Indian Institute of Science, Banglore
Department of Computational and Data Sciences (CDS)

**DS284: Numerical Linear Algebra**

Assignment 3 [Posted Oct 4, 2022]

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Submission Deadline: Oct 18, 2022 23:59 hrs | Max Points: 100

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**Notations:** Vectors and matrices are denoted below by bold faced lower case and upper case alphabets respectively.

## Problem 1 [15 marks]

(a) Geometrically, the orthogonal matrix is a matrix transformation that preserves 2-Norm of a matrix and causes rotation / reflection.

Can you justify $\mathbf{I} - 2\mathbf{P}$ is orthogonal matrix if $\mathbf{P}$ is orthogonal projector?

Prove the same algebraically as well.

(b) Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and its projector $\mathbf{P}$ which projects all vectors orthogonally on to column space of $\mathbf{A}$, then answer the following questions:

- If $\mathbf{A}$ is full rank, what is $\mathbf{P}$?
- Given $\mathbf{P}$ is there any way to find out the null space of $\mathbf{A}$?
- What can you say about the eigen-values of $\mathbf{P}$?

(c) If $\mathbf{P} \in \mathbb{R}^{m \times m}$ be a non-zero projection. Show that $\|\mathbf{P}\|_2 \geqslant 1$ with equality, if and only if $\mathbf{P}$ is orthogonal projector.

## Problem 2 [10 marks]

Given below is a magic matrix of size 3.

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

Find $\mathbf{Q}$ and $\mathbf{R}$ from QR factorization of given matrix by hand. Now do the following in Matlab/ Octave/ Python. For Matlab the command is:

[$\mathbf{Q}$, $\mathbf{R}$] = qr(magic(3))

Do these $\mathbf{Q}$ and $\mathbf{R}$ match your $\mathbf{Q}$ and $\mathbf{R}$? Is the QR factorization unique? If not unique, can you impose a condition on $\mathbf{R}$ to make the factorization unique?

## Problem 3 [10 marks]

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 \mid & x \mid & x^2 \mid & \dots \mid & x^{n-1} \end{bmatrix}$$

Each column is a function in $L^2[-1,1]$ i.e., a vector space of real-valued function on $[-1,1]$ which has inner-product of two functions $f$ and $g$ defined as:

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \tag{1}$$

If the QR factorization of $\mathbf{A}$ using the above definition of inner product can be written as

$$\mathbf{A} = \mathbf{QR} = \begin{bmatrix} q_0(x) \mid & q_1(x) \mid & q_2(x) \mid & \dots \mid & q_{n-1}(x) \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ 0 & 0 & \dots & r_{3n} \\ . & & & \\ . & & & \\ 0 & 0 & \dots & r_{mn} \end{bmatrix}$$

where columns of $\mathbf{Q}$ are functions of $x$, and are orthonormal with respect to inner product defined in equation (1). Now answer the following:

(a) Consider $n = 4$, derive expressions of $q_0(x), q_1(x), q_2(x), q_3(x)$ by using Gram Schmidt orthogonalization procedure.

(b) Show that $\int_{-1}^{1} q_{n-1}(x)dx = 0$ for $n \geqslant 2$.

Note that these $q_{n-1}(x)$ are called Legendre polynomials and the roots of these polynomials are called Gauss Legendre points and play a very important role in numerical integration as quadrature rules.

## Problem 4 [15 marks]

Let matrix:

$$\mathbf{A} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

(a) Consider a computer which rounds all computed results to five digits of relative accuracy. Using CGS or MGS, what will be the matrix $\mathbf{Q}$ associated with QR decomposition of $\mathbf{A}$ assuming that you are working on such a computer.

(b) Apply Householder's method to compute QR factorization of $\mathbf{A}$ using the same 5 digit arithmetic.

Compare the $\mathbf{Q}'s$ obtained in (a) and (b) and comment on orthogonal nature of the $\mathbf{Q}$ matrix.

## Problem 5 [20 marks]

Assuming that floating point properties (i) and (ii) described in Problem 4 of assignment 2 hold good, let us analytically find out the loss of orthogonality between two linearly independent unit normalized vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^m$, during Gram-Schmidt Orthogonalization.

(a) If $\mathbf{q}_1 = \mathbf{a}_1$, then mathematically we know that the vector orthogonal to $\mathbf{q}_1$ can be computed as $\mathbf{w}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1$. Denoting $\overline{r}_{12} = fl(\mathbf{q}_1^T \mathbf{a}_2)$, show that

$$|\overline{r}_{12} - r_{12}| \leqslant m\, \epsilon_{machine} + O(\epsilon_{machine}^2)$$

(b) If $\overline{\mathbf{w}}_2 = fl(\mathbf{a}_2 - fl(\overline{r}_{12}\mathbf{q}_1))$, Show that error in $\overline{\mathbf{w}}_2$ i.e.

$$|\overline{\mathbf{w}}_2 - \mathbf{w}_2| \leqslant (m+3)\epsilon_{machine} + O(\epsilon_{machine}^2)$$

(c) Assuming that normalization $\overline{\mathbf{q}}_2 = \frac{\overline{w}_2}{\overline{r}_{22}}$ and $\overline{r}_{22} = \|\overline{w}_2\|_2$ is carried out without error, show that

$$|\mathbf{q}_1^T \overline{\mathbf{q}}_2| \leqslant \frac{(m+3)\epsilon_{machine}}{\overline{r}_{22}}$$

[Note: If $\overline{r}_{22}$ is small, we can see from the equation that we incur considerable loss of orthogonality.]

## Problem 6 [20 marks]

In this problem you will test different algorithms for the least squares problem to approximate the function $f(t) = \sin(10t)$ for $t \in [0, 1]$ using a polynomial fit . To this end, first generate $m = 100$ data points using the above function which forms your given data i.e $(t_i, f(t_i)$ for $i = 1 \cdots m$. Using this data, we would like to construct a 14th degree least squares polynomial fit to $f(t)$. Determine its least square it using the following methods.

(a) Use QR Factorization with your implementation of Modified Gram Schmidt. You should write your own back substitution code for solving the resulting triangular system.

(b) Using QR Factorization with your implementation of Householder factorization.

(c) Using SVD (Computed with any inbuilt libraries in MATLAB/Python/Octave)

(d) Using normal equations, you can use backslash command in MATLAB to solve this system.

Accept the MATLAB/Octave/Python least squares solution (given by backslash "\" in MATLAB) as the truth. Display and plot the approximation given by this "true" solution and compare it with $f(t)$. Compare with the solution given by 4 methods described above. Explain the results.

## Problem 7 [10 marks]

Given below is the matrix **A** and vector **b**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 8 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the least square solution for the above. Now do the same for the below matrix also.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

Were you able to find the least square solution? If not explain why? Must there be a restriction on **A** for a least square solution to exist or will it always exist?