Lecture 18. Nash Bargaining Problem

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Two Person Bargaining Problem

- Two players have the possibility of concluding a mutually beneficial agreement
- There is a conflict about which agrement to conclude
- No agreement may be imposed without the approval of the players

The solution should depend only on

- the payoffs they would expect if the negotiations were to fail
- the payoffs that are feasible in the process of negotiation

Applications of the Bargaining Problem

1. Management – Labour arbitration
2. International bilateral relations
3. Duopoly markets
4. Property settlement disputes
5. Supply chain contracts

⋮

The Nash Bargaining Problem

(F, v)

F: feasible set $\subset \mathbb{R}^2$ – closed and convex

- F : feasible set $\subseteq \mathbb{R}^2$ – closed and convex
- $v = (v_1, v_2) \in \mathbb{R}^2$ Disagreement point
  aka Default point; status-quo point
  solution that will be implemented in case no agreement is reached
- $F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1 ; x_2 \geq v_2\}$
  is non-empty and bounded.

Convexity of F

If $(x_1, x_2), (y_1, y_2) \in F$, then, for any $0 \leq \lambda \leq 1$,

$$\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \in F$$

This is a natural requirement since the convex combination can be achieved by the correlated strategy

$$\left((x_1, x_2) : \lambda \; ; \; (y_1, y_2) : (1-\lambda) \right)$$

F is closed

The limit of a sequence of feasible allocations is also feasible.

It will be odd if the limit lies outside the feasible set.

$F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1 ; x_2 \geq v_2\}$ is non-empty and bounded.

this means $\exists$ at least one feasible this means $\exists$ at least one feasible allocation that is as good as (v_1, v_2).

Further, unbounded gains over (v_1, v_2) are not feasible.

Typical choices for F

$$F = \{(u_1(\alpha), u_2(\alpha)) : \alpha \in \Delta(S_1 \times S_2)\}$$

$$F = \{(u_1(\alpha), u_2(\alpha)) : \alpha \in \Delta(S_1 \times S_2) \text{ and } \alpha \text{ is IR}\}$$

$$F = \{(u_1(\alpha), u_2(\alpha)) : \alpha \text{ is a correlated equilibrium}\}$$

$\vdots$

Disagreement Point: Typical choices

v_i = maxmin value of player i (security value) (no regret value)

v_i = utility of player i under a focal Nash equilibrium

v_i = utility of player i under a rational threat equilibrium

$\cdot$

⋮

The Nash Axioms

Nash brilliantly proposed five axioms that a solution to the bargaining problem should ideally satisfy and showed that such a solution exists and is unique.

1. Individual Rationality
2. Strong Efficiency
3. Scale Co-Variance
4. Independence of Irrelevant Alternatives
5. Symmetry

Suppose a candidate solution is given by

$$\left(f_1(F, v), \; f_2(F, v) \right) \in \mathbb{R}^2$$

1. Individual Rationality

$$f_1(F, v) \geq v_1$$
$$f_2(F, v) \geq v_2$$

2. Strong Efficiency Axiom

$(x_1, x_2) \in F$ is strongly (Pareto) efficient if $\nexists \; y = (y_1, y_2) \in F \ni$

$y_1 \geq x_1$ and

if $\nexists\ y = (y_1, y_2) \in F \ni$

$y_1 \geq x_1$; $y_2 \geq x_2$; and

$y_1 > x_1$ or $y_2 > x_2$ or both

strong efficiency axiom says that

$f(F, v) \in F$ and is strongly efficient.

This means there exists no other feasible solution that is strictly better for one player and not worse for the other player.

<u>Example</u> Suppose $f(F, v) = (4, 4)$

It should not happen that

$(5, 5) \in F$

$(5, 4) \in F$

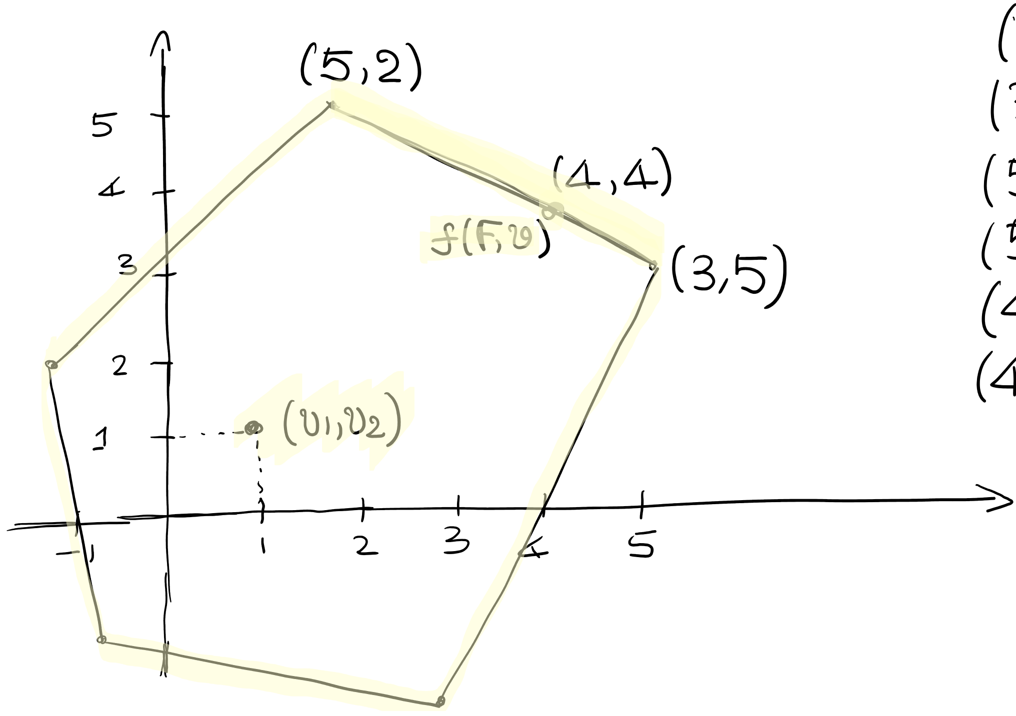
$(4, 5) \in F$

on the other hand, it can happen that

$(5, 2) \in F$

$(3, 5) \in F$

$(3, 3) \in F$



$(5, 2) \in F$

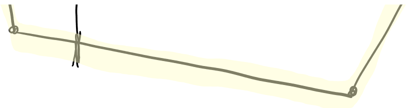
$(3, 5) \in F$

$(5, 5) \notin F$

$(5, 4) \notin F$

$(4, 5) \notin F$

$(4, 4) \geq (1, 1)$



3. Scale co-variance

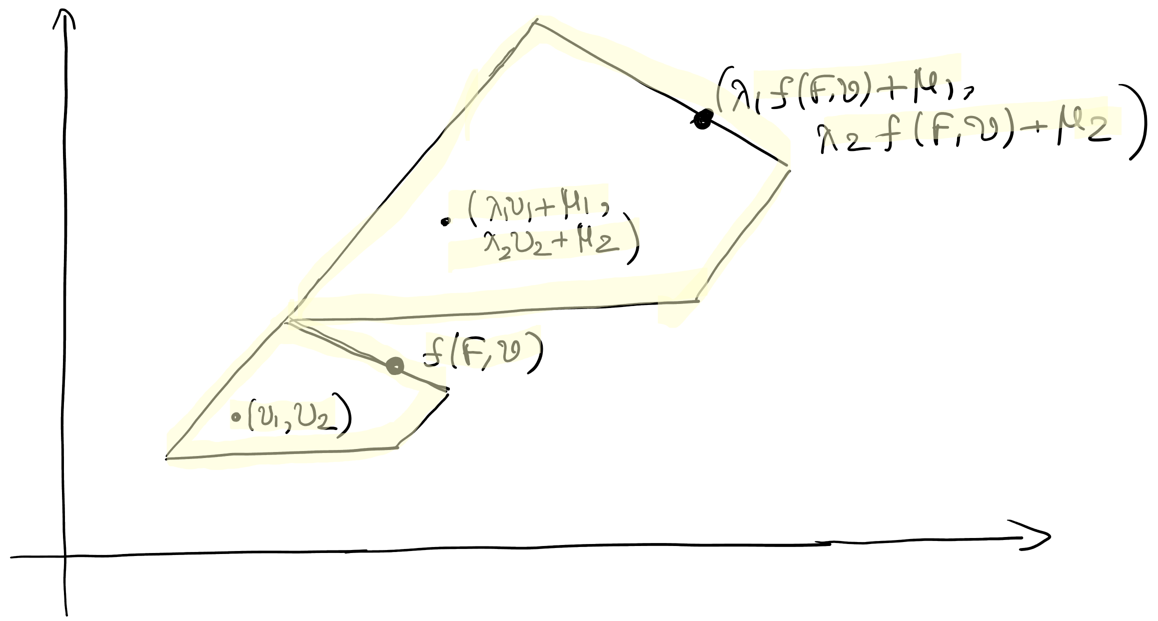
For any $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1 > 0$, $\lambda_2 > 0$, define

$$G = \left\{ (\lambda_1 x_1 + \mu_1, \ \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F \right\}$$

$$w = (\lambda_1 v_1 + \mu_1, \ \lambda_2 v_2 + \mu_2)$$

then the solution will also follow the same scaling:

$$f(G, w) = (\lambda_1 f_1(F, v) + \mu_1, \ \lambda_2 f_2(F, v) + \mu_2)$$



4) Independence of Irrelevant Alternatives

If G is closed and convex, then

$G \subseteq F$ and $f(F, v) \in G$

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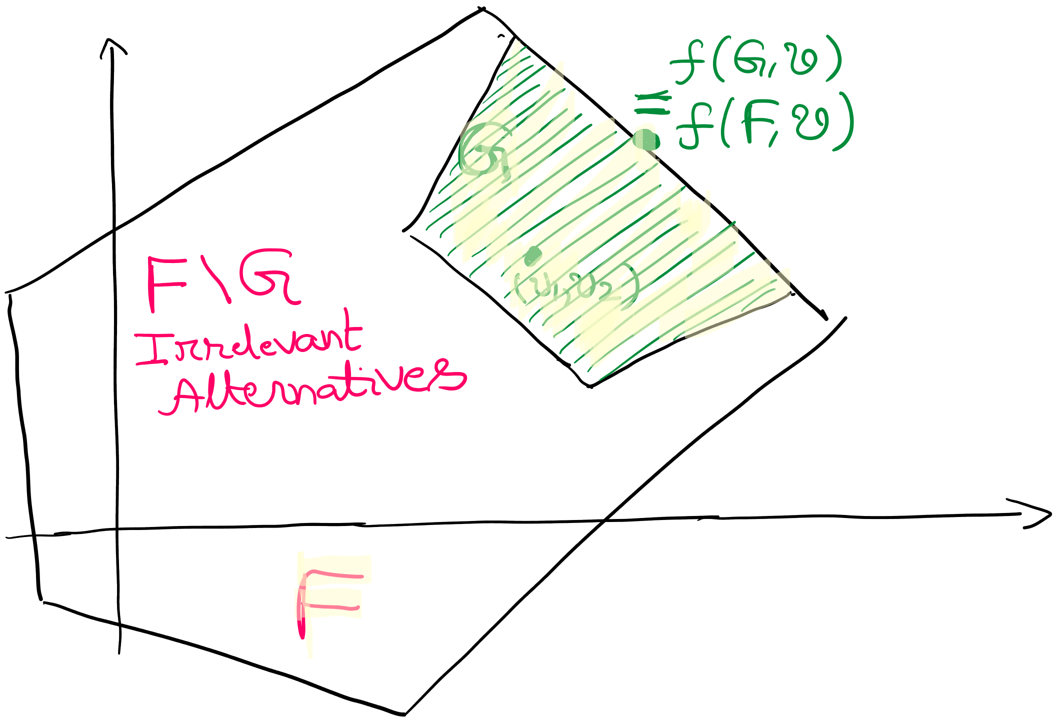
$\Rightarrow f(G, v) = f(F, v)$

If two persons, after careful negotiations are able to come up with a "sound" shortlisting of feasible alternatives, then they don't have to look beyond this subset for the final solution.



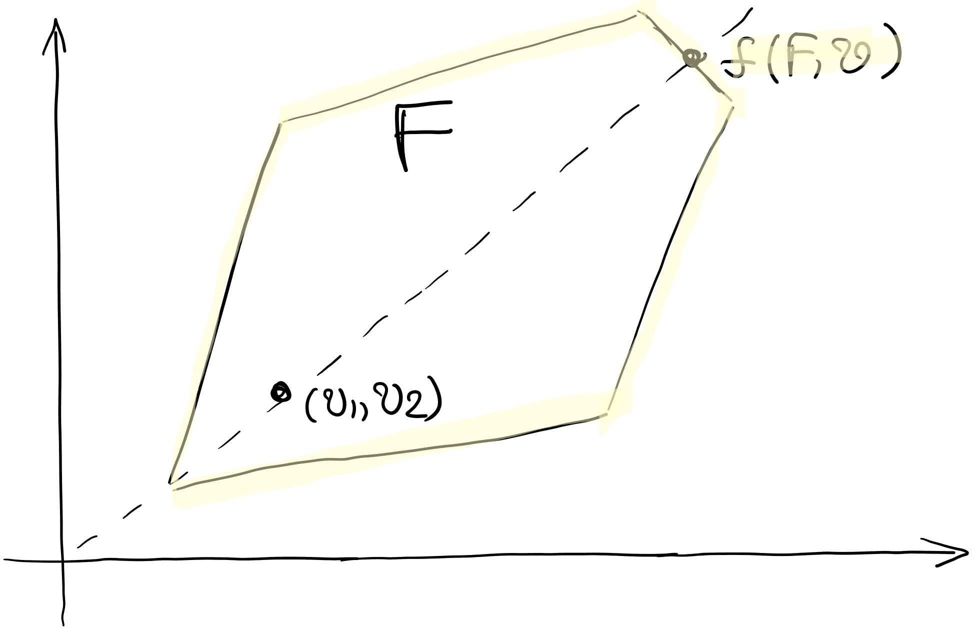
5. Symmetry

$\{(x_2, x_1) : (x_1, x_2) \in F\} = F$ and $v_1 = v_2$

$\Rightarrow f_1(F, v) = f_2(F, v)$

If the feasible set is "symmetric" and $v_1 = v_2$, then the solution will also be symmetric.

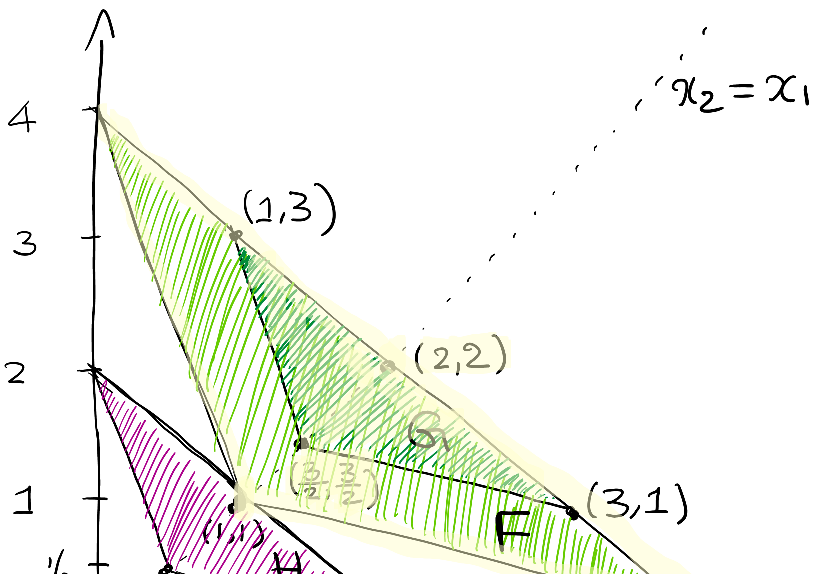
symmetric.

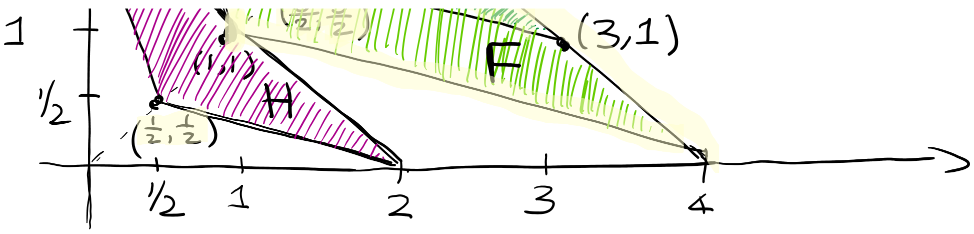


The Nash Bargaining Solution

There exists a unique function f that satisfies axioms (1), (2), (3), (4), (5), given by

$$f(F,v) \in \underset{\substack{(x_1,x_2)\in F \\ x_1 \geq v_1 \\ x_2 \geq v_2}}{\operatorname{argmax}} (x_1-v_1)(x_2-v_2)$$





F = closed, convex set — convex hull enclosing points $(1,1), (4,0), (0,4)$

$v = (1,1)$

$f(F, v) = (2,2)$ IR, Efficiency, Symmetry

Suppose $\lambda_1 = \lambda_2 = \frac{1}{2}$; $\mu_1 = \mu_2 = 1$
then we get the space (convex and closed):

$$G = \{(\lambda_1 x_1 + \mu_1 , \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F\}$$

G is the convex hull enclosed by the points $(\frac{3}{2}, \frac{3}{2})$, $(1,3)$, and $(3,1)$.

Scaled default point $w = (\frac{3}{2}, \frac{3}{2})$.
By scale covariance, we get
$f(G, w) = (2,2)$
this can also be derived from IIR since
$G \subseteq F$ and $f(F, v) \in G$.

Suppose $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\mu_1 = \mu_2 = 0$, we get
H which is the convex hull enclosed by
$(\frac{1}{2}, \frac{1}{2})$, $(2,0)$, and $(0,2)$. Now

$f(H, (\frac{1}{2}, \frac{1}{2})) = (1,1)$ by scale covariance.