Lecture 19. Coalitional Games. (TU Games)

09 May 2021 17:08

Coalitional Games
characteristic form Games
Tranferrable Utility Games (TU games)

Motivating Example: Divide_the_Dollar

$N = \{1, 2, 3\}$

Total wealth of 300 to be divided among the players

$$S_1 = S_2 = S_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 \leq 300 ; x_1 \geq 0 ; x_2 \geq 0 ; x_3 \geq 0\}$$

Version 1:

All players get a zero payoff unless all of them propose the same allocation.

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ &= 0 \quad \text{else} \end{aligned}$$

Version 2:

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ &= 0 \quad \text{else} \end{aligned}$$

Version 3:

$$\begin{aligned} u_i(s_1, s_2, s_3) = x_i \quad &\text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ &\text{or } s_1 = s_3 = (x_1, x_2, x_3) \end{aligned}$$

or $s_1 = s_3 = (x_1, x_2, x_3)$

$= 0$ else

Version 4 (Majority Voting Game)

$$u_i(s_1, s_2, s_3) = x_i \quad \text{if } s_1 = s_2 = (x_1, x_2, x_3)$$
$$\text{or } s_1 = s_3 = (x_1, x_2, x_3)$$
$$\text{or } s_2 = s_3 = (x_1, x_2, x_3)$$
$$= 0 \quad \text{else}$$

The four versions represent different bargaining powers for different players. The final payoffs to the players after the negotiations conclude will be different for different versions.

If the Nash bargaining theory is naively extended to this case with $(v_1, v_2, v_3) = (0, 0, 0)$, the solution for all the versions turns out to be $(100, 100, 100)$.

Analysis of the coalitional dynamics is interesting and challenging. For example, in version 4,

$$(150, 150, 0) \Longrightarrow (0, 250, 50)$$
$$\Longrightarrow (200, 0, 100)$$
$$\Longrightarrow (0, 100, 200)$$
$$\Longrightarrow (190, 110, 0)$$

$\Rightarrow (190, 110, 0)$
$\Rightarrow (0, 120, 180) \ldots$

In order to capture coalitional dynamics in a succinct way, TU games or characteristic form games have been proposed.

Transferable Utility (TU) Games
Characteristic Form Games

(N, v)

$N = \{1, 2, \ldots, n\}$

$v : 2^N \to \mathbb{R}$ with $v(\phi) = 0$

"worth" or "value"

$v(C)$ for any $C \subseteq N$ is the total value that players of C are capable of raising by themselves, without any assistance from players in $N \setminus C$.

Example 1: Divide-the-Dollar

Version 1:

$$v(\phi) = v(\{1\}) = v(\{2\}) = v(\{3\})$$
$$= v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\})$$
$$= 0$$

$= v(\{1,2\}) = v(\{1,3\}) = \dots$

$= 0$

$v(\{1,2,3\}) = 300$

version 2:

$v(\phi) = v(1) = v(2) = v(3) = v(13) = v(23) = 0$

$v(12) = v(123) = 300$

version 3:

$v(\phi) = v(1) = v(2) = v(3) = v(23) = 0$

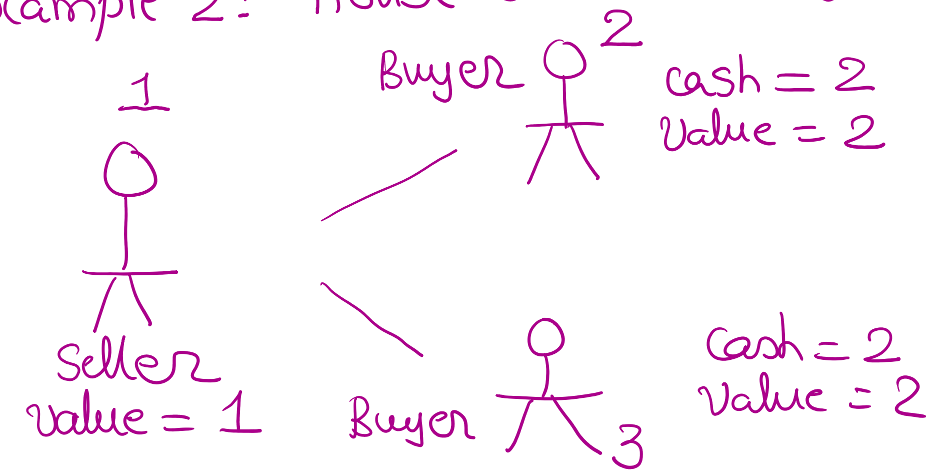
$v(12) = v(13) = v(123) = 300$

version 4:

$v(\phi) = v(1) = v(2) = v(3) = 0$

$v(12) = v(13) = v(23) = v(123) = 300$

Example 2: House Allocation



$1 \leq p \leq 2$

p = price at which house is sold

$v(1) = 1$

$v(2) = 2$

$v(3) = 2$

$v(12) = p + (2 - p + 2) = 4$

$v(13) = p + (2 - p + 2) = 4$

$v(23) = 2 + 2 = 4$

$v(123) = p + 2 + 2 + (2 - p) = 6$

Example 3: Glove Market

$N = \{1, 2, 3, 4, 5\}$

$N_L = \{1, 2\}$ Left gloves

$N_R = \{3, 4, 5\}$ Right gloves

$v(C) = \#$ of matched pairs of gloves

$= \min\left(|N_L \cap C|, |N_R \cap C|\right)$

$v(1) = v(2) = v(3) = v(4) = v(5) = v(12) = v(34)$
$= v(35) = v(45) = 0$

$v(13) = v(14) = v(15) = v(23) = v(24) = v(25) = 1$

$\vdots$

$v(1234) = v(1245) = v(1235) = v(12345) = 2$

$$v(1234) = v(1245) = v(1235) = v(12345) = 2$$

Monotone Games

$$v(C) \leq v(D) \quad \forall\, C \subseteq D \subseteq N$$

eg: Majority voting, House allocation

Superadditive Games

$$v(C \cup D) \geq v(C) + v(D) \qquad \forall\, C, D \subseteq N, \; C \cap D = \phi$$

Union of non-overlapping coalitions produces more value than the individual coalitions combined.

eg: Majority voting, House Allocation

Convex Games

$$v(C \cup D) \geq v(C) + v(D) - v(C \cap D) \quad \forall\, C, D \subseteq N$$

Alternately,

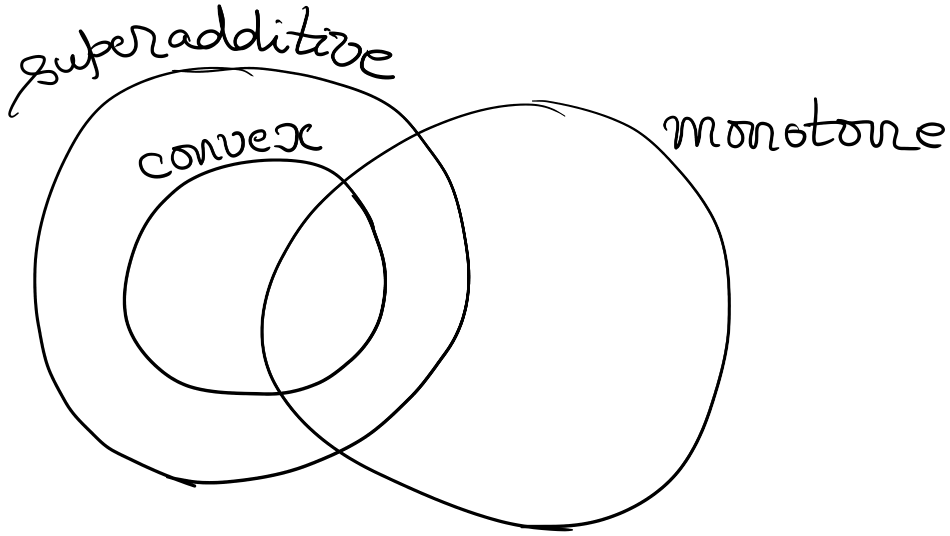
$$v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D) \quad \forall\, C \subseteq D \subseteq N \quad \forall\, i \notin D$$

Marginal contribution of a new player to a larger coalition is larger than that of the player for a smaller coalition

eg: House Allocation

player for a smaller coalition

eg: House Allocation



Majority voting game is not convex since:

$$v(13) - v(1) = 300 - 0 = 300$$

$$v(123) - v(12) = 300 - 300 = 0$$