Indian Institute of Science Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Assignment 1 [Posted Aug 20, 2022]

Faculty Instructor: Dr. Phani Motamarri
TAs: Ashish Rout, Dibya Nayak, Gourab Panigrahi, Nikhil Kodali

Submission Deadline: Sept. 4, 2022 23:59 hrs | Max Points: 100

Notations: Vectors and matrices are denoted below by bold-faced lower case and upper case alphabets, respectively.

Problem 1 [30 marks]

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$, $R(.)$ denotes the range of the matrix, $N(.)$ denotes the null space of a given matrix, $dim(.)$ denotes the dimension of a vector space, then prove the following:

(a) $dim[R(\mathbf{AB})] \leq dim[R(\mathbf{A})]$ [3 marks]

(b) If the matrix $\mathbf{B}$ is non-singular then $dim[R(\mathbf{AB})] = dim[R(\mathbf{A})]$ [3 marks]

(c) $dim[N(\mathbf{AB})] \leq dim[N(\mathbf{A})] + dim[N(\mathbf{B})]$ [4 marks]

(d) $dim[R(\mathbf{A})] + dim[N(\mathbf{A})] = n$ [4 marks]

(e) $rank(\mathbf{A}) + rank(\mathbf{B}) - n \leq rank(\mathbf{AB}) \leq min(rank(\mathbf{A}), rank(\mathbf{B}))$ [5 marks]
Hint: Use the result in (a)

(f) Given a vector $\mathbf{u} \in \mathbb{R}^n$, $rank\,(\mathbf{u}\mathbf{u}^T)$ is 1. [5 marks]
Hint: Use the result in (a)

(g) Row rank always equals column rank. [6 marks]

Problem 2 [10 marks]

Suppose there always exists a set of real coefficients $c_1, c_2, c_3, ...c_{10}$ for any set of real numbers $d_1, d_2, d_3, ...d_{10}$

$$\sum_{j=1}^{10} c_j f_j(i) = d_i \qquad \text{for } \; i \in \{1, 2, ...10\}$$

where $f_1, f_2, f_3, ...f_{10}$ are a set of functions defined on the interval [1,10]

(a) Use the concepts discussed in class to show that $d_1, d_2, d_3, ...d_{10}$ determine $c_1, c_2, c_3, ...c_{10}$ uniquely.
[6 marks]

(a) Let $\mathbf{A}$ be a 10×10 matrix representing the linear mapping from data $d_1, d_2, d_3, ...d_{10}$ to coefficients $c_1, c_2, c_3, ...c_{10}$. What is the i,j th entry of $\mathbf{A}^{-1}$? [4 marks]

Problem 3 [15 marks]

A matrix $\mathbf{S}$ is said to be symmetric if $\mathbf{S}^T = \mathbf{S}$ and skew-symmetric if $\mathbf{S}^T = -\mathbf{S}$. Now verify the following:

(a) The matrix $\mathbf{Q} = (\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S})$ is an orthogonal matrix for any skew-symmetric matrix $\mathbf{S}$. [3 marks]

(b) Note that a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ can be decomposed as $\mathbf{QDQ}^T$ where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{D}$ is a diagonal matrix. Using this result, show that $\mathbf{u}^T\mathbf{A}\mathbf{u} = 0 \ \forall \ \mathbf{u} \in \mathbb{R}^m$, if and only if $\mathbf{A} = 0$. [5 marks]

(c) Show that "$\mathbf{u}^T\mathbf{S}\mathbf{u} = 0 \ \forall \ \mathbf{u} \in \mathbb{R}^m$, if and only if $\mathbf{S}$ is a skew-symmetric matrix."
[7 marks]

Problem 4 [35 marks]

If $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$, then show the following.

(a) $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$ [5 marks]

(b) $\|\mathbf{x}\|_2 \leq \sqrt{m}\|\mathbf{x}\|_\infty$ [5 marks]

(c) $\|\mathbf{A}\|_\infty \leq \sqrt{n}\|\mathbf{A}\|_2$ [5 marks]

(d) $\|\mathbf{A}\|_2 \leq \sqrt{m}\|\mathbf{A}\|_\infty$ [5 marks]

(e) $\|\mathbf{A}\|_F = \sqrt{tr(\mathbf{A}^T\mathbf{A})}$ [5 marks]

(f) $\frac{1}{\sqrt{m}}\|\mathbf{A}\|_1 \leq \|\mathbf{A}\|_2 \leq \sqrt{n}\|\mathbf{A}\|_1$ [5 marks]

(g) $\|\mathbf{A}\|_2 \leq \sqrt{\|\mathbf{A}\|_1\|\mathbf{A}\|_\infty}$ [5 marks]

Give an example of a non-zero vector or matrix for which equality is achieved in the above inequalities.

Problem 5 [10 marks]

Induced matrix norm is defined as $\|\mathbf{A}\|^{(m,n)} = \max_{\mathbf{x}} \|\mathbf{Ax}\|^{(m)}$, where $\mathbf{x} \in \mathbb{R}^n$ and is a unit vector. $\|.\|$ corresponds to p-norm $(1 \leq p < \infty)$. For this exercise, let us consider p to be a natural number.

Using MATLAB/Octave/Python programming environment, create a matrix using "$\mathbf{A} = randn(100, 2)$". Subsequently, create random unit vectors $\mathbf{x}$ using "$temp = randn(2, 1)$" and normalize $\mathbf{x}$ using "$\mathbf{x} = temp/norm(temp)$". Check for multiple random vectors $\mathbf{x}$ (use a loop, and check for about 1000 random vectors $\mathbf{x}$) using "$norm\_of\_Ax = norm(\mathbf{Ax}, p)$" for $p = 1, 2, 3, 4, 5, 6, \infty$. What is the maximum value of p-norm for the vector $\mathbf{Ax}$? Now calculate p-norm of $\mathbf{A}$ using "$norm\_of\_A = norm(\mathbf{A}, p)$" for $p = 1, 2, \infty$ within the same programming environment you used before. Verify the equality $\|\mathbf{A}\|^{(m,n)} = \max_{\mathbf{x}} \|\mathbf{Ax}\|^{(m)}$ for $p = 1, 2, \infty$. Note that this equality is true for other values of p as well but you are restricting to $p = 1, 2, \infty$ in this exercise.