Conditioning and Stability

→ Conditioning pertains to sensitivity of a mathematical problem to perturbations in inputs.

→ Stability pertains to perturbation behavior of an algorithm used to solve the mathematical problem on a computer.

(*) Conditioning of a problem :-

· Solving a problem is like evaluating a function

$$\underset{\sim}{y} = f(\underset{\sim}{x})$$

Here $\underset{\sim}{x}$ represents input to the problem (the data), $\underset{\sim}{f}$ represents problem itself and $\underset{\sim}{y}$ represents its solution

* What happens to $\underset{\sim}{y}$ when given $\underset{\sim}{x}$ is perturbed slightly ?

If small changes in $\underset{\sim}{x}$ leads to large changes in $\underset{\sim}{y}$, we say the problem is ill-conditioned and usually we are interested in solving well-conditioned problem.

* <u>Absolute</u> <u>condition</u> <u>number</u>:-

If a small perturbation of $\underset{\sim}{x}$ is denoted by $\delta \underset{\sim}{x}$, then let the resulting perturbation in the solution be represented as δf

i.e $\delta f = f(\underset{\sim}{x} + \delta \underset{\sim}{x}) - f(\underset{\sim}{x})$

then the absolute condition number $\hat{K} = K(x)$ of the problem f at x is given by

$$K(x) = \max_{\delta x} \left(\frac{\|\delta f\|}{\|\delta x\|} \right) \quad — (1)$$

for infinitesimally small $\underset{\sim}{\delta f}$ and $\underset{\sim}{\delta x}$

If $\underset{\sim}{f}$ has a derivative, we can evaluate the Jacobian matrix $\underset{\sim}{J}(x)$ as $J_{ij} = \dfrac{\partial f_i}{\partial x_j}$

We have $\underset{\sim}{\delta f} \approx \underset{\sim}{J}(x)\, \underset{\sim}{\delta}(x)$ with equality as $\|\delta x\| \to 0$

$$K(x) = \max_{\delta x} \frac{\|J(x)\,\delta x\|}{\|\delta x\|}$$

$$\hat{K} = K(x) = \|J(x)\|$$

<u>Relative condition number</u>:-

Assume $\underset{\sim}{\delta f}$ and $\underset{\sim}{\delta x}$ are infinitesimal

$$\hat{K}^R = \max_{\underset{\sim}{\delta x}} \left[\frac{\frac{\|\delta f\|}{\|f(x)\|}}{\frac{\|\delta x\|}{\|x\|}} \right]$$

$$= \max_{\underset{\sim}{\delta x}} \left(\frac{\frac{\|\delta f\|}{\|\delta x\|}}{\frac{\|f(x)\|}{\|x\|}} \right)$$

$$\boxed{\hat{K}^R = \frac{\|J(x)\|}{\|f(x)\| / \|x\|}}$$

A problem is well-conditioned if $\underset{\sim}{K}$ is small (eg:- 1, 10, 10^2). and ill-conditioned if K is large (eg: 10^6, 10^{16} ..)

Examples :-

(1) $f(x) = \frac{x}{2} \qquad x \in \mathbb{R}$

Input : x $\qquad J = \frac{df}{dx} = \frac{1}{2}$

output : $\frac{x}{2}$

$$\hat{K} = \frac{\|J\|}{\frac{\|f(x)\|}{\|x\|}} = \frac{1/2}{\frac{(x/2)}{x}} = 1$$

well conditioned problem.

(2) $f(\underset{\sim}{x}) = x_1 - x_2$ where $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = [1 \ -1]_{1\times 2}$$

$$\hat{K} = \frac{\|J\|_\infty}{\|f(\underset{\sim}{x})\|_\infty / \|\underset{\sim}{x}\|_\infty}$$

$\|J\|_\infty$ is max row sum $= 2$

$$\hat{K} = \frac{2}{\frac{|x_1 - x_2|}{\max\{|x_1|, |x_2|\}}}$$

$$= \frac{2 \max\{|x_1|, |x_2|\}}{|x_1 - x_2|}$$

if $|x_1 - x_2|$ is small ≈ 0, K is large and is not well conditioned when you are subtracting two numbers which are very close by.

$$\hat{K} = \frac{\|J\|_1}{\frac{\|f(x)\|_1}{\|x\|_1}} = \frac{1}{\frac{|x_1 - x_2|}{(|x_1|+|x_2|)}} = \frac{(|x_1|+|x_2|)}{|x_1 - x_2|}$$

Root finding of quadratic equation :-

$$x^2 - 2xp + 1 = 0$$

$$x_1 = p - \sqrt{p^2-1}$$
$$x_2 = p + \sqrt{p^2-1}$$

Input :- value of p

output :- x_1, x_2 which are the roots of quadratic equation.

Examine sensitivity of x_2 w.r.t p

$$\hat{K} = K(p) = \frac{\|J\|}{\frac{\|f(p)\|}{\|p\|}} \quad ; \quad J = \frac{dx_2}{dp}$$

$$K(p) = \left|\frac{dx_2}{dp}\right| \times \frac{|p|}{|x_2|}$$

$$= \left|1 + \frac{p}{\sqrt{p^2-1}}\right| \times \frac{|p|}{|p + \sqrt{p^2-1}|}$$

$$= \left| \frac{p+\sqrt{p^2-1}}{\sqrt{p^2-1}} \right| \times \frac{|p|}{|p+\sqrt{p^2-1}|}$$

$$= \frac{|p|}{\sqrt{p^2-1}}$$

The root x_2 is sensitive for p close to 1 with $K(p) \to \infty$ and insensitive for large p with $K \simeq 1$

<u>Eigenvalues of matrix</u> :-

$$\underset{\sim}{A}\,\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$$

<u>Input</u> :- $\underset{\sim}{A}$

<u>output</u> : eigenvalues λ

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 1000 \\ 0 & 1 \end{pmatrix}$$ eigenvalues of $A = \{1, 1\}$

$$\hat{\underset{\sim}{A}} = \begin{pmatrix} 1 & 1000 \\ 0.001 & 1 \end{pmatrix}$$ eigenvalues of A after perturbation $\{0, 2\}$

Consider a symmetric matrix $\underset{\sim}{A} = \underset{\sim}{A}^T$

λ and $\lambda + \delta\lambda$ are corresponding eigenvalues $\underset{\sim}{A}$ and $\underset{\sim}{A} + \underset{\sim}{\delta A}$, then $\underbrace{|\delta\lambda| \leq \|\delta A\|_2}$ — (1)

Relative condition number = $\max_{\|\delta A\|} \dfrac{\frac{|\delta \lambda|}{|\lambda|}}{\frac{\|\delta A\|_2}{\|A\|_2}}$

use ① $\rightarrow$ $\boxed{\text{Relative condition number } \hat{k} = \dfrac{\|A\|_2}{|\lambda|}}$

Conditioning of matrix-vector multiplication:-

$y = A x$

Fixed A

Input :- x

Output :- Ax

Consider the problem of computing Ax for fixed A and input x

$$\boxed{k = \max_{\delta x} \dfrac{\frac{\|\delta f\|}{\|f\|}}{\frac{\|\delta x\|}{\|x\|}}}$$

$$k = \max_{\delta x} \dfrac{\frac{\|A(x+\delta x) - Ax\|}{\|Ax\|}}{\frac{\|\delta x\|}{\|x\|}}$$

$$= \max_{\delta x} \dfrac{\frac{\|A\delta x\|}{\|Ax\|}}{\frac{\|\delta x\|}{\|x\|}} = \underbrace{\left(\max_{\delta x} \frac{\|A\delta x\|}{\|\delta x\|}\right) \times \frac{\|x\|}{\|Ax\|}}$$

$$\hat{k} = \frac{\|A\| \, \|x\|}{\|Ax\|} \quad — ②$$

$\hat{k}$ for a give A and at x

To loosen above equality to get a bound of $\underset{\sim}{x}$, let us assume $\underset{\sim}{A}$ is square and non-singular

$$\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{A}\underset{\sim}{x}$$

$$\|\underset{\sim}{x}\| = \|\underset{\sim}{A}^{-1}\underset{\sim}{A}\underset{\sim}{x}\| \leq \|\underset{\sim}{A}^{-1}\|\,\|\underset{\sim}{A}\underset{\sim}{x}\|$$

$$\frac{\|\underset{\sim}{x}\|}{\|\underset{\sim}{A}\underset{\sim}{x}\|} \leq \|\underset{\sim}{A}^{-1}\| \quad — \text{(3)}$$

From eqn (2) $\hat{K} = \dfrac{\|\underset{\sim}{A}\|\,\|\underset{\sim}{x}\|}{\|\underset{\sim}{A}\underset{\sim}{x}\|}$ ✓

using (3) $\hat{K} \leq \|\underset{\sim}{A}\|\,\|\underset{\sim}{A}^{-1}\| \quad — \text{(4)}$

What about for a give $\underset{\sim}{A}$, if you want to compute $\underset{\sim}{A}^{-1}\underset{\sim}{b}$ from a given input $\underset{\sim}{b}$?

Input :- $\underset{\sim}{b}$ Fixed $\underset{\sim}{A}$, $\underset{\sim}{A}^{-1}$

Output :- $\underset{\sim}{A}^{-1}\underset{\sim}{b} = \underset{\sim}{x}$

$$\hat{K} = \frac{\|\underset{\sim}{A}^{-1}\|\,\|\underset{\sim}{b}\|}{\|\underset{\sim}{A}^{-1}\underset{\sim}{b}\|}$$

$$\hat{K} = \frac{\|A^{-1}\|\,\|b\|}{\|\underset{\sim}{x}\|}$$

using (4), even for the problem of $\underset{\sim}{A}^{-1}\underset{\sim}{b}$

$$\hat{K} \leq \|\underset{\sim}{A}^{-1}\|\,\|\underset{\sim}{A}\|$$

Result:-

Let $\underset{\sim}{A} \in \mathbb{R}^{m \times m}$ be non-singular and consider the equation $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$, the problem of computing $\underset{\sim}{b}$ given $\underset{\sim}{x}$ has condition number

$$\hat{K} = \frac{\|\underset{\sim}{A}\| \|\underset{\sim}{x}\|}{\|\underset{\sim}{b}\|} \leq \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\|$$

with respect to perturbations of $\underset{\sim}{x}$.

The problem of computing $\underset{\sim}{x}$ given $\underset{\sim}{b}$ has condition number

$$\hat{K} = \frac{\|\underset{\sim}{A}^{-1}\| \|\underset{\sim}{b}\|}{\|\underset{\sim}{x}\|} \leq \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\|$$

Condition number of a matrix:-

The condition number of $\underset{\sim}{A}$ (relative to norm $\|\cdot\|$) denoted by $\hat{K}(\underset{\sim}{A}) = \|\underset{\sim}{A}\| \|\underset{\sim}{A}^{-1}\|$

If $\hat{K}(\underset{\sim}{A})$ is small, $\underset{\sim}{A}$ is said to be well-conditioned

if $\hat{K}(A)$ is large, $\underset{\sim}{A}$ is said to be ill conditioned.

If $\underset{\sim}{A}$ is singular, or close to singular $K(A) \to \infty$

In the 2-norm

$$K(\underset{\sim}{A}) = \|\underset{\sim}{A}\|_2 \|\underset{\sim}{A}^{-1}\|_2$$

$\sigma_1 \rightarrow$ max singular value of $\underset{\sim}{A}$

$(1/\sigma_m) \rightarrow$ max singular value of $\underset{\sim}{A}^{-1}$

$\sigma_m \rightarrow$ min singular of $\underset{\sim}{A}$

$$K(\underset{\sim}{A}) = \frac{\sigma_1}{\sigma_m}$$

For a square matrix, non zero singular values are square roots of non-zero eigenvalues of A^TA or AA^T.

$$K(A) = \frac{\sqrt{\lambda_{max}(A^TA)}}{\sqrt{\lambda_{min}(A^TA)}}$$

$$K(A) = \frac{|\lambda_{max}(\underset{\sim}{A})|}{|\lambda_{min}(\underset{\sim}{A})|} \quad (\text{If } \underset{\sim}{A} \text{ is symmetric})$$

If $\underset{\sim}{A} \in \mathbb{R}^{m \times n}$ $(m > n)$, $K(\underset{\sim}{A})$ is defined in terms of pseudoinverse A^+.

$$K(A) = \|A\| \|A^+\|$$

$$A^+ = (A^TA)^{-1} A^T$$

<u>Conditioning of system of equations</u>:-

So far we fixed $\underset{\sim}{A}$ and perturbed $\underset{\sim}{x}$ or $\underset{\sim}{b}$

→ What about if we perturb A?

Fix $\underset{\sim}{b}$ and consider $f : A \to x = \underset{\sim}{A}^{-1}\underset{\sim}{b}$

inputs :- $\underset{\sim}{A}$

output :- $\underset{\sim}{x}$

$\underset{\sim}{A}$ is perturbed by $\delta \underset{\sim}{A}$ and let $\underset{\sim}{x}$, the output be perturbed by $\delta \underset{\sim}{x}$

$$(\underset{\sim}{A} + \delta\underset{\sim}{A})(\underset{\sim}{x} + \delta\underset{\sim}{x}) = b \quad - (1)$$

$$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b} \quad - (2)$$

using (1) and (2) above, we have

$$(\delta\underset{\sim}{A})\underset{\sim}{x} + \underset{\sim}{A}(\delta\underset{\sim}{x}) = 0$$

$$\delta\underset{\sim}{x} = -\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{x}$$

$$\|\delta\underset{\sim}{x}\| = \|-\underset{\sim}{A}^{-1}(\delta\underset{\sim}{A})\underset{\sim}{x}\|$$

$$\leq \|\underset{\sim}{A}^{-1}\| \, \|(\delta\underset{\sim}{A})\underset{\sim}{x}\|$$

$$\leq \|\underset{\sim}{A}^{-1}\| \, \|\delta\underset{\sim}{A}\| \, \|\underset{\sim}{x}\| \quad - (3)$$

$$\hat{K} = \max_{\delta\underset{\sim}{A}} \frac{\frac{\|\delta\underset{\sim}{x}\|}{\|\underset{\sim}{x}\|}}{\frac{\|\delta\underset{\sim}{A}\|}{\|\underset{\sim}{A}\|}}$$

using eq 3

$$\underbrace{\left(\frac{\frac{\|\delta x\|}{\|x\|}}{\frac{\|\delta A\|}{\|A\|}} \right) \leq \|\underset{\sim}{A}^{-1}\| \, \|A\|}$$

If perturbation $\underset{\sim}{\delta A}$ exists which makes the above inequality an equality

then

$\hat{K}$ of solving system of equations

$$\hat{K} = \max_{\underset{\sim}{\delta A}} \left(\frac{\frac{\|\underset{\sim}{\delta x}\|}{\|\underset{\sim}{x}\|}}{\frac{\|\underset{\sim}{\delta A}\|}{\|\underset{\sim}{A}\|}} \right) = K(\underset{\sim}{A})$$

Result :-

For a fixed $\underset{\sim}{b}$, Input $\to \underset{\sim}{A}$, output $\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}$

condition number of this problem with respect to perturbations in $\underset{\sim}{A}$ is

$$\hat{K} = \|\underset{\sim}{A}\|\|\underset{\sim}{A}^{-1}\| = K(\underset{\sim}{A})$$

eg :-

$$K(\underset{\sim}{A}) = 10^{6}$$

$$\frac{\|\underset{\sim}{\delta x}\|}{\|\underset{\sim}{x}\|} \le 10^{6} \times \frac{\|\underset{\sim}{\delta A}\|}{\|\underset{\sim}{A}\|} \qquad \varepsilon_M \approx 10^{-8}$$

$$\frac{\|\underset{\sim}{\delta A}\|}{\|\underset{\sim}{A}\|} \le O(\varepsilon_M)$$

$$\le 10^{6} \times 10^{-8}$$

$$\approx O(10^{-2})$$