

Contract Theory:

Economic tool to incentivize an (strategic) agent to put in effort.

"How to incentivize people to put in effort"

Examples:

1. Employment Contract
2. VC Investment Contract
3. Insurance Contract
4. Freelance Contract
5. Government Procurement Contract
- ...

{ Paying for the agent's outcomes }

Hart & Holmström
2016 Nobel

"Modern Economies held together by innumerable contracts"

Outline & Increasingly becoming complex.
"Modern" Examples

1. Crowdsourcing Platforms
2. Online Marketing
3. Complex Supply Chains
4. Pay-for-Performance Medicare.

Basic Model

[Holmström '79]

- Two Players: Principal & Agent
- Ingredients: Private Information & Incentives.

F_n : Website Owner
hires. : Principal.

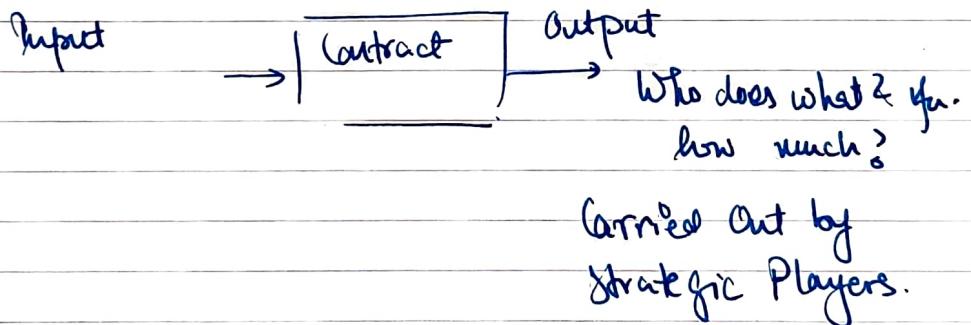
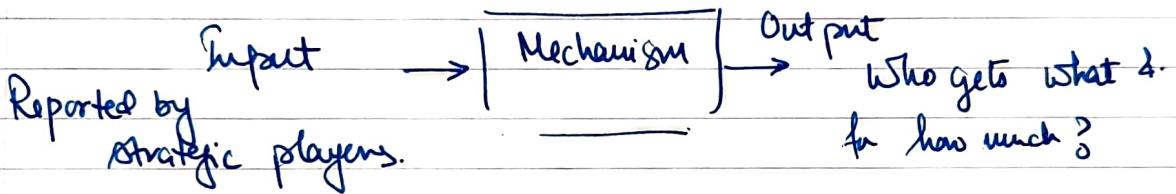
Right level of compensation. to get a balance between how much the principal has to pay & the agent's effort.

Marketing : Agent Exec. → attract ~~visitors~~ visitors.

Two defining features. (illustrated by the example):

- (1) Agent's actions are hidden - "Moral Hazard"
- (2) Principal never changes (only pays) agent - "limited liability"

Moral hazard: Situation in which the agent is creating some action, but not reaping the reward of the action.
The principal is getting the reward.



Design Goal: Maximize objective (welfare/revenue +.) subject to incentive constraints.
(truth telling or effort).

(2)

Contract Design under Uncertainty : Model

- Agent has n actions (effort levels) with costs c_1, c_2, \dots, c_n .
- Principal has m rewards $r_1, r_2, \dots, r_m \in \mathbb{R}_+$
- Every action i induces distribution \vec{f}_i over \vec{r}
 - $f_{i,j} = \text{prob. that action } a_i \text{ yields reward } r_j$
 - $R_i = \sum_j f_{i,j} r_j = \text{Expected reward from action } i$

$$\begin{matrix} & r_1 & r_2 & \dots & r_m \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix} & \xrightarrow{\vec{f}_1} & \xrightarrow{\vec{f}_2} & \dots & \xrightarrow{\vec{f}_n} \end{matrix}$$

$(\vec{c}, \vec{r}, (\vec{f}_1, \vec{f}_2, \dots, \vec{f}_n))$

Contract: \vec{T}

Specifies a payment $t_j \geq 0$ per reward r_j

Exp. payment
for action i^*

$$T_i^* = \sum_j t_j f_{i,j}$$

Imp.: Transfer depends
on reward. NOT.
on effort level.

$$R_i = \sum_j t_j f_{i,j}$$

Effect is unobservable.

Agent selects effort i^* that maximizes her exp. utility

$$T_{i^*} - c_{i^*}$$

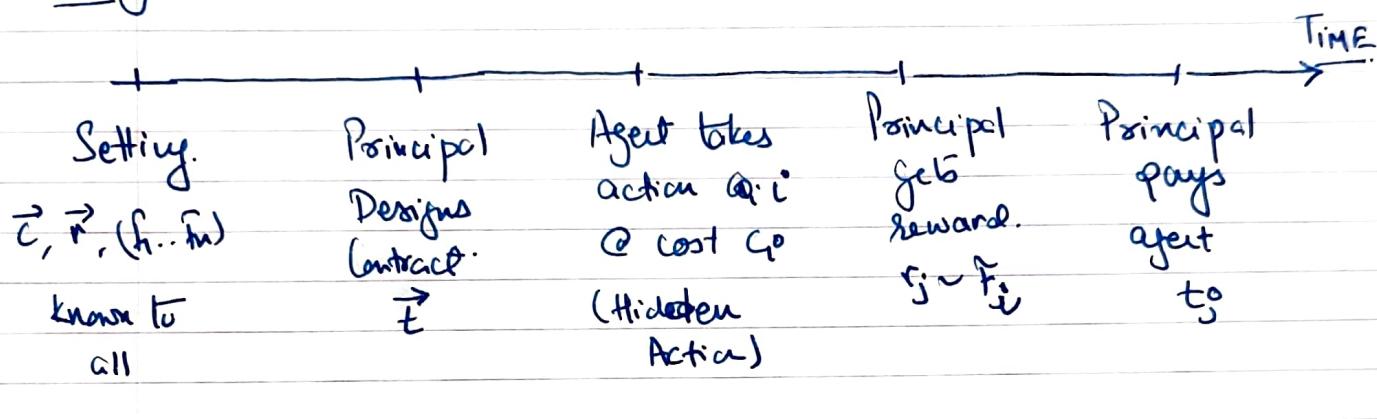
Exp. Payment Cost

- Principal's Exp. Revenue from agent's choice $\forall i \in [n]$

$$R_i = T_i$$

w
 w.
 Expected
Reward.
 Exp
Payment.

Timing:



We say that a contract is optimal if it maximizes principal's exp revenue.

Landscape of Micro Econ.

(Other Incentive Problems
in the field).

[Salanie]

	Uninformed players sets the designs	Informed Player sets the design.
Private Info is hidden type	Mech. Design.	Signaling (Persuasion).
Private Info is hidden action	Contract Design.	-

Contract settings

- Parameter n, m
 - Agent has actions $1, 2, \dots, n$ with costs $0 = c_1 \leq c_2 \leq c_3 \leq \dots \leq c_n$
 - Principal has rewards $0 \leq r_1 \leq r_2 \leq \dots \leq r_m$
 - Action i induces dist. f_i (called "technology"). sometimes with expectation R_i .
- Assumption: $R_1 \leq R_2 \dots \leq R_n$.

- Contract = vector of transfers.

$$\vec{t} = (t_1, t_2, \dots, t_m)$$

Recall two defining features

Example 0

	$t_1 = 0$	$t_2 = 3$	$t_3 = 7$	$t_4 = 10$
① "Low Effort" $C_1 = 0$	0.72	0.18	0.08	0.02
② "Medium Effort" $C_2 = 1$	0.12	0.48	0.08	0.32
③ "High Effort" $C_3 = 2$	0	0.4	0	0.6

$$R_1 = 1.3$$

$$R_2 = 5.2$$

$$R_3 = 7.2.$$

Example Contract

$$t_1 = 0 \quad t_2 = 1 \quad t_3 = 2 \quad t_4 = 5$$

Example 1

$$n = m = 2.$$

Action 1.
Cost $G = 0$

$$t_1 = 1$$

$$t_2 = 3$$

$$R_1 = 1$$

$$\Gamma_{f_1, f_2}$$

| Point Masses |

Action 2
Cost $G_2 = 4/3$

$$0$$

$$1.$$

$$R_2 = 3.$$

(4)

- Best way to incentivize action 1. is to have
(Min Pay) $\vec{t} = (0, 0) \rightarrow \text{Expected Revenue} = 1$.
- Best way to incentivize action 2. is to pay.
 $\vec{t} = (0, 4/3) \rightarrow \text{Expected Revenue} = 3 - 4/3 = 5/3.$

Here, Revenue Opt. Contract = $(0, 4/3)$.

Contracting the Optimal Contract

Min-Pay Problem.

- Input: Contract Setup ($\vec{c}, \vec{r}, \vec{f}$); an action i^* .
- Output: Minimum T_{i^*} that incentivizes action i^* .

linear program:

$$\min_{(t_1, t_2, \dots, t_m)} \sum_{j=1}^m f_{i,j} t_j$$

st.

$$\sum_{j=1}^m f_{i,j} t_j - c_i \geq \sum_{j=1}^m f_{a,j} t_j - c_a$$

$$t_1, t_2, \dots, t_m \geq 0.$$

Actions
 $a \in [n]$

Optimal contract solvable via n Min Pay problems.

Example Non Monotonicity ($n, m > 2$)

$$r_1 = 1 \quad r_2 = 1.1 \quad r_3 = 4.9 \quad r_4 = 5 \quad r_5 = 5.1 \quad r_6 = 5.2$$

$C_1 = 0$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	0	0	0
$C_2 = 1$	0	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	0	0
$C_3 = 2$	0	0	0	$\frac{3}{8}$	$\frac{3}{8}$	0
$C_4 = 2.2$	0	0	0	0	$\frac{3}{8}$	$\frac{3}{8}$

Optimal
Contract

$$\vec{t}^* = (t_1^* = 0 \quad t_2^* = 0 \quad t_3^* = 0.15 \quad t_4^* = 3.9 \quad t_5^* = 2 \quad t_6^* = 0)$$

Unusual

(5)

Example (1) with contract

$$t_1 = 0 \quad t_2 = 1 \quad t_3 = 2 \quad t_4 = 5$$

<u>Expected Transfers</u>	Action 1 :	0.44	(dow)
	2 :	2.24	(Med)
	3 :	3.4.	(High).

∴ Agent selects.

$$\begin{aligned} \text{action 3. with emp utility} &= 3.4 - 2 \\ &= 1.4. \end{aligned}$$

$$\begin{aligned} \text{Principal's Revenue.} &= 7.2 - 3.4 \\ (\text{with agent selecting action 3}) &= 3.8. \end{aligned}$$

Recall	$R_1 = 1.3$
	$R_2 = 5.2$
	$R_3 = 7.2$

first Best = Solution ignoring IC constraints

$$\text{(Opt. Welf.)} \quad \text{first Best} = \max_{1 \leq i \leq n} \{ R_i - c_i \}$$

∴ Opt ≠ first Best.
(Revenue).

Implementation Problem.

Given action i ,
Does there exist a contract t^* such that
under it, action i is utility maximizing for the agent.

Dual of Min Pay.

Lemma: Action i^* is implementable (upto tie breaking) iff. no convex combination of the other actions has same or better rewards at a lower cost.

Pf: Action i^* implementable

iff. LP feasible

following

LP feasible.

m decision variables

$\{t_1, t_2, \dots, t_m\}; m = \text{constraints}$

$$\min 0$$

s.t. m

$$\sum_{j=1}^m f_{i,j} t_j - c_i \geq \sum_{a=1}^m f_{a,j} t_j - c_a \quad \forall a \in (m) \quad a \neq i$$

$$t_j \geq 0 \quad \forall j \in (m)$$

Original Infeasible iff. \exists feasible dual prob. with objective > 0

$$\max. \quad c_i - \sum_{a \neq i} \lambda_a c_a \quad \begin{matrix} \xrightarrow{\text{contained}} \\ \text{const.} \end{matrix}$$

s.t.

$$\sum_{a \neq i} \lambda_a f_{a,j} \leq f_{i,j} \quad \forall j \in (m)$$

Convex Comb
of actions

$$\lambda_a \geq 0 \quad \sum_{a \neq i} \lambda_a = 1.$$

(6)

Ques:

There exists an optimal contract with $\leq (n-1)$ transfers.

Criticism of LP-Based Approach

1. Requires perfect knowledge of T . "More Normative than Positive"
2. Dependence on $n-m$ might be prohibitive.
3. The contract that comes out of LP might be arbitrary.

Structure of Optimal Contracts

I Optimal contracts for 2 actions & 2 rewards.

$$(n=m=2)$$

$$\begin{array}{ccc} \text{Failure } r_1 & < & \text{Success } r_2 \\ 1-p & & p \end{array} \quad (r_2 > r_1)$$

(Low) Cost $G_1 = 0$

$$1-p \quad p$$

(High) Cost $G_2 = c > 0$

$$1-\pi \quad \pi$$

$$\text{first-Best} = R_2 - c.$$

$$R_1(1-\pi) + \pi R_2 - c$$

$$\boxed{\begin{array}{l} \text{Principal can always} \\ \text{contract} \\ R_1 = (1-p)r_1 + p \cdot R_2 \end{array}}$$

Assume that opt. contract incentivizes action 2 (high).

Notation: Contract pays t_1 for failure,
 t_2 for success.

It for working to .

$$t_1(1-\pi) + \pi t_2 - c \geq t_1(1-p) + t_2 p.$$

$$(\pi - p)(t_2 - t_1) \geq c \quad \rightarrow$$

④ binds at optimal contract.

Also, to minimize transfer
(opt wst revenue).

must have. $t_1 = 0$.

⑤ gives us $t_2 = \frac{c}{\pi - p}$.

Hence, Exp Rev of Principal = $R_2 - \frac{c}{\pi - p} \cdot \pi$

Opt Contract has a "closed form" here.

$$t_1 = 0 \quad t_2 = \frac{c}{\pi - p} \pi$$

$$R_2 - c.$$

Monotonicity holds here $t_2 > t_1$.

(transfer increases with reward).

↑ Generalizes to
any m as long as
 $m = \alpha$

(7)

(II) Optimal Contract for $n=2$ actions, m Rewards.

	r_1	r_2	...	r_m
Action 1	$f_{1,1}$	$f_{1,2}$...	$f_{1,m}$
Action 2	$f_{2,1}$	$f_{2,2}$...	$f_{2,m}$

Recall: There exists an optimal contract with $(n-1)$ non-zero transfers.

Binding IC constraint for action 2
(the action with non-zero rest).

$$t_j^o f_{2,j} - c = t_j^o F_{2,j}$$

for some t_j^o

Optimal contract is $t_j^o = \frac{c}{f_{2,j} - F_{2,j}}$

Principal contracts

$$R_2 = \frac{c}{\frac{f_{2,j}}{F_{2,j} - F_{2,j}}}$$

That is, principal contracts.

$$R_2 = c \cdot \frac{1}{\left(1 - \frac{f_{1,j}}{F_{2,j}}\right)}$$

Hence, choose j^* that maximizes

minimizes $\left\{ \frac{f_{1,j}}{F_{2,j}} \right\}$.

This ratio is called the likelihood ratio. (by actions 1 & 2).

Takeaway: Optimal contract pays for reward with min likelihood ratio.

Recap:

Q: Which reward r_j gets the non-zero transfer t_j^0 if opt. contract.

Ans: Pay for r_j with min likelihood ratio $\frac{f_{1,j}^0}{f_{2,j}^0}$

Statistical Inference intuition:

Principal is inferring agent's action from the reward.

→ Pays more for reward which can infer agent is working



• Extreme Example. ($m=2$ $m > 2$).

Assume r_j^0 has non-zero prob ϵ only if agent "works" (chooses action 2) outcome

That is,

r_j^0 occurs, "gives away" agent's actions.

- optimal contract has single non-zero transfer $t_j^* = \frac{c}{\epsilon}$.

The good: Principal contracts the first best $R_2 - c$.

The bad: Contract is non-monotone.

Recap:

Role of rewards in the model is two-fold:

(1) Represent surplus to be shared.

(2) Signal to the principal the agent's action.

- The optimal contract is shaped by (2).
- Can be mismatched with (1).

Optimal
Contract

φ rewards

n rewards

φ actions.

Monotone & Pay-f.
reward with
min likelihood ratio

Pay for reward
with min likelihood
ratio

n actions

Monotonicity

Straps assumed
needed.