भारतीय विज्ञान संस्थान

Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)
DS284: Numerical Linear Algebra
Final Exam – Aug 2021 Term
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Duration: 14:00 hrs to 17:00 hrs — Max Points: 100

Notations: (i) Vectors and matrices are denoted by bold faced lower case and upper case alphabets respectively. (ii) Set of all real numbers is denoted by $\mathbb{R}$ (iii) Set of all n dimensional vectors is denoted by $\mathbb{R}^n$ and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$. (iv) $\mathbf{I}_m$ denotes the identity matrix of order m. (v) ϵ_{mach} denotes machine epsilon. (vi) $\mathbf{0}_n$ denotes the n-dimensional column vector with each element being zero.
Start each problem on a new page.
There are 4 pages.

Problem 1 [6x3=18 points]

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) Let $(\mathbf{A} + \mathbf{A}^T)$ be a positive definite matrix for a non-zero matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$. Then $\mathbf{A}$ is always non-singular.

(b) Oblique projectors have one of its eigenvalues as 0.

(c) For a matrix $\mathbf{F} = \mathbf{I}_m - 2\mathbf{q}\mathbf{q}^T$, where $\mathbf{q} \in \mathbb{R}^m$ and m is an odd integer. Then the determinant of $\mathbf{F}$ is 1

(d) A matrix $\mathbf{Q} \in \mathbb{R}^{m\times n}$ has n orthonormal columns, then m has to be always lesser than n.

(e) If $\hat{\mathbf{x}} \in \mathbb{R}^n$ satisfies the system of equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a full rank matrix with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$, then $\hat{\mathbf{x}}$ always satisfies the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$

(f) Consider a non-zero symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ and a non-zero vector $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{K}_n = \mathbf{Q}_n\mathbf{R}_n$ be the QR factorization of the n-dimensional Krylov subspace $\mathbf{K}_n = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \mathbf{A}^2\mathbf{b} \ \dots \ \mathbf{A}^{n-1}\mathbf{b}]$ for $n < m$. As $n \to m$, columns of $\mathbf{Q}_n$ approach the eigenvectors of $\mathbf{A}$

Problem 2 [3+3+3 =9 points]

Let us consider some data about a product sold in Walmart. The data is related to normalised price and fiber content of various cornflakes brands available in Walmart. There are 4 brands of cornflakes. A data matrix $\mathbf{X} \in \mathbb{R}^{4\times 2}$ is constructed with each of these feature vectors forming the two columns of the matrix $\mathbf{X}$. Upon computing the SVD of $\mathbf{X}$, one finds that the matrix $\mathbf{U}$ formed by the left singular vectors to be $\mathbf{U} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$ and the matrix $\mathbf{V}$ formed by the right singular vectors to be $\mathbf{V} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Further the singular values are found to be $\sqrt{2}$ and 0.5

(a) Reconstruct the data matrix $\mathbf{X}$ from the singular values and singular vectors given above. Further compute the Frobenius norm of the data matrix $\mathbf{X}$. What will be the rank of this matrix $\mathbf{X}$? Justify.

(b) Construct the rank one approximation of $\mathbf{X}$ using the dominant singular vector and the corresponding singular value.

(c) Compute the matrix $\mathbf{C} = \mathbf{X}^T\mathbf{X}$. Verify that the given non-zero singular values of $\mathbf{X}$ are square roots of the eigenvalues of the matrix $\mathbf{C}$ by computing the eigenvalues of the matrix $\mathbf{C}$.

Problem 3 [3+7+6=16 points]

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric matrix. Now answer the following questions:

(a) Show that eigenvectors corresponding to different eigenvalues of $\mathbf{A}$ are orthogonal to each other.

(b) Let λ_1 be an eigenvalue of $\mathbf{A}$ with algebraic multiplicity 1 and $\mathbf{u}_1$ be the corresponding normalized eigenvector. We then pick the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{m-1}\}$ in $\mathbb{R}^m$ such that $\mathbf{Q}_1 = [\mathbf{u}_1, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{m-1}]$ is an orthogonal matrix. Show that $\mathbf{Q}_1^T\mathbf{A}\mathbf{Q}_1 = \begin{bmatrix} \lambda_1 & \mathbf{0}_{m-1}^T \\ \mathbf{0}_{m-1} & \mathbf{A}_1 \end{bmatrix}$, where $\mathbf{A}_1$ is a symmetric matrix of order $m-1$. Furthermore, show that the remaining $m-1$ eigenvalues of $\mathbf{A}$ are the $m-1$ eigenvalues of $\mathbf{A}_1$.

(c) Let $\|\mathbf{A}\|_1$ denote vector induced matrix 1-norm of $\mathbf{A}$. Show that $\rho(\mathbf{A}) \leqslant \|\mathbf{A}\|_1$ where, ρ denotes spectral radius i.e. largest absolute eigen value $|\lambda|$ of $\mathbf{A}$.
(Hint:- Relate eigenvalues of $\mathbf{A}$ and $\mathbf{A}^k$ subsequently show that
$|\rho(\mathbf{A})|^k = \rho(\mathbf{A}^k) \leqslant \|\mathbf{A}^k\|_1 \leqslant \|\mathbf{A}\|_1^k$)

Problem 4 [3+7=10 points]

Consider the system of linear equations $(\mathbf{A} - \mu\,\mathbf{I}_m)\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{m\times m}$ being a non-zero symmetric matrix and $\mathbf{b} \in \mathbb{R}^m$ is a non-zero vector.

(a) Show that there exists a unique solution $\mathbf{x}$ for the above system of linear equations if $\mu \neq \lambda_j$, where λ_j is the j^{th} eigenvalue of $\mathbf{A}$ and $1 \leqslant j \leqslant m$.

(b) Show that the solution $\mathbf{x}$ can be written as $\mathbf{x} = \sum_{j=1}^{m} \frac{\mathbf{u}_j^T\mathbf{b}}{\lambda_j - \mu}\mathbf{u}_j$, where $(\mathbf{u}_j, \lambda_j)$ denotes the j^{th} eigenvector-eigenvalue pair of $\mathbf{A}$. [**Note:** Mere substitution of $\mathbf{x} = \sum_{j=1}^{m} \frac{\mathbf{u}_j^T\mathbf{b}}{\lambda_j - \mu}\mathbf{u}_j$ in $(\mathbf{A} - \mu\,\mathbf{I}_m)\mathbf{x} = \mathbf{b}$ to show that it satisfies the linear system of equations will not fetch any points]

Problem 5 [3+4+5+7=19 points]

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a full rank matrix, and $\mathbf{A} = \mathbf{QR}$ denote the full QR decomposition of $\mathbf{A}$, where $\mathbf{Q} \in \mathbb{R}^{m\times m}$ is an orthogonal matrix and $\mathbf{R} \in \mathbb{R}^{m\times m}$ is an upper triangular matrix. Let $\mathbf{q}_i$ and $\mathbf{a}_i$ denote the i^{th} column of the matrices $\mathbf{Q}$ and $\mathbf{A}$ respectively for $1 \leqslant i \leqslant m$. Now, answer the following questions with clear arguments:

(a) Construct the projector matrix $\mathbf{P}$ that orthogonally projects a vector onto a subspace spanned by the set of vectors $< \mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_{j-1} >$ for some j ($\leqslant m$).

(b) Write down the expression for the complimentary projector to $\mathbf{P}$ obtained above in (a). Show that this complimentary projector is symmetric. Let $\tilde{\mathbf{a}}_j$ be the projection of $\mathbf{a}_j$ obtained using this complementary projector. To this end, write the expression for $\tilde{\mathbf{a}}_j$ in terms of $< \mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_{j-1} >$ and $\mathbf{a}_j$. What subspace does $\tilde{\mathbf{a}}_j$ belong?

(c) Show that the absolute value of diagonal entry of the $\mathbf{R}$ matrix i.e $|\mathbf{R}_{jj}|$ is related to the 2-norm of the vector $\tilde{\mathbf{a}}_j$ obtained in (b).

(d) Using the above results and $\mathbf{QR}$ decomposition of $\mathbf{A}$. Show that:
$|det(\mathbf{A})| \leqslant \prod_{j=1}^{m} ||\mathbf{a}_j||_2$

Problem 6 [3+4+8+3+3+7 =28 points]

Consider a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$. Answer the following 6 questions:

(a) Show that the singular values of $\mathbf{A}$ are absolute values of eigenvalues of $\mathbf{A}$. What can you say about the vector induced matrix norm $\|\mathbf{A}\|_2$ in terms of eigenvalues of $\mathbf{A}$? Support your argument.

(b) Show that $|\mathbf{x}^T\mathbf{A}\mathbf{x}| \leqslant \|\mathbf{A}\|_2$ for any non-zero unit vector $\mathbf{x} \in \mathbb{R}^m$.

(c) Let the vector $\mathbf{u} \in \mathbb{R}^m$ be an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue λ i.e. $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$. Further, let the matrix $\mathbf{A}$ undergo a symmetric matrix perturbation by $\delta\mathbf{A}$ such that $\dfrac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = O(\epsilon_{mach})$. Let, $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ and $\tilde{\lambda} = \lambda + \delta\lambda$ be the eigenvector-eigenvalue pair of the perturbed matrix $\tilde{\mathbf{A}} = \mathbf{A} + \delta\mathbf{A}$. Now, show that

$$|\delta\lambda| \leqslant \|\delta\mathbf{A}\|_2$$

(Hint:- Note that the perturbed matrix $\tilde{\mathbf{A}}$ is symmetric and start with the eigenvalue problem corresponding to $\tilde{\mathbf{A}}$ to first show that $|\delta\lambda| = |\mathbf{u}^T\,\delta\mathbf{A}\,\mathbf{u}|$. You may need to use the fact that the eigenvectors of a symmetric matrix are orthogonal and hence form a basis for $\mathbb{R}^m$)

(d) Deduce the relative condition number for the problem of computing the eigenvalue λ of our symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ using the inequality derived in part (c).

(e) Consider the problem of computing eigenvalues of the matrix $\mathbf{M} = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}$. As you can see the eigenvalues of this matrix $\mathbf{M}$ are 2, 2. Find the condition number for the problem of computing the eigenvalue 2 for the above matrix $\mathbf{M}$ using the result obtained in part(d).

(f) There are two algorithms designed – Algorithm S and Algorithm U to compute the eigenvalues of the above matrix $\mathbf{M} = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}$. Algorithm S is designed to be a backward stable algorithm. To this end, comment on the relative forward error incurred in computing an eigenvalue of $\mathbf{M}$ by employing this backward stable Algorithm S. Furthermore, the Algorithm U is designed to compute eigenvalues of the above matrix $\mathbf{M}$ by solving the roots of the characteristic polynomial of the matrix $\mathbf{M}$ i.e. $p(z) = z^2 - 4z + 4$. Compute the forward relative error incurred in computing the eigenvalue 2 of $\mathbf{M}$ using the Algorithm U (**Note:** Assume that the floating point approximation errors incurred in the coefficient of z and constant term in $p(z)$ are both ϵ ($\epsilon \leqslant \epsilon_{\text{mach}}$)). Using this estimate of error, argue that Algorithm U is unstable with proper reasoning.