Stability of Algorithms:-

(1) Desirable:- Exact solutions to numerical problems

(2) Reality:- Problems are continuous while computer arithmetic is discrete

(3) Stability tells us what it means to get the best answer even if this is not the exact answer!

Algorithm:-

An abstract way to think about solving a problem is evaluating function

$f: X \to Y$ $\quad$ X: vector space of data

Y: vector space of solutions

$y = f(x)$

where $x \in X$, $y \in Y$

An algorithm can be viewed as a different

function $\tilde{f}$ that usually takes the same data $x \in X$ and maps it to a result which is a collection of floating point numbers that belongs to Y.

Accuracy:- A good algo should have an $\tilde{f}$ that closely approximates the underlying problem f.

→ Absolute error of computation:-

$$\|\tilde{f}(x) - f(x)\|$$

→ Relative error of computation:-

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$$

We say that $\tilde{f}$ is an accurate algorithm for f if for all relevant input data x,

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\varepsilon_M)$$

Forward relative error

"On the order of machine epsilon"

If f is ill-conditioned

i.e $\max_{\delta x} \frac{\frac{\|\delta f\|}{\|f\|}}{\frac{\|\delta x\|}{\|x\|}} = \hat{K}$ is very large.

$$\frac{\|\delta x\|}{\|x\|} \simeq O(\varepsilon_M)$$

$$\frac{\|\delta f\|}{\|-\|} \leq K\, O(\varepsilon_M)$$

Very ambitious to design an algo $\tilde{f}$ such that relative error in computation is $O(\varepsilon_M)$

Instead of aiming for accuracy of an algo, the most we can aim is for stability!

We can say that an algorithm $\tilde{f}$ for solving a problem f is stable if for all (relevant) input data x

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_M)$$

for some $\tilde{x}$ satisfying $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_M)$

i.e A stable algorithm gives nearly right answer to nearly right question.

Backward stability :-

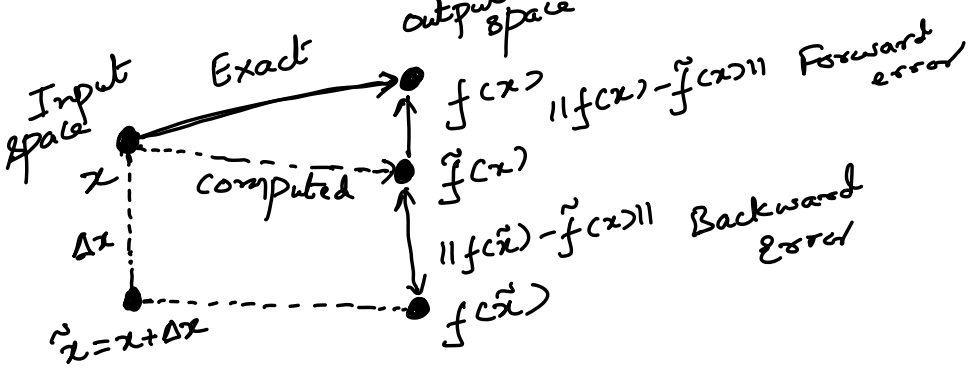
Backward error :- $\|f(\tilde{x}) - \tilde{f}(x)\|$

What is the input $\tilde{x}$ for which my algo with input x has exactly computed a solution for

$$f(\tilde{x}) = \tilde{f}(x)$$

Then such an algo $\tilde{f}$ for problem f is backward stable. (i.e Backward error is zero)

i.e Exactly right answer to a nearly right question!



Stability of Floating point arithmetic operations!

$+, -, \times, /$ (Classical arithmetic operations)

Floating point analogues $\oplus, \ominus, \otimes, \oslash$

Recall Floating point axioms :-

$$fl(x) = x(1+\varepsilon) \text{ where } |\varepsilon| < \varepsilon_M$$

$$x \circledast y = x * y(1+\varepsilon) \text{ where } |\varepsilon| < \varepsilon_M$$

Example 1 :-

Floating point arithmetic for $\ominus$

$\underset{\sim}{x} \in \mathbb{R}^2$ $\quad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$f(\underset{\sim}{x}) = x_1 - x_2$

$f(x_1, x_2) = x_1 - x_2$

$x_1 \rightarrow fl(x_1)$; $x_2 \rightarrow fl(x_2)$

$fl(x_1) = x_1(1+\varepsilon_1)$; $fl(x_2) = x_2(1+\varepsilon_2)$

$\varepsilon_1, \varepsilon_2$ satisfy $|\varepsilon_1|, |\varepsilon_2| = O(\varepsilon_M)$

Algo :- $fl(x_1) \ominus fl(x_2)$

$\Rightarrow [x_1(1+\varepsilon_1)] \ominus [x_2(1+\varepsilon_2)]$

$$fl(x_1) \ominus fl(x_2) = [x_1(1+\varepsilon_1) - x_2(1+\varepsilon_2)](1+\varepsilon_3)$$

where $|\varepsilon_3| = O(\varepsilon_M)$

$$= x_1(1+\varepsilon_1)(1+\varepsilon_3) - x_2(1+\varepsilon_2)(1+\varepsilon_3)$$

$$= x_1(1+\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3) - x_2(1+\varepsilon_2+\varepsilon_3+\varepsilon_2\varepsilon_3)$$

$$= x_1(1+\varepsilon_4) - x_2(1+\varepsilon_5)$$

where $|\varepsilon_4|, |\varepsilon_5| \leq 2\varepsilon_M + O(\varepsilon_M^2) = O(\varepsilon_M)$

$$\tilde{f}(x_1, x_2) = f(x_1) \ominus f(x_2)$$

$$= \tilde{x}_1 - \tilde{x}_2 = f(\tilde{x}_1, \tilde{x}_2)$$

Algo is backward stable!

Example 2:-

Outer product between two vectors :-

$\underset{\sim}{x} \in \mathbb{R}^m$, $\underset{\sim}{y} \in \mathbb{R}^n$

compute outer product $\underset{\sim}{A} = \underset{\sim}{x}\,\underset{\sim}{y}^T$

$\boxed{A_{ij} = x_i y_j}$ = $m \times 1 \; 1 \times n$

$\underset{\sim}{A}_{m \times n}$

↑ rank 1 matrix

$$\tilde{f}(\underset{\sim}{x}, \underset{\sim}{y}) = \tilde{A}_{ij}$$

$$= f(x_i) \otimes f(y_j)$$

$$\tilde{A}_{ij} = f(x_i) \otimes f(y_j)$$

$$= [f(x_i) \times f(y_j)](1+\varepsilon_3^{ij})$$

$$= [x_i(1+\varepsilon_1^i)\, y_j(1+\varepsilon_2^j)](1+\varepsilon_3^{ij})$$

$$= x_i y_j (1+\varepsilon_1^i)(1+\varepsilon_2^j)(1+\varepsilon_3^{ij})$$

$$= x_i y_j (1+\varepsilon_1^{i}+\varepsilon_2^{j}+\varepsilon_1^{i}\varepsilon_2^{j})(1+\varepsilon_3^{ij})$$

$$= x_i y_j (1+\varepsilon_4^{ij})(1+\varepsilon_3^{ij}) \leftarrow (1)$$

I want to verify

$$\tilde{f}(\underset{\sim}{x},\underset{\sim}{y}) = \tilde{\underset{\sim}{x}}\,\tilde{\underset{\sim}{y}}^{T} \leftarrow (*)$$

$$= (\underset{\sim}{x}+\delta\underset{\sim}{x})(\underset{\sim}{y}+\delta\underset{\sim}{y})^{T}$$

This algo is not backward stable because from eqn (1), it is not possible for my algo always to give a perturbed rank 1 matrix of the form $\tilde{\underset{\sim}{x}}\,\tilde{\underset{\sim}{y}}^{T}$

Is this algo stable? Exercise!

Example 3:-

Adding 1 to floating point number!

Let $x \in \mathbb{R}$ and $f(x) = x+1$

$$\tilde{f}(x) = fl(x) \oplus 1$$

$$\tilde{f}(x) = (fl(x)+1)(1+\varepsilon_1)$$

$$= (x(1+\varepsilon_2)+1)(1+\varepsilon_1)$$

$$= (1+x+x\varepsilon_2)(1+\varepsilon_1)$$

$$= 1 + x + x\varepsilon_2 + \varepsilon_1 + \varepsilon_1 x + x\varepsilon_2\varepsilon_1$$

$$= 1 + x\left[1 + \varepsilon_1 + \varepsilon_2 + \frac{\varepsilon_1}{x}\right] \quad - (2)$$

For backward stability is $\tilde{f}(x) = f(\tilde{x})$

$$= \tilde{x} + 1$$

$$= x(1+\varepsilon) + 1 \; ?$$

From eq (2)

$$1 + x\left[1 + \varepsilon_1 + \varepsilon_2 + \frac{\varepsilon_1}{x}\right]$$

As $x \to 0$, $\varepsilon \neq O(\varepsilon_M)$

This is not backward stable

Unstable Algos :-

→ Calculation of eigenvalues of a matrix by finding roots of characteristic polynomial

→ Since λ is eigenvalue of a matrix $\underset{\sim}{A}$, then the characteristic polynomial is $p(\lambda) = \det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$

roots $p(\lambda) = 0$ are eigenvalues of A

Algo :-

(1) Find the coefficients of $p(\lambda) = \det(A - \lambda I)$

(2) Find its roots

The problem of finding roots of a polynomial given its coefficients is an ill-conditioned problem!

i.e roots are sensitive to small errors in polynomial coefficients!

Eg :- $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Recall $K = \frac{\|A\|_2}{\lambda}$

Eigenvalues of A are not sensitive to perturbations of the entries. If we design a stable algo, we should be able to compute eigenvalues with relative errors of $O(\varepsilon_M)$

Let us see the order of error in roots if we compute the roots of characteristic polynomial for the above problem:

$\Rightarrow \quad x^2 - 2x + 1$

Let us analyse the error in roots for perturbation in the coefficient p for $x^2 - px + 1$

$$\text{Roots} :- \quad \frac{p \pm \sqrt{p^2 - 4}}{2}$$

Let us say my error in estimating the coefficient p be $|\varepsilon| < \varepsilon_m$

$$\tilde{p} = p(1+\varepsilon)$$

$$\text{New Roots} :- \quad \frac{p(1+\varepsilon) \pm \sqrt{[p(1+\varepsilon)]^2 - 4}}{2}$$

$$= \frac{p(1+\varepsilon) \pm \sqrt{p^2(1+\varepsilon)^2 - 4}}{2}$$

For our problem at hand $p = 2$

$$\text{New roots for } p=2 :- \quad \frac{2(1+\varepsilon) \pm \sqrt{4(1+\varepsilon)^2 - 4}}{2}$$

$$= (1+\varepsilon) \pm \sqrt{1 + \varepsilon^2 + 2\varepsilon - 1}$$

$$\Rightarrow (1+\varepsilon) \pm \sqrt{\varepsilon^2 + 2\varepsilon}$$

Error in root :- $\frac{|(1+\varepsilon) \pm \sqrt{2\varepsilon - 1}|}{|1|}$

Dominant Error in my root $\simeq O(\sqrt{\varepsilon_m})$

$\varepsilon_m \simeq 10^{-16}$

$\sqrt{\varepsilon_m} \simeq 10^{-8}$

Accuracy of Backward stable Algo :-

Thm :- If a backward stable algo is applied to solve a problem f with condition number K, the relative forward errors satisfy

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(K(x)\varepsilon_m)$$

Pf :- By definition of backward stability we have $\tilde{f}(x) = f(\tilde{x})$

for $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_m)$

we also know

$$K(x) = \max_{\delta x} \frac{\frac{\|\delta f\|}{\|f\|}}{\frac{\|\delta x\|}{\|x\|}}$$

$$\frac{\left(\frac{\|f(\tilde{x}) - f(x)\|}{\|f(x)\|} \right)}{\frac{\|\tilde{x} - x\|}{\|x\|}} \leq K(x)$$

But B.S, $f(\tilde{x}) = \tilde{f}(x)$

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} \leq K(x) \frac{\|\tilde{x} - x\|}{\|x\|}$$

$\uparrow$ $O(\varepsilon_m)$

$$O(K(x)\,\varepsilon_m)$$