Simple Near-Optimal Auctions

Exp Revenue = Exp Virtual Welfare.

$$\mathbb{E}_{v\sim f} \sum_i p_i(v) = \mathbb{E}_{v\sim f} \sum_i \varphi_i(v_i)\, x_i(v)$$

Virtual Welf. Maximizer. Mech $\ni$

$$x(v) = \arg\max_x \sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)$$

for each v

$$\varphi(z) = z - \frac{1-F_i(z)}{f_i(z)}$$

THM: For regular dists. f_i-s, the virtual welf. maximizing mech is DSIC & revenue opt.

Revenue Optimal Auctions can be complex.

Single Item: When val. distributions are identical. (v_i-s are iids).
Revenue opt. auction = Second Price Auction with reserve price ($r = \varphi^{-1}(0)$).

HOWEVER, even for single item setting with non identical bidders the auction is relatively complex.
- requires detailed info about the val dist.
- does not resemble auction in practice.

Pursue approx opt. auctions that are simpler, more practical, & more robust.

[The complexity is inevitable if one wants to maximize (exp) revenue up to the last bit]

Hence, with "simplicity" we consider approximately optimal auctions.

Prophet Inequality

(Samuel-Cahn '84)

Optimal Stopping Rules.

- Known Distributions $G_1, G_2 \dots, G_n$ (Independent).

  Number of Rounds n

- In round $i \in \{1, 2, \dots n\}$.

  Observe realization $\pi_i \sim G_i$

- After seeing π_i, irrevocable decision
  - Accept π_i, Stop.
  - Discard π_i, Continue (permanently)

[Trade off under uncertainty:

Risk of accepting a reasonable prize π_i early & then missing out later vs.

Risk of having to settle for a lousy prize in end stages]

Prophet Ineq. offers a simple strategy that performs as well as a fully clairvoyant 'prophet'.

↓

Benchmark $\mathbb{E}_{\pi_i \sim G_i} \max_{1 \le i \le n} \pi_i$

THM: (Prophet Ineq.) For every sequence of ind. distrs. $G_1, G_2 \ldots G_n$, there is a strategy that guarantees exp. reward at least $\frac{1}{2} \mathbb{E}_{\pi \sim G} [\max_i \pi_i]$. Moreover, there is such a threshold strategy, which accepts prize i iff. π_i is some threshold t.

Pf: Write $z^+ = \max\{0, z\}$.

Consider any threshold t. We derive an upper bound on $\mathbb{E} \max$ & a lower bound on the exp. reward of threshold t.

Write $q(t)$ = prob. that all π_i-s are less than t.
(i.e., the threshold strategy accepts no reward)

Exp Reward of threshold-t strategy $\begin{cases} 0 & \text{w.p. } q(t) \\ \ge t & \text{w.p. } 1-q(t) \end{cases}$

We refine the second case ↑

Event E_i: Exactly prize $\pi_i \ge t$, then, beyond t, the reward additionally has $(\pi_i - t)$.

[When two rewards π_i & π_j are more than t, then the reward gained depends on which of π_i / π_j observed before.]

In the analysis, we avoid this complication and only go beyond t when exactly one of the rewards are greater than t.

$$\mathop{\mathbb{E}}_{\pi \leftarrow G}\left[\begin{matrix}\text{payoff of}\\ t\text{-threshold}\\ \text{strategy}\end{matrix}\right] \geq (1-q(t))t + \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\pi}\left[\pi_i - t \mid E_i\right] \Pr\{E_i\}.$$

$$= (1-q(t))t + \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\pi}\left[\pi_i - t \,\middle|\, \begin{matrix}\pi_i \geq t\\ \pi_j < t \ \forall j \neq i\end{matrix}\right] \Pr\left\{\begin{matrix}\pi_i \geq t\\ \pi_j < t \ \forall j \neq i\end{matrix}\right\}.$$

$$= (1-q(t))t + \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\pi}\left[\pi_i - t \,\middle|\, \begin{matrix}\pi_i \geq t\\ \pi_j < t \ \forall j \neq i\end{matrix}\right] \Pr\{\pi_i \geq t\} \Pr\left\{\pi_j < t \ \forall j \neq i\right\}.$$

fn. only of π_i ind of π_j-s.

$$= (1-q(t))t + \sum_{i=1}^{n} \underbrace{\mathop{\mathbb{E}}_{\pi}\left[\pi_i - t \mid \pi_i \geq t\right] \Pr\{\pi_i \geq t\}}_{= \mathbb{E}[(\pi_i - t)^+]} . \underbrace{\Pr\left\{\pi_j < t \ \forall j \neq i\right\}}_{\geq q(t)}$$

$$\geq (1-q(t)).t + q(t) \sum_{i=1}^{n} \mathbb{E}\left[(\pi_i - t)^+\right]$$

[NOTE

$q(t) := \Pr\{\pi_j < t \ \forall j\}$

$\leq \Pr\{\pi_j < t \ \forall j \neq i\}$.

Therefore,

$$\mathbb{E}_{\pi\sim G}\left[\begin{matrix}\text{Reward of}\\ \text{t-threshold}\\ \text{strategy}\end{matrix}\right] \geq (1-q(t))t + q(t)\sum_{i=1}^{n} \mathbb{E}_{\pi}\left[(\pi_i - t)^+\right]$$

Upper Bound on prophet's exp. reward.

$$\mathbb{E}_{\pi}\left[\max_i \pi_i\right] = \mathbb{E}_{\pi}\left[t + \max_{1\leq i\leq n} (\pi_i - t)\right]$$

$$\leq \mathbb{E}_{\pi}\left[t + \max_{1\leq i\leq n} (\pi_i - t)^+\right]$$

$$\leq t + \mathbb{E}_{\pi}\left[\sum_{i=1}^{n} (\pi_i - t)^+\right]$$

$$= t + \sum_{i=1}^{n} \mathbb{E}_{\pi}\left[(\pi_i - t)^+\right]$$

Comparing the LB & UB, we can set t such that $q(t) = 1/2$.

to obtain the threshold that satisfies the theorem statement. $\boxed{\mathbb{Z}}$

Remark: Works with adversarial tie breaking.

Application of Prophet Ineq : Simple Single-Item Auction.

Idea: $\pi_i = \varphi_i^+(v_i)$ G_i is the corresponding dist. induced by F_i.

In the current context of single item auction. (with regular F_i-s)

$$\text{Exp revenue of opt auction} = \mathbb{E}_{v \sim f}\left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)\right]$$

$$= \mathbb{E}_{v \sim f}\left[\max_i \varphi_i^+(v_i)\, x_i(v)\right]$$

Hence, the (exp) optimal revenue

= Exp prize obtained by a prophet.

Mech: $\boxed{\text{Virtual Threshold Alloc. Rule}}$

1. Choose t such that $\Pr[\max_i \varphi_i^+(v_i) \geq t] = 1/2$.
2. Give the item to bidder i with $\varphi_i(v_i) \geq t$, if any, breaking ties arbitrarily.

COR: (Virtual Threshold Alloc Rules are Near Optimal)

If x is virtual threshold alloc rule, then.

$$\mathbb{E}_v \sum_{i=1}^{n} \varphi_i^+(v_i)\, x_i(v) \geq \frac{1}{2}\, \mathbb{E}_v\left[\max_i \varphi_i^+(v_i)\right].$$

Specific form of Virtual Threshold Alloc Rule.

Second-Price with Bidder-Specific Reserves

1. Set a reserve price $r_i = \varphi_i^{-1}(t)$ for each bidder i
with t defined as for virtual threshold alloc rules.

2. Give the item to the highest bidder that meets her reserve, if any.

Under Regular Distrs.

this is a monotone allocation rule.

Check.

THM: (Simple vs Optimal Auction)
For all $n \geq 1$ and regular distrs. $F_1, F_2 \ldots F_n$, the exp. revenue of a second-price auction, with suitable reserve prices is at least $1/2$ of that of the optimal auction.