Mixed Strategy Nash Equilibrium

$$\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

$\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a MSNE if $\forall i \in N$,

$$u_i(\sigma_i^*, \underline{\sigma}_{i}^*) \geq u_i(\sigma_i, \underline{\sigma}_{i}^*) \quad \forall \sigma_i \in \Delta(S_i)$$

We now present a Necessary and Sufficient condition (NASC) for a mixed strategy profile to be an MSNE.

We start with an important property of a mixed strategy profile.

Convex Combination

Suppose $y_1, y_2, \ldots, y_n \in \mathbb{R}$.
A convex combination is an expression
$\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$ where
$\lambda_i \in \mathbb{R}$ $\lambda_i \geq 0$ $\forall i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$u_i(\sigma_i, \underline{\sigma}_i) = \sum_{(s_i, \underline{s}_i) \in S} \left(\prod_{j \in N} \sigma_j(s_j) \right) u_i(s_i, \underline{s}_i)$$

$$= \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} \left(\prod_{j \in N} \sigma_j(s_j) \right) u_i(s_i, \underline{s}_i)$$

$$= \sum_{s_i \in S_i} \sum_{\underline{s}_i \in \underline{S}_i} \left(\prod_{\substack{j \neq i \\ j \in N}} \sigma_j(s_j) \right) \sigma_i(s_i)\, u_i(s_i, \underline{s}_i)$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i) \sum_{\underline{s}_i \in \underline{S}_i} \left(\prod_{\substack{j \neq i \\ j \in N}} \sigma_j(s_j) \right) u_i(s_i, \underline{s}_i)$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \underline{\sigma}_i)$$ convex combination

An Illustrative Example

$N = \{1, 2\}$ $\quad S_1 = \{x_1, x_2, x_3, x_4, x_5\}$

$$u_1(\sigma_1, \sigma_2) = \sigma_1(x_1)\, u_1(x_1, \sigma_2) + \cdots + \sigma_1(x_5)\, u_1(x_5, \sigma_2)$$

Suppose

$u_1(x_1, \sigma_2) = 5$

$u_1(x_2, \sigma_2) = u_1(x_3, \sigma_2) = 10$

$u_1(x_4, \sigma_2) = u_1(x_5, \sigma_2) = 20$

Max value of $u_1(\sigma_1, \sigma_2) = 20$
attained when $\sigma_1(x_4) + \sigma_1(x_5) = 1$
or when $\sigma_1(x_1) = \sigma_1(x_2) = \sigma_1(x_3) = 0$

$$\max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2) = 20$$

observe that

$$\max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2) = \max_{s_1 \in S_1} u_1(s_1, \sigma_2)$$

Suppose

$$p \in \{\sigma_1 \in \Delta(S_1) : u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1', \sigma_2) \;\; \forall \sigma_1' \in \Delta(S_1)\}$$

Then we have that the max value is attained when

$$p(x_4) + p(x_5) = 1$$

$$\Leftrightarrow p(x_1) + p(x_2) + p(x_3) = 0$$

$$\Leftrightarrow p(x_1) = p(x_2) = p(x_3) = 0$$

$$\Leftrightarrow p(y) = 0 \quad \forall y \notin \operatorname*{argmax}_{s_1 \in S_1} u_1(s_1, \sigma_2)$$

This leads to the following Lemma:

Given $\Pi = \langle N, (S_i), (u_i) \rangle$, then for any profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and for any $i \in N$,

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \underline{\sigma}_{-i}) = \max_{s_i \in S_i} u_i(s_i, \underline{\sigma}_{-i})$$

Furthermore,

$$p_i \in \underset{\sigma_i \in \Delta(S_i)}{\operatorname{argmax}} u_i(\sigma_i, \underline{\sigma}_{-i}) \text{ iff}$$

$$p_i(y) = 0 \quad \forall y \notin \underset{s_i \in S_i}{\operatorname{argmax}} u_i(s_i, \underline{\sigma}_{-i})$$

NASC for a Mixed Strategy Profile to be a MSNE

Definition:

Support of a Mixed strategy σ_i

$$\delta(\sigma_i) = \{ s_i \in S_i : \sigma_i(s_i) > 0 \}$$

NASC: Given $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

$(\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ is a MSNE iff $\forall i \in N$:

(1) $u_i(s_i, \underline{\sigma}_{-i}^*)$ is the same $\forall s_i \in \delta(\sigma_i^*)$

(2) $u_i(s_i, \underline{\sigma}_{-i}^*) \geq u_i(s_i', \underline{\sigma}_{-i}^*)$
$\forall s_i \in \delta(\sigma_i^*) \ \forall s_i' \in S_i \setminus \delta(\sigma_i^*)$

Proof of Necessity:

$(\sigma_i^*, \underline{\sigma}_{-i}^*)$ is a MSNE

$$\Rightarrow u_i(\sigma_i^*, \underline{\sigma}_{-i}^*) = \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \underline{\sigma}_{-i}^*)$$
$$= \max_{s_i \in S_i} u_i(s_i, \underline{\sigma}_{-i}^*)$$

$$\Rightarrow \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \underline{\sigma}_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \underline{\sigma}_{-i}^*)$$

$$\Rightarrow \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \underline{\sigma}_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \underline{\sigma}_{-i}^*)$$

$s_i \in \delta(\sigma_i^*)$

The above is similar to

$$\pi_1 x_1 + \cdots + \pi_k x_k = \max(x_1, \ldots, x_k, \ldots, x_n)$$

with

$$\pi_1 + \cdots + \pi_k = 1 \quad \text{and } \pi_j \neq 0 \text{ for } j = 1, \ldots, k$$

The above implies that

$$x_1 = x_2 = \cdots = x_k = \max(x_1, x_2, \ldots, x_k)$$

This means

$$u_i(s_i, \underline{\sigma}_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \underline{\sigma}_{-i}^*) \quad \forall s_i \in \delta(\sigma_i^*)$$

clearly this also means that

$$u_i(s_i, \underline{\sigma}_{-i}^*) \geq u_i(s_i', \underline{\sigma}_{-i}^*) \quad \forall s_i \in \delta(\sigma_i^*) \; \forall s_i' \in S_i \setminus \delta(\sigma_i^*)$$

Proof of Sufficiency

We are given (1) and (2) and we have to show $(\sigma_i^*, \underline{\sigma}_{-i}^*)$ is a MSNE.

$$u_i(\sigma_i^*, \underline{\sigma}_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i)\, u_i(s_i, \underline{\sigma}_{-i}^*)$$

$$= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\, u_i(s_i, \underline{\sigma}_{-i}^*)$$

$$= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\, w \quad \text{where } w = u_i(s_i, \underline{\sigma}_{-i}^*) \text{ for any } s_i \in \delta(\sigma_i^*) \text{ (by virtue of (1))}$$

$$= w \quad \text{since } \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) = 1$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i)\, w \quad \text{for any mixed strategy } \sigma_i$$

$$\geq \sum \sigma_i(s_i)\, u_i(s_i, \underline{\sigma}_{-i}^*) \quad \text{since } w = u_i(s_i, \underline{\sigma}_{-i}^*)$$

$$\geqslant \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \underline{\sigma}_{-i}^{*})$$

since $w = u_i(s_i, \underline{\sigma}_{-i}^{*})$ for any $s_i \in \delta(\sigma_i^*)$ and is max (by virtue of (2))

$$= u_i(\sigma_i, \underline{\sigma}_{-i}^{*})$$

Thus
$u_i(\sigma_i^*, \underline{\sigma}_{-i}^{*}) \geqslant u_i(\sigma_i, \underline{\sigma}_{-i}^{*}) \quad \forall \sigma_i \in \Delta(S_i)$
which means $(\sigma_i^*, \underline{\sigma}_{-i}^{*})$ is a MSNE.

Example: Derive all Nash equilibria of:

1 \ 2	A	B
A	10,10	0,0
B	0,0	1,1

Support of a Mixed Strategy profile
$$\delta(\sigma_1, \ldots, \sigma_n) = \delta(\sigma_1) \times \cdots \times \delta(\sigma_n)$$

For the above example, the set of all possible supports for all mixed strategy profiles includes:

$\{A\} \times \{A\}$; $\{A\} \times \{B\}$; $\{A\} \times \{A,B\}$
$\{B\} \times \{A\}$; $\{B\} \times \{B\}$; $\{B\} \times \{A,B\}$
$\{A,B\} \times \{A\}$; $\{A,B\} \times \{B\}$; $\{A,B\} \times \{A,B\}$

Any MSNE will correspond to some support. So let us investigate if each of the above supports may lead to a MSNE.

$\{A\} \times \{A\}$
this leads to a PSNE (A, A) which is a degenerate MSNE $((1,0),(1,0))$.

$\{A\} \times \{B\}$

this cannot lead to a NE. Why ?

$\{A\} \times \{A, B\}$

this cannot lead to a NE. Why ?

$\{B\} \times \{A\}$

Cannot lead to a NE. Why ?

$\{B\} \times \{B\}$

Leads to a PSNE $(B, B) \equiv ((0,1), (0,1))$

$\{B\} \times \{A, B\}$

Cannot lead to a NE. Why ?

$\{A, B\} \times \{A\}$

cannot lead to a NE. Why ?

$\{A, B\} \times \{B\}$

Cannot lead to a NE. Why ?

$\{A, B\} \times \{A, B\}$

Let's investigate this further.

Suppose $\sigma_1^* = (x, 1-x) \quad \sigma_2^* = (y, 1-y)$

where $x \neq 0 \quad x \neq 1 \quad y \neq 0 \quad y \neq 1$

Suppose (σ_1^*, σ_2^*) is a MSNE.

Let's apply the NASC.

Condition 1: for player 1

$u_1(A, (y, 1-y)) = u_1(B, (y, 1-y))$

$\Rightarrow 10y = 1-y$

$\Rightarrow y = \frac{1}{11}$

Condition 1 for player 2

$u_2((x, 1-x), A) = u_2((x, 1-x), B)$

$\Rightarrow 10x = 1-x$

$\Rightarrow x = \frac{1}{11}$

Condition (2) is vacuously true since

$$\delta(\sigma_1^*) = \delta(\sigma_2^*) = \{A, B\}$$

$$S_1 \setminus \sigma_1^* = S_2 \setminus \sigma_2^* = \phi$$

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Thus NASC is satisfied and the profile

$\left(\left(\frac{1}{11}, \frac{10}{11}\right), \left(\frac{1}{11}, \frac{10}{11}\right)\right)$ is a MSNE.