QR Algorithm

Pure QR algorithms :- $A \in \mathbb{R}^{m \times m}$

$A^{(0)} = A$

for $k = 1, 2, \ldots$ do

$\rightarrow Q^{(k)} R^{(k)} = A^{(k-1)}$ % compute QR factorization

$\rightarrow \underbrace{A^{(k)} = R^{(k)} Q^{(k)}}$ % Recombine factors in reverse order

end for

Under suitable assumptions, this simple algorithm converges to a Schur form for A ; upper triangular for arbitrary A, or diagonal form if A is real symmetric matrix. $A^{(k)}$

$A^{(0)} = A$

$\underline{k=1}$

$$Q^{(1)} R^{(1)} = A^{(0)} \quad \text{--- (1)}$$

$$\underbrace{A^{(1)} = R^{(1)} Q^{(1)}} \quad \text{--- (2)}$$

From ① $\underline{R}^{(1)} = \underline{Q}^{(1)T} \underline{A}^{(0)}$

From ② $\underline{A}^{(1)} = \underline{Q}^{(1)T} \underline{A}^{(0)} \underline{Q}^{(1)}$

for generic k

$$\underset{\sim}{R}^{(k)} = \left(\underset{\sim}{Q}^{(k)}\right)^T \underline{A}^{(k-1)}$$

then $$\underset{\sim}{A}^{(k)} = \underset{\sim}{R}^{(k)} \underline{Q}^{(k)}$$

$$= \underbrace{\left(\underset{\sim}{Q}^{(k)}\right)^T \underline{A}^{(k-1)} \underset{\sim}{Q}^{(k)}}$$

Unnormalized simultaneous iteration :-

To explain how $\underset{\sim}{Q}\underset{\sim}{R}$ algorithm works, we relate to another algorithm called simultaneous iteration (subspace iteration)!

Idea :- Apply power iteration to

several vectors at once.

Suppose we start with a set of n linearly independent vectors at once $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$

$$\underset{\sim}{v}_1^{(0)}, \underset{\sim}{v}_2^{(0)}, \ldots \underset{\sim}{v}_n^{(0)} \qquad n < m$$

The hypothesis is that $\underset{\sim}{A}^k v_1^{(0)}$ converges (as $k \to \infty$) to the eigen vector corresponding to the largest eigenvalue of $\underset{\sim}{A}$, what will happen to space $\langle \underset{\sim}{A}^k \underset{\sim}{v}_1^{(0)}, \underset{\sim}{A}^k \underset{\sim}{v}_2^{(0)}, \ldots \underset{\sim}{A}^k \underset{\sim}{v}_n^{(0)} \rangle$?

Under suitable assumptions, this space $\langle \underset{\sim}{A}^k \underset{\sim}{v}_1^{(0)}, \underset{\sim}{A}^k \underset{\sim}{v}_2^{(0)}, \ldots \underset{\sim}{A}^k v_n^{(0)} \rangle$ converge to the space $\langle \underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots \underset{\sim}{q}_n \rangle$

corresponding to eigenvectors of n largest eigenvalues of $\underset{\sim}{A}$.

In matrix notation

$$\underset{\sim}{V}^{(0)} = \left[\underset{\sim}{V}_1^{(0)} \middle| \underset{\sim}{V}_2^{(0)} \cdots \middle| V_n^{(0)} \right]$$

and define

$$\boxed{\underset{\sim}{V}^{(k)} = \underset{\sim}{A}^k \underset{\sim}{V}^{(0)}}$$

$$= \left[\underset{\sim}{V}_1^{(k)} \middle| \underset{\sim}{V}_2^{(k)} \middle| \cdots \middle| V_n^{(k)} \right]$$

Because we are interested in the space spanned by columns of $\underset{\sim}{V}^{(k)}$, we can extract a nice basis for this space from the reduced $\underset{\sim}{Q}\underset{\sim}{R}$

factorization of $\underset{\sim}{V}^{(k)}$

$$\underset{\sim}{\hat{Q}}^{(k)} \underset{\sim}{R}^{(k)} = \underset{\sim}{V}^{(k)}$$

$\underset{\sim}{\hat{Q}}^{(k)} \in \mathbb{R}^{m \times n}$

$\underset{\sim}{R}^{(k)} \in \mathbb{R}^{n \times n}$

Can we hope that as $k \to \infty$ the columns of $\underset{\sim}{\hat{Q}}^{(k)}$ converge to the eigenvectors $(\pm \underset{\sim}{q_1}, \pm \underset{\sim}{q_2}, \dots \pm q_n)$?

Expand $v_j^{(0)}$ and $v_j^{(k)}$ in terms of eigenvectors of $\underset{\sim}{A}$

$$v_j^{(0)} = a_{1j} \underset{\sim}{q_1} + a_{2j} \underset{\sim}{q_2} + \dots a_{mj} \underset{\sim}{q_m}$$

$$v_j^{(k)} = \lambda_1^k a_{1j} \underset{\sim}{q_1} + \lambda_2^k a_{2j} \underset{\sim}{q_2} + \dots \lambda_m^k a_{mj} \underline{q_m}$$

As before like power iteration the convergence happens provided the following two conditions are satisfied :-

1. The first $n+1$ eigenvalues are distinct in absolute value:

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \dots > |\lambda_n| > |\lambda_{n+1}| \geq |\lambda_{n+2}| \geq \dots \geq |\lambda_m|$$

2. Let $\bar{Q} = [\underset{\sim}{q_1}, \dots \underset{\sim}{q_n}] \in \mathbb{R}^{m \times n}$

i.e $\bar{Q}$ is matrix formed from eigenvectors of $\underset{\sim}{A}$, all the leading principal submatrices of $\hat{Q}^T V^{(0)}$ are non singular.

(The upper left square submatrices of dimensions $1\times1, 2\times2, \ldots n\times n$) are non singular!

Thm: Let the above two assumptions hold good, suppose we perform iteration for the given $V^{(0)}$,

$$\underbrace{V^{(k)} = A^k V^{(0)}}, \quad \hat{Q}^{(k)} R^{(k)} = V^{(k)}$$

Then $k \to \infty$, the columns of matrices $\hat{Q}^{(k)}$ converge linearly to the eigenvectors of A:

$\| q_j^{(k)} - \pm q_j \| = O(C^k)$ for each $j = 1, 2, \dots n$ where

$$C = \max_{1 \le k \le n} \left| \frac{\lambda_{k+1}}{\lambda_k} \right| < 1$$

* As $k \to \infty$, the $v_1^{(k)}, v_2^{(k)}, \dots v_n^{(k)}$ all converge to multiples of $\underline{q}_1$.

Although $\langle \underline{v}_1^{(k)}, \underline{v}_2^{(k)}, \dots v_j^{(k)} \rangle$ converges to something useful, these vectors form a highly ill-conditioned basis for that space!

→ Simultaneous iteration as we described is practically not useful!

Hence we orthonormalize at every k^{th} step instead of once at the end, so we do not construct

$V^{(k)}$ but of a different sequence of matrices $Z^{(k)}$ with the same column space.

Algo:- choose $Q^{(0)} \in \mathbb{R}^{m\times n}$ with orthonormal columns.

$\hat{Q}^{(0)} = Q^{(0)}$ $\qquad AQ^{(0)}$

for $k = 1, 2, \ldots$ do

$Z^{(k)} = A\hat{Q}^{(k-1)}$.

$\hat{Q}^{(k)}\hat{R}^{(k)} = Z^{(k)}$ % reduced QR factorization of $Z^{(k)}$

end for

The column space $\hat{Q}^{(k)}$ and $Z^{(k)}$ are the same and also same for $A^k\hat{Q}^{(0)}$

and hence we expect $\hat{Q}^{(k)}$ converge to the largest n eigenvectors under same two conditions as before.

Thm:- Algorithm for simultaneous iteration with QR decomposition at every k step generates the same matrix $\hat{Q}^{(k)}$ as the iteration without QR decomposition at every step.

" Simultaneous iteration $\Longleftrightarrow$ QR algorithm "

Simultaneous iteration algorithm :-

You start with $n=m$ linearly independent vectors. Infact you start with canonical basis

$$\bar{Q}^{(0)} = I = [e_1 | e_2 | \cdots | e_m]$$

Algo :- $\bar{Q}^{(0)} = I$ ✓

for $k = 1, 2, \ldots$

$$Z = A\bar{Q}^{(k-1)}$$

$$Z = \bar{Q}^{(k)} R_s^{(k)}$$

$$A^{(k)} = [\bar{Q}^{(k)}]^T A \bar{Q}^{(k)}$$

$$\underbrace{\bar{R}_s^{(k)} = R_s^{(k)} R_s^{(k-1)} \cdots R_s^{(1)}}$$

What will be A^k?

(i) $A = AI = A\bar{Q}^{(0)} = \bar{Q}^{(1)} R_s^{(1)}$

$$= \underbrace{\bar{Q}^{(1)} \bar{R}_s^{(1)}}$$

(ii) $A^2 = A(A)$

$$= A(\bar{Q}^{(1)} \bar{R}_s^{(1)}) = \bar{Q}^{(2)} R_s^{(2)} \bar{R}_s^{(1)}$$

$$= \underbrace{\bar{Q}^{(2)} \bar{R}_s^{(2)}}$$

(iii) $$\underset{\sim}{A}^3 = \underset{\sim}{A}(\underset{\sim}{A}^2) = \underbrace{\bar{Q}^{(3)} \bar{R}_s^{(3)}}$$

$\vdots$

$$\downarrow \quad \boxed{\underset{\sim}{A}^k = \bar{\underset{\sim}{Q}}^{(k)} \bar{\underset{\sim}{R}}_s^{(k)}} \quad \boxed{\underset{\sim}{A}^{(k)} = \left(\bar{\underset{\sim}{Q}}^{(k)}\right)^T A \bar{Q}^{(k)}}$$

QR Algorithm :-

$$A^{(0)} = A$$

for $k = 1, 2, \ldots$

$$\underset{\sim}{Q}^{(k)} \underset{\sim}{R}^{(k)} = \underset{\sim}{A}^{(k-1)} \leftarrow$$

$$\underset{\sim}{A}^{(k)} = \underset{\sim}{R}^{(k)} \underset{\sim}{Q}^{(k)}$$

$$\left.\begin{aligned} &\rightarrow \underset{\sim}{Q}_R^{(k)} = \underset{\sim}{Q}^{(1)} \underset{\sim}{Q}^{(2)} \cdots \underset{\sim}{Q}^{(k)} \\ &\rightarrow \underset{\sim}{R}_R^{(k)} = \underset{\sim}{R}^{(k)} \underset{\sim}{R}^{(k-1)} \cdots \underset{\sim}{R}^{(1)} \end{aligned}\right\} \text{Defining } \underset{\sim}{Q}_R^{(k)}, \underset{\sim}{R}_R^{(k)}$$

What happens to A^k in this QR setting?

(i) $A = A^{(0)} = Q^{(1)} R^{(1)} = Q_R^{(1)} R_R^{(1)}$

(ii)
$$
\begin{aligned}
A^2 = A(A) &= A(Q^{(1)} R^{(1)}) \\
&= Q^{(1)} \underbrace{R^{(1)} (Q^{(1)}} R^{(1)}) \\
&= \underbrace{Q^{(1)} Q^{(2)}} \underbrace{R^{(2)} R^{(1)}} \\
&= Q_R^{(2)} R_R^{(2)}
\end{aligned}
$$

(iii)
$$
\begin{aligned}
A^3 &= A(A^2) \\
&= A(Q^{(1)} Q^{(2)} R^{(2)} R^{(1)}) \\
&= Q^{(1)} \underbrace{R^{(1)} (Q^{(1)}} \underbrace{Q^{(2)} R^{(2)} R^{(1)}}) \\
&= Q^{(1)} Q^{(2)} \underbrace{R^{(2)} Q^{(2)}} R^{(2)} R^{(1)} \\
&= Q^{(1)} Q^{(2)} Q^{(3)} R^{(3)} R^{(2)} R^{(1)}
\end{aligned}
$$

$$= Q_R^{(3)} R_R^{(3)}$$

$$\vdots$$

$$A^k = Q_R^{(k)} R_R^{(k)}$$

Let us look at $A^{(k)}$

$$A^{(k)} = R^{(k)} Q^{(k)}$$

$$A^{(1)} = R^{(1)} Q^{(1)} = I R^{(1)} Q^{(1)}$$

$$= \left(Q^{(1)}\right)^T \underbrace{Q^{(1)} R^{(1)}} Q^{(1)}$$

$$= \left(Q^{(1)}\right)^T A \, Q^{(1)}$$

$$= \underbrace{\left(Q_R^{(1)}\right)^T A \, Q_R^{(1)}}$$

$$A^{(2)} = R^{(2)} Q^{(2)} = \left(Q^{(2)}\right)^T \underbrace{Q^{(2)} R^{(2)}} Q^{(2)}$$

$$= \left(Q^{(2)}\right)^T A^{(1)} Q^{(2)} = \left(Q^{(2)}\right)^T \left(Q_R^{(1)}\right)^T A \, Q_R^{(1)} Q^{(2)}$$

$$= \left(Q_R^{(2)}\right)^T A\, Q_R^{(2)}$$

$$\vdots$$

$$A^{(k)} = \underbrace{\left(Q_R^{(k)}\right)^T A\, Q_R}$$

Simultaneous iteration :- $A^k = \bar{Q}^{(k)} \bar{R}_s^{(k)}$

QR Algorithm :- $\underbrace{A^k = Q_R\, R_R}_{}^{(k)}$

Since QR decomposition of A^k has to unique,

$$\bar{Q}^{(k)} = Q_R^{(k)}$$

$$\bar{R}_s^{(k)} = R_R^{(k)} = \bar{R}^{(k)}$$

Simultaneous iteration :- $A^{(k)} = (\bar{Q}^k)^T A \bar{Q}^{(k)}$

QR algo : $A^{(k)} = (Q_R^{(k)})^T A\, Q_R^{(k)}$

$$= (\bar{Q}^{(k)})^T A \bar{Q}^{(k)}$$

Convergence of QR algorithm :-

(1) $\underset{\sim}{A}^k = \underset{\sim}{\bar{Q}}^{(k)} \underset{\sim}{\bar{R}}^{(k)}$ as $k \to \infty$

$\bar{Q}^{(k)}$ converges to eigen subspace

i.e it constructs orthonormal eigenbasis for successive powers of A^k

(2) $\underset{\sim}{A}^{(k)} = (\underset{\sim}{\bar{Q}}^{(k)})^T \underline{A}\, \underline{\bar{Q}}^{(k)}$

→ diagonal elements of $\underset{\sim}{A}^{(k)}$ are the Rayleigh quotients of $\underset{\sim}{A}$ corresponding to the colums of $\bar{Q}^{(k)}$ i.e $\underset{\sim}{\bar{q}}_i^T \underline{A}^{(k)} \underline{q}_i$

columns of $\bar{Q}_k$ → eigenvectors as $k \to \infty$

these rayleigh quotients → eigenvalues

off diagonal entries of $\underset{\sim}{A}^{(k)}$ converge to 0 as $k \to \infty$, as the columnvectors become eigenvectors and the approximation become orthogonal

i.e
$$A_{ii}^{(k)} = \bar{q}_i^{(k)} \underset{\sim}{A}^{(k)} \underset{\sim}{q}_i^{(k)}$$
$$\Downarrow$$
$$\lambda_i^{(k)}$$

$$A_{ij}^{(k)} = (\underset{\sim}{q}_i^{(k)})^T \underset{\sim}{A}^{(k)} \underset{\sim}{q}_j^{(k)}$$
$$\Downarrow$$
$$0$$

as $\underset{\sim}{q}_i^{(k)}$ and $\underset{\sim}{q}_j^{(k)}$ become more and more orthogonal as $k \to \infty$

Thm:- Let the pure QR algorithm be applied to a real symmetric matrix $\underset{\sim}{A}$ with eigenvalues satisfying $|\lambda_1| > |\lambda_2| \ldots > |\lambda_m|$ and whose corresponding eigenvector matrix $\underset{\sim}{Q}$

has nonsingular leading principal minors, then as $k \to \infty$,

$\underset{\sim}{A}^{(k)} \to \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_m)$ and

$\underset{\sim}{\bar{Q}}^{(k)} \to Q$ at the linear rate $\max_k \frac{|\lambda_{k+1}|}{|\lambda_k|}$

<u>QR</u> algorithm <u>with</u> shifts

$\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ real and symmetric

we have real eigenvalues $\{\lambda_j\}$ and orthonormal eigenvectors $\{\underset{\sim}{q}_j\}$.

We have just seen

Pure QR algo $\to$ Simultaneous iteration applied to $\underset{\sim}{I}$

$\Rightarrow$ first column evolves as a power iteration applied to $\underset{\sim}{e}_1$

$\boxed{\text{We can also view pure QR as a simultaneous inverse iteration applied to a flipped identity matrix.}}$

As before let $Q^{(k)}$ be the orthogonal matrix generated at step k of the QR algorithm!

$$\boxed{\bar{Q}^{(k)} = \prod_{j=1}^{k} Q^{(j)} = \left[q_1^{(k)} \middle| q_2^{(k)} \middle| \ldots \middle| q_m^{(k)} \right]}$$

is same as the orthogonal matrix at step (k) of the simultaneous iteration.

$$\boxed{A^k = \bar{Q}^{(k)} \bar{R}^{(k)}}$$

$$\Rightarrow (A^k)^{-1} = \left(\bar{Q}^{(k)} \bar{R}^{(k)} \right)^{-1}$$

$$\Rightarrow \underset{\sim}{A}^{-k} = \left(\bar{R}^{(k)}\right)^{-1}\left(\bar{Q}^{(k)}\right)^{-1}$$

$$= \underbrace{\left(\bar{R}^{(k)}\right)^{-1}\left(\bar{Q}^{(k)}\right)^{T}}$$ ✓

Using symmetry of A^{-k}, we can write

$$\underset{\sim}{A}^{-k} = \left(\underset{\sim}{A}^{-k}\right)^{T} = \left(\underbrace{\left(\bar{R}^{(k)}\right)^{-1}\left(\bar{Q}^{(k)}\right)^{T}}\right)^{T}$$

$$= \underbrace{\bar{Q}^{(k)}\left(\bar{R}^{(k)}\right)^{-T}}$$

$$\boxed{\underset{\sim}{A}^{-k} = \underset{\sim}{\bar{Q}}^{(k)}\left(\underset{\sim}{\bar{R}}^{(k)}\right)^{-T}} \quad \text{— (1)}$$

Now Define $P \in \mathbb{R}^{m\times m}$, a flipped identity matrix

$$\boxed{\underset{\sim}{P} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \qquad \underset{\sim}{P}^{2} = \underset{\sim}{I}}$$

Using (1)

$$\underset{\sim}{A}^{-k}\underset{\sim}{P} = \bar{Q}^{(k)}\left(\bar{R}^{(k)}\right)^{-T} P$$

$$= [\bar{Q}^{(k)}]\, I\, (\bar{R}^{(k)})^{-T} P \qquad A[\;]$$

$$A^{k}[\;]$$

$$\boxed{A^{-k} \underset{\sim}{P} = (\underbrace{\bar{Q}^{(k)} P}) \boxed{(\underbrace{P(\bar{R}^{(k)})^{-T} P})}} \qquad = \bar{Q}^{(k)} R$$

$\underbrace{\bar{Q}^{(k)}} P$ is orthogonal matrix

and $\underbrace{P(\bar{R}^{(k)})}^{-T} P$ is also upper triangular, we have a

a QR factorization for $A^{-k} \underset{\sim}{P}$

$$(A^{-1})^{k} \underset{\sim}{P} = [\underbrace{\bar{Q}^{(k)} \underset{\sim}{P}}][\underbrace{P(\bar{R}^{(k)})^{-T} \underset{\sim}{P}}]$$

i.e we are carrying out simultaneous iteration on $\underset{\sim}{A}^{-1}$ applied to intial matrix $\underset{\sim}{P}$ (Flipped identity matrix)

This means, we are doing simultaneous inverse iteration $\underset{\sim}{A}$.

In particular, first column of $\underset{\sim}{\bar{Q}}^{(k)} \underset{\sim}{P}$ which is the last column of $\underset{\sim}{\bar{Q}}^{(k)}$ evolves as k-steps of inverse iteration to $\underset{\sim}{e}_m$.

<u>Shifted inverse iteration within the framework of QR algorithm</u>:-

QR algo is both simultaneous and inverse simultaneous iteration

<u>Algo</u>:- $[Q^{(0)}]^T \underset{\sim}{A}^{(0)} Q^{(0)} = \underset{\sim}{A}$ ($\underset{\sim}{A}^{(0)}$ is tridiagonal reduction of A)

for $k = 1, 2 \ldots$

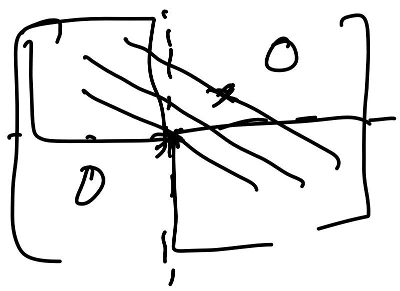
Pick a shift $\mu^{(k)}$ eg:- $\mu^{(k)} = A^{(k-1)}_{mm}$

$\longrightarrow \quad \underset{\sim}{Q}^{(k)} \underset{\sim}{R}^{(k)} = \underset{\sim}{A}^{(k-1)} - \mu^{(k)} \underset{\sim}{I}$

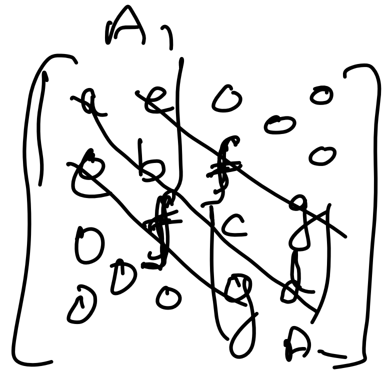
$\longrightarrow \quad \underset{\sim}{A}^{(k)} - \mu^{(k)} \underset{\sim}{I} = R^{(k)} Q^{(k)}$

$\longrightarrow$ If any off diagonal $\left[\underset{\sim}{A}^{(k)} = \underset{\sim}{R}^{(k)} \underset{\sim}{Q}^{(k)} + \mu^{(k)} \underset{\sim}{I}\right]$ element $A^{(k)}_{j,j+1}$ is sufficiently close to 0, we set

$$A_{j,j+1} = A_{j+1,j} = 0$$



$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = A^{(k)}$$



and keep QR algo to $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$

$\longrightarrow \quad \left(\underset{\sim}{A}^{(k-1)} - \mu^{(k)} \underset{\sim}{I}\right) = \underset{\sim}{Q}^{(k)} \underset{\sim}{R}^{(k)}$

$\longrightarrow \quad \underset{\sim}{A}^{(k)} = R^{(k)} Q^{(k)} + \mu^{(k)} \underset{\sim}{I}$

$$\Rightarrow \quad \underset{\sim}{A}^{(k)} = \left(\underset{\sim}{Q}^{(k)}\right)^T \underset{\sim}{A}^{(k-1)} \underset{\sim}{Q}^{(k)}$$

$$\Rightarrow \quad \boxed{\underset{\sim}{A}^{(k)} = \left(\bar{\underset{\sim}{Q}}^{(k)}\right)^T \underset{\sim}{A}^{(0)} \bar{\underset{\sim}{Q}}^{(k)}}$$

We can also show shifted QR algorithm

$$(\underset{\sim}{A} - \mu^{(k)} \underset{\sim}{I})(\underset{\sim}{A} - \mu^{(k-1)} \underset{\sim}{I}) \cdots (\underset{\sim}{A} - \mu^{(1)} \underset{\sim}{I}) = \bar{Q}^{(k)} \bar{R}^{(k)}$$

i.e $\bar{Q}^{(k)} = \prod_{j=1}^{k} Q^{(j)}$ is an orthogonalization of $\prod_{j=k}^{1} (\underset{\sim}{A} - \mu^{(j)} \underset{\sim}{I})$

The first column of $\bar{Q}^{(k)}$ is the result of applying shifted power iteration to $\underset{\sim}{e}_1$ with shifts $\mu^{(j)}$ and

Last column of $\bar{Q}^{(k)}$ evolves as shifted inverse iteration applied to $\underset{\sim}{e_m}$ with shifts $\mu^{(j)}$.

If these shifts are good eigenvalue estimates, the last column of $\bar{Q}^{(k)}$ converge quickly to ~~an~~ an eigenvector!

$\longrightarrow$ Choose shifts in the spirit of rayleigh qustient iteration applied to the last column of $\bar{Q}^{(k)}$.

$$\mu^{(k)} = \frac{\left(q_m^{(k)}\right)^T A\, q_m^{(k)}}{q_m^{(k)T} q_m^{(k)}}$$

$$= \boxed{\left(q_m^{(k)}\right)^T A\, q_m^{(k)}}$$

If we choose this shift at every step $\mu^{(k)}$, $q_m^{(k)}$ are the same as those computed by Rayleigh quotient iteration starting with $\underset{\sim}{e_m}$.

QR algorithm has cubic convergence in the sense $q_m^{(k)}$ converge cubically to the eigenvector!

$r(q_m^{(k)}) \longrightarrow m, m$ entry of $A^{(k)}$

$A_{mm}^{(k)}$

$$\boxed{A_{mm}^{(k)}} = \underline{e_m}^T \underline{A}^{(k)} \underline{e_m}$$

$$= \underset{\sim}{e_m}^T (\underline{\bar{Q}}^{(k)})^T \underline{A} \, \underline{\bar{Q}}^{(k)} e_m$$

$$= (\underline{q_m}^{(k)})^T \underline{A} \, \underline{q_m}^{(k)}$$

This is called Rayleigh quotient shift. (RQS)

<u>Wilkinson shifts</u> are usually used in case RQS fails to converge.

<u>Thm</u> :- Let $\underset{\sim}{A} \in \mathbb{R}^{m\times m}$ a real, symmetric and triagonal. Suppose we diagonalize $\underset{\sim}{A}$ by the QR algorithm. Let $\tilde{\Lambda}$ be computed diagonalized $\underset{\sim}{A}$, and $\tilde{\underset{\sim}{Q}}$ be the exact orthogonal matrix resulting

Then $\tilde{\underset{\sim}{Q}}\tilde{\underset{\sim}{\Lambda}}\tilde{\underset{\sim}{Q}}^T = \underset{\sim}{A} + \underset{\sim}{\delta A}$

where $\dfrac{\|\delta A\|}{\|A\|} = O(\varepsilon_M)$

for some $\underset{\sim}{\delta A} \in \mathbb{R}^{m\times m}$

i.e QR algorithm produces an

exact solution to a slightly perturbed problem!

We can conclude that tridiagonal reduction followed by QR algorithm is a backward stable way of computing eigenvalues of a matrix!

Also computed eigenvalues $\tilde{\lambda}_j$ satisfy

$$\boxed{\frac{|\tilde{\lambda}_j - \lambda_j|}{\|A\|} = O(\varepsilon_M)}$$