Pairwise Ranking via Stable Committee Selection

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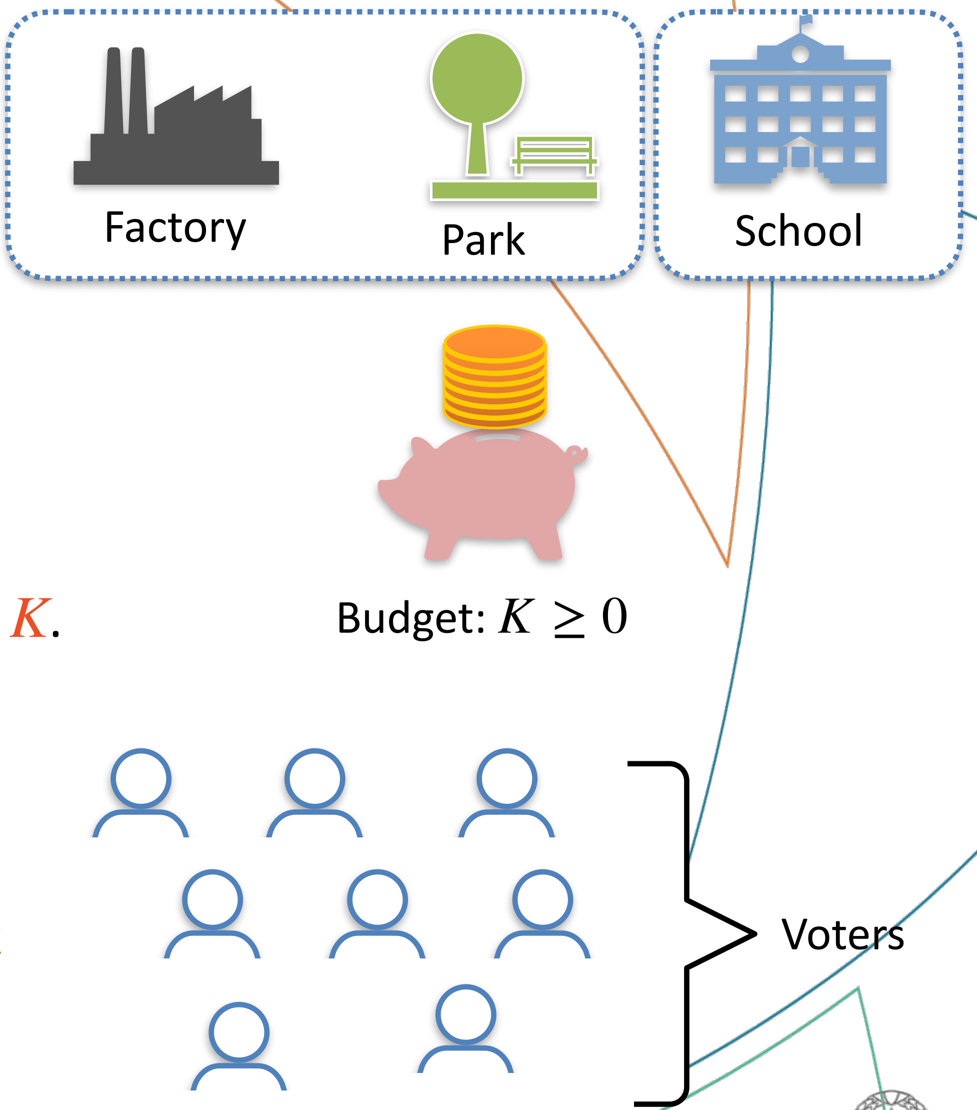
Committee Selection

- Voters: $V = \{1,2,\ldots,n\}$.
- Candidates: $C = \{1,2,\ldots,m\}$.
- Each candidate $i \in C$ is associated with a non-zero weight s_i.
- A committee is a subset of candidates.
- The weight of the committee is $w(S) = \sum_{i \in S} s_i$.
- Goal: Given a weight limit $K \geq 0$, find a committee of weight $\leq K$.
- The voters have preference over committees.
  - Example: {School, Park} $\succ$ {Factory}
- Satisfies monotonicity: $S \succeq_v S'$ if $S \supseteq S'$
- **Applications to:** Multi-winner voting, Participatory budgeting, network design etc.



STABLE COMMITTEE

- Every demographic of voters should feel that they have been fairly represented. They should not deviate and choose their own committee of proportionally smaller size.
- Given two committees $S_1, S_2 \subseteq C$, the pairwise score of S_2 over S_1 is the number of voters who strictly prefer S_2 to S_1:

$$V(S_1, S_2) := |\{v \in V : S_2 \succ_v S_1\}|$$

- Given a committee $S \subseteq C$ of weight at most K, we say that the committee S' blocks S if,

$$\frac{V(S, S')}{n} \geq \frac{w(S')}{K}$$

- Committee S is stable if there is no blocking S'.

Non Existence [Cheng et al 2020]

- Y. Cheng, Z. Jiang, K. Munagala, and K. Wang. Group fairness in committee selection. *ACM Trans. Econ. Comput.*, 8(4), Oct. 2020.
- 6 voters and 6 candidates, $K = 3$.
- Preference model given by Ranking.

Voters	Ranking
1	$a \succ b \succ c \succ d \succeq e \succeq f$
2	$b \succ c \succ a \succ d \succeq e \succeq f$
3	$c \succ a \succ b \succ d \succeq e \succeq f$
4	$d \succ e \succ f \succ a \succeq b \succeq c$
5	$e \succ f \succ d \succ a \succeq b \succeq c$
6	$f \succ d \succ e \succ a \succeq b \succeq c$

Not Stable

- W.l.o.g., choose $S = \{a, d, e\}$.
- Then voters 2 and 3 are not satisfied.
- Can deviate and choose $S' = \{c\}$, then,

$$2 = V(S, S') = \frac{K'}{K} \cdot n = \frac{1}{3} \cdot 6 = 2.$$

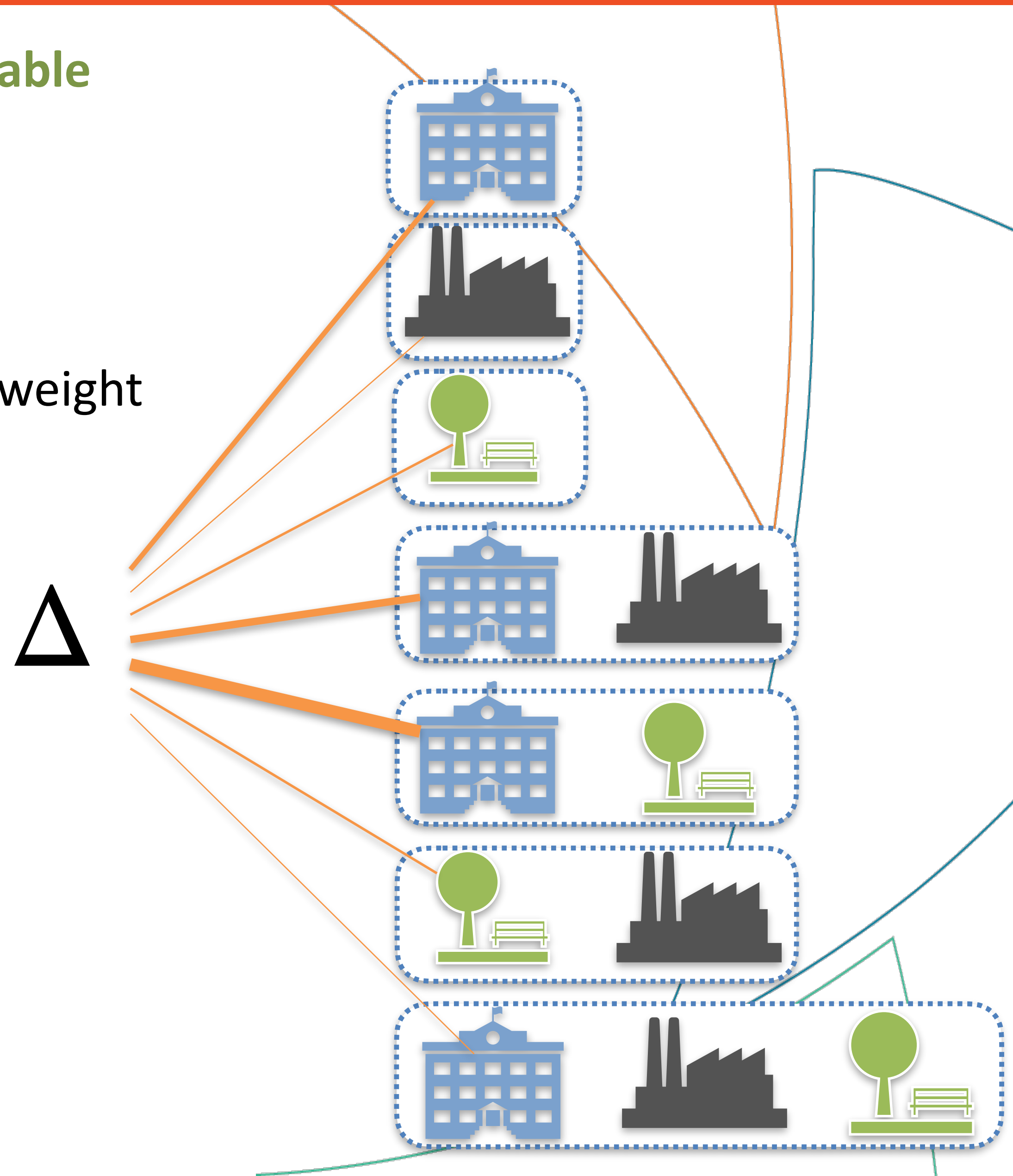
$\implies$ $\{c\}$ blocks $\{a, d, e\}$

Key Idea [Jiang et al]: Randomised Committee

- Z. Jiang, K. Munagala, and K. Wang. **Approximately stable committee selection**, STOC 2020.
- STABLE LOTTERY: Nothing but a randomised stable committee.
  - A distribution (or lottery) Δ over committees of weight at most K is stable if for all committees $S' \subseteq C$:

$$\mathbf{E}_{S\sim\Delta}\left[\frac{V(S,S')}{n}\right] < \frac{w(S')}{K}$$

Factory Park School Budget: $K \geq 0$



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Approximate Stability

- A committee $S \subseteq C$ of weight at most K is c-approximately stable if for all committees $S' \subseteq C$

$$\frac{V(S, S')}{n} < c \cdot \frac{w(S')}{K}$$

- A lottery Δ over the committees of weight at most K is c-approximately stable if for all committees $S' \subseteq C$

$$\mathbf{E}\left[\frac{V(S, S')}{n}\right] < c \cdot \frac{w(S')}{K}$$

- **Main Theorem:** A 32-approximately stable committee always exists.
- **Lemma:** A 2-approximately stable lottery always exists.
- **Proof Idea of Main Theorem:**
  1. Construct a 2-approximately stable lottery.
  2. Construct a deterministic solution from the lottery found in step 1, using iterated rounding.

Duality View [Cheng et. al., 2020]

- Stability as a two player zero-sum game.
  - **Players:** { Attacker (a), Defender (d) }.
  - **Strategies**: Pure: Committees S_a and S_d Mixed: Lotteries Δ_a and Δ_d
  - **Utilities:** $$u_a(S_a, S_d) = V(S_d, S_a) - c \cdot \frac{w(S_a)}{K} \cdot n = -u_d(S_a, S_d)\,.$$
  - **Defender plays first:** $$\min_{\Delta_d} \max_{S_a} \mathbf{E}_{S_d \sim \Delta_d}\left[V(S_d, S_a) - c \cdot \frac{w(S_a)}{K} \cdot n\right] < 0$$
  - Because of the duality of the zero-sum games, this is equivalent to the following.
  - **Attacker plays first:** $$\max_{\Delta_a} \min_{S_d : w(S_d) \le K} \mathbf{E}_{S_a \sim \Delta_a}\left[V(S_d, S_a) - c \cdot \frac{w(S_a)}{K} \cdot n\right] < 0$$
  - **Lemma:** A 2-approximately stable lottery always exists.
  - **Proof Idea:** Straight forward using randomised dependent rounding.

Stable committee exists if this is feasible

From Randomised to Deterministic

- For any instance, we have a 2-approximately stable randomised committee Δ.
- Let x_i be the probability that Δ realises S_i.
- Then for a voter v, the committee S is good if S is within the top 75% of her preferences measured by Δ

$$\mathcal{G}_v(\Delta) = \left\{ S \subseteq C : \sum_{S_i \succeq_v S} x_i \leq 1 - \beta \right\} \quad \text{and} \quad \mathcal{B}_v(\Delta) = \left\{ S \subseteq C : \sum_{S_i \preceq_v S} x_i \leq \beta \right\}$$

Good committees

Bad committees

- **Properties of good committees:**

1. Any committee S_a is considered good by no more than $8 \cdot \dfrac{w(S_a)}{K} \cdot n$ voters.
2. There is a committee S_d considered good by at least $0.75n$ voters.

Recursion

- **Properties of good committees:**

1. Any committee S_a is considered good by no more than $8 \cdot \dfrac{w(S_a)}{K} \cdot n$ voters.
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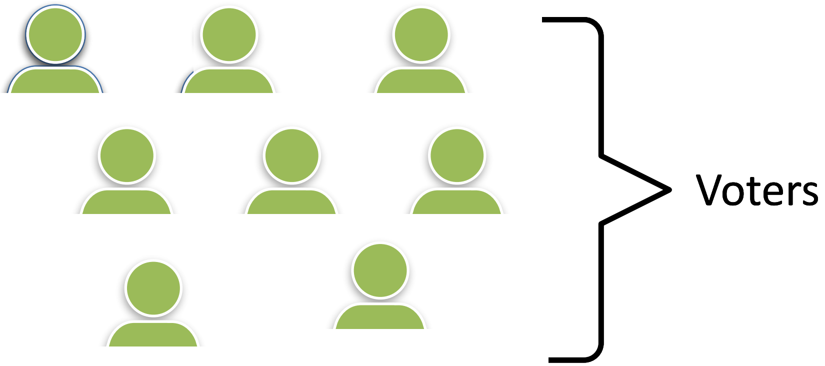


<u>Selected committee</u>

ALGORITHM: Iterated Rounding

1. $t \leftarrow 0; V^{(0)} \leftarrow [n]; T^{(0)} \leftarrow \phi; K^{(0)} \leftarrow (1-\alpha)K$.
2. While $V^{(t)} \neq \phi$ do
   1. $\Delta^{(t)} \leftarrow$ Lottery$(V^{(t)}, K^{(t)})$ — Gives 2-approximately stable lottery
   2. Let $S^{(t)}$ be any committee such that $\left|\left\{v \in V^{(t)} : S^{(t)} \notin \mathscr{B}_v\left(\Delta^{(t)}\right)\right\}\right| \geq (1-\beta) \cdot |V^{(t)}|$. — Not a bad committee for majority
   3. $W^{(t)} \leftarrow \left\{v \in V^{(t)} \mid S^{(t)} \notin \mathscr{B}_v\left(\Delta^{(t)}\right)\right\}$
   4. $V^{(t+1)} \leftarrow V^{(t)} \backslash W^{(t)}$ — Remove the users that are satisfied
   5. $T^{(t+1)} \leftarrow T^{(t)} \cup S^{(t)}$ — Add committee to the final output
   6. $K^{(t+1)} \leftarrow \alpha K^{(t)}$ — Decrease the budget
   7. $t \leftarrow t+1$
3. Return $T^f \leftarrow T^{(t)}$

Open Questions

1. Deterministic exact committees in more specific settings — **approval voting.**
2. Do exactly stable lotteries exist?
3. Closing the gap.
    - [2,32].
4. Computing stable committees efficiently.
    - Currently the running time is exponential in the number of committees.

Other Problems

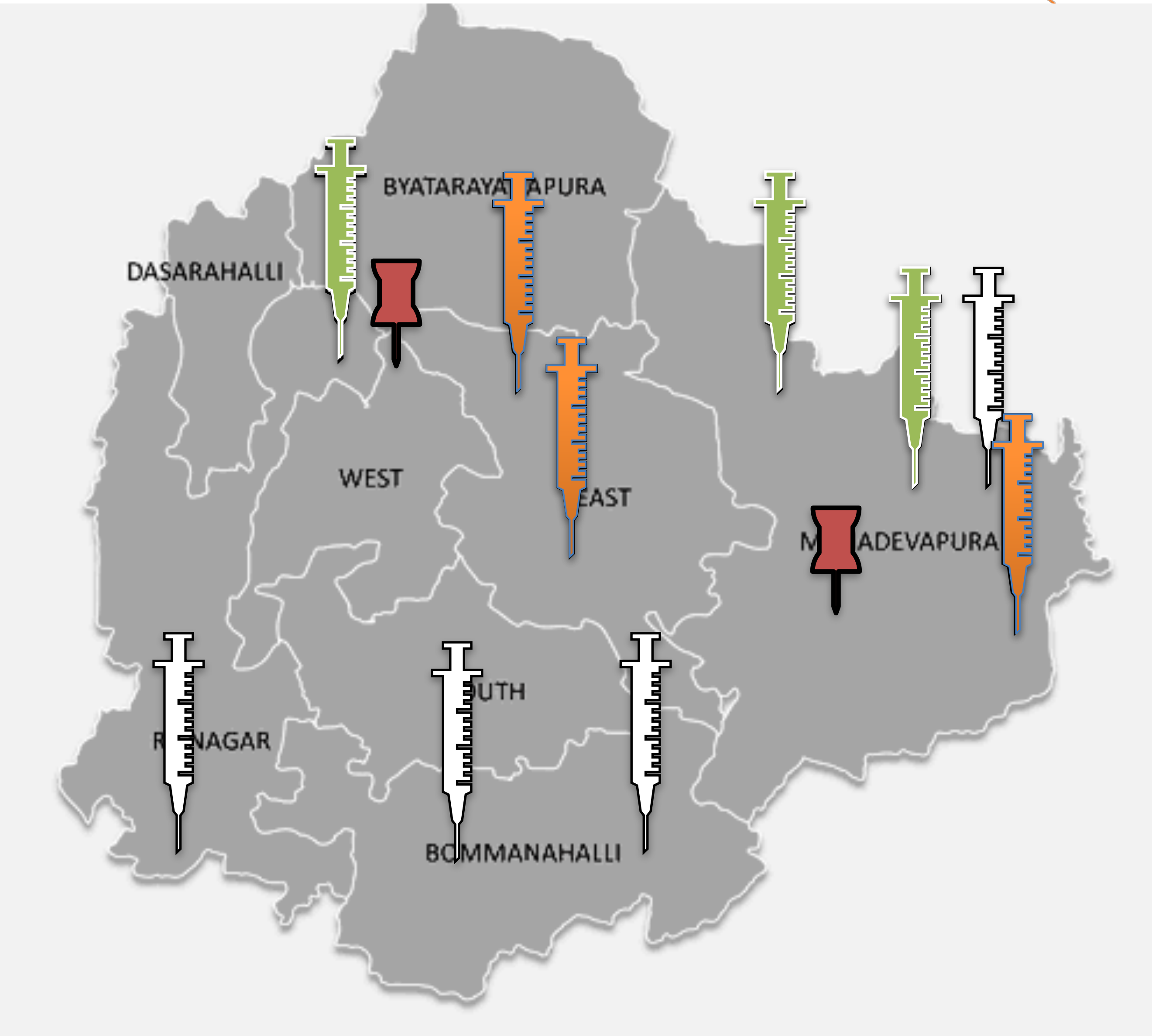
Problem	Preference Model
Ranking	Voter has a ranking over the candidates. Prefers committees in which top-1 candidate is ranked higher.
Approval Voting	Voter approves a subset of candidates. Prefers committees with more number of approved candidates.
Participatory Budgeting	Voter has a utility for every candidate. Prefers the committee with higher utility.
Facility Location	Voter has a distance metric over the candidates. Prefers the committee with closest distance candidate.

My Ideas: PAIRWISE RANKING

Which committee would you prefer?

Orange or Green?

In Ranking, the green committee is preferred over the orange committee.



In Pairwise Ranking, I propose to look at the number of pairwise winners.

So, orange committee Is preferred.

My Ideas: PAIRWISE RANKING

- [Def] **Pairwise majority Score:** Let each voter v have a ranking $\sqsupset_v$ over all the m candidates in the set $[m]$. Let $S, S' \in 2^{[m]}$ be any two committees. Let $k = \min\{\,|S|, |S'|\,\}$. Then for voter v, the pairwise majority score function $h_v : 2^{[m]} \times 2^{[m]} \to \mathbf{Z}_{\geq 0}$ is given by the following,

$$h_v(S, S') = \sum_{i \in S_k} \sum_{j \in S'_k} \mathbf{I}\left[i \sqsupset_v j\right]$$

Here S_k is the top k candidates in S

- [Def] Pairwise Ranking: Each candidate has unit weight. Each voter v has a a ranking over candidates,

$$S \geq_v S' \iff h_v(S, S') \geq h_v(S', S)$$

Pairwise Ranking is Monotone

- **Lemma 1:** The preference model in Pairwise Ranking is complete.
- **Proof:** Fix a voter v. For $S, S' \in 2^{[m]}$ either $h_v(S, S') \geq h_v(S', S)$ or $h_v(S', S) \geq h_v(S, S')$. Therefore, either $S \succeq_v S'$ or $S' \succeq_v S$. ■

- **Lemma 2:** The preference model in Pairwise Ranking is monotone.
- **Proof:** Fix a voter v. For $S, T \in 2^{[m]}$ such that $T \supseteq S$. Then,
  - $\forall k \in \{1, \ldots, |S|\}, \quad i \in T_k \backslash T_{k-1}, \quad j \in S_k \backslash S_{k-1},$
    - either $\quad i = j \quad$ or $\quad i \sqsupset_v j \quad \implies h_v(T, S) > h_v(S, T) \implies T \succ_v S.$
  - It is easy to see that this is always true because otherwise, $\exists k \in \{1, \ldots, |S|\}$ such that $i \in T_k \backslash T_{k-1}, \quad j \in S_k \backslash S_{k-1},$ and $i \sqsubset_v j,$
  - $\implies j \notin T_k \backslash T_{k-1} \implies S \not\subseteq T$ (Contradiction). ■

Approximate Pairwise Ranking

- **Theorem:** For the preference model given by Pairwise Ranking with n voters and m candidates, unit weights and the cost-threshold $K \geq 0$, a 32-approximately stable committee of weight at most K always exists.
- **Proof:** Follows by Lemma 1, Lemma 2, and the main theorem of Jiang et al. Monotonicity is key. ∎

CONCLUSIONS:

1. Pairwise Ranking gives a more practical preference model.
2. A 32-approximate stable committee for pairwise ranking always exits.
3. Interesting open direction to close the gap [2,32].

Thank you!