Contract Theory

Economic tool to incentivize an (strategic) agent to put in effort.

"How to incentivize people to put in effort"

[Paying for the effort's outcomes]

Examples:

1. Employment Contract
2. VC investment Contract
3. Insurance Contract
4. Freelance Contract
5. Government Procurement Contract

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Hart & Holmström
2016 Nobel
"Modern Economies held together by innumerable contracts"

Online & Increasingly becoming complex.
"Modern" Examples

1. Crowdsourcing Platforms
2. Online Marketing
3. Complex Supply Chains
4. Pay-for-Performance Medicare.

Basic Model

[Holmström '79].

- Two Players: Principal & Agent
- Ingredients: Private Information & Incentives.

Ex: Website Owner hires : Principal.
Marketing Exec. to attract ~~Vis~~ visitors : Agent

Right level of compensation to get a balance between how much the principal has to pay & the agent's effort.

Two defining features. (illustrated by the example):

① Agent's actions are hidden - "Moral Hazard"

② Principal never charges (only pays) agent - "Limited liability"

[Moral Hazard: Situation in which the agent is exerting some action, but not reaping the reward of the action.
The principal is getting the reward.]

Input → $\boxed{\text{Mechanism}}$ → Output

Reported by strategic players.

Output: Who gets what & for how much?

Input → $\boxed{\text{Contract}}$ → Output

Output: Who does what & for how much?

Carried out by strategic Players.

Design Goal: Maximize objective (welfare/revenue etc.) subject to incentive constraints.
(truth telling or effort).

Contract Design under Uncertainty : Model

- Agent has n actions (effort levels) with costs $c_1, c_2, \dots, c_n$.
- Principal has m rewards $r_1, r_2, \dots r_m \in \mathbb{R}_+$
- Every action i induces distribution F_i over $\vec{r}$
  - $F_{i,j}$ = prob. that action a_i yields reward r_j
  - $R_i = \sum_j F_{i,j} r_j$ = Expected reward from action i.

$$\begin{array}{c|cccc} & r_1 & r_2 & \dots & r_m \\ \hline 1 & & -F_1- & & \\ 2 & & -F_2- & & \\ 3 & & \vdots & & \\ \vdots & & & & \\ n & & -F_n- & & \end{array}$$

$(\vec{c}, \vec{r}, (F_1, F_2 \dots F_n))$

Contract: $\vec{t}$

Specifies a payment $t_j \geq 0$ per reward r_j

Exp. payment for action i
- $T_i = \sum_j t_j F_{i,j}$
- $R_i = \sum_j r_j F_{i,j}$

[Imp: Transfer depends on reward. NOT on effort level.]

Effort is unobservable.

Agent selects effort i that maximizes her exp. utility

$$\underset{\text{Exp. payment}}{T_i} - \underset{\text{cost}}{c_i}$$

- Principal's Exp. Revenue from agent's choice ~~of~~ $i \in [n]$

$$\underbrace{R_i}_{\text{Expected Reward.}} - \underbrace{T_i}_{\text{Exp Payment.}}$$

Timing:

				TIME →
Setting. $\vec{c}, \vec{r}, (f_1 .. f_n)$ known to all	Principal Designs Contract. $\vec{t}$	Agent takes action i @ cost c_i (Hidden Action)	Principal gets reward. $r_j \sim F_i$	Principal pays agent t_j

We say that a contract is optimal. if it maximizes principal's exp revenue.

Landscape of Micro Econ (Other Incentive Problems in the field).

[Salanie]

	Uninformed player sets the designs	Informed Player sets the design.
Private Info is hidden type	Mech. Design.	Signaling (Persuasion).
Private Info is hidden action	Contract Design.	—.

Contract Setting:

- Parameter n, m
- Agent has actions $1, 2, \ldots n$ with costs

$$0 = c_1 \le c_2 \le c_3 \le \ldots \le c_n$$

- Principal has rewards

$$0 \le r_1 \le r_2 \le \ldots \le r_m$$

- Action i induces dist. F_i (sometimes called "technology"). with expectation R_i.

Assumption: $R_1 \le R_2 \ldots \le R_n$.

- Contract := vector of transfers.

$$\vec{t} = (t_1\ t_2 \ldots t_m)$$

[Recall two defining features]

Example:

	$r_1 = 0$	$r_2 = 3$	$r_3 = 7$	$r_4 = 10$
① "Low Effort" $c_1 = 0$	0.72	0.18	0.08	0.02
② "Medium Effort" $c_2 = 1$	0.12	0.48	0.08	0.32
③ "High Effort" $c_3 = 2$	0	0.4	0	0.6

$R_1 = 1.3$
$R_2 = 5.2$
$R_3 = 7.2.$

Example Contract t $\quad t_1 = 0 \quad t_2 = 1 \quad t_3 = 2 \quad t_4 = 5$

Example

$n = m = 2.$

	$r_1 = 1$	$r_2 = 3$	
Action 1. Cost $c_1 = 0$	1	0	$R_1 = 1$
Action 2 Cost $c_2 = 4/3$	0	1.	$R_2 = 3.$

[f_1, f_2 Point Masses]

- Best way to incentivize action 1. is to have (Min Pay)

  $\vec{t} = (0, 0) \rightarrow$ Expected Revenue $= 1$.

- Best way to incentivize action 2. is to pay.

  $\vec{t} = (0, 4/3) \rightarrow$ Expected. Revenue $= 3 - 4/3 = 5/3$.

Here, Revenue opt. Contract $= (0, 4/3)$.

Computing the Optimal Contract

Min-Pay Problem.

- Input: Contract setting $(\vec{c}, \vec{r}, \vec{F})$; an action i.
- Output: Minimum T_i that incentivizes action i.

Linear Program:

$$\min_{(t_1, t_2 .. t_m)} \sum_{j=1}^{m} F_{i,j}\, t_j$$

s.t.

$$\sum_{j=1}^{m} F_{i,j}\, t_j - c_i \geq \sum_{j=1}^{m} F_{a,j}\, t_j - c_a \qquad \forall \text{ actions } a \in [n]$$

$$t_1, t_2 .. t_m \geq 0.$$

Optimal Contract solvable via n Min Pay problems.

Example Non Monotonicity $(n, m > 2)$

	$r_1 = 1$	$r_2 = 1.1$	$r_3 = 4.9$	$r_4 = 5$	$r_5 = 5.1$	$r_6 = 5.2$
$c_1 = 0$	$3/8$	$3/8$	$2/8$	0	0	0
$c_2 = 1$	0	$3/8$	$3/8$	$2/8$	0	0
$c_3 = 2$	0	0	$3/8$	$3/8$	$2/8$	0
$c_4 = 2.2$	0	0	0	$3/8$	$3/8$	$2/8$

Optimal Contract

$$\vec{t}^* = (t_1^* = 0 \quad t_2^* = 0 \quad t_3^* = 0.15 \quad t_4^* = 3.9 \quad t_5^* = 2 \quad t_6^* = 0)$$

Unusual

Example (1) with contract

$t_1 = 0 \quad t_2 = 1 \quad t_3 = 2 \quad t_4 = 5$

Expected Transfers Action 1 : 0.44 (Low)
2 : 2.24 (Med)
3 : 3.4. (High).

$\therefore$ Agent selects.
action 3. with exp utility $= 3.4 - 2$
$= 1.4.$

Principal's Revenue. $= 7.2 - 3.4$
(with agent selecting action 3) $= 3.8.$

Recall
$R_1 = 1.3$
$R_2 = 5.2$
$R_3 = 7.2$

First Best = Solution ignoring IC constraints

$$\text{First Best (Opt. Welf.)} = \max_{1 \le i \le n} \{ R_i - c_i \}$$

! Opt (Revenue) $\neq$ First Best.

Implementability Problem.

Given action i, Does there exist a contract $\vec{t}$ such that under it, action i is utility maximizing for the agent.

Dual of Min Pay

· **Lemma:** Action i is implementable (upto tie breaking) iff. no convex combination of the other actions has same distribution over the rewards at a lower cost.

Pf: Action i implementable iff. following LP feasible.

m decision variables $\{t_1, t_2, \ldots, t_m\}$; $n-1$ constraints

$$\min \; 0$$

s.t.

$$\sum_{j=1}^{m} F_{i,j}\, t_j - c_i \geq \sum_{j=1}^{m} F_{a,j}\, t_j - c_a \quad \forall a \in [n],\ a \neq i$$

$$t_j \geq 0 \quad \forall j \in [m]$$

Primal Infeasible iff. $\exists$ feasible dual soln. with objective > 0

$$\max. \quad c_i - \sum_{a \neq i} \lambda_a c_a$$

Combined cost.

s.t.

$$\sum_{a \neq i} \lambda_a F_{a,j} \leq F_{i,j} \quad \forall j \in [m]$$

$$\lambda_a \geq 0 \qquad \sum_{a \neq i} \lambda_a = 1.$$

Convex Comb of actions

COR: There exists an optimal contract with $\leq (n-1)$ nonzero transfers.

Criticism of LP-Based Approach

"More Normative than Positive"

1. Requires perfect knowledge of F.
2. Dependence on $n \cdot m$ might be prohibitive.
3. The contract that comes out of LP might be arbitrary.

$\Longrightarrow$ **Structure of Optimal Contracts**

(I) Optimal Contracts for 2 actions & 2 rewards.

$(n = m = 2)$

		Failure r_1 <	Success r_2.	$(r_2 > r_1)$
(low)	Cost $c_1 = 0$	$1-p$	p.	
(High)	Cost $c_2 = c > 0$	$1-\pi$.	π	

Principal can always contract.

$R_1 = (1-p) r_1 + p \cdot r_2$.

First-Best $= R_2 - c$.

$= r_1(1-\pi) + \pi r_2 - c$

Assume that opt. contract incentivizes action 2 (High).

<u>Notation:</u> Contract pays t_1 for failure, t_2 for success.

IC for working is:

$$t_1(1-\pi) + \pi t_2 - c \geq t_1(1-p) + t_2 p.$$

$$(\pi - p)(t_2 - t_1) \geq c \quad \text{——} \quad (*)$$

(*) binds at optimal contract.

Also, to minimize transfer
(opt wrt revenue).

must have $\underline{t_1 = 0}$.

(*) gives us $t_2 = \dfrac{c}{\pi - p}$.

Hence, Exp Rev of Principal $= R_2 - \dfrac{c}{\pi - p} \cdot \pi$

[Compare to first best $R_2 - c$.]

Opt Contract has a "closed form" here.

$t_1 = 0 \qquad t_2 = \dfrac{c\pi}{\pi - p}$.

Monotonicity holds here $t_2 > t_1$.
(transfer increases with reward).

[Generalizes to any n as long as $m = 2$]

II Optimal Contract for $n=2$ actions, m Rewards.

	r_1	r_2	$\cdots$	r_m
Action 1	$F_{1,1}$	$F_{1,2}$	$\cdots$	$F_{1,m}$
Action 2	$F_{2,1}$	$F_{2,2}$	$\cdots$	$F_{2,m}$

Recall: There exists an optimal contract with $(n-1)$ nonzero transfers.

Binding IC constraint for action 2 (the action with non zero cost).

$$t_{j^\circ} F_{2,j^\circ} - c = t_{j^\circ} F_{1,j^\circ}$$

for some j° Optimal Contract is $t_{j^\circ} = \dfrac{c}{F_{2,j^\circ} - F_{1,j^\circ}}$.

Principal extracts

$$= R_2 - c \cdot \frac{F_{2,j^\circ}}{F_{2,j^\circ} - F_{1,j^\circ}}$$

That is, principal extracts.

$$R_2 - c \frac{1}{\left(1 - \frac{F_{1,j^\circ}}{F_{2,j^\circ}}\right)}$$

Hence, choose j^* that ~~maximizes~~ minimizes $\left\{ \dfrac{F_{1,j}}{F_{2,j}} \right\}$.

This ratio is called the likelihood ratio. (b/w actions 1 & 2).

Takeaway: optimal contract pays for reward with min likelihood ratio.

Recap:

Q: Which reward r_j gets the nonzero transfer t_j in opt. contract.

Ans: Pay for r_j with min likelihood ratio $\frac{f_{1,j}}{f_{2,j}}$

[Statistical Inference intuition:

Principal is inferring agent's action from the reward.

→ Pays more for reward which can infer agent is working

- Extreme example. ($n=2$ $m>2$).

Assume $r_{\hat{j}}$ has nonzero prob ε only if agent "works" (chooses action 2) outcome

That is,

$r_{\hat{j}}$ occurs, "gives away" agent's actions.

- optimal contract has single nonzero transfer $t_{j^*} = \frac{c}{\varepsilon}$.

The good: Principal extracts the first best $R_2 - c$.

The bad: Contract is non-monotone.

Recap: Role of rewards in the model is two-fold:

(1.) Represent surplus to be shared.
(2) Signal to the principal the agent's action.

- The optimal contract is shaped by (2).
- Can be mismatched with (1).

Optimal Contract	2 rewards	m rewards
2 actions.	Monotone & Pay for reward with min likelihood ratio	Pay for reward with min likelihood ratio
n actions	Monotonicity	Strong assumption needed.