Indian Institute of Science Banglore
Department of Computational and Data Sciences (CDS)
DS284: Numerical Linear Algebra
Assignment 4 [Posted Oct 25, 2022]
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Submission Deadline: Nov 10, 2022 23:59 hrs Max Points: 100

Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively.

Problem 1 [15 marks]

Gaussian Elimination allows us to compute determinant of a square matrix. Recall the following points about the determinant.

- Swapping 2 rows multiplies the determinant by -1.
- Multiplying a row by a non-zero scalar multiplies the determinant by the same scalar.
- Adding one row, which is a scalar multiple of the other does not change the determinant.

Now describe the procedure how Gaussian Elimination with partial pivoting can be used to find the determinant of a general square matrix. Also comment on the number of floating point operations in this procedure!

Problem 2 [20 marks]

Let $\mathbf{A}$ be a matrix defined in MATLAB as:

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A = rand(N)
A = A - diag(diag(A)) + diag(0.001 * ones(N,1))
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Compute LU Decomposition of $\mathbf{A}$ with and without partial pivoting. Plot $\|\mathbf{LU} - \mathbf{A}\|_F$ versus N for $N = 5, 6, 7..., 20$. For LU Decomposition with partial pivoting, use built in LU function. For LU Decomposition without pivoting, write your own function.

Problem 3 [24 marks]

(a) Let $\mathbf{A}$ be a non-singular square matrix and let $\mathbf{A} = \mathbf{QR}$ be its QR factorization. Let also $\mathbf{A}^T\mathbf{A} = \mathbf{U}^T\mathbf{U}$ be the Cholesky factorization of $\mathbf{A}^T\mathbf{A}$. Can you conclude that $\mathbf{R} = \mathbf{U}$? If yes, prove it; if not, why not?

(b) Recall that by $\mathbf{A} \in \mathbb{R}^{m\times m}$, being symmetric and strictly positive definite, we mean $\mathbf{A} = \mathbf{A}^T$ and $\forall \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq 0$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is positive semi-definite if $\forall \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq 0$, we have $\mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0$

If $\{\phi_i(x)\}_{i=1\ldots m}$ denote m linearly independent basis functions (non-zero) defined over $[-1, 1]$ in an m-dimensional vector space then show that the matrix $\mathbf{M} = \int_{-1}^{1} \phi_i(x)\phi_j(x)dx$ for $i, j = 1, 2\cdots m$ is a symmetric positive definite matrix.

Similarly show that the matrix $\mathbf{K} = \int_{-1}^{1} \frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx$ for $i, j = 1, 2\cdots m$ is a symmetric positive semi-definite matrix.

(c) Show that a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, strictly positive definite if and only if there exists a matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ of rank n, where $n \leqslant m$, such that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$. Assuming that $\mathbf{A}$ is of this form, is there a unique such $\mathbf{B}$?

Problem 4 [18 marks]

For each of the following statements prove that it is true or give an example to show that it is false. Assume $\mathbf{A} \in \mathbb{C}^{m\times m}$ unless otherwise indicated.

(a) If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.

(b) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$ then so it $-\lambda$.

(c) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is λ^*. (λ^* is the complex conjugate of λ).

(d) If λ is an eigen value of $\mathbf{A}$ and $\mathbf{A}$ is non-singular, then λ^{-1} is the eigenvalue of $\mathbf{A}^{-1}$.

(e) If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = 0$.

(f) If $\mathbf{A}$ is diagonalizable and all its eigenvalues are equal, then $\mathbf{A}$ is diagonal.

Problem 5 [23 marks]

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ with entries a_{ij} for $i, j = 1, 2, ..., n$ and define the closed disks $D(a_{ii}, r_i)$ centered at the diagonal entries a_{ii} of $\mathbf{A}$ of radius $r_i = \sum_{j=1}^{n}(1-\delta_{ij})|a_{ij}|$ for $i = 1, 2, ..., n$. Note that δ_{ij} represents Kronecker delta i.e $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The above disks are called Greshgorin's disks.

(a) Prove that every eigenvalue of $\mathbf{A}$ lies in a Greshgorin disk.
(*Hint*: Let λ be any eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ be the corresponding eigenvector with largest entry 1.)

(b) Suppose that $\mathbf{A}$ is diagonally dominant i.e. $|a_{ii}| > \sum_{j=1}^{n}(1-\delta_{ij})|a_{ij}|$ for all $i = 1, 2, ..., n$. Prove that $\mathbf{A}$ is invertible.

(c) Give estimates based on (a), for the eigenvalues of:

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix} \qquad \text{where } |\epsilon| < 1$$