 1: Set 1: *We can see that the 2 equations $x + 2y = 3$ and $4x + 5y = 6$ intersect at $(-1, 2)$. Since they intersect at exactly one point and diverge before and after it, we can deduce that this system of equations has exactly one (i.e., unique) solution.*

(b)

- Set 1

$$
\begin{aligned}
& x + 2y = 3, 4x + 5y = 6 \\
\implies & x = 3 - 2y \implies 4(3 - 2y) + 5y = 6 \\
\implies & y = 2 \text{ and } x = -1 \\
\implies & \boxed{\text{Unique Solution}}
\end{aligned}
$$

Algebraically we see that only one pair of (x,y) satisfies both equations which gives us an unique solution. The observation matches with what we deduced from the graph of the equations.

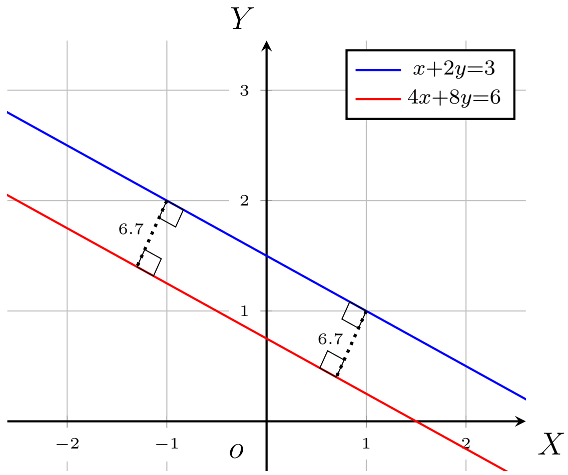

Figure 2: Set 2: *We can see that the 2 equations $x + 2y = 3$ and $4x + 8y = 6$ are always equi-distant to each other (i.e., they are parallel). Hence we can deduce that this system of equations has no solution.*

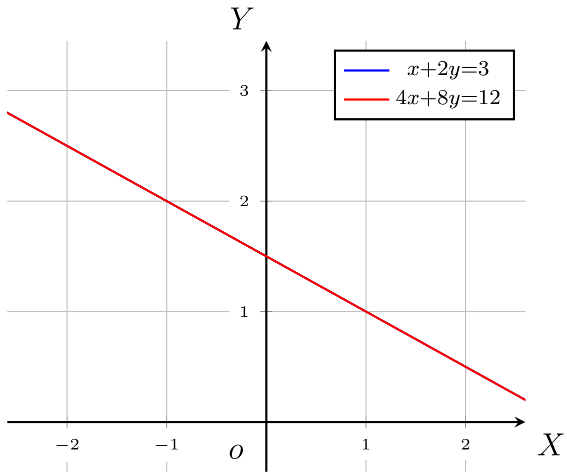

Figure 3: Set 3: *We can see that the 2 equations $x + 2y = 3$ and $4x + 8y = 12$ coincide. Hence wee can deduce that this system of equations has infinite solutions (i.e., all the points on the lines).*

- Set 2

$$
\begin{aligned}
& x + 2y = 3, 4x + 8y = 6 \\
\implies & x = 3 - 2y \implies 4(3 - 2y) + 8y = 6 \\
\implies & 12 - 8y + 8y = 6 \text{ but } 12 \neq 6 \\
\implies & \boxed{\text{No Solution}}
\end{aligned}
$$

We can see that there is no solution for this system of equations as they both have the same slope but different intercepts. The observation is same as what we find from the graphical solution.

- Set 3

$$
\begin{aligned}
& x + 2y = 3, 4x + 8y = 12 \\
\implies & x = 3 - 2y \implies 4(3 - 2y) + 8y = 12 \\
\implies & 12 - 8y + 8y = 12 \text{ but } 12 = 12 \\
\implies & \text{Any (x,y) satisfying x + 2y = 3 is a Solution} \\
\implies & \boxed{\text{Infinite Solutions}}
\end{aligned}
$$

We can see that by multiplying with 4 on both sides to the first equation, we get the second equation. Hence algebraically both equations are the same. From the graph we saw that both the equations concide which is the same as what we observe algebraically.

(c)

- Set 1: $x + 2y = 3, 4x + 5y = 6$ in $\mathbf{Ax} = \mathbf{b}$ format can be expressed as

$$
\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \implies x \begin{pmatrix} 1 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}
$$

We can see that $(1, 4)$ and $(2, 5)$ are linearly independent and all linear combinations of them span $\mathbb{R}^2$ (i.e,. they form the basis of $\mathbb{R}^2$). Hence any vector in $\mathbb{R}^2$ can be expressed as a linear combination of the 2 vectors with unique coefficients (i.e., unique solution).

- Set 2: $x + 2y = 3, 4x + 8y = 6$ in $\mathbf{Ax} = \mathbf{b}$ format can be expressed as

$$
\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \implies x \begin{pmatrix} 1 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}
$$

We can see that $(1, 4)$ and $(2, 8)$ are linearly dependent. Hence all linear combinations of them don't span $\mathbb{R}^2$. Hence only the vector which is a scalar multiple of them can be expressed as a linear combination of them with non-unique coefficients (i.e., no solution or infinite solutions). But since the vector $(3, 6)$ is not a scalar multiple, this has no solution.

- Set 3: $x + 2y = 3, 4x + 8y = 12$ in $\mathbf{Ax} = \mathbf{b}$ format can be expressed as

$$
\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix} \implies x \begin{pmatrix} 1 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}
$$

We can see that $(1, 4)$ and $(2, 8)$ are linearly dependent. Hence all linear combinations of them don't span $\mathbb{R}^2$. Hence only the vector which is a scalar multiple of them can be expressed as a linear combination of them with non-unique coefficients (i.e., no solution or infinite solutions). But since the vector $(3, 12)$ is a scalar multiple, this has infinite solutions.

5. Review of Matrices:

$$\mathbf{a} = \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix}$$

(a) Given that,

$$\mathbf{a}.\mathbf{b} = \sum_{i=1}^{n} a_i b_i \text{ where } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \tag{34}$$

From equation 34

$$\begin{aligned}
\mathbf{a}.\mathbf{b} &= \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix} . \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix} \\
&= 0 + \left(\frac{209 \times 408}{362 \times 577}\right) - \left(\frac{209 \times 408}{362 \times 577}\right) \\
&= 0 \\
\mathbf{a}.\mathbf{c} &= \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix} . \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix} \\
&= -\left(\frac{209 \times 396}{362 \times 485}\right) + \left(\frac{209 \times 198}{362 \times 485}\right) + \left(\frac{209 \times 198}{362 \times 485}\right) \\
&= 0 \\
\mathbf{b}.\mathbf{c} &= \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix} . \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix} \\
&= 0 + \left(\frac{408 \times 198}{577 \times 485}\right) - \left(\frac{408 \times 198}{577 \times 485}\right) \\
&= 0 \\
\implies & \boxed{\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c} = \mathbf{b}.\mathbf{c} = 0}
\end{aligned}$$

(b) Given that,

$$\text{geometric length of a vector } \mathbf{a} = \sqrt{\mathbf{a}.\mathbf{a}} \text{ where } \mathbf{a} \in \mathbb{R}^n \tag{35}$$

From equation 35

$$\begin{aligned}
\text{geometric length of a vector } \mathbf{a} &= \sqrt{\begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix} . \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix}} \\
&= \sqrt{\left(\frac{209}{362}\right)^2 + \left(\frac{209}{362}\right)^2 + \left(\frac{209}{362}\right)^2} \\
&= \left(\frac{209}{362}\right)\sqrt{3} \approx 1 \\
\implies & \boxed{\text{geometric length of } \mathbf{a} = 1}
\end{aligned}$$

$$\text{geometric length of a vector } \mathbf{b} = \sqrt{\begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix}}$$

$$= \sqrt{0 + \left(\frac{408}{577}\right)^2 + \left(\frac{408}{577}\right)^2}$$

$$= \left(\frac{408}{577}\right)\sqrt{2} \approx 1$$

$$\Longrightarrow \boxed{\text{geometric length of } \mathbf{b} = 1}$$

$$\text{geometric length of a vector } \mathbf{c} = \sqrt{\begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix} \cdot \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix}}$$

$$= \sqrt{\left(\frac{396}{485}\right)^2 + \left(\frac{198}{485}\right)^2 + \left(\frac{198}{485}\right)^2}$$

$$= \left(\frac{198}{485}\right)\sqrt{6} \approx 1$$

$$\Longrightarrow \boxed{\text{geometric length of } \mathbf{c} = 1}$$

$$\text{geometric length of a vector } \mathbf{x} = \sqrt{\begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix}}$$

$$= \sqrt{(2)^2 + \left(\frac{40}{57}\right)^2 + \left(\frac{8}{77}\right)^2}$$

$$= \sqrt{4 + 0.4925 + 0.0108} = 2.1221$$

$$\Longrightarrow \boxed{\text{geometric length of } \mathbf{x} = 2.1221}$$

(c)

$$\mathbf{A} = \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix}$$

$$\Longrightarrow \mathbf{Ax} = \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix} \begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix}$$

$$= \begin{pmatrix} -1.0699 \\ -0.7009 \\ 1.6933 \end{pmatrix}$$

$$\Longrightarrow \text{ geometric length of } \mathbf{Ax} = \sqrt{\begin{pmatrix} -1.0699 \\ -0.7009 \\ 1.6933 \end{pmatrix} \cdot \begin{pmatrix} -1.0699 \\ -0.7009 \\ 1.6933 \end{pmatrix}} = 2.122$$

$$\Longrightarrow \text{ geometric length of } \mathbf{Ax} = \text{geometric length of } \mathbf{x}$$

$\boxed{\therefore \text{ We observe that the geometric length of } \mathbf{x} \text{ is same as that of } \mathbf{Ax}}$

(d)

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix} \\
\mathbf{A}^T &= \begin{pmatrix} -209/362 & -209/362 & 209/362 \\ 0 & -408/577 & -408/577 \\ 396/485 & -198/485 & 198/485 \end{pmatrix} \\
\implies \mathbf{A}^T\mathbf{A} &= \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix} \begin{pmatrix} -209/362 & -209/362 & 209/362 \\ 0 & -408/577 & -408/577 \\ 396/485 & -198/485 & 198/485 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\implies \mathbf{A}\mathbf{A}^T &= \begin{pmatrix} -209/362 & -209/362 & 209/362 \\ 0 & -408/577 & -408/577 \\ 396/485 & -198/485 & 198/485 \end{pmatrix} \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix} \\
&= \begin{pmatrix} 3(209/362)^2 & 0 & 0 \\ 0 & 2(408/577)^2 & 0 \\ 0 & 0 & 6(198/485)^2 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\boxed{\therefore \mathbf{A}\mathbf{A}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}
$$

(e) Given that,

$$
\mathbf{A}\mathbf{A}^T = \mathbf{I}, \text{ where } \mathbf{A} \text{ is a full rank matrix} \tag{36}
$$

We know that geometric length of a vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x, y, z \in \mathbb{R}$ is $\sqrt{\mathbf{v}.\mathbf{v}}$ i.e., $\sqrt{x^2+y^2+z^2}$ which can be written as $\sqrt{\mathbf{v}^T\mathbf{v}}$, as

$$
\begin{aligned}
\mathbf{v}^T\mathbf{v} &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(x^2+y^2+z^2\right) \\
&\implies \boxed{\text{geometric length of a vector} = \sqrt{\mathbf{v}^T\mathbf{v}}}
\end{aligned} \tag{37}
$$

Let $\mathbf{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$, then

$$
\begin{aligned}
(\mathbf{v}_1+\mathbf{v}_2)^T &= \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{pmatrix}^T = \begin{pmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \end{pmatrix} \\
\mathbf{v}_1^T+\mathbf{v}_2^T &= \begin{pmatrix} a_1 & b_1 & c_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \end{pmatrix} \\
\implies & \boxed{(\mathbf{v}_1+\mathbf{v}_2)^T = \mathbf{v}_1^T+\mathbf{v}_2^T}
\end{aligned} \tag{38}
$$

We know that,

$$
\begin{aligned}
\mathbf{A}\mathbf{v} &= x \begin{pmatrix} a \\ d \\ g \end{pmatrix} + y \begin{pmatrix} b \\ e \\ h \end{pmatrix} + z \begin{pmatrix} c \\ f \\ i \end{pmatrix} \text{, where } \mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \\
\implies (\mathbf{A}\mathbf{v})^T &= x \begin{pmatrix} a \\ d \\ g \end{pmatrix}^T + y \begin{pmatrix} b \\ e \\ h \end{pmatrix}^T + z \begin{pmatrix} c \\ f \\ i \end{pmatrix}^T \quad \text{[from equation 38]} \\
&= x \begin{pmatrix} a & d & g \end{pmatrix} + y \begin{pmatrix} b & e & h \end{pmatrix} + z \begin{pmatrix} c & f & i \end{pmatrix} \\
&= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \\
\implies & \boxed{(\mathbf{A}\mathbf{v})^T = \mathbf{v}^T \mathbf{A}^T}
\end{aligned}
\tag{39}
$$

$\therefore$ From equations 37 and 39, geometric length of $\mathbf{A}\mathbf{v}$ is

$$
\begin{aligned}
\sqrt{(\mathbf{A}\mathbf{v}).(\mathbf{A}\mathbf{v})} &= \sqrt{(\mathbf{A}\mathbf{v})^T \mathbf{A}\mathbf{v}}, \text{ where } \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \sqrt{\mathbf{v}^T \mathbf{A}^T \mathbf{A}\mathbf{v}} = \sqrt{\mathbf{v}^T (\mathbf{A}^T \mathbf{A}) \mathbf{v}} \\
&= \sqrt{\mathbf{v}^T (\mathbf{I}) \mathbf{v}} \quad \text{[from equation 36]} \\
&= \sqrt{\mathbf{v}^T \mathbf{v}} = \sqrt{\mathbf{v}.\mathbf{v}} \\
&= \text{geometric length of } \mathbf{v} \\
\text{geometric length of } \mathbf{A}\mathbf{v} &= \text{geometric length of } \mathbf{v}
\end{aligned}
$$

$$\boxed{\text{Hence observation in part(c) will always be true for any } \mathbf{x} \text{, given equation 36}}$$