Lecture 2: Introduction to Games

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This course is in three parts:

- Noncooperative game theory
- Cooperative game theory
- Mechanism Design

Game — refers to an interaction involving decision makers (players) (agents) who are rational and intelligent.

Game Theory = (Equilibrium) analysis of games

Mechanism Design = design of games having a certain (equilibrium) behaviour

Strategic Form Game

$$\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

$N = \{1, 2, \ldots, n\}$ Set of players

S_i = Action set or strategy set of player i where $i \in N$

$u_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$

utility function or payoff function of player $i \in N$

We can have infinite number of players and each player could have (possibly uncountably) infinite number of strategies

Finite (strategic form) game

n is finite and each S_i ($i \in N$) is finite

Example 1: Student Coordination Game

$N = \{1, 2\}$

$S_1 = S_2 = \{\text{IISc}, \text{MG Road}\} = \{A, B\}$

Payoff Matrix : $S_1 \times S_2 \longrightarrow \mathbb{R}$

1 \ 2	A	B
A	100, 100	0, 0
B	0, 0	10, 10

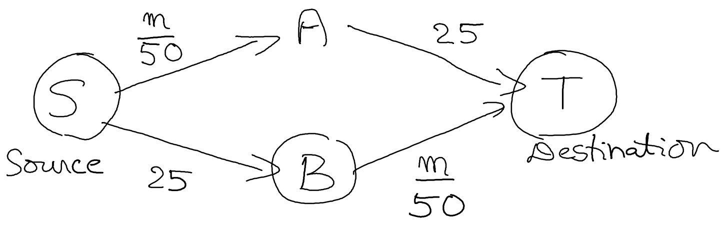
$u_1(A, A) = 100$; $u_1(A, B) = 0$; $u_1(B, A) = 0$; $u_1(B, B) = 10$

$u_2(A, A) = 100$; $u_2(A, B) = 0$; $u_2(B, A) = 0$; $u_2(B, B) = 10$

Rules of the game

(1) Players are rational
(2) Players are intelligent
(3) The payoff matrix is common knowledge
(4) Simultaneous move game

Example 2: Braess Paradox



Suppose $n = 1000$
1000 vehicles wish to move from S to T

$$\Gamma = \left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$$

$$S_1 = S_2 = \cdots = S_n = \{A, B\}$$

Suppose $(s_1, s_2, \ldots, s_n)$ is a strategy profile

$n_A(s_1, s_2, \ldots, s_n) =$ # of vehicles passing through A given the strategy profile $(s_1, s_2, \ldots, s_n)$

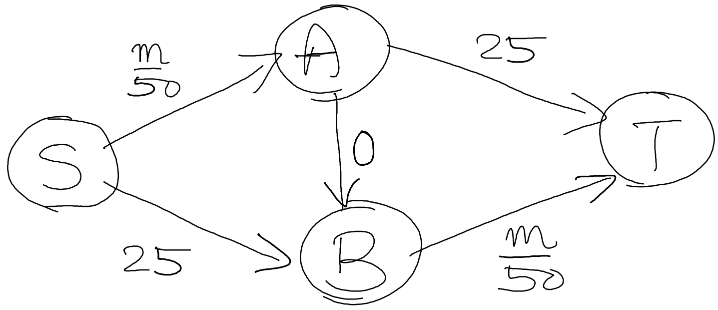
$n_B(s_1, s_2, \ldots, s_n) =$ # of vehicles passing through B

$$u_i(s_1, \ldots, s_n) = -\left(\frac{n_A(s_1, \ldots, s_n)}{50} + 25\right) \quad (s_i = A)$$
$$= -25 - \frac{n_B(s_1, \ldots, s_n)}{50} \quad (s_i = B)$$

Note that

$$u_i(A, A, \ldots, A) = -20 - 25 = -45 \quad \forall i$$
$$u_i(\underbrace{A, \ldots, A}_{500}; \underbrace{B, \ldots, B}_{500}) = -10 - 25 = -35 \quad \forall i$$

(Paradoxical) Variation of this



Now $S_i = \{A, B, AB\} \quad \forall i \in N$

$s = (s_1, \ldots, s_n)$ $\qquad n_A(s) + n_{AB}(s) \qquad (s_i = A)$

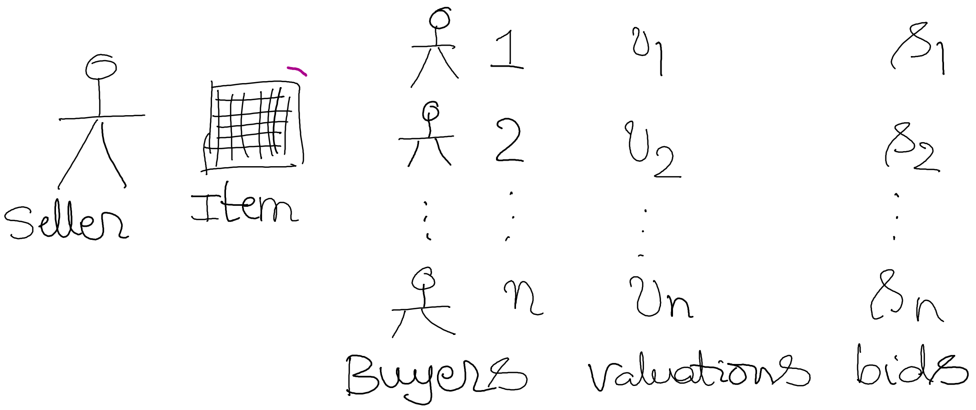
$$u_i(s_1,\ldots,s_n) = -25 - \frac{n_A(s) + n_{AB}(s)}{50} \quad (s_i = A)$$

$$= -25 - \frac{n_B(s) + n_{AB}(s)}{50} \quad (s_i = B)$$

$$= -\left(\frac{n_A(s) + n_{AB}(s)}{50}\right) - \left(\frac{n_B(s) + n_{AB}(s)}{50}\right)$$

Example 3: Sealed Bid Auction Game

Auction for a single indivisible item



$N = \{1, 2, \ldots, n\}$

$S_1 = S_2 = \cdots = S_n = (0, \infty)$

$s = (s_1, \ldots, s_n)$ strategy profile or bid profile

We need to define $u_i(s_1, \ldots, s_n)$

Suppose the highest bidder wins with ties resolved in favour of the bidder with lowest index

Auction consists of
- winner determination
- payment by the winner

Given a strategy profile, suppose

$y_i(s_1, \ldots, s_n)$ is the winning bidder and

$t_i(s_1, \ldots, s_n)$ is the payment by the bidder

For this auction mechanism,

$$y_i(s_1, \ldots, s_n) = 1 \quad \text{if } s_i > s_j \text{ for } j = 1, 2, \ldots, i-$$
$$\text{and } s_i \geq s_j \text{ for } j = i+1, \ldots,$$
$$= 0 \quad \text{else}$$

$$u_i(s_1, \ldots, s_n) = y_i(s_1, \ldots, s_n)(v_i - t_i(s_1, \ldots$$
$$\text{for } i = 1, 2, \ldots, n$$

Now we will examine

(1) Preferences and Utilities

(2) Rationality

(3) Intelligence

(4) Common knowledge

Preferences and Utilities

Let us recall

1 \ 2	A	B
A	100,100	0,0
B	0,0	10,10

The outcomes are

$S_1 \times S_2 = \{(A,A), (A,B), (B,A), (B,B)\}$

Player 1:

Prefers (A,A) to $(A,B), (B,A), (B,B)$

$(A,A) \succ (A,B)$

$(A,A) \succ (B,A)$

$(A,A) \succ (B,B)$

Prefers (B,B) to $(A,B), (B,A)$

Indifferent between (A,B) and (B,A)

Similarly Player 2

Similarly Player 2.

Preferences are reflexive, transitive, and complete in game theory.

Utilities are real numbers

Question: Can preferences be always transformed into real numbers?

von Neumann and Oskar Morgenstern showed, under weak conditions, that preferences can be mapped to real numbers in a way that is consistent with expected utility maximization.

Rationality

refers to always making decisions in pursuit of one's individual objectives

- maximize an expected value of individual utility function

"expected utility maximization"

Rationality may imply

- selfishness
- altruism
- ...

Expected utility maximization need not imply maximizing monetary returns

Intelligence

Each player knows everything about the game that a game theorist knows

All players have the competence to make informed inferences about the strategic play by the other players

Each player can compute what her

best responses are, imagining the best response strategies ~~the~~ other players are going to play.

Roger Myerson

The assumptions of rationality and intelligence are reasonable and logical.

Any theory that is (not consistent with these assumptions will lose credibility because:

If a theory predicts that some individuals will be systematically fooled into making mistakes, the theory loses validity when individuals learn to better understand situations.

Common knowledge

A fact is common knowledge if
every player knows it,
every player knows that every player knows it,
every player knows that every player knows that every player knows it,
etc...

This notion is a consequence of "Intelligence"