Correlated Strategies

$\Gamma = \langle N, (S_i), (u_i) \rangle$

$C \subseteq N$ coalition of players

$S_C = \times_{i \in C} S_i$

$\tau_C \in \Delta(S_C)$ represents a prob. dist. on the joint actions of players in coalition C

Example

$N = \{1, 2, 3\}$

$S_1 = \{x_1, y_1\}$; $S_2 = \{x_2, y_2\}$; $S_3 = \{x_3, y_3, z_3\}$

$S_2 \times S_3 = \{(x_2, x_3), (x_2, y_3), (x_2, z_3), (y_2, x_3), (y_2, y_3), (y_2, z_3)\}$

A correlated strategy of this coalition would be

$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$

Another correlated strategy is

$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}\right)$

Suppose $\alpha \in \Delta(S_N)$

We can define in a natural way

$$u_i(\alpha) = \sum_{s \in S} \alpha(s) u_i(s)$$

$$u(\alpha) = (u_1(\alpha), \ldots, u_n(\alpha))$$

Expected payoff allocations to players when they play

Expected payoff allocations to players when they play a correlated strategy

Note the difference between a correlated strategy and a mixed strategy profile.

Correlated strategy $\alpha \in \Delta\left(\times S_i\right)$

Mixed strategy profile $\sigma \in \times\left(\Delta(S_i)\right)$

Example

$N = \{1,2\}$; $S_1 = \{x_1, y_1\}$; $S_2 = \{x_2, y_2\}$

Suppose $\sigma = (\sigma_1, \sigma_2)$ with

$\sigma_1 = (\delta_1, 1-\delta_1)$; $\sigma_2 = (\delta_2, 1-\delta_2)$

Then:

$u_i(\sigma) = \delta_1\delta_2 u_i(x_1, x_2) + \dots + (1-\delta_1)(1-\delta_2) u_i(y_1, y_2)$

This $u_i(\sigma)$ will be the same as $u_i(\alpha)$ where

$\alpha = \left(\delta_1\delta_2, \delta_1(1-\delta_2), (1-\delta_1)\delta_2, (1-\delta_1)(1-\delta_2)\right)$

Given any mixed strategy profile σ, we can find a correlated strategy α such that

$u_i(\alpha) = u_i(\sigma) \quad \forall i \in N$

The vice-versa is not always possible. Thus correlated strategies produce a richer space of utility allocations. Correlated strategies are key to the study of cooperative games.

We study two classes of cooperative games:

- Games with contracts
- Games with communication

Then we define the notion of Correlated Equilibrium.

We next look at the space of utility allocations under correlated strategies, correlated equilibria, etc.

This paves the way for discussing solution concepts for cooperative game theory, such as:

- Nash Bargaining Solution
- the Core
- The Shapley value

Games with Contracts

Example: Modified Version of Prisoner's Dilemma

	x_2	y_2
x_1	2, 2	0, 6
y_1	6, 0	1, 1

- (y_1, y_2) is a SDSE and unique NE
- (x_1, x_2) yields better payoffs

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Question: Can rational players achieve (x_1, x_2) ?

Not in noncooperative games. We will see two ways in which players can cooperate and realise this:

- cooperation through contracts
- cooperation through communication

Example of a Contract:

- If both sign, players play (x_1, x_2)
- If only 1 signs, 1 plays y_1 (SDS)
- If only 2 signs, 2 plays y_2 (SDS)
- If neither signs, business as usual (noncooperative game)

We now expand the strategy sets as:

$S_1 = \{x_1, y_1, a_1\}$ $\quad a_1$: Player 1 signs contract

$S_2 = \{x_2, y_2, a_2\}$ $\quad a_2$: Player 2 signs contract

	x_2	y_2	a_2
x_1	2,2	0,6	0,6
y_1	6,0	1,1	1,1
a_1	6,0	1,1	2,2

In this expanded game,

- (a_1, a_2) is a WDSE
- (y_1, y_2) is a PSNE

- (y_1, y_2) is a PSNE

Note that both the players signing the contract is a correlated strategy:

$$\big((x_1,x_2) : 1 \,;\, (x_1,y_2) : 0 \,;\, (y_1,x_2) : 0 \,;\, (y_1,y_2) : 0\big)$$

One more Contract

The players will be able to even better by signing one more contract.

- If both sign, they toss a coin and play (x_1, y_2) or (y_1, x_2).
- If only 1 signs, player 1 plays y_1
- If only 2 signs, player 2 plays y_2
- If neither signs, business as usual with only contract a_1

Now

$S_1 = \{x_1, y_1, a_1, b_1\}$; $S_2 = \{x_2, y_2, a_2, b_2\}$

	x_2	y_2	a_2	b_2
x_1	2,2	0,6	0,6	0,6
y_1	6,0	1,1	1,1	1,1
a_1	6,0	1,1	2,2	1,1
b_1	6,0	1,1	1,1	3,3

- (b_1, b_2) is a PSNE with payoffs $(3,3)$

- (b_1, b_2) is a PSNE with payoffs $(3,3)$
- (a_1, a_2) is a PSNE with payoffs $(2,2)$
- (y_1, y_2) is a PSNE with payoffs $(1,1)$
- $\left(\left(0,0,\frac{2}{3},\frac{1}{3}\right), \left(0,0,\frac{2}{3},\frac{1}{3}\right)\right)$ is a MSNE with payoffs $\left(\frac{5}{3}, \frac{5}{3}\right)$

Each contract can be represented by correlated strategies played by all coalitions.

Contract 1: $\tau = (\tau_1, \tau_2, \tau_{12})$

$\tau_1 : (x_1 : 0 ; y_1 : 1)$

$\tau_2 : (x_2 : 0 ; y_2 : 1)$

$\tau_{12} : ((x_1, x_2) : 1 ; (x_1, y_2) : 0 ; (y_1, x_2) : 0 ; (y_1, y_2) : 0)$

Contract 2: $\tau = (\tau_1, \tau_2, \tau_{12})$

$\tau_1 : (x_1 : 0 ; y_1 : 1)$

$\tau_2 : (x_2 : 0 ; y_2 : 1)$

$\tau_{12} : ((x_1, x_2) : 0 ; (x_1, y_2) : \frac{1}{2} ; (y_1, x_2) : \frac{1}{2} ; (y_1, y_2) : 0)$

A contract can therefore be defined as

$$\tau = (\tau_C)_{\substack{C \subseteq N \\ C \neq \emptyset}}$$

$\tau_C \in \Delta(S_C)$ is the correlated strategy that would be implemented if only the players in C have signed the contract.

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Contract Signing Game

A game in which signing a contract is included as an additional strategy.

Q: Which contracts would all the players be willing to sign?

Guess: When $(a_1, a_2, \ldots, a_n)$ is a NE of the contract signing game.

Q: When will $(a_1, a_2, \ldots, a_n)$ be a NE?

Individually Rational Correlated Strategy

$\alpha \in \Delta(S)$ is said to be IR if

$u_i(\alpha) \geq v_i \quad \forall\, i \in N$ where

v_i is an acceptable lower bound.

maxmin value of player i

minmax value of player i

$$\underline{v_i} = \max_{\tau_i \in \Delta(S_i)} \min_{\tau_{-i} \in \Delta(S_{-i})} u_i(\tau_i, \tau_{-i})$$

$$\overline{v_i} = \min_{\tau_{-i} \in \Delta(S_{-i})} \max_{\tau_i \in \Delta(S_i)} u_i(\tau_i, \tau_{-i})$$

It can be shown that $\underline{v_i} = \overline{v_i}$ $(= v_i)$

In fact,

$$\underline{v_i} = \max_{\tau_i \in \Delta(S_i)} \min_{\underline{s_i} \in \underline{S_i}} u_i(\tau_i, \underline{s_i})$$

$$\overline{v_i} = \min_{\underline{\tau_i} \in \Delta(\underline{S_i})} \max_{s_i \in S_i} u_i(s_i, \underline{\tau_i})$$

We now prove two results which establish the connection between IR of correlated strategies and NE of contract signing games.

Result 1:

Suppose α is IR. Define contract $\tau = (\tau_C)_{C \subseteq N}$:

$\tau_N = \alpha$

$\underline{\tau_i}$ = minmax correlated strategy of $-i$ against player i

τ_C for any other C — arbitrary

Suppose a_i denotes that player i signs contract.

$$u_i(a_1, a_2, \ldots, a_n) = u_i(\alpha) \geq v_i$$

Consider

$$u_i(s_i, \underline{a_i}) = u_i(s_i, \underline{\tau_i}) \leq v_i$$

since $\underline{\tau_i}$ is minmax strategy against player i

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Thus $u_i(a_i, \underline{a}_i) \geq u_i(s_i, \underline{a}_i)$ $\forall s_i \in S_i$ $\forall i \in N$

Hence $(a_1, a_2, \ldots, a_n)$ is a PSNE.

We have shown: If α is an IR correlated strategy, we can find a contract such that all players signing the contract is a NE. Players would be willing to sign such a contract.

Result 2

Suppose we are given a contract τ such that $(a_1, a_2, \ldots, a_n)$ is a NE of the contract signing game. Can

$$u_i(a_1, a_2, \ldots, a_n) < v_i \quad \text{for any } i \in N \,?$$

Suppose the above is true. Then choose the maxmin strategy (say s_i) for player i. Since v_i is the maxmin value of the player i,

$$u_i(s_i, \underline{a}_i) \geq v_i$$

However $(a_i, \underline{a}_i)$ being a NE,

$$u_i(a_i, \underline{a}_i) \geq u_i(s_i, \underline{a}_i) \geq v_i$$

contradiction!

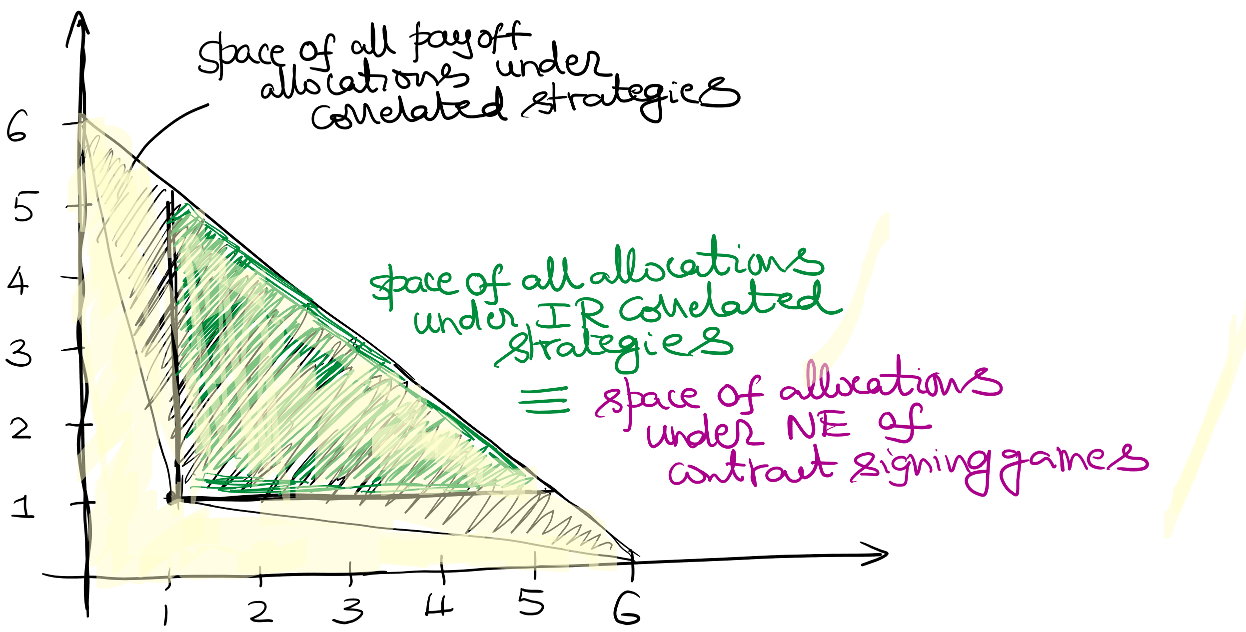
Therefore

$$u_i(a_1, a_2, \ldots, a_n) \geq v_i \quad \forall i \in N$$

The correlated strategy corresponding to the contract is

The correlated strategy corresponding to all players signing the contract is therefore IR.

We have shown: Given a contract $\tau = (\tau_C)_{C \subseteq N}$, if all players signing the contract $(a_1, a_2, \ldots, a_n)$ is a NE, then τ_N is an IR correlated strategy.



Both the spaces above are

- convex
- closed
- bounded

Correlated Equilibrium

This is a correlated strategy $\alpha \in \Delta(S)$ which when proposed by a trusted mediator is acceptable to all the (rational) players — the players are happy to play the strategy recommended by the mediator.

happy ... by the mediator.

Recall that PSNE (MSNE) is a pure strategy profile (mixed strategy profile) which when proposed by a trusted mediator is acceptable to all the (rational) players.

Example of a Game with Communication

1 \ 2	x_2	y_2
x_1	5, 1	0, 0
y_1	4, 4	1, 5

The above game has three NE.

- (x_1, x_2) yielding payoffs $(5, 1)$
- (y_1, y_2) yielding payoffs $(1, 5)$
- $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ yielding $\left(\frac{5}{2}, \frac{5}{2}\right)$

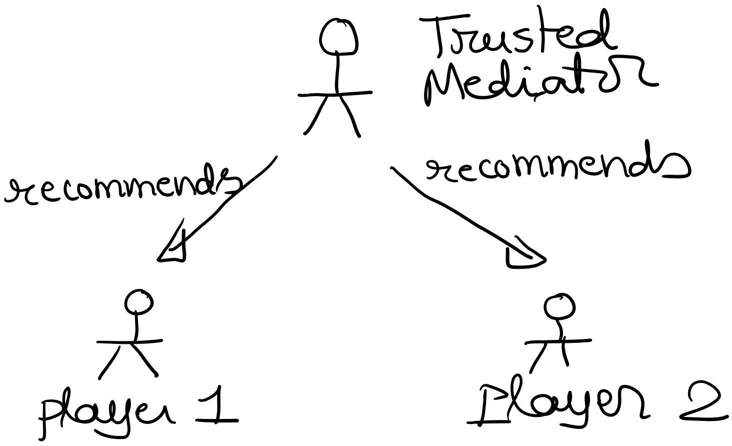
(y_1, x_2) is desirable but not a NE.
Can be realized using a contract.
If contracts are not allowed, what do we do?

Can we recommend a correlated strategy which yields good payoffs and which the players are willing to follow under assumptions of rationality and intelligence?

Correlated Strategy 1

$$\alpha = \left((x_1, x_2) : \tfrac{1}{2} ; (x_1, y_2) : 0 ; (y_1, x_2) : 0 ; (y_1, y_2) : \tfrac{1}{2} \right)$$

Here $u_1(\alpha) = u_2(\alpha) = 3$



Mediator recommends (x_1, x_2) or (y_1, y_2) with equal probability.

this correlated strategy is known to the players.

Player 1's Logic:

x_1 is recommended $\rightarrow$ thinks 2 is recommended x_2 $\rightarrow$ plays x_1 thinking 2 plays x_2

y_1 is recommended $\rightarrow$ thinks 2 is recommended y_2 $\rightarrow$ plays y_1 thinking 2 plays y_2

Player 2's Logic:

x_2 recommended $\rightarrow$ thinks 1 is recommended x_1 $\rightarrow$ plays x_2

$y_2 \longrightarrow y_1 \longrightarrow y_2$

Correlated Strategy 2:

Correlated Strategy 2 :

$$\alpha = \left((x_1, x_2) : \tfrac{1}{3} ; (x_1, y_2) : 0 ; (y_1, x_2) : \tfrac{1}{3} ; (y_1, y_2) : \tfrac{1}{3}\right)$$

$$u_1(\alpha) = u_2(\alpha) = \frac{10}{3}$$

Player 1's Logic :

$x_1 \longrightarrow$ 1 thinks 2 is recommended x_2
$\longrightarrow$ 1 plays x_1 thinking 2 plays x_2

$y_1 \longrightarrow$ 1 thinks 2 is recommended x_2 with prob $= \frac{1}{2}$ and y_2 with prob $= \frac{1}{2}$
$\longrightarrow$ 1 thinks 2 will play a mixed strategy $(x_2 : \frac{1}{2} ; y_2 : \frac{1}{2})$; computes payoff as $\frac{5}{2}$ by playing x_1 and again $\frac{5}{2}$ by playing y_1.
So happy to play y_1.

Player 2's Logic :

$x_2 \longrightarrow$ 2 thinks 1 will play a mixed strategy $(x_1 : \frac{1}{2} ; y_1 : \frac{1}{2})$; computes payoff as $\frac{5}{2}$ by playing x_2 and again $\frac{5}{2}$ by playing y_2.
So happy to play x_2

$y_2 \longrightarrow y_1 \longrightarrow$ happy to play y_2

Correlated Strategy 3

$$\alpha = \left((x_1, x_2) : 0 ; (x_1, y_2) : 0 ; (y_1, x_2) : 1 ; (y_1, y_2) : 0\right)$$

$$u_1(\alpha) = u_2(\alpha) = 4$$

Player 1's logic :

Player 1's logic :

$y_1 \longrightarrow$ 1 knows 2 is recommended x_2
$\longrightarrow$ 1 plays x_1 since x_1 fetches 5 while y_1 fetches 4

this means player 1 does not obey the mediator's recommendation.

Player 2's logic :

$x_2 \longrightarrow$ 2 knows 1 is recommended y_1
$\longrightarrow$ 2 plays y_2 since y_2 fetches 5 and x_2 fetches 4

this means player 2 does not obey.

Conclusion

Correlated strategies 1 and 2 are "correlated equilibria"

Correlated strategy 3 is not a correlated equilibrium.

Setup for Correlated Equilibrium

$$\Gamma = \langle N, (S_i), (u_i) \rangle$$

Mediator announces to all players a correlated strategy $\alpha \in \Delta(S)$ (common knowledge)

Mediator picks a pure strategy profile $s \in S$. Reveals to each player $i \in N$, the strategy s_i but not s_{-i}.

Player i either obeys to play strategy s_i or chooses any other strategy according to is chooses any other strategy according to

$$\delta_i : S_i \to S_i$$

$\delta_i(s_i)$ = strategy chosen by player i when the mediator recommends s_i

$\delta_i(s_i) = s_i$ means player i "obeys"

$\alpha \in \Delta(S)$ is called a correlated equilibrium if

$$u_i(\alpha) \geq \sum_{(s_i, \underline{s}_{-i}) \in S} \alpha(s_i, \underline{s}_{-i})\, u_i(\delta_i(s_i), \underline{s}_{-i})$$

$$\forall \delta_i : S \to S \quad \forall i \in N \quad \ldots \circledast$$

Note: A correlated equilibrium is any correlated strategy which can be implemented self-enforcingly by a mediator who makes non-binding recommendations.

$\circledast$ is equivalent to:

$$\sum_{\underline{s}_{-i} \in \underline{S}_{-i}} \alpha(s_i, \underline{s}_{-i}) \left\{ u_i(s_i, \underline{s}_{-i}) - u_i(s_i', \underline{s}_{-i}) \right\} \geq 0$$

$$\forall s_i \in S_i \quad \forall s_i' \in S_i \quad \forall i \in N$$

$$\ldots \circledast'$$

Note that a correlated equilibrium α is a feasible solution of the following LP:

max 1 subject to

$\circledast'$

$$\alpha(s) \geq 0 \quad \forall s \in S$$

$$\sum_{s \in S} \alpha(s) = 1$$

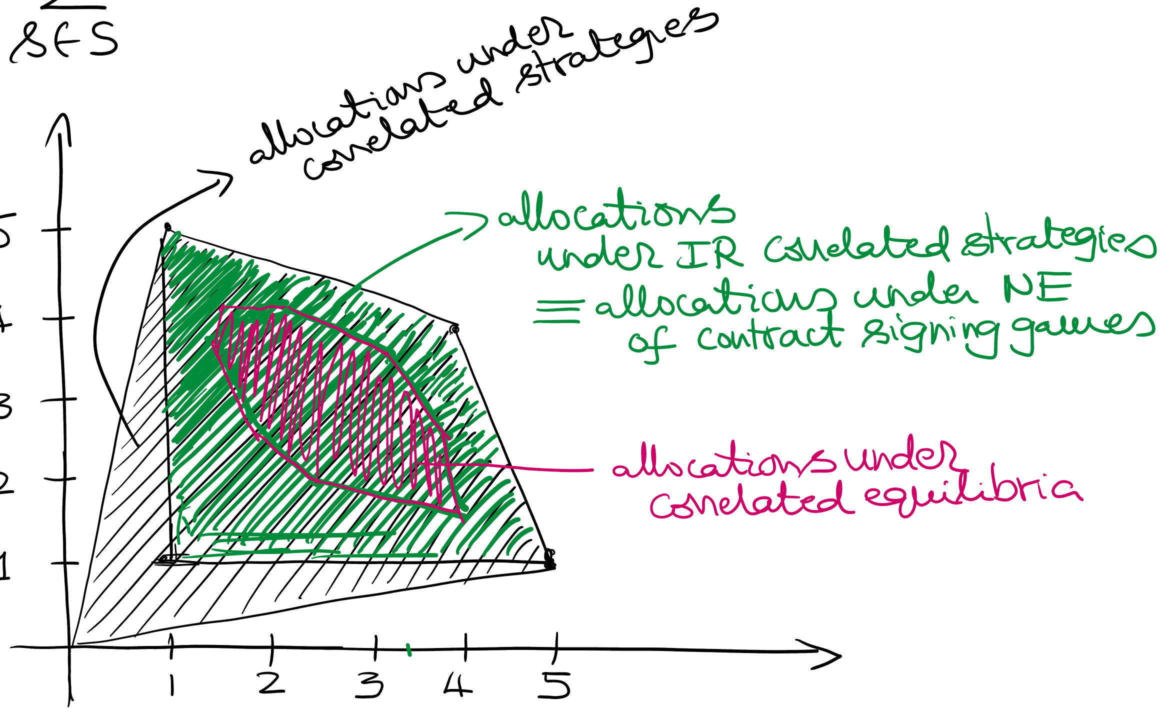
If we wish to find a welfare maximizing correlated equilibrium, we need to solve

$$\max \sum_{i \in N} u_i(\alpha) \quad \text{subject to}$$

$\circledast'$;

$$\alpha(s) \geq 0 \quad \forall s \in S$$

$$\sum_{s \in S} \alpha(s) = 1$$



All the above sets are convex, closed, and bounded.

Cooperative game theory looks at all these possibilities and comes up with solution concepts that describe logical final outcomes of a cooperative game.