# Solutions to Test(2

05 March 2022 11:53

## Question 1

The matrix given is $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

The primal LP is

maximize $z$ s.t.

$$z \leq 2x_1 - 3x_2$$
$$z \leq -x_1 + 4x_2$$
$$x_1 \geq 0 \; ; \; x_2 \geq 0 \; ;$$
$$x_1 + x_2 = 1$$

Using $x_2 = 1 - x_1$, we obtain the constraints as

$$z \leq 5x_1 - 3$$
$$z \leq -5x_1 + 4$$
$$0 \leq x_1 \leq 1$$

Note that $(5x_1 - 3)$ increases with $x_1$ while $(-5x_1 + 4)$ decreases with $x_1$

Since we are trying to maximize $z$, the optimal solution is obtained when

$$5x_1 - 3 = -5x_1 + 4$$

This yields $10x_1 = 7$ & $x_1^* = \frac{7}{10}$

The optimal solution is therefore

$$\left(\frac{7}{10}, \frac{3}{10}\right) \text{ with}$$

optimal value $= 5x_1 - 3 - 2$

---

The Dual LP is

minimize $w$ subject to

$$w \geq 2y_1 - y_2$$
$$w \geq -3y_1 + 4y_2$$

$$y_1 \geq 0 \; ; \; y_2 \geq 0$$
$$y_1 + y_2 = 1$$

Using $y_2 = 1 - y_1$, we obtain the constraints

$$w \geq 3y_1 - 1$$
$$w \geq -7y_1 + 4$$
$$0 \leq y_1 \leq 1$$

Here again, we find that $3y_1 - 1$ increase with $y_1$ and $-7y_1 + 4$ decreases with $y$ and the optimal solution is obtained wh

$$3y_1 - 1 = -7y_1 + 4$$

which yields

$$y_1^* = \frac{1}{2}$$

and $y_2^* = \frac{1}{2}$

with optimal value $= 3y_1^* - 1 = \frac{1}{2}$

Thus the optimal solution of the dual is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the optimal value is $\frac{1}{2}$.

clearly, the MSNE here is

$$\left(\left(\frac{7}{10}, \frac{3}{10}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$$

---

Question 2

We are supposed to find the space of correlated equilibria of the matrix ga

$$\begin{array}{c|cc} & A & B \\ \hline A & 1,-1 & -1,1 \\ B & -1,1 & 1,-1 \end{array}$$

We know that correlated equilibria are obtained as the feasible solutions of the following set of constraints:

$$\sum_{\underline{s}_{-i} \in \underline{S}_{-i}} \alpha(s_i, \underline{s}_{-i}) \left\{ u_i(s_i, \underline{s}_{-i}) - u_i(s_i', \underline{s}_{-i}) \right\}$$

$$\forall s_i \in S_i \;\; \forall s_i' \in S_i \;\; \forall i$$

Here $N = \{1, 2\}$ $\quad S_1 = S_2 = \{A, B\}$

By systematically enumerating all the equations, we get the following four equations.

$$\alpha(A, A)(2) + \alpha(A, B)(-2) \geq 0$$

$$\alpha(A,A)(2) + \alpha(A,B)(-2) \geq 0$$
$$\alpha(B,A)(-2) + \alpha(B,B)(2) \geq 0$$
$$\alpha(A,A)(-2) + \alpha(B,A)(2) \geq 0$$
$$\alpha(A,B)(2) + \alpha(B,B)(-2) \geq 0$$

This immediately leads to

$$\alpha(A,A) \geq \alpha(A,B)$$
$$\alpha(B,B) \geq \alpha(B,A)$$
$$\alpha(B,A) \geq \alpha(A,A)$$
$$\alpha(A,B) \geq \alpha(B,B)$$

The above inequalities are satisfied iff

$$\alpha(A,A) = \alpha(A,B) = \alpha(B,A) = \alpha(B,B)$$

which means $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is the only correlated equilibrium for this game. This is consistent with the fact that $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is the unique MSNE for this game.

---

Question 3

Convexity of a TU game $(N, v)$ means

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$$

$v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$

$\forall C, D \subseteq N$

Version 2 of the game.

We know $v(12) = v(123) = 300$ with the rest of the values equal to zero.

In the above definition of convexity, the only RHS values possible are 0, 300, 6 there is nothing to show if the RHS is If the RHS is 600, the implication is th

$C = \{1, 2\}$ and $D = \{1, 2, 3\}$ or

$C = \{1, 2, 3\}$ and $D = \{1, 2\}$

In either case, LHS = RHS = 600 and the inequality holds.

In the case when RHS = 300, the meani is that exactly one of C or D is eith $\{1, 2\}$ or $\{1, 2, 3\}$ in which case the teri $v(C \cup D)$ in the LHS will be 300 anc the inequality holds.

Thus Version 2 game is convex.

Version 3

$v(12) = v(13) = v(123) = 300$

choose $C = \{1, 2\}$ $D = \{1, 3\}$

LHS = 300

$RHS = 300 + 300 = 600$

Thus Version 3 is not convex.

Version 4

$v(12) = v(13) = v(23) = v(123) = 300$

choose $C = \{1,2\}$ $D = \{1,3\}$ & $\{2,3$

and we are done. In fact there are multiple such choices.

Version 4 is also not convex.

---

Glove Market

$N_L = \{1, 2, \ldots, k\}$

$N_R = \{k+1, k+2, \ldots, 2k\}$

Suppose

$(x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}) \in Core(N$

then note that

$$\sum_{i \in N_L \cup N_R} x_i = k$$

Suppose $l \in N_L$ and $r \in N_R$. Then

$v(\{l, r\}) = 1$

$$x_l + x_r \geq v(\{l, r\}) = 1$$

Also note that

$$\sum_{\substack{i \neq l \\ i \neq r \\ i \in N_L \cup N_R}} x_i = k-1$$

Using all of the above equations, one can show that

$$x_1 = x_2 = \cdots = x_k \quad (= \alpha, \text{ say})$$
$$x_{k+1} = x_{k+2} = \cdots = x_{2k}$$

then the Core will be the set:

$$\mathbb{C}(N, v) = \{(\underbrace{\alpha, \alpha, \ldots, \alpha}_{i \in N_L}, \underbrace{1-\alpha, 1-\alpha, \ldots, 1-\alpha}_{i \in N_R}) : 0$$

Please verify the above with a simple example such as

$$N_L = \{1, 2\}$$
$$N_R = \{3, 4\}$$

then
$(0, 0, 1, 1)$, $(1, 1, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \cdot$
are all in the core of this game.

Intuitively, one can see that the Shapley Va is given by

$$\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\right)$$