Indian Institute of Science Banglore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Assignment 5 [Posted Nov 15, 2022]

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Strict Submission Deadline: Nov 25 2022 23:59 hrs | Max Points: 100

Notations: Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively.

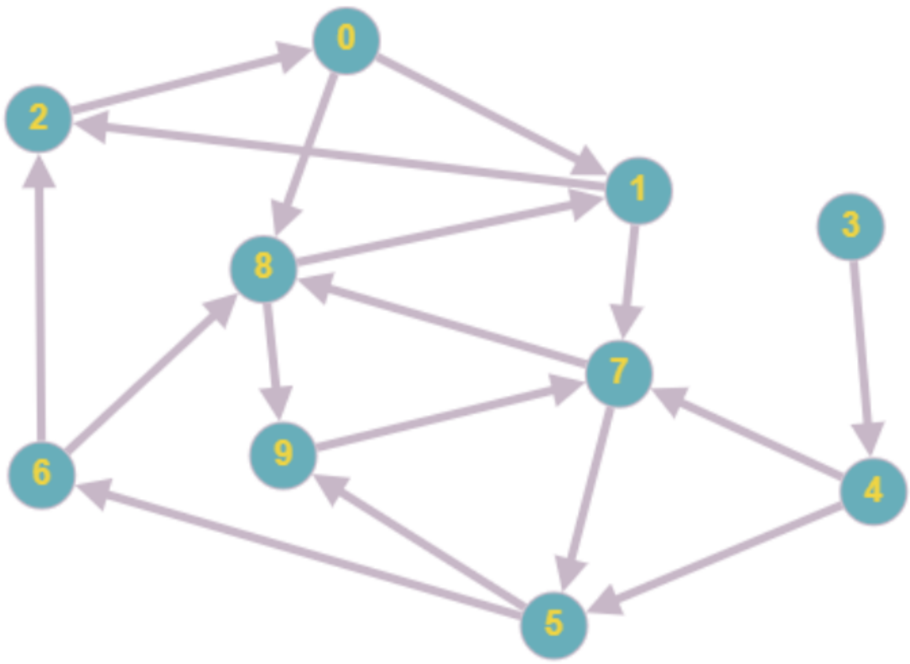
Problem 1 — 4 x 3 = 12 Marks

This question will help you appreciate the usefulness of power iteration in a real world application. In this question, you need to calculate the Page rank of each of the nodes of the given graph by first constructing its Markov transition matrix and subsequently, calculating the most dominant eigenvector of this transition matrix i.e the eigenvector corresponding to the largest eigenvalue. Use the power iteration approach discussed in the class to compute the dominant eigenvector.

Justify your result showing the following data/figures:

- Plot of the 2-norm of the residual of eigenvalue problem involving the dominant eigenvector with iteration number. (Use the unit vector generated at the end of each iteration in your power iteration algorithm to compute the eigenvalue problem residual at each iteration)
- Plot of the 2-norm of the difference between the vectors corresponding to the successive iterates of your power iteration with number of iterations.
- Compute the Rayleigh quotient at each iteration of your algorithm and plot the convergence with respect to iteration number.
- State the node numbers with least and highest page ranks.

Refer to the document attached with the assignment for description about building the Markov Transition matrix of a given graph.



Problem 2 **[4 x 3 = 12 marks]**

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) Pure QR algorithm is equivalent to the Simultaneous iteration applied to an initial guess of vectors which are columns of a square full rank matrix.

(b) Pure QR algorithm used to solve the eigenvalues and eigenvectors of a matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ generates a sequence of matrices $\mathbf{Q}^{(k)}$ which converges to the eigenvector matrix of $\mathbf{A}$ as $k \to \infty$.

(c) Pure QR algorithm used to solve the eigenvalues and eigenvectors of a matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ generates QR factorization to be $\mathbf{A}^k = \mathbf{Q}^{(k)}\mathbf{R}^{(k)}$ generated at the k^{th} iteration of the algorithm.

(d) Computational complexity for finding the eigenvectors using Pure QR algorithm and the Simultaneous iteration is same.

Problem 3 **[4 x 3 = 12 marks]**

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) An eigenvalue solver can be designed to compute eigenvalues and eigenvectors of a given matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ in a finite number of steps, using exact arithmetic.

(b) An eigenvalue solver designed to compute all eigenvalues and eigenvectors of a symmetric dense matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ requires at most $O(m^3)$ work, if it is not initially reduced to tri-diagonal form in Phase 1.

(c) Power iteration produces a sequence of vectors $\mathbf{v}^{(i)}$ that converges to the eigenvector corresponding to the largest eigenvalue of $\mathbf{A} \in \mathbb{R}^{m\times m}$ starting with any initial guess vector $\mathbf{v}^{(0)} \neq \mathbf{0}$.

(d) Let $\mathbf{F} \in \mathbb{R}^{m\times m}$ denote the Householder reflector that introduces zeros below the diagonal entry of symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ in the 1st column when pre-multiplied with $\mathbf{A}$. Then eigenvalues of $\mathbf{FAF}^T$ and $\mathbf{A}$ are the same.

Problem 4 [3+4+8+3 =18 points]

Consider a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$. Answer the following questions:

(a) Show that the singular values of $\mathbf{A}$ are absolute values of eigenvalues of $\mathbf{A}$. What can you say about the vector induced matrix norm $\|\mathbf{A}\|_2$ in terms of eigenvalues of $\mathbf{A}$? Support your argument.

(b) Show that $|\mathbf{x}^T\mathbf{A}\mathbf{x}| \leqslant \|\mathbf{A}\|_2$ for any non-zero unit vector $\mathbf{x} \in \mathbb{R}^m$.

(c) Let the vector $\mathbf{u} \in \mathbb{R}^m$ be an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue λ i.e. $\mathbf{Au} = \lambda\mathbf{u}$. Further, let the matrix $\mathbf{A}$ undergo a symmetric matrix perturbation by $\delta\mathbf{A}$ such that $\dfrac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = O(\epsilon_{mach})$. Let, $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ and $\tilde{\lambda} = \lambda + \delta\lambda$ be the eigenvector-eigenvalue pair of the perturbed matrix $\tilde{\mathbf{A}} = \mathbf{A} + \delta\mathbf{A}$. Now, show that

$$|\delta\lambda| \leqslant \|\delta\mathbf{A}\|_2$$

(d) Deduce the relative condition number for the problem of computing the eigenvalue λ of our symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ using the inequality derived in part (c).

Problem 5 [4 x 3 = 12 marks]

Assert if the following statements are True or False. Give a detailed reasoning for your assertion. Marks will be awarded only for your reasoning.

(a) Let, $\mathbf{M} \in \mathbb{R}^{n\times n}$ be a non-zero matrix with $n \geqslant 2$ and $\mathbf{M}^2 = \mathbf{0}$ where, $\mathbf{0}$ is a null matrix. Let, $\alpha \in \mathbb{R}\backslash\{0\}$ then, the algebraic multiplicity of the eigenvalue α for $(\alpha\mathbf{I} - \mathbf{M})$ is n.

(b) Let, $\mathbf{M} \in \mathbb{R}^{n\times n}$ be a non-zero matrix with $n \geqslant 2$ and $\mathbf{M}^2 = \mathbf{0}$ where, $\mathbf{0}$ is a null matrix. Let, $\alpha \in \mathbb{R}\backslash\{0\}$ then, $-\alpha$ is one of the k distinct eigenvalues of $(\mathbf{M} - \alpha\mathbf{I})$ with $k > 1$.

(c) If λ is an eigenvalue with both algebraic multiplicity and geometric multiplicity to be 3 then the dimension of column space of $\mathbf{A} - \lambda\mathbf{I}_m$ is 3.

(d) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be any non-zero diagonalizable matrix then rank of $\mathbf{A}$ = number of non-zero eigenvalues of $\mathbf{A}$.

Problem 6 [4 + 5 + 5 = 14 Marks]

If $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric matrix with one eigenvalue much smaller than others, i.e. if $\lambda_1, \lambda_2, ..., \lambda_m$ are eigenvalues of $\mathbf{A}$ in increasing order, then $\lambda_1 <<< \lambda_2 \leqslant \lambda_3 ... \leqslant \lambda_m$ (This means that $\mathbf{A}$ is an ill-conditioned matrix). Let the corresponding eigenvectors be the $\mathbf{q}_1, \mathbf{q}_2, ...\mathbf{q}_m$. Now answer the following questions:

(a) If the system of equations $\mathbf{Aw} = \mathbf{v}$ is solved using a backward stable algorithm for some $\mathbf{v} \in \mathbb{R}^m$ yielding a computed vector $\tilde{\mathbf{w}} = \mathbf{w} + \delta\mathbf{w}$, show that $\delta\mathbf{w} = -(\mathbf{A} + \delta\mathbf{A})^{-1}(\delta\mathbf{A})\mathbf{w}$ for some $\delta\mathbf{A}$ such that $\|\delta\mathbf{A}\| = O(\epsilon_M)\|\mathbf{A}\|$.

(b) Assuming that $\mathbf{v}$ is a vector with components in the directions of all eigenvectors $\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_m$ of $\mathbf{A}$, show that $\frac{\mathbf{w}}{\|\mathbf{w}\|} \approx \mathbf{q}_1$, where $\mathbf{w} = \mathbf{A}^{-1}\mathbf{v}$ is the exact solution of the system of equations given in (a). [**Hint**: First show that $\mathbf{w}$ is approximately in the direction of $\mathbf{q}_1$]

(c) Using the Taylor series expansion $(\mathbf{A}+\delta\mathbf{A})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\delta\mathbf{A})\mathbf{A}^{-1} + O(\epsilon_M^2)$, show that $\frac{\tilde{\mathbf{w}}}{\|\tilde{\mathbf{w}}\|} \approx \mathbf{q}_1$. [**Hint**: Using the expression $\delta\mathbf{w}$ derived in (a) and the fact $\delta\mathbf{A}$ is random roundoff perturbation, show that $\delta\mathbf{w}$ is in the direction of $\mathbf{q}_1$]

[This sequence of steps in (a), (b) and (c) show that, though the computed solution $\tilde{\mathbf{w}}$ is far away from $\mathbf{w}$ for the ill-conditioned system $\mathbf{Aw} = \mathbf{v}$, $\frac{\tilde{\mathbf{w}}}{\|\tilde{\mathbf{w}}\|}$ need not be far from $\frac{\mathbf{w}}{\|\mathbf{w}\|}$.]

Problem 7 [4 + 5 + 5 + 6 = 20 Marks]

If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$, answer the following questions:

(a) Write the characteristic equation associated with the above matrix $\mathbf{A}$ and subsequently compute its eigenvalues

(b) Set up the Arnoldi iteration with the starting vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to find the orthonormal basis vectors $\{\mathbf{q}_1, \mathbf{q}_2\}$ spanning the two dimensional Krylov subspace $\mathcal{K}_2 =< \mathbf{b}, \mathbf{Ab} >$

(c) Find the orthogonal projection $\mathbf{H}$ of $\mathbf{A}$ onto $\mathcal{K}_2$ represented in the basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ obtained in (b) above, and then compute the eigenvalues of this $\mathbf{H}$ (also called Ritz values). Find the absolute error between the smallest Ritz value and the smallest eigenvalue of $\mathbf{A}$ and similarly compute the absolute error between the largest Ritz value and largest eigenvalue of $\mathbf{A}$.

(d) Consider the system of equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$ as given above and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as given in (b). Find the exact solution $\mathbf{x}^*$ which solves $\mathbf{Ax} = \mathbf{b}$

using forward substitution. Subsequently find the vector $\hat{\mathbf{x}} \in \mathcal{K}_2$ that minimizes the norm $\|\mathbf{Ac} - \mathbf{b}\|_2$ over all possible vectors $\mathbf{c} \in \mathcal{K}_2$, where $\mathcal{K}_2$ is the Krylov subspace constructed in (b). Finally, find the norm of error between exact solution $\mathbf{x}^*$ and $\hat{\mathbf{x}}$.