Revenue Maximization: Single Item Two Bidders.

In the Bayesian Setup. Consider two bidders with.

$F_1 = F_2 = \text{Unif}[0,1]$.

The Second Price Auction.
(Vickery Auction) has nonzero expected revenue.

Specifically, revenue of second price auction $= \sum_{i=1}^{2} p_i(b)$

Assuming, truth telling, revenue $=$ ~~min~~ second highest valuation.

$\therefore$ Expected Revenue $= \mathbb{E}_{X,Y \sim \text{Unif}(0,1)} [\min\{X, Y\}]$

(*) $= 1/3$

[Calculation: left as an exercise]

Another option is to supplement second price auction with a reserve price. r.

$\downarrow$

In this auction, if the bids are both less than r then no one gets the good.

Else, highest bidder gets the good at price $\max\{r,$ second highest bid$\}$.

[This is equivalent to an auction where we introduce a fictitious bid of ~~[illegible]~~ r & charge it zero in case r wins.]

For $F_1 = F_2 = \text{Unif}[0,1]$.
& reserve price r.

Note Exp value of n iid unif rvs
$X_1, X_2 \ldots X_n \underset{iid}{\sim} \text{Unif}(a,b)$ $Y = \min x_i$
$\mathbb{E}\,Y = \frac{b+na}{n+1}$

$\mathbb{E}$ revenue $= \frac{5}{12}$.

Details: Iterated Expectation.

$\mathbb{E}\left[\min\{X,Y\} \,\middle|\, \begin{matrix} X \geq 1/2 \\ Y \geq 1/2 \end{matrix}\right] \cdot \Pr\left\{\begin{matrix} X \geq 1/2 \\ Y \geq 1/2 \end{matrix}\right\}$

$+ \; 0 \cdot \Pr\left\{\begin{matrix} X \leq 1/2 \\ Y \leq 1/2 \end{matrix}\right.$ and $\left.\right\}$.

$+ \; \frac{1}{2} \Pr\left\{ X \geq 1/2 \right.$ and $\left. Y < 1/2 \right\}$.

$+ \; \frac{1}{2} \cdot \Pr\left\{ Y \geq 1/2 \right.$ and $\left. X < 1/2 \right\}$.

$$\begin{aligned} \text{Exp Rev} &= \frac{1+2\cdot\frac{1}{2}}{3}\cdot\frac{1}{4} + \frac{1}{8}+\frac{1}{8} \\ &= \frac{2}{12} + \frac{1}{4} \\ &= \frac{5}{12}. \quad \square \end{aligned}$$

Hence $\mathbb{E}$xp Revenue with reserve price $1/2$ $>$ $\mathbb{E}$xp Revenue of Second Price Auction.

GOAL: Description of a revenue maximizing auction for any single param. env. & any. Distributions $F_1, F_2, \ldots, F_n$. (independent).

Throughout we focus on DSIC mechanisms. Hence, assume truthful bids $b_i = v_i \;\forall i$

By defn, the exp. revenue is

$$\mathbb{E}_{(v_i \sim F_i)_i} \left[\sum_{i=1}^{n} p_i (v_1, v_2 \ldots, v_n) \right].$$

Shorthand.

$v \sim F.$

Product Distribution $F_1 \times F_2 \cdots \times F_n$

We will develop a formula for exp. revenue that only references the allocation rule x & hence is far easier to maximize.

Defn: [Virtual Valuation]. For bidder i with value distribution F_i and valuation v_i, her virtual valuation v_i is defined as

$$\varphi_i (v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Ex. if $F_i(z) = z$ in $[0,1]$ Unif Distributi

Then,

$$\varphi_i (z) = z - \left(\frac{1-z}{1}\right)$$
$$= 2z - 1 \quad \text{on } [0,1]$$

Note $\varphi_i(z)$ can be negative.

$\varphi_i(z) \leq z$ and $\frac{1-F_i(z)}{f_i(z)}$ is sometimes called the "information rent".

Expected Revenue Equals Expected Virtual Welfare.

$\boxed{\text{Lemma}}$ for every single param env. with valuation distributions $F_1, F_2, \ldots F_n$, every DSIC mechanism (x, p), every i, every value v_{-i} of the other agents,

$F_1, \ldots F_n$ independent

$$\mathbb{E}_{v_i \sim F_i}[p_i(v)] = \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) x_i(v)]$$

NOTE: $v = (v_i, v_{-i})$

Pf: We start with Myerson's lemma.

$$p_i(v) = \int_0^{v_i} z \cdot x_i'(z, v_{-i})\, dz.$$

$\boxed{\text{Step 1:}}$

$$\mathbb{E}_{v_i \sim F_i}[p_i(v)] = \int_0^{v_{max}} p_i(v) f_i(v_i)\, dv_i$$

Using Independence of F_i-s.

$$= \int_0^{v_{max}} \left[\int_0^{v_i} z\, x_i'(z, v_{-i})\, dz\right] f_i(v_i)\, dv_i$$

$\boxed{\text{Step 2}}$ Reversing the order of integration. i

$$\int_0^{v_{max}} \left[\int_0^{v_i} z\, x_i'(z, v_{-i})\, dz\right] f_i(v_i)\, dv_i$$

yields

$$\int_0^{v_{max}} \left[\int_z^{v_{max}} f_i(v_i)\, dv_i \right] z \cdot x_i'(z, v_{-i})\, dz.$$

This simplifies to

$$\int_0^{v_{max}} \underbrace{[1 - F_i(z)]\, z}_{g} \cdot \underbrace{x_i'(z, v_{-i})}_{h'}\, dz.$$

Step 3 Integration by parts yields

Recall

$$\int_a^b g h' = g h \Big|_a^b - \int_a^b g' h$$

$$\underbrace{(1 - F_i(z))\, z \cdot x_i(z, v_{-i}) \Big|_0^{v_{max}}}_{0 - 0}$$

$$- \int_0^{v_{max}} x_i(z, v_{-i}) \left(1 - F_i(z) - z f_i(z)\right) dz.$$

Hence, we get.

$$\int_0^{v_{max}} \underbrace{\left(z - \frac{1 - F_i(z)}{f_i(z)} \right)}_{\varphi_i(z)} x_i(z, v_{-i})\, f_i(z, \cancel{v_{-i}})\, dz.$$

This quantity corresponds to $\mathbb{E}_{v_i \sim F_i} \left[\varphi_i(v_i) \cdot x_i(v_i, v_{-i}) \right]$

Therefore. $\mathbb{E}_{v_i \sim F_i} [p_i(v)] = \mathbb{E}_{v_i \sim F_i} [\varphi_i(v_i)\, x_i(v_i, v_{-i})]$

The lemma stands proved.

The lemma directly leads to the following important theorem.

THM: Exp Rev = Exp. Virtual Welfare.

For every single param env. with valuation distributions $f_1, f_2, \ldots f_n$, and every DSIC mech. (x, p),

$$\underset{v\sim f}{\mathbb{E}}\left[\sum_{i=1}^{n} p_i(v)\right] = \underset{v\sim f}{\mathbb{E}}\left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)\right]$$

Exp Rev. Exp. Virtual Welf.

Pf: Taking exp. wrt $v_{-i} \sim f_{-i}$ in the previous Lemma's equality

$$\underset{v\sim f}{\mathbb{E}}[p_i(v)] = \underset{v\sim f}{\mathbb{E}}[\varphi_i(v_i)\, x_i(v)]$$

Linearity of exp. gives us.

$$\underset{v\sim f}{\mathbb{E}}\left[\sum_{i=1}^{n} p_i(v)\right] = \sum_{i=1}^{n} \underset{v\sim f}{\mathbb{E}}\, p_i(v)$$

$$= \sum_{i=1}^{n} \underset{v\sim f}{\mathbb{E}}\, \varphi_i(v_i) \cdot x_i(v)$$

$$= \underset{v\sim f}{\mathbb{E}}\left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)\right]$$

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