* Restriction to real symmetric matrices :-

If $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ and $\underset{\sim}{A} = \underset{\sim}{A}^T$, $\underset{\sim}{A}$ has real eigenvalues and a complete set of orthogonal eigenvectors i.e $\underset{\sim}{A} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^T$

$\lambda_1, \lambda_2 \dots \lambda_m \rightarrow$ real eigenvalues of $\underset{\sim}{A}$

$\underset{\sim}{q}_1, \underset{\sim}{q}_2, \dots \underset{\sim}{q}_m \rightarrow$ orthonormal eigenvectors

(I) The Rayleigh quotient of a vector $\underset{\sim}{x} \in \mathbb{R}^m$ for a given real symmetric matrix $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ is the scalar

$$\boxed{r(\underset{\sim}{x}) = \frac{\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}}}$$

Remarks:-

(i) If $\underset{\sim}{x}$ is an eigenvector then $r(x) = \lambda$

the corresponding eigenvalue of $\underset{\sim}{A}$ for that eigenvector $\underset{\sim}{x}$.

(ii) Given $\underset{\sim}{x} \in \mathbb{R}^m$ (which is not necessarily an eigenvector), what scalar α minimizes

$$\|\underset{\sim}{A}\underset{\sim}{x} - \alpha \underset{\sim}{x}\|_2$$

i.e what scalar acts like an eigenvalue

$$\underset{\sim}{x}\alpha = \underset{\sim}{A}\underset{\sim}{x}$$

This is an $m \times 1$ least squares problem!

$\underset{\sim}{x}$ is known matrix $m \times 1$

α is the unknown,

$\underset{\sim}{A}\underset{\sim}{x}$ is basically the known right hand side.

$$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$$

$$\|\underset{\sim}{A}\underset{\sim}{x} - \underset{\sim}{b}\|$$

$$\boxed{(A^T A)\underset{\sim}{x} = A^T b}$$

$\rightarrow$ m equations for 1 unknown!

Normal equations for our least squares problem

$$(\underset{\sim}{x}^T \underset{\sim}{x})\alpha = \underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x}$$

$$\boxed{\alpha = \frac{\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}}}$$

$$\alpha = r(x) = \frac{x^T A x}{x^T x} \quad \text{which minimizes} \quad \|\alpha x - A x\|_2$$

This "α" is the natural eigenvalue estimate to consider if x is approximately equal to an eigenvector

Given with a vector x, $\boxed{x = \sum_{j=1}^{m} c_j q_j}$

Then $r(x) = \dfrac{x^T A x}{x^T x} = \dfrac{\sum_{j=1}^{m} c_j^2 \lambda_j}{\sum_{j=1}^{m} c_j^2}$ ✓

If x is close to one of the eigenvectors q_J

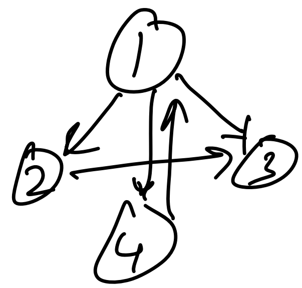
$$\frac{|c_j|}{|c_J|} < \varepsilon \quad \text{for all } j \neq J$$

we can show that $\boxed{|r(x) - r(q_J)| = O(\|x - q_J\|^2)}$, $\boxed{\|x - q_J\| < \varepsilon}$

Rayleigh quotient is quadratically

an accurate estimate of an eigenvalue

Power iteration :-



Suppose $\underset{\sim}{v}^{(0)}$ is a vector such that $\|\underset{\sim}{v}^{(0)}\| = 1$, the power iteration produces a sequence of vectors $\underset{\sim}{v}^{(i)}$ that converges to an eigenvector corresponding to the largest eigenvalue of $\underset{\sim}{A}$.

Algo :- Power iteration

Initialize $\underset{\sim}{v}^{(0)}$ to some vector $\|v^{(0)}\| = 1$

for $k = 1, 2, \ldots$

$$\underset{\sim}{w} = \underset{\sim}{A}\,\underset{\sim}{v}^{(k-1)}$$

$$\underset{\sim}{v}^{(k)} = \frac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|}$$

$$\lambda^{(k)} = \left[\underset{\sim}{v}^{(k)}\right]^T A\, v^{(k)}$$

$$v^{(0)}$$

$$\frac{Av^{(0)}}{\|Av^{(0)}\|}$$

$$\underbrace{A\left[\frac{Av^{(0)}}{\|Av^{(0)}\|}\right]}$$

$$\frac{A^k v^{(0)}}{\|A^k v^{(0)}\|}$$

$$v^{(0)} = a_1 q_1 + a_2 q_2 + \dots a_m q_m$$

$$A^k v^{(0)} = a_1 A^k q_1 + a_2 A^k q_2 + \dots a_m A^k q_m$$

$$= a_1 \lambda_1^k q_1 + a_2 \lambda_2^k q_2 + \dots a_m \lambda_m^k q_m$$

$$= a_1 \lambda_1^k \left[q_1 + \left(\frac{a_2}{a_1}\right) \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)}^k q_2 + \dots + \left(\frac{a_m}{a_1}\right)\left(\frac{\lambda_m}{\lambda_1}\right)^k q_m \right] \quad (*)$$

$$\left|\frac{\lambda_j}{\lambda_1}\right| < 1 \quad \text{for } j > 1 \qquad |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots |\lambda_m| \geq 0$$

as $k \to \infty$, $A^k v^{(0)} \to \underbrace{a_1 \lambda_1^k q_1}$

$$v^{(k)} = \frac{A^k v^{(0)}}{\|A^k v^{(0)}\|} \longrightarrow q_1 \quad \text{as } k \to \infty$$

<u>Thm</u>:- Let $|\lambda_1| > |\lambda_2| \geq \dots |\lambda_m| \geq 0$ and $q_1^T v^{(0)} \neq 0$, then iterates of the power iteration satisfy

$$\boxed{\|v^{(k)} - (\pm q_1)\| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)} \text{ and}$$

$$|\lambda^{(k)} - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right) \text{ as } k \to \infty$$

If $\lambda_1 > 0$, $\dfrac{\underset{\sim}{A}^k \underset{\sim}{v}^{(0)}}{\|\underset{\sim}{A}^k \underset{\sim}{v}^{(0)}\|} \rightarrow \dfrac{a_1 \lambda_1^k \underset{\sim}{q}_1}{|a_1 \lambda_1^k|} \rightarrow \underset{\sim}{q}_1$

If $\lambda_1 < 0$, $\dfrac{\underset{\sim}{A}^k \underset{\sim}{v}^{(0)}}{\|\underset{\sim}{A}^k \underset{\sim}{v}^{(0)}\|} \rightarrow \dfrac{a_1 \lambda_1^k \underset{\sim}{q}_1}{|a_1 \lambda_1^k|}$

if k is even

$$\frac{a_1 \lambda_1^k \underset{\sim}{q}_1}{|a_1 \lambda_1^k|} = \frac{\cancel{a_1 \lambda_1^k} \underset{\sim}{q}_1}{\cancel{a_1 \lambda_1^k}}$$

if k is odd, $|a_1 \lambda_1^k| = -a_1 \lambda_1^k$

$$\frac{a_1 \lambda_1^k \underset{\sim}{q}_1}{|a_1 \lambda_1^k|} = -\underset{\sim}{q}_1$$

Shortcomings :-

(a) It can only find eigenvector corresponding to largest eigenvalue

(b) Convergence is linear with error being reduced by a constant factor $\approx \left|\frac{\lambda_2}{\lambda_1}\right|$ at each iteration

(c) If $\lambda_2 \simeq \lambda_1$ convergence can be very slow!

Inverse iteration:-

* We amplify the differences between eigenvalues and hence accelerate the convergence!

* We pick $\mu \in \mathbb{R}$ that is not eigenvalue of $\underset{\sim}{A}$, the eigenvectors of $(\underset{\sim}{A} - \mu \underset{\sim}{I})^{-1}$ are same as eigenvectors of $\underset{\sim}{A}$, and the corresponding eigenvalues are $\left\{\frac{1}{\lambda_j - \mu}\right\}_{j=1}^{m}$

where $\{\lambda_j\}_{j=1}^{m}$ are eigenvalues of $\underset{\sim}{A}$.

Now suppose μ is close eigenvalue λ_J of $\underset{\sim}{A}$, then $\frac{1}{\lambda_J - \mu}$ will be much larger than $\frac{1}{\lambda_j - \mu}$ for all $j \neq J$

$\rightarrow$ If we apply power iteration to $(\underset{\sim}{A} - \mu \underset{=}{I})^{-1}$, the process would converge rapidly to $\underset{\sim}{q}_J$.

The ideas is called inverse iteration

Algo:- Initialize $\underset{\sim}{v}^{(0)}$ to some vector with $\|\underset{-}{v}^{(0)}\| = 1$

Initialize μ to some value near λ_J

for $k = 1, 2 \ldots$

$\rightarrow$ Solve $(A - \mu \underset{=}{I})\underset{\sim}{w} = v^{(k-1)}$ for $\underset{\sim}{w}$ | % $(A - \mu I)^{-1}$ on $v^{(k-1)}$

$\rightarrow$ $v^{(k)} = \frac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|}$

$\rightarrow$ $\lambda^{(k)} = (v^{(k)})^T \underset{-}{A} \underset{\sim}{v}^{(k)}$

<u>Thm</u> :- Suppose λ_J is the closest eigenvalue to μ and λ_K is the second closest

i.e $|\mu-\lambda_J| < |\mu-\lambda_K| \le |\mu-\lambda_j|$ for all $j \neq J$

Suppose $q_J^T v^{(0)} \neq 0$, then the iterates of the inverse iteration satisfy

$$\|v^{(k)} - (\pm q_J)\| = O\left(\left(\frac{|\mu-\lambda_J|}{|\mu-\lambda_K|}\right)^k\right)$$

and $$|\lambda^{(k)} - \lambda_J| = O\left(\left|\frac{\mu-\lambda_J}{\mu-\lambda_K}\right|^{2k}\right)$$

Power iteration on A

(i) eigenvector estimate $\rightarrow$ eigenvalue estimate

Inverse iteration

(i) eigenvalue estimate $\rightarrow$ eigenvector estimate

Algo:- Rayleigh quotient iteration.

Initialize $\underset{\sim}{v}^{(0)}$ to some vector $\|\underset{\sim}{v}^{(0)}\| = 1$

Initialize $\lambda^{(0)} = (\underset{\sim}{v}^{(0)})^T \underset{\sim}{A} \underset{\sim}{v}^{(0)}$

for $k = 1, 2, \ldots$

Solve $(\underset{\sim}{A} - \lambda^{(k-1)} \underset{\sim}{I}) \underset{\sim}{w} = v^{(k-1)}$;

$v^{(k)} = \frac{w}{\|w\|}$

$\lambda^{(k)} = (v^{(k)})^T \underset{\sim}{A} v^{(k)}$

end for.

Each iteration triples the number of digits of accuracy

When it converges, convergence is cubic i.e if λ_J is an eigenvalue of

$\underset{\sim}{A}$ and $\underset{\sim}{v}^{(0)}$ is sufficiently close to the eigenvector $\underset{\sim}{q}_J$, then as $k \to \infty$

$$\|\underset{\sim}{v}^{(k+1)} - (\pm \underset{\sim}{q}_J)\| = O(\|\underset{\sim}{v}^{(k)} - \underset{\sim}{q}_J\|^3)$$

and $|\lambda^{(k+1)} - \lambda_J| = O(|\lambda^{(k)} - \lambda_J|^3)$