Lecture 8 Two Player Zerosum Games

21 March 2021 09:55

Two Player Zerosum Games (TPZS games)
Also called Strictly Competitive Games

$\langle \{1,2\}, S_1, S_2, u_1, u_2 \rangle$ such that

$u_1(s) + u_2(s) = 0 \quad \forall s \in S_1 \times S_2$

Constant Sum Game

$u_1(s) + u_2(s) = k \quad \forall s \in S_1 \times S_2 \,;\; k \in \mathbb{R}$

Matrix Game

$S_1 = \{s_{11}, s_{12}, \ldots, s_{1m}\} = \{1, 2, \ldots, m\}$

$S_2 = \{s_{21}, s_{22}, \ldots, s_{2n}\} = \{1, 2, \ldots, n\}$

$A = [a_{ij}] = [u_1(i,j)] \quad \forall i \in S_1 \quad \forall j \in S_2$

$u_2(i,j) = -a_{ij} \quad \forall i \in S_1 \quad \forall j \in S_2$

Example 1 :

Matching Pennies

1 \ 2	1	2
1	1, −1	−1, 1
2	−1, 1	1, −1

$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

1 – Row Player
2 – Column Player

First we consider pure strategy play.

Suppose row player plays $i \in S_1$.
Column player will play a strategy so as to

min

Column player will play a strategy ...

$$\min_{j \in S_2} a_{ij}$$

Row player now looks to maximize the above:

$$\max_{i \in S_1} \min_{j \in S_2} a_{ij}$$

This is called maximinimisation which is a consequence of rationality in matrix games.

Suppose column player chooses $j \in S_2$. Player 1 rationally chooses a strategy to

$$\max_{i \in S_1} a_{ij}$$

This is the maximum loss for player 2 who now rationally tries to minimize this loss:

$$\min_{j \in S_2} \max_{i \in S_1} a_{ij}$$

This is minmaximisation.

Thus, in matrix games, rationality of player 1 translates to maxminimization while the rationality of player 2 translates to minmaximization.

Example

$$\begin{bmatrix} 0,0 & -1,1 & 1,-1 \\ 1,-1 & 0,0 & -1,1 \\ --,1 & 1,-1 & 0,0 \end{bmatrix}$$

$\underline{v_1} = \max(-1,-1,-1) = -1$ All are maxmin strategies

$= 1$ All are minmax strategies

$\underline{v}_1 = \max(-1, -1, -1) = -1$ All are maximin strategies

$\overline{v}_1 = \min(1, 1, 1) = 1$ All are minmax strategies

$\underline{v}_2 = \max(-1, -1, -1) = -1$ All are maxmin strategies

$\overline{v}_2 = \min(1, 1, 1) = 1$ All are minmax strategies

Example 2

$$\begin{bmatrix} 100 & 200 & 100 \\ 0 & -100 & 200 \\ -100 & 0 & -200 \end{bmatrix}$$

$\underline{v}_1 = \max(100, -100, -200) = 100$ maxmin str $= 1$

$\overline{v}_1 = \min(100, 200, 200) = 100$ minmax str $= 1$ against 1

$\underline{v}_2 = \max(-100, -200, -200) = -100$ maxmin str $= 1$

$\overline{v}_2 = \min(-100, 100, 200) = -100$ $\left.\begin{matrix}\text{minmax} \\ \text{against } 2\end{matrix}\right\} = 1$

We denote

$$\underline{v}_1 = -\underline{v}_2 = \underline{v} = \max_{i \in S_1} \min_{j \in S_2} a_{ij}$$

$$\overline{v}_1 = -\overline{v}_2 = \overline{v} = \min_{j \in S_2} \max_{i \in S_1} a_{ij}$$

In general, we have seen that $\underline{v} \neq \overline{v}$

Of course, $\overline{v} \geq \underline{v}$.

If $\underline{v} = \overline{v}$, then the common number $v = \underline{v} = \overline{v}$ ... the value of the game.

of $\underline{v} = \overline{v}$, ...
is called the value of the game.

Saddle Point of a Matrix Game

In a matrix game A, a saddle point is any element a_{ij} which is simultaneously a column maximum and a row minimum:

$$a_{ij} \geq a_{kj} \quad \forall k = 1, 2, \ldots, m \quad \text{and}$$
$$a_{ij} \leq a_{i\ell} \quad \forall \ell = 1, 2, \ldots, n$$

Saddle Point $\equiv$ PSNE

Suppose a_{ij} is a saddle point:

$$\iff u_1(i, j) \geq u_1(i', j) \quad \forall i' \in S_1$$
$$\& \; u_2(i, j) \geq u_2(i, j') \quad \forall j' \in S_2$$

$\iff (i, j)$ is a PSNE.

Examples:

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

No Saddle points
$\underline{v} = -1 \qquad \overline{v} = 1$

$$\begin{bmatrix} \boxed{100} & 200 & 100 \\ 0 & -100 & 200 \end{bmatrix}$$

a_{11} is a saddle point
$\underline{v} = 100 \qquad \overline{v} = 100$

$$\begin{bmatrix} 0 & -100 & 200 \\ -100 & 0 & -200 \end{bmatrix} \qquad \underline{v} = 100 \qquad \overline{v} = 100$$

$$\begin{bmatrix} 5 & \boxed{3} & 5 & \boxed{3} \\ 2 & 1 & -1 & -2 \\ 4 & \boxed{3} & 5 & \boxed{3} \end{bmatrix}$$

$a_{12}, a_{14}, a_{32}, a_{34}$ are all saddle points.

$\underline{v} = 3 \qquad \overline{v} = 3$

An important result.

A matrix game has a saddle point iff $\underline{v} = \overline{v}$, that is, maxmin = minmax.

Another important result

If a_{ij} and a_{hk} are saddle points, then a_{ik} and a_{hj} are also saddle points.

An Important observation

$$\begin{bmatrix} \boxed{100} & 200 & 100 \\ 0 & -100 & 200 \\ -100 & 0 & -200 \end{bmatrix}$$

Saddle point (PSNE) is also robust to deviations by the other player.. This happens because a saddle point is also a row minimum. When the other player deviates, the current player is not worse off.

minimum. When the other player deviates,
the current player is not worse off.

Let's move on to mixed strategies.

$x = (x_1, x_2, \ldots, x_m)$ Mixed Strategy of row player

$y = (y_1, y_2, \ldots, y_n)$ Mixed strategy of Column player

$$u_1(x, y) = -u_2(x, y)$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij}$$
$$= xAy \quad \text{(in vector notation)}$$

When row player plays x, she is assured of

$$\min_{y \in \Delta(S_2)} xAy$$

The row player should therefore choose x such that

$$\max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_1)} xAy$$

Thus maxminimization is row player's rational (optimal) strategy.

When Column player plays y, he assures himself of losing no more than

$$\max_{x \in \Delta(S_1)} xAy$$

The column player should therefore choose a strategy y so as to minimize this loss:

$$\min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} xAy$$

Thus minmaximization is column player's rational (optimal) strategy.

We have already seen a Lemma which showed that max over mixed strategies is equal to max over pure strategies. We use that lemma to derive:

$$\max_{x \in \Delta(S_1)} xAy = \max_{i \in S_1} \sum_{j=1}^{n} a_{ij} y_j$$

$$\min_{y \in \Delta(S_2)} xAy = \min_{j \in S_2} \sum_{i=1}^{m} a_{ij} x_i$$

Thus the row player's optimization problem becomes:

$$\text{maximize} \quad \min_{j} \sum_{i=1}^{m} a_{ij} x_i$$

subject to

$$x_i \geq 0 \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_i = 1$$

P_1

The column player's optimization problem

The column player's optimization problem becomes:

$$\text{minimize} \quad \max_{i} \sum_{j=1}^{n} a_{ij} y_j$$

subject to:

$$y_j \geq 0 \qquad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} y_j = 1$$

P_2

Using a simple trick, these two problems P_1 and P_2 can be transformed into equivalent Linear Programming problems LP_1 and LP_2.

Row Player's optimization Problem

maximize z subject to

$$z \leq \sum_{i=1}^{m} a_{ij} x_i \qquad j = 1, 2, \ldots, n$$

$$x_i \geq 0 \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_i = 1$$

LP_1

Column Player's optimization problem:

minimize w subject to

$$w \geq \sum^{n} a_{ij} y_i \qquad i = 1, 2, \ldots, m$$

$$w \geq \sum_{j=1}^{n} a_{ij} y_j \qquad i = 1, 2, \dots, m$$

$$y_j \geq 0 \qquad j = 1, 2, \dots, n \qquad LP_2$$

$$\sum_{j=1}^{n} y_j = 1$$

Example : Rock – Paper – Scissors

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

LP_1 in this case would be

maximize z subject to

$z \leq x_2 - x_3$

$z \leq -x_1 + x_3$

$z \leq x_1 - x_2$

$x_1 + x_2 + x_3 = 1$

$x_1 \geq 0$; $x_2 \geq 0$; $x_3 \geq 0$

LP_2 in this case would be :

minimize w subject to

$w \geq -y_2 + y_3$

$w \geq y_1 - y_3$

$w \geq -y_1 + y_2$

$y_1 \geq 0 \; ; \; y_2 \geq 0 \; ; \; y_3 \geq 0$

$y_1 + y_2 + y_3 = 1$

An Impotant observation:

LP_1 and LP_2 are duals of each other

We invoke Strong Duality Theorem:

If an LP has an optimal solution, then its dual also has an optimal solution and moreover, the optimal values of the primal and dual are the same.

Minimax Theorem
(von Neumann and Oskar Morgenstern)

Given a matrix game A, there exist

$x^* = (x_1^*, x_2^*, \ldots, x_m^*)$

$y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ such that

$$\max_{x \in \Delta(S_1)} x A y^* = \min_{y \in \Delta(S_2)} x^* A y$$

Moreover, the profile (x^*, y^*) is a randomized saddle point (MSNE).

Proof

Consider the primal LP, LP_1, and its dual LP_2.

Easy to prove that LP_1 has an optimal solution (by looking at its structure). Suppose an optimal solution is given by

$z^*, x_1^*, x_2^*, \ldots, x_m^*$

LP_2 also has an optimal solution and suppose an optimal solution is given by

$w^*, y_1^*, y_2^*, \ldots, y_n^*$

By strong duality theorem, the optimal values are the same: That is,

$$z^* = w^*$$

Since $z^*, x_1^*, x_2^*, \ldots, x_m^*$ is an optimal solution of LP_1, we have

$$z^* = \sum_{i=1}^{m} a_{ij^*} x_i^* \quad \text{for some } j^* \in \{1, 2, \ldots, n\}$$

By feasibility,

$$\sum_{i=1}^{m} a_{ij^*} x_i^* \leq \sum_{i=1}^{m} a_{ij} x_i^* \quad \text{for } j = 1, 2, \ldots, n$$

Therefore

$$\sum_{i=1}^{m} a_{ij^*} x_i^* = \min_{j \in S} \sum_{i=1}^{m} a_{ij} x_i^*$$

$$\sum_{i=1}^{m} a_{ij^*} x_i^* = \min_{j \in S_2} \sum_{i=1} a_{ij} x_i^*$$

$$= \min_{y \in \Delta(S_2)} x^* \underline{A} y$$

Thus

$$z^* = \min_{y \in \Delta(S_2)} x^* \underline{A} y$$

on similar lines, we can show:

$$w^* = \max_{x \in \Delta(S_1)} x \underline{A} y^*$$

Since $z^* = w^*$, we have proved that

$$\max_{x \in \underline{\Delta}(S_1)} x \underline{A} y^* = \min_{y \in \underline{\Delta}(S_2)} x^* \underline{A} y$$

Now we show that (x^*, y^*) is a MSNE:

$$u_1(x^*, y^*) = x^* \underline{A} y^*$$

$$\geq \min_{y \in \Delta(S_2)} x^* \underline{A} y$$

$$= \max_{x \in \Delta(S_1)} x A y^*$$

$$\geq u_1(x, y^*) \quad \forall x \in \underline{\Delta}(S_1)$$

Similarly

$$u_2(x^*, y^*) = -x^* \underline{A} y^*$$

Similarly

$$
\begin{aligned}
u_2(x^*, y^*) &= -x^* A y^* \\
&\geq -\max_{x \in \Delta(S_1)} x A y^* \\
&= -\min_{y \in \Delta(S_2)} x^* A y \\
&\geq u_2(x^*, y) \quad \forall y \in \Delta(S_2)
\end{aligned}
$$

therefore (x^*, y^*) is a MSNE.

Example: Rock – Paper – Scissors

$$x_1^* = x_2^* = x_3^* = \frac{1}{3}$$

$$y_1^* = y_2^* = y_3^* = \frac{1}{3}$$

$$w^* = z^* = 0$$

Final Result:

Given a matrix game A, (x^*, y^*) is a MSNE iff

$$x^* \in \operatorname*{argmax}_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} x A y \quad \text{and}$$

$$y^* \in \operatorname*{argmin}_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} x A y$$

Moreover,

$$u_1(x^*, y^*) = -u_2(x^*, y^*)$$

Moreover,

$$u_1(x^*, y^*) = -u_2(x^*, y^*)$$

$$= \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} x \underline{A} y$$

$$= \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} x \underline{A} y$$

$\square$