This say that projector P seperates $\mathbb{R}^m$ into two subspaces.

Orthogonal Projectors:-

An orthogonal projector is one that projects onto a subspace S_1 along subspace S_2 where S_1 and S_2 are orthogonal subspaces.

Thm:- A projector $\underset{\sim}{P}$ is orthogonal projector if and only if $\underset{\sim}{P} = \underset{\sim}{P}^T$

Pf:- step 1 If $\underbrace{\underset{\sim}{P} = \underset{\sim}{P}^T}$, we need to show projector $\underset{\sim}{P}$ is orthogonal

Consider an inner product between a vector in S_1 i.e $\underset{\sim}{P}\underset{\sim}{x} \in S_1$ and vector $(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y} \in S_2$

$$(\underset{\sim}{P}\underset{\sim}{x}, (\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}) = (\underset{\sim}{P}\underset{\sim}{x})^T(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T\underset{\sim}{P}^T(\underset{\sim}{I}-\underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T \underset{\sim}{P}(\underset{\sim}{I} - \underset{\sim}{P})\underset{\sim}{y}$$

$$= \underset{\sim}{x}^T [\underset{\sim}{P} - \underset{\sim}{P}^2] \underset{\sim}{y} = \underset{\sim}{0}$$

Step 2 :-

To prove :→ An orthogonal projector $\underset{\sim}{P} \in \mathbb{R}^{m \times m}$ ($\underset{\sim}{P}$ projects onto S_1 along S_2 where $S_1 \perp S_2$)

satisfies $\underset{\sim}{P} = \underset{\sim}{P}^T$

Let S_1 have dimension $n < m$ and let $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_m\}$ be the basis for $\mathbb{R}^m$ where $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_n\}$ be the basis for S_1 and $\{\underset{\sim}{q}_{n+1}, \dots \underset{\sim}{q}_m\}$ be the basis for S_2.

Let us try to construct SVD for $\underset{\sim}{P}$.

for $j < n$, $\underset{\sim}{P} \underset{\sim}{q}_j = \underset{\sim}{q}_j$ ✓

and $j > n$, $\underset{\sim}{P} \underset{\sim}{q}_j = \underset{\sim}{0}$

Now let us construct a matrix $\underset{\sim}{Q}$

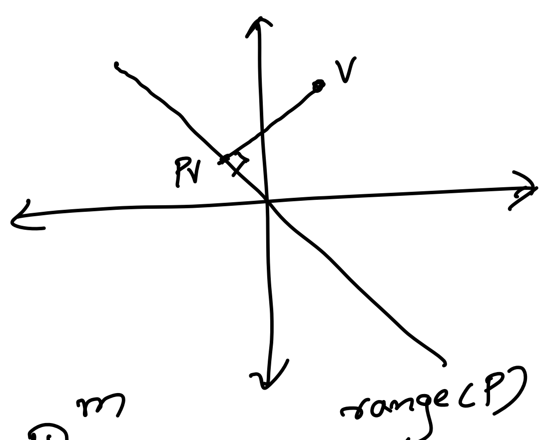
$$\underset{\sim}{Q} = \begin{bmatrix} | & | & | & & | \\ \underset{\sim}{q_1} & \underset{\sim}{q_2} & \underset{\sim}{q_3} & \cdots & \underset{\sim}{q_m} \\ | & | & | & & | \end{bmatrix}$$

$$\underset{\sim}{P}\,\underset{\sim}{Q} = \begin{bmatrix} | & | & & | & | & | & & | \\ \underset{\sim}{q_1} & q_2 & \cdots & q_n & 0 & 0 & \cdots & 0 \\ | & | & & | & | & | & & | \end{bmatrix}$$

$$\underset{\sim}{Q}^T \underset{\sim}{P}\,\underset{\sim}{Q} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} = \underset{\sim}{\Sigma}$$

$$\underset{\sim}{P} = \underset{\sim}{Q}\,\underset{\sim}{\Sigma}\,\underset{\sim}{Q}^T \quad \text{we constructed}$$

$$\text{SVD of } \underset{\sim}{P} \Rightarrow \quad \underset{\sim}{P} = \underset{\sim}{P}^T$$



Orthogonal Projectors corresponding to a subspace spanned by orthonormal basis

$V^n \subseteq \mathbb{R}^m$

Let us consider an n-dimensional subspace in $\mathbb{R}^m$ and $\{\underset{\sim}{q_1}, \underset{\sim}{q_2}, \ldots \underset{\sim}{q_n}\}$ be the set of n orthonormal vectors in $\mathbb{R}^m$ spanning our n-dimensional subspace.

Let $\hat{Q} = \begin{bmatrix} | & | & | & & | \\ \underset{\sim}{q}_1 & \underset{\sim}{q}_2 & \underset{\sim}{q}_3 & \cdots & \underset{\sim}{q}_n \\ | & | & | & & | \end{bmatrix}_{m \times n}$

Let $\underset{\sim}{v} \in \mathbb{R}^m$ can be decomposed into a component in the column space $\hat{Q}$ plus a component $\underset{\sim}{r}$ perpendicular to column space of $\hat{Q}$

$$\underset{\sim}{v} = \underset{\sim}{r} + \underbrace{\sum_{i=1}^{n} (\underset{\sim}{q}_i^T \underset{\sim}{v})\, \underset{\sim}{q}_i}$$

The map $\underset{\sim}{v} \mapsto \sum_{i=1}^{n} (\underset{\sim}{q}_i^T \underset{\sim}{v}) \underset{\sim}{q}_i$ is an orthogonal projection onto range $(\hat{Q})$

$$\underset{\sim}{y} = \underset{\sim}{P}\underset{\sim}{v} = \sum_{i=1}^{n} \underbrace{(\underset{\sim}{q}_i^T \underset{\sim}{v})\, \underset{\sim}{q}_i}$$

$$\underset{\sim}{y} = \underset{\sim}{P}\underset{\sim}{v} = \sum_{i=1}^{n} \underbrace{\underset{\sim}{q}_i (\underset{\sim}{q}_i^T \underset{\sim}{v})}$$

$$= \underbrace{\sum_{i=1}^{n} \underset{\sim}{q}_i \underset{\sim}{q}_i^T}_{P} \underset{\sim}{v}$$

$$\underset{\sim}{y} = \underset{\sim}{\hat{Q}} \underset{\sim}{\hat{Q}}^T \underset{\sim}{v}$$

where $\underset{\sim}{\hat{Q}} = \begin{bmatrix} | & | & & | \\ \underset{\sim}{q}_1 & \underset{\sim}{q}_2 & \cdots & \underset{\sim}{q}_n \\ | & | & & | \end{bmatrix}$

$$\boxed{\underset{\sim}{P} = \underset{\sim}{\hat{Q}} \underset{\sim}{\hat{Q}}^T}$$

$$\tilde{Q} = \begin{bmatrix} | & | & | & & | \\ \tilde{q}_1 & \tilde{q}_2 & \tilde{q}_3 & \cdots & \tilde{q}_n \\ | & | & | & & | \end{bmatrix}$$

Show $P = \tilde{P}$? $\quad \tilde{P} = \tilde{Q}\tilde{Q}^T$

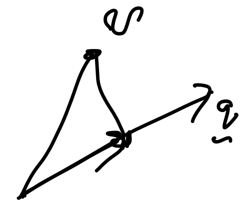
Exercise

* Complement of orthogonal projector is also orthogonal projector.

i.e $\underset{\sim}{P} = \underset{\sim}{P}^T$ then $(\underset{\sim}{I} - \underset{\sim}{P})^T = (\underset{\sim}{I} - \underset{\sim}{P})$

The complements projects onto space orthogonal to range ($\underset{\sim}{P}$)

* Eg:- Rank 1 orthogonal projector that isolates component of a vector $\underset{\sim}{v}$ in a single direction $P_q = \underset{\sim}{q}\,\underset{\sim}{q}^T$



$(q^T v)\, q$

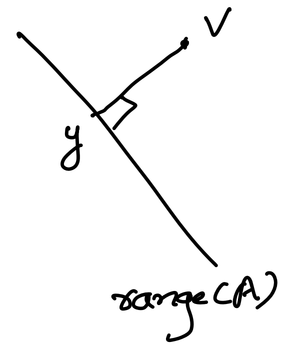
$$(\underset{\sim}{q}\,\underset{\sim}{q}^T)\,\underset{\sim}{v} = \bar{v}$$
$$= (\underset{\sim}{q}^T \underset{\sim}{v})\,\underset{\sim}{q}$$

<u>Projection onto</u> n-dimensional <u>subspace</u> represented <u>by arbitrary</u> <u>basis</u> :-

Let the subspace be spanned by the linearly independent vectors $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots \underset{\sim}{a}_n\}$. $\underset{\sim}{A} \in \mathbb{R}^{m\times n}$ have the columns $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots \underset{\sim}{a}_n\}$

$$A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}_{m \times n}$$

Let $y \in \text{range}(A)$ be the projected vector. We know that $(y - v) \perp \text{range}(A)$



$$a_j^T (y - v) = 0 \quad \text{— (1)}$$

for every $j = 1 \ldots n$

Since $y \in \text{range}(A)$, we can write

$$y = Ax \text{ for some } x.$$

and hence (1) becomes

$$a_j^T (Ax - v) = 0 \quad \forall j = 1 \ldots n$$

$$A^T [Ax - v] = 0$$

$$\Rightarrow A^T A x = A^T v$$

$$x = (A^T A)^{-1} A^T v$$

(show $(A^T A)^{-1}$ exists?)

$$y = Pv = Ax$$

$$\Rightarrow Pv = A (A^T A)^{-1} A^T v \quad \forall v \in \mathbb{R}^m$$

$$\Rightarrow \boxed{\underset{\sim}{P} = \underset{\sim}{A}(\underset{\sim}{A}^T\underset{\sim}{A})^{-1}\underset{\sim}{A}^T}$$

"Show that this $\underset{\sim}{P}$ is same as $\underset{\sim}{P}$ obtained by $\hat{Q}\hat{Q}^T$ before?"

Exercise!